Fakultät für Naturwissenschaften

Institute of Theoretical Physics

Title of the work

Bachelor Thesis

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Abstract

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When is the system entangled (distance-measures, fidelity ??)

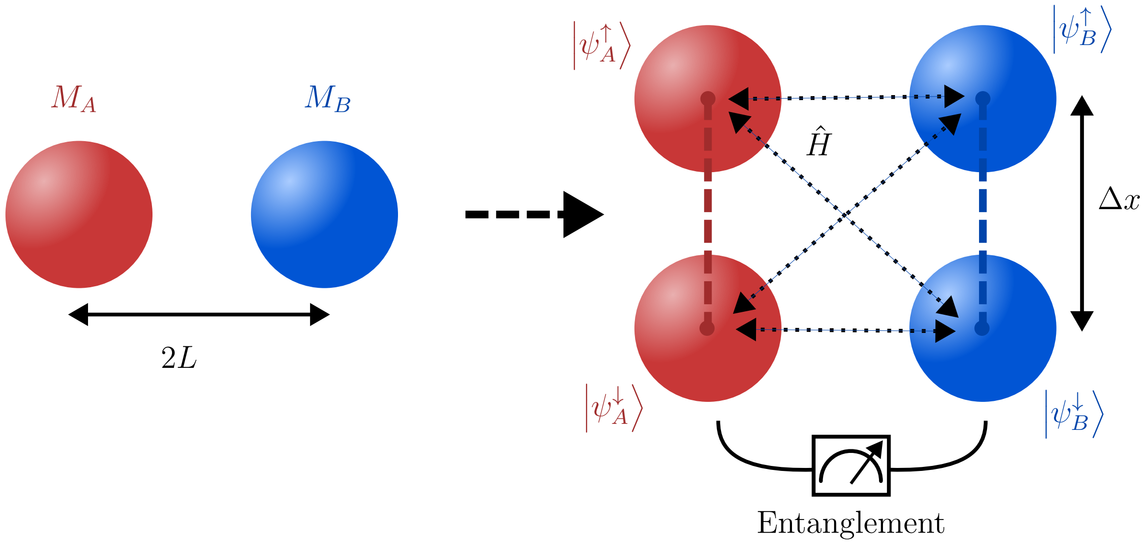

Figure 2.1: Simple Problem from Julens paper

Ignoring (for now) all other forms of interaction between the masses except gravity

The density $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ can be represented in the basis $\left\{\left|\psi_A^{\uparrow}\right\rangle, \left|\psi_A^{\downarrow}\right\rangle\right\} \otimes \left\{\left|\psi_B^{\uparrow}\right\rangle, \left|\psi_B^{\downarrow}\right\rangle\right\}$ as

$$\rho(t) = \frac{1}{4}\begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix} \tag{2.1}$$

2.1 Entanglement measures

Why are they needed, what can one do?

