Optimal Control

Wintersemester 2025/26

Dozent: Prof. Dr. Andrea Iannelli

Autor: Jan Fehnl

Freundliche Unterstützung: Alexander Schütz & Gemini

Credits gehen raus an:
Gabriel Schneider & Luka Radenović & Patrick Böpple

February 19, 2026

Contents

0 Introduction

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0, \quad t \in [t_0, t_f]$$

$$f : [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$$

$$x = \text{state}, \quad u = \text{input}$$

Initial Value Problem (IVP)

Given $x_0, u(\cdot)$ we can compute $x(\cdot)$

functions of time

When is this possible? It depends on f.

Lemma 0.1 (Sufficient conditions)

Existence & Uniqueness of solutions of ODEs.
Assume that

- f is piecewise continuous in t and u
- f is globally Lipschitz in x

$$\exists\, k(t, u) \text{ s.t. } \|f(t, x_1, u) - f(t, x_2, u)\| \leq k(t, u)\|x_1 - x_2\|, \ \forall x_1, x_2 \in \mathbb{R}^{n_x}$$

Then $x(\cdot)$ exists for all t and is unique.

Remarks

- Lipschitz continuous $\Rightarrow$ continuous, but not the converse
- $\sqrt{x}$ is continuous but not Lipschitz, $\dot{x} = \sqrt{x}$ does not have a unique solution
- Continously differentiable $(\mathcal{C}^1) \Rightarrow$ locally Lipschitz continous $\forall x_1, x_2 \in \mathcal{X} \subset \mathbb{R}^{n_x}$
- Locally Lipschitz continuous x guarantees existence & uniqueness for small enough times

In this course we will assume $f \in \mathcal{C}^1$ and implicitely assume that t_f is chosen such that $x(\cdot)$ exists in $[t_0, t_f]$.

We do not need to worry about existence & uniqueness!

Goal in Optimal Control: Design u such that

1. $u(t) \in \underset{\uparrow}{\mathcal{U}}(t),\ x(t) \in \underset{\uparrow}{\mathcal{X}}(t) \quad \forall t \in [t_0, t_f], \quad \mathcal{X} \subseteq \mathbb{R}^{n_x},\ \mathcal{U} \subseteq \mathbb{R}^{n_u}$
   sets defining constraints on $u \& x$
   $\Rightarrow$ Admissible input/state trajectories

2. The system behaves optimally according to

$$\underset{\uparrow}{J}(u) = \int_{t_0}^{t_f} \underset{\uparrow}{l}(t, x(t), u(t))\, dt + \underset{\uparrow}{\varphi}(t_f, x(t_f))$$

Cost function running cost terminal cost

$\Rightarrow$ optimal behaviour

Formally, we can state the goal as follows:
Find an admissible input $u^\star$ which causes the dynamics to follow an admissible trajectory $x^\star$ which minimizes J, that is

$$\int_{t_0}^{t_f} l(t, x^\star(t), u^\star(t))\, dt + \varphi(t_f, x^\star(t_f)) \leq \int_{t_0}^{t_f} l(t, x(t), u(t))\, dt + \varphi(t_f, x(t_f))$$

$\forall$ admissible x, u

Examples of cost functions

1) Minimum-time problem
   Goal: transfer the system from x_0 to a set $\mathcal{S}$ in the minimum time

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \qquad (l = 1, \varphi = 0)$$

$$x(t_f) \in \mathcal{S}$$

   Note: t_f is also a decision variable! The unknowns are (u, t_f).

2) Minimum control-effort problem

$$J = \int_{t_0}^{t_f} \|u(t)\|^2\, dt$$

$$x(t_f) \in \mathcal{S}$$

3) Tracking problem

$$J = \int_{t_0}^{t_f} (x(t) - r(t))^T Q (x(t) - r(t))\, dt$$

$Q > 0$ (positive definit matrix: symmetric & all eigenvalues positive)
$r(t)$ given signal

