## Classical deterministic complexity of Edmonds' problem and Quantum Entanglement

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#### Abstract

Generalizing a decision problem for bipartite perfect matching , J. Edmonds introduced in [15] the problem (now known as the Edmonds Problem) of deciding if a given linear subspace of M(N) contains a nonsingular matrix, where M(N) stands for the linear space of complex  $N \times N$  matrices. This problem led to many fundamental developments in matroid theory etc.

Classical matching theory can be defined in terms of matrices with nonnegative entries. The notion of Positive operator, central in Quantum Theory, is a natural generalization of matrices with nonnegative entries. (Here operator refers to maps from matrices to matrices.) First, we reformulate the Edmonds Problem in terms of of completely positive operators, or equivalently, in terms of bipartite density matrices . It turns out that one of the most important cases when Edmonds' problem can be solved in polynomial deterministic time, i.e. an intersection of two geometric matroids, corresponds to unentangled (aka separable) bipartite density matrices . We introduce a very general class (or promise) of linear subspaces of M(N) on which there exists a polynomial deterministic time algorithm to solve Edmonds' problem .

The algorithm is a thoroughgoing generalization of algorithms in [29], [38], and its analysis benefits from an operator analog of permanents, so called Quantum Permanents . Finally, we prove that the weak membership problem for the convex set of separable normalized bipartite density matrices is NP-HARD.

#### 1 Introduction and Main Definitions

Let M(N) be the linear space of  $N \times N$  complex matrices . The following fundamental problem has been posed by J. Edmonds in [15]:

**Problem 1.1:** Given a linear subspace  $V \subset M(N)$  to decide if there exists a nonsingular matrix  $A \in V$ .

We will assume throughout the paper that the subspace V is presented as a finite spanning k-tuple of rational matrices  $S(V) = \{A_1, ..., A_k\} (k \leq N^2)$ , i.e. the linear space generated by them is equal to V. As usual, the complexity parameter of the input  $\langle S(V) \rangle$  is equal to V is equal to V in the input V is equal to V.

Thus Edmonds' problem is equivalent to checking if the following determinantal polynomial

$$P_{\mathbf{A}}(x_1,...,x_k) = \det(\sum_{1 \le i \le k} x_i A_i)$$

is not identically equal to zero.

This determinantal polynomial can be efficiently evaluated, hence randomized poly-time algorithms, based on Schwartz's lemma or its recent improvements, are readily available (notice that our problem is defined over infinite field with infinite characteristic).

But for general linear subspaces of M(N), i.e. without an extra assumption (promise), poly-time deterministic algorithms are not known and the problem is believed to be "HARD".

Like any other homogeneous polynomial,  $P_{\mathbf{A}}(x_1,...,x_k)$  is a weighted sum of monomials of degree N, i.e.

$$P_{\mathbf{A}}(x_1, ..., x_k) = \sum_{(r_1, ..., r_k) \in I_{k,N}} a_{r_1, ..., r_k} x_1^{r_1} x_2^{r_2} ... x_k^{r_k}, \tag{1}$$

where  $I_{k,N}$  stands for a set of vectors  $r=(r_1,...,r_k)$  with nonnegative integer components and  $\sum_{1\leq i\leq k}r_i=N$ .

We will make substantial use of the following (Hilbert) norm of determinantal polynomial P(.)

:

$$||P||_G^2 =: \sum_{(r_1, \dots, r_k) \in I_{k,N}} |a_{r_1, \dots, r_k}|^2 r_1! r_2! \dots r_k!$$
(2)

It is easy to show that the determinantal polynomial  $P_{\mathbf{A}}(x_1,...,x_k) \equiv 0$  iff  $P_{\mathbf{A}}(r_1,...,r_k) = 0$  for all  $(r_1,...,r_k) \in I_{k,N}$ , which amounts to  $|I_{k,N}| = \frac{(N+k-1)!}{N!(k-1)!}$  computations of determinants .

We will show that  $||P||_G^2$  can be evaluated in  $O(2^N N!)$  computations of determinants .

More importantly,  $||P||_G^2$  serves as a natural tool to analyze our main algorithm.

The algorithm to solve Edmonds' problem, which we introduce and analyze later in the paper, is a rather thoroughgoing generalization of the recent algorithms [29], [38] for deciding the existence of perfect matchings. They are based on so-called Sinkhorn's iterative scaling. The algorithm in [38] is a greedy version of Sinkhorn's scaling and has been analyzed using KLD-divergence; the algorithm in [29] is a standard Sinkhorn's scaling and a "potential" used for its analysis is the permanent. Our analysis is a sort of combination of techniques from [29], [38] . Most importantly,  $\|P\|_G^2$  can be viewed as a generalization of the permanent.

The organization of this paper proceeds as follows. In Section 2 we will recall fundamental notions from Quantum Information Theory such as bipartite density matrix, positive and completely positive operator, separability and entanglement. After that we will rephrase Edmonds' problem using those notions and reformulate the famous Edmonds-Rado theorem on the rank of intersection of two geometric matroids in terms of the rank non-decreasing property of the corresponding (separable) completely positive operator. We will end Section 2 by introducing a property, called the Edmonds-Rado property, of linear subspaces of M(N) which allows a polytime deterministic algorithm to solve Edmonds' problem and will explain how is this property is related to quantum entanglement.

In Section 3 we will express G-norm of a determinantal polynomial  $P_{\mathbf{A}}(x_1,...,x_k)$  in terms of the associated bipartite density matrix, and we will prove various inequalities and properties of G-norm which will be needed later on for the analysis of the main algorithm.

In Section 4 we will introduce and analyze the main algorithm of the paper, Operator Sinkhorn Scaling.

In Section 5 we will apply this algorithm to solve Edmonds' problem for linear subspaces of M(N) having the Edmonds-Rado property. In Section 6 we will prove NP-HARDNESS of the weak membership problem for the compact convex set of separable normalized density matrices.

Finally, in the Conclusion section we will pose several open problems and directions for future research.

We would like to stress that our paper does not contain explicit connections to Quantum Computing. It rather aims to study quantum entanglement from the point of view of classical computational complexity and computational geometry and to use some ideas and structures from Quantum Information Theory to construct and analyse classical algorithms.

The main algorithm of this paper is a third "generation" of scalings applications to computer science problems, starting with ([29], [38]; applied to bipartite perfect matchings and an approximation of the permanent) and ([35], [36]; applied to an approximation of the mixed discriminant and mixed volume).

And here it is used to solve very non-trivial, important and seemingly different problem.

# 2 Bipartite density matrices, completely positive operators and Edmonds Problem

**Definition 2.1:** A positive semidefinite matrix  $\rho_{A,B}:C^N\otimes C^N\to C^N\otimes C^N$  is called a bipartite unnormalized density matrix

(**BUDM**). If  $tr(\rho_{A,B}) = 1$  then this  $\rho_{A,B}$  is called a bipartite density matrix.

It is convenient to represent a bipartite  $\rho_{A,B} = \rho(i_1,i_2,j_1,j_2)$  as the following block matrix:

$$\rho_{A,B} = \begin{pmatrix}
A_{1,1} & A_{1,2} & \dots & A_{1,N} \\
A_{2,1} & A_{2,2} & \dots & A_{2,N} \\
\dots & \dots & \dots & \dots \\
A_{N,1} & A_{N,2} & \dots & A_{N,N}
\end{pmatrix},$$
(3)

where  $A_{i_1,j_1} =: \{ \rho(i_1, i_2, j_1, j_2) : 1 \leq i_2, j_2 \leq N \}, 1 \leq i_1, j_1 \leq N.$ A (**BUDM**)  $\rho$  is called **separable** if

$$\rho = \rho_{(X,Y)} =: \sum_{1 \le i \le K} x_i x_i^{\dagger} \otimes y_i y_i^{\dagger}, \tag{4}$$

and **entangled** otherwise.

If the vectors  $x_i, y_i; 1 \le i \le K$  in (6) are real then  $\rho$  is is called **real separable**.

The quantum marginals are defined as  $\rho_A = \sum_{1 \leq i \leq N} A_{i,i}$  and  $\rho_B(i,j) = tr(A_{i,j}); 1 \leq i,j \leq N$ .

Next we define the (**BUDM**)  $\rho_{\mathbf{A}}$  associated with the k-tuple

 $\mathbf{A} = (A_1, ..., A_k):$ 

$$\rho_{\mathbf{A}}(i_1, i_2, j_1, j_2) =: \sum_{1 \le l \le k} A_l(i_1, i_2) \overline{A_l(j_1, j_2)}, \tag{5}$$

where for a complex number z = x + iy its conjugate  $\bar{z} = x - iy$ .

Rewriting expression (5) in terms of blocks of  $\rho_{\mathbf{A}}$  as in (3), we get that

$$A_{i,j} = \sum_{1 \le l \le k} A_l e_i e_j^{\dagger} A_l^{\dagger}, 1 \le i, j \le N.$$

(In quantum physics language, one can view a tuple  $\mathbf{A}=(A_1,...,A_k)$  of complex matrices as a tuple of unnormalized bipartite "wave functions"; and (**BUDM**)  $\rho_{\mathbf{A}}$  as a corresponding mixed bipartite state.)

We will call (**BUDM**)  $\rho$  weakly separable if there exists a separable  $\rho'_{(X,Y)}$  with the same image as  $\rho$ :  $Im(\rho) = Im(\rho'_{(X,Y)})$ .

(Recall that in this finite dimensional case  $Im(\rho)$  is the linear subspace formed by all linear combinations of columns of matrix  $\rho$ .)

A linear operator  $T: M(N) \to M(N)$  is called positive if  $T(X) \succeq 0$  for all  $X \succeq 0$ , and strictly positive if  $T(X) \succeq \alpha tr(X)I$  for all  $X \succeq 0$  and some  $\alpha > 0$ . A positive operator T is called completely positive if

$$T(X) = \sum_{1 \le i \le N^2} A_i X A_i^{\dagger}; A_i, X \in M(N)$$

$$\tag{6}$$

Choi's representation of the linear operator  $T: M(N) \to M(N)$  is a block matrix  $CH(T)_{i,j} =: T(e_i e_j^{\dagger})$ . The dual to T with respect to the inner product  $\langle X, Y \rangle = tr(XY^{\dagger})$  is denoted as  $T^*$ . A very useful and easy result of Choi states that T is completely positive iff CH(T) is (**BUDM**). Using this natural (linear) correspondence between completely positive operators and (**BUDM**), we will freely "transfer" properties of (**BUDM**) to completely positive operators. For example, a linear operator T is called separable iff CH(T) is separable, i.e.

$$T(Z) = T_{(X,Y)}(Z) = \sum_{1 \le i \le K} x_i y_i^{\dagger} Z y_i x_i^{\dagger}$$

$$\tag{7}$$

Notice that  $CH(T_{(X,Y)}) = \rho_{(\bar{Y},X)}$  and  $T^*_{(X,Y)} = T_{(Y,X)}$ .

