Calculus II

Bhashkar Paudyal

June 24, 2025

Abstract

This is the continuation of the course **Calculus I** taught by **Dr Ganga Ram Phaijoo**. The textbook used in this course is [Geo15]

Contents

1 Polar Coordinate System **8**
1.1 Setting up a Polar Coordinate System . . . 8
1.2 Uniqueness of Polar Coordinates . . . 8
1.3 Signs of r and θ . . . 11
1.4 Graphing Polar Equations or Inequalities . . . 12
1.5 Examples of Polar Graphing . . . 12
1.6 Exercise 1 . . . 13

2 Plotting Polar Coordinates and Transforming them **17**
2.1 Graph of Polar Coordinate System . . . 17
2.2 Deceptive Coordinates . . . 18
2.3 Relating Cartesian and Polar Coordinates . . . 19

3 Polar Curve **20**
3.1 Symetricity of $r = f(\theta)$ . . . 20
3.2 Slope of $r = f(\theta)$ . . . 21
3.3 Graphing a Polar Curve . . . 22

4 Cardiod Limacon and Lemniscates **24**
4.1 Cardioid . . . 24
4.2 Limacon . . . 25
4.3 Graphing of Limacons . . . 26
4.4 Rose Petals or Flowers . . . 30
4.5 Lemniscates . . . 32

5 Line, Circle, Cylinder and Sphere **34**
5.1 Straight Lines in Polar Form . . . 34
5.2 Equation of a Circle in Polar Form . . . 36
5.2.1 Different cases of Circle . . . 36
5.3 Cylindrical Coordinates . . . 36
5.3.1 Some Useful Equations . . . 36
5.4 Spherical Coordinates . . . 37
5.4.1 Some Useful Equations . . . 37

6 Relation Between Cartesian, Polar, Cylindrical and Spherical Coordinates **38**

7 Polar Integration **39**
7.1 Area Bounded by Polar Curve . . . 39

8 Multivariate Functions, Limit and Continuity **41**
8.1 Functions with Several Variables . . . 41
8.2 Level Curve and Surface . . . 41
8.3 Limits in Higher Dimensions . . . 42
8.4 Continuity of a Multivarite Function . . . 44
8.4.1 Two Path Test for Non-existence of a Limit . . . 44

9 Partial Differentiation **46**
9.1 Partial Derivatives . . . 46
9.2 First Principle . . . 46
9.3 Second Order Partial Derivative . . . 47
9.4 Linearization . . . 47
9.5 Total Differential . . . 48

10 Chain Rule of Partial Differentiation **49**
10.1 Chain Rule for Functions with Multiple Variables . . . 49
10.1.1 General Form . . . 49
10.1.2 The 2-2 Case: Two Variables, Two Parameters . . . 49
10.1.3 Additional Cases . . . 50
10.2 Implicit Differentiation . . . 51
10.2.1 Theory . . . 51
10.3 Directional Derivatives . . . 52
10.3.1 Definition and Derivation . . . 52
10.3.2 Computational Formula . . . 54
10.3.3 Connection to Partial Derivatives . . . 54

11 Critical Points in Two Variable Functions **57**
11.1 Critical Points . . . 57
11.2 Saddle Points . . . 57
11.3 Hessian/Discriminant of f: D_f . . . 57
11.4 Confirming Extremes or Saddle . . . 58
11.5 Finding Absolute Extremas . . . 58

12 Double Integrals and Application using Fubini's Theorem **59**
12.1 Review of the Definite Integrals . . . 59
12.2 Double Integrals . . . 59
12.2.1 Double Integrals as Area, Volume and Average . . . 59
12.3 Fubini's Theorem . . . 60

12.3.1 In Rectangular Region 60
12.3.2 In Any Region 61

13 Double Integrals in Polar Form **63**
13.1 Double Integrals 63
13.2 Relating the Double Integrals in Cartesian and Polar Forms 63
13.3 Converting Double Integrals from Cartesian to Polar Forms 64

List of Tables

3.1 Summary of Symmetricity of $r = f(\theta)$ 21
3.2 r-θ Table 22

4.1 Summary Table of Limacon Types 25

List of Theorems

Note (Polar Coordinates) . . . 8
Example . . . 8
Note . . . 9
Note . . . 12
Example (Plot the graph of: $1 \leq r \leq 2$, $0 \leq \theta \leq \pi/2$) . . . 12
Answer . . . 12
Example (Plot the graph of: $-3 \leq r \leq 2$, $\theta = \pi/4$) . . . 12
Answer . . . 12
Example (Plot the graph of: $\frac{\pi}{3} \leq \theta \leq \frac{5\pi}{6}$) . . . 13
Answer . . . 13
Exercise . . . 13
Answer . . . 14
Answer . . . 14
Answer . . . 14
Answer . . . 15
Answer . . . 15
Answer . . . 15
Answer . . . 16
Exercise . . . 18
Answer . . . 18
2.2.1 Problem (Same point in different form give different analysis) . . . 18
Answer . . . 18
Note (Solution to Deceptive Coordinates) . . . 18
Example (Cartesian to Polar) . . . 19
Example (Polar to Cartesian) . . . 19
Remark . . . 19
3.0.1 Definition (Polar Curve) . . . 20
Example . . . 20
Example . . . 20
Example . . . 21
Exercise . . . 22
Answer . . . 22
4.1.1 Definition (Cardioid) . . . 24

4.2.1 Definition (Limacon) . . . 25
Exercise . . . 26
Answer . . . 26
Exercise . . . 27
Answer . . . 27
Exercise . . . 28
Answer . . . 28
Exercise . . . 29
Answer . . . 29
4.4.1 Definition . . . 30
Remark . . . 30
Example . . . 31
Remark . . . 32
4.5.1 Definition . . . 32
Remark . . . 32
Example . . . 33
Note . . . 35
Remark (Main Point) . . . 40
8.1.1 Definition (Multivarite Function) . . . 41
8.2.1 Definition (Level Curve) . . . 42
Example . . . 42
8.2.2 Definition (Level Surface) . . . 42
8.3.1 Definition (Limit) . . . 43
8.3.1 Theorem . . . 43
8.4.1 Definition (Continuity of a Multivarite Function) . . . 44
Remark . . . 45
Example . . . 45
Answer . . . 45
9.1.1 Definition (Partial Derivative) . . . 46
9.2.1 Definition (First Principle of Partial Derivatives) . . . 46
Example . . . 46
9.3.1 Definition (Second Order Partial Derivatives) . . . 47
Example . . . 47
9.4.1 Definition (Linearization) . . . 47
Example . . . 48
9.5.1 Definition (Total Differential) . . . 48
Example . . . 48
Example (2-2 Case Example) . . . 50
Example (3-2 Case: Three Variables, Two Parameters) . . . 50
Example (3-1 Case: Three Variables, One Parameter) . . . 51
Example (2-1 Case: Two Variables, One Parameter) . . . 51
Example (3-3 Case: Three Variables, Three Parameters) . . . 51
Example (Implicit Differentiation Example) . . . 52

Remark 52
Example (Directional Derivative Example) 54
Exercise 55
Answer 55
Remark 55
12.1.1Definition (Definite Integral as Area) 59
Remark (Geometric Interpretation) 59
12.2.1Definition (Double Integral over Region) 59
Remark (Area from Double Integral) 59
Proof 60
12.2.2Definition (Average Value Over Region) 60
Example (Average of $f(x, y) = x + y$ over Unit Square) 60
12.3.1Theorem 60
Example 60
12.3.1Definition (Fubini's Theorem on General Regions) 61
Example (Triangular Region Bounded by $x = 0$, $y = 0$, $x + y = 1$) 61
13.1.1Definition (Double Integral in Polar Coordinates) 63
Remark (Why r Appears) 63
Proof 63
Exercise 64
Answer 64
Remark (When to Use Polar Coordinates) 64

Chapter 1

Polar Coordinate System

Lecture 1: Polar Coordinate System

1.1 Setting up a Polar Coordinate System

9 Apr. 14:00

To setup a polar coordinate system, we need a **pole**(origin) and a reference direction called **polar axis** usually in the direction of the +ve x-axis in the cartesian coordinate plane.

Note (Polar Coordinates). Polar Coordinates help you to locate a point using a distance and an angle.

1. **Radial Coordinate** (r) This is the distance from a fixed reference point, known as the pole (or origin), to the point in question.
2. **Angular Coordinate** (θ) This is the angle between the line connecting the pole to the point and a reference direction, typically the positive x-axis, measured counterclockwise.

The polar coordinates of a point are denoted as (r, θ).

1.2 Uniqueness of Polar Coordinates

In contrast to Cartesian Coordinate, polar coordinates are not unique. It is because one can rotate many times and face the same directions.

Example. $361°$ and $1°$ label as same angles.
Also, the following coordinates are the same:

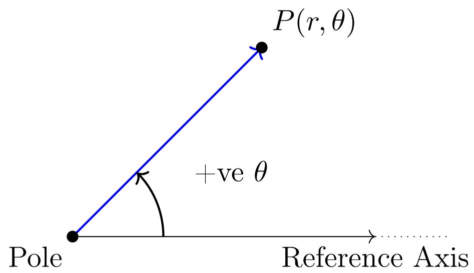
1. $(5, \frac{3\pi}{4})$

2. $(-5, -\frac{\pi}{4})$
3. $(5, \frac{11\pi}{4})$
4. $(5, -\frac{5\pi}{4})$
5. $(5, \frac{11\pi}{4} + 2\pi)$

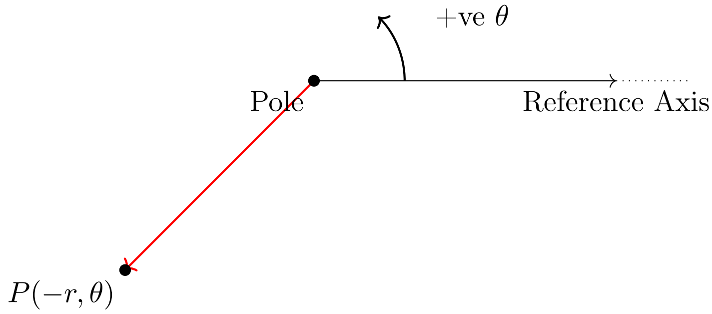
- Something is very good to be true.