1 Nonlinear Programming

Nonlinear Programs (NLP) are general finite-dimensional optimization problems:

$$\min_x f(x)$$

$$\text{s.t. } g(x) \leq 0, \quad h(x) = 0$$

$f : \mathbb{R}^n \to \mathbb{R}$, objective function
$g : \mathbb{R}^n \to \mathbb{R}^{n_g}$, inequality constraints
$h : \mathbb{R}^n \to \mathbb{R}^{n_h}$, equality constraints
Feasible set:

$$D = \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}$$

$\bar{x} \in D$ feasible point

Definition 1.1 (Global, local Minimizers)

$x^\star \in \mathcal{D}$ Global Minimizer of the NLP if

$$f(x^\star) \leq f(x) \quad \forall x \in \mathcal{D}$$

$f(x^\star)$ is the Global Minimum (or Minimum)
Nomenclature: $x^\star$ is also called (optimal) solution, $F(x^\star)$ is optimal value
$x^\star$ is a strict global minimizer if $f(x^\star) < f(x) \quad \forall x \in \mathcal{D}$
$x^\star \in \mathcal{D}$ Local Minimizer if

$$\exists \varepsilon > 0, \text{ s.t. } f(x^\star) \leq f(x) \quad \forall x \in B_\varepsilon(x^\star) \cap \mathcal{D}$$

$$B_\varepsilon(x) := \{y \mid \|x - y\| \leq \varepsilon\} \qquad \|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ any norm in } \mathbb{R}^n$$

Strict local Minimizer if inequality holds strictly
Global min $\overset{\rightarrow}{\not\leftarrow}$ local min

Solving an NLP boils down to finding global or local minimizers.
Does a solution always exist? No.

Definition 1.2 (infimum)

Given $\mathcal{S} \subseteq \mathbb{R}$, $\inf(\mathcal{S})$ is the greatest lower bound of $\mathcal{S}$:

- $z \geq \inf(\mathcal{S}), \quad \forall z \in \mathcal{S} \quad$ (lower bound)
- $\forall \bar{\alpha} > \inf(\mathcal{S}) \quad \exists z \in \mathcal{S}$ s. t. $\bar{\alpha} > z \quad$ (greatest bound)

Example $\mathcal{S} = [-1, 1]$, $-50 = \inf(\mathcal{S})$? $\quad\rightarrow\quad$ No, $\inf(\mathcal{S}) = -1$

- Analogous: $\sup(\mathcal{S})$ is smallest upper bound.
- inf and sup always exist if $\mathcal{S} \neq \emptyset$
- $\inf(\mathcal{S})$ does not have to be an element of $\mathcal{S}$
- If $\mathcal{S}$ unbounded from below $\rightarrow \inf(\mathcal{S}) = -\infty$
- $\inf([a, b]) = \inf((a, b]) = a$

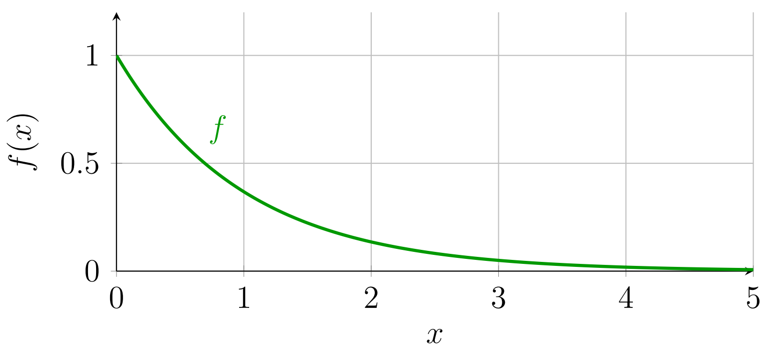
Connections with NLP?

$$f : \mathcal{D} \rightarrow \mathbb{R}$$

$$\inf(\underbrace{f(x) \,|\, x \in \mathcal{D}}_{\mathcal{S}}) \coloneqq \bar{f} = \inf_{x \in \mathcal{D}} f(x) \quad \text{(similar to NLP)}$$

Whenever NLP has solution, then NLP ist equivalent to this, but $\nexists x^\star \in \mathcal{D}$ s. t. $f(x^\star) = \bar{f} \quad \rightarrow \quad$ infimum exists, but not minimum

Examples $f(x) = e^{-x}, \quad \mathcal{D} = [0, \infty), \quad \inf(\mathcal{S}) = 0$
$f(x) = x, \quad \mathcal{D} = \mathbb{R}, \quad \inf(\mathcal{S}) = -\infty$, min doesn't exist!