(The components of the vector  $\bar{y}$  are the complex conjugates of corresponding components of y ).  $\blacksquare$ 

**Remark 2.2:** There is a natural (column by column ) correspondence between M(N) and  $C^{N^2} \cong C^N \otimes C^N$ . It works as follows

$$\{A(i,j), 1 \le i, j \le N\} \in M(N) \Leftrightarrow (A(1,1), ..., A(1,N); ......; A(1,N), ..., A(N,N))^T \in C^{N^2}$$

In light of definition (2.1), we will represent a linear subspace  $V \subset M(N) \cong C^N \otimes C^N$  in Edmonds Problem as the image of the (**BUDM**)  $\rho$ . And as the complexity measure we will use the number of bits of (rational) entries of  $\rho$  plus the dimension N.

**Definition 2.3:** A positive linear operator  $T:M(N)\to M(N)$  is called rank non-decreasing iff

$$Rank(T(X)) \ge Rank(X) \text{ if } X \succeq 0;$$
 (8)

and is called indecomposable iff

$$Rank(T(X)) > Rank(X) \text{ if } X \succeq 0 \text{ and } 1 \le Rank(X) < N.$$
 (9)

A positive linear operator  $T: M(N) \to M(N)$  is called doubly stochastic iff T(I) = I and  $T^*(I) = I$ ; called  $\epsilon$  - doubly stochastic iff  $DS(T) =: tr((T(I) - I)^2) + tr((T^*(I) - I)^2) \le \epsilon^2$ .

The next Proposition (2.4) is a slight generalization of the corresponding result in [29].

**Proposition 2.4:** Doubly stochastic operators are rank non-decreasing. If either T(I) = I or  $T^*(I) = I$  and  $DS(T) \leq N^{-1}$  then T is rank non-decreasing. If  $DS(T) \leq (2N+1)^{-1}$  then T is rank non-decreasing.

Let us consider a completely positive operator  $T_{\mathbf{A}}: M(N) \to M(N)$ ,

 $T(X) = \sum_{1 \leq i \leq k} A_i X A_i^{\dagger}$ , and let  $L(A_1, A_2, ..., A_K)$  be a linear subspace of M(N) generated by matrices  $\{A_i, 1 \leq i \leq k\}$ . It is easy to see that if  $\hat{A} \in L(A_1, A_2, ..., A_k)$  then  $\hat{A}(Im(X)) \subset Im(T(X))$  for all  $X \succeq 0$ .

Therefore, if  $L(A_1, A_2, ..., A_k)$  contains a nonsingular matrix then the operator T is rank non-decreasing.

This simple observation suggested the following property of linear subspaces of M(N):

#### Edmonds-Rado Property (ERP):

A linear subspace  $V = L(A_1, A_2, ..., A_k)$  has the **(ERP)** property if the existence of nonsingular matrix in V is equivalent to the fact that the associated completely positive operator  $T_{\mathbf{A}}$  is rank non-decreasing. In other words, a linear subspace  $V \subset M(N)$  has the **(ERP)** property if the fact that all matrices in V are singular is equivalent to the existence of two linear subspaces  $X, Y \subset C^N$  such dim(Y) < dim(X) and  $A(X) \subset Y$  for all matrices  $A \in V$ .

The main "constructive" result of this paper is that for linear subspaces of M(N) having the **ERP** there is a deterministic poly-time algorithm to solve Edmonds' problem.

In the rest of this section we will explain why we chose to call this property Edmonds-Rado, will describe a rather wide class of linear subspaces with **(ERP)** property and will give an example of a subspace without it.

#### 2.1 Examples of linear subspaces of M(N) having Edmonds-Rado Property

Let us first list some obvious but useful facts about the Edmonds-Rado property.

- 1. Suppose that  $V = L(A_1, A_2, ..., A_k) \subset M(N)$  has the **(ERP)** and  $C, D \in M(N)$  are two nonsingular matrices. Then linear subspace  $V_{C,D} =: L(CA_1D, CA_2D, ..., CA_kD)$  also has the **(ERP)**.
- 2. If  $V=L(A_1,A_2,...,A_k)\subset M(N)$  has the **(ERP)** then both  $V^\dagger=:L(A_1^\dagger,A_2^\dagger,...,A_k^\dagger)$  and  $V^T=L(A_1^T,A_2^T,...,A_k^T)$  have the **(ERP)** .
- 3. Any linear subspace  $V = L(A_1, A_2, ..., A_k) \subset M(N)$  with matrices  $\{A_i, 1 \leq i \leq k\}$  being positive semidefinite has the **(ERP)**.
- 4. Suppose that linear subspaces  $V = L(A_1, A_2, ..., A_k) \subset M(N_1)$  and  $W = L(B_1, B_2, ..., B_k) \subset M(N_2)$  both have the **(ERP)**. Define the following matrices  $C_i \in M(N_1 + N_2), 1 \leq i \leq k$ :

$$C_i = \left(\begin{array}{cc} A_i & D_i \\ 0 & B_i \end{array}\right)$$

Then the linear subspace  $L(C_1, C_2, ..., C_k) \subset M(N_1 + N_2)$  also has the **(ERP)**. A particular case of this fact is that any linear subspace of M(N) which has a basis consisting of upper diagonal matrices has the **(ERP)**.

5. Any 1-dimensional subspace of M(N) has the **(ERP)** property.

The next theorem gives the most interesting example which motivated the name "Edmonds-Rado Property". Let us first recall one of the most fundamental results in matroids theory, i.e. the Edmonds-Rado characterization of the rank of the intersection of two geometric matroids.

**Definition 2.5:** The intersection of two geometric matroids  $MI(X,Y) = \{(x_i,y_i), 1 \leq i \leq K\}$  is a finite family of distinct 2-tuples of non-zero N-dimensional complex vectors, i.e.  $x_i, y_i \in C^N$ . The rank of MI(X,Y), denoted by Rank(MI(X,Y)) is the largest integer m such that there exist  $1 \leq i_1 < ... < i_m \leq K$  with both sets  $\{x_{i_1},...,x_{i_m}\}$  and  $\{y_{i_1},...,y_{i_m}\}$  being linearly independent.  $\blacksquare$ 

The Edmonds-Rado theorem ([25]) states (in the much more general situation of the intersection of any two matroids with a common ground set) that

$$Rank(MI(X,Y)) = (10)$$

$$\min_{S \subset \{1,2,\dots,K\}} dim L(x_i; i \in S) + dim L(y_i; j \in \bar{S})$$

$$\tag{11}$$

It is easy to see that Rank(MI(X,Y)) is the maximum rank achieved in the linear subspace  $L(x_1y_1^{\dagger},...,x_Ky_K^{\dagger})$ ; and Rank(MI(X,Y)) = N iff  $L(x_1y_1^{\dagger},...,x_Ky_K^{\dagger})$  contains a nonsingular matrix.

**Theorem 2.6:** Suppose that  $T: M(N) \to M(N)$ ,  $T(X) = \sum_{1 \leq j \leq l} A_i X A_i^{\dagger}$ , is a completely positive weakly separable operator, i.e. there exists a family of rank one matrices  $\{x_1 y_1^{\dagger}, ..., x_l y_l^{\dagger}\} \subset M(N)$  such that  $L(A_1, ..., A_L) = L(x_1 y_1^{\dagger}, ..., x_l y_l^{\dagger})$ .

Then the following conditions are equivalent:

- **Fact 1** T is rank non-decreasing.
- **Fact 2** The rank of intersection of two geometric matroids MI(X,Y) is equal to N.
- **Fact 3** The exists a nonsingular matrix A such that  $Im(AXA^{\dagger}) \subset Im(T(X)), X \succeq 0$ .
- Fact 4 The exists a nonsingular matrix A such that the operator  $T'(X) = T(X) AXA^{\dagger}$  is completely positive.

**Proof:**  $[2 \Longrightarrow 1]$ 

Suppose that the rank of MI(X,Y) is equal to N. Then

$$RankT(X) = dim(L(x_i; i \in S)) \text{ where } S =: \{i : y_i^{\dagger} X y_i \neq 0\}$$

As  $dim(L(y_j; j \in \bar{S}) \leq dim(Ker(X)) = N - Rank(X)$  hence, from the Edmonds-Rado Theorem we get that

 $Rank(T(X)) \ge N - (N - Rank(X)) = Rank(X).$ 

 $[1 \Longrightarrow 2]$  Suppose that T is rank non-decreasing and for any  $S \subset \{1, 2, ..., l\}$  consider an orthogonal projector  $P \succeq 0$  on

 $L(y_j; j \in \bar{S})^{\perp}$ . Then

$$dim(L(x_i:i\in S))\geq \\ Rank(T(P))\geq Rank(P)=N-dim(L(y_i;j\in \bar{S})).$$

It follows from the Edmonds-Rado Theorem that the rank of MI(X,Y) is equal to N. All other "equivalences" follow now directly.

**Remark 2.7:** Theorem 2.6 makes the Edmonds-Rado theorem sound like Hall's theorem on bipartite perfect matchings.

Indeed, consider a weighted incidence matrix  $A_{\Gamma}$  of a bipartite graph  $\Gamma$ , i.e.  $A_{\Gamma}(i,j) > 0$  if i from the first part is adjacent to j from the second part and equal to zero otherwise. Then Hall's theorem can be immediately reformulated as follows:

A perfect matching, which is just a permutation in this bipartite case, exists iff

 $|A_{\Gamma}x|_{+} \geq |x|_{+}$  for any vector x with nonnegative entries, where  $|x|_{+}$  stands for the number of positive entries of a vector x.

All known algorithms (for instance, linear programming based on [25]) to compute the rank of the intersection of two geometric matroids require an explicit knowledge of pairs of vectors  $(x_i, y_i)$ , or, in other words, an explicit representation of the rank one basis  $\{x_i y_i^{\dagger}, 1 \leq i \leq l\}$ . The algorithm in this paper requires only a promise that such a rank one basis (not necessarily rational!) does exist.