Logarithmic negativity, properties, calculation

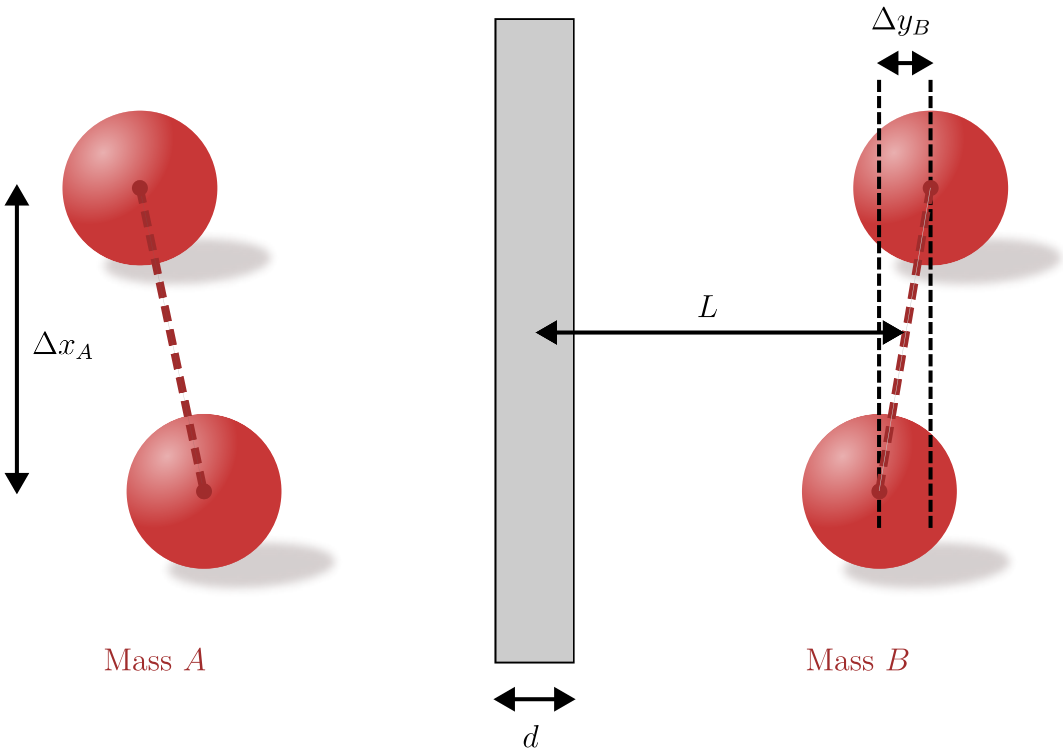

Figure 2.2: My problem

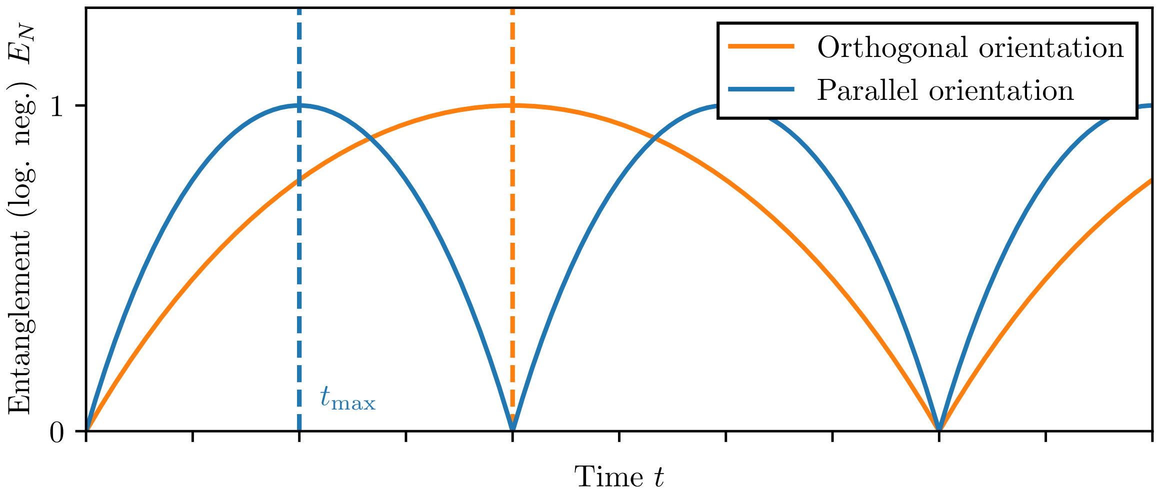

Figure 2.3: Entanglement dynamics quantified by the logarithmic negativity for two different orientations of the spatial superpositions relative to each other. The time of maximum entanglement t_{max} for the parallel configuration is given by $t_{\text{max}} = 8\pi\hbar L^3/(GM_A M_B d^2) \simeq 258\,\text{ms}$.

