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# Differential Equations

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## Abstract

This course was taught by Prof. Dil Bahadur Gurung. It teaches Differential Equations

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# 1 Introduction to Differential Equations

## 1.1 Need of Differential Equations

Differential equations are essential tools for modeling systems that involve **change** or **motion**. They help us describe the relationships between a function and its derivatives — enabling prediction of future behavior of systems based on current conditions.

Some common applications include:
- Modeling population growth and decay
- Describing motion of objects (Newton's laws)
- Representing electrical circuits
- Modeling spread of diseases or epidemics
- Predicting stock prices and economic changes

## 1.2 Use of Differential Equation in Data Science

In **Data Science**, differential equations play a crucial role in:
- Modeling **continuous-time processes** such as temperature change or chemical concentration.
- Describing **dynamic systems** used in time series forecasting and control systems.
- Optimizing **machine learning models** (e.g., gradient descent as an iterative differential process).
- Defining **neural differential equations**, where model parameters evolve according to differential laws.

Mathematical models derived from data often lead to **ordinary differential equations (ODEs)** or **partial differential equations (PDEs)** that capture relationships among variables.

## 1.3 Differential Equations

**Definition 1.3.1 (Differential Equation)**

A **Differential Equation** is an equation that relates a function with one or more of its **derivatives**. It expresses how a quantity changes with respect to another.

Mathematically, a differential equation can be represented as:

$$F\left(x, y, y', y'', \dots, y^{\{(n)\}}\right) = 0 \tag{1}$$

where:
- $x$ is the **independent variable**,
- $y$ is the **dependent variable** (function of $x$),
- $y', y'', \dots, y^{\{(n)\}}$ are the **derivatives** of $y$ with respect to $x$.

*Example.* $y'' + 3y' + 2y = 0$ is a **second-order linear homogeneous differential equation**.

♣

## 1.4 Classification of Differential Equations

Differential equations can be classified based on the following criteria:

- **Order**: The highest derivative present in the equation.
- **Degree**: The power of the highest derivative (when the equation is polynomial in derivatives).
- **Linearity**: Whether the dependent variable and its derivatives appear linearly.

For example:
- First-order ODE: $y' + 2y = 5$
- Second-order nonlinear ODE: $y'' + y(y')^2 = 0$

# 2 Solution of Differential Equations

## 2.1 Types of Variables in Differential Equations

Differential equations involve different types of variables that define how the function behaves and changes. These variables are essential for setting up the correct model.

### 2.1.1 Independent Variables

The **independent variable** is the quantity upon which other quantities depend. It is typically represented as **x**, **t**, or **r** depending on the context.

Example: In $y' = 3x^2$, the variable **x** is the independent variable.

### 2.1.2 Dependent Variables

The **dependent variable** is the function that depends on the independent variable. It represents the unknown quantity we aim to find.

Example: In $y' = 3x^2$, **y** is the dependent variable because its value changes according to **x**.

### 2.1.3 Parameters

A **parameter** is a constant that affects the behavior or family of solutions but remains fixed for a particular solution. Parameters determine the shape or scale of the solution curve.

Example: In $y = Ce^{2x}$, **C** is a parameter.

## 2.2 Solution of Differential Equation

**Definition 2.2.1 (Solution of Differential Equation)**

The **solution of a differential equation** is a function that satisfies the equation for all values of the independent variable within a given interval.

Formally, if a function $y = \varphi(x)$ satisfies the equation

$$F\left(x, y, y', y'', ..., y^{\{(n)\}}\right) = 0 \tag{2}$$

then, the function $y = \varphi(x)$ is called the **solution** of the differential equation.

## 2.3 General Solution

A **general solution** of a differential equation includes **arbitrary constants** that represent an infinite family of solutions.

> Note
>
> **A general solution of a differential equation contains as many arbitrary constants as the order of the differential equation.**

*Example.*

$$\frac{dy}{dx} = 2x \tag{3}$$

Using separation of variables,

$$dy = 2x dx \tag{4}$$

Integrating both sides gives,

$$y = x^2 + c_1 \tag{5}$$

where **C** is an arbitrary constant representing the general solution.

## 2.4 Order of Differential Equation

The **order** of a differential equation is the **highest derivative** of the dependent variable present in the equation.

Example:

- **First order:**

$$\frac{dy}{dx} + y = 0 \tag{6}$$

- **Second order:**

$$\frac{d^2y}{dx^2} + 3y = 0 \tag{7}$$

Mathematically, for a differential equation

$$a_{n(x)} \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x) \frac{dy}{dx} + a_0(x) y = 0 \tag{8}$$

the **order** is **n**.