Another example comes from [16]. Consider pairs of matrices  $(A_i, B_i \in M(N); 1 \le i \le K$ . Let  $V_i \subset M(N)$  be the linear subspace of all matrix solutions of the equation  $XA_i = B_iX$ . One of the problems solved in [16] is to decide if  $W = V_1 \cap ... \cap V_K$  contains a nonsingular matrix.

It is not clear to the author whether the class of such linear subspaces W satisfies the **(ERP)** property.

But suppose that  $A_1$  is similar to  $B_1$  ( $V_1$  contains a nonsingular matrix ) and, additionally, assume that  $dim(Ker(A_1 - \lambda I) = dim(Ker(B_1 - \lambda I) \le 1 \text{ for all complex } \lambda \in C.$ 

(I.e. just one Jordan block for each eigenvalue.)

It is not difficult to show that in this case there exist two nonsingular matrices D, Q and upper diagonal matrices  $(U_1, ..., U_r)$  such that  $V_1 = L(DU_1Q, ..., DU_rQ)$ . It follows, using Facts (1, 4) above, that  $V_1$  as well as any of its linear subspaces satisfy (**ERP**).

**Example 2.8:** Consider the following completely positive doubly stochastic operator  $Sk_3: M(3) \to M(3):$ 

$$Sk_3(X) = \frac{1}{2} (A_{(1,2)} X A_{(1,2)}^{\dagger} + A_{(1,3)} X A_{(1,3)}^{\dagger} + A_{(2,3)} X A_{(2,3)}^{\dagger})$$
(12)

Here  $\{A_{(i,j)}, 1 \leq i < j \leq 3\}$  is a standard basis in the linear subspace  $K(3) \subset M(3)$  consisting of all skew-symmetric matrices, i.e.  $A_{(i,j)} =: e_i e_j^{\dagger} - e_j e_i^{\dagger}$  and  $\{e_i, 1 \leq i \leq 3\}$  is a standard orthonormal basis in  $C^3$ .

It is clear that all  $3 \times 3$  skew-symmetric matrices are singular. As  $Sk_3$  is a completely positive doubly stochastic operator, and, thus, is rank non-decreasing, therefore  $K(3) \subset M(3)$  is an example of a linear subspace not having **(ERP)** property.

More "exotic" properties of this operator can be found in [7]. ■

# 3 Quantum permanents and G-norms of determinantal polynomials

Consider a k-tuple of  $N \times N$  complex matrices  $\mathbf{A} = (A_1, ..., A_k)$ . Our first goal here is to express the square of the G-norm of a determinantal polynomial  $P_{\mathbf{A}}(x_1, ..., x_k)$  in terms of the associated bipartite density matrix (**BUDM**)  $\rho_{\mathbf{A}}$ , which is defined as in (5).

Consider an N-tuple of complex  $N \times N$  matrices,  $\mathbf{B} = (B_1, ..., B_N)$ . Recall that the mixed discriminant  $M(\mathbf{B}) = M(B_1, ..., B_N)$  is defined as follows:

$$M(B_1, ...B_N) = \frac{\partial^n}{\partial x_1 ... \partial x_N} \det(x_1 B_1 + .... + x_N B_N).$$
(13)

Or equivalently:

$$M(B_1, ...B_N) = \sum_{\sigma, \tau \in S_N} (-1)^{sign(\sigma\tau)} \prod_{i=1}^N B_i(\sigma(i), \tau(i)),$$
(14)

where  $S_n$  is the symmetric group, i.e. the group of all permutations of the set  $\{1, 2, \dots, N\}$ . If matrices  $B_i, 1 \leq i \leq N$  are diagonal then their mixed discriminant is equal to the corresponding permanent ([35]).

**Definition 3.1:** Let us consider a block matrix  $\rho$  as in (3) (not necessarily positive semidefinite). We define the quantum permanent,  $QP(\rho)$ , by the following equivalent formulas:

$$QP(\rho) =: \sum_{\sigma \in S_N} (-1)^{sign(\sigma)} M(A_{1,\sigma(1)}, ..., A_{N,\sigma(N)}); \tag{15}$$

$$QP(\rho) = \frac{1}{N!} \sum_{\tau_1, \tau_2, \tau_3, \tau_4 \in S_N} (-1)^{sign(\tau_1 \tau_2 \tau_3 \tau_4)}$$

$$\prod_{i=1}^{N} rho(\tau_1(i), \tau_2(i), \tau_3(i), \tau_4(i)).$$
(16)

Straight from this definition, we get the following inner product formula for quantum permanents:

$$QP(\rho) = <\rho^{\otimes N} Z, Z>, \tag{17}$$

where  $\rho^{\otimes N}$  stands for a tensor product of N copies of  $\rho, < .,. >$  is a standard inner product and

$$Z(j_1^{(1)}, j_2^{(1)}; ...; j_1^{(N)}, j_2^{(N)}) = \frac{1}{N!^{\frac{1}{2}}} (-1)^{sign(\tau_1 \tau_2)}$$

if  $j_k^{(i)} = \tau_k(i) (1 \le i \le N); \tau_k \in S_N(k = 1, 2)$  and zero otherwise.

**Remark 3.2:** Notice that the equality (17) implies that if  $\rho_1 \succeq \rho_2 \succeq 0$  then  $QP(\rho_1) \geq QP(\rho_2) \geq 0$ .

The standard norm of  $N^{2N}$ -dimensional vector Z defined above is equal to 1. Thus, if  $\rho$  is a normalized bipartite density matrix then  $QP(\rho)$  can be viewed as the probability of a particular outcome of some (von Neumann) measurement. Unfortunately, in this case  $QP(\rho) \leq \frac{N!}{N^N}$ . Consider an arbitrary permutation  $\sigma \in S_4$  and for a block matrix (or tensor)  $\rho = \{\rho(i_1, i_2, i_3, i_4); 1 \leq i_1, i_2, i_3, i_4 \leq N\}$  define  $\rho^{\sigma} = \{\rho(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)})\}$ . It is easy to see that  $QP(\rho) = QP(\rho^{\sigma})$ . Another simple but important fact about quantum permanents is the following identity:

$$QP((A_1 \otimes A_2)\rho(A_3 \otimes A_4)) = \det(A_1 A_2 A_3 A_4)QP(\rho) \tag{18}$$

The author clearly (and sympathetically ) realizes that some readers might object to (or ridicule) the name "quantum permanent". The next example, hopefully, will explain possible motivations.  $\blacksquare$ 

**Example 3.3:** Let us present a few cases when Quantum Permanents can be computed "exactly ". They will also illustrate how universal this new notion is.

- 1. Let  $\rho_{A,B}$  be a product state, i.e.  $\rho_{A,B} = C \otimes D$ . Then  $QP(C \otimes D) = N!Det(C)Det(D)$ .
- 2. Let  $\rho_{A,B}$  be a pure state, i.e. there exists a matrix  $(R = R(i,j) : 1 \le i, j \le N)$  such that  $\rho_{A,B}(i_1,i_2,j_1,j_2) = R(i_1,i_2)\overline{R(j_1,j_2)}$ . In this case  $QP(\rho_{A,B}) = N!|Det(R)|^2$ .
- 3. Define blocks of  $\rho_{A,B}$  as  $A_{i,j} = R(i,j)e_ie_i^{\dagger}$ . Then  $QP(\rho_{A,B}) = Per(R)$ .

The following propositions provide important upper bounds for quantum permanents of positive semidefinite matrices.

**Proposition 3.4:** Suppose that  $\rho_{A,B}$  is a (BUDM). Then

$$\max_{\sigma \in S_N} |M(A_{1,\sigma(1)}, ..., A_{N,\sigma(N)})| = M(A_{1,1}, ..., A_{N,N})$$
(19)

**Proof:** For  $\tau, \sigma \in S_N$  define a matrix

$$B_{\tau,\sigma} =: A_{\tau(1),\sigma(1)} \otimes A_{\tau(2),\sigma(2)} \otimes \dots \otimes A_{\tau(N),\sigma(N)}$$

Since  $\rho_{A,B}$  is positive semidefinite hence the block matrix  $\{B_{\tau,\sigma}: \tau, \sigma \in S_N\}$  is also positive semidefinite. It is well known ([11]) and easy to prove that

$$M(A_1,...,A_N) = tr((A_1 \otimes ... \otimes A_N)VV^{\dagger})$$

for some universal  $N^N$ -dimensional vector V. It follows that the following  $N! \times N!$  matrix C

$$C_{\tau,\sigma} = tr(B_{\tau,\sigma}VV^{\dagger}) = M(A_{\tau(1),\sigma(1)}, A_{\tau(2),\sigma(2)}, ..., A_{\tau(N),\sigma(N)})$$

is also positive semidefinite. Thus

$$|C_{\tau,\sigma}| \le (C_{\tau,\tau}C_{\sigma,\sigma})^{\frac{1}{2}} = M(A_{1,1}, ..., A_{N,N})$$

Corollary 3.5: If  $\rho_{A,B}$  is (BUDM) then

$$QP(\rho_{A,B}) \le N!M(A_{1,1}, ..., A_{N,N}) \le N!Det(\rho_A).$$
 (20)

The permanental part of Example (3.3) shows that N! is the exact constant in both parts of (20), i.e. if blocks  $A_{i,j} = e_i e_j^{\dagger}, 1 \leq i, j \leq N$  then  $QP(\rho_{A,B}) = N!$  and  $M(A_{1,1}, ..., A_{N,N}) = Det(\rho_A) = 1$ .

The next proposition follows from Hadamard's inequality : if  $X \succ 0$  is  $N \times N$  matrix then  $Det(X) \leq \prod_{i=1}^{N} X(i,i)$ .

**Proposition 3.6:** If  $X \succ 0$  then the following inequality holds:

$$Det(\sum_{i=1}^{K} x_i y_i^{\dagger} X y_i x_i^{\dagger}) \ge Det(X) M P_{(X,Y)}.$$
(21)

Corollary 3.7: Suppose that a separable (BUDM)  $\rho_{A,B}$  is Choi's representation of the completely positive operator T.

Then for all  $X \succ 0$  the following inequality holds:

$$Det(T(X)) \ge QP(\rho_{A,B})Det(X)$$
 (22)

Since  $\rho_A = T(I)$ , hence  $QP(\rho_{A,B}) \leq Det(\rho_A)$  in the separable case. (Notice that Corollary 3.5 provides an example of an entangled (**BUDM**) which does not satisfy (22).)

