# Quadratic forms and Galois cohomology

## The connection (Milnor's conjectures)

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# Description via symbols

Let $F$ be a field with $\mathrm{char}(F) \neq 2$.

$$
\begin{tikzcd}
& \overbrace{(F^\times)/(F^\times)^2 \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} (F^\times)/(F^\times)^2}^{n \text{ times}} \arrow[dl] \arrow[d] \arrow[dr, "\text{induced by the cup product}"] & \\
I^n(F)/I^{n+1}(F) & K_n^M(F)/2 \arrow[l, dashed, "s_n"'] \arrow[r, dashed, "h^n_{F,2}"'] & H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n})
\end{tikzcd}
$$

with

$$s_n \colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F), \quad \{a_1, \ldots, a_n\} \mapsto (\langle a_1 \rangle - \langle 1 \rangle) \cdot \ldots \cdot (\langle a_n \rangle - \langle 1 \rangle)$$

**Question**

Are $s_n$ and $h^n_{F,2}$ isomorphisms?

# Proof that $s_n$ is well-defined and surjective

## Proposition

The map $s_n\colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$ is well-defined and surjective.

## Proof.

Consider the map $F^\times \times \ldots \times F^\times \to I^n(F)/I^{n+1}(F), \quad (a_1, \ldots, a_n) \mapsto (\langle a_1\rangle - \langle 1\rangle) \cdot \ldots \cdot (\langle a_n\rangle - \langle 1\rangle)$.

$n$-linear: $\langle a\rangle - \langle 1\rangle + \langle b\rangle - \langle 1\rangle = \langle ab\rangle - \langle 1\rangle \mod I^2(F)$, because
$\langle a\rangle + \langle b\rangle - \langle ab\rangle - \langle 1\rangle = -(\langle a\rangle - \langle 1\rangle)(\langle b\rangle - \langle 1\rangle) \in I^2(F)$.

Steinberg: $(\langle a\rangle - \langle 1\rangle)(\langle 1-a\rangle - \langle 1\rangle) = \langle a(1-a)\rangle - \langle a\rangle - \langle 1-a\rangle + \langle 1\rangle = 0$, because
$\langle a\rangle + \langle 1-a\rangle = \langle a + (1-a)\rangle + \langle a(1-a)(a+1-a)\rangle = \langle 1\rangle + \langle a(1-a)\rangle$.

mod 2: $2\{a_1, \ldots, a_n\} = \{a_1^2, a_2, \ldots, a_n\} \mapsto \underbrace{(\langle a_1^2\rangle - \langle 1\rangle)}_{=\langle 1\rangle - \langle 1\rangle = 0}(\langle a_2\rangle - \langle 1\rangle) \cdot \ldots \cdot (\langle a_n\rangle - \langle 1\rangle) = 0$.

surjective: $I(F)$ is additively generated by Pfister forms $\langle a\rangle - \langle 1\rangle$.

□

# Splitting the problem into two

Let $F$ be a field with $\mathrm{char}(F) \neq 2$.

**Conjecture ('Milnor conjecture on norm residue symbol', 'Bloch-Kato conjecture for prime 2')**

The map $h^*_{F,2} \colon K^M_*(F)/2 \to H^*_{\mathrm{Gal}}(F, \mu_2^{\otimes *})$ is an isomorphism.

**Conjecture ('Milnor conjecture on quadratic forms')**

The map $s_* \colon K^M_*(F)/2 \to \mathrm{gr}_I(W(F))$ is an isomorphism.

# Proof strategy for the Milnor conjecture on norm residue symbol

Want to show: The map $h^n_{F,2}\colon K^M_n(F)/2 \to H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n})$ is an isomorphism.

1) Induction on $n$: If the statement hold for all fields $F$ and $n < N$ then it holds for $n = N$.
2) $h^n_{F,2}$ is an isomorphism for certain 'big' fields $F$
(i.e. $F$ has no odd degree extensions and $K^M_n(F) = 2K^M_n(F)$)
3) Assume there is a field $F$ for which $h^n_{F,2}$ is not an isomorphism, then there is an extension providing a counter example to the previous step.

Details: For any $\{a_1, \dots, a_n\} \in K^M_n(F)$ there is a field extension $F \hookrightarrow F'$ such that $\{a_1, \dots, a_n\} \in 2K^M_n(F')$ and $K^M_n(F')/2 \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$ is not an isomorphism
(take a big colimit to get a single field providing a counter example).

# On the third step: how to find a good field extension

Suppose there is a field $F$ for which $h^n_{F,2}$ is not an isomorphism.

