AC-2
August 2010 QE

LTI and LT Systems – State-Space Approach

Unless otherwise stated, you need to justify your answers.

Problem 1. (25 points) Suppose a LTI system is given by

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 \end{bmatrix} x(t).$$

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(a) (5 points) Is the autonomous system (i.e. $u(t) \equiv 0$) stable?

(b) (5 points) Is the system controllable?

(c) (5 points) Is the system stabilizable?

(d) (5 points) Is the system observable?

(e) (5 points) Is the system detectable?

Problem 2. (25 points) Consider the LTI system $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(a) (15 points) Compute the matrix exponential e^{At}, and find $x(t)$ under the initial state $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the input $u(t) \equiv 1$, $t \geq 0$.

(b) (5 points) Is it possible to find a state feedback controller $u = -Kx$ for some $K \in \mathbb{R}^{1\times 2}$ so that the closed-loop system dynamic matrix $\bar{A} = A - BK$ has exactly the eigenvalues $\{-1, -2\}$? If so, find K; otherwise, state your reason.

(c) (5 points) Is it possible to find a state feedback controller $u = -Kx$ for some $K \in \mathbb{R}^{1\times 2}$ so that the closed-loop system dynamic matrix $\bar{A} = A - BK$ has exactly the eigenvalues $\{0, -2\}$? If so, find K; otherwise, state your reason.

Problem 3. (25 points) Consider the following single-input single-output discrete-time system:

$$
\begin{aligned}
x(k+1) &= bcx(k) + bu(k), \\
y(k) &= cx(k),
\end{aligned}
$$

where $x(k) \in \mathbb{R}^3$, $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $c = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$.

(a) (10 points) Find the set of reachable states from the origin.

(b) (10 points) Find the set of unobservable states.

(c) (5 points) Find the set of states controllable to the origin (namely, the set of all $x(0)$ for which there is an input $u(k)$, $k = 0, 1, \ldots$, such that $x(T) = 0$ for some $T \geq 0$)?

Problem 4. (25 points) Consider the following linear time-varying system:

$$
\dot{x}(t) = A(t)x(t) = \begin{bmatrix} -e^{-t} & \frac{1}{t+1} \\ 0 & -e^{-t} \end{bmatrix} x(t), \quad x(t) \in \mathbb{R}^2.
$$

(a) (15 points) Find the state transition matrix $\Phi(t, \tau)$ of the system.

(b) (5 points) Is the system asymptotically stable?

(c) (5 points) Given $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, will the system trajectory $x(t)$ stay bounded as $t \to \infty$?

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