3 Casimir effect

General introduction and comparison with retarded van der Waals forces

$$F_{\text{Casimir}} = -\frac{\hbar c \pi^2}{240 L^4} A \tag{3.1}$$

$$V_{\text{Casimir}} = \frac{\hbar c \pi^2}{720 L^3} A \tag{3.2}$$

Lifshitz:

$$F_{\text{DD}} = \frac{\hbar c \pi^2}{240 L^4} \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right)^2 \varphi(\varepsilon_r) \tag{3.3}$$

$$F_{\text{DM}} = \frac{\hbar c \pi^2}{240 L^4} \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \varphi(\varepsilon_r) \tag{3.4}$$

The numeric function φ is shown in fig. 3.1.

3.1 Proximity force approximation

3.2 Imperfect plate and spheres

Python numerical approach, gaussian modes (vibration modes of a spherical plane), perlin noise

3.3 Casimir forces between a conducting plate and a dielectric sphere

3.3.1 Polarizability of a dielectric sphere

The polarizability α is defined via

$$\mathbf{E}_\infty \alpha = \mathbf{p}, \tag{3.5}$$

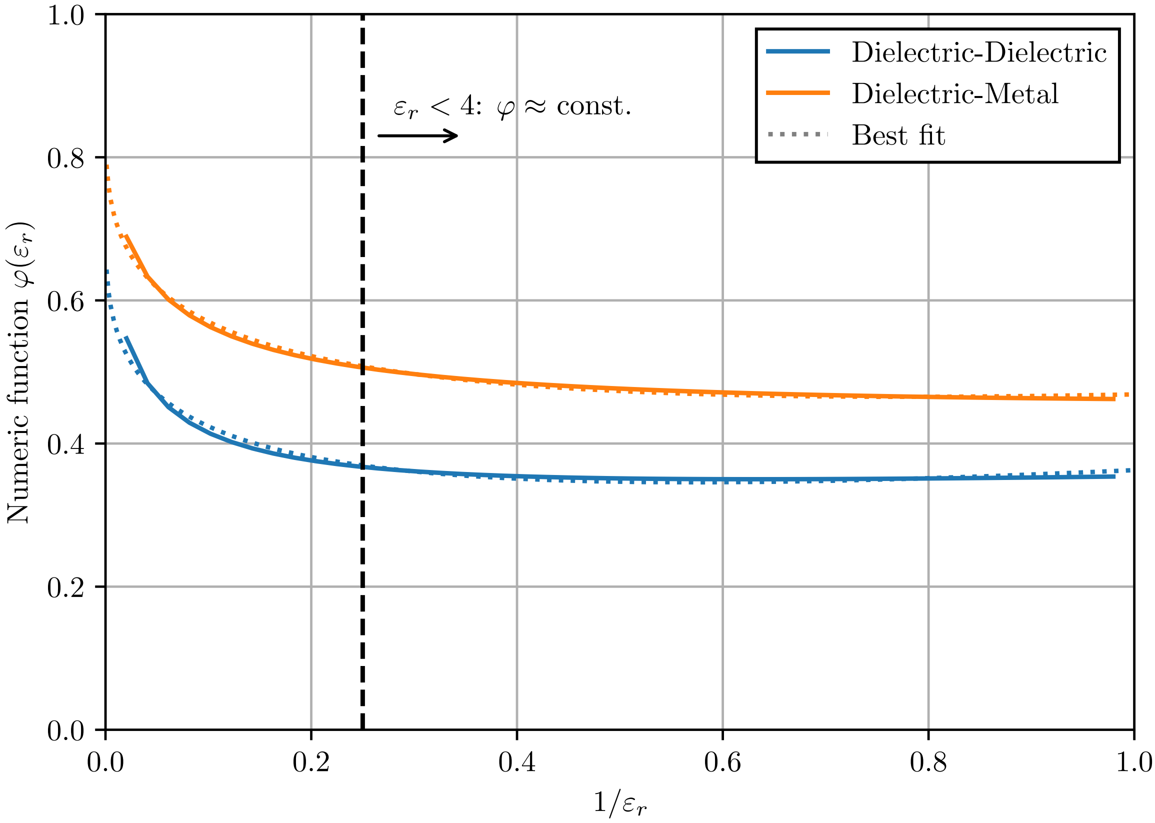

Figure 3.1: Numeric casimir interaction $\varphi(\epsilon_r)$ between **(blue)** two dielectric plates and **(orange)** a dielectric and a conductor.