When does the infimum coincide with the minimum?

Theorem 1.1 (Extreme value problem) (Weierstrass Theorem)

$f : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$

If:

- $f \in \mathcal{C}$ on $\mathcal{D}$
- $\mathcal{D}$ is compact
- $\mathcal{D} \neq \emptyset$

Then f attains a minimum on $\mathcal{D}$.

Definition 1.3 (Continuous function)

$f : \mathcal{D} \to \mathbb{R}$ is continuous at $x \in \mathcal{D}$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s. t. } \|x - x'\| < \delta \quad \Rightarrow \quad \|f(x) - f(x')\| < \varepsilon$$

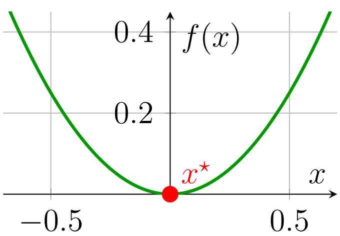
If f is continuous $\forall x \in \mathcal{D}$ then f is continuous on $\mathcal{D} \to f \in \mathcal{C}$

Implication for NLP: If f is $\mathcal{C}$ on $\mathcal{D}$ and $\mathcal{D}$ is compact and non-empty then [NLP] has a solution!

- $\mathcal{D} \subseteq \mathbb{R}^n$: in finite-dimensional spaces: compact = closed and bounded
  Not compact:
  - $(a, b]$ (not closed)
  - $(-\infty, b]$ (unbounded)

  Compact set:
  - $[a, b]$ $-\infty < a < b < \infty$

  Warning: $\mathcal{D}$ infinite dimensional (e. g. function space) then
  compact $\underset{\not\Leftarrow}{\Rightarrow}$ bounded and closed



Theorem 1.1 is restrictive e. g. $f(x) = x^2, \mathcal{D} = (-\infty, \infty)$ has unique minimum

- Notation convention: Technically it is "wrong" to write

$$\min_{x \in \mathcal{D}} f(x)$$

more compact is:

$$\underset{x \in \mathcal{D}}{\text{minimize}}\ f(x) \quad \text{or} \quad \inf_{x \in \mathcal{D}} f(x)$$

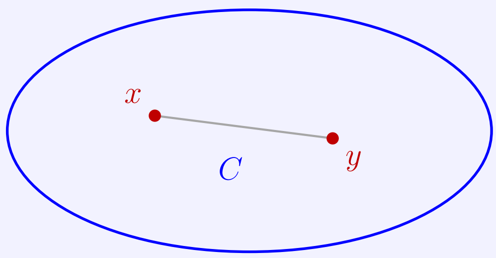
Goal of the Chapter: characterize necessary and sufficient conditions for $x^\star$ to be global minimizer of NLP.

Convexity

Definition 1.4 (Convex sets & functions)

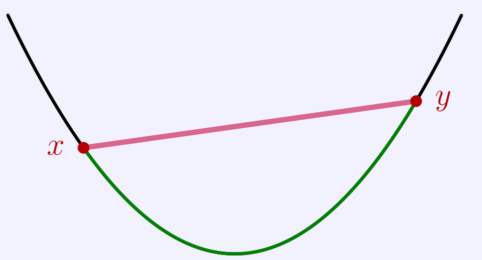
- A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> (cvx) if $\forall x, y \in C$

$$\{z \mid z = \lambda x + (1-\lambda)\lambda,\ \lambda \in [0,1]\} \subseteq C$$



- Given a cvx set C, a function $f : C \to \mathbb{R}$ is cvx if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in C, \quad \lambda \in (0,1)$$



- f is strictly cvx if the inequality holds strictly.

Remarks

- The definition extends to vector functions $f : C \to \mathbb{R}^n$ for convex f_i
- $f : C_1 \times C_2 \to \mathbb{R}$
  $f(x, y)$ is jointly cvx, in x, y if $z := \begin{bmatrix} x \\ y \end{bmatrix}$, $f(z)$ is in cvx in z.