Note. Generally, $\theta' = 2n\pi + \theta$ where $n \in \mathbb{Z}$, θ is the angle something rotates and θ' is the Angular Coordinate.
Hence, Polar Coordinates are also called **deceptive coordinates**. Section 2.2

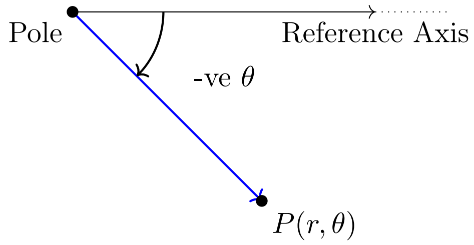
1.3 Signs of r and θ



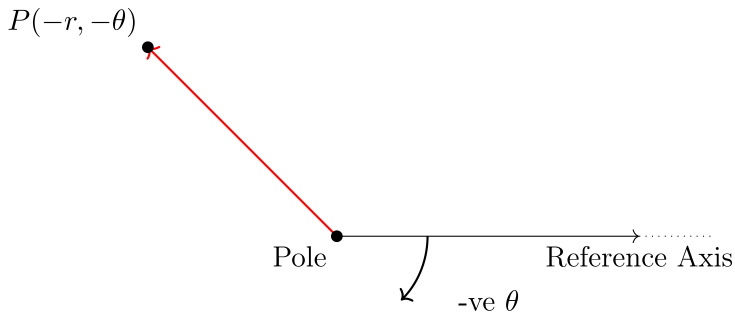
(a) +ve r, +ve θ (standard)



(b) -ve r, +ve θ (point in opposite direction)



(c) +ve r, -ve θ (clockwise)



(d) -ve r, -ve θ (point in opposite direction)

Figure 1.1: All four sign combinations of polar coordinates with correct angle and direction of r

The direction of +ve θ is by convention counter-clockwise.
The direction of +ve r is by convention in the direction of theta.
So, the direction of -ve θ is clockwise.
And, the direction of -ve r is in the opposite direction of θ.
So, if θ is clockwise, r is in the clockwise direction and $-r$ is in the counter-clockwise direction.

1.4 Graphing Polar Equations or Inequalities

Polar equations or Set of polar inequalities can be used to plot graphs of usually circular figures.

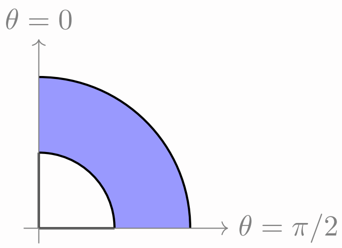
Note. Some special cases:

1. $r = |a|$ gives a circle with center as pole and radius as a.
2. $\theta = \theta_o$ gives a straight line passing through the pole.
3. $\cancel{r = a}$ $r = |a|$, $\theta = \theta_o$ gives a point $P(a, \theta_o)$.

1.5 Examples of Polar Graphing

Example (Plot the graph of: $1 \leq r \leq 2$, $0 \leq \theta \leq \pi/2$).

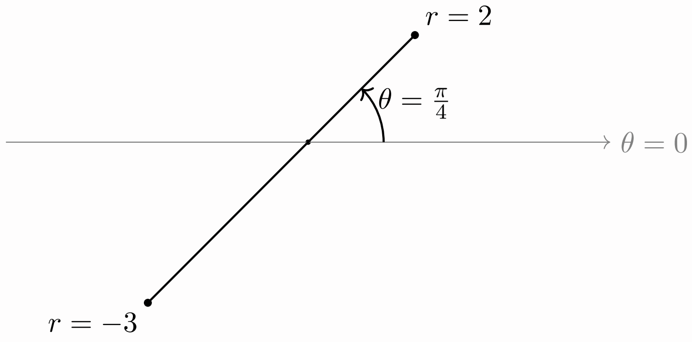
Answer.



⊛

Example (Plot the graph of: $-3 \leq r \leq 2$, $\theta = \pi/4$).

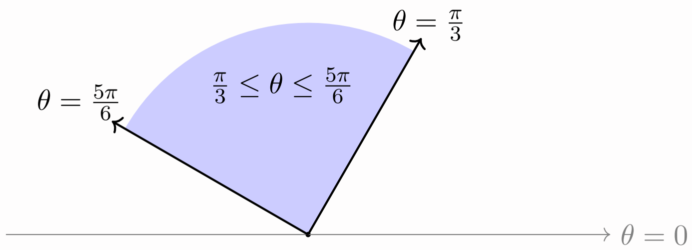
Answer.



⊛

Example (Plot the graph of: $\frac{\pi}{3} \leq \theta \leq \frac{5\pi}{6}$).

Answer.

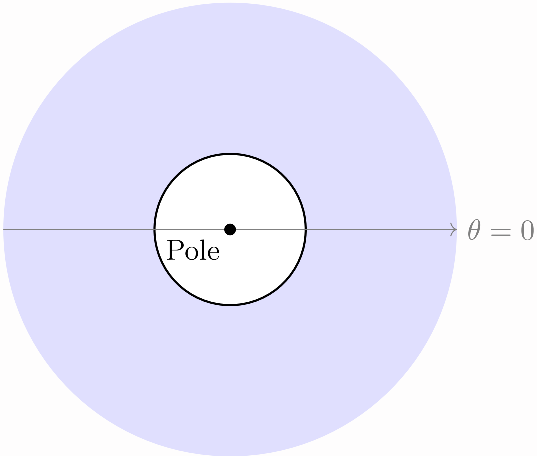


⊛

1.6 Exercise 1

Exercise. **Plot the graph of the following:**

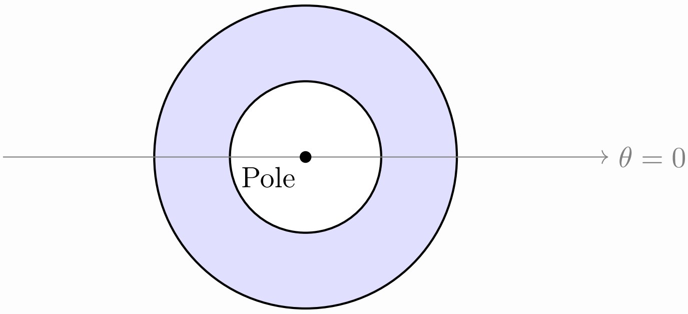
1. $r \geq 1$



Answer.

⊛

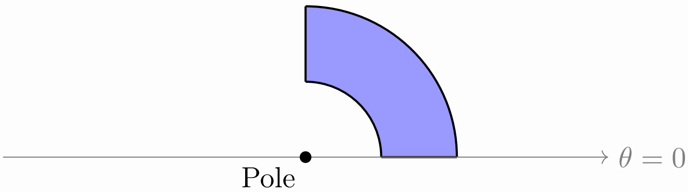
2. $1 \le r \le 2$



Answer.

⊛

3. $0 \le \theta \le \pi/2$, $1 \le r \le 2$

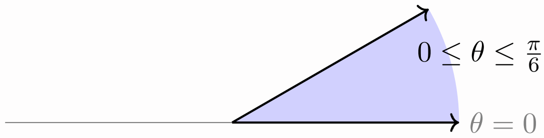


Answer.

⊛

4. $0 \le \theta \le \pi/6$, $r \ge 0$

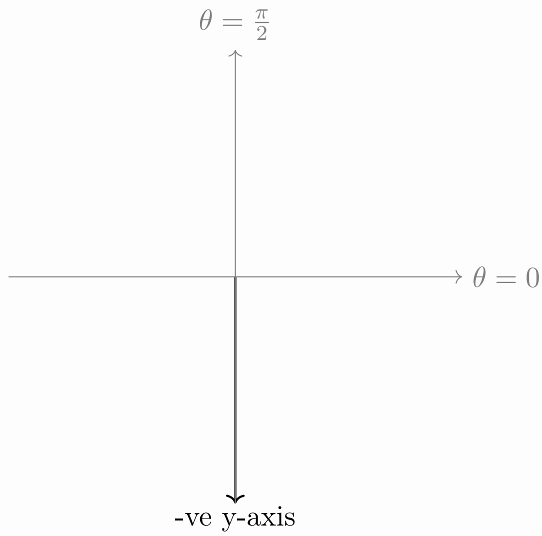
Answer.



⊛

5. $\theta = \pi/2$, $r \leq 0$

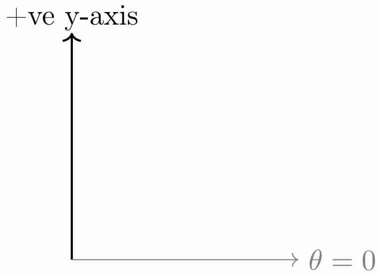
Answer.



⊛

6. $\theta = \pi/2$, $r \geq 0$

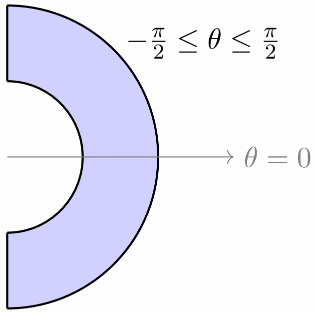
Answer.



⊛

7. $-\pi/2 \leq \theta \leq \pi/2$, $1 \leq r \leq 2$

Answer.



⊛

Chapter 2

Plotting Polar Coordinates and Transforming them

Lecture 2: Handling Polar Coordinates

2.1 Graph of Polar Coordinate System

16 Apr. 14:16

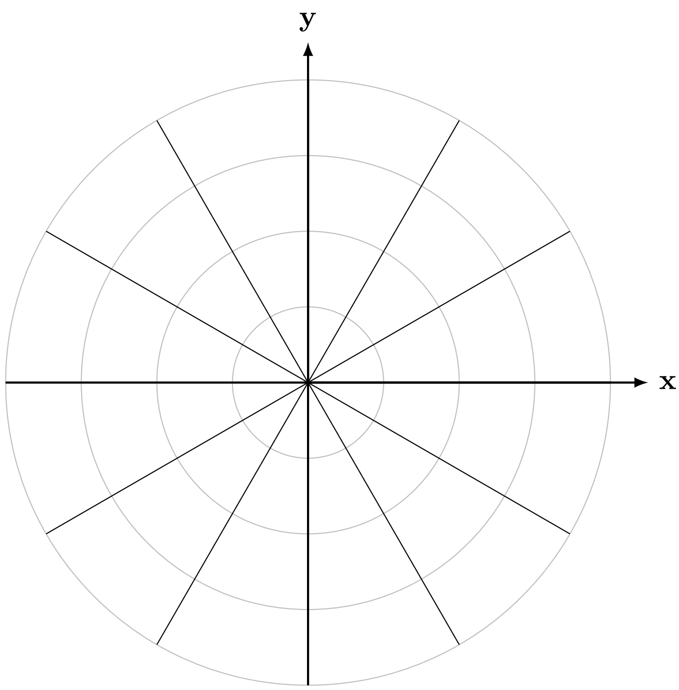
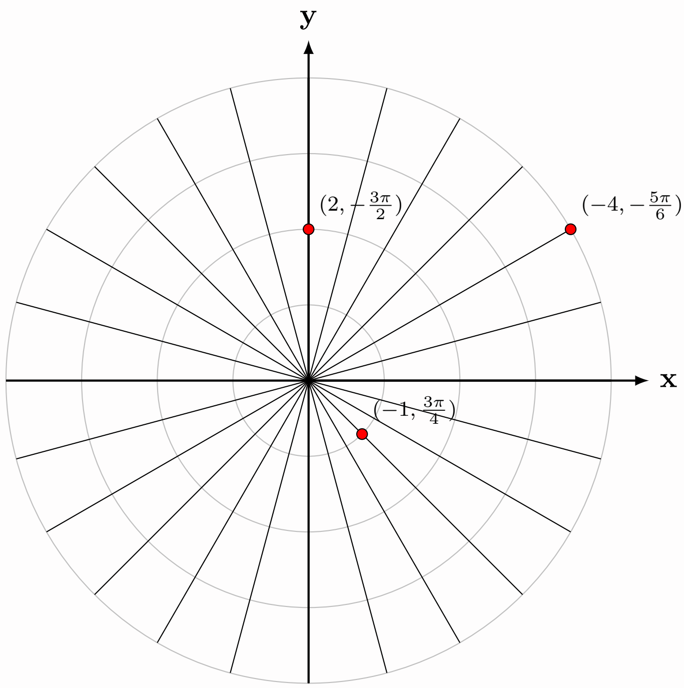


