Fakultät für Naturwissenschaften

Institute of Theoretical Physics

Title of the work

Bachelor Thesis

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Abstract

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$$F_{\text{Casimir}} = -\frac{\hbar c \pi^2}{240 L^4} A \tag{3.1}$$

$$V_{\text{Casimir}} = \frac{\hbar c \pi^2}{720 L^3} A \tag{3.2}$$

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3.3.1 Polarizability of a dielectric sphere

To quantify the polarizability $\alpha = |\mathbf{p}| \,/\, |\mathbf{E}_\infty|$ of a homogenous sphere with $\varepsilon_r > 1$, the influence of an electric field $\mathbf{E}_\infty$ in (without loss of generality) is assumed to be in $\mathbf{e_z}$ direction. The electrostatic boundary conditions for the problem are given by

$$V_{\text{in}}\Big|_{r=R} = V_{\text{out}}\Big|_{r=R} \quad \text{and} \quad \varepsilon_r \varepsilon_0 \frac{\partial V_{\text{in}}}{\partial r}\bigg|_{r=R} = \varepsilon_0 \frac{\partial V_{\text{out}}}{\partial r}\bigg|_{r=R} \tag{3.3}$$

and the electric potential outside of the sphere at $r \to \infty$ should be equal to the homogenous electric field $V_{\text{out}} = -\mathbf{E}_\infty \cdot \mathbf{r} = -E_\infty r \cos\theta$. The electric potential inside and outside the sphere can be calculated using the spherical decomposition of the general electric potential $V \propto 1/|\mathbf{r}-\mathbf{r}'|$ into Legendre Polynomials P_l [1, p. 188-190]:

$$V_{\text{in}}(r,\theta) = -E_\infty r\cos\theta + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{3.4}$$

$$V_{\text{out}}(r,\theta) = -E_\infty r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.5}$$

Applying both boundary conditions, it follows that [1, p. 249-251]

$$\begin{cases} A_l = B_l = 0 & \text{for } l \neq 1, \\ A_1 = -\frac{3}{\varepsilon_r+2}E_\infty, \quad B_1 = \frac{\varepsilon_r-1}{\varepsilon_r+2}R^3E_\infty \end{cases} \tag{3.6}$$

and the resulting homogenous electric field $\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}}$ inside the sphere is given as

$$\mathbf{E}_{\text{in}} = \frac{3}{\varepsilon_r+2}\mathbf{E}_\infty. \tag{3.7}$$

The polarizability α can be now calculated using $\alpha\mathbf{E}_\infty = \mathbf{p} = 4/3\pi R^3\mathbf{P} = 4/3\pi R^3\varepsilon_0(\varepsilon_r-1)\mathbf{E}_{\text{in}}$ to

$$\alpha_{\text{sphere}} = 4\pi\varepsilon_0 R^3\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right). \tag{3.8}$$

