CS-2
August 2010 QE

Problem 1. [50 pts]

Equation 1 below is the formula for reconstructing the DTFT, $X(\omega)$, from N equi-spaced samples of the DTFT over $0 \le \omega < 2\pi$. $X_N(k) = X(\frac{2\pi k}{N}), k = 0, 1, ..., N-1$ is the N-pt DFT of $x[n]$, which corresponds to N equi-spaced samples of the DTFT of $x[n]$ over $0 \le \omega < 2\pi$.

$$X_r(\omega) = \sum_{k=0}^{N-1} X_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)} \tag{1}$$

(a) Let $x[n]$ be a discrete-time rectangular pulse of length $L = 12$ as defined below:

$$x[n] = \{-1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1\}$$

(i) $X_N(k)$ is computed as a 16-point DFT of $x[n]$ and used in Eqn (1) with $N = 16$. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(ii) $X_N(k)$ is computed as a 12-point DFT of $x[n]$ and used in Eqn (1) with $N = 12$. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(iii) $X_N(k)$ is computed as an 8-point DFT of $x[n]$ and used in Eqn (1) with $N = 8$. That is, $X_N(k)$ is obtained by sampling the DTFT of $x[n]$ at 8 equi-spaced frequencies between 0 and 2π. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(b) Let $x[n]$ be a discrete-time sinewave of length $L = 12$ as defined below. For all sub-parts of part (b), $X_N(k)$ is computed as a 12-pt DFT of $x[n]$ and used in Eqn (1) with $N = 12$.

$$x[n] = \cos\left(\frac{\pi}{3}n\right)\{u[n] - u[n-12]\}$$

(i) Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.

(ii) What is the numerical value of $X_r(\frac{\pi}{3})$? The answer is a number and you do not need a calculator to determine the value; this also applies to the next 2 parts.

(iii) What is the numerical value of $X_r(\frac{5\pi}{3})$?

(iv) What is the numerical value of $X_r(\frac{\pi}{2})$?

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Problem 2. [50 pts] Consider a finite-length sinewave of the form below where k_o is an integer in the range $0 \le k_o \le N-1$.

$$x[n] = e^{j2\pi \frac{k_o}{N} n}\{u[n] - u[n-N]\} \tag{2}$$

In addition, $h[n]$ is a causal FIR filter of length L, where $L < N$. In this problem $y[n] = x[n] * h[n]$ is the linear convolution of the causal sinewave of length N in Equation (1) with a causal FIR filter of length L, where $L < N$.

$$y[n] = x[n] * h[n]$$

(a) The region $0 \le n \le L-1$ corresponds to *partial overlap*. The convolution sum can be written as:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{partial overlap: } 0 \le n \le L-1 \tag{3}$$

Determine the upper and lower limits in the convolution sum above for $0 \le n \le L-1$.

(b) The region $L \le n \le N-1$ corresponds to *full overlap*. The convolution sum is:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{full overlap: } L \le n \le N-1 \tag{4}$$

(i) Determine the upper & lower limits in the convolution sum for $L \le n \le N-1$.

(ii) Substituting $x[n]$ in Eqn (1), show that for this range $y[n]$ simplifies to:

$$y[n] = H_N(k_o)e^{j2\pi \frac{k_o}{N} n} \quad \text{for} \quad L \le n \le N-1 \tag{5}$$

where $H_N(k)$ is the N-point DFT of $h[n]$ evaluated at $k = k_o$. To get the points, you must show all work and explain all details.

(c) The region $N \le n \le N+L-2$ corresponds to *partial overlap*. The convolution sum:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{partial overlap: } N \le n \le N+L-2 \tag{6}$$

Determine the upper & lower limits in the convolution sum for $N \le n \le N+L-2$.

(d) Add the two regions of partial overlap at the beginning and end to form:

$$z[n] = y[n] + y[n+N] = \sum_{k=??}^{??} h[k]x[n-k] \quad \text{for: } 0 \le n \le L-1 \tag{7}$$

(i) Determine the upper and lower limits in the convolution sum above.

(ii) Substituting $x[n]$ in Eqn (1), show that for this range $z[n]$ simplifies to:

$$z[n] = y[n] + y[n+N] = H_N(k_o)e^{j2\pi \frac{k_o}{N} n} \quad \text{for } 0 \le n \le L-1 \tag{8}$$

where $H_N(k)$ is the N-point DFT of $h[n]$ evaluated at $k = k_o$ as defined previously.

(e) $y_N[n]$ is formed by computing $X_N(k)$ as an N-pt DFT of $x[n]$ in Eqn 2, $H_N(k)$ as an N-pt DFT of $h[n]$, and then $y_N[n]$ as the N-pt inverse DFT of $Y_N(k) = X_N(k)H_N(k)$. Write a simple, closed-form expression for $y_N[n]$. Is $z[n] = y_N[n] = y[n] + y[n+N]$ for $0 \le n \le N-1$?