Finally, in the next theorem we connect quantum permanents with G-norms of determinantal polynomials.

#### Theorem 3.8:

1. Consider an arbitrary polynomial

$$P(x_1, x_2, ..., x_k) = \sum_{(r_1, ..., r_k) \in F} a_{r_1, ..., r_k} x_1^{r_1} x_2^{r_2} ... x_k^{r_k}, |F| < \infty$$

where F is some finite set of vectors with nonnegative integer components and define its G-norm as follows

$$||P||_G^2 =: \sum_{(r_1,\dots,r_k)\in F} |a_{r_1,\dots,r_k}|^2 r_1! r_2! \dots r_k!$$

Then the following identity holds:

$$||P||_G^2 = E_{\xi_1,\dots,\xi_k}(|P(\xi_1,\dots,\xi_k)|^2), \tag{23}$$

where  $(\xi_1,...,\xi_k)$  are independent identically distributed zero mean gaussian complex random variables and the covariance matrix of  $\xi_1$ , viewed as a 2-dimensional real vector, is equal to  $\frac{1}{2}I$ .

2. Consider a k-tuple of  $N \times N$  complex matrices  $\mathbf{A} = (A_1, ..., A_k)$  and the corresponding determinantal polynomial  $P_{\mathbf{A}}(x_1, ..., x_k) =: \det(\sum_{1 \leq i \leq k} x_i A_i)$ . Then the following identity holds

$$||P_{\mathbf{A}}||_G^2 = QP(\rho_{\mathbf{A}}) \tag{24}$$

**Proof:** The proof is in Appendix 1.

**Remark 3.9:** Theorem 3.8, more precisely the combinations of its two parts, can be viewed as a generalization of the famous Wick formula [28]

It seems reasonable to predict that formula(24) might be of use in the combinatorics described in [28].

It is well known (see, for instance, [17]) that the mixed discriminant  $M(A_1,...,A_N)$  can be evaluated by computing  $2^N$  determinants. Therefore there the quantum permanent  $QP(\rho)$  can be evaluated by computing  $N!2^N$  determinants. Now, formula (24) suggests the following algorithm to compute

$$\|\det(\sum_{1\leq i\leq k} x_i A_i)\|_G^2:$$

first, construct the associated bipartite density matrix  $\rho_{\mathbf{A}}$ , which will take  $O(N^4k)$  additions and multiplications; secondly, compute  $QP(\rho_{\mathbf{A}})$ .

Total cost is  $Cost(N) = O(N!2^N N^3)$ . On the other hand, just the number of monomials in  $\det(\sum_{1 \le i \le k} x_i A_i)$  is equal to  $|I_{k,N}| = \frac{(N+k-1)!}{N!(k-1)!}$ . If  $k-1 = aN^2$  then  $|I_{k,N}| \ge \frac{a^N N^{2N}}{N!}$ . Thus,

$$\frac{|I_{k,N}|}{Cost(N)} \ge \frac{a^N N^{2N}}{O(N!^2 2^N N^3)} \ge \approx \frac{(ae^2)^N}{2^N N^3}$$

We conclude that if  $a > \frac{2}{e^2}$  our approach is exponentially faster than the "naive" one, i.e. than evaluating  $\det(\sum_{1 \le i \le k} x_i A_i)$  at all vectors  $(x_1, ..., x_k) \in I_{k,N}$ .

Our approach provides an  $O(N!2^NN^3)$  - step deterministic algorithm to solve a general case of Edmonds' Problem .

# 4 Operator Sinkhorn's iterative scaling

Recall that for a square matrix  $A = \{a_{ij} : 1 \leq i, j \leq N\}$  row scaling is defined as

$$R(A) = \{ \frac{a_{ij}}{\sum_{j} a_{ij}} \},\,$$

column scaling as  $C(A) = \{\frac{a_{ij}}{\sum_i a_{ij}}\}$  assuming that all denominators are nonzero.

The iterative process ...CRCR(A) is called Sinkhorn's iterative scaling (SI). There are two main, well known, properties of this iterative process, which we will generalize to positive operators.

#### Proposition 4.1:

- 1. Suppose that  $A = \{a_{i,j} \geq 0 : 1 \leq i, j \leq N\}$ . Then (SI) converges iff A is matching, i.e., there exists a permutation  $\pi$  such that  $a_{i,\pi(i)} > 0 \ (1 \leq i \leq N)$ .
- 2. If A is indecomposable, i.e., A has a doubly-stochastic pattern and is fully indecomposable in the usual sense, then (SI) converges exponentially fast. Also in this case there exist unique positive diagonal matrices  $D_1, D_2, \det(D_2) = 1$  such that the matrix  $D_1^{-1}AD_2^{-1}$  is doubly stochastic.

**Definition 4.2:** [Operator scaling ] Consider a positive linear operator  $T: M(N) \to M(N)$ . Define a new positive operator, Operator scaling,  $S_{C_1,C_2}(T)$  as :

$$S_{C_1,C_2}(T)(X) =: C_1 T(C_2^{\dagger} X C_2) C_1^{\dagger}$$
 (25)

Assuming that both T(I) and  $T^*(I)$  are nonsingular we define analogs of row and column scalings:

$$R(T) = S_{T(I)^{-\frac{1}{2}},I}(T), C(T) = S_{I,T^*(I)^{-\frac{1}{2}}}(T) \tag{26}$$

Operator Sinkhorn's iterative scaling (OSI) is the iterative process ...CRCR(T)

**Remark 4.3:** Using Choi's representation of the operator T as in Definition(2.1), we can define analogs of operator scaling (which are exactly so called local transformations in Quantum

Information Theory) and (OSI) in terms of (BUDM):

$$S_{C_1,C_2}(\rho_{A,B}) = C_1 \otimes C_2(\rho_{A,B}) C_1^{\dagger} \otimes C_2^{\dagger};$$

$$R(\rho_{A,B}) = \rho_A^{-\frac{1}{2}} \otimes I(\rho_{A,B}) \rho_A^{-\frac{1}{2}} \otimes I,$$

$$C(\rho_{A,B}) = I \otimes \rho_B^{-\frac{1}{2}}(\rho_{A,B}) I \otimes \rho_B^{-\frac{1}{2}}.$$

$$(27)$$

The standard ("classical") Sinkhorn's iterative scaling is a particular case of Operator Sinkhorn's iterative scaling (OSI) when the initial Choi's representation of the operator T is a diagonal (BUDM).

Let us introduce a class of locally scalable functionals (**LSF** ) defined on a set of positive linear operators, i.e. functionals satisfying the following identity :

$$\varphi(S_{C_1,C_2}(T)) = Det(C_1C_1^{\dagger})Det(C_2C_2^{\dagger})\varphi(T)$$
(28)

We will call an (**LSF**) bounded if there exists a function f such that  $|\varphi(T)| \leq f(tr(T(I)))$ . It is clear that bounded (**LSF**) are natural "potentials" for analyzing (OSI). Indeed, Let  $T_n, T_0 = T$  be a trajectory of (OSI). T is a positive linear operator. Then  $T_i(I) = I$  for odd i and  $T_{2i}(I)^* = I, i \geq 1$ . Thus if  $\varphi(.)$  is (**LSF**) then

$$\varphi(T_{i+1}) = a(i)\varphi(T_i), a(i) = Det(T_i^*(I))^{-1} \text{ if } i \text{ is odd },$$

$$a(i) = Det(T_i(I))^{-1} \text{ if } i > 0 \text{ is even.}$$
(29)

As  $tr(T_i(I)) = tr(T_i^*(I)) = N, i > 0$ , thus by the arithmetic/geometric means inequality we have that  $|\varphi(T_{i+1})| \ge |\varphi(T_i)|$  and if  $\varphi(.)$  is bounded and  $|\varphi(T)| \ne 0$  then  $DS(T_n)$  converges to zero.

To prove a generalization of Statement 1 in Prop.(4.1) we need to "invent" a bounded (**LSF**)  $\varphi(.)$  such that  $\varphi(T) \neq 0$  iff the operator T is rank non-decreasing. We call such functionals "responsible for matching". It follows from (10) and (20) that QP(CH(T)) is a bounded (**LSF**). Thus if  $QP(CH(T)) \neq 0$  then  $DS(T_n)$  converges to zero and, by Prop. (2.4), T is rank non-decreasing. On the other hand,  $QP(CH(Sk_3)) = 0$  and  $Sk_3$  is rank non-decreasing (even indecomposable). This is another "strangeness" of entangled operators. We wonder if it is possible to have a "nice", say polynomial with integer coefficients, responsible for matching (**LSF**)? We introduce below a responsible for matching bounded (**LSF**) which is continuous but non-differentiable.

**Definition 4.4:** For a positive operator  $T: M(N) \to M(N)$ , we define its capacity as  $Cap(T) = \inf\{Det(T(X)) : X \succ 0, Det(X) = 1\}$ .

It is easy to see that Cap(T) is  $(\mathbf{LSF})$ . Since  $Cap(T) \leq Det(T(I)) \leq (\frac{tr(T(I))}{N})^N$ , hence Cap(T) is a bounded  $(\mathbf{LSF})$ . **Lemma 4.5:** A positive operator  $T: M(N) \to M(N)$  is rank non-decreasing iff Cap(T) > 0.

**Proof:** Let us fix an orthonormal basis (unitary matrix)  $U = \{u_1, ..., u_N\}$  in  $C^N$  and associate with a positive operator T the following positive operator:

$$T_U(X) =: \sum_{1 \le i \le N} T(u_i u_i^{\dagger}) tr(X u_i u_i^{\dagger}). \tag{30}$$

(In physics terms,  $T_U$  represents decoherence with respect to the basis U, i.e. in this basis applying  $T_U$  to matrix X is the same as applying T to the diagonal restriction of X.)

It is easy to see that a positive operator T is rank non-decreasing iff the operators  $T_U$  are rank non-decreasing for all unitary U.

And for fixed U all properties of  $T_U$  are defined by the following N-tuple of  $N \times N$  positive semidefinite matrices:

$$\mathbf{A}_{T,U} =: (T(u_1 u_1^{\dagger}), ..., T(u_N u_N^{\dagger}). \tag{31}$$

Importantly for us,  $T_U$  is rank non-decreasing iff the mixed discriminant  $M(T(u_1u_1^{\dagger}), ..., T(u_Nu_N^{\dagger})) > 0$ .

