1 Simple Hamiltonian

The Hamiltonian of the system is given by

$$H = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \tilde{g}_A(\hat{z}) \otimes |\psi_A\rangle\langle\psi_A| + \tilde{g}_B(\hat{z}) \otimes |\psi_B\rangle\langle\psi_B| \tag{1}$$

where $\hat{z} = \sqrt{\hbar/2m\omega}(\hat{a} + \hat{a}^\dagger)$ is the amplitude of the vibration. The coupling between the plate and the cat-state $|\Psi\rangle = 1/\sqrt{2}(|\psi_A\rangle + |\psi_B\rangle)$ is given by the function $\tilde{g}_{A,B}$. Linearized, this coupling looks like

$$\tilde{g}_{A,B} \approx V_{\mathrm{PFA}} \left(\frac{1}{\mathscr{L}^2} + \frac{2\hat{z}u_{A,B}}{\mathscr{L}^3} \right) \tag{2}$$

where $V_{\mathrm{PFA}} = \hbar c \pi^3 R/720$ is the proximity-force-approximation of the Casimir force. After substitution, the Hamiltonian takes the form

$$H = H_0 + g_A(a + a^\dagger) |\psi_A\rangle\langle\psi_A| + g_B(a + a^\dagger) |\psi_B\rangle\langle\psi_B| + V_{\mathrm{PFA}} \frac{1}{\mathscr{L}^2} \tag{3}$$

which can be transformed into the interaction picture. Using $a \to a \exp\{-i\omega t\}$, this yields

$$\begin{aligned} H_{\mathrm{int}}(t) &= e^{i/\hbar H_0 t} H_{\mathrm{int}} e^{-i/\hbar H_0 t} && (4) \\ &= g_A(a e^{-i\omega t} + a^\dagger e^{i\omega t}) |\psi_A\rangle\langle\psi_A| + g_B(a e^{-i\omega t} + a^\dagger e^{i\omega t}) |\psi_B\rangle\langle\psi_B| + g_0 && (5) \end{aligned}$$

Using a *magnus expansion*, the time evolution unity

$$U(t) = \exp\left\{ -\frac{i}{\hbar} H_{\mathrm{int}}(t) t \right\} = \exp\left\{ \sum_{k=1}^{\infty} \Omega_k(t) \right\} \tag{6}$$

can be easily calculated. The coefficients $\Omega_k(t)$ are given by

$$\begin{aligned} \Omega_1(t) &= -\frac{i}{\hbar} \int_0^t \mathrm{d}t_1 \, H_{\mathrm{int}}(t) && (7) \\ &= \Big[g_A(f_1 a^\dagger - f_1^* a) |\psi_A\rangle\langle\psi_A| + g_B(f_1 a^\dagger - f_1^* a) |\psi_B\rangle\langle\psi_B| \Big] - \frac{i}{\hbar} g_0 t && (8) \end{aligned}$$

with $f_1 = (1 - e^{i\omega t})/\hbar\omega$.

$$\begin{aligned} \Omega_2(t) &= -\frac{1}{2\hbar^2} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, [H_{\mathrm{int}}(t_1), H_{\mathrm{int}}(t_2)] && (9) \\ &= -\frac{i}{\hbar^2\omega^2} (\sin(\omega t) - \omega t) \left(g_A^2 |\psi_A\rangle\langle\psi_A| + g_B^2 |\psi_B\rangle\langle\psi_B| \right) && (10) \\ &= f_2 \left(g_A^2 |\psi_A\rangle\langle\psi_A| + g_B^2 |\psi_B\rangle\langle\psi_B| \right) && (11) \end{aligned}$$

For this calculation, $[a,a^\dagger]=1$, $[\,.\,,g_0]=0$ and $\langle\psi_A|\psi_B\rangle=0$ has been used. It follows directly by the fact that $[H,|\psi_{A,B}\rangle\langle\psi_{A,B}|]=0$ that

$$\Omega_k=0,\quad k\geq 3. \tag{12}$$

The time evolution is therefore given by $\Omega_{1,2}(t)$ and can be rewritten as

$$\begin{aligned}U(t)&=D(g_Af_1)e^{f_2g_A^2}\,|\psi_A\rangle\langle\psi_A|+D(g_Bf_1)e^{f_2g_B^2}\,|\psi_B\rangle\langle\psi_B| &(13)\\ &=D(\zeta_A)e^{\varphi_A}\,|\psi_A\rangle\langle\psi_A|+D(\zeta_B)e^{\varphi_B}\,|\psi_B\rangle\langle\psi_B| &(14)\end{aligned}$$

where

$$D(\alpha)=\exp\Big\{\alpha a^\dagger-\alpha^* a\Big\} \tag{15}$$

is the so-called *displacement operator* of the oscillator.