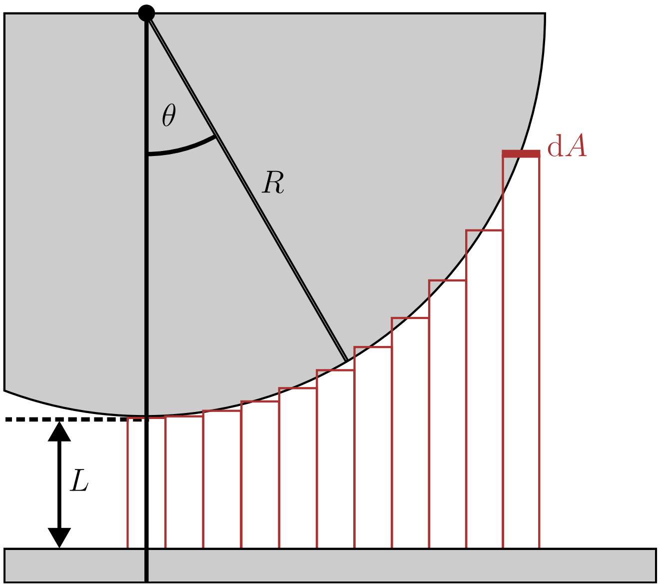

Figure 3.2: In the proximity force approximation the sphere is divided into infinitesimal plane areas $\mathrm{d}A$ which all exert a force $\mathrm{d}F$ according to eq. (3.1). All the contributions are added up together.

where $\mathbf{p}$ is the induced dipole moment and $\mathbf{E}_\infty$ is the external electric field that induces the dipole moment. For a linear and uniform dielectric, it is given as $\mathbf{p} = \mathcal{V}\varepsilon_0(\varepsilon_r - 1)\mathbf{E}_{\text{in}}$ [1, p. 220-226]. Here, $\mathcal{V}$ is the volume of the object and $\mathbf{E}_{\text{in}}$ is the electric field inside the dielectric. The electrostatic boundary conditions for the problem are given by

$$V_{\text{in}}\Big|_{r=R} = V_{\text{out}}\Big|_{r=R} \quad \text{and} \quad \varepsilon_r\varepsilon_0 \frac{\partial V_{\text{in}}}{\partial r}\Big|_{r=R} = \varepsilon_0 \frac{\partial V_{\text{out}}}{\partial r}\Big|_{r=R} \tag{3.6}$$

and the electric potential outside of the sphere at $r \to \infty$ should be equal to the external dipole-inducing field $V_{\text{out}}|_{r\to\infty} = -\mathbf{E}_\infty \cdot \mathbf{r} = -E_\infty r \cos\theta$. The electric potential inside and outside the sphere can be calculated using the spherical decomposition of the general electric potential $V \propto 1/|\mathbf{r} - \mathbf{r}'|$ into Legendre Polynomials P_l [1, p. 188-190]:

$$V_{\text{in}}(r,\theta) = -E_\infty r\cos\theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{3.7}$$

$$V_{\text{out}}(r,\theta) = -E_\infty r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.8}$$