Figure 2.1: Polar Coordinate System

Exercise. Plot:

1. $(-1, \frac{3\pi}{4})$
2. $(2, -\frac{3\pi}{2})$
3. $(-4, -\frac{5\pi}{6})$



Answer. ⊛

2.2 Deceptive Coordinates

Problem 2.2.1 (Same point in different form give different analysis). Does the point $(2, \frac{\pi}{2})$ lie on the curve $r = 2\cos 2\theta$?
How do you check it?

Answer. By substituting the value of r and θ in the above equation. The answer is **NO**! ⊛

But take the same point but written as: $(-2, \frac{\pi}{2})$. Now does it lie on the curve $r = 2\cos 2\theta$? **YES**!

Note (Solution to Deceptive Coordinates). **Use Cartesian Coordinates**!!
To solve this duality of polar coordinates, simply change the polar coordinates to Cartesian Coordinate Section 2.3.

2.3 Relating Cartesian and Polar Coordinates

Polar and Cartesian Coordinates relate by the following formulae:

1. $x^2 + y^2 = r^2$
2. $\tan\theta = \frac{y}{x}$
3. $x = r\cos\theta$
4. $y = r\sin\theta$

Examples of Coordinate Conversion

Example (Cartesian to Polar). Convert the Cartesian coordinates $(3, 4)$ to polar coordinates (r, θ).
Solution:

- Calculate radius: $r = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$
- Calculate angle: $\theta = \arctan\left(\frac{4}{3}\right) \approx 53.13°$ or 0.93 radians

Result: $(5, 53.13°)$ or $(5, 0.93 \text{ rad})$

Example (Polar to Cartesian). Convert the polar coordinates $(2, \pi/3)$ to Cartesian coordinates (x, y).
Solution:

- Calculate x: $x = 2\cos\left(\frac{\pi}{3}\right) = 2 \times \frac{1}{2} = 1$
- Calculate y: $y = 2\sin\left(\frac{\pi}{3}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

Result: $(1, \sqrt{3})$

Remark. Remember these key points:

- For Cartesian $\rightarrow$ Polar, always consider the **quadrant** when calculating θ
- For Polar $\rightarrow$ Cartesian, exact values are preferred when angles are special fractions of π

Chapter 3

Polar Curve

Lecture 3: Polar Curves

Definition 3.0.1 (Polar Curve).

$$r = f(\theta) \tag{3.1}$$

28 Apr. 22:52

3.1 Symetricity of $r = f(\theta)$

1. Given (r, θ) lies in $r = f(\theta)$,

$$(r, -\theta) \text{ or } (-r, \pi - \theta) \text{ in } r = f(\theta) \Rightarrow r = f(\theta) \text{ is symmetric abt. x-axis.}$$

> **Example.** $r = 1 + \cos\theta$ (true)
> $r = 2a\sin\theta$ (false)
> $r = 2a\sin 2\theta$ (false)

2. Given (r, θ) lies in $r = f(\theta)$,

$$(-r, -\theta) \text{ or } (r, \pi - \theta) \text{ in } r = f(\theta) \Rightarrow r = f(\theta) \text{ is symmetric abt. y-axis.}$$

> **Example.** $r = 1 + \cos\theta$ (false)
> $r = 2a\sin\theta$ (true)
> $r^2 = \cos 2\theta$ (true)

3. Given (r, θ) lies in $r = f(\theta)$,

$$(-r, \theta) \text{ or } (r, \pi + \theta) \text{ in } r = f(\theta) \Rightarrow r = f(\theta) \text{ is symmetric abt. pole.}$$

Example. $r = 1 + \cos\theta$ (false)
$r = \sin\theta$ (false)
$r = \sin 2\theta$ (true)
$r^2 = \sin 2\theta$ (true)

x-axis	$(r, -\theta)$	$(-r, \pi - \theta)$
y-axis	$(r, \pi - \theta)$	$(-r, -\theta)$
pole	$(r, \pi + \theta)$	$(-r, \theta)$

Table 3.1: Summary of Symmetricity of $r = f(\theta)$

3.2 Slope of $r = f(\theta)$

Given a polar curve $r = f(\theta)$, we can express the Cartesian coordinates as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

The slope of the tangent to the curve at a point is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Differentiating x and y with respect to θ (using the product rule):

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\cos\theta\right) = f'(\theta)\cos\theta - f(\theta)\sin\theta,$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\sin\theta\right) = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

Therefore, the slope at $\theta = \theta_0$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0} = \left.\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\right|_{\theta=\theta_0}.$$

Slope at the Pole:
The *pole* corresponds to $r = f(\theta_0) = 0$. At the pole, the expressions simplify:

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{r=0,\,\theta=\theta_0} &= \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} \\ &= \tan\theta_0 \qquad \text{(provided } f'(\theta_0) \neq 0).\end{aligned}$$

Thus, **the slope of the tangent at the pole is** $\tan\theta_0$, provided the curve passes through the pole at θ_0 and $f'(\theta_0) \neq 0$.

3.3 Graphing a Polar Curve

Exercise. Plot the curve $r = 1 + \cos\theta$, a cardiod(heart-shaped).

Answer. The following is the process for graphing given curve.

1. Find the slope at pole:
   At pole, $r = 0$
   We have,

$$\begin{aligned} r &= 1 + \cos\theta \\ \text{Set } r = 0: \quad 0 &= 1 + \cos\theta \\ \cos\theta &= -1 \\ \theta &= \pi, \quad n \in \mathbb{Z} \\ \therefore \tan\theta_o &= 0 \end{aligned}$$

2. Find the Symmetricity of the curve:
   The curve is symmetric about only x-axis.

3. Make $r - \theta$ table for specific points.

⊛

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 1 + \cos(\theta)$	2	$1 + \frac{\sqrt{3}}{2}$	$1 + \frac{\sqrt{2}}{2}$	$1 + \frac{1}{2}$	1	$1 - \frac{1}{2}$	$1 - \frac{\sqrt{2}}{2}$	$1 - \frac{\sqrt{3}}{2}$	0

Table 3.2: r-θ Table

Plot:

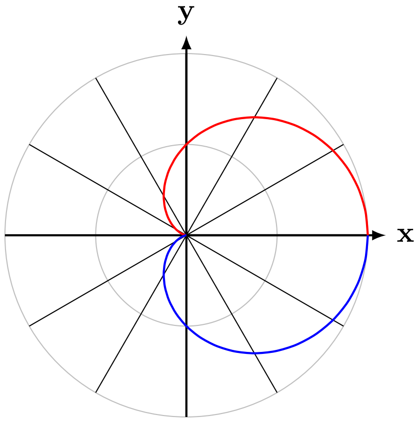


Figure 3.1: Plot of $r = 1 + \cos\theta$

Chapter 4

Cardiod Limacon and Lemniscates

Lecture 4

4.1 Cardioid

6 May. 17:29

Definition 4.1.1 (Cardioid). A **cardioid** is a plane curve locus of a point on the perimeter of a circle as it rolls around a fixed circle of the same radius. In polar coordinates, the cardioid has the standard form:

$$r = a(1 \pm \cos\theta) \quad \text{or} \quad r = a(1 \pm \sin\theta)$$

where $a > 0$ is a constant that determines the size of the cardioid. Locus Animation of Cardioid Formation

Sign Variations and Their Effects The sign in front of the trigonometric function and the choice between sine and cosine determine the orientation of the cardioid:

- $r = a(1 + \cos\theta)$: cardioid tail to the right (along the positive x-axis).
- $r = a(1 - \cos\theta)$: cardioid tail to the left (along the negative x-axis).
- $r = a(1 + \sin\theta)$: cardioid tail upward (along the positive y-axis).
- $r = a(1 - \sin\theta)$: cardioid tail downward (along the negative y-axis).

Symmetry Cardioids exhibit reflective symmetry:

- Equations involving $\cos\theta$ are symmetric about the horizontal axis (x-axis).
- Equations involving $\sin\theta$ are symmetric about the vertical axis (y-axis).

Play with it

Geometric Properties

- The curve has a cusp at the origin (pole).
- The maximum radius is $2a$, and the minimum radius is 0.

4.2 Limacon

Definition 4.2.1 (Limacon). A **Limacon** is a polar curve defined by the equation:

$$r = a \pm b\cos\theta \quad \text{or} \quad r = a \pm b\sin\theta$$

where $a, b \in \mathbb{R}$, and the shape of the curve depends on the ratio $\frac{a}{b}$. Detailed Explanation and Graphing

Types of Limacons Based on $\frac{a}{b}$ Let us analyze the curve based on the relative magnitudes of a and b:

- **Case 1:** $a < b$ (inner loop limacon)

  The curve has an inner loop. It crosses the origin and forms a loop inside the main body. The smaller the ratio $\frac{a}{b}$, the larger the inner loop.

- **Case 2:** $a = b$ (cardioid)

  When $a = b$, the limacon becomes a cardioid. See Definition 4.1.1.

- **Case 3:** $a > b$ (dimpled or convex limacon)

  - If $\frac{a}{b} > 2$, the curve resembles a nearly circular convex limacon.
  - If $\frac{a}{b} = 2$, the curve develops a dimple(flat) at the pole.
  - If $\frac{a}{b} < 2$, the curve has a pronounced dimple but no inner loop.