Define the capacity of  $\mathbf{A}_{T,U}$ ,

$$Cap(\mathbf{A}_{T,U}) =: \inf\{Det(\sum_{1 \leq i \leq N} T(u_i u_i^{\dagger}) \gamma_i) : \gamma_i > 0, \prod_{1 \leq i \leq N} \gamma_i = 1\}.$$

It is clear from the definitions that Cap(T) is equal to infimum of  $Cap(\mathbf{A}_{T,U})$  over all unitary U.

One of the main results of [35] states that

$$M(\mathbf{A}_{T,U}) =: M(T(u_1 u_1^{\dagger}), ..., T(u_N u_N^{\dagger})) \le Cap(\mathbf{A}_{T,U}) \le \frac{N^N}{N!} M(T(u_1 u_1^{\dagger}), ..., T(u_N u_N^{\dagger})).$$
(32)

As the mixed discriminant is a continuous (analytic) functional and the group SU(N) of unitary matrices is compact, we get the next inequality:

$$\min_{U \in SU(N)} M(\mathbf{A}_{T,U}) \le Cap(T) \le \frac{N^N}{N!} \min_{U \in SU(N)} M(\mathbf{A}_{T,U})$$
(33)

The last inequality proves that Cap(T) > 0 iff positive operator T is rank non-decreasing.

So, the capacity is a bounded ( $\mathbf{LSF}$  ) responsible for matching, which proves the next theorem :

#### Theorem 4.6:

- 1. Let  $T_n, T_0 = T$  be a trajectory of (OSI), where T is a positive linear operator. Then  $DS(T_n)$  converges to zero iff T is rank non-decreasing.
- 2. A positive linear operator  $T: M(N) \to M(N)$  is rank non-decreasing iff for all  $\epsilon > 0$  there exists an  $\epsilon$ -doubly stochastic operator scaling of T.

3. A positive linear operator T is rank non-decreasing iff there exists  $\frac{1}{N}$ -doubly stochastic operator scaling of T.

The next theorem generalizes second part of Prop. (4.1) and is proved on almost the same lines as Lemmas 24,25,26,27 in [35].

#### Theorem 4.7:

1. There exist nonsingular matrices  $C_1, C_2$  such that  $S_{C_1,C_2}(T)$  is doubly stochastic iff the infimum in Definition 4.4 is attained.

Moreover, if 
$$Cap(T) = Det(T(C))$$
 where  $C \succ 0$ ,  $Det(C) = 1$  then  $S_{T(C)^{\frac{-1}{2}}, C^{\frac{1}{2}}}(T)$  is doubly stochastic.

Positive operator T is indecomposable iff the infimum in Definition 4.4 is attained and unique.

- 2. A doubly stochastic operator T is indecomposable iff  $tr(T(X))^2 \le a \ tr(X)^2$  for some  $0 \le a < 1$  and all traceless hermitian matrices X.
- 3. If a positive operator T is indecomposable then  $DS(T_n)$  converges to zero with the exponential rate, i.e.  $DS(T_n) \leq Ka^n$  for some K and  $0 \leq a < 1$ .

**Remark 4.8:** Consider an  $N \times N$  matrix A with nonnegative entries. Similarly to (30), define its capacity as follows:

$$Cap(A) = \inf \{ \prod_{1 \le i \le N} (Ax)_i : x_i > 0, 1 \le i \le N; \prod_{1 \le i \le N} x_i = 1 \}$$

Recall that the KLD-divergence between two matrices is defined as

$$KLD(A||B) = \sum_{1 \le i,j \le N} B(i,j) \log(\frac{B(i,j)}{A(i,j)})$$

It is easy to prove (see, for instance, [38]) that

$$-\log(Cap(A)) = \inf\{KLD(A||B) : B \in D_N\},\$$

where  $D_N$  is the convex compact set of  $N \times N$  doubly stochastic matrices.

Of course, there is a quantum analog of KLD-divergence, the so called von Neumann divergence. It is not clear whether there exists a similar "quantum" characterization of the capacity of completely positive operators.

The inequality (20) can be strengthen to the following one:

$$QP(CH(T)) \leq N!Cap(T)$$

And N! is also an exact constant in this inequality above.

# 5 Polynomial time deterministic algorithm for the Edmonds Problem

Let us consider the following three properties of (**BUDM**)  $\rho_{A,B}$ . (We will view this  $\rho_{A,B}$  as Choi's representation of a completely positive operator T, i.e.  $\rho_{A,B} = CH(T)$ .)

**P1**  $Im(\rho_{A,B})$  contains a nonsingular matrix.

**P2** The Quantum permanent  $QP(\rho_{A,B}) > 0$ .

**P3** Operator T is rank non-decreasing.

Part 2 of theorem (3.8) proves that  $P1 \Leftrightarrow P2$  and Example(2.8) illustrated that that the implication  $P2 \Rightarrow P3$  is strict. It is not clear whether either P1 or P3 can be checked in deterministic polynomial time.

Next, we will describe and analyze Polynomial time deterministic algorithm to check whether P3 holds provided that it is promised that  $Im(\rho_{A,B})$ , viewed as a linear subspace of M(N), has the **Edmonds-Rado Property**. Or, in other words, that it is promised that  $P1 \Leftrightarrow P3$ . In terms of Operator Sinkhorn's iterative scaling (OSI) we need to check if there exists n such that  $DS(T_n) \leq \frac{1}{N}$ . If  $L =: \min\{n : DS(T_n) \leq \frac{1}{N}\}$  is bounded by a polynomial in N and number of bits of  $\rho_{A,B}$  then we have a Polynomial time Deterministic algorithm to solve Edmonds' problem provided that it is promised that  $P1 \Leftrightarrow P3$ . Algorithms of this kind for "classical" matching problem appeared independently in [29] and [38]. In the "classical" case they are just another, conceptually simple, but far from optimal, poly-time algorithms to check whether a perfect matching exists. But in this general Edmonds Problem setting, our, Operator Sinkhorn's iterative scaling based approach seems perhaps to be the only possibility.

Assume, without loss of generality, that all entries of  $\rho_{A,B}$  are integer numbers and their maximum magnitude is  $\mathbf{M}$ . Then  $Det(\rho_A) \leq (\mathbf{M}\mathbf{N})^{\mathbf{N}}$  by Hadamard's inequality. If  $QP(\rho_{A,B}) > 0$  then necessary  $QP(\rho_{A,B}) \geq 1$  for it is an integer number. Thus

$$QP(CH(T_1)) = \frac{QP(CH(T))}{Det(\rho_A)} \ge (\mathbf{MN})^{-\mathbf{N}}.$$

Each nth iteration  $(n \leq L)$  after the first one will multiply the Quantum permanent by  $Det(X)^{-1}$ , where  $X \succ 0, tr(X) = N$  and  $tr((X - I)^2) > \frac{1}{N}$ . Using results from [29],  $Det(X)^{-1} \geq (1 - \frac{1}{3N})^{-1} =: \delta$ . Putting all this together, we get the following upper bound on L, the number of steps in (OSI) to reach the "boundary"  $DS(T_n) \leq \frac{1}{N}$ :

$$\delta^{L} \le \frac{QP(CH(T_{L}))}{(\mathbf{M}\mathbf{N})^{-\mathbf{N}}} \tag{34}$$

It follows from (20) that  $QP(CH(T_L)) \leq N!$ 

Taking logarithms we get that

$$L \le 3N(N\ln(N) + N(\ln(N) + \ln(\mathbf{M})); \tag{35}$$

Thus L is polynomial in the dimension N and the number of bits  $\log(\mathbf{M})$ .

To finish our analysis, we need to evaluate the complexity of each step of (OSI).

Recall that  $T_n(X) = L_n(T(R_n^{\dagger}XR_n))L_n^{\dagger}$ 

for some nonsingular matrices  $L_n$  and  $R_n$ ,

 $T_n(I) = L_n(T(R_n^{\dagger}R_n))L_n^{\dagger}$  and  $T_n^*(I) = R_n(T^*(L_n^{\dagger}L_n))R_n^{\dagger}$ .

To evaluate  $DS(T_n)$  we need to compute  $tr((T_n^*(I) - I)^2)$  for odd

n and  $tr((T_n(I)-I)^2)$  for even n.

Define  $P_n = L_n^{\dagger} L_n$ ,  $Q_n = R_n^{\dagger} R_n$ . It is easy to see that the matrix  $T_n(I)$  is similar to  $P_n T(Q_n)$ , and  $T_n^*(I)$  is similar to  $Q_n T^*(P_n)$ .

As traces of similar matrices are equal, to evaluate  $DS(T_n)$  it is sufficient to compute matrices  $P_n, Q_n$ .

But,  $P_{n+1} = (T(Q_n))^{-1}$  and  $Q_{n+1} = (T^*(P_n))^{-1}$ .

And this leads to standard rational matrix operations with  $O(N^3)$  per iteration in (OSI).

Notice that our original definition of (OSI) requires computation of an operator square root. It can be replaced by the Cholesky factorization, which still requires computing scalar square roots. But our final algorithm is rational!

Remark 5.1: To ensure that all the matrices we need to invert along the algorithm (OSI) are nonsingular indeed it is sufficient that both  $T(I) \succ 0$  and  $T^*(I) \succ 0$  (strictly positive definite). It is easy to see that if positive operator T is rank non-decreasing then its dual  $T^*$  is also rank non-decreasing. Thus if positive operator T is rank non-decreasing then necessarily  $T(X) \succ 0$  and  $T^*(X) \succ 0$  for all  $X \succ 0$ .

# 6 Weak Membership Problem for the convex compact set of normalized bipartite separable density matrices is NP-HARD

One of the main research activities in Quantum Information Theory is a search for "operational" criterion for the separability. We will show in this section that, in a sense defined below, the problem is NP-HARD even for bipartite normalized density matrices provided that each part is large (each "particle" has large number of levels). First, we need to recall some basic notions from computational convex geometry.

#### 6.1 Algorithmic aspects of convex sets

We will follow [25].