Applying both boundary conditions, it follows that [1, p. 249-251]

$$\begin{cases} A_l = B_l = 0 & \text{for } l \neq 1, \\ A_1 = -\frac{3}{\varepsilon_r+2}E_\infty, \quad B_1 = \frac{\varepsilon_r-1}{\varepsilon_r+2}R^3E_\infty \end{cases} \tag{3.9}$$

and the resulting homogenous electric field $\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}}$ inside the sphere is given as

$$\mathbf{E}_{\text{in}} = \frac{3}{\varepsilon_r + 2}\mathbf{E}_\infty. \tag{3.10}$$

The field is shown on the right in fig. 3.3.The polarizability α of the sphere can be now be determined to

$$\alpha_{\text{sphere}} = 4\pi\varepsilon_0 R^3 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right). \tag{3.11}$$

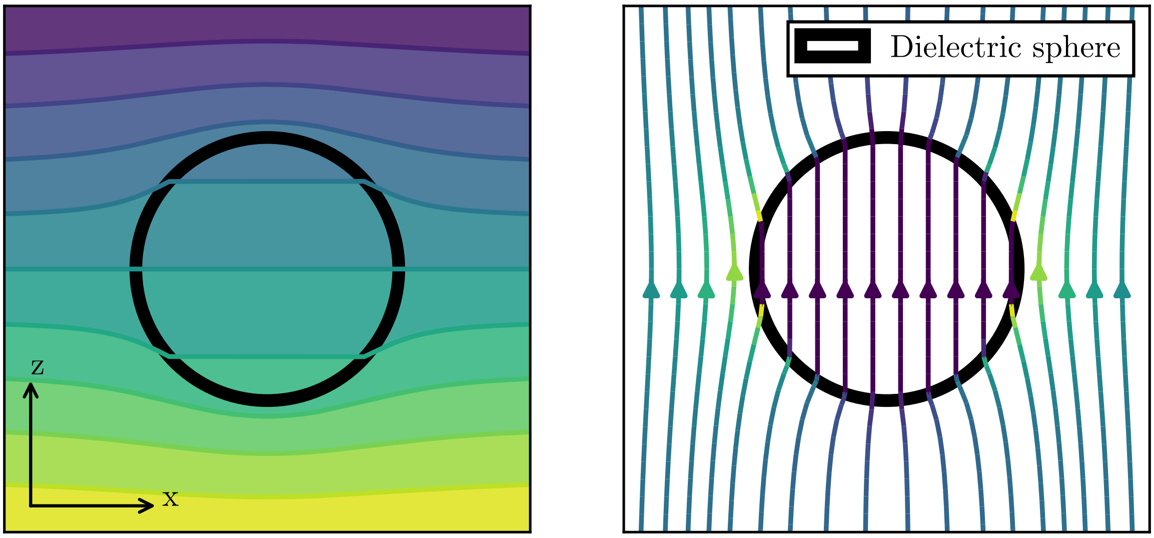

Figure 3.3: left: Electric potential V of a dielectric sphere in a external electric field $\mathbf{E}_\infty \parallel \mathbf{e_z}$. **right:** The corresponding electric field lines inside and outside the dielectric sphere.