Type	Relation between a and b
Inner Loop	$a < b$
Cardioid	$a = b$
Dimpled	$b < a < 2b$
Convex	$a \geq 2b$

Table 4.1: Summary Table of Limacon Types

Orientation Based on Trigonometric Function and Sign

- $r = a + b\cos\theta$: curve tail toward the right.
- $r = a - b\cos\theta$: curve tail toward the left.
- $r = a + b\sin\theta$: curve tail upward.
- $r = a - b\sin\theta$: curve tail downward.

Play with it.

Symmetry Limacons, like cardioids, exhibit reflective symmetry:

- Equations with $\cos\theta$ are symmetric about the x-axis.
- Equations with $\sin\theta$ are symmetric about the y-axis.

Geometric Notes

- The limacon can have an inner loop, be dimpled, or convex depending on parameters.
- It has reflective symmetry based on its trigonometric form.

4.3 Graphing of Limacons

Exercise. Plot the curve $r = 2 + 3\sin\theta$, an inner loop limacon.

Answer.

1. At pole: $r = 0 \Rightarrow 2 + 3\sin\theta = 0 \Rightarrow \sin\theta = -\frac{2}{3}$
2. Symmetry: about vertical (y) axis.
3. Use polar coordinates to plot.

⊛

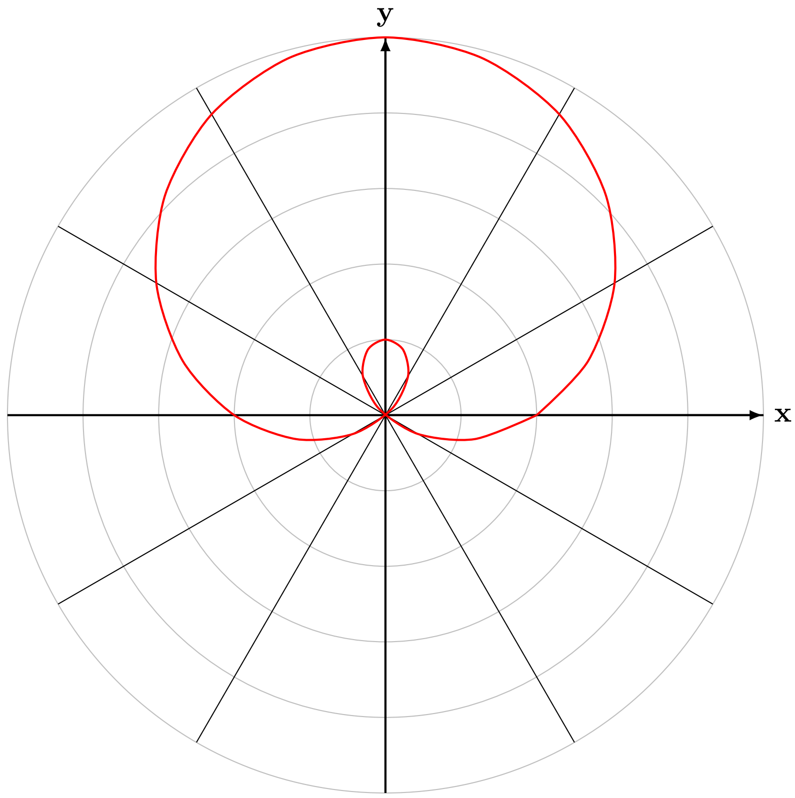


Figure 4.1: Plot of $r = 2 + 3\sin\theta$

Exercise. Plot the curve $r = 3 + 2\cos\theta$, a dimpled limacon.

Answer.

1. At pole: $r = 0 \Rightarrow \cos\theta = -\frac{3}{2}$ (no solution).
2. Symmetry: about horizontal (x) axis.

⊛

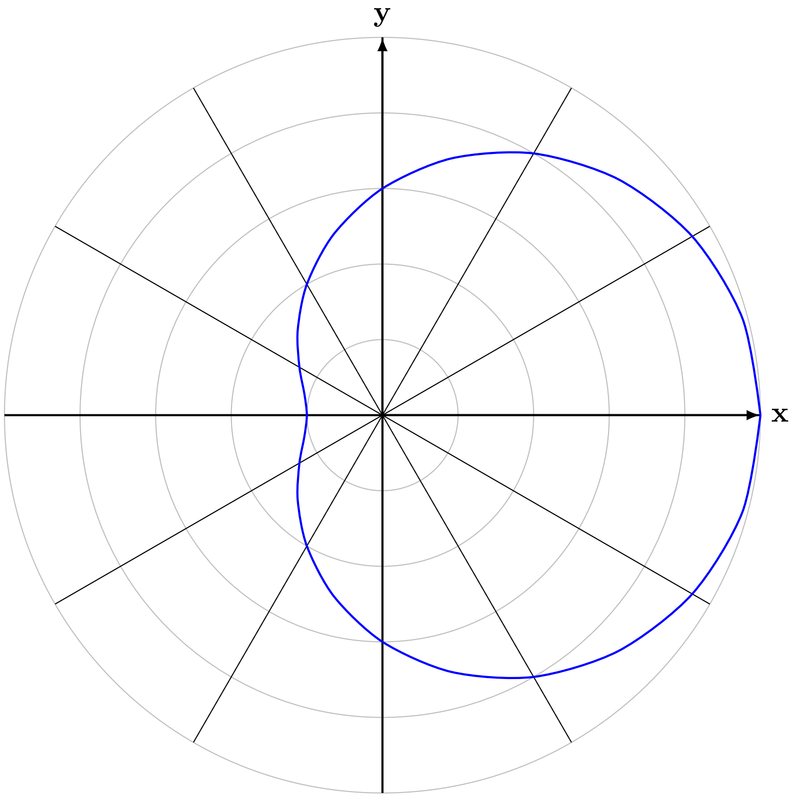


Figure 4.2: Plot of $r = 3 + 2\cos\theta$

Exercise. Plot the curve $r = 4 + 2\sin\theta$, a convex limacon.

Answer.

1. At pole: $r = 0 \Rightarrow \sin\theta = -2$ (no solution).
2. Symmetry: about vertical (y) axis.

⊛

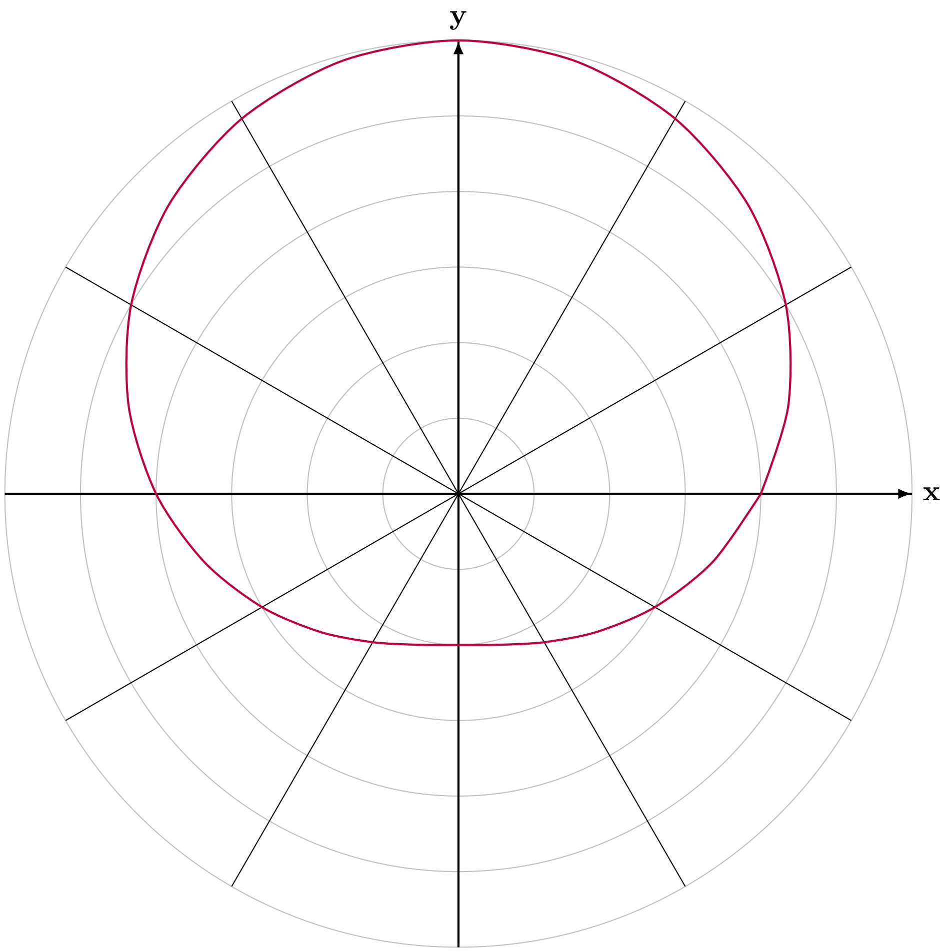


Figure 4.3: Plot of $r = 4 + 2\sin\theta$

Exercise. Plot the curve $r = 6 + 2\cos\theta$, a convex limacon.

Answer.

