Optimization

1. (10 pts) Consider the function

$$f(x_1, x_2) = x_1 + \frac{3x_2}{x_1}$$

and the point $\boldsymbol{x}^{(0)} = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$. Use Taylor's theorem to construct

(i) (5 pts) a linear approximation, $l(x_1, x_2)$, of $f(x_1, x_2)$ at $\boldsymbol{x}^{(0)}$;

(ii) (5 pts) a quadratic approximation, $q(x_1, x_2)$, of $f(x_1, x_2)$ at $\boldsymbol{x}^{(0)}$.

2. (10 pts) Is $\boldsymbol{d} = \begin{bmatrix} -2 & 1 \end{bmatrix}^\top$ a direction of descent of

$$f(x_1, x_2) = x_1 + \frac{3x_2}{x_1}$$

at the point $\boldsymbol{x}^{(0)} = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ or not? Justify your answer.

3. (30 pts) Convert the following optimization problem into a linear programming problem and solve it,

$$\begin{array}{ll} \text{maximize} & -|x_1| - |x_2| - |x_3| \\ \text{subject to} & \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Write in Exam Book Only

4. (15 pts) Consider the following primal problem:

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 7 \\ & x_1 + x_5 = 3 \\ & x_i \geq 0, \quad i = 1, 2, 3, 4, 5. \end{array}$$

(i) (5 pts) Construct the dual problem corresponding to the above primal problem.

(ii) (10 pts) It is known that the solution to the above primal is $\boldsymbol{x}^* = \begin{bmatrix} 3 & 5 & 3 & 0 & 0 \end{bmatrix}^\top$.

Find the solution to the dual.

5. (20 pts) Solve the optimization problem,

$$\begin{array}{ll} \text{optimize} & -\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{p}^\top \boldsymbol{x} + 2\pi \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{array}$$

where $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$, $\boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, $\operatorname{rank} \boldsymbol{A} = m$. Is the obtained solution a minimizer or maximizer? Justify your answer.

6. (15 pts) Use the **Lagrange's conditions** to solve the optimization problem,

$$\begin{array}{ll} \text{optimize} & x_1 x_2 \\ \text{subject to} & x_1 + x_2 + x_3 = 1 \\ & x_1 + x_2 - x_3 = 0. \end{array}$$

(**Warning:** Other approaches will not be graded.)