The Hamiltonian of the system is given by

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \tilde{g}_A(\hat{z}) \otimes |\psi_A\rangle\langle\psi_A| + \tilde{g}_B(\hat{z}) \otimes |\psi_B\rangle\langle\psi_B| \tag{1}$$

where $\hat{z} = \sqrt{\hbar/2m\omega}(\hat{a} + \hat{a}^\dagger)$ is the amplitude of the vibration. The coupling between the plate and the cat-state $|\Psi\rangle = 1/\sqrt{2}(|\psi_A\rangle + |\psi_B\rangle)$ is given by the function $\tilde{g}_{A,B}$. Linearized, this coupling looks like

$$\tilde{g}_{A,B} \approx V_{\text{PFA}}\left(\frac{1}{\mathscr{L}^2} + \frac{2\hat{z}u_{A,B}}{\mathscr{L}^3}\right) \tag{2}$$

where $V_{\text{PFA}} = \hbar c\pi^3 R/720$ is the proximity-force-approximation of the Casimir force. After substitution, the Hamiltonian takes the form

$$H = H_0 + g_A(a + a^\dagger)\,|\psi_A\rangle\langle\psi_A| + g_B(a + a^\dagger)\,|\psi_B\rangle\langle\psi_B| + V_{\text{PFA}}\frac{1}{\mathscr{L}^2} \tag{3}$$

which can be transformed into the interaction picture. Using $a \to a\exp\{-i\omega t\}$, this yields

$$\begin{aligned} H_{\text{int}}(t) &= e^{i/\hbar H_0 t} H_{\text{int}} e^{-i/\hbar H_0 t} && (4)\\ &= g_A(ae^{-i\omega t} + a^\dagger e^{i\omega t})\,|\psi_A\rangle\langle\psi_A| + g_B(ae^{-i\omega t} + a^\dagger e^{i\omega t})\,|\psi_B\rangle\langle\psi_B| + g_0 && (5) \end{aligned}$$

Using a *magnus expansion*, the time evolution unity

$$U(t) = \exp\left\{-\frac{i}{\hbar}H_{\text{int}}(t)t\right\} = \exp\left\{\sum_{k=1}^{\infty}\Omega_k(t)\right\} \tag{6}$$

can be easily calculated. The coefficients $\Omega_k(t)$ are given by

$$\begin{aligned} \Omega_1(t) &= -\frac{i}{\hbar}\int_0^t \mathrm{d}t_1\, H_{\text{int}}(t) && (7)\\ &= \Big[g_A(f_1 a^\dagger - f_1^* a)\,|\psi_A\rangle\langle\psi_A| + g_B(f_1 a^\dagger - f_1^* a)\,|\psi_B\rangle\langle\psi_B|\Big] - \frac{i}{\hbar}g_0 t && (8) \end{aligned}$$

with $f_1 = (1 - e^{i\omega t})/\hbar\omega$.

$$\begin{aligned} \Omega_2(t) &= -\frac{1}{2\hbar^2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, [H_{\text{int}}(t_1), H_{\text{int}}(t_2)] && (9)\\ &= -\frac{i}{\hbar^2\omega^2}\left(\sin(\omega t) - \omega t\right)\left(g_A^2\,|\psi_A\rangle\langle\psi_A| + g_B^2\,|\psi_B\rangle\langle\psi_B|\right) && (10)\\ &= f_2\left(g_A^2\,|\psi_A\rangle\langle\psi_A| + g_B^2\,|\psi_B\rangle\langle\psi_B|\right) && (11) \end{aligned}$$

For this calculation, $[a, a^\dagger] = 1$, $[\,.\,, g_0] = 0$ and $\langle\psi_A|\psi_B\rangle = 0$ has been used. It follows directly by the fact that $[H, |\psi_{A,B}\rangle\langle\psi_{A,B}|] = 0$ that

$$\Omega_k = 0, \quad k \geq 3. \tag{12}$$

The time evolution is therefore given by $\Omega_{1,2}(t)$ and can be rewritten as

$$U(t) = D(g_A f_1) e^{f_2 g_A^2} |\psi_A\rangle\langle\psi_A| + D(g_B f_1) e^{f_2 g_B^2} |\psi_B\rangle\langle\psi_B| \tag{13}$$
$$= D(\zeta_A) e^{\varphi_A} |\psi_A\rangle\langle\psi_A| + D(\zeta_B) e^{\varphi_B} |\psi_B\rangle\langle\psi_B| \tag{14}$$

where

$$D(\alpha) = \exp\left\{\alpha a^\dagger - \alpha^* a\right\} \tag{15}$$

is the so-called *displacement operator* of the oscillator.