The initial state consists of the thermal state of the shield vibration ρ_{th} and the cat-state ρ_{cat}

$$\rho(t=0)=\rho_{\mathrm{th}}\otimes\frac{1}{2}\left(|\psi_A\rangle\langle\psi_A|+|\psi_B\rangle\langle\psi_B|+|\psi_A\rangle\langle\psi_B|+|\psi_B\rangle\langle\psi_A|\right) \tag{16}$$

After time evolution, this evolves into

$$\begin{aligned}\rho(t)=D(\zeta_A)\rho_{\mathrm{th}}D^\dagger(\zeta_A)&\otimes\frac{1}{2}e^{\varphi_A}\,|\psi_A\rangle\langle\psi_A|\,e^{\varphi_A^*} &(17)\\ +D(\zeta_B)\rho_{\mathrm{th}}D^\dagger(\zeta_B)&\otimes\frac{1}{2}e^{\varphi_B}\,|\psi_B\rangle\langle\psi_B|\,e^{\varphi_B^*} &(18)\\ +D(\zeta_A)\rho_{\mathrm{th}}D^\dagger(\zeta_B)&\otimes\frac{1}{2}e^{\varphi_A}\,|\psi_A\rangle\langle\psi_B|\,e^{\varphi_B^*} &(19)\\ +D(\zeta_B)\rho_{\mathrm{th}}D^\dagger(\zeta_A)&\otimes\frac{1}{2}e^{\varphi_B}\,|\psi_B\rangle\langle\psi_A|\,e^{\varphi_A^*} &(20)\end{aligned}$$

I am interested in the evolution of the cat-state. This subsystem can be retrieved by tracing out the state of the thermal oscillator:

$$\rho_{\mathrm{cat}}(t)=\mathrm{tr}_{\mathrm{th}}\,\rho(t)=\frac{1}{2}\begin{pmatrix}\mathrm{tr}\{\ldots\rho_{\mathrm{th}}\ldots\} & e^{\varphi_A+\varphi_B^*}\,\mathrm{tr}\{\ldots\}\\ e^{\varphi_A^*+\varphi_B}\,\mathrm{tr}\{\ldots\} & \mathrm{tr}\{\ldots\}\end{pmatrix} \tag{21}$$

The general form of

$$\mathrm{tr}\Big\{D(\zeta_i)\rho_{\mathrm{th}}D^\dagger(\zeta_j)\Big\} \tag{22}$$

needs therefore be evaluated. The thermal state can be expanded into coherent states like

$$\rho_{\mathrm{th}}=\sum\frac{1}{Z}e^{-\beta\hbar\omega(n+1/2)}\,|n\rangle\langle n|=\int\mathrm{d}\alpha^2\frac{1}{\pi\,\langle n\rangle}e^{-\frac{|\alpha^2|}{\langle n\rangle}}\,|\alpha\rangle\langle\alpha| \tag{23}$$

where $\langle n\rangle=1/(e^{\beta\hbar\omega}-1)$ is the average occupation number and $\mathrm{d}\alpha^2=\mathrm{d}\,\mathrm{Re}(\alpha)\mathrm{d}\,\mathrm{Im}(\alpha)$. It turns out, that the trace is given by

$$\mathrm{tr}\Big\{D(\zeta_i)\rho_{\mathrm{th}}D^\dagger(\zeta_j)\Big\}=\exp\bigg\{\phi-|\Delta\zeta|^2\left(\langle n\rangle+\frac{1}{2}\right)\bigg\} \tag{24}$$

with $\Delta\zeta = \zeta_i - \zeta_j$ and $\phi = (\zeta_j^* \zeta_i - \zeta_j \zeta_i^*)/2$. Substituting everything back into this equation, it turns out that $\phi = 0$ and

$$|\Delta\zeta|^2 = \frac{(g_A - g_B)^2}{\hbar^2\omega^2} 4\sin^2\left(\frac{\omega t}{2}\right). \tag{25}$$

With this, the final evolution is given as

$$\rho_{\text{cat}}(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\varphi}e^{-\gamma} \\ e^{i\varphi}e^{-\gamma} & 1 \end{pmatrix} \tag{26}$$

with

$$\gamma = \frac{4(g_A - g_B)^2}{\hbar^2\omega^2}\sin^2\left(\frac{\omega t}{2}\right)\left[\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right] \tag{27}$$

$$\varphi = \frac{1}{\hbar^2\omega^2}\left(\sin(\omega t) - \omega t\right)\left(g_A^2 - g_B^2\right) \tag{28}$$