Example $f(x, y) = x^2 + y^2$, $z = \begin{bmatrix} x \\ y \end{bmatrix} \to f(z) = z_1^2 + z_2^2$

Definition 1.5

An NLP is a convex program if
- f is convex function,
- $\mathcal{D}$ is convex set.

Lemma 1.1

Let $x^\star$ be a local minimizer of cvx program. Then $x^\star$ is also global minimizer.

Proof: try as an erxercise

Minimizers of convex NLP form a convex set.
This set might be empty (Convex NLPs not guaranteed to have solution).
However: Unique solution for strictly convex NLPs, if a solution exists.

Lemma 1.2 (First/Second order conditions for convexity)

1. $f : C \to \mathbb{R}$ continuously differentiable on C. Then f is cvx iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in C$$

$$(\nabla f)_i = \frac{\partial f}{\partial x_i} \text{ is gradient (sometimes } f_{x_i}\text{)}$$

2. f twice differentiable on C, then f convex iff

$$\nabla^2_{xx} f(x) \succeq 0 \quad \forall x \in C$$

$$(\nabla^2_{xx})_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{(Hessian)}$$

- $A \succeq 0$ means that: $A = A^T$ and pos semi-definite, i. e. all eigenvalues non-negative
- f strictly cvx if $\nabla^2_{xx} f(x) \succ 0 \quad \forall x \in C$ with $A \succ 0$ meaning pos definite and symmetric
- Interpretation: Curvature of function should be non-negative/positive
- For exercises to check convexity, the second condition is generelly useful. First condition is useful for proofs.

For $\mathcal{D} = \{x \mid g(x) \leq 0,\, h(x) = 0\}$ the following holds: If

$$\left.\begin{array}{l} \bullet\ g \text{ are convex functions,} \\ \bullet\ h \text{ are affine functions (i.e. } h(x) = 0 \Leftrightarrow Ax = b\text{),} \end{array}\right\} \text{sufficient}$$

then $\mathcal{D}$ is a convex set.

Example $a, b \in \mathbb{R}$

$$\begin{array}{ccc} \min\limits_x f & & \min\limits_x f \\ \text{s.t. } x^3 - 1 \leq 0 & \leftrightarrow & \text{s.t. } x - 1 \leq 0 \\ (ax+b)^2 = 0 & & ax + b = 0 \\ \text{non-convex} & & \text{convex} \\ \text{non-affine} & & \text{affine} \end{array}$$

Moral to recognize convexity of NLP:

1. Use definition of cvx NLP, cvx f, convex $\mathcal{D}$
2. If $\mathcal{D}$ written as equality/inequality-constraints, check g convex/h affine.
   If not, check further whether the feasible set is cvx or not (e.g. can be written equivalently with cvx g/affine h).

1.1 Unconstrained Problems

$$\mathcal{D} = \mathbb{R}^n$$

Assume throughout that $f \in \mathcal{C}^1$ (continuously differentiable).

Definition 1.6 (Descent Direction)

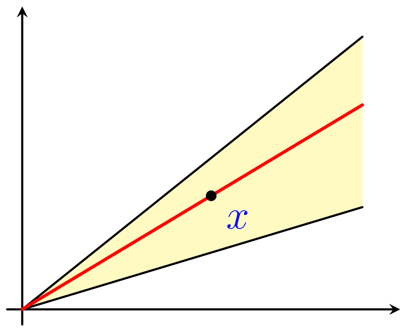
$d \in \mathbb{R}^n$ is a descent direction for f at $\bar{x} \in \mathbb{R}^n$ if

$$\exists \delta > 0 \quad \text{s.t.} \quad f(\bar{x} + \lambda d) < f(\bar{x}) \quad \forall \lambda \in (0, \delta).$$

$F(\bar{x})$: Cone of descent directions
Set of all descent directions of f at $\bar{x}$

A set $K \subseteq \mathbb{R}^n$ is a cone if it contains the full ray through any point in the set.

$$K \text{ cone if } \forall x \in K \text{ and } \rho \geq 0, \quad \rho x \in K$$



This is a geometric characterization of descent direction. It gives us a geometric condition for $x^\star$ to be a local minimizer.

Lemma 1.3 (Geometric Condition for local minimum)

$x^\star$ is a local minimizer iff

$$\mathcal{F}(x^\star) = \emptyset.$$

We want an algebraic condition to be able to compute or look for $x^\star$.

