I. Fourier-Based Imaging (60 points)

For the following, consider an image f(x,y) with a (continuous two-dimensional) Fourier transform given by F(u,v) and a forward projection

$$p_{\theta}(r) = \mathcal{FP} \{ f(x,y) \}$$

=
$$\int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)) dz$$

and let $P_{\theta}(\rho)$ denote the (continuous time) Fourier transform of $p_{\theta}(r)$.

Define the functions

$$\delta(x,y) = \delta(x)\delta(y)$$

$$\operatorname{rect}(x) = \begin{cases} 1 & \text{if } |x| \le 1/2 \\ 0 & \text{if } |x| > 1/2 \end{cases}$$

Sampling of the signal f(x,y) may be (ideally) represented as multiplication by the *comb* function:

$$\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

In this case, the sampled signal is

$$f_s(x,y) = f(x,y) \times \delta_s(x,y;\Delta x,\Delta y)$$

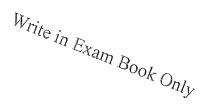
(20 pts) 1. Symbolically derive $F_s(u, v)$, the 2D Fourier transform of $f_s(x, y)$. (Show all intermediate steps.)

(10 pts) 2. For f(x,y) frequency band-limited to |u| < U and |v| < V, derive and graphically illustrate the Nyquist sampling criteria related to Δx and Δy .

(10 pts) 3. Assume f(x,y) is spatially limited to $|x| < \frac{X}{2}$ and $|y| < \frac{Y}{2}$. If a $X \times Y$ image of f(x,y) is generated using $n_x \times n_y$ pixels, what is the set of points in (u,v) space that have been sampled?

(10 pts) 4. If the "field-of-view" in part c is doubled, to produce a $2X \times 2Y$ image, without increasing the number of pixels, how does the set of sampled points in (u, v) space change?

(10 pts) 5. Now assume that the "field-of-view" remains $X \times Y$, but the number of pixels is doubled to $2n_x \times 2n_y$. How does the set of sampled points in (u, v) space change?



II. Projection/Emission Tomographic Imaging (40 points)

(20 pts) 6. Assume that you have received projection data from a tomographic imaging system. In this case, an array of detectors lies along a line that is tangential to the radius of the imaged object. Each detector results in a single recorded value per measurement, with this value being proportional to the accumulated effect of a property (either absorption or emission) that varies along a line through the imaged object — i.e., the value represents a line integral. The "lines of response" as measured across the detector array may be reformed into a sinogram, which is then used to reconstruct the distribution of the given property within the target by one of a number of methodologies. One of the conceptually simplest techniques is the algebraic reconstruction technique (ART). Illustrate the ART reconstruction procedure associated with two projection angles obtained from a 3 × 3 image, given the following sinogram components:

$$p_{\theta=0}(r) = [11 \ 14 \ 8]$$

 $p_{\frac{\pi}{2}}(r) = [14 \ 10 \ 9]$

Note: Assume that $\theta = 0$ implies a projection line orthogonal to the x-axis.

(20 pts) 7. Suppose that for an emission tomographic system, detector units are tightly packed in a ring around the to-be-imaged object. Assume in this case that each detector unit comprises four square photomultiplier tubes (PMTs) (2 × 2 matrix) fronted by a single scintillation crystal with slits made in such a way that it is divided into an 8 × 8 matrix of individual detectors. *Note:* Assume that the center of the $(i, j)^{th}$ PMT is at (x_i, y_j) , and that the PMTs and the detectors cover the exact same square area.

In this case, we will assume that the response of a PMT to an event occurring in a particular subcrystal may be modeled as

$$a_{PMT} = e^{-\frac{r}{\tau}}$$

where r is the distance from the center of the PMT to the center of the subcrystal, and τ is the spatial length constant of propagation of the light in the crystal.

Find a general expression for the response in the $(i,j)^{th}$ PMT to an event in the $(k,l)^{th}$ subcrystal. Remember to provide a diagram to indicate the numeric ordering of your PMTs and subcrystals.