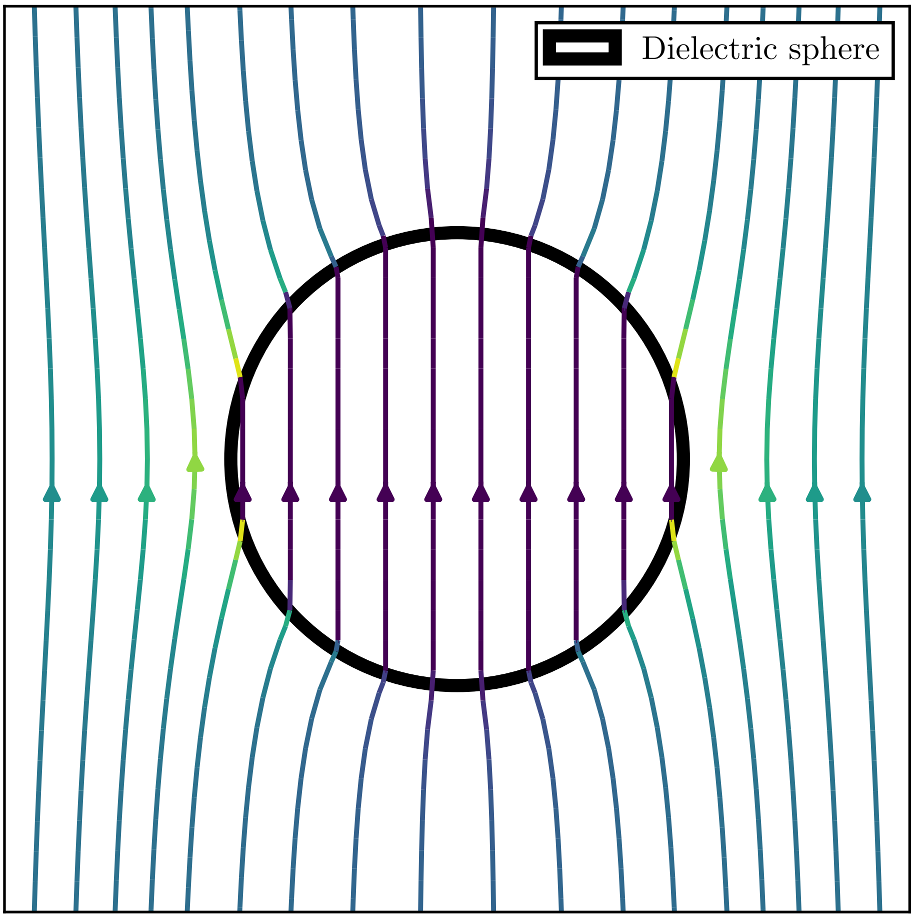

Figure 3.1: Electric field lines through an dielectric sphere

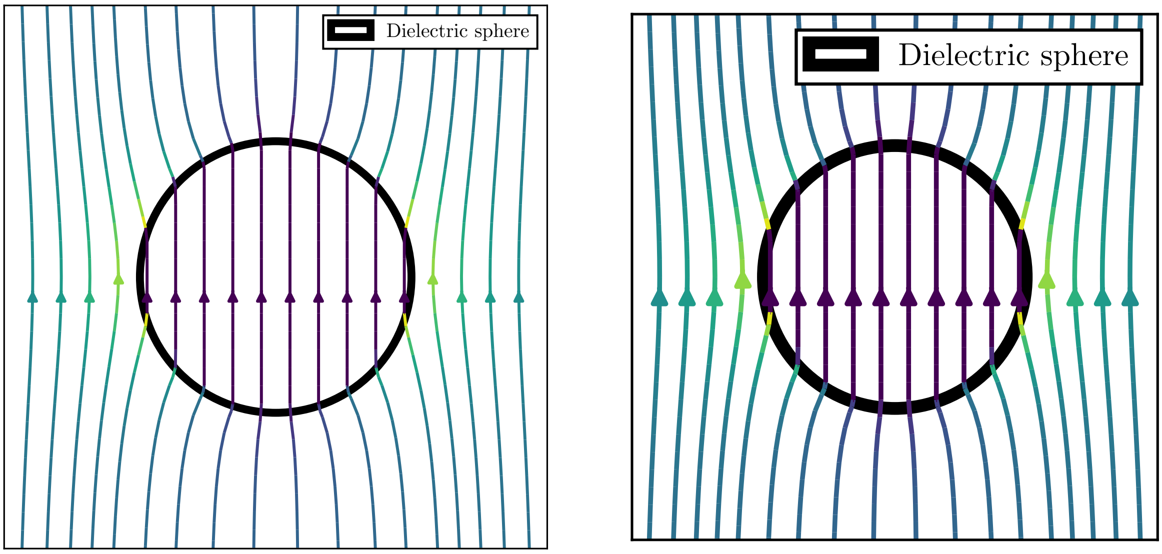

Figure 3.2: Three simple graphs

Bibliography

[1] D. J. Griffiths, ***Elektrodynamik, Eine Einführung***, edited by U. Schollwöck, 4th edition (Pearson, Hallbergmoos, 2018), 1711 pages.

