AC-3
August 2015 QE

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1. (20 pts) Solve the following linear program,

$$\begin{aligned} \text{maximize} \quad & -x_1 - 2x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + 2x_2 - x_3 = 5 \\ & 2x_1 + 3x_2 - x_3 = 6 \\ & x_1 \text{ free}, x_2 \geqslant 0, x_3 \leqslant 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 3 & -1 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 4 \end{array}\right)$$

The answer is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 1-\alpha \\ -3-\alpha \end{bmatrix}$$

So the function to be maximized is

$$-\alpha - 2(1-\alpha) + 4(-3-\alpha) = -14 - 3\alpha$$

Since $x_2 \geqslant 0$ and $x_3 \leqslant 0$,

$$\begin{cases} 1-\alpha \geqslant 0 \\ -3-\alpha \leqslant 0 \end{cases} \to \begin{cases} \alpha \leqslant 0 \\ \alpha \geqslant -3 \end{cases} \to -3 \leqslant \alpha \leqslant 0$$

To maximize the function, *alpha* is chosen to be -3. Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}.$$

2. (20 pts) Formulate the first-order necessary conditions for the quadratic program,

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}, \end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{c} \in \mathbb{R}^m, m \leq n$, and $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$.

(a) (15 pts) Represent the obtained conditions as a system of linear equations and **write down the solution to the problem.**

(b) (5 pts) What is the condition, involving $\boldsymbol{A}$ and $\boldsymbol{Q}$, that must be satisfied for the solution to be unique?

(a)

$$\begin{aligned}
\mathcal{L}(x,\lambda) &= \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x} + \lambda\left(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{c}\right) \\
\nabla_{\boldsymbol{x}}\mathcal{L}(x,\lambda) &= \boldsymbol{x}^{*T}\boldsymbol{Q} - \boldsymbol{b}^T + \lambda\boldsymbol{A} = 0 \\
\boldsymbol{x}^{*T} &= (\boldsymbol{b}^T - \lambda\boldsymbol{A})\boldsymbol{Q}^{-1} \\
\boldsymbol{x}^* &= \boldsymbol{Q}^{-1}(\boldsymbol{b} - \lambda\boldsymbol{A}^T) \\
\boldsymbol{A}\boldsymbol{x}^* &= \boldsymbol{c} \\
\boldsymbol{A}\boldsymbol{Q}^{-1}(\boldsymbol{b} - \lambda^*\boldsymbol{A}^T) &= \boldsymbol{c} \\
\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{b} - \lambda^*\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T &= \boldsymbol{c} \\
\lambda^* &= \left(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\right)^{-1}\left(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{b} - \boldsymbol{c}\right) \\
\boldsymbol{x}^* &= \boldsymbol{Q}^{-1}\left(\boldsymbol{b} - \left(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\right)^{-1}\left(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{b} - \boldsymbol{c}\right)\boldsymbol{A}^T\right)
\end{aligned}$$

(b) A and Q need to be full rank for the solution to be unique.

3. (20 pts) Consider the optimization problem,

$$\begin{aligned} &\text{maximize} \quad && -x_1^2 + x_1 - x_2 - x_1x_2 \\ &\text{subject to} \quad && x_1 \geqslant 0, x_2 \geqslant 0 \end{aligned}$$

(a) (10 pts) Characterize feasible directions at the point

$$\boldsymbol{x}^* = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

(b) (10 pts) Write down the second-order necessary condition for $\boldsymbol{x}^*$. Does the point $\boldsymbol{x}^*$ satisfy this condition?

Rewrite the problem,

$$\begin{aligned} &\text{minimize} \quad && x_1^2 - x_1 + x_2 + x_1x_2 \\ &\text{subject to} \quad && x_1 \geqslant 0, x_2 \geqslant 0 \end{aligned}$$

(a) For $\boldsymbol{d}$ to be feasible, we need $d_2 \geqslant 0$ and d_1 can take an arbitrary value in $\mathbb{R}$.

