Adaptive posterior contraction results for Bayesian methods for diffusions

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Bayesian vs. frequentist statistics

- Frequentist statistics: one true parameter, recover the parameter from the data.
- Bayesian statistics: no 'true' parameter. Believe expressed in probability distribution on the parameter space.
- 'Frequentist behaviour' of Bayesian methods.
- Bayesian estimators:
  - Posterior mean
  - Posterior mode
- Bayesian uncertainty quantification: credible sets

Parametric vs. nonparametric situation

- 🙂 Parametric Bayesian models: $\sqrt{n}$-consistent.
- 😐 Nonparametric models:
  - Inconsistent posteriors are possible.
  - There is a gap between priors with good theoretical performance and priors with good numerical performance.

Frequentist analysis of Bayesian methods

- Consistency
- Posterior contraction rates
- Coverage of credible sets
- Bernstein-von Mises 'central limit' theorems

Subject of today

- Posterior contraction rates for diffusion processes.
- Adaptation to unknown smoothness of the function.
- Empirical Bayes.

Statistical inference for diffusions

- Diffusions on the line: real-valued strong Markov processes with continuous paths,
- Under weak conditions a diffusion is described via an SDE

$$dX_t = \theta(X_t)dt + \sigma(X_t)dW_t,$$

- $\theta : \mathbb{R} \to \mathbb{R}$ is measurable, 1-periodic and $\int_0^1 \theta(x)^2 dx < \infty$.
- Observations $X^T = \{X_t : t \in [0, T]\}$ of

$$dX_t = \theta(X_t)dt + dW_t,$$

- Goal: estimate θ.

Key ingredients for posterior convergence

$$\mathbb{E}_{\theta_0} \Pi(\{\theta : \|\theta - \theta_0\|_2 \geq \varepsilon_n\} \mid X^n) \to 0$$

$$\|$$

$$\mathbb{E}_{\theta_0} \left[\frac{\int_{\{\theta:\|\theta-\theta_0\|_2 \geq \varepsilon_n\}^c} p_\theta(X^T) d\Pi(\theta)}{\int p_\theta(X^T) d\Pi(\theta)} \right]$$

$$\mathbb{E}_{\theta_0} \left[\frac{\int_{\left\{\theta \in \Theta^T:\|\theta-\theta_0\|_2 \geq \varepsilon_n\right\}^c} p_\theta(X^T) d\Pi(\theta)}{\int p_\theta(X^T) d\Pi(\theta)} \right] + \Pi(\Theta_T^c \mid X^T)$$

$$\leq \frac{e^{-CT\varepsilon_T^2}}{e^{-cT\varepsilon_T^2}} + \mathbf{o}(1)$$

1. Tests,
2. Enough prior mass around true parameter.
3. Model is not too big.

Posterior convergence

When

$$\Pi(\theta : \|\theta - \theta_0\| < \varepsilon_T) \geq e^{-\xi T \varepsilon_T^2},$$

For every $K > 0$, there are measurable sets Θ_T so that

$$\Pi(\Theta_T) = 1 - e^{-KT\varepsilon_T^2}$$

and for every $a \in (0, 1)$, there is a $C > 0$

$$N\left(a\varepsilon_T, \{\theta \in \Theta_T : \|\theta - \theta_0\|_2 < \varepsilon_T\}\right) \leq e^{CT\varepsilon_T^2},$$

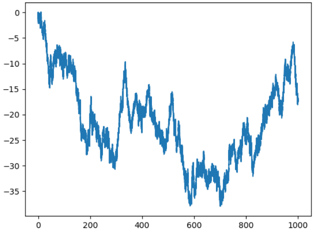
then for some $M > 0$,

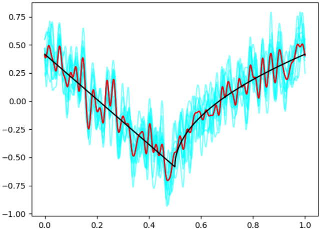
$$\mathbb{E}_{\theta_0} \Pi(\theta : \|\theta - \theta_0\|_2 \leq M\varepsilon_T \mid X^T) \to 1, \text{ as } T \to \infty.$$