The initial state consists of the thermal state of the shield vibration ρ_{th} and the cat-state ρ_{cat}

$$\rho(t=0) = \rho_{\text{th}} \otimes \frac{1}{2} \left(|\psi_A\rangle\langle\psi_A| + |\psi_B\rangle\langle\psi_B| + |\psi_A\rangle\langle\psi_B| + |\psi_B\rangle\langle\psi_A|\right) \tag{16}$$

After time evolution, this evolves into

$$\rho(t) = D(\zeta_A)\rho_{\text{th}} D^\dagger(\zeta_A) \otimes \frac{1}{2} e^{\varphi_A} |\psi_A\rangle\langle\psi_A| e^{\varphi_A^*} \tag{17}$$
$$+ D(\zeta_B)\rho_{\text{th}} D^\dagger(\zeta_B) \otimes \frac{1}{2} e^{\varphi_B} |\psi_B\rangle\langle\psi_B| e^{\varphi_B^*} \tag{18}$$
$$+ D(\zeta_A)\rho_{\text{th}} D^\dagger(\zeta_B) \otimes \frac{1}{2} e^{\varphi_A} |\psi_A\rangle\langle\psi_B| e^{\varphi_B^*} \tag{19}$$
$$+ D(\zeta_B)\rho_{\text{th}} D^\dagger(\zeta_A) \otimes \frac{1}{2} e^{\varphi_B} |\psi_B\rangle\langle\psi_A| e^{\varphi_A^*} \tag{20}$$

I am interested in the evolution of the cat-state. This subsystem can be retrieved by tracing out the state of the thermal oscillator:

$$\rho_{\text{cat}}(t) = \operatorname{tr}_{\text{th}} \rho(t) = \frac{1}{2} \begin{pmatrix} \operatorname{tr}\{\dots\rho_{\text{th}}\dots\} & e^{\varphi_A + \varphi_B^*} \operatorname{tr}\{\dots\} \\ e^{\varphi_A^* + \varphi_B} \operatorname{tr}\{\dots\} & \operatorname{tr}\{\dots\} \end{pmatrix} \tag{21}$$

The general form of

$$\operatorname{tr}\left\{D(\zeta_i)\rho_{\text{th}} D^\dagger(\zeta_j)\right\} \tag{22}$$

needs therefore be evaluated. The thermal state can be expanded into coherent states like

$$\rho_{\text{th}} = \sum \frac{1}{Z} e^{-\beta\hbar\omega(n+1/2)} |n\rangle\langle n| = \int \mathrm{d}\alpha^2 \frac{1}{\pi\langle n\rangle} e^{-\frac{|\alpha^2|}{\langle n\rangle}} |\alpha\rangle\langle\alpha| \tag{23}$$

where $\langle n\rangle = 1/(e^{\beta\hbar\omega} - 1)$ is the average occupation number and $\mathrm{d}\alpha^2 = \mathrm{d}\,\mathrm{Re}(\alpha)\mathrm{d}\,\mathrm{Im}(\alpha)$. It turns out, that the trace is given by

$$\operatorname{tr}\left\{D(\zeta_i)\rho_{\text{th}} D^\dagger(\zeta_j)\right\} = \exp\left\{\phi - |\Delta\zeta|^2 \left(\langle n\rangle + \frac{1}{2}\right)\right\} \tag{24}$$

with $\Delta\zeta = \zeta_i - \zeta_j$ and $\phi = (\zeta_j^* \zeta_i - \zeta_j \zeta_i^*)/2$. Substituting everything back into this equation, it turns out that $\phi = 0$ and

$$|\Delta\zeta|^2 = \frac{(g_A - g_B)^2}{\hbar^2\omega^2} 4 \sin^2\left(\frac{\omega t}{2}\right). \tag{25}$$

With this, the final evolution is given as

$$\rho_{\text{cat}}(t) = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\varphi}e^{-\gamma} \\ e^{i\varphi}e^{-\gamma} & 1 \end{pmatrix} \tag{26}$$

with

$$\gamma = \frac{4(g_A - g_B)^2}{\hbar^2\omega^2} \sin^2\left(\frac{\omega t}{2}\right) \left[\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right] \tag{27}$$

$$\varphi = \frac{1}{\hbar^2\omega^2} \left(\sin(\omega t) - \omega t\right) \left(g_A^2 - g_B^2\right) \tag{28}$$

For small times $\omega t \ll 1$ and $\hbar\omega \ll k_B T$, this can be approximated as

$$\gamma = \frac{4(g_A - g_B)^2}{\hbar^2} t^2 \left[\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right] \approx \frac{4(g_A - g_B)^2}{\hbar^2} t^2 \left[\frac{k_B T}{\hbar\omega}\right] \tag{29}$$

$$\varphi = 0 \tag{30}$$

which is precisely (differ only by a factor of 4 ???) the more simple expression computed just by the integral over the probability distribution $p(\hat{z})$ with the result

$$\gamma = 4\Delta\phi^2 \left(\sqrt{\frac{k_B T}{m\omega^2}}\right) \approx \Delta\phi^2 \left(\sqrt{(\Delta z)_T^2}\right) \tag{31}$$

with

$$\Delta\phi(z) \approx \frac{c\pi^3 R t}{720} \cdot \frac{2(u_A - u_B)z}{\mathscr{L}^3} = \frac{g_A - g_B}{\hbar} tz \cdot \sqrt{\frac{2m\omega}{\hbar}} \tag{32}$$

and the variance of the oscillator

$$(\Delta z)_T^2 = \frac{\hbar}{2m\omega} \coth\left(\frac{\hbar\omega}{2k_B T}\right) \approx \frac{k_B T}{m\omega_{kl}^2}. \tag{33}$$