**Definition 6.1:** A proper (i.e. with nonempty interior) convex set  $K \subset \mathbb{R}^n$  is called well-bounded a-centered if there exist a rational vector  $a \in K$  and positive (rational) numbers r, R such that  $B(a,r) \subset K$  and  $K \subset B(a,R)$  (here  $B(a,r) = \{x : ||x-a|| \le r\}$  and ||.|| is a standard euclidean norm in  $\mathbb{R}^n$ ). The encoding length of such a convex set K is

$$< K > = n + < r > + < R > + < a >,$$

where  $\langle r \rangle, \langle R \rangle, \langle a \rangle$  are the number of bits of corresponding rational numbers and rational vector.

Following [25] we define  $S(K, \delta)$  as a union of all  $\delta$ -balls with centers belonging to K; and  $S(K, -\delta) = \{x \in K : B(x, \delta) \subset K\}.$ 

**Definition 6.2:** The Weak Membership Problem  $(WMEM(K, y, \delta))$  is defined as follows: Given a rational vector  $y \in \mathbb{R}^n$  and a rational number  $\delta > 0$  either

- (i) assert that  $y \in S(K, \delta)$ , or
- (ii) assert that  $y \notin S(K, -\delta)$ .

The Weak Validity Problem  $(WVAL(K,c,\gamma,\delta))$  is defined as follows :

Given a rational vector  $c \in \mathbb{R}^n$ , rational number  $\gamma$  and a rational number  $\delta > 0$  either

- (i) assert that  $\langle c, x \rangle =: c^T x \leq \gamma + \delta$  for all  $x \in S(K, -\delta)$ , or
- (ii) assert that  $c^T x \ge \gamma \delta$  for some  $x \in S(K, \delta)$ .

**Remark 6.3:** Define  $M(K,c) =: \max_{x \in K} \langle c, x \rangle$ . It is easy to see that

$$M(K,c) \ge M(S(K,-\delta),c) \ge M(K,c) - \|c\| \delta \frac{R}{r};$$
  
$$M(K,c) \le M(S(K,\delta),c) \le M(K,c) + \|c\| \delta$$

Recall that the seminal Yudin-Nemirovski theorem ([14], [25]) implies that if there exists a deterministic algorithm solving  $WMEM(K,y,\delta)$  in  $Poly(< K > + < y > + < \delta >)$  steps then there exists a deterministic algorithm solving  $WVAL(K,c,\gamma,\delta)$  in  $Poly(< K > + < c > + < \delta > + < c > + < \delta >)$  steps.

Let us denote as SEP(M, N) a compact convex set of separable density matrices  $\rho_{A,B}: C^M \otimes C^N \to C^M \otimes C^N$ ,  $tr(\rho_{A,B}) = 1, M \geq N$ . Recall that

$$SEP(M,N) = \\ CO(\{xx^{\dagger} \otimes yy^{\dagger}: x \in C^M, y \in C^N; \|x\| = \|y\| = 1\}),$$

where CO(X) stands for the convex hull generated by a set X.

Our goal is to prove that the Weak Membership Problem for SEP(M,N) is NP-HARD. As we are going to use the Yudin–Nemirovski theorem, it is sufficient to prove that  $WVAL(SEP(M,N),c,\gamma,\delta)$  is NP-HARD with respect to the complexity measure  $(M+\langle c \rangle+\langle \delta \rangle+\langle \gamma \rangle)$  and to show that  $\langle SEP(M,N) \rangle$  is polynomial in M.

## **6.2** Geometry of SEP(M, N)

First, SEP(M,N) can be viewed as a compact convex subset of the hyperplane in  $R^D, D =: N^2M^2$ . The standard euclidean norm in  $R^{N^2M^2}$  corresponds to the Frobenius norm for density matrices, i.e.  $\|\rho\|_F = tr(\rho\rho^\dagger)$ . The matrix  $\frac{1}{NM}I \in SEP(M,N)$  and  $\|\frac{1}{NM}I - xx^\dagger \otimes I\|_{L^2(M,N)}$ 

 $yy^{\dagger}|_F = \sqrt{\frac{D-1}{D}} < 1$  for all norm one vectors x, y. Thus SEP(M, N) is covered by the ball  $B(\frac{1}{NM}I, \sqrt{\frac{D-1}{D}})$ .

The following result was recently proved in [40].

**Theorem 6.4:** Let  $\Delta$  be a block hermitian matrix as in (5). If  $tr(\Delta) = 0$  and  $\|\Delta\|_F \leq \sqrt{\frac{1}{D(D-1)}}$  then the block matrix  $\frac{1}{D}I + \Delta$  is separable.

Summarizing, we get that for D = MN

$$B(\frac{1}{D}I,\sqrt{\frac{1}{D(D-1)}})\subset SEP(M,N)\subset B(\frac{1}{D}I,\sqrt{\frac{D-1}{D}}),$$

(balls are restricted to the corresponding hyperplane ) and conclude that  $\langle SEP(M,N) \rangle \leq Poly(MN)$ . It is left to prove that  $WVAL(SEP(M,N),c,\gamma,\delta)$  is NP-HARD with respect to the complexity measure  $(MN+\langle c \rangle + \langle \delta \rangle + \langle \gamma \rangle)$ .

#### 6.3 Proof of Hardness

Let us consider the following hermitian block matrix:

$$C = \begin{pmatrix} 0 & A_1 & \dots & A_{M-1} \\ A_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{M-1} & 0 & \dots & 0 \end{pmatrix}, \tag{36}$$

i.e. its (i, j) blocks are zero if either  $i \neq 1$  or  $j \neq 1$  and (1, 1) block is also zero;  $A_1, ..., A_{M-1}$  are real symmetric  $N \times N$  matrices.

#### Proposition 6.5:

$$\max_{\rho \in SEP(M,N)} (tr(C\rho))^2 = \max_{y \in R^N, ||y|| = 1} \sum_{1 \le i \le M-1} (y^T A_i y)^2.$$

**Proof:** First, by linearity and the fact that the set of extreme points

$$Ext(SEP(M,N)) =$$
  
$$\{xx^{\dagger} \otimes yy^{\dagger} : x \in C^M, y \in C^N; ||x|| = ||y|| = 1\}$$

we get that

$$\max_{\rho \in SEP(M,N)} tr(C\rho) =$$
$$\max_{xx^{\dagger} \otimes yy^{\dagger}: x \in C^{M}, y \in C^{N}; ||x|| = ||y|| = 1} tr(C(xx^{\dagger} \otimes yy^{\dagger})).$$

But  $tr(C(yy^{\dagger} \otimes xx^{\dagger})) = tr(A(y)xx^{\dagger})$ , where real symmetric  $M \times M$  matrix A(y) is defined as follows:

$$A(y) = \begin{pmatrix} 0 & a_1 & \dots & a_{M-1} \\ a_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{M-1} & 0 & \dots & 0 \end{pmatrix};$$

 $a_i = tr(A_i y y^{\dagger}), 1 \le i \le M - 1.$ 

$$\max_{\rho \in SEP(M,N)} tr(C\rho) =$$

$$\max_{yy^{\dagger} \otimes xx^{\dagger}: x \in C^{M}, y \in C^{N}; ||x|| = ||y|| = 1} tr(C(xx^{\dagger} \otimes yy^{\dagger})) =$$

$$\max_{||y|| = 1} \lambda_{max} A(y).$$

(Above  $\lambda_{max}A(y)$  is the maximum eigenvalue of A(y)) It is easy to see A(y) has only two real non-zero eigenvalues (d,-d), where  $d=\sum_{1\leq i\leq M-1}(tr(A_iyy^{\dagger}))^2$ . As  $A_i,1\leq i\leq N-1$  are real symmetric matrices we finally get that

$$\max_{\rho \in SEP(M,N)} (tr(C\rho))^2 = \max_{y \in R^N, ||x|| = 1} \sum_{1 \le i \le N-1} (y^T A_i y)^2.$$

Proposition (6.5) and Remark (6.3) suggest that in order to prove NP-HARDness of  $WVAL(SEP(M,N),c,\gamma,\delta)$  with respect to the complexity measure  $M+\langle c\rangle+\langle \delta\rangle+\langle c\rangle$  $\gamma$  > it is sufficient to prove that the following problem is NP-HARD :

**Definition 6.6:** (RSDF problem) Given  $k \mid l \times l$  real rational symmetric matrices  $(A_i, 1 \le i \le l)$ and rational numbers  $(\gamma, \delta)$  to check whether

$$\gamma + \delta \ge \max_{x \in R^l, \|x\| = 1} f(x) \ge \gamma - \delta, f(x) =: \sum_{1 \le i \le l} (x^T A_i x)^2.$$

respect to the complexity measure

$$(lk + \sum_{1 \le i \le l} \langle A_i \rangle + \langle \delta \rangle + \langle \gamma \rangle).$$

It was shown in [13], by a reduction from KNAPSACK, that the RSDF problem is NP-HARD  $\begin{array}{l} \text{provided} \\ k \geq \frac{l(l-1)}{2} + 1. \end{array}$ 

We summarize all this in the following theorem

**Theorem 6.7:** The Weak Membership Problem for SEP(M,N) is NP-HARD if  $N \leq M \leq N$  $\frac{N(N-1)}{2} + 2.$ 

**Remark 6.8:** It is easy exercise to prove that (**BUDM**)  $\rho_{A,B}$  written in block form (3) is real separable iff it is separable and all the blocks in (3) are real symmetric matrices. It follows that, with obvious modifications, Theorem 6.7 is valid for real separability too.

The construction (37) was inspired by Arkadi Nemirovski's proof of the NP-HARDness of checking the positivity of a given operator [6].

## 7 Concluding Remarks

Many ideas of this paper were suggested by [35]. The world of mathematical interconnections is very unpredictable (and thus is so exciting). The main technical result in a very recent breakthrough in Communication Complexity [39] is a rediscovery of particular, rank one, case of a general, matrix tuples scaling, result proved in [35] with much simpler proof than in [39]. Perhaps this paper will produce something new in Quantum Communication Complexity.

We still don't know whether there is a deterministic poly-time algorithm to check whether a given completely positive operator is rank non-decreasing. And this question is related to lower bounds on Cap(T) provided that Choi's representation CH(T) is an integer semidefinite matrix.

Theorem(6.7) together with other results from our paper gives a new, classical complexity based, insight on the nature of quantum entanglement and, in a sense, closes a long line of research in Quantum Information Theory.

Also, this paper suggests a new way to look at "the worst entangled" bipartite density matrices (or completely positive operators). For instance, the operator  $Sk_3$  from Example (2.8) seems to be "the worst entangled" and it is not surprising that it appears in many counterexamples. We hope that the constructions introduced in this paper, especially the Quantum Permanent, will have a promising future.