## 2.5 Integral Curves

**Definition 2.5.1 (Integral Curves)**

The **integral curve** of a differential equation,

$$F\left(x, y, y', y'', ..., y^{\{(n)\}}\right) = 0 \tag{9}$$

is the **graph of its solution function**,

$$y = \varphi(x) + c_1 + c_2 + ... + c_n = \varphi(x) + C \tag{10}$$

in the coordinate plane.

It represents the set of all points $(x, y)$ that satisfy the differential equation in Equation 9. Different values of the arbitrary constant **C** produce different integral curves from the general solution. The set of all such curves forms a **family of integral curves**.

*Example.* For $y = x^2 + C$, each value of **C** gives a distinct **parabola**, and all such parabolas together form a **family of integral curves**.

> ⓘ Note
>
> No two integral curves in the family of curves intersect each other.

♣

## 2.6 Initial Value Problem

**Definition 2.6.1 (Initial Value Problem (IVP))**

An **Initial Value Problem (IVP)** consists of a differential equation together with an **initial condition** that specifies the value of the dependent variable (and possibly its derivatives) at a particular point.

Formally,

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{11}$$

is a common form of an IVP of first order differential equation, where $y(x_0) = y_0$ is the initial condition and $f(x, y) = \frac{dy}{dx}$ is the differential equation.

The goal is to find a **particular solution y(x)** that satisfies both the differential equation and the initial condition.

> ⓘ Note
>
> **An Initial Value Problem (IVP) has a unique particular solution that satisfies both the differential equation and the initial condition.**

♣

#### 2.6.1 Initial Condition

The **initial condition** provides specific values for the dependent variable (and possibly its derivatives) at a given value of the independent variable.

#### 2.6.2 Solution Steps

To solve an **Initial Value Problem (IVP)**:

1) Input: Differential equation $\frac{dy}{dx} = f(x, y)$ and initial condition $y(x_0) = y_0$
2) Find the general solution of the differential equation.
3) Substitute initial condition in general solution.
4) Output: The particular solution satisfying both the differential equation and the initial condition

*Example.* Solve the IVP: $\frac{dy}{dx} = 2x, \quad y(0) = 3$

The general solution for given differential equation is

$$y = x^2 + C \tag{12}$$

Substituting the initial condition $y(0) = 3$ gives,

$$3 = 0 + C \tag{13.1}$$

$$\therefore C = 3 \tag{13.2}$$

Thus, the particular solution is

$$y = x^2 + 3 \tag{14}$$

### 2.7 Particular Solution

A **particular solution** of a differential equation is a specific solution obtained by assigning particular values to the arbitrary constants in the general solution.

*Example.* For the differential equation

$$\frac{dy}{dx} = 2x \tag{15}$$

the general solution is

$$y = x^2 + C \tag{16}$$

By choosing **C = 5**, we get a particular solution:

$$y = x^2 + 5 \tag{17}$$

## 3 Real World Models using Differential Equations

### 3.1 Bacteria Growth Model

The growth of a bacteria population can be modeled using a differential equation. Let **P(t)** be the population at time **t**. The rate of change of the population is proportional to the current population:

$$\frac{dP}{dt} = kP \tag{18}$$

where **k** is the growth rate constant.

### 3.2 Radioactive Decay Model

The decay of a radioactive substance can be described by a differential equation. Let **N(t)** be the quantity of the substance at time **t**. The rate of change of the quantity is proportional to the current quantity:

$$\frac{dN}{dt} = -\lambda N \tag{19}$$

where **λ** is the decay constant.

### 3.3 Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient temperature. Let **T(t)** be the temperature of the object at time **t**, and **T_a** be the ambient temperature. The differential equation is:

$$\frac{dT}{dt} = -k(T - T_a) \tag{20}$$

where **k** is a positive constant related to the cooling rate.

### 3.4 Population Growth Model

The growth of a population can be modeled using a differential equation. Let **P(t)** be the population at time **t**. The rate of change of the population can be modeled as:

$$\frac{dP}{dt} = rP\left(1 - \left(\frac{P}{K}\right)\right) \tag{21}$$

where **r** is the intrinsic growth rate and **K** is the carrying capacity of the environment.

### 3.5 Chemical Reaction Model

The rate of a chemical reaction can be described by a differential equation. Let **C(t)** be the concentration of a reactant at time **t**. The rate of change of the concentration can be modeled as:

$$\frac{dC}{dt} = -kC^n \tag{22}$$

where **k** is the rate constant and **n** is the order of the reaction.