1. At pole: $r = 0 \Rightarrow \cos\theta = -3$ (no solution).
2. Symmetry: about horizontal (x) axis.

⊛

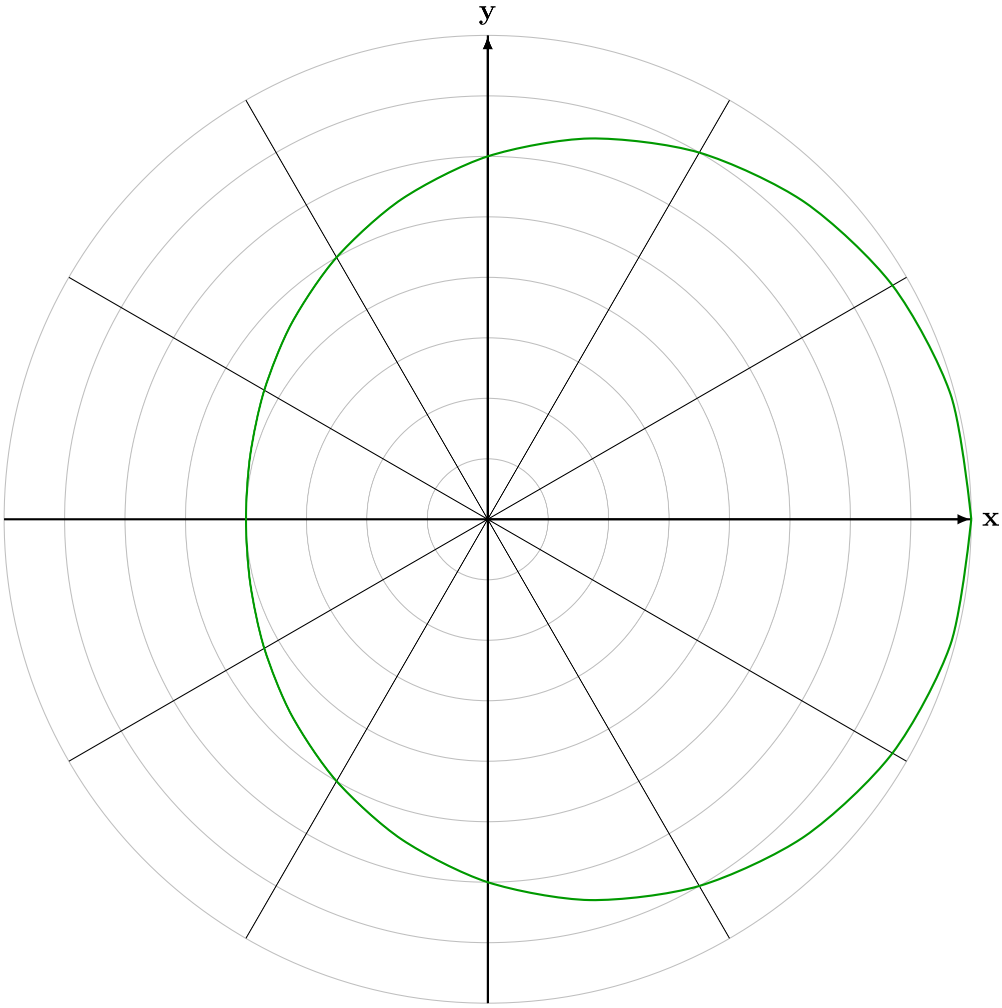


Figure 4.4: Plot of $r = 6 + 2\cos\theta$

4.4 Rose Petals or Flowers

Definition 4.4.1. A rose curve is a polar graph defined by equations of the form:

$$r = a\cos(m\theta) \quad \text{or} \quad r = a\sin(m\theta)$$

where a and m are real constants. The shape resembles petals of a flower.
Play with it.

Remark. The number of petals in a rose curve depends on whether m is odd or even:

- If m is **odd** , the curve has exactly m petals.
- If m is **even** , the curve has $2m$ petals.

This rule applies for both sine and cosine cases.

Example. Let $r = 2\cos(3\theta)$. Here, $m = 3$ is odd, so the graph has 3 petals.

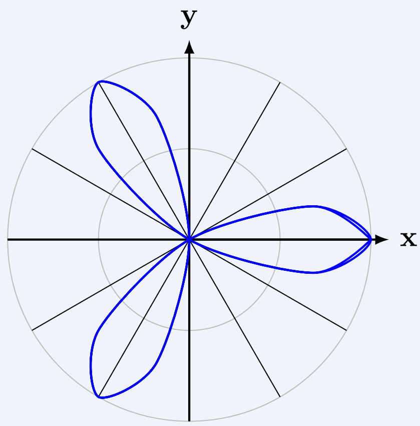


Figure 4.5: Plot of $r = 2\cos(3\theta)$

Let $r = 4\sin(2\theta)$. Here, $m = 2$ is even, so the graph has $2 \times 2 = 4$ petals.

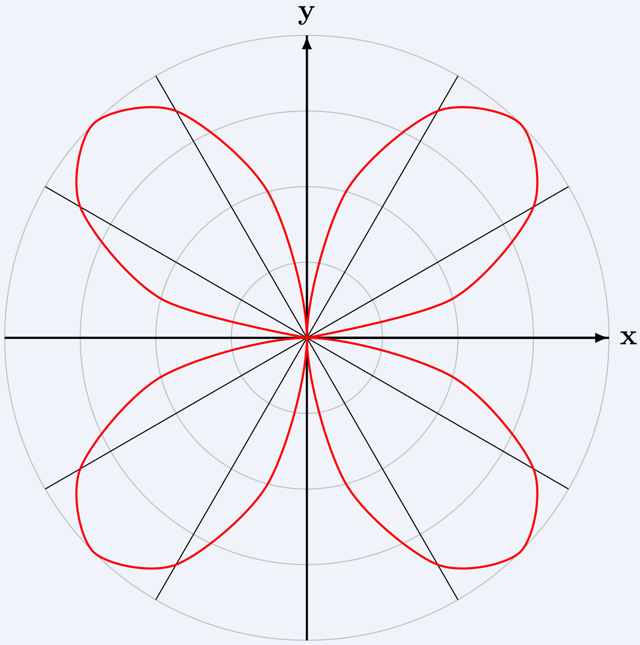


Figure 4.6: Plot of $r = 4\sin(2\theta)$

Remark. The symmetry of rose curves also depends on the trigonometric function:

- $\cos(m\theta)$ curves are symmetric about the x-axis.
- $\sin(m\theta)$ curves are symmetric about the y-axis (for odd m) or origin (for even m).

4.5 Lemniscates

Definition 4.5.1. A lemniscate is a polar curve that resembles a figure-eight or infinity symbol. The general form is:

$$r^2 = a^2\cos(2\theta) \quad \text{or} \quad r^2 = a^2\sin(2\theta)$$

These curves are symmetric and bounded, typically having two loops.
Desmos Graph
Play with it.

Remark. The symmetry of lemniscates depends on the trigonometric function:

- $r^2 = a^2\cos(2\theta)$: symmetric about the x-axis.

- $r^2 = a^2 \sin(2\theta)$: symmetric about the origin or y = x line.

Example. Let $r^2 = 4\sin(2\theta)$, which implies $a^2 = 4 \Rightarrow a = 2$. This is a lemniscate symmetric about the origin.

Let $r^2 = 4\cos(2\theta)$, again with $a^2 = 4 \Rightarrow a = 2$. This is a lemniscate symmetric about the x-axis.

Chapter 5

Line, Circle, Cylinder and Sphere

Lecture 5: Fifth Lecture

5.1 Straight Lines in Polar Form

9 May. 00:51

In Cartesian Coordinate system, a straight line can be representation in many form. One of them is slope form $y = mx + c$. But the one that mostly relates with Polar System is the normal form of representing a straight line.

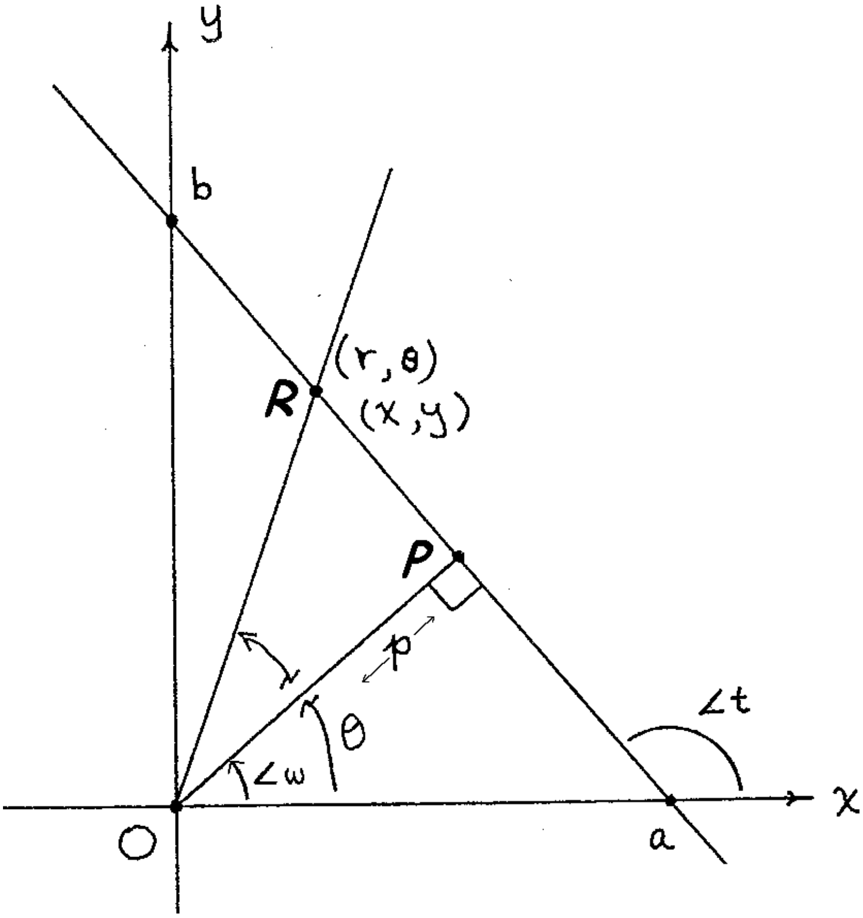


Figure 5.1: Figure for derivation of Straight Line

$$x\cos(\alpha) + y\sin(\alpha) = r_o \tag{5.1}$$

Now we subsititute the value of x and y so that the equation becomes:

$$r\cos\theta\cos\theta_o + r\sin(\theta)\sin(\theta_o) = r_o$$

$$r_o = r(\cos(\theta - \theta_o)) \tag{5.2}$$

Note. Here, r_o is the radial coordinate of the point of intersection between the perpendicular to the line in question from the origin.

5.2 Equation of a Circle in Polar Form

From above figure, Using cosine law on triange . . . ,

$$\cos(\theta - \theta_o) = \frac{r^2 + {r_o}^2 - a^2}{2rr_o}$$

$$\therefore \quad r^2 + {r_o}^2 - 2rr_o \cos(\theta - \theta_o) = a^2 \tag{5.3}$$

5.2.1 Different cases of Circle

Circle passes through pole

$$r^2 = 2ar\cos(\theta - \theta_o) \tag{5.4}$$

Circle passes through pole and center in x-axis Here, $\theta_o = 0$. Then from Equation 5.4,

$$r = 2a\cos\theta$$

where, $a \in \mathbb{Z}$.

Circle passes through pole and center in y-axis Here, $\theta_o = \frac{\pi}{2}$. Then from Equation 5.4,

$$r = 2a\sin\theta$$

where, $a \in \mathbb{Z}$.

Circle and center at pole Here, $r = 0$. Then from Equation 5.3,

$$r = a$$

where, $a \in \mathbb{Z}$.

5.3 Cylindrical Coordinates

A point in cylindrical coordinate system is represented with (r, θ, z).
where (r, θ) is the point in *x-y plane* And z is the translation of that point in the z-direction.