Example

- SDE $dX_t = \theta_0(X_t)dt + dW_t$,
- Prior:

$$\theta = \sum_{k=1}^{100} k^{-1} Z_k \phi_k$$

- 



Posterior contraction for the Gaussian process prior

- Continuous stochastic process with fdd are multivariate Gaussian.
- Prior:

$$\theta = \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k.$$

- ☺ Posterior = Gaussian process + explicit formula
- True function $\theta_0 = \sum_{k=1}^{\infty} \theta_k \phi_k$ satisfies $\sum_{k=1}^{\infty} k^{2\beta} \theta_k^2 < \infty$.
- 😐 Minimax posterior convergence rate $T^{-\frac{\beta}{1+2\beta}}$ if and only if $\alpha = \beta$.
- ☹ Not adaptive!

Solution I: Hierarichal Bayes

- Let the posterior choose the right smoothness.
- Hyperpriors on the hyperparameters.

$$1 \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k$$

- Requirements:
  - Hyperprior should give enough mass range of optimal hyperparameters (easy).
  - Remaining mass condition (hard).

Scaling parameter

- Hyperprior on the scaling.

$$
\begin{aligned}
E &\sim \mathrm{Exp}(1), \\
S &= \frac{E^{1/2+\alpha}}{\sqrt{T}}, \\
\theta \mid S &= S \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k
\end{aligned}
$$

- When $0 < \beta \leq \alpha + 1/2$, and $\theta \in H^{\beta}$, then the posterior contracts with rate $T^{-\frac{\beta}{1+2\beta}}$.
- 😐 Optimal rates for the most import range, suboptimal for supersmooth functions.
- ☹ Prior on L intricate.

Prior on the baseline smoothness

- Hyperprior on α

$$\pi(\alpha) \propto e^{-T^{\frac{1}{1+2\alpha}}}, \alpha \in (0, \log T]$$

$$\theta \mid \alpha = \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k$$

- ☺ Adaptivity to every Sobolev smoothness!
- ☹ Are other possibilities on α possible?

A superior solution!

-

$$J \sim \text{geometric},$$
$$L^2 \sim \text{inverse gamma},$$
$$\theta \mid J, L \sim L \sum_{j=1}^{J} j^{-1/2-\alpha} Z_k \phi_k.$$

- When $0 < \beta \leq \alpha + 1/2$ the posterior contracts with rate $T^{-\frac{\beta}{1+2\beta}}$ and when $\beta > \alpha + 1/2$, the posterior contracts with rate $\left(\frac{T}{\log T}\right)^{-\frac{\beta}{1+2\beta}}$.
- ☺ (Nearly) optimal rates for every smoothness.
- ☺ Has good numerical properties!


It's Tea Time!

Empirical Bayes

- Use prior Π_s
$$\theta = s \sum_{k=1}^{\infty} k^{-1/2-\alpha} Z_k \phi_k,$$
- estimate s from the data,
- use
$$\Pi_{\hat{s}} = \Pi_s(\cdot \mid X^T)\Big|_{s=\hat{s}}$$
for the inference.

Hierarchical vs. empirical Bayes

- ☹ Gaussian process prior is not adaptive.
- Hierarchical Bayes solution: equip hyperparameters with additional prior.
  - ☹ Prior is not Gaussian
  - ☺ Adaptivity + (near) optimal rates.
- Empirical Bayes solution: estimate hyperparameter from the data and use plug-in posterior for inference.
  - ☺ the (data-driven) prior still Gaussian.
  - 😐 But a lot unknown about the theoretical and computational performance...
  - The analysis is considerably harder.

