I. Fourier-Based Imaging (60 points)

For the following, consider an image $f(x,y)$ with a (continuous two-dimensional) Fourier transform given by $F(u,v)$ and a forward projection

$$\begin{aligned} p_\theta(r) &= \mathcal{FP}\{f(x,y)\} \\ &= \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta))\, dz \end{aligned}$$

and let $P_\theta(\rho)$ denote the (continuous time) Fourier transform of $p_\theta(r)$.

Define the functions

$$\begin{aligned} \delta(x,y) &= \delta(x)\delta(y) \\ \mathrm{rect}(x) &= \begin{cases} 1 & \text{if } |x| \le 1/2 \\ 0 & \text{if } |x| > 1/2 \end{cases} \end{aligned}$$

Sampling of the signal $f(x,y)$ may be (ideally) represented as multiplication by the *comb* function:

$$\delta_s(x,y;\Delta x,\Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

In this case, the sampled signal is

$$f_s(x,y) = f(x,y) \times \delta_s(x,y;\Delta x,\Delta y)$$

(20 pts) 1. Symbolically derive $F_s(u,v)$, the 2D Fourier transform of $f_s(x,y)$. (Show all intermediate steps.)

(10 pts) 2. For $f(x,y)$ frequency band-limited to $|u| < U$ and $|v| < V$, derive and graphically illustrate the Nyquist sampling criteria related to Δx and Δy.

(10 pts) 3. Assume $f(x,y)$ is spatially limited to $|x| < \frac{X}{2}$ and $|y| < \frac{Y}{2}$. If a $X \times Y$ image of $f(x,y)$ is generated using $n_x \times n_y$ pixels, what is the set of points in (u,v) space that have been sampled?

(10 pts) 4. If the "field-of-view" in part **c** is doubled, to produce a $2X \times 2Y$ image, without increasing the number of pixels, how does the set of sampled points in (u,v) space change?

(10 pts) 5. Now assume that the "field-of-view" remains $X \times Y$, but the number of pixels is doubled to $2n_x \times 2n_y$. How does the set of sampled points in (u,v) space change?

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II. Projection/Emission Tomographic Imaging (40 points)

(20 pts) 6. Assume that you have received projection data from a tomographic imaging system. In this case, an array of detectors lies along a line that is tangential to the radius of the imaged object. Each detector results in a single recorded value per measurement, with this value being proportional to the accumulated effect of a property (either absorption or emission) that varies along a line through the imaged object — i.e., the value represents a line integral. The "lines of response" as measured across the detector array may be reformed into a sinogram, which is then used to reconstruct the distribution of the given property within the target by one of a number of methodologies. One of the conceptually simplest techniques is the algebraic reconstruction technique (ART). Illustrate the ART reconstruction procedure associated with two projection angles obtained from a 3×3 image, given the following sinogram components:

$$\begin{aligned} p_{\theta=0}(r) &= [11\ 14\ 8] \\ p_{\frac{\pi}{2}}(r) &= [14\ 10\ 9] \end{aligned}$$

Note: Assume that $\theta = 0$ implies a projection line orthogonal to the x-axis.

(20 pts) 7. Suppose that for an emission tomographic system, detector units are tightly packed in a ring around the to-be-imaged object. Assume in this case that each detector unit comprises four square photomultiplier tubes (PMTs) (2×2 matrix) fronted by a single scintillation crystal with slits made in such a way that it is divided into an 8×8 matrix of individual detectors. *Note:* Assume that the center of the $(i,j)^{th}$ PMT is at (x_i, y_j), and that the PMTs and the detectors cover the exact same square area.

In this case, we will assume that the response of a PMT to an event occurring in a particular subcrystal may be modeled as

$$a_{PMT} = e^{-\frac{r}{\tau}}$$

where r is the distance from the center of the PMT to the center of the subcrystal, and τ is the spatial length constant of propagation of the light in the crystal.

Find a general expression for the response in the $(i,j)^{th}$ PMT to an event in the $(k,l)^{th}$ subcrystal. Remember to provide a diagram to indicate the numeric ordering of your PMTs and subcrystals.