4 Plate vibrations

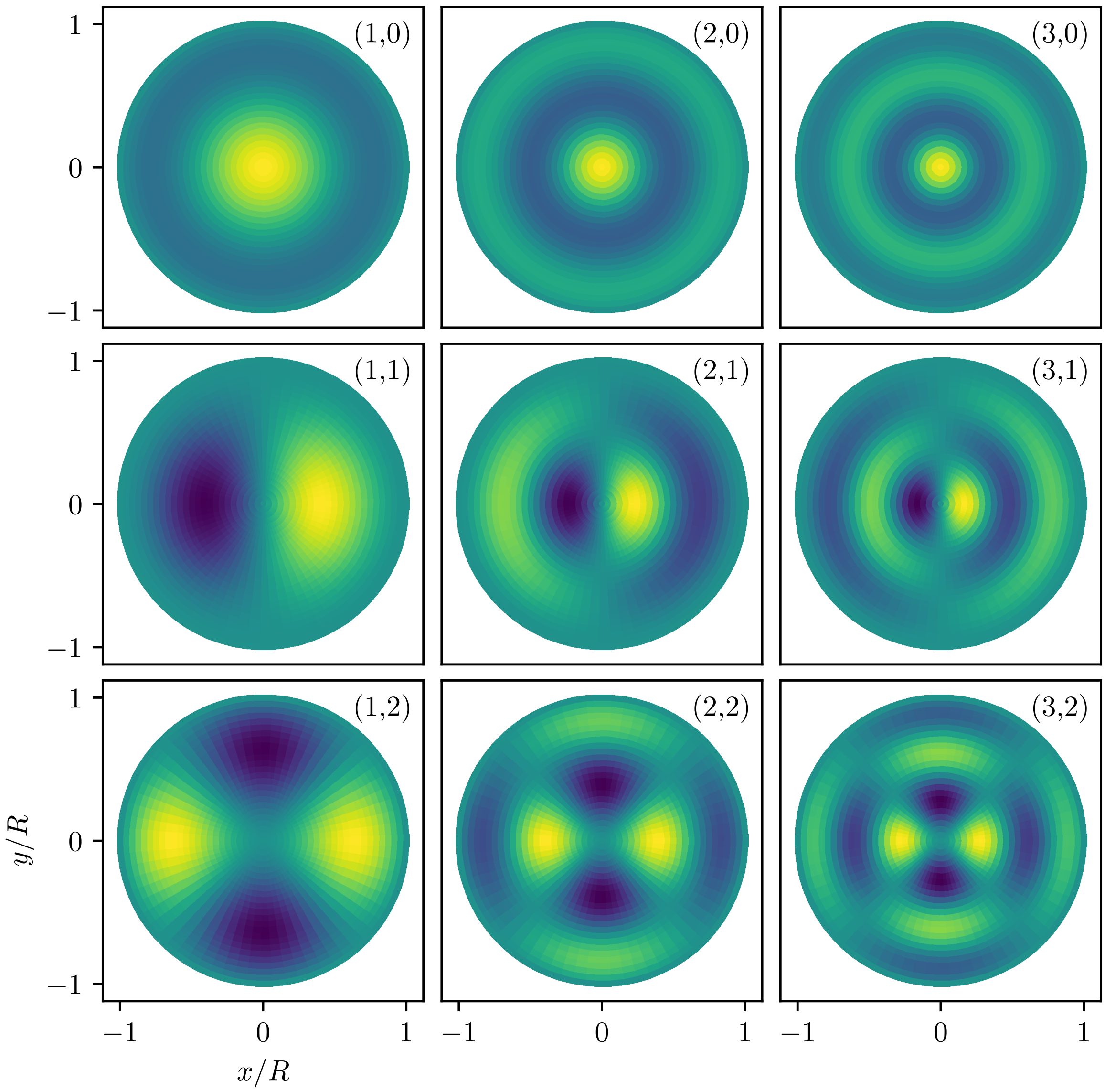

Figure 4.1: Vibrational modes of a spherical plate fixed at the edge with $R/d = 1000$.

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A Proofs and other stuff

A.1 Negativity

Lemma A.1. *The trace norm* $\|A\|_1 \equiv \operatorname{tr}\sqrt{A^\dagger A}$ *of a hermitian matrix* A *is equal to the sum of the absolute eigenvalues of* A*.*

Proof. This can be immediately seen by the spectral theorem:

$$\operatorname{tr}\sqrt{A^\dagger A} = \operatorname{tr}\sqrt{A^2} = \operatorname{tr}\left\{U\sqrt{\operatorname{diag}(\lambda_1,\dots)^2}U^\dagger\right\} = \sum_i \sqrt{\lambda_i^2} = \sum_i |\lambda_i| \,.$$