- Prior Π_s defined by $s\sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k\phi_k$.
- "Prior behaves best when it puts a lot of prior mass around θ_0."
- That is when $s \asymp T^{\frac{\alpha-\beta}{1+2\beta}}$ for $0 < \beta \leq \alpha + 1/2$.
- Optimise $\Pi_s\left(\|\theta - \theta_0\|_2 \leq \varepsilon_T\right)$ over

$$\Lambda = \left\{kT^{-\frac{1}{4+4\alpha}} : k \in \mathbb{N}, kT^{-\frac{1}{4+4\alpha}} \leq T^{\alpha}\right\}.$$

- Marginal maximum likelihood estimator (MMLE)

$$\hat{s} = \mathrm{argmax}_{s\in\Lambda} \int p_\theta(X^T) d\Pi_s(\theta),$$

$$p_\theta(X^T) = \exp\left\{\int_0^T \theta(X_t)dX_t - \frac{1}{2}\int_0^T \theta(X_t)^2 dt\right\}$$

Theorem

When θ_0 is β-Sobolev smooth, $0 < \beta \leq \alpha + 1/2$, then for some $M > 0$,

$$\Pi_{\hat{s}}\left(\theta : \|\theta - \theta_0\|_2 \leq M T^{-\frac{\beta}{1+2\beta}} \;\middle|\; X^T\right) \to 1$$

in $\mathbb{P}_{\theta_0}$-probability as $T \to \infty$.

Outline of the proof

Show $\Pi_{\hat{s}}\left(\|\theta-\theta_0\| > MT^{-\frac{\beta}{1+2\beta}} \mid X^T\right) \to 0.$

Ingredients of the proof:

1. Determine $\Lambda_0 \subseteq \Lambda$ where $\Pi_s(\cdot \mid X^T)$ enjoys good rates.
2. $\mathbb{P}_{\theta_0}(\hat{s} \in \Lambda_0) \to 1$, as $T \to \infty$,
3.

$$
\begin{aligned}
&\mathbb{P}_{\theta_0}\left(\Pi_{\hat{s}}\left(\|\theta-\theta_0\|_2 \geq MT^{-\frac{\beta}{1+2\beta}} \mid X^T\right)\right) \\
\leq&\mathbb{P}_{\theta_0}\left(\Pi_{\hat{s}}\left(\|\theta-\theta_0\|_2 \geq MT^{-\frac{\beta}{1+2\beta}} \mid X^T\right)\mathbb{I}_{\{\hat{s}\in\Lambda_0\}}\right) \\
&+\mathbb{P}_{\theta_0}\left(\Pi_{\hat{s}}\left(\|\theta-\theta_0\|_2 \geq MT^{-\frac{\beta}{1+2\beta}} \mid X^T\right)\mathbb{I}_{\{\hat{s}\notin\Lambda_0\}}\right) \\
\leq&\mathbb{P}_{\theta_0}\left(\max_{s\in\Lambda_0}\Pi_s\left(\|\theta-\theta_0\|_2 \geq MT^{-\frac{\beta}{1+2\beta}} \mid X^T\right)\right) \\
&+\mathbb{P}_{\theta_0}(\hat{s}\notin\Lambda_0) \to 0.
\end{aligned}
$$

Determining Λ_0

Let $K > 0$ be constant. There is a unique $\varepsilon_s > 0$ so that

$$\Pi_s(\|\theta - \theta_0\|_2 < K\varepsilon_s) = e^{-T\varepsilon_s^2}.$$

Let

$$\varepsilon_0 = \min_{s \in \Lambda} \varepsilon_s.$$

Let $L > 1$ be a constant and

$$\Lambda_0 = \{s \in \Lambda : \varepsilon_s \leq L\varepsilon_0\}.$$

Lemma

For $L > 1$ big enough, with $\mathbb{P}_{\theta_0}$-probability converging to one $\hat{s} \in \Lambda_0$.

Step 1 Take p_θ / p_{θ_0} instead.

$$\begin{aligned} &\mathrm{argmax}_{s \in \Lambda} \int p_\theta(X^T) d\Pi_s(\theta) \\ =&\, \mathrm{argmax}_{s \in \Lambda} \int p_\theta(X^T)/p_{\theta_0}(X^T) d\Pi_s(\theta). \end{aligned}$$