[2] J. S. Pedernales and M. B. Plenio, "On the origin of force sensitivity in tests of quantum gravity with delocalised mechanical systems", Contemporary Physics **64**, 147–163 (2023) 10.1080/00107514.2023.2286074, arXiv:2311.04745.

[3] M. Plenio, "Logarithmic negativity: a full entanglement monotone that is not convex.", Physical Review Letters **95**, 090503 (2005) 10.1103/PhysRevLett.95.090503, arXiv:quant-ph/0505071.

[4] M. B. Plenio and S. Virmani, "An introduction to entanglement measures", Quantum Information & Computation **7**, 1–51 (2005), arXiv:quant-ph/0504163.

[5] M. Hartmann, "Casimir effect in the plane-spheregeometry: Beyond the proximityforce approximation", PhD thesis (Universität Augsburg, July 2018).

[6] A. Canaguier-Durand, R. Guérout, P. A. M. Neto, A. Lambrecht, and S. Reynaud, "The Casimir effect in the sphere-plane geometry", International Journal of Modern Physics Conference Series **14**, 250–259 (2012) 10.1142/s2010194512007374, arXiv:1202.3272.

[7] G. Vidal and R. F. Werner, "A computable measure of entanglement", Phys. Rev. A **65**, 032314 (2001) 10.1103/physreva.65.032314, arXiv:quant-ph/0102117.

[8] M. A. Nielsen and I. L. Chuang, ***Quantum computation and quantum information***, 10th anniversary ed. (Cambridge University Press, Cambridge, 2010), 1676 pages.

[9] G. A. E. Vandenbosch, "The basic concepts determining electromagnetic shielding", American Journal of Physics **90**, 672–681 (2022) 10.1119/5.0087295.

[10] J. S. Pedernales, G. W. Morley, and M. B. Plenio, "Motional Dynamical Decoupling for Matter-Wave Interferometry", Phy. Rev. Lett. **125**, 023602 (2019) 10.1103/physrevlett.125.023602, arXiv:1906.00835.

[11] D. Carney, P. C. E. Stamp, and J. M. Taylor, "Tabletop experiments for quantum gravity: a user's manual", Classical and Quantum Gravity **36**, 034001 (2018) 10.1088/1361-6382/aaf9ca, arXiv:1807.11494.

[12] L. H. Ford, "Casimir Force between a Dielectric Sphere and a Wall: A Model for Amplification of Vacuum Fluctuations", Phys. Rev. A **58**, 4279–4286 (1998) 10.1103/physreva.58.4279, arXiv:quant-ph/9804055.

[13] T. W. van de Kamp, R. J. Marshman, S. Bose, and A. Mazumdar, "Quantum Gravity Witness via Entanglement of Masses: Casimir Screening", Phys. Rev. A **102**, 062807 (2020) `10.1103/physreva.102.062807`, arXiv:`2006.06931`.

A Proofs and other stuff

A.1 Negativity

Lemma A.1. *The trace norm* $\|A\|_1 \equiv \operatorname{tr}\sqrt{A^\dagger A}$ *of a hermitian matrix* A *is equal to the sum of the absolute eigenvalues of* A.

