Part I: Sampling and the Fourier Domain (60 points)

For the following, consider an image $f(x, y)$ with a (continuous two-dimensional) Fourier transform given by $F(u, v)$ and a forward projection

$$\begin{aligned} p_\theta(r) &= \mathcal{FP}\{f(x,y)\} \\ &= \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta))\, dz \end{aligned}$$

and let $P_\theta(\rho)$ denote the (continuous time) Fourier transform of $p_\theta(r)$.

Define the functions

$$\begin{aligned} \delta(x,y) &= \delta(x)\delta(y) \\ \text{rect}(x) &= \begin{cases} 1 & \text{if } |x| \le 1/2 \\ 0 & \text{if } |x| > 1/2 \end{cases} \end{aligned}$$

Sampling of the signal $f(x, y)$ may be (ideally) represented as multiplication by the *comb* function:

$$\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

In this case, the sampled signal is

$$f_s(x, y) = f(x, y) \times \delta_s(x, y; \Delta x, \Delta y)$$

(20 pts) 1. Symbolically derive $F_s(u, v)$, the 2D Fourier transform of $f_s(x, y)$.

(10 pts) 2. For $f(x, y)$ frequency band-limited to $|u| < U$ and $|v| < V$, derive and graphically illustrate the Nyquist sampling criteria related to Δx and Δy.

(10 pts) 3. Assume $f(x, y)$ is spatially limited to $|x| < \frac{X}{2}$ and $|y| < \frac{Y}{2}$. If a $X \times Y$ image of $f(x, y)$ is generated using $n_x \times n_y$ pixels, what is the set of points in (u, v) space that have been sampled?

(10 pts) 4. If the "field-of-view" in part **c** is doubled, to produce a $2X \times 2Y$ image, without increasing the number of pixels, how does the set of sampled points in (u, v) space change?

(10 pts) 5. Now assume that the "field-of-view" remains $X \times Y$, but the number of pixels is doubled to $2n_x \times 2n_y$. How does the set of sampled points in (u, v) space change?

Part II: Fourier-Based Reconstruction (40 pts)

For each of the following errors (assumed to take place during gridding of acquired data into the Fourier domain), describe and provide sample illustrations of the expected and obtained reconstructions. (Assume all acquisitions obtain 2^N samples to be gridded in a rectilinear fashion for subsequent reconstruction via FFT.)

(10 pts) 6. The u and v axes are swapped.

(10 pts) 7. Samples in u, intended to be Δu apart, are gridded at a distance of $2\Delta u$, but the FFT is performed over the original fixed limits of $(-u_{max}, u_{max})$

(10 pts) 8. The even rows in v are omitted, but the odd rows are gridded at the appropriate locations, and the FFT is performed over the original fixed limits of $(-v_{max}, v_{max})$.

(10 pts) 9. Odd rows of u are reflected across the v axis (i.e., the value intended for u is gridded at $-u$), but the even rows are gridded correctly.