Step 2 Let $s_0 \in \Lambda, \varepsilon_{s_0} = \varepsilon_0$. There are constants $0 < A < B$ so that with $\mathbb{P}_{\theta_0}$-probability converging to one,

$$\begin{aligned} &\int p_\theta(X^T)/p_{\theta_0}(X^T) d\Pi_{s_0}(\theta) \geq e^{-AT\varepsilon_0^2} \\ >& e^{-BT\varepsilon_0^2} \geq \max_{s \in \Lambda \setminus \Lambda_0} \int p_\theta(X^T)/p_{\theta_0}(X^T) d\Pi_s(\theta) \end{aligned}$$

From

$$
\begin{aligned}
&\int p_\theta(X^T)/p_{\theta_0}(X^T)\,d\Pi_{\hat{s}}(\theta) \\
\geq &\int p_\theta(X^T)/p_{\theta_0}(X^T)\,d\Pi_{s_0}(\theta) \geq e^{-AT\varepsilon_0^2} \\
> &e^{-BT\varepsilon_0^2} \geq \max_{s\in\Lambda\setminus\Lambda_0} \int p_\theta(X^T)/p_{\theta_0}(X^T)\,d\Pi_s(\theta)
\end{aligned}
$$

follows that $\hat{s} \in \Lambda_0$ (on this event).

Goal: show that

$$\mathbb{P}_{\theta_0}\left(\max_{s\in\Lambda\setminus\Lambda_0}\int p_\theta(X^T)/p_{\theta_0}(X^T)d\Pi_s(\theta)\geq e^{-BT\varepsilon_0^2}\right)\to 0.$$

$$\begin{aligned}
&\mathbb{P}_{\theta_0}\left(\max_{s\in\Lambda\setminus\Lambda_0}\int p_\theta(X^T)/p_{\theta_0}(X^T)d\Pi_s(\theta)\geq e^{-BT\varepsilon_0^2}\right)\\
\leq& T^{\alpha+\frac{1}{4+4\alpha}}\max_{s\in\Lambda\setminus\Lambda_0}\mathbb{P}_{\theta_0}\left(\int p_\theta(X^T)/p_{\theta_0}(X^T)d\Pi_s(\theta)\geq e^{-BT\varepsilon_0^2}\right).
\end{aligned}$$

Let $s \in \Lambda \backslash \Lambda_0$. Consider

$$
\begin{aligned}
& \mathbb{P}_{\theta_0}\left(\int p_\theta(X^T)/p_{\theta_0}(X^T) d\Pi_s(\theta) \geq e^{-BT\varepsilon_0^2}\right) \\
= & \mathbb{E}_{\theta_0}\left[\mathbb{I}\left\{\int p_\theta(X^T)/p_{\theta_0}(X^T) d\Pi_s(\theta) \geq e^{-BT\varepsilon_0^2}\right\}(\varphi + 1 - \varphi)\right] \\
\leq & \mathbb{E}_{\theta_0}\varphi + e^{BT\varepsilon_0} \int \mathbb{E}_\theta[1-\varphi] d\Pi_s(\theta).
\end{aligned}
$$

Use $\varepsilon_s \geq L\varepsilon_0$ for $s \in \Lambda \backslash \Lambda_0$.

$$
\begin{aligned}
& \mathbb{E}_{\theta_0}\varphi \\
\leq & \ e^{-C_1 T \varepsilon_s^2} \\
\leq & \ e^{-C_1 L^2 T \varepsilon_0^2}.
\end{aligned}
$$

$$
\begin{aligned}
& \int \mathbb{E}_\theta[1-\varphi] d\Pi_s(\theta) \\
\leq & \int_{\|\theta-\theta_0\| \leq K\varepsilon_s} d\Pi_s(\theta) \\
& + \int_{\|\theta-\theta_0\| > K\varepsilon_s} \mathbb{E}_\theta[1-\varphi] d\Pi_s(\theta) \\
\leq & \ e^{-T\varepsilon_s^2} + e^{-C_2 T \varepsilon_s^2} \\
\leq & \ e^{-L^2 T \varepsilon_0^2} + e^{-C_2 L^2 T \varepsilon_0^2}
\end{aligned}
$$

Future work

- The asymptotic behaviour of credible sets.
- “Empirical Bayes” as tool to show rates for hierarchical Bayes priors.
- Simulation studies.

Thank you!