$\mathcal{B}(\mathbb{R})$: σ-field generated by interval (a,b)

$\sigma(\mathcal{G})$: smallest field containing all elements of $\mathcal{G}$

non-empty set of outcomes

collection of subsets of S

Session 1 Random experiment characterized by $\{S, F, P\}$

Session 2 Event space F satisfies closure property $\longrightarrow$ $\left\{ \begin{array}{l} \boxed{1}\ A \in F(S) \rightarrow \bar{A} \in F(S) \\ \boxed{2}\ \bigcup_{i=1}^{n} A_i \in F(S) \\ \boxed{3}\ \bigcup_{i=1}^{\infty} A_i \in F(S) \end{array} \right.$

Probability mapping P satisfies axiom of probability $\left\{ \begin{array}{l} P(A_1) \geq 0 \quad P(S) = 1 \\ P(A_1 \cup A_2) = P(A_1) + P(A_2) \text{ for disjoint } A_1, A_2 \in F(S) \end{array} \right.$

Session 3 $F(S) \longrightarrow [0,1]$

Session 5 In case of countable S, P is called probability mass function (PMF)

In case of uncountable S, P is called probability density function (PDF)

Session 6 Conditional probability $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, Total probability law $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$

Bayes formula $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Session 7 Given (S_1, F_1, P_1) and (S_2, F_2, P_2).

We can form combined experiment (S, F, P) with $\left\{ \begin{array}{l} S = S_1 \times S_2 \quad F = \sigma(\{A \times B \mid \forall A \in F_1 \text{ and } \forall B \in F_2\}) \\ P \text{ is generated according to axiom of probability} \\ \text{such that } P(A \times S_1) = P_1(A) \text{ for } \forall A \in F_1 \\ \qquad P(S_2 \times B) = P_2(B) \text{ for } \forall B \in F_2 \end{array} \right.$

Extension to n-combined experiment is straightforward

Session 10 Random variable $X(\cdot)$ is a mapping from S to $\mathbb{R}$. $(S, F, P) \xrightarrow{X(\cdot)} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$

Being Borel measurable function (such that $X^{-1}(A) = \{w \in S \mid X(w) = A \in F\}$ for $\forall A \in \mathcal{B}(\mathbb{R})$)

ensures that we can compute $P_X(A) = P(X \in A) = P(\{w \in S \mid X(w) \in A\})$

Session 11 Cumulative distribution function (CDF) $F_X(x) = P_X((-\infty, x]) = P(X^{-1}((-\infty, x]))$

Session 12 RV is continuous if $F_X(x)$ is a continuous function over x and $\dfrac{dF_X(x)}{dx} = f_X(x)$

Session 15 Function of RV $Y = g(X)$ is valid RV if $\boxed{1}$ domain of g contains range of X

$\boxed{2}$ R_Y is Borel set $\boxed{3}$ $P(\{g(X = +\infty)\}) = 0$

Session 16 $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$E[X|M] = \int_{-\infty}^{\infty} x f_X(x|M)\,dx \qquad Var[X] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_X(x)\,dx \qquad f_X(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx} \Phi_X(w)\,dw$

Session 18 Characteristic function of X $\Phi_X(w)$ is $E[e^{iwx}] = \int_{-\infty}^{\infty} e^{iwx} f_X(x)\,dx$

$\Phi_X(w) = \Phi_Y(w) \rightarrow X$ and Y are identical but $f_X(x) = f_Y(y) \not\longrightarrow X$ and Y are identical

If $Y = aX + b$, $\Phi_Y(w) = e^{iwb}\Phi_X(aw)$

Moment generating function of X $\Phi_X(s)$ is $E[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx \qquad E[X^n] = \left.\dfrac{d^n \Phi_X(s)}{ds^n}\right|_{s=0}$

mean	variance	Type	PMF
		Uniform	$\frac{1}{n}$
np	$np(1-p)$	Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$
$\frac{1}{p}$	$\frac{(1-p)}{p^2}$	Geometric	$p^k(1-p)$
λ	λ	Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}$

