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**Fakultät für
Naturwissenschaften**

Institute of Theoretical
Physics

Title of the work

Bachelor Thesis

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Abstract

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1 Introduction

1.1 Feynman's Gedankenexperiment

2 A first look at the problem

Problem, general calculations from Julens paper without casimir

When is the system entangled (distance-measures, fidelity ??)

2.1 Entanglement measures

Why are they needed, what can one do?

Logarithmic negativity, properties, calculation

3 Casimir effect

General introduction and comparison with retarded van der Waals forces

$$F_{\text{Casimir}} = -\frac{\hbar c \pi^2}{240 L^4} A \quad (3.1)$$

$$V_{\text{Casimir}} = \frac{\hbar c \pi^2}{720 L^3} A \quad (3.2)$$

3.1 Proximity force approximation

3.2 Imperfect plate and spheres

Python numerical approach, gaussian modes (vibration modes of a spherical plane), perlin noise

3.3 Casimir forces between a conducting plate and a dielectric sphere

3.3.1 Polarizability of a dielectric sphere

The polarizability α is defined via

$$\mathbf{E}_{\infty} \alpha = \mathbf{p}, \quad (3.3)$$

where \mathbf{p} is the induced dipole moment and \mathbf{E}_{∞} is the external electric field that induces the dipole moment. For a linear and uniform dielectric, it is given as $\mathbf{p} = \mathcal{V} \varepsilon_0 (\varepsilon_r - 1) \mathbf{E}_{\text{in}}$ [1, p. 220-226]. Here, \mathcal{V} is the volume of the object and \mathbf{E}_{in} is the electric field inside the dielectric. The electrostatic boundary conditions for the problem are given by

$$V_{\text{in}} \Big|_{r=R} = V_{\text{out}} \Big|_{r=R} \quad \text{and} \quad \varepsilon_r \varepsilon_0 \frac{\partial V_{\text{in}}}{\partial r} \Big|_{r=R} = \varepsilon_0 \frac{\partial V_{\text{out}}}{\partial r} \Big|_{r=R} \quad (3.4)$$

and the electric potential outside of the sphere at $r \rightarrow \infty$ should be equal to the external dipole-inducing field $V_{\text{out}}|_{r \rightarrow \infty} = -\mathbf{E}_{\infty} \cdot \mathbf{r} = -E_{\infty} r \cos \theta$. The electric potential inside

3 Casimir effect

and outside the sphere can be calculated using the spherical decomposition of the general electric potential $V \propto 1/|\mathbf{r} - \mathbf{r}'|$ into Legendre Polynomials P_l [1, p. 188-190]:

$$V_{\text{in}}(r, \theta) = -E_{\infty} r \cos \theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta), \quad (3.5)$$

$$V_{\text{out}}(r, \theta) = -E_{\infty} r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta). \quad (3.6)$$

Applying both boundary conditions, it follows that [1, p. 249-251]

$$\begin{cases} A_l = B_l = 0 \\ A_1 = -\frac{3}{\varepsilon_r + 2} E_{\infty}, \quad B_1 = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_{\infty} \end{cases} \quad \text{for } l \neq 1, \quad (3.7)$$

and the resulting homogenous electric field $\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}}$ inside the sphere is given as

$$\mathbf{E}_{\text{in}} = \frac{3}{\varepsilon_r + 2} \mathbf{E}_{\infty}. \quad (3.8)$$

The field is shown on the right in fig. 3.1. The polarizability α of the sphere can be now be determined to

$$\alpha_{\text{sphere}} = 4\pi\varepsilon_0 R^3 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right). \quad (3.9)$$