5.3.1 Some Useful Equations

- $r = a$ represents a cylinder with infinite height and depth.

- $\theta = \theta_o$ represents a plane containing z-axis.

- $z = z_o$ represents a plane roof parallel to *x-y plane*.
- $z = r^2$ represents a **paraboloid**.
- $z = r$ represents a cone.

5.4 Spherical Coordinates

A point in spherical coordinate system is represented with (ρ, ϕ, θ).
where,
ρ is the distance of the point from the pole.
ϕ is the angle between the projection of the line (joining pole to our point) in the *y-z plane* and the z-axis.
θ is the angle between the projection of the line (joining pole to our point) in the *x-y plane* and the x-axis.

5.4.1 Some Useful Equations

- $\rho = a \rightarrow$ Sphere
- $\phi = \phi_o \rightarrow$ Cone
  - $\phi_o = \frac{\pi}{2} \rightarrow$ *x-y plane*
  - $\phi_o = 0 \rightarrow$ *+ve z-axis*
  - $\phi_o = \pi \rightarrow$ *-ve z-axis*
  - $0 < \phi_o < \frac{\pi}{2} \rightarrow$ cone, widens at $2\phi_o$ angle in the positive z direction.
  - $\frac{\pi}{2} < \phi_o < \pi \rightarrow$ cone, widens at $2(\pi - \phi_o)$ angle in the negative z direction.
- $\theta = \theta_o \rightarrow$ plane containing z-axis.

Chapter 6

Relation Between Cartesian, Polar, Cylindrical and Spherical Coordinates

Lecture 6: Sixth Lecture

$$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2 \tag{6.1}$$

10 May. 22:42

$$\begin{aligned} z &= \rho \cos\phi \\ r &= \rho \sin\phi \end{aligned} \tag{6.2}$$

Chapter 7

Polar Integration

7.1 Area Bounded by Polar Curve

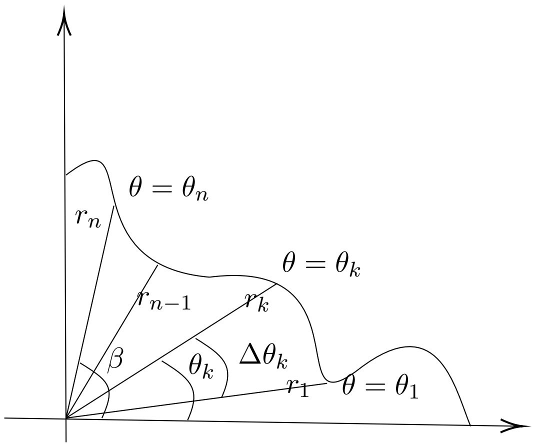


Figure 7.1: Area Bounded by Polar Curves

For the k^{th} sector the area of the sector is:

$$A_k = \frac{1}{2} r_k^2 \Delta\theta$$

Then the area of all the sectors is:

$$\sum_{k=1}^{n} A_k$$

Then replacing the summation with limit as $n \to \infty$ then, the area bounded by the curve is:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} A_k = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

> **Remark** (Main Point). We need to know what the α and β is.
> This type of integration is useful for finding the shared area by two curve.

Think and visualize of a situation where there are two curves and we need to find the area bounded by one curve but not the other.
Let's say that the curve that is the one we need to find the area bounded by it is $r = f(\theta)$ and the other one is $r = g(\theta)$.
Then the area bounded by $r = f(\theta)$ is:

$$A_f = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 d\theta$$

$$A_g = \int_{\alpha}^{\beta} \frac{1}{2}(g(\theta))^2 d\theta$$

Then, the area bounded by f but not g is:

$$A = \int_{\alpha}^{\beta} f(\theta)^2 - g(\theta)^2 d\theta$$

Chapter 8

Multivariate Functions, Limit and Continuity

Lecture 8: Eight Lecture

8.1 Functions with Several Variables

21 May. 23:18

Definition 8.1.1 (Multivarite Function). Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's domain. The set of w-values take on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables x_1 to x_n. We also call the x_j's the function's input variables and call w the function's output variable.

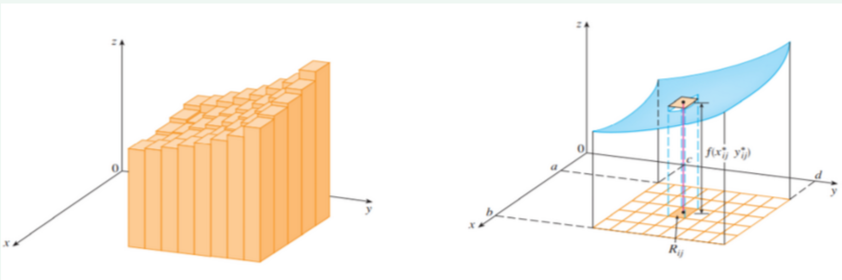


Figure 8.1: Multivarite Function

8.2 Level Curve and Surface

Definition 8.2.1 (Level Curve). The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a level curve of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Example. Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

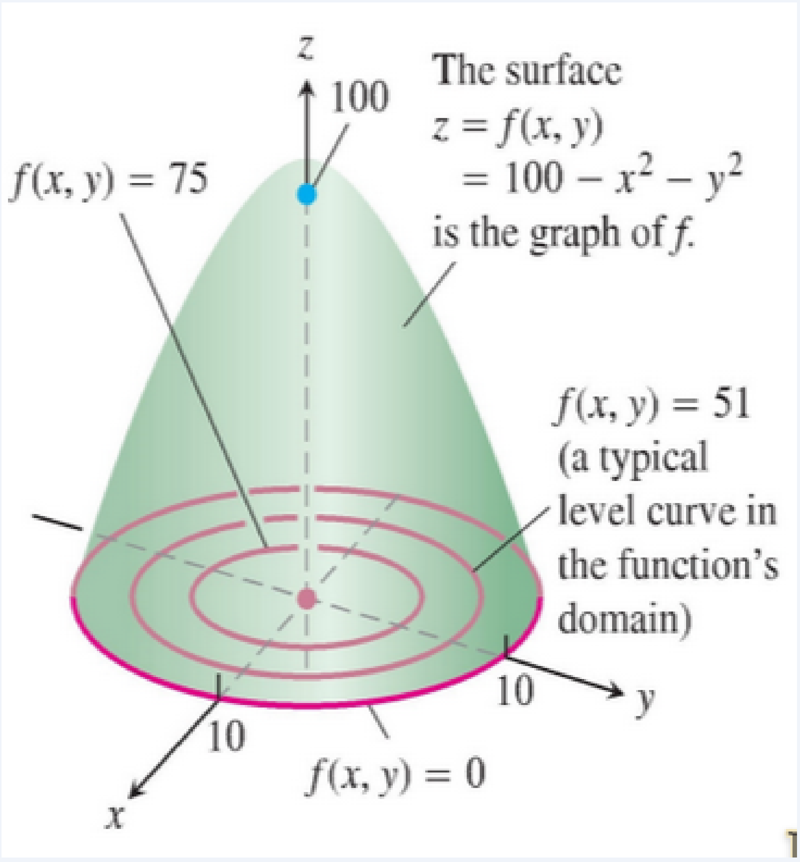


Figure 8.2: The graph and selected level curve of the function $f(x, y)$

Definition 8.2.2 (Level Surface). The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called level surface of f.

8.3 Limits in Higher Dimensions

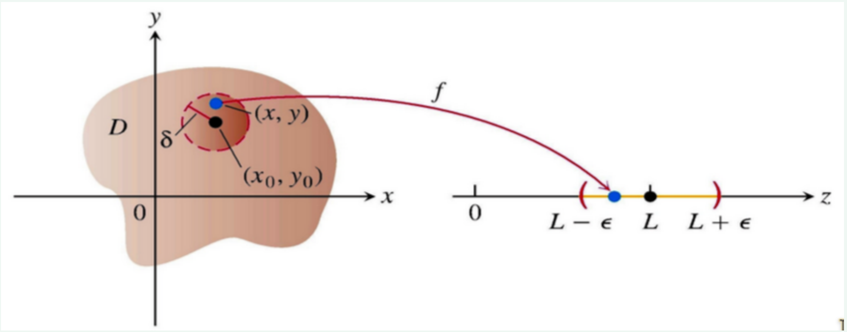
Definition 8.3.1 (Limit). We say that a function $f(x,,y)$ approaches the **limit** L as (x,y) approaches (x_o,y_o), and write:

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y) = L$$

$\forall\ (x,y) \in \text{Domain}(f)$:

$$\forall\ \epsilon > 0, \exists\delta > 0 \Rightarrow (\sqrt{(x-x_o)^2+(y-y_o)^2} < \delta \Rightarrow |f(x,y)-L| < \epsilon)$$

.



1

Figure 8.3: In the limit definition, δ is the radius of a disk centered at (x_o,y_o). $\forall\ (x,y)$ withint this disk, the function values $f(x,y)$ lie inside the corresponding interval $(L-\epsilon, L+\epsilon)$

Theorem 8.3.1. The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. **Sum Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)+g(x,y)] = L+M$$

2. **Difference Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)-g(x,y)] = L-M$$

3. **Constant Multiple Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL \quad \text{(any number } k\text{)}$$

4. **Product Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\cdot g(x,y)] = L\cdot M$$

5. **Quotient Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$$

6. **Power Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n, \quad n \text{ a positive integer}$$

7. **Root Rule:**

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n},$$

n a +ve integer, and if n is even, assume that $L > 0$.

8.4 Continuity of a Multivarite Function

Definition 8.4.1 (Continuity of a Multivarite Function). A function $f(x,y)$ is continuous at the point (x_0,y_0) if

1. f is defined at (x_0,y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

A function is continuous if it is continuous at every point of the domain.

8.4.1 Two Path Test for Non-existence of a Limit

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

does not exist.

Remark. It is easy to prove non-existence of a limit rather that existence.

Example. Show that:

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuos at origin.

Answer. **Along the path** $y = mx$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x.mk}{x^2 + (mx)^2} \tag{8.1}$$

$$= \frac{2m}{1+m^2} \tag{8.2}$$

For $k = 1$ i.e, along the path $y = x$, the limit is 1 and for $k = -1$, the limit is -1.

So by two path test, limit of $f(x,y)$ does not exist at origin. $\therefore$ The function is not continuous at origin through $f(0,0) = 0$ $\circledast$

Chapter 9

Partial Differentiation

Lecture 9: Ninth Lecture

9.1 Partial Derivatives

22 May. 00:38

Definition 9.1.1 (Partial Derivative). Let $f(x, y)$ be a function of two variables. The **partial derivative** of f with respect to x, denoted f_x or $\frac{\partial f}{\partial x}$, is found by differentiating f with respect to x while keeping y constant. Similarly, the partial derivative with respect to y, denoted f_y or $\frac{\partial f}{\partial y}$, is found by differentiating f with respect to y while keeping x constant.

