ECE QE CS-1 2018

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1 Question 1

(a)

$$\begin{aligned}
P(min(X,Y)=k) &= P(X=k\cap Y>k)+P(Y=k\cap X>k)+P(X=k\cap Y=k)\\
&= \frac{1}{2^k}*\sum_{i=k+1}^{+\infty}\frac{1}{2^i}+\frac{1}{2^k}*\sum_{i=k+1}^{+\infty}\frac{1}{2^i}+\frac{1}{2^k}*\frac{1}{2^k}\\
&= (\frac{1}{2})^k*2(\frac{1}{2})^{k+1}+(\frac{1}{2})^k*2(\frac{1}{2})^{k+1}+(\frac{1}{2})^{2k}\\
&= 3*(\frac{1}{2})^{2k}=3(\frac{1}{4})^k
\end{aligned} \tag{1}$$

(b)

$$\begin{aligned}
P(X=Y) &= \sum_{i=1}^{+\infty}\frac{1}{2^i}*\frac{1}{2^i}\\
&= \sum_{i=1}^{+\infty}(\frac{1}{4})^i=\frac{(1/4)*(1-0)}{1-(1/4)}\\
&= \frac{1}{3}
\end{aligned} \tag{2}$$

(c)

$$\begin{aligned}
P(Y>X) &= \sum_{i=1}^{+\infty}\sum_{j=i+1}^{+\infty}\frac{1}{2^i}*\frac{1}{2^j}\\
&= \sum_{i=1}^{+\infty}(\frac{1}{2})^i*\frac{(1/2)^{i+1}}{1-(1/2)}\\
&= \sum_{i=1}^{+\infty}(\frac{1}{4})^i\\
&= \frac{1}{3}
\end{aligned} \tag{3}$$

(d)

$$\begin{aligned}
P(Y=kX) &= \sum_{i=1}^{+\infty}\frac{1}{2^i}*\frac{1}{2^{ki}}\\
&= \frac{1}{1-(\frac{1}{2})^{k+1}}
\end{aligned} \tag{4}$$

2 Question 2

(a)

$$\begin{aligned}
P(N=n) &= \int_{-\infty}^{+\infty}p_N(n|X=x)f_X(x)dx=\int_0^1 x^n(1-x)dx\\
&= \frac{1}{n+1}x^{n+1}-\frac{1}{n+2}x^{n+2}\ |_0^1=\frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{(n+1)(n+2)}
\end{aligned} \tag{5}$$

(b) Using Bayes' theorem,

$$f_X(x|N=n) = \frac{p_N(n|X=x)f_X(x)}{p_N(n)} = (n+1)(n+2)x^n(1-x) \tag{6}$$

,when $0 \leq x \geq 1$
(c)

$$\begin{aligned} MMSE = E(X|N=n) &= \int_{-\infty}^{+\infty} x * f_X(x|N=n)dx = \int_0^1 (n+1)(n+2)x^{n+1}(1-x)dx \\ &= (n+1)(n+2)(\frac{1}{n+2}x^{n+2} - \frac{1}{n+3}x^{n+3}) \mid_0^1 \\ &= \frac{n+1}{n+3} \end{aligned} \tag{7}$$

3 Question 3

(a) If $X(t)$,$Y(t)$ is WSS, $\mu_{(X(t))} = \mu_X$ and $\mu_{(Y(t))} = \mu_Y$. Therefore,$\mu_{(Z(t))} = \mu_X\mu_Y$.

$$\begin{aligned} r_{ZZ}(t_1, t_2) &= E(X(t_1)Y(t_1)X(t_2)Y(t_2)) \\ &= E(X(t_1)X(t_2)) * E(Y(t_1)Y(t_2)) \\ &= r_{XX}(l) * r_{YY}(l) \end{aligned} \tag{8}$$

This result only depends on th time difference. Therefore, Z is WSS.
(b) Since Y is WSS. We only need to verify X is WSS.

$$\begin{aligned} E(X(t)) &= \int_{-\infty}^{+\infty} \theta cos(\omega_0 t + \theta)d\theta \\ &= \theta sin(\omega_0 t + \theta) \mid_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} sin(\omega_0 t + \theta)d\theta \\ &= 0 \end{aligned} \tag{9}$$

$$\begin{aligned} r_{XX}(t_1, t_2) = E(X(t_1)X(t_2)) &= E(cos(\omega_0 t_1 + \theta) * cos(\omega_0 t_2 + \theta)) \\ &= \frac{1}{2}E(cos(\omega_0 t_1 - \omega_0 t_2) + cos(\omega_0 t_1 + \omega_0 t_2 + 2\theta)) \\ &= \frac{1}{2}(cos(\omega_0 t_1 - \omega_0 t_2) + \frac{1}{2}E(cos(\omega_0 t_1 + \omega_0 t_2)cos(2\theta) - sin(\omega_0 t_1 + \omega_0 t_2)sin(2\theta)) \\ &= \frac{1}{2}(cos(\omega_0 t_1 - \omega_0 t_2) + \int_0^{2\pi} \theta cos(2\theta)d\theta * cos(\omega_0 t_1 + \omega_0 t_2) - \int_0^{2\pi} \theta sin(2\theta)d\theta * sin(\omega_0 t_1 + \omega_0 t_2) \end{aligned} \tag{10}$$

Since two integral part are both 0. The result is $\frac{1}{2}(cos(\omega_0 t_1 - \omega_0 t_2)$. It is only depends on the difference of t_1 and t_2. So Z is also WSS.

Then we will find the power spectral density of Z:

$$\begin{aligned} R_Z = R_X R_Y &= \frac{1}{2}cos(\omega_0 t)e^{-\alpha|t|} \\ &= \frac{1}{2}\frac{e^{iw_0 t} + e^{-iw_0 t}}{2}e^{-\alpha|t|} \end{aligned} \tag{11}$$

$$\begin{aligned} F\{e^{-\alpha|t|}\} &= \int_{-\infty}^{+\infty} e^{-\alpha|t|}e^{iwt}dt \\ &= \int_{-\infty}^{0} e^{\alpha t}e^{iwt}dt + \int_0^0 e^{-\alpha t}e^{iwt}dt \\ &= \frac{1}{\alpha - iw} + \frac{1}{\alpha + iw} \\ &= \frac{2\alpha}{\alpha^2 + w^2} \end{aligned} \tag{12}$$

$$S_Z(t) = F\{R_Z(t)\} = \frac{\alpha}{2(\alpha^2 + (w - w_0)^2)} + \frac{\alpha}{2(\alpha^2 + (w + w_0)^2)} \tag{13}$$

4 Question 4

$$\begin{aligned}\Phi_Y(y) = E(e^{iwy}) = E(e^{iw*(1/n)\sum_{i=1}^{n} X_i}) = \prod_{i=1}^{n} E(e^{iw\frac{X_i}{n}}) \\ = \prod_{i=1}^{n} e^{\frac{-|w|}{n}} = e^{-|w|}\end{aligned} \tag{14}$$

$$\begin{aligned}f_Y(y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iwy}e^{-|w|}dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{w(-iy+1)}dw + \int_{0}^{+\infty} e^{w(-iy-1)}dw \\ &= \frac{1}{2\pi}(\frac{1}{1-iy} + \frac{-1}{-1-iy}) \\ &= \frac{1}{\pi}\frac{1}{1+y^2}\end{aligned} \tag{15}$$

Therefore, the pdf shows that $Y_1, Y_2, Y_3...$ converge in distribution. And it is converage to a Cauchy distribution.