(b)

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= \begin{bmatrix} 2x_1 - 1 + x_2 \\ 1 + x_1 \end{bmatrix} \\ \boldsymbol{F}(\boldsymbol{x}) &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ \boldsymbol{d}^T \nabla f(\boldsymbol{x}^*) &= \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} = \frac{3}{2} d_2 = 0 \\ d_2 &= 0 \\ \boldsymbol{d}^T \boldsymbol{F}(\boldsymbol{x}) \boldsymbol{d} &= \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= 2d_1^2 + 2d_1d_2 \\ &= 2d_1^2 \\ &\geqslant 0 \end{aligned}$$

Therefore, the point $\boldsymbol{x}^*$ satisfy the SONC.

4. (20 pts) Consider the following primal problem:

$$\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\
& -x_1 + 2x_2 + x_4 = 7 \\
& x_1 + x_5 = 3 \\
& x_i \geqslant 0, \quad i = 1, 2, 3, 4, 5
\end{aligned}$$

(a) (5 pts) Construct the dual problem corresponding to the above primal function.

(b) (15 pts) It is known that the solution to the above primal is $\boldsymbol{x}^* = \begin{bmatrix} 3 & 5 & 3 & 0 & 0 \end{bmatrix}^T$. Find the solution to the dual.

(a) Rewrite the problem,

$$\begin{aligned}
\text{minimize} \quad & -x_1 - 2x_2 \\
\text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\
& -x_1 + 2x_2 + x_4 = 7 \\
& x_1 + x_5 = 3 \\
& x_i \geqslant 0, \quad i = 1, 2, 3, 4, 5
\end{aligned}$$

so the dual problem is

$$\begin{aligned}
\text{maximize} \quad & \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} \\
\text{subject to} \quad & \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \leqslant \begin{bmatrix} -1 & -2 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\lambda}^T \boldsymbol{a}_i = \boldsymbol{c}_i \\
\rightarrow & \begin{cases} -2\lambda_1 - \lambda_2 + \lambda_3 = -1 \\ \lambda_1 + 2\lambda_2 = -2 \\ \lambda_1 = 0 \end{cases} \\
\rightarrow & \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -1 \\ \lambda_3 = -2 \end{cases}
\end{aligned}$$

Therefore, the solution to the dual is

$$\boldsymbol{\lambda} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$$

5. (20 pts) Find the minimizer of

$$f(x_1, x_2) = \frac{1}{2}x_1^2 + x_2^2 + x_1 + \frac{1}{2}x_2 + 3$$

using the conjugate gradient algorithm. The starting point is $\boldsymbol{x}^{(0)} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

$$\begin{aligned}
\nabla f(\boldsymbol{x}) &= \begin{bmatrix} x_1 + 1 \\ 2x_2 + \frac{1}{2} \end{bmatrix} \\
\boldsymbol{Q} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\
\boldsymbol{g}^{(0)} &= \nabla f(\boldsymbol{x}^{(0)}) \\
&= \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \\
\boldsymbol{d}^{(0)} &= -\boldsymbol{g}^{(0)} \\
&= \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} \\
\alpha_0 &= -\frac{\begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}}{\begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}} \\
&= \frac{5}{6} \\
\boldsymbol{x}^{(1)} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{5}{6} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{5}{6} \\ -\frac{5}{12} \end{bmatrix} \\
\boldsymbol{g}^{(1)} &= \nabla f(\boldsymbol{x}^{(1)}) \\
&= \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} \\
\beta_0 &= \frac{\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}}{\begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}} \\
&= \frac{1}{9} \\
\boldsymbol{d}^{(1)} &= -\boldsymbol{g}^{(1)} \\
&= \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} -\frac{5}{18} \\ \frac{5}{18} \end{bmatrix} \\
\alpha_1 &= -\frac{\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{5}{18} \\ \frac{5}{18} \end{bmatrix}}{\begin{bmatrix} -\frac{5}{18} & \frac{5}{18} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{5}{18} \\ \frac{5}{18} \end{bmatrix}} \\
&= \frac{3}{5} \\
\boldsymbol{x}^{(2)} &= \begin{bmatrix} -\frac{5}{6} \\ -\frac{5}{12} \end{bmatrix} + \frac{3}{5} \begin{bmatrix} -\frac{5}{18} \\ \frac{5}{18} \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ -\frac{1}{4} \end{bmatrix} \\
\boldsymbol{g}^{(2)} &= \nabla f(\boldsymbol{x}^{(2)}) \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Therefore, $\boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ -\frac{1}{4} \end{bmatrix}$ is the minimizer.