Goal: for a symbol $\{a_1, \dots, a_n\} \in K^M_n(F)$ find a field extension $F'$ such that $\{a_1, \dots, a_n\} \in 2K^M_n(F')$ and $K^M_n(F')/2 \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$ is not an isomorphism.

The first part is easy: take $F' = F[X]/(X^2 - a_i)$ for any $i = 1, \dots, n$.
The problem is to control $K^M_n(F')/2 \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$.
Instead, use $F(Q_{\{a_1,\dots,a_n\}})$ with

$$Q_{\{a_1,\dots,a_n\}} = \{q_{\langle\langle a_1\rangle\rangle\otimes\dots\otimes\langle\langle a_{n-1}\rangle\rangle}(x_0, \dots, x_{2^{n-1}-1}) - a_n x^2_{2^{n-1}} = 0\} \subseteq \mathbb{P}^{2^{n-1}}_F$$

i) Introduce motivic cohomology to study the behaviour of $K^M_n(F)/2 \to K^M_n(F')/2$ and $H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n}) \to H^n_{\mathrm{Gal}}(F', \mu_2^{\otimes n})$

ii) Algebraic topological input: motivic Steenrod operations

iii) Algebraic geometry input: identify a direct summand of the motive of $Q_{\{a_1,\dots,a_n\}}$

# Starting the proof of the Milnor conjecture on quadratic forms

Want to show: The map $s_n \colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$ is an isomorphism.

Known cases are:

i) The map $s_n$ is surjective for all $n \in \mathbb{N}$ (see before).
ii) The maps $s_0 \colon \mathbb{Z}/2\mathbb{Z} \to W(F)/I(F)$ and $s_1 \colon F^\times/(F^\times)^2 \to I(F)/I^2(F)$ are isomorphisms
iii) The map $s_2$ is an isomorphism (by writing down an inverse)

Using 'standard' facts about quadratic forms (Arason-Pfister Hauptsatz, ...) one can show:

**Proposition**

$$s_n(\{a_1, \dots, a_n\}) = s_n(\{b_1, \dots, b_n\}) \quad \Leftrightarrow \quad \{a_1, \dots, a_n\} = \{b_1, \dots, b_n\} \in K_n^M(F)/2$$

This does not show the injectivity of $s_n$. It shows injectivity for pure symbols.

# Proof strategy of the Milnor conjecture on quadratic forms

Have: Injectivity of $s_n \colon K_n^M(F)/2 \to I^n(F)/I^{n+1}(F)$ on pure symbols

Idea: Find a field extension $F'$ such that $\alpha \in K_n^M(F)/2$ becomes a pure symbol in $K_n^M(F')/2$.
Observation: By going to $F(Q_{\{a_1,\ldots,a_n\}})$ the symbol $\{a_1,\ldots,a_n\}$ vanishes.
Key: The kernel $K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2$ is a s nice as possible.

## Proposition

If $\alpha = \{a_1,\ldots,a_n\} \neq 0 \in K_n^M(F)/2$, then

$$\ker\left(K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2\right) = \mathbb{Z}/2\mathbb{Z}\cdot\alpha.$$

For $\alpha = \alpha_1 + \ldots + \alpha_k \in K_n^M(F)/2$ take $F' = F(Q_{\alpha_1})(Q_{\alpha_2})\ldots(Q_{\alpha_i})$ such that

$$\alpha \neq 0 \in K_n^M(F')/2 \text{ and } \alpha = 0 \in K_n^M(F'(Q_{\alpha_{i+1}}))/2.$$

# Understanding the kernel

**Proposition**

If $\alpha = \{a_1, \ldots, a_n\} \neq 0 \in K_n^M(F)/2$, then

$$\ker\left(K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2\right) = \mathbb{Z}/2\mathbb{Z} \cdot \alpha.$$

The maps $K_n^M(F)/2 \to K_n^M(F(Q_\alpha))/2$ already played a crucial role in the previous proof.

i) Describe the kernel in terms of a motivic cohomology group.

ii) Again use the splitting of the motive of $Q_\alpha$.

# Summary

1) The connection between quadratic forms and Galois cohomology has the form

$$I^n(F)/I^{n+1}(F) \cong K_n^M(F)/2 \cong H^n_{\mathrm{Gal}}(F, \mu_2^{\otimes n}),$$

2) These objects have a nice description via symbols modulo a single relation in degree 2

$$K_n^M(F) = (F^\times)^{\otimes n}/\langle \ldots \otimes a \otimes 1 - a \otimes \ldots \rangle,$$

3) The proofs require heavy, but interesting, machinery (e.g. motivic cohomology).