We think, that in general, mixed discriminants and mixed volumes [1] should be studied and used more enthusiastically in the Quantum context. After all, they are noncommutative generalizations of the permanent....

The G-norm defined in (2) appears in this paper mainly because of formula (24). It is called by some authors ([2]) Bombieri's norm (see also, [3], [4], [5]).

Also, the G-norm arises naturally in quantum optics and the study of quantum harmonic oscillators.

This norm satisfies some remarkable properties ([3], [4]) which, we think, can be used in quantum/linear optics computing research.

Combining formulas (23) and (24), one gets an unbiased nonnegative valued random estimator for quantum permanents of bipartite unnormalized density matrices. But, as indicated in [9], it behaves rather badly for the entangled bipartite unnormalized density matrices.

From the other hand, there is a hope, pending on a proof of a "third" generation of van der Waerden

conjecture (([23], [22], [21]), ([35], [26])), to have even a deterministic polynomial time algorithm to approximate within a simply exponential factor quantum permanents of separable unnormalized bipartite density matrices (more details on this matter can be found in [7]). It is my great pleasure to thank my LANL colleagues Manny Knill and Howard Barnum .

Many thanks to Marek Karpinski and Alex Samorodnitsky for their comments on this paper . Finally, I would like to thank Arkadi Nemirovski for many enlightening discussions.

# A Proof of Theorem (3.8) and a permanental corollary

The main goal of this Appendix is a "direct proof" of formula (24). A much shorter probabilistic proof is presented in Appendix  ${\bf C}$ .

**Proof:** [ Proof of formula (23) ]

It is sufficent to prove that for any monomial

$$\frac{1}{\pi^k} \int \dots \int |z_1^{r_1} \dots z_k^{r_k}|^2 e^{-(x_1^2 + y_1^2)} \dots e^{-(x_k^2 + y_k^2)} dx_1 dy_1 \dots dx_k dy_k = r_1! r_2! \dots r_k! (z_l = x_l + iy_l, 1 \le l \le k).$$
(37)

And that distinct monomials are orthogonal, i.e.

$$\int \dots \int (z_1^{r_1} \dots z_k^{r_k} \overline{z_1^{h_1} \dots z_k^{h_k}}) e^{-(x_1^2 + y_1^2)} \dots e^{-(x_k^2 + y_k^2)} dx_1 dy_1 \dots dx_k dy_k = 0 (r \neq h)$$
(38)

Notice that both 2k-dimensional integrals (37) and (38) are products of corresponding 2-dimensional integrals. Thus (37) is reduced to the fact that

$$\frac{1}{\pi} \int \int (x_1^2 + y_1^2)^{2r_1} e^{-(x_1^2 + y_1^2)} dx_1 dy_1 = r_1!.$$

Using polar coordinates in a standard way, we get that

$$\frac{1}{\pi} \int \int (x_1^2 + y_1^2)^{2r_1} e^{-(x_1^2 + y_1^2)} dx_1 dy_1 = \int_0^\infty R^{r_1} e^{-R} dR = r_1!.$$

Similarly (38) is reduced to

$$\int \int (x_1 + iy_1)^m (x_1^2 + y_1^2)^k e^{-(x_1^2 + y_1^2)} dx_1 dy_1 = 0,$$

where m is positive integer and k is nonnegative integer.

But

$$\int \int (x_1 + iy_1)^m (x_1^2 + y_1^2)^k e^{-x_1^2 + y_1^2} dx_1 dy_1 = \int_0^\infty R^{2k} e^{-R^2} (\int_0^{2\pi} e^{-im\phi} d\phi) dR = 0.$$

**Proof:** [Proof of formula (24)]

First , let us recall how coefficients of  $\det(\sum_{1 \leq i \leq k} x_i A_i)$  can be expressed in terms of the corresponding mixed discriminants . Let us associate a vector  $r \in I_{k,N}$  an N-tuple of  $N \times N$  complex matrices  $\mathbf{B}_r$  consisting of  $r_i$  copies of  $A_i (1 \leq i \leq k)$  . Notice that

$$\mathbf{B}_r = (B_1, ..., B_N); B_i \in \{A_1, ..., A_k\}, 1 < i < k.$$

It is well known and easy to check that for this particular determinantal polynomial its coefficients satisfy the following identities:

$$a_{r_1,...,r_k} = \frac{M(\mathbf{B}_r)}{r_1! r_2! ... r_k!}; (r_1, ..., r_k) \in I_{k,N}$$
(39)

We already defined mixed discriminants by two equivalent formulas (13), (14). The next equivalent definition is handy for our proof:

$$M(B_1, ...B_N) = \sum_{\sigma \in S_N} \det([B_1(e_{\sigma(1)}|B_2(e_{\sigma(2)}|...|B_N(e_{\sigma(N)}]).$$
(40)

In the formula (40) above,  $(e_1,...,e_N)$  is a canonical basis in  $\mathbb{C}^N$ , and for a  $N \times N$  complex matrix B a column vector  $B(e_i)$  is an i-th column of B.

We will use in this proof three basic elementary facts about mixed discriminants. First is "local additivity", i.e.

$$M(A_1 + B, A_2, ..., A_N) = M(A_1, A_2, ..., A_N) + M(B, A_2, ..., A_N).$$

Second is permutation invariance, i.e.

$$M(A_1, A_2, ..., A_N) = M(A_{\tau(1)}, A_{\tau(2)}, ..., A_{\tau(N)}), \tau \in S_N.$$

And the third one is easy formula for the rank one case:

$$M(x_1y_1^T, ..., x_Ny_N^T) = \det(x_1y_1^T + ... + x_Ny_N^T),$$

where  $(x_i, y_i; 1 \le i \le N)$  are N-dimensional complex column-vectors. Recall that blocks of  $\rho_{\mathbf{A}}$  are defined as

$$A_{i,j} = \sum_{1 \le i \le k} A_k e_i e_j^{\dagger} A_k^{\dagger}, 1 \le i, j \le N.$$

Let us rewrite formula (15) as follows:

$$QP(\rho) =: \frac{1}{N!} \sum_{\sigma, \tau \in S_N} (-1)^{sign(\sigma)} M(A_{\tau(1), \sigma(1)}, ..., A_{\tau(N), \sigma(N)}); \tag{41}$$

Using this formula (41) we get the following expression for quantum permanent of bipartite density matrix  $\rho_{\mathbf{A}}$  using "local" additivity of mixed dicriminant in each matrix component :

$$QP(\rho_{\mathbf{A}}) = \frac{1}{N!} \sum_{t_1, \dots, t_N} \sum_{\tau_1, \tau_2 \in S_N} M(A_{t_1} e_{\tau_1(1)} e_{\tau_2(1)}^{\dagger} A_{t_1}^{\dagger}, \dots, A_{t_N} e_{\tau_1(N)} e_{\tau_2(N)}^{\dagger} A_{t_N}^{\dagger}).$$

Using rank one formula above and formula (40), we get that

$$\sum_{\tau_1,\tau_2 \in S_N} M(A_{t_1} e_{\tau_1(1)} e_{\tau_2(1)}^{\dagger} A_{t_1}^{\dagger},...,A_{t_N} e_{\tau_1(N)} e_{\tau_2(N)}^{\dagger} A_{t_N}^{\dagger}) = |M(A_{t_1},...,A_{t_N})|^2.$$

The last formula gives the following, intermediate, identity:

$$QP(\rho_{\mathbf{A}}) = \frac{1}{N!} \sum_{t_1, \dots, t_N} |M(A_{t_1}, \dots, A_{t_N})|^2.$$
(42)

What is left is to "collect" in (42), using invariance of mixed discriminants respect to permutations, all occurances of  $M(\mathbf{B}_r)$  (as defined in (39)), where  $r = (r_1, ..., r_k) \in I_{k,N}$ .

It is easy to see that this number  $N(r_1,...,r_k)$  of occurances of  $M(\mathbf{B}_r)$  is equal to the coefficient of monomial  $x_1^{r_1}x_2^{r_2}...x_k^{r_k}$  in the polynomial  $(x_1+...+x_k)^N$ . In other words ,  $N(r_1,...,r_k)=\frac{N!}{r_1!...r_k!}$  , which finally gives that

$$QP(\rho_{\mathbf{A}}) = \sum_{r \in I_k} \frac{|M(\mathbf{B}_r)|^2}{r_1! ... r_k!}.$$

Using formula (39) for coefficients of determinantal polynomial  $\det(\sum_{1 \le i \le k} x_i A_i)$  we get that

$$\|P_{\mathbf{A}}\|_G^2 = \sum_{(r_1, \dots, r_k) \in I_{k,N}} |a_{r_1, \dots, r_k}|^2 r_1! r_2! \dots r_k! = QP(\rho_{\mathbf{A}})$$

■Putting Part 1 and Part 2 together we get in the next corollary a formula expressing permanents of positive semidefinite matrices as squares of G-norms of multilinear polynomials . A particular , rank two case , of this formula was (implicitly) discovered in [4] .

Corollary A.1: Consider complex positive semidefinite  $N \times N$  matrix  $Q = DD^{\dagger}$ , where a "factor" D is  $N \times M$  complex matrix. Define a complex gaussian vector  $z = D\xi$ , where  $\xi$  is an M-dimensional complex gaussian vector as in theorem 3.8.

The following formula provides unbiased nonnegative valued random estimator for Per(Q):

$$Per(Q) = E_{\xi_1, \dots, \xi_N}(|z_1|^2 \dots |z_N|^2). \tag{43}$$

**Proof:** Consider the following m-tuple of complex  $N \times N$  matrices:

$$\mathbf{Diag} = (Diag_1, ..., Diag_m); Diag_j = Diag(D(1, j), ..., D(N, j)), 1 \le j \le M.$$

Then  $P_{\mathbf{Diag}}(x_1,...,x_m) = \prod_{1 \leq i \leq N} (Dx)_i$ , where  $(Dx)_i$  is ith component of vector Dx.

Thus Part 1 of theorem 3.8 gives that  $||P_{\mathbf{Diag}}||_G^2 = E_{z_1,...,z_N}(|z_1|^2...|z_N|^2)$ . It is easy to see that the block representation of bipartite density matrix  $\rho_{\mathbf{Diag}}$  associated with m-tuple  $\mathbf{Diag}$  is as follows:

$$\rho_{\mathbf{Diag}} = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \dots & \dots & \dots & \dots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{pmatrix}, A_{i,j} = Q(i,j)e_ie_j^T.$$

Therefore  $QP(\rho_{\mathbf{Diag}}) = Per(Q)$ .

