QE 2015 AC-3

Problem 1

minimize $x_1 + 2x_2 + 4x_3$, subject to $x_1 + 2x_2 + x_3 = 5$, $2x_1 + 3x_2 + x_3 = 6$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

1. if $x_1 x_2$,

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1/(3-4) \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

   $f = -3 + 8 = 5$ minimum.

2. if $x_1 x_3$,

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = 1/(1-2) \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

   $f = 1 + 4 \times 4 = 16$

3. if $x_2 x_3$,

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 1/(2-3) \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

   $f = 2 + 12 = 14$.

Problem 2

$$
\begin{gathered}
1.\ l = \frac{1}{2}x^TQx - b^Tx + \lambda(Ax - c) \\
Dl = Qx - b^T + A\lambda^T = 0 \\
x = Q^{-1}(b^T - A\lambda^T) \\
Ax = c \\
AQ^{-1}(b^T - A\lambda^T) = c \\
b^T - A\lambda^T = QA^{-1}c \\
x = A^{-1}c
\end{gathered}
$$

2. A should be m by n, the rank of A is m.

Problem 3

1. minimize $x_1^2 - x_1 + x_2 + x_1x_2$,

$$
Df = \begin{bmatrix} 2x_1 - 1 + x_2 \\ 1 + x_1 \end{bmatrix}
$$

Basic feasible direction is

$$
\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} d^1 \\ d^2 \end{bmatrix} \geq 0
$$

$$
\begin{cases} \frac{1}{2} + \alpha d^1 \geq 0 \\ \alpha d^2 \geq 0. \end{cases} \tag{1}
$$

$$
\begin{cases} d^1 \geq -\frac{1}{2\alpha} \\ d^2 \geq 0. \end{cases} \tag{2}
$$

is feasible direction

2. For SONC,

$$
Df(x_0) = \begin{bmatrix} 1 - 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix}
$$

is feasible direction.

$$
D^2f(x_0) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}
$$

$$
d^TD^2f(x_0)d = \begin{bmatrix} d^1 & d^2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d^1 \\ d^2 \end{bmatrix} = \begin{bmatrix} 2d^1 + d^2 \\ d^1 \end{bmatrix} \begin{bmatrix} d^1 \\ d^2 \end{bmatrix}
$$

$$= 2(d^1)^2 + d^1d^2 + d^1d^2$$
$$= 2((d^1)^2 + d^1d^2)$$

since $d^1, d^2 \geq 0$, so $d^T D^2 f(x_0) d \geq 0$ is satisfied.

Problem 4

1. maximize $x_1 + 2x_2$,

$$\text{subject to } -2x_1 + x_2 + x_2 = 2$$
$$-x_1 + 2x_2 + x_4 = 7$$
$$x_1 + x_5 = 3$$

$$\begin{array}{rrrrr|r|} -2 & 1 & 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 0 & 1 & 3 \\ 1 & 2 & 0 & 0 & 0 & 0 \end{array}$$

change the matrix to

$$\begin{array}{rrr|r|} -2 & -1 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 7 & 3 & 0 \end{array}$$

Then we got minimize $2\lambda_1 + 7\lambda_2 + 3\lambda_3$,
subject to

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

2.

$$x = \begin{bmatrix} 3 & 5 & 3 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

After we solve the matrix, we got,

$$\lambda = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Notice that $3 \times 1 + 2 \times 5 = 3 + 10 = 0 + 7 \times 1 + 2 \times 3$

Problem 5

$$f = \frac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + 3$$

$$x^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$g^0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$d^0 = -g^0 = -\frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\alpha^0 = \frac{-g^{0T} d^0}{d^{0T} Q d^0} = \frac{- \begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \frac{5}{\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \frac{5}{6}$$

$$x^1 = x^0 + \alpha^0 d^0 = \frac{5}{6} \frac{-1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\frac{5}{12} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$g^1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} (-\frac{5}{12}) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\beta_k = \frac{\frac{1}{6} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{-\frac{1}{2} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} = -\frac{1}{3} \frac{\begin{bmatrix} 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \frac{1}{9}$$

$$d^1 = -g^1 + \beta d^0 = \frac{1}{18} \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

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$$\alpha^1 = \frac{-g^{1T}d^1}{d^{1T}Qd^1} = \frac{-\frac{1}{6}\begin{bmatrix} 1 & -2 \end{bmatrix}\frac{1}{18}\begin{bmatrix} -5 \\ 5 \end{bmatrix}}{\frac{1}{18}\begin{bmatrix} -5 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\frac{1}{18}\begin{bmatrix} -5 \\ 5 \end{bmatrix}} = -3\frac{-5-10}{\begin{bmatrix} -5 & 10 \end{bmatrix}\begin{bmatrix} -5 \\ 5 \end{bmatrix}} = \frac{3}{5}$$

$$x^2 = x^1 + \alpha^1 d^1 = \frac{1}{4}\begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

$$g^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\frac{1}{4}\begin{bmatrix} -4 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

x^2 is optimal.