Type	CDF	characteristic	mean	variance
Uniform	$\frac{1}{b-a}$			
Exponential	$\lambda e^{-\lambda x}$	$\frac{\lambda}{\lambda - iw}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gaussian	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$	$e^{iw\mu} e^{-\frac{1}{2}w^2\sigma^2}$		
Cauchy	$\frac{1}{\pi(1+x^2)}$	$e^{-\lvert w \rvert}$		

$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

$\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$

$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$

Fourier Transform

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{iwt}\, dw$ $\quad 1 \longleftrightarrow \delta(f)$ $\quad \mathrm{rect}(t) \longleftrightarrow \mathrm{sinc}(f)$ $\quad u(t) \longleftrightarrow \frac{1}{iw} + \frac{1}{2}\delta(f)$

$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt$ $\quad \delta(t) \longleftrightarrow 1$ $\quad \cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}(\delta(f-f_0)+\delta(f+f_0))$ $\quad e^{-at}u(t) \longleftrightarrow \frac{1}{a+iw}$

$\delta(t-t_0) \longleftrightarrow e^{-iwt_0}$ $\quad \sin(2\pi f_0 t) \longleftrightarrow \frac{1}{2i}(\delta(f-f_0)-\delta(f+f_0))$

$e^{i2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$

Poisson process $\quad P\{N(t_1,t_2)=k\} = \frac{e^{-\lambda(t_2-t_1)}[\lambda(t_2-t_1)]^k}{k!}$

Session 19 $\quad f_{XY}(x,y) = \frac{1}{2\pi \cancel{\sigma_1\sigma_2}\sqrt{1-r^2}} \exp\left(-\frac{1}{2(1-r^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2r(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right)$

$\Phi_{XY}(x,y) = e^{(iw_1\mu_1+iw_2\mu_2)}\, e^{-\frac{1}{2}[\sigma_x^2 w_1^2 + 2r\sigma_x\sigma_y w_1 w_2 + \sigma_y^2 w_2^2]}$

Session 20 $\quad$ Two RVs are statistically independent if $f_{XY}(x,y) = f_X(x) f_Y(y)$

Session 21 $\quad$ Joint density determination $\quad Z = f(X,Y)$ and $W = g(X,Y) \rightarrow f_{ZW}(z,w) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(z,w)}\right| = f_{XY}(x,y)\begin{vmatrix}\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}\end{vmatrix}$

Correlation $\quad E[XY]$ $\qquad$ correlation coefficient $= \frac{E[(X-\bar{X})(Y-\bar{Y})]}{\sigma_x\sigma_y}$

Covariance $\quad E[(X-\bar{X})(Y-\bar{Y})] = E[XY] - E[X]E[Y]$ $\quad$ This is 0 if RVs are uncorrelated.

Session 23 $\quad E[X^j Y^k] = \frac{\partial^j \partial^k}{\partial s_1^j \partial s_2^k}\Phi_{XY}(s_1,s_2)\Big|_{s_1=0,\, s_2=0}$ $\qquad E_{XY}[g(X,Y)] = E_X[E_Y[g(X,Y)|X]]$

Session 24 $\quad$ Random vector $\underline{X} = (X_1, X_2, X_3, \dots)$ is a mapping from S to $\mathbb{R}^n$

Given $\underline{X} = (x_1, x_2, x_3, \dots)$ and $\underline{Y} = (y_1, y_2, y_3, \dots)$ $\quad f_{\underline{Y}}(y_1,y_2,y_3,\dots) = f_{\underline{X}}(x_1,x_2,x_3,\dots)\left|\frac{\partial(x_1,x_2,x_3,\dots)}{\partial(y_1,y_2,y_3,\dots)}\right| = f_X(x_1,x_2,\dots)\begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \ddots & & \\ & & & \frac{\partial x_n}{\partial y_n}\end{vmatrix}$

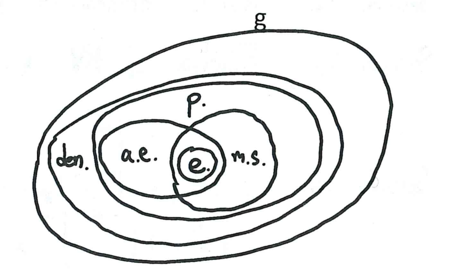
Correlation matrix $\quad R = \begin{pmatrix} E[X_1X_1] & E[X_1X_2] & \cdots & E[X_1X_n] \\ E[X_2X_1] & \ddots & & \\ & & & E[X_nX_n] \end{pmatrix}$