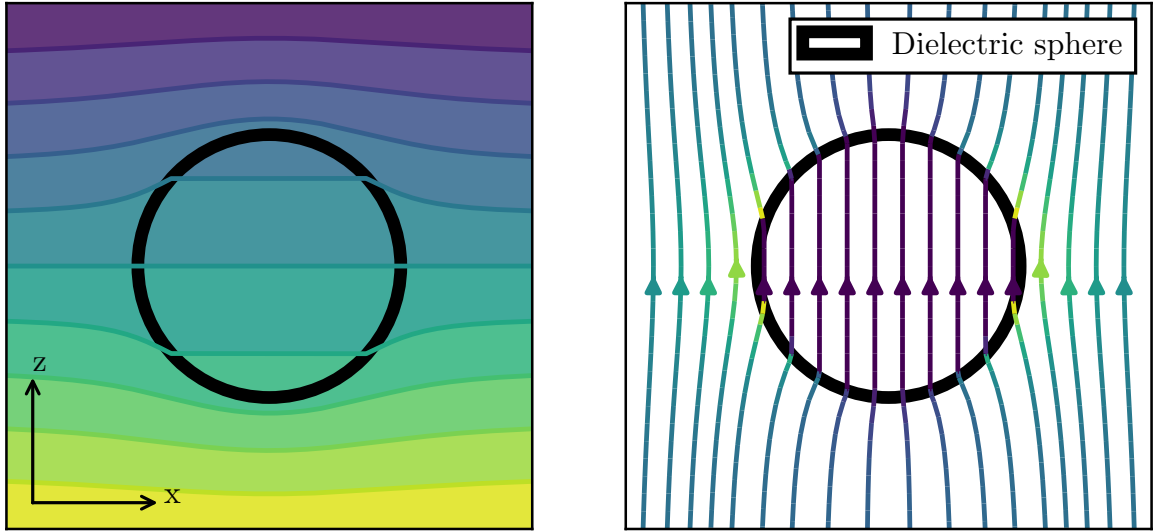


Figure 3.1: **left:** Electric potential V of a dielectric sphere in a external electric field $\mathbf{E}_{\infty} \parallel \mathbf{e}_z$. **right:** The corresponding electric field lines inside and outside the dielectric sphere.

4 Plate vibrations

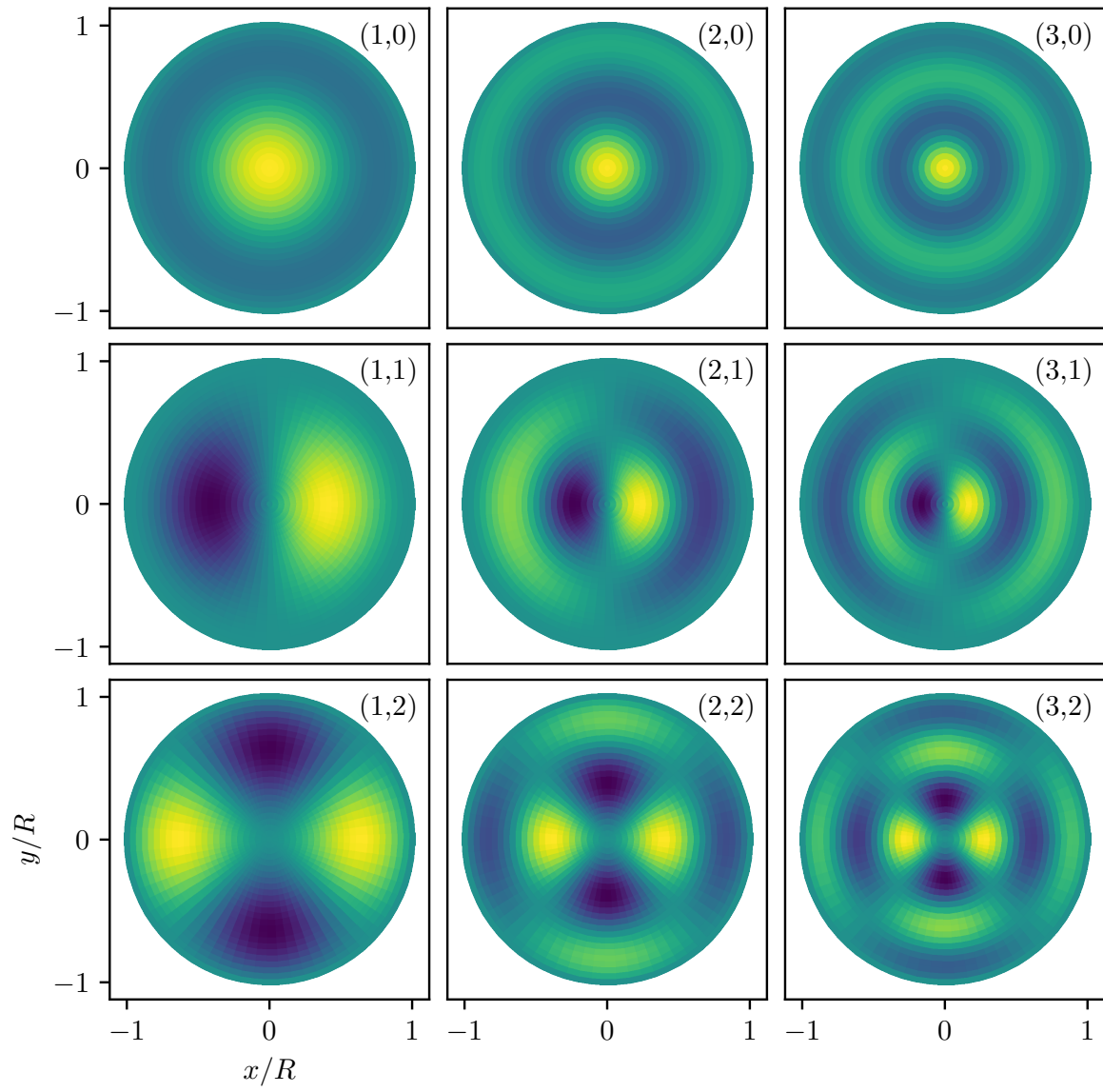


Figure 4.1: Vibrational modes of a spherical plate fixed at the edge with $R/d = 1000$.