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}$$

9.2 First Principle

Definition 9.2.1 (First Principle of Partial Derivatives). Let $f(x, y)$ be a function of two variables. Then the partial derivative of f with respect to x at the point (x_0, y_0) is defined as:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Similarly, the partial derivative of f with respect to y at (x_0, y_0) is:

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Example. Let $f(x, y) = x^2 y$. Find f_x and f_y at $(1, 2)$ using the first principle.

For f_x:

$$f_x(1,2) = \lim_{h\to 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim_{h\to 0} \frac{(1+h)^2 \cdot 2 - 1^2 \cdot 2}{h}$$

$$= \lim_{h\to 0} \frac{(1+2h+h^2)\cdot 2 - 2}{h} = \lim_{h\to 0} \frac{2+4h+2h^2-2}{h} = \lim_{h\to 0} \frac{4h+2h^2}{h} = \lim_{h\to 0}(4+2h) = 4$$

For f_y:

$$f_y(1,2) = \lim_{h\to 0} \frac{f(1,2+h) - f(1,2)}{h} = \lim_{h\to 0} \frac{1^2 \cdot (2+h) - 1^2 \cdot 2}{h} = \lim_{h\to 0} \frac{2+h-2}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

9.3 Second Order Partial Derivative

Definition 9.3.1 (Second Order Partial Derivatives). The second order partial derivatives are defined as:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

Note: If the mixed partial derivatives f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$ (**Clairaut's Theorem**).

Example. Let $f(x,y) = x^2y + 3xy^2$. Then,

$$f_x = \frac{\partial}{\partial x}(x^2y + 3xy^2) = 2xy + 3y^2$$

$$f_y = \frac{\partial}{\partial y}(x^2y + 3xy^2) = x^2 + 6xy$$

$$f_{xy} = \frac{\partial}{\partial y}(2xy + 3y^2) = 2x + 6y, \quad f_{yx} = \frac{\partial}{\partial x}(x^2 + 6xy) = 2x + 6y$$

9.4 Linearization

Definition 9.4.1 (Linearization). The linearization of a function $f(x,y)$ near a point (x_0, y_0) is the linear function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

This is the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

Example. Let $f(x, y) = x^2 + y^2$ and consider the point $(1, 2)$. Then

$$f(1, 2) = 1^2 + 2^2 = 5, \quad f_x = 2x, \quad f_y = 2y$$

$$f_x(1, 2) = 2, \quad f_y(1, 2) = 4$$

$$L(x, y) = 5 + 2(x - 1) + 4(y - 2)$$

9.5 Total Differential

Definition 9.5.1 (Total Differential). The total differential df of a function $f(x, y)$ is given by:

$$df = f_x(x, y)\, dx + f_y(x, y)\, dy$$

It gives an approximation of the change in f when x and y change by small amounts.

Example. Let $f(x, y) = x^2 y + y^3$. Then

$$f_x = 2xy, \quad f_y = x^2 + 3y^2$$

$$df = 2xy\, dx + (x^2 + 3y^2)\, dy$$

Chapter 10

Chain Rule of Partial Differentiation

The chain rule for partial differentiation is a fundamental tool in multivariable calculus that allows us to compute derivatives of composite functions. This chapter explores various forms of the chain rule, from simple cases to more complex scenarios involving multiple variables and intermediate functions.

10.1 Chain Rule for Functions with Multiple Variables

10.1.1 General Form

Consider a function $z = f(x_1, x_2, \ldots, x_n)$ where each variable x_i is itself a function of m other variables $t_1, t_2, \ldots, t_m$. The **chain rule** states that:

$$\frac{\partial z}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t_j}$$

for $j = 1, 2, \ldots, m$.

10.1.2 The 2-2 Case: Two Variables, Two Parameters

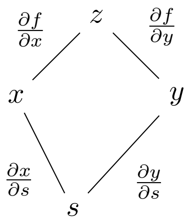
When $z = f(x, y)$ and both $x = x(s, t)$ and $y = y(s, t)$, we have:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

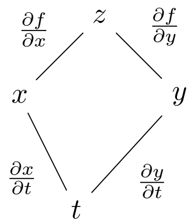
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Chain Rule Diamond Trees

The chain rule can be visualized using **diamond trees** that show the dependency relationships:



For $\frac{\partial z}{\partial s}$



For $\frac{\partial z}{\partial t}$

Example (2-2 Case Example). Let $z = x^2 + y^2$, where $x = s + t$ and $y = s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.
Solution: First, compute the partial derivatives:

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y \tag{10.1}$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial x}{\partial t} = 1 \tag{10.2}$$

$$\frac{\partial y}{\partial s} = 1, \quad \frac{\partial y}{\partial t} = -1 \tag{10.3}$$

Using the chain rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \tag{10.4}$$

$$= 2x \cdot 1 + 2y \cdot 1 = 2(x + y) = 2(2s) = 4s \tag{10.5}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \tag{10.6}$$

$$= 2x \cdot 1 + 2y \cdot (-1) = 2(x - y) = 2(2t) = 4t \tag{10.7}$$

10.1.3 Additional Cases

Example (3-2 Case: Three Variables, Two Parameters). Let $w = f(x, y, z)$ where $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$. Then:

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

Example (3-1 Case: Three Variables, One Parameter). If $w = f(x, y, z)$ where $x = x(t)$, $y = y(t)$, and $z = z(t)$, then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Note: This is an ordinary derivative since w depends on only one independent variable t.

Example (2-1 Case: Two Variables, One Parameter). For $z = f(x, y)$ where $x = x(t)$ and $y = y(t)$:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example (3-3 Case: Three Variables, Three Parameters). When $w = f(x, y, z)$ and each variable depends on three parameters u, v, s:

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

with similar expressions for $\frac{\partial w}{\partial v}$ and $\frac{\partial w}{\partial s}$.

10.2 Implicit Differentiation

When a relationship between variables is given implicitly by an equation $F(x, y) = 0$, we can find $\frac{dy}{dx}$ without explicitly solving for y.

10.2.1 Theory

For an equation $F(x, y) = c$ that implicitly defines y as a function of x, the **implicit differentiation formula** is:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

provided that $\frac{\partial F}{\partial y} \neq 0$.
Derivation: If $F(x, y) = 0$ and $y = y(x)$, then by the chain rule:

$$\frac{d}{dx}[F(x, y(x))] = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ gives the formula above.

Example (Implicit Differentiation Example). Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 25$.
Solution: Let $F(x, y) = x^2 + y^2 - 25$. Then:

$$\frac{\partial F}{\partial x} = 2x \tag{10.8}$$

$$\frac{\partial F}{\partial y} = 2y \tag{10.9}$$

Therefore:

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

provided $y \neq 0$.

Remark. Implicit differentiation is particularly useful when solving for y explicitly would be difficult or impossible, such as with equations like $xy + \sin(xy) = x^2 + y^3$.

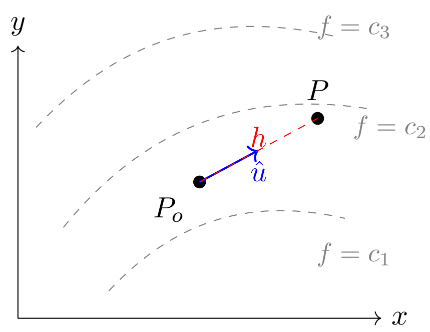
10.3 Directional Derivatives

The **directional derivative** measures the rate of change of a function in a specific direction, generalizing the concept of partial derivatives.

10.3.1 Definition and Derivation

Let $f(x, y)$ be a function and $\mathbf{u} = \langle a, b \rangle$ be a **unit vector** (i.e., $|\mathbf{u}| = 1$). The directional derivative of f at point (x_0, y_0) in the direction of $\mathbf{u}$ is defined as:

Geometric Intuition and Figure



Let point P be (x, y). We know,

$$\vec{P_oP} = h\hat{u} \tag{10.10}$$
$$\langle x - x_o, y - y_o\rangle = \langle ha, hb\rangle \tag{10.11}$$
$$\therefore \ \langle x, y\rangle = \langle x_o + ha, y_o + hb\rangle \tag{10.12}$$

Then the differentiation in $\hat{u}$ direction is given by:

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{10.13}$$

Derivation Using the Chain Rule

Step 1: Parameterize the path.
Let $g(t) = f(x_0 + ta, y_0 + tb)$. At $t = 0$, we are at (x_0, y_0), and at $t = h$, we are at $(x_0 + ha, y_0 + hb)$.
Step 2: Recognize the derivative.

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

Step 3: Apply the chain rule.
Let $x(t) = x_0 + ta$ and $y(t) = y_0 + tb$. Then:

$$\frac{dx}{dt} = a \quad \text{and} \quad \frac{dy}{dt} = b$$

By the chain rule:

$$g'(t) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b$$

Step 4: Evaluate at the point of interest.
At $t = 0$:

$$g'(0) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot a + \frac{\partial f}{\partial y}(x_0, y_0) \cdot b$$

Step 5: Express in terms of gradient and dot product.
The gradient of f at (x_0, y_0) is:

$$\nabla f(x_0, y_0) = \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle$$

Therefore:

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

Final Formula

For a differentiable function $f(x, y)$ and unit vector $\mathbf{u} = \langle a, b \rangle$:

$$\boxed{D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = \frac{\partial f}{\partial x}(x_0, y_0) \cdot a + \frac{\partial f}{\partial y}(x_0, y_0) \cdot b}$$

Geometric Interpretation

The directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ represents:

- The **slope** of the tangent line to the curve formed by intersecting the surface $z = f(x, y)$ with a vertical plane containing the direction vector $\mathbf{u}$
- The **instantaneous rate of change** of f per unit distance moved in the direction $\mathbf{u}$

Special Cases

- When $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$: $D_{\mathbf{i}} f = \frac{\partial f}{\partial x}$
- When $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$: $D_{\mathbf{j}} f = \frac{\partial f}{\partial y}$

This shows that partial derivatives are special cases of directional derivatives along the coordinate axes.

10.3.2 Computational Formula

For a differentiable function $f(x, y)$, the directional derivative can be computed using:

$$D_{\mathbf{u}} f(x, y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b = \nabla f \cdot \mathbf{u}$$

where $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ is the **gradient** of f.

10.3.3 Connection to Partial Derivatives

Special Cases: - When $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$: $D_{\mathbf{i}} f = \frac{\partial f}{\partial x}$ - When $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$: $D_{\mathbf{j}} f = \frac{\partial f}{\partial y}$ This shows that **partial derivatives are special cases of directional derivatives** along the coordinate axes.