$\square$

Proposition A.1. *The* ***negativity*** $\mathcal{N}(\rho)$ *of a state* ρ *is given as the absolute sum of all negative eigenvalues of* ρ*:*

$$\mathcal{N}(\rho) \equiv \frac{\left\|\rho^{\Gamma_A}\right\|_1 - 1}{2} = \left|\sum_{\lambda_i<0} \lambda_i\right| . \tag{A.1}$$

Proof. The proof is in parts given by Vidal [7]. It is known that the density matrix is hermitian: $\rho = \rho^\dagger$. Using lemma A.1, the trace norm of the density matrix is is given as $\|\rho\|_1 = \sum \lambda_i = \operatorname{tr}\rho = 1$. The partial transpose ρ^{Γ_A} obviously also satisfies $\operatorname{tr}\rho^{\Gamma_A} = 1$ but might have negative eigenvalues. Since ρ^{Γ_A} is still hermitian, the trace norm is given by

$$\left\|\rho^{\Gamma_A}\right\|_1 = \sum_i |\lambda_i| = \sum_{\lambda_i\geq 0} \lambda_i + \sum_{\lambda_i<0} |\lambda_i| = \sum_i \lambda_i + 2\sum_{\lambda_i<0} |\lambda_i| = 1 + 2\sum_{\lambda_i<0} |\lambda_i| \,,$$

where in the last step $\sum \lambda_i = \operatorname{tr}\rho^{\Gamma_A} = 1$ was used. The negativity can be defined as $\mathcal{N}(\rho) = \left|\sum_{\lambda_i<0}\lambda_i\right|$ and the statement is shown. $\square$

Remark. The ***logarithmic negativity*** [3] relates to the negativity as follows

$$E_N(\rho) = \log_2 \left\|\rho^{\Gamma_A}\right\|_1 = \log_2\left(2\mathcal{N}(\rho) + 1\right) \tag{A.2}$$

and can therefore be easily calculated by using the above proposition A.1. In comparison to the negativity, logarithmic negativity has additive properties [4]:

$$E_N\left(\rho \otimes \sigma\right) = E_N(\rho) + E_N(\sigma)$$

A.2 Fidelity

The ***fidelity*** of two quantum states ρ and σ is defined as [8, p. 409-412]

$$F(\rho,\sigma) = \operatorname{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \tag{A.3}$$

and can be used as a distance measurement between quantum states. It is monotonic, concave and bounded between 0 and 1. If both states are equal $\rho = \sigma$, it is clear that $F(\rho,\sigma) = 1$, by using $\sqrt{\rho}\rho\sqrt{\rho} = \rho^2$. If both states commute, i.e. they are diagonalizable in the same orthogonal basis $\{|i\rangle\}$,

$$\rho = \sum_i r_i \, |i\rangle\langle i| \, ; \quad \sigma = \sum_i s_i \, |i\rangle\langle i| \, ,$$

the fidelity is given as [8]

$$F(\rho,\sigma) = \operatorname{tr}\sqrt{\sum_i r_i s_i \, |i\rangle\langle i|} = \sum_i \sqrt{r_i s_i}.$$

This can be seen immediately by the use of the spectral theorem $\operatorname{tr}\sqrt{\rho} = \operatorname{tr}\Big\{U\sqrt{\operatorname{diag}(r_i)}U^\dagger\Big\} = \operatorname{tr}\operatorname{diag}(\sqrt{r_i})$. Another special case is given for the fidelity of a pure state $\rho = |\psi\rangle\langle\psi|$ and an arbitrary state σ [8]:

$$F(|\psi\rangle\,,\sigma) = \operatorname{tr}\sqrt{\langle\psi|\,\sigma\,|\psi\rangle\,|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\,\sigma\,|\psi\rangle}.$$

If the state $\sigma = |\phi\rangle\langle\phi|$ is also pure, the fidelity reduces to

$$F(|\psi\rangle\,,|\phi\rangle) = |\langle\psi|\phi\rangle| \leq 1, \tag{A.4}$$

with equality being attained if the states are the same and only differ by a phase.