Lemma 1.4 (Algebraic first-order characterization of $\mathcal{F}$)

If $\nabla f(\bar{x}) \neq 0$, then

$$\mathcal{F}_0(\bar{x}) = \{d \mid \nabla f(\bar{x})^T d < 0\} = \mathcal{F}(\bar{x}).$$

Otherwise

$$\mathcal{F}_0(\bar{x}) \subseteq \mathcal{F}(\bar{x}).$$

Proof: try Taylor-series expansion of f at $\bar{x}$

Graphical interpretation:

∇f forms angles greater or equal than $90°$ with <u>all</u> descent directions.

Lemma 1.5 (First-order necessary condition for local minimum)

If $x^\star$ is a local minimizer, then

$$\underbrace{\nabla f(x^\star) = 0}_{\text{“stationary point”}} .$$

Proof: Contradiction
If $\nabla f(x^\star) \neq 0$, then $d = -\nabla f(x^\star) \neq 0$. Therefore there exists a descent direction $d \in \mathcal{F}(x^\star)$ by Lemma 1.4. Thus $\exists \delta > 0$ s.t. $f(x^\star + \lambda d) < f(x^\star) \quad \forall \lambda \in (0, \delta)$.
This is a contradiction with the fact, that $x^\star$ is a minimizer. □

Why only necessary?
It can't be a sufficient condition because in case where $\nabla f(x^\star) = 0$ we cannot use Lemma 1.4, e.g. $f_1(x) = -x^2$, $f_2(x) = x^3$, $\nabla f_1(0) = \nabla f_2(0) = 0$.

Lemma 1.6 (second order necessary condition)

Assume f is twice continuously differentiable $f \in \mathcal{C}^2$

$$x^\star \text{ local minimizer } \Rightarrow \nabla^2_{xx} f(x^\star) \succeq 0$$

Note: the condition on the Hessian of f can be interpreted as a local convexity property (around $x^\star$).

Proof: 2^{nd} order Taylor expansion around $x^\star$ in direction $d \in \mathbb{R}^n$:

$$f(x^\star + \lambda d) = f(x^\star) + \lambda \nabla f(x^\star)^T d + \frac{\lambda^2}{2} d^T \nabla^2_{xx} f(x^\star) d + \lambda^2 \|d\|^2 \alpha(\lambda d)$$

$(\rightarrow \alpha(\cdot))$ is a function that is order 1 or higher in $\lambda d)$

1. If $x^\star$ is local minimzer $\Rightarrow \nabla f(x^\star) = 0$

2. Divide by λ^2:

$$\frac{f(x^\star + \lambda d) - f(x^\star)}{\lambda^2} = \frac{1}{2} d^T \nabla^2_{xx} f(x^\star) d + \|d\| \alpha(\lambda d)$$

3. $\lambda \to 0$ on the right-hand-side the first term dominates

$$\frac{f(x^\star + \lambda d) - f(x^\star)}{\lambda^2} \approx \frac{1}{2} d^T \nabla^2_{xx} f(x^\star) d$$

4. For $x^\star$ is a local minimizer, the left-hand-side must be ≥ 0 for any $d \in \mathbb{R}^n$
$\Rightarrow \quad d^T \nabla^2_{xx} f(x^\star) d \geq 0 \quad \forall d \quad \Rightarrow \quad \nabla^2_{xx} f(x^\star) \succeq 0$ □

Only a necessary condition, because when $\nabla^2_{xx} f(x^\star)$ is singular, we need to use higher-order information.

Generally it is hard to get (global) sufficient conditions. $\to$ convexity to the rescue!

Lemma 1.7 (First order N&S condition for global minimizers)

Assume f is convex.

$$\exists x^\star \text{ s.t. } \nabla f(x^\star) = 0 \quad \Leftrightarrow \quad x^\star \text{ is a global minimizer}$$

If f is strictly convex, then the minimizer is unique.

Proof: ($\nabla f = 0 \Rightarrow$ global minimum)

First order condition for convexity:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in \mathbb{R}^n$$

Pick $x = x^\star$:

$$f(y) \geq f(x^\star), \quad \forall y \in \mathbb{R}^n$$

(Other direction holds because of Lemma 1.5)

What if we do not have global convexity?