Covariance matrix $\quad C = \begin{pmatrix} E[(X_1-\bar{X}_1)(X_1-\bar{X}_1)] & E[(X_1-\bar{X}_1)(X_2-\bar{X}_2)] & \cdots \\ E[(X_2-\bar{X}_2)(X_1-\bar{X}_1)] & \ddots & \end{pmatrix}$

Characteristic function $\quad \Phi_{\underline{X}}(\Omega) = E[e^{i\Omega \underline{X}^T}]$

Session 27 $\quad$ Sequence of RVs $\{X_n\} = X_1, X_2, X_3, \dots$ converges to X if there exist $n \in \mathbb{N}$ such that $|X_n - X| < \varepsilon$ for $\forall \varepsilon > 0$

Converge
- everywhere $\quad X_1(w), X_2(w), X_3(w), \dots$ converges to $X(w)$ for each w
- almost everywhere $\quad P(\{X_n \rightarrow X\}) = 1$
- mean square $\quad E[|X_n - X|^2] \longrightarrow 0$
- in probability $\quad P(\{|X_n - X| > \varepsilon\}) \longrightarrow 0$ as $n \rightarrow \infty$
- in distribution $\quad F_{X_n}(x) \longrightarrow F_X(x)$ as $n \rightarrow \infty$
- in density $\quad f_{X_n}(x) \longrightarrow f_X(x)$ as $n \rightarrow \infty$



Chebyshev inequality $\quad P(\{|X-\mu| > \varepsilon\}) \le \sigma^2/\varepsilon^2$

Session 28 $\quad$ Random Stochastic Process $X(t)$ $\quad \mathbb{R}\times S \rightarrow \mathbb{R}$

RP is strict sense stationary if n-order are invariant to time $\quad f_{X(t_1),X(t_2),\dots}(x_1,x_2,x_3,\dots) = f_{X(t_1+c),X(t_2+c),\dots}(x_1,x_2,x_3,\dots)$

RP is wide sense stationary if n-order depends on time difference

$\boxed{1}\ E[X(t)] = \bar{X}$ $\qquad \boxed{2}\ E[X(t_1)X(t_2)] = R_{XX}(t_1-t_2)$

Autocorrelation of RP $X(t)$ is $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

Autocovariance of RP $X(t)$ is $C_{XX}(t_1,t_2) = E[(X(t_1)-\mu_1(t_1))(X(t_2)-\mu_2(t_2))]$

Characteristic function Gaussian RP is $\Phi_{X(t_1)X(t_2)\dots}(w_1,\dots,w_n) = e^{i\mu_X\sum_{k=1}^{n} w_k - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_X(t_k-t_j)w_j w_k}$

Session 29 $\quad$ Linear Time Invariant System $\quad Y(t) = h(t) * X(t)$

$\left.\begin{array}{l} x(t) \rightarrow y(t) \\ x(t+c) \rightarrow y(t+c) \end{array}\right\}$ satisfying $\left\{\begin{array}{l} \boxed{1}\ L[X_1(t)+X_2(t)] = L[X_1(t)] + L[X_2(t)] \\ \boxed{2}\ L[A\cdot X(t)] = A\cdot L[X(t)] \end{array}\right.$

WSS/SSS input $\xrightarrow{LTI}$ WSS/SSS

Gaussian input $\longrightarrow$ Gaussian

$E[L[X(t)]] = L[E[X(t)]]$

Session 30 $\quad$ Power spectral density $\quad S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(t) e^{-iwt}\, dt$ $\qquad R_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w) e^{iwt}\, dw$

If $H(w) = \int_{-\infty}^{\infty} h(t) e^{-iwt}\, dt$, $\quad S_{YY}(w) = S_{XX}(w)|H(w)|^2$

white noise process has $C_{ww}(t_1,t_2) = 0$