

[3(a)] Given that  $X(t)$  and  $Y(t)$  are independent random processes.

Their autocovariance  $C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - E[X(t_1)]E[Y(t_2)] = 0$  because they are uncorrelated.

Thus,  $E[X(t)Y(t)] = E[X(t)]E[Y(t)] = \text{constant}$ .

$$\begin{aligned} E[Z(t_1)Z(t_2)] &= E[X(t_1)Y(t_1)X(t_2)Y(t_2)] \\ &= E[X(t_1)X(t_2)]E[Y(t_1)Y(t_2)] \\ &= R_{XX}(t_1 - t_2)R_{YY}(t_1 - t_2) \end{aligned}$$

Therefore,  $Z$  is WSS.

[3(b)] First, we show that  $X(t)$  is WSS

$$\begin{aligned} E[X(t_1)X(t_2)] &= E[\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta)] \\ &= E\left[\frac{\cos(\omega_0 t_1 - \omega_0 t_2) + \cos(\omega_0 t_1 + \omega_0 t_2 + 2\theta)}{2}\right] \\ &= \frac{1}{2}\cos(\omega_0 t_1 - \omega_0 t_2) + \frac{1}{2}E[\cos(\omega_0 t_1 + \omega_0 t_2 + 2\theta)] \\ &= \square + \frac{1}{2}E[\cos(\omega_0 t_1 + \omega_0 t_2)\cos 2\theta - \sin(\omega_0 t_1 + \omega_0 t_2)\sin 2\theta] \\ &= \square + \frac{1}{2}\left(\int_0^{2\pi} \cos 2\theta d\theta \cdot \cos(\omega_0 t_1 + \omega_0 t_2) - \int_0^{2\pi} \sin 2\theta d\theta \cdot \sin(\omega_0 t_1 + \omega_0 t_2)\right) \\ &= \frac{1}{2}\cos(\omega_0 t_1 - \omega_0 t_2) \end{aligned}$$

$$\begin{aligned} E[X(t)] &= E[\cos(\omega_0 t + \theta)] \\ &= E[\cos(\omega_0 t)\cos(\theta) - \sin(\omega_0 t)\sin(\theta)] \\ &= \text{constant} \end{aligned}$$

Thus,  $X$  is WSS.

$$\begin{aligned} R_Z(t) &= R_X(t)R_Y(t) \\ &= \frac{1}{2}\cos(\omega_0 t)e^{-\alpha|t|} \\ &= \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})e^{-\alpha|t|} \end{aligned}$$

$$\begin{aligned} F\{e^{-\alpha|t|}\} &= \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{i\omega t} dt \\ &= \int_0^{\infty} e^{-\alpha t} e^{i\omega t} dt + \int_{-\infty}^0 e^{\alpha t} e^{i\omega t} dt \end{aligned}$$

$$= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$S_Z(\omega) = F\{R_Z(t)\} = \frac{\alpha}{2(\alpha^2 + (\omega - \omega_0)^2)} + \frac{\alpha}{2(\alpha^2 + (\omega + \omega_0)^2)}$$

[4(a)]  $E[|Y_m - \mu|^2]$

$$\begin{aligned} &= E[Y_m^2] - 2\mu E[Y_m] + \mu^2 \\ &= \left(\frac{1}{m^2}(E[X^2]m + E[X]^2 m(m-1))\right) - \mu^2 \\ &= \frac{(\sigma^2 + \mu^2)}{m} + \frac{\mu^2(m-1)}{m} - \mu^2 = \frac{\sigma^2}{m} \rightarrow 0 \end{aligned}$$

when  $m \rightarrow \infty$

Thus,  $Y_m$  converges to  $\mu$  in MS sense.

$$\begin{aligned} [4(b)] \quad \sigma_Y^2 &= E[Y_m^2] - E[Y_m]^2 \\ &= \frac{1}{m^2}(E[X^2]m + E[X]^2 m(m-1)) - \mu^2 \\ &= \frac{1}{m^2}((\sigma^2 + \mu^2)m + \mu^2(m-1)) - \mu^2 \\ &= \frac{\sigma^2}{m} + \mu^2 - \mu^2 = \frac{\sigma^2}{m} \end{aligned}$$

From Chebyshev inequality,

$$\begin{aligned} P\{|X - \mu| \geq \epsilon\} &\leq \frac{\sigma_Y^2}{\epsilon^2} = \frac{\sigma^2}{m\epsilon^2} \rightarrow 0 \\ &\text{when } m \rightarrow \infty \end{aligned}$$