Lemma 1.8 (Second order sufficient condition for local minimizer)

Assume $f \in \mathcal{C}^2$.
If $\nabla f(x^\star) = 0$ and $\nabla^2_{xx} f(x^\star) \succ 0 \quad \Rightarrow \quad x^\star$ is strict local minimizer.

Proof: Taylor expansion (Similar to Lemma 1.6)

1.2 Constrained Problems

$$\mathcal{D} \subseteq \mathbb{R}^n, \quad \mathcal{D} = \{x \mid g_i(x) \leq 0,\ i = 1, \ldots, n_g \quad h_j(x) = 0,\ j = 1, \ldots, n_h\}$$

We assume throughout g_i, h_j are all $\mathcal{C}^1$ functions.

Definition 1.7 (Tangent vector, tangent cone)

$p \in \mathbb{R}^n$ is a tangent vector to $\mathcal{D}$ at $\bar{x} \in \mathcal{D}$ if $\exists$ differential curve $\bar{x}(s) : [0, \varepsilon) \to \mathcal{D}$ with $\varepsilon > 0$ such that $\bar{x}(0) = \bar{x}$, $\frac{d\bar{x}}{ds}\Big|_{s=0} = p$.

Tangent cone $\mathcal{T}_{\mathcal{D}}(\bar{x})$ to $\bar{x}$ is the set of all tangent vectors

$$\mathcal{T}_{\mathcal{D}}(\bar{x}) := \{p \mid p \text{ tangent vector to } \mathcal{D} \text{ at } \bar{x}\}$$

Graphical representation:

Set of directions that make us stay feasible (at least infinitesimally)

When it comes to geometric conditions for optimality in constrained problems, we now have 2 sets/2 directions:

- $d \in \mathcal{F}(x) \to$ descent direction: objective improves
- $d \in \mathcal{T}_D(\bar{x}) \to$ tangent vector: we stay feasible

Lemma 1.9 (Geometric condition for local minimizer, $\mathcal{D} \subseteq \mathbb{R}^n$)

$x^\star$ is a local minimizer iff $\mathcal{F}(x^\star) \cap \mathcal{T}_{\mathcal{D}}(x^\star) = \emptyset$

It is basically says that "any improving direction can't be feasible".

As in the unconstrained case, we want to turn geometric conditions to algebraic ones.

Lemma 1.10 (1st order Nec. condition - semi-algebraic)

If $x^\star$ is a local minimizer. Then:

1. $x^\star \in \mathcal{D}$
2. $\underbrace{\forall p \in \mathcal{T}_{\mathcal{D}}(x^\star)}_{\text{geometric}}$, it holds $\underbrace{p^T \nabla f(x^\star) \geq 0}_{\text{algebraic}}$

Proof:

Item 1 $\to$ feasibility

Item 2: Assume there is a p s.t. $p^T \nabla f(x^\star) < 0$. Then

$$\exists \text{ curve } \bar{x}(s) \in \mathcal{D} \text{ s.t. } \left.\frac{df(\bar{x})}{ds}\right|_{s=0} = p^T \nabla f(x^\star) < 0 \quad \text{chain rule}$$

which would mean that p is descent direction. Contradicts $x^\star$ local minimizer.

This is almost a translation of Lemma 1.9 because we replaced $\mathcal{F}(x^\star)$ with its algebraic form "$d^T \nabla f(x^\star) < 0$".

To obtain a fully algebraic test, we need a few more concepts.

Definition 1.8 (Active constraints, active set, regular points)

$\bar{x} \in \mathcal{D}$

- g_i is active at $\bar{x}$ if $g_i(\bar{x}) = 0$
- $A(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\}$ set of active constraints at $\bar{x}$
- $\bar{x} \in \mathcal{D}$ is a regular point if $\nabla g_i(\bar{x})$, $i \in \mathcal{A}(\bar{x})$ and $\nabla h_j(\bar{x})$, $j = 1, \ldots, n_h$ are linearly independent.