### 3.6 Falling Object Model

The motion of a falling object under the influence of gravity and air resistance can be modeled using a differential equation. Let **v(t)** be the velocity of the object at time **t**. The differential equation is:

$$\frac{dv}{dt} = g - \left(\frac{\gamma}{m}\right)v \tag{23}$$

where **g** is the acceleration due to gravity, $\gamma$ is the drag coefficient, and **m** is the mass of the object.

Variables: t, m, g, $\gamma$, $v(t)$
Units:
- **t**: seconds (s)
- **m**: kilograms (kg)
- **g**: meters per second squared ($m/s^2$)
- $\gamma$: kilograms per meter (kg/m)
- **v(t)**: meters per second (m/s)

### 3.7 Real World Modelling Scenario

Consider the following assumptions:
1) There is a crop field where there is a certain population of pests that damage the crops.
2) To control the pest population, a pesticide is sprayed at regular intervals.
3) At one such interval, the pesticide kills a constant number of pests in the whole field called the predation rate.
4) The pests reproduce at a rate proportional to their current population.
5) time is measured in the intervals between pesticide applications.

**Modelling:**

Let **P(t)** be the pest population at time **t**.

Then according to Assumption 4, the growth of the pest population can be modeled as:

$$\frac{dP}{dt}\alpha P \tag{24}$$

According to Assumption 3, the pesticide kills a constant number of pests, say **k**, at each interval.

Thus, the overall model becomes:

$$\frac{dP}{dt} = \gamma P - k \tag{25}$$

where $\gamma$ is the reproduction rate constant.

## 4 Qualitative Analysis

Qualitative Study of Differential Equations involves analyzing the behavior of solutions without finding explicit solutions.

### 4.1 Autonomous and Non-Autonomous Equations

Consider the differential equation:

$$\frac{dy}{dx} = f(x, y) \tag{26}$$

This is a non-autonomous first-order differential equation.

Consider the differential equation:

$$\frac{dy}{dx} = g(y) \tag{27}$$

This is an autonomous first-order differential equation.

> **Note**
>
> Only autonomous equations are considered for qualitative analysis.

## 4.2 Steps for Qualitative Analysis

1) Identify Equilibrium/Critical/Fixed Points:

   Set $\frac{dy}{dx} = 0$ and solve for $\mathbf{y}$ to find equilibrium points.

2) Phase Diagram:

   Plot the equilibrium points on the $\mathbf{y}$-axis and indicate the direction of change (increasing or decreasing) in each interval.

3) Slope field / direction fields:

   A direction field is a graphical representation of the slopes of solutions at various points in the plane.

   3.a) To draw a direction field, select a grid of points in the $\mathbf{xy}$-plane.
   3.b) At each point, compute the slope given by the differential equation.
   3.c) Draw a small line segment at each point with the computed slope.

4) Bifurcation Diagram:

   A bifurcation diagram shows how the equilibrium points change as a parameter in the differential equation is varied. (y vs bifurcation parameter)

   4.a) Identify the bifurcation parameter in the differential equation.
   4.b) Vary the bifurcation parameter and find the equilibrium points for each value.
   4.c) Plot the equilibrium points against the bifurcation parameter to visualize the bifurcation diagram.

5) Stability Analysis:

   Analyze the stability of equilibrium points by examining the sign of the derivative of the right-hand side of the differential equation at those points.

   5.a) If the derivative is negative at an equilibrium point, it is stable (attracting).
   5.b) If the derivative is positive at an equilibrium point, it is unstable (repelling).
   5.c) If the derivative is zero at an equilibrium point, it is neither stable nor unstable (neutral).
   5.d) Use the phase diagram and direction fields to visually confirm the stability of equilibrium points.

### 4.3 Example of Qualitative Analysis

*Example.* Discuss the qualitative analysis of the falling object model given by the differential equation:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{28}$$

*Solution.* Rewrite the equation as,

$$\frac{dv}{dt} = \frac{49 - v}{5} \tag{29}$$

1) Equilibrium solution:

   Set $\frac{dv}{dt} = 0$

   => $49 - v = 0$

   => $v = 49$ m/s

2) Phase Diagram:(dt vs v)

   The equilibrium point is at **v** = 49 m/s.

$$f(v) = \frac{49 - v}{5} \tag{30}$$

   Rearranging, we have:

$$\frac{v}{49} + \frac{f(v)}{9.8} = 1 \tag{31}$$

Figure 1: Phase diagram for the D.E,
x-axis -> v,
y-axis -> f(v)

3) Slope field/ Direction field table

| v | sign of v | Behaviour of v | Arrow Direction |
|---|---|---|---|
| v < 49 | +ve | increasing | ↗ |
| v=49 | zero | no change | - |
| v>49 | -ve | decreasing | ↘ |

> **Note**
>
> Detail Study of slope field gives the nature of the solution.