Proof. This can be immediately seen by the spectral theorem:

$$\operatorname{tr}\sqrt{A^\dagger A} = \operatorname{tr}\sqrt{A^2} = \operatorname{tr}\left\{U\sqrt{\operatorname{diag}(\lambda_1,\dots)^2}U^\dagger\right\} = \sum_i \sqrt{\lambda_i^2} = \sum_i |\lambda_i| \,.$$

$\square$

Proposition A.1. *The* ***negativity*** $\mathcal{N}(\rho)$ *of a state* ρ *is given as the absolute sum of all negative eigenvalues of* ρ*:*

$$\mathcal{N}(\rho) \equiv \frac{\left\|\rho^{\Gamma_A}\right\|_1 - 1}{2} = \left|\sum_{\lambda_i<0} \lambda_i\right| . \tag{A.1}$$

Proof. The proof is in parts given by Vidal [7]. It is known that the density matrix is hermitian: $\rho = \rho^\dagger$. Using lemma A.1, the trace norm of the density matrix is is given as $\|\rho\|_1 = \sum \lambda_i = \operatorname{tr}\rho = 1$. The partial transpose ρ^{Γ_A} obviously also satisfies $\operatorname{tr}\rho^{\Gamma_A} = 1$ but might have negative eigenvalues. Since ρ^{Γ_A} is still hermitian, the trace norm is given by

$$\left\|\rho^{\Gamma_A}\right\|_1 = \sum_i |\lambda_i| = \sum_{\lambda_i\geq 0} \lambda_i + \sum_{\lambda_i<0} |\lambda_i| = \sum_i \lambda_i + 2\sum_{\lambda_i<0} |\lambda_i| = 1 + 2\sum_{\lambda_i<0} |\lambda_i| \,,$$

where in the last step $\sum \lambda_i = \operatorname{tr}\rho^{\Gamma_A} = 1$ was used. The negativity can be defined as $\mathcal{N}(\rho) = \left|\sum_{\lambda_i<0} \lambda_i\right|$ and the statement is shown. $\square$

Remark. The ***logarithmic negativity*** [3] relates to the negativity as follows

$$E_N(\rho) = \log_2 \left\|\rho^{\Gamma_A}\right\|_1 = \log_2 \left(2\mathcal{N}(\rho) + 1\right) \tag{A.2}$$

and can therefore be easily calculated by using the above proposition A.1. In comparison to the negativity, logarithmic negativity has additive properties [4]:

$$E_N\left(\rho \otimes \sigma\right) = E_N(\rho) + E_N(\sigma)$$

A.2 Fidelity

The ***fidelity*** of two quantum states ρ and σ is defined as [8, p. 409-412]

$$F(\rho,\sigma) = \operatorname{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \tag{A.3}$$

and can be used as a distance measurement between quantum states. It is monotonic, concave and bounded between 0 and 1. If both states are equal $\rho = \sigma$, it is clear that $F(\rho,\sigma) = 1$, by using $\sqrt{\rho}\rho\sqrt{\rho} = \rho^2$. If both states commute, i.e. they are diagonalizable in the same orthogonal basis $\{|i\rangle\}$,

$$\rho = \sum_i r_i \, |i\rangle\langle i| \, ; \quad \sigma = \sum_i s_i \, |i\rangle\langle i| \, ,$$

the fidelity is given as [8]

$$F(\rho,\sigma) = \operatorname{tr}\sqrt{\sum_i r_i s_i \, |i\rangle\langle i|} = \sum_i \sqrt{r_i s_i}.$$

This can be seen immediately by the use of the spectral theorem $\operatorname{tr}\sqrt{\rho} = \operatorname{tr}\Big\{U\sqrt{\operatorname{diag}(r_i)}U^\dagger\Big\} = \operatorname{tr}\operatorname{diag}(\sqrt{r_i})$. Another special case is given for the fidelity of a pure state $\rho = |\psi\rangle\langle\psi|$ and an arbitrary state σ [8]:

$$F(|\psi\rangle\,,\sigma) = \operatorname{tr}\sqrt{\langle\psi|\,\sigma\,|\psi\rangle\,|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\,\sigma\,|\psi\rangle}.$$

If the state $\sigma = |\phi\rangle\langle\phi|$ is also pure, the fidelity reduces to

$$F(|\psi\rangle\,,|\phi\rangle) = |\langle\psi|\phi\rangle| \leq 1, \tag{A.4}$$

with equality being attained if the states are the same and only differ by a phase.