Problem 1. [50 pts]

Equation 1 below is the formula for reconstructing the DTFT, $X(\omega)$, from N equi-spaced samples of the DTFT over $0 \le \omega < 2\pi$. $X_N(k) = X(\frac{2\pi k}{N}), k = 0, 1, ..., N-1$ is the N-pt DFT of x[n], which corresponds to N equi-spaced samples of the DTFT of x[n] over $0 \le \omega < 2\pi$.

$$X_r(\omega) = \sum_{k=0}^{N-1} X_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N\sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)}$$
(1)

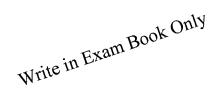
(a) Let x[n] be a discrete-time rectangular pulse of length L=12 as defined below:

$$x[n] = \{-1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1\}$$

- (i) $X_N(k)$ is computed as a 16-point DFT of x[n] and used in Eqn (1) with N=16. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.
- (ii) $X_N(k)$ is computed as a 12-point DFT of x[n] and used in Eqn (1) with N=12. Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.
- (iii) $X_N(k)$ is computed as an 8-point DFT of x[n] and used in Eqn (1) with N=8. That is, $X_N(k)$ is obtained by sampling the DTFT of x[n] at 8 equi-spaced frequencies between 0 and 2π . Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.
- (b) Let x[n] be a discrete-time sinewave of length L=12 as defined below. For all subparts of part (b), $X_N(k)$ is computed as a 12-pt DFT of x[n] and used in Eqn (1) with N=12.

$$x[n] = \cos\left(\frac{\pi}{3}n\right) \left\{ u[n] - u[n-12] \right\}$$

- (i) Write a closed-form expression for the resulting reconstructed spectrum $X_r(\omega)$.
- (ii) What is the numerical value of $X_r(\frac{\pi}{3})$? The answer is a number and you do not need a calculator to determine the value; this also applies to the next 2 parts.
- (iii) What is the numerical value of $X_r(\frac{5\pi}{3})$?
- (iv) What is the numerical value of $X_r(\frac{\pi}{2})$?



Problem 2. [50 pts] Consider a finite-length sinewave of the form below where k_o is an integer in the range $0 \le k_o \le N - 1$.

$$x[n] = e^{j2\pi \frac{k_0}{N}n} \{ u[n] - u[n-N] \}$$
 (2)

In addition, h[n] is a causal FIR filter of length L, where L < N. In this problem y[n] = x[n] * h[n] is the linear convolution of the causal sinewave of length N in Equation (1) with a causal FIR filter of length L, where L < N.

$$y[n] = x[n] * h[n]$$

(a) The region $0 \le n \le L-1$ corresponds to partial overlap. The convolution sum can be written as:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad partial \ overlap: \ 0 \le n \le L-1$$
 (3)

Determine the upper and lower limits in the convolution sum above for $0 \le n \le L-1$.

(b) The region $L \leq n \leq N-1$ corresponds to full overlap. The convolution sum is:

$$y[n] = \sum_{k=27}^{??} h[k]x[n-k] \quad \text{full overlap: } L \le n \le N-1$$
 (4)

- (i) Determine the upper & lower limits in the convolution sum for $L \leq n \leq N-1$.
- (ii) Substituting x[n] in Eqn (1), show that for this range y[n] simplifies to:

$$y[n] = H_N(k_o)e^{j2\pi\frac{k_o}{N}n} \quad \text{for} \quad L \le n \le N - 1$$
(5)

where $H_N(k)$ is the N-point DFT of h[n] evaluated at $k = k_o$. To get the points, you must show all work and explain all details.

(c) The region $N \leq n \leq N + L - 2$ corresponds to partial overlap. The convolution sum:

$$y[n] = \sum_{k=2}^{??} h[k]x[n-k] \quad partial \ overlap: \quad N \le n \le N+L-2$$
 (6)

Determine the upper & lower limits in the convolution sum for $N \leq n \leq N + L - 2$.

(d) Add the two regions of partial overlap at the beginning and end to form:

$$z[n] = y[n] + y[n+N] = \sum_{k=??}^{??} h[k]x[n-k] \quad \text{for: } 0 \le n \le L-1$$
 (7)

- (i) Determine the upper and lower limits in the convolution sum above.
- (ii) Substituting x[n] in Eqn (1), show that for this range z[n] simplifies to:

$$z[n] = y[n] + y[n+N] = H_N(k_o)e^{j2\pi \frac{k_o}{N}n} \quad \text{for } 0 \le n \le L-1$$
 (8)

where $H_N(k)$ is the N-point DFT of h[n] evaluated at $k = k_o$ as defined previously.

(e) $y_N[n]$ is formed by computing $X_N(k)$ as an N-pt DFT of x[n] in Eqn 2, $H_N(k)$ as an N-pt DFT of h[n], and then $y_N[n]$ as the N-pt inverse DFT of $Y_N(k) = X_N(k)H_N(k)$. Write a simple, closed-form expression for $y_N[n]$. Is $z[n] = y_N[n] = y[n] + y[n+N]$ for $0 \le n \le N-1$?