> Example (Directional Derivative Example). Find the directional derivative of $f(x, y) = x^2 + xy + y^2$ at point $(1, 2)$ in the direction of vector $\mathbf{v} = \langle 3, 4 \rangle$.
> **Solution:** First, find the unit vector:
>
> $$|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5 \quad \Rightarrow \quad \mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

Compute the partial derivatives:

$$\frac{\partial f}{\partial x} = 2x + y \tag{10.14}$$
$$\frac{\partial f}{\partial y} = x + 2y \tag{10.15}$$

At point $(1, 2)$:

$$\frac{\partial f}{\partial x}(1,2) = 4, \quad \frac{\partial f}{\partial y}(1,2) = 5$$

Therefore:

$$D_{\mathbf{u}} f(1,2) = 4 \cdot \frac{3}{5} + 5 \cdot \frac{4}{5} = \frac{12}{5} + \frac{20}{5} = \frac{32}{5}$$

Exercise. Let $z = \sin(xy) + x^2 y$ where $x = e^t$ and $y = t^2$. Find $\frac{dz}{dt}$ using the chain rule.

Answer. Using the chain rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Where:

$$\frac{\partial z}{\partial x} = y\cos(xy) + 2xy \tag{10.16}$$
$$\frac{\partial z}{\partial y} = x\cos(xy) + x^2 \tag{10.17}$$
$$\frac{dx}{dt} = e^t \tag{10.18}$$
$$\frac{dy}{dt} = 2t \tag{10.19}$$

Substituting $x = e^t$ and $y = t^2$:

$$\frac{dz}{dt} = (t^2\cos(e^t \cdot t^2) + 2e^t \cdot t^2) \cdot e^t + (e^t\cos(e^t \cdot t^2) + e^{2t}) \cdot 2t$$

⊛

Remark. The chain rule for partial derivatives extends naturally to functions of more variables and more complex dependency structures. The key insight is to trace all possible paths from the dependent variable to the independent variable through the dependency tree, multiplying derivatives along each path and summing the results.

For more detailed treatments of these topics, see Paul's Online Math Notes and Libre-

Texts Calculus III.

Chapter 11

Critical Points in Two Variable Functions

Lecture 11: Eleventh Lecture

A function $z = f(x, y)$ is a function with two variables. It may have *Local or Global* ***Minima*** *or* ***Maxima*** *or* ***Saddle Point***. One can also check the concavity of the surface. 4 Jun. 11:39

11.1 Critical Points

A critical point is a point $(a, b, f(a, b))$ where $f_x = f_y = 0$ or f_x or f_y is undefined.

11.2 Saddle Points

If a point $(a, b, f(a, b))$ is a stationary point and if,

$$\exists(x, y) \text{ s.t. } f(x, y) > f(a, b) \text{ somewhere } \wedge f(x, y) < f(a, b) \text{ elsewhere}$$

Then, $(a, b, f(a, b))$ is a saddle point.

11.3 Hessian/Discriminant of f: D_f

$$\mid \mathbf{H}_f \mid = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - {f_{xy}}^2$$

where $\mathbf{H}_f$ is the *Hesian Matrix* of f such that:

$$(\mathbf{H}_f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Here, at (a, b),

$$f_{xy} = f_{yx}$$

Since f is continuos at (a, b). See Clairaut's Theorem.

11.4 Confirming Extremes or Saddle

Check at $(x, y) = (a, b)$.

$$\textbf{Local Maxima} : f_{xx} < 0 \text{ and } D_f > 0) \tag{11.1}$$
$$\textbf{Local Minima} : f_{xx} > 0 \text{ and } D_f < 0) \tag{11.2}$$
$$\textbf{Saddle} : f_{xx} = 0 \text{ and } D_f < 0) \tag{11.3}$$
$$\textbf{Inconclusive} : D_f = 0) \tag{11.4}$$

11.5 Finding Absolute Extremas

A boundry will be given. To find the extremas,

1. find all the critical points inside the boundry.
2. find all the critical points in the boundry(single variable function)
3. Compare the values of the function for all candidates
4. List the absolute extremas(points of maximum and minimum value of the function).

Chapter 12

Double Integrals and Application using Fubini's Theorem

12.1 Review of the Definite Integrals

$$\int_a^b f(x)\,dx \tag{12.1}$$

Definition 12.1.1 (Definite Integral as Area). The definite integral of a function $f(x)$ over the interval $[a,b]$ represents the net area under the curve from $x = a$ to $x = b$.

Remark (Geometric Interpretation). If $f(x) \geq 0$ on $[a,b]$, then the definite integral represents the area under the curve.

12.2 Double Integrals

12.2.1 Double Integrals as Area, Volume and Average

Definition 12.2.1 (Double Integral over Region). The double integral of $f(x,y)$ over region R is denoted by:

$$\iint_R f(x,y)\,dA \tag{12.2}$$

where $dA = dx\,dy$ or $dA = dy\,dx$ depending on the order of integration.

Remark (Area from Double Integral). When $f(x,y) = 1$, the double integral $\iint_R 1\,dA$ gives the area of region R.

Proof. If $f(x,y) \geq 0$, then $\iint_R f(x,y)\,dA$ represents the volume under the surface $z = f(x,y)$ over region R. ⊛

Definition 12.2.2 (Average Value Over Region). The average value of a function $f(x,y)$ over a region R is given by:

$$\bar{f} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA \tag{12.3}$$

Example (Average of $f(x,y) = x + y$ over Unit Square). Let $f(x,y) = x + y$, over the region $R = [0,1] \times [0,1]$.

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^1 \int_0^1 (x+y)\,dx\,dy \\
&= \int_0^1 \left[\int_0^1 x\,dx + \int_0^1 y\,dx\right] dy \\
&= \int_0^1 \left[\frac{1}{2} + y\right] dy \\
&= \left[\frac{1}{2}y + \frac{1}{2}y^2\right]_0^1 \\
&= \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

12.3 Fubini's Theorem

12.3.1 In Rectangular Region

Theorem 12.3.1. For a rectangular region $R : a \leq x \leq b$ and $c \leq y \leq d$,

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy \tag{12.4}$$

Fubini's Theorem for Rectangles. Since the region R is rectangular and the limits are constant, the order of integration can be changed without affecting the value of the double integral. This follows from iterated integral theory and Tonelli's theorem for non-negative functions or Fubini's theorem for integrable functions. ■

Example. Fubini: Rectangular Region $R = [0,1] \times [0,2]]$ Evaluate $\iint_R xy\,dA$, where

$R = [0, 1] \times [0, 2]$.

$$\begin{aligned}
\iint_R xy\, dA &= \int_0^1 \int_0^2 xy\, dy\, dx \\
&= \int_0^1 x \left[\int_0^2 y\, dy \right] dx \\
&= \int_0^1 x \cdot \left[\frac{1}{2} y^2 \right]_0^2 dx \\
&= \int_0^1 x \cdot 2\, dx = \left[x^2\right]_0^1 = 1
\end{aligned}$$

12.3.2 In Any Region

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$
2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Figure 12.1: Fubini's General Theorem

Definition 12.3.1 (Fubini's Theorem on General Regions).

Example (Triangular Region Bounded by $x = 0$, $y = 0$, $x + y = 1$). Let R be the triangular region bounded by $x = 0$, $y = 0$, and $x + y = 1$. Compute $\iint_R (x + y)\, dA$. The region $R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}$

$$\begin{aligned}
\iint_R (x+y)\,dA &= \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx \\
&= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^{1-x} dx \\
&= \int_0^1 \left[x(1-x) + \frac{1}{2}(1-x)^2\right] dx \\
&= \int_0^1 \left[x - x^2 + \frac{1}{2}(1-2x+x^2)\right] dx \\
&= \int_0^1 \left[x - x^2 + \frac{1}{2} - x + \frac{1}{2}x^2\right] dx \\
&= \int_0^1 \left[-\frac{1}{2}x^2 + \frac{1}{2}\right] dx \\
&= \left[-\frac{1}{6}x^3 + \frac{1}{2}x\right]_0^1 = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}
\end{aligned}$$

Chapter 13

Double Integrals in Polar Form

13.1 Double Integrals

Definition 13.1.1 (Double Integral in Polar Coordinates). If $f(x, y)$ is continuous on a region R, and we convert to polar coordinates using:

$$x = r\cos\theta, \quad y = r\sin\theta$$

then the double integral becomes:

$$\iint_R f(x, y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{13.1}$$

Remark (Why r Appears). The extra factor r accounts for the Jacobian determinant when changing from rectangular to polar coordinates.

13.2 Relating the Double Integrals in Cartesian and Polar Forms

$$dx\,dy = r\,dr\,d\theta \tag{13.2}$$

Proof. The Jacobian determinant for the transformation from (x, y) to (r, θ) is:

$$J = \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence, $dx\,dy = r\,dr\,d\theta$. ⊛

13.3 Converting Double Integrals from Cartesian to Polar Forms

Exercise. Convert the integral

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} 1 \, dy \, dx$$

to polar form.

Answer. The given region is the upper half of the unit disk:

$$R = \{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2}\}$$

This corresponds to:

$$R = \{(r, \theta) : 0 \leq r \leq 1, \ 0 \leq \theta \leq \pi\}$$

Converting the integral:

$$\begin{aligned}
\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} 1 \, dy \, dx &= \iint_R 1 \, dx \, dy \\
&= \int_0^{\pi} \int_0^1 1 \cdot r \, dr \, d\theta \\
&= \int_0^{\pi} \left[\frac{1}{2} r^2 \right]_0^1 d\theta \\
&= \int_0^{\pi} \frac{1}{2} \, d\theta = \left[\frac{1}{2} \theta \right]_0^{\pi} = \frac{\pi}{2}
\end{aligned}$$

⊛

Remark (When to Use Polar Coordinates). Use polar form when the region is circular or radial symmetry is present, as integration becomes simpler.

Equivalence of Cartesian and Polar Forms. Let R be a region described easily in polar coordinates. Then:

$$\iint_R f(x, y) \, dx \, dy = \iint_R f(r \cos \theta, r \sin \theta) \cdot r \, dr \, d\theta$$

This is justified by the substitution rule in multiple integrals and the Jacobian determinant of the coordinate transformation. ■

Appendix

Bibliography

[Geo15] Joel R. Hass George B. Thomas Jr. Maurice D. Weir. *Thomas' Calculus*. Pearson Education, 2015. URL: https://books.google.com.np/books/about/Thomas_Calculus.html?id=DdtQCwAAQBAJ&redir_esc=y.