Now Part 2 of theorem 3.8 gives that

$$Per(Q) = QP(\rho_{\mathbf{Diag}}) = ||P_{\mathbf{Diag}}||_G^2 = E_{z_1,...,z_N}(|z_1|^2...|z_N|^2).$$
 (44)

Remark A.2: Corollary (A.1) together with a remarkable supermultiplicative inequality for the G-norm ([3], [4]) give a completely new look at many nontrivial permanental inequalities, such as famous Leib's inequality [10] etc, and allow new corellational inequalities for analytic functions of complex gaussian vectors and new ("short") characterizations of independence of analytic functions of complex gaussian vectors. More on this will be described in [8].

## B Wick formula

In the next theorem we recall famous Wick formula (see , for instance , [28] ) .

**Theorem B.1:** Consider complex  $2N \times M$  matrix A and a real M-dimensional gaussian vector x with zero mean and covariance matrix  $E(xx^T) = I$ . Define  $(y_1, ..., y_{2N})^T = Ax$ . Then the following Wick formula holds

$$W(A) =: E(\prod_{1 \le i \le 2N} y_i) = Haf(AA^T),$$
 (45)

where hafnian Haf(B) of  $2N \times 2N$  matrix B is defined as follows:

$$Haf(B) = \sum_{1 \le p_1 < p_2 < \dots p_N; p_1 < q_1, \dots, p_N < q_N \le 2N} \prod_{1 \le i \le N} B(p_i, q_i)$$
(46)

Let us show how formula (43) follows from (45).

**Proposition B.2:** Suppose that complex  $N \times M$  matrix D in Theorem 1.4 can be written as D = C + iB. Consider the following complex  $2N \times 2M$  matrix A:

$$\sqrt{2}A = \begin{pmatrix} C + iB & iC - B \\ C - iB & -B - iC \end{pmatrix}.$$

Then

$$AA^T = \left( \begin{array}{cc} 0 & DD^{\dagger} \\ DD^{\dagger} & 0 \end{array} \right),$$

and  $W(A)=E_{z_1,\dots,z_N}(|z_1|^2...|z_N|^2)$ , where the expression  $E_{z_1,\dots,z_N}(|z_1|^2...|z_N|^2)$  is the same as in Corollary (A.1).

As it easy to see that  $Haf(AA^T)=Per(DD^{\dagger})$  , thus , using Wick formula (45) we reprove formula (43) .

Summarizing, we can say at this point that formula(43) is essentially a different way to write Wick formula. (We thank A.Barvinok for pointing at this observation and reference [28]). From the other hand formula(43) is a direct corrolary of formula(24) for the case of tuples of diagonal matrices. It is easy to see that one can also consider upper triangular matrices. More generally consider the following group action on tuples of square complex matrices  $\mathbf{A} = (A_1, ..., A_k)$ :

$$\mathbf{A}_{X,Y} = (XA_1Y, ..., XA_kY), \det(XY) = \det(X)\det(Y) = 1 \tag{47}$$

As

$$P_{\mathbf{A}_{X,Y}}(x_1,...,x_k) = \det(\sum_{1 \le i \le k} x_i X A_i Y) = \det(X) \det(Y) P_{\mathbf{A}}(x_1,...,x_k)$$

this group action does not change corresponding determinantal polynomial.

Finally , it follows that Wick formula is a particular case of formula (24) when there exist two matrices X and Y such that  $\det(XY) = \det(X) \det(Y) = 1$  and matrices  $(XA_1Y, ..., XA_kY)$  are all upper triangular , or , in Lie-algebraic language , there exists two nonsingular matrices X and Y such that the Lie algebra generated by  $(XA_1Y, ..., XA_kY)$  is solvable .

It seems reasonable to predict that formula(24) might be of good use in combinatorics desribed in [28].

## C Short probabilistic proof of formula (24)

## C.1 Hilbert space of analytical functions

Consider a Hilbert space  $L_{k,G}$  of analytic functions

$$f(x_1, x_2, ..., x_k) = \sum_{(r_1, ..., r_k)} a_{r_1, ..., r_k} x_1^{r_1} x_2^{r_2} ... x_k^{r_k},$$

where the G- inner product is defined as

$$\langle f, g \rangle_G = \sum_{(r_1, \dots, r_k)} a_{r_1, \dots, r_k} \overline{b_{r_1, \dots, r_k}} r_1! \dots r_k!$$
 (48)

It is easy to see that  $L_{k,G}$  is a closed proper subspace of  $L_2(C^k,\mu)$ , where  $\mu$  is a gaussian measure on  $C^k$ , i.e. its density function

$$p(z) = \frac{1}{\pi^k} e^{-|z|^2}$$

**Proposition C.1:** Suppose that  $f, g \in L_2(C^k, \mu)$  and the matrix  $U : C^k \to C^k$  is unitary, i.e.  $UU^* = I$ . Then

$$< f(Ux), g>_{L_2(C^k,\mu)} = < f, g(U^*x)>_{L_2(C^k,\mu)}$$

**Proof:** This is just a reformulation of a well known obvious fact that p(z) = p(Uz)  $(e-|z|^2 = e-|Uz|^2)$  for unitary U.

**Lemma C.2:** Let  $P(x_1, x_2, ..., x_k)$  be a homogeneous polynomial of total degree N and  $g \in L_2(C^k, \mu)$ . Then for any matrix  $A: C^k \to C^k$  the following identity holds:

$$< P(Ax), g>_{L_2(C^k,\mu)} = < f, g(A^*x)>_{L_2(C^k,\mu)}$$
 (49)

**Proof:** First, there is an unique decomposition  $g = Q + \delta$ , where  $Q(x_1, x_2, ..., x_k)$  is a homogeneous polynomial of total degree N and  $\langle R, \delta \rangle_{L_2(C^k, \mu)} = 0$  for any homogeneous polynomial R of total degree N.

As P(Ax) is a homogeneous polynomial of total degree N for all A thus  $< P(Ax), \delta >_{L_2(C^k,\mu)} \equiv 0$ 

It is left to prove (49) only when g is a homogeneous polynomial of total degree N. We already know that (49) holds for unitary A. Also , because of formula(23) , in this homogeneous case (49) holds for diagonal A. To finish the proof , we use the singular value decomposition A = VDiagU , where U, V are unitary and Diag is a diagonal matrix with a nonnegative entries .  $\blacksquare$ 

**Remark C.3:** A homogeneous part of Lemma has been proved in [4] using the fact that the linear space of homogeneous polynomials of total degree N is spanned by N powers of linear forms .  $\blacksquare$ 

## C.2 Unbiased estimators for Quantum permanents

**Remark C.4:** Consider a four-dimensional tensor  $\rho(i_1, i_2, i_3, i_4), 1 \leq i_1, i_2, i_3, i_4 \leq N$ . One can view it as a block matrix as in (3), where the blocks are defined by

$$A_{i_1,j_1} =: \{ \rho(i_1,i_2,j_1,j_2) : 1 \le i_2, j_2 \le N \}, 1 \le i_1, j_1 \le N$$

We also can permute indices :  $\rho(i_{\pi(1)},i_{\pi(2)},i_{\pi(3)},i_{\pi(4)})$ , and get another block matrix . The main point is that it follows from formula(16) that a permutation of indices does not change the quantum permanent  $QP(\rho)$ . In what follows below we will use the following simple and natural trick : permute indices and use mixed discriminants based equivalent formula(15) for  $QP(\rho)$  based on the corresponding block structure .

The next proposition follows directly from the definition .

#### Proposition C.5:

1. Consider a block matrix  $\rho$  as in (3) and associate with it the following operator  $T: M(N) \to M(N)$ ,  $T(X) = \sum_{1 \le i,j \le N} X(i,j) A_{i,j}$ .

Let X be a random complex zero mean matrix such that  $E(|X(i,j)|^2 \equiv 1$  and for any two permutations  $\tau_1, \tau_2$  the set of entries

 $\{X_{i,j}: j=\tau_1(i) \text{ or } j=\tau_2(i)\}$  consists of independent random variables . Then

$$QP(\rho) = E(\det(T(X))\overline{\det(X)}). \tag{50}$$

2. Consider a  $N \times N$  matrix A and a random zero mean vector  $z \in C^N$  such that  $E(z_i\overline{z_j}) = 0$  for all  $i \neq j$ . Then

$$Per(A) = E(\prod_{1 \le i \le N} (Tz)_i \overline{\prod_{1 \le i \le N} z_i})$$
 (51)

Let us present now a promised short probabilistic proof of (24): Consider without loss of generality a  $N^2$ -tuple of  $N \times N$  complex matrices  $\mathbf{A} = (A_{(1,1)}, ..., A_{(N,N)})$ . Recall that the (**BUDM**)  $\rho_{\mathbf{A}}$  associated with the k-tuple  $\mathbf{A} = (A_1, ..., A_k)$  is defined as:

$$\rho_{\mathbf{A}}(i_1, i_2, j_1, j_2) =: \sum_{1 \le l \le k} A_l(i_1, i_2) \overline{A_l(j_1, j_2)}$$

Define the following operator  $T(X) = \sum_{1 \leq i,j \leq N} X(i,j) B_{i,j}$ , where  $N \times N$  matrix  $B_{i,j} = \{\rho_{\mathbf{A}}(i,j,m,l): 1 \leq m,l \leq N\}$ . It is easy to see that  $T(X) = C^*C(X)$ , where  $C(X) = \sum_{1 \leq i,j \leq N} X(i,j) A_{(i,j)}$ . Thus  $QP(\rho_{\mathbf{A}}) = E_X(\det(T(X)\overline{\det(X)})$ , where random gaussian matrix X has the density

$$p(X) = \frac{1}{\pi^{N^2}} e^{-(tr(XX^*))}.$$

I.e. the entries X(i, j) are IID canonical complex gaussian random variables. Finally, we get that (using for the first identity (50) and for the second (49))

$$QP(\rho_{\mathbf{A}}) = E_X(\det(T(X)\overline{\det(X)}) = E_X(|\det(C(X)|^2))$$

But,  $E_X(|\det(C(X)|^2) = ||P_A||_G^2 \text{ from (23)}.$ 

Similarly, the permanental formula (43) can be proved using (51).

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