2025-11-10

Differential Equations

Bhashkar Paudyal

Abstract
This course was taught by Prof. Dil Bahadur Gurung. It teaches Differential Equations

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1 Introduction to Differential Equations

1.1 Need of Differential Equations

Differential equations are essential tools for modeling systems that involve **change** or **motion**. They help us describe the relationships between a function and its derivatives — enabling prediction of future behavior of systems based on current conditions.

Some common applications include:

- Modeling population growth and decay
- Describing motion of objects (Newton's laws)
- Representing electrical circuits
- Modeling spread of diseases or epidemics
- Predicting stock prices and economic changes

1.2 Use of Differential Equation in Data Science

In **Data Science**, differential equations play a crucial role in:

- Modeling **continuous-time processes** such as temperature change or chemical concentration.
- Describing **dynamic systems** used in time series forecasting and control systems.
- Optimizing **machine learning models** (e.g., gradient descent as an iterative differential process).
- Defining **neural differential equations**, where model parameters evolve according to differential laws.

Mathematical models derived from data often lead to **ordinary differential equations (ODEs)** or **partial differential equations (PDEs)** that capture relationships among variables.

1.3 Differential Equations

Definition 1.3.1 (Differential Equation)

A **Differential Equation** is an equation that relates a function with one or more of its **derivatives**. It expresses how a quantity changes with respect to another.

Mathematically, a differential equation can be represented as:

$F\left(x, y, y', y'', ..., y^{\{(n)\}}\right) = 0$

where:

- x is the **independent variable**,
- y is the **dependent variable** (function of x),
- $y', y'', ..., y^{\{(n)\}}$ are the **derivatives** of y with respect to x.

For example: $y'' + 3y' + 2y = 0$

is a **second-order linear homogeneous differential equation**.

♣

1.4 Classification of Differential Equations

Differential equations can be classified based on the following criteria:

- **Order**: The highest derivative present in the equation.
- **Degree**: The power of the highest derivative (when the equation is polynomial in derivatives).
- **Linearity**: Whether the dependent variable and its derivatives appear linearly.

For example:
- First-order ODE: $y' + 2y = 5$
- Second-order nonlinear ODE: $y'' + y(y')^2 = 0$

2 Solution of Differential Equations

2.1 Types of Variables in Differential Equations

Differential equations involve different types of variables that define how the function behaves and changes. These variables are essential for setting up the correct model.

2.1.1 Independent Variables

The **independent variable** is the quantity upon which other quantities depend. It is typically represented as **x**, **t**, or **r** depending on the context.

Example: In $y' = 3x^2$, the variable **x** is the independent variable.

2.1.2 Dependent Variables

The **dependent variable** is the function that depends on the independent variable. It represents the unknown quantity we aim to find.

Example: In $y' = 3x^2$, **y** is the dependent variable because its value changes according to **x**.

2.1.3 Parameters

A **parameter** is a constant that affects the behavior or family of solutions but remains fixed for a particular solution. Parameters determine the shape or scale of the solution curve.

Example: In $y = Ce^{2x}$, **C** is a parameter.

Definition 2.1.1 (Solution of Differential Equation)

The **solution of a differential equation** is a function that satisfies the equation for all values of the independent variable within a given interval.

Formally, if a function $y = \varphi(x)$ satisfies the equation

$$F\left(x, y, y', y'', ..., y^{\{(n)\}}\right) = 0 \tag{1}$$

, then $y = \varphi(x)$ is called the **solution** of the differential equation.

♣

2.2 General Solution

A **general solution** of a differential equation includes **arbitrary constants** that represent an infinite family of solutions. It contains as many arbitrary constants as the **order** of the differential equation.

Example: For $\frac{dy}{dx} = 2x$, Integrating both sides gives $y = x^2 + C$, where **C** is an arbitrary constant representing the general solution.

2.3 Order of Differential Equation

The **order** of a differential equation is the **highest derivative** of the dependent variable present in the equation.

Example:
- $\frac{dy}{dx} + y = 0 \longrightarrow$ **First order**
- $\frac{d^2y}{dx^2} + 3y = 0 \longrightarrow$ **Second order**

Mathematically, for a differential equation

$$a_{n(x)}\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{2}$$

the **order** is **n**.

2.4 Integral Curves

Definition 2.4.1 (Integral Curves)

The **integral curve** of a differential equation is the **graph of its solution function** in the coordinate plane.

It represents the set of all points (x, y) that satisfy the differential equation. Different values of the arbitrary constant **C** produce different integral curves from the general solution.

Example: For $y = x^2 + C$, each value of **C** gives a distinct **parabola**, and all such parabolas together form a **family of integral curves**.

♣

2.5 Initial Value Problem

Definition 2.5.1 (Initial Value Problem (IVP))

An **Initial Value Problem (IVP)** consists of a differential equation together with an **initial condition** that specifies the value of the dependent variable (and possibly its derivatives) at a particular point.

Formally,

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{3}$$

The goal is to find a particular solution **y(x)** that satisfies both the differential equation and the initial condition.

♣

2.5.1 Initial Condition

The **initial condition** provides specific values for the dependent variable (and possibly its derivatives) at a given value of the independent variable.

Example: For $\frac{dy}{dx} = 2x$, if $y(0) = 3$, then substituting into $y = x^2 + C$ gives $3 = 0 + C \to C = 3$. Thus, the particular solution is $y = x^2 + 3$.

2.5.2 Solution Steps

To solve an **Initial Value Problem (IVP)**:

1) Input: Differential equation $\frac{dy}{dx} = f(x, y)$ and initial condition $y(x_0) = y_0$
2) Step 1: Separate variables (if possible)
3) Step 2: Integrate both sides to find the general solution
4) Step 3: Apply the initial condition to determine the value of the constant
5) Step 4: Substitute the constant into the general solution to obtain the particular solution

6) Output: The particular solution satisfying both the differential equation and the initial condition