2 Considering all modes

Considering all modes (k, l), the total Hamiltonian looks like

$$H = \sum_{n\in\{(k,l)\}} \hbar\omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right) + g_{A,n}(a_n + a_n^\dagger)\,|\psi_A\rangle\langle\psi_A| + g_{B,n}(a_n + a_n^\dagger)\,|\psi_B\rangle\langle\psi_B| + g_0 \tag{29}$$

In the interaction picture, this Hamiltonian is given as

$$H_{\text{int}}(t) = \sum_n g_{A,n}\left(a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}\right)|\psi_A\rangle\langle\psi_A| + \dots \tag{30}$$

The Magnus expansion terms can be similarly calculated as before. Using $[a_i, a_j^\dagger] = \delta_{ij}$, $[H, |\psi_i\rangle\langle\psi_i|] = 0$ and $\langle\psi_A|\psi_B\rangle = 0$, one arrives at

$$\Omega_i(t) = \sum_n \mathbb{1}_\infty^{\otimes n} \otimes \Omega_{i,n}(t) \otimes \mathbb{1}_\infty^{\otimes\dots} \tag{31}$$

where $\Omega_{i,n}(t)$ $(i = 1, 2)$ are the same coefficients as before. It is important to note, that the dimensions are explicitly written out here to make the next steps easier. In all previous steps, they were neglected because of triviality. The time evolution can now be expressed as

$$\begin{aligned}
U &= \exp\left\{\sum_n \Omega_{1,n}(t) + \Omega_{2,n}(t)\right\} && (32)\\
&= \prod_n \exp\left\{\mathbb{1}^{\otimes n} \otimes \left[g_{A,1}(f_{1,n}a_n^\dagger - f_{1,n}^* a)\,|\psi_A\rangle\langle\psi_A| + \dots + f_{2,n}\dots\right] \otimes \mathbb{1}^{\otimes\dots}\right\} && (33)\\
&= \prod_n \left[\mathbb{1}^{\otimes n} \otimes \exp\{\dots\} \otimes \mathbb{1}^{\otimes\dots}\right] && (34)\\
&= \bigotimes_n \{D(\zeta_{A,n})e^{\varphi_{A,n}}\,|\psi_A\rangle\langle\psi_A|\} + \bigotimes_n \{D(\zeta_{B,n})e^{\varphi_{B,n}}\,|\psi_B\rangle\langle\psi_B|\} && (35)
\end{aligned}$$

where in the last step the property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ was exploited. This interaction can therefore not induce entanglement. The initial state is given by

$$\rho(t=0) = \left(\bigotimes_{n \in \{(k,l)\}} \rho_{\text{th,n}} \right) \otimes \frac{1}{2} \left(|\psi_A\rangle\langle\psi_A| + |\psi_B\rangle\langle\psi_B| + |\psi_A\rangle\langle\psi_B| + |\psi_B\rangle\langle\psi_A| \right) \tag{36}$$

and the time evolved state is therefore simply given by

$$\rho(t) = \left(\bigotimes_n D(\zeta_{A,n}) e^{\varphi_{A,n}} \rho_{\text{th,n}} e^{\varphi^*_{A,n}} D^\dagger(\zeta_{A,n}) \right) \otimes \frac{1}{2} |\psi_A\rangle\langle\psi_A| \tag{37}$$

$$+ \left(\bigotimes_n D(\zeta_{B,n}) e^{\varphi_{B,n}} \rho_{\text{th,n}} e^{\varphi^*_{B,n}} D^\dagger(\zeta_{B,n}) \right) \otimes \frac{1}{2} |\psi_B\rangle\langle\psi_B| \tag{38}$$

$$+ \left(\bigotimes_n D(\zeta_{A,n}) e^{\varphi_{A,n}} \rho_{\text{th,n}} e^{\varphi^*_{B,n}} D^\dagger(\zeta_{B,n}) \right) \otimes \frac{1}{2} |\psi_A\rangle\langle\psi_B| \tag{39}$$

$$+ \left(\bigotimes_n D(\zeta_{B,n}) e^{\varphi_{B,n}} \rho_{\text{th,n}} e^{\varphi^*_{A,n}} D^\dagger(\zeta_{A,n}) \right) \otimes \frac{1}{2} |\psi_B\rangle\langle\psi_A| \tag{40}$$

The evolved qubit state $\rho_{\text{qubit}}(t)$ is now given by the elements

$$\text{tr}\left\{ \bigotimes_n D(\zeta_{i,n}) \rho_{\text{th,n}} D^\dagger(\zeta_{j,n}) e^{\varphi_{i,n} + \varphi^*_{j,n}} \right\} = \prod_n \text{tr}\{\dots\} \tag{41}$$

Thus, each element in the density matrix is given as the product of all individual elements for all modes (k,l).