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A Proofs and other stuff

A.1 Negativity

Lemma A.1. *The trace norm $\|A\|_1 \equiv \text{tr} \sqrt{A^\dagger A}$ of a hermitian matrix A is equal to the sum of the absolute eigenvalues of A .*

Proof. This can be immediately seen by the spectral theorem:

$$\text{tr} \sqrt{A^\dagger A} = \text{tr} \sqrt{A^2} = \text{tr} \left\{ U \sqrt{\text{diag}(\lambda_1, \dots)^2} U^\dagger \right\} = \sum_i \sqrt{\lambda_i^2} = \sum_i |\lambda_i|.$$

□

Proposition A.1. *The **negativity** $\mathcal{N}(\rho)$ of a state ρ is given as the absolute sum of all negative eigenvalues of ρ :*

$$\mathcal{N}(\rho) \equiv \frac{\|\rho^{\Gamma_A}\|_1 - 1}{2} = \left| \sum_{\lambda_i < 0} \lambda_i \right|. \quad (\text{A.1})$$

Proof. The proof is in parts given by Vidal [7]. It is known that the density matrix is hermitian: $\rho = \rho^\dagger$. Using lemma A.1, the trace norm of the density matrix is given as $\|\rho\|_1 = \sum \lambda_i = \text{tr} \rho = 1$. The partial transpose ρ^{Γ_A} obviously also satisfies $\text{tr} \rho^{\Gamma_A} = 1$ but might have negative eigenvalues. Since ρ^{Γ_A} is still hermitian, the trace norm is given by

$$\|\rho^{\Gamma_A}\|_1 = \sum_i |\lambda_i| = \sum_{\lambda_i \geq 0} \lambda_i + \sum_{\lambda_i < 0} |\lambda_i| = \sum_i \lambda_i + 2 \sum_{\lambda_i < 0} |\lambda_i| = 1 + 2 \sum_{\lambda_i < 0} |\lambda_i|,$$

where in the last step $\sum \lambda_i = \text{tr} \rho^{\Gamma_A} = 1$ was used. The negativity can be defined as $\mathcal{N}(\rho) = \left| \sum_{\lambda_i < 0} \lambda_i \right|$ and the statement is shown. □

Remark. The **logarithmic negativity** [3] relates to the negativity as follows

$$E_N(\rho) = \log_2 \|\rho^{\Gamma_A}\|_1 = \log_2 (2\mathcal{N}(\rho) + 1) \quad (\text{A.2})$$

and can therefore be easily calculated by using the above proposition A.1. In comparison to the negativity, logarithmic negativity has additive properties [4]:

$$E_N(\rho \otimes \sigma) = E_N(\rho) + E_N(\sigma)$$

A.2 Fidelity

The *fidelity* of two quantum states ρ and σ is defined as [8, p. 409-412]

$$F(\rho, \sigma) = \text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \quad (\text{A.3})$$

and can be used as a distance measurement between quantum states. It is monotonic, concave and bounded between 0 and 1. If both states are equal $\rho = \sigma$, it is clear that $F(\rho, \sigma) = 1$, by using $\sqrt{\rho} \rho \sqrt{\rho} = \rho^2$. If both states commute, i.e. they are diagonalizable in the same orthogonal basis $\{|i\rangle\}$,

$$\rho = \sum_i r_i |i\rangle\langle i|; \quad \sigma = \sum_i s_i |i\rangle\langle i|,$$

the fidelity is given as [8]

$$F(\rho, \sigma) = \text{tr} \sqrt{\sum_i r_i s_i |i\rangle\langle i|} = \sum_i \sqrt{r_i s_i}.$$

This can be seen immediately by the use of the spectral theorem $\text{tr} \sqrt{\rho} = \text{tr} \{U \sqrt{\text{diag}(r_i)} U^\dagger\} = \text{tr} \text{diag}(\sqrt{r_i})$. Another special case is given for the fidelity of a pure state $\rho = |\psi\rangle\langle\psi|$ and an arbitrary state σ [8]:

$$F(|\psi\rangle, \sigma) = \text{tr} \sqrt{|\psi\rangle\langle\psi| \sigma |\psi\rangle\langle\psi|} = \sqrt{\langle\psi| \sigma |\psi\rangle}.$$

If the state $\sigma = |\phi\rangle\langle\phi|$ is also pure, the fidelity reduces to

$$F(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle| \leq 1, \quad (\text{A.4})$$

with equality being attained if the states are the same and only differ by a phase.