Handout

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1 Model

We observe a sample path $X^T = (X_t : t \in [0, T])$ of

$$dX_t = \theta_0(X_t)dt + dW_t,$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is measurable, one-periodic and $\int_0^1 \theta(x)^2 dx < \infty$.

Definition 1. Let $(\varepsilon_T)_{T>0}$ be a sequence of positive numbers converging to zero. We say that the posterior contracts with rate ε_T when for some constant $M > 0$,

$$\mathbb{E}_{\theta_0} \Pi(\theta : \|\theta - \theta_0\| \leq M\varepsilon_T \mid X^T) \to 1, \text{ as } T \to \infty.$$

2 Specification of the priors

2.1 Gaussian process prior

$$\begin{aligned}
\theta =& \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k, \\
\phi_{2k}(x) =& \sqrt{2}\cos(2\pi kx), \\
\phi_{2k-1}(x) =& \sqrt{2}\sin(2\pi kx), \\
Z_k \overset{iid}{\sim}& N(0, 1), \\
\alpha >& 0 \text{ is constant.}
\end{aligned}$$

2.2 Hierarchical prior

2.2.1 Prior on scaling

$$\begin{aligned}
E \sim& \mathrm{Exp}(1), \\
S =& \frac{E^{\alpha+1/2}}{\sqrt{T}} \\
\theta \mid S =& S \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k.
\end{aligned}$$

2.2.2 Prior on baseline smoothness

$$\begin{aligned}
\pi(\alpha) \propto& e^{-T^{\frac{1}{1+2\alpha}}}, \alpha \in (0, \log T], \\
\theta \mid \alpha =& \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k.
\end{aligned}$$

3 Rates of convergence

Theorem 2. *When θ_0 is β-Sobolev smooth, $\beta > 0$, then the Gaussian process prior contracts with rate $T^{-\frac{\beta}{1+2\beta}}$ when $\alpha = \beta$, the hierarchical prior with the hyperprior on the scaling achieves this rate when $\beta \leq \alpha + 1/2$ and the hierarchical prior with the hyperprior on α attains it for every $\beta > 0$.*

4 Empirical Bayes

Likelihood estimator for scaling.

$$\hat{s} = \operatorname{argmax}_s \int p^\theta(X^T) d\Pi_s(\theta), \quad s \in \left\{ kT^{-\frac{1}{4+4\alpha}} : k \in \left\{1, \dots, \left\lfloor T^{\alpha + \frac{1}{4+4\alpha}} \right\rfloor \right\}\right\}$$

Inference with plug-in posterior $\Pi_{\hat{s}}(\,\cdot \mid X^T) = \Pi_s(\,\cdot \mid X^T)\Big|_{s=\hat{s}}$, with

$$\Pi_s \sim \theta = s \sum_{k=1}^{\infty} k^{-\alpha-1/2} Z_k \phi_k.$$

Theorem 3. *When θ_0 is β-Sobolev smooth, $0 < \beta \leq \alpha + 1/2$ then for some constant $M > 0$,*

$$\mathbb{E}_{\theta_0}\left[\Pi_{\hat{s}}\left(\theta : \|\theta - \theta_0\|_2 \leq MT^{-\frac{\beta}{1+2\beta}} \mid X^T\right)\right] \to 1, \ \text{as } T \to \infty.$$

5 More information

- J. van Waaij (2018). “Adaptive posterior contraction rates for diffusions”. PhD thesis. University of Amsterdam. ISBN: 978-94-028-0883-4. eprint: http://hdl.handle.net/11245.1/6b890997-2565-4718-9b0f-719c755fa0d9
- J. van Waaij. *Slides and additional material of my talk on 18 July 2018*. https://github.com/Jan-van-Waaij/wias
- J. van Waaij (2017–2018). *BayesianNonparametricStatistics.jl, a Julia package to simulate SDE's and sample from the posterior.* URL: https://github.com/Jan-van-Waaij/BayesianNonparametricStatistics.jl