Lemma 1.11 (Algebraic first-order characterization of target set)

If $\bar{x}$ is regular point. Then

$$\mathcal{T}_{\mathcal{D}}(\bar{x}) = \{p \mid \nabla h(\bar{x})^p = 0,\ \nabla g_i(\bar{x})^p \leq 0, \quad \forall i \in \mathcal{A}(\bar{x})\} \qquad ①$$

where $\nabla h(\bar{x}) := [\nabla h_1(\bar{x}), \ldots, \nabla h_{n_k}(\bar{x})] \in \mathbb{R}^{n \times n_h}$.
① can be written equivalently as $\mathcal{T}_{\mathcal{D}}(\bar{x}) = \{p \mid \mathcal{A}(\bar{x})p \geq 0\}$

$$A(\bar{x}) := \begin{bmatrix} \nabla h(\bar{x})^T \\ -\nabla h(\bar{x})^T \\ \vdots \\ -\nabla g_i(\bar{x})^T \\ \vdots \end{bmatrix} \in \mathbb{R}^{(2n_h + |\mathcal{A}(\bar{x})|) \times n} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} i \in \mathcal{A}(\bar{x})$$

In other words, item 2 of Lemma 1.10 can be written as follows:

$$p \in \mathbb{R}^n : \mathcal{A}(x^\star)p \geq 0,\ p^T \nabla f(x^\star) < 0$$

still not very tractable?

Farkas Lemma to the rescue:

Lemma 1.12 (Farkas Lemma)

For any matrix $A \in \mathbb{R}^{m \times n}$, vector $b \in \mathbb{R}^n$.
Exactly one of the following holds:

1. $\exists y \in \mathbb{R}^m \; y \geq 0$, such that $A^T y = b$

2. $\exists p \in \mathbb{R}^m$, such that $A^T p \geq 0$, $p^T b < 0$

Take $A \equiv A(x^\star)$ and $b \equiv \nabla f(x^\star)$

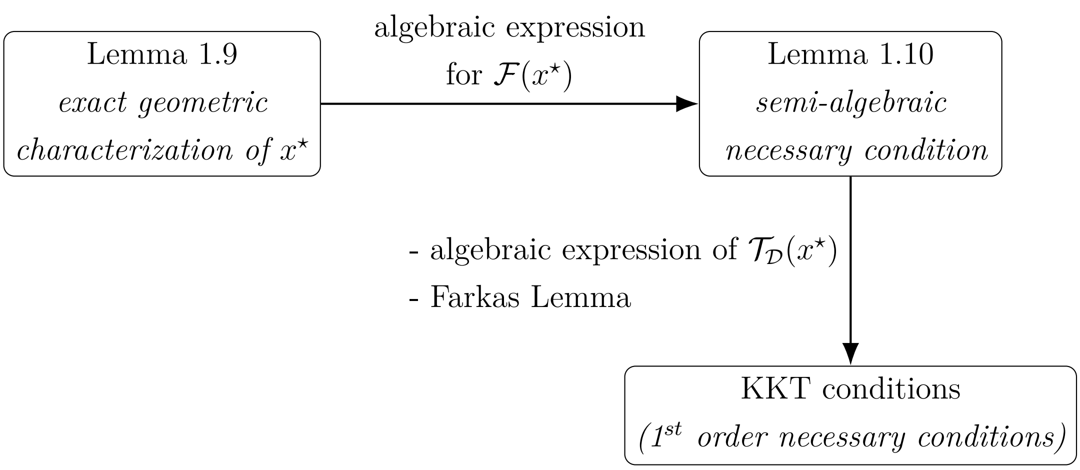
If we find y satisfying 1., then 2. can't hold $\Rightarrow$ item 2 of Lemma 1.10 is verified $\Rightarrow$ $x^\star \in \mathcal{D}$ is a local minimizer.

KTK-conditions just follow from imposing

item 1 of Lemma 1.10 $\rightarrow x^\star \in \mathcal{D}$

item 2 of Lemma 1.10 $\rightarrow \exists y \in \mathbb{R}^m$, $y \geq 0$ s.t. $\mathcal{A}(x^\star)^T y = \nabla f(x^\star)$

Conceptual summary:



Informal recap:

$x^\star$ local minimizer $\Rightarrow x^\star \in \mathcal{D}, \quad \forall p \in \mathcal{T}_D(x^\star), \quad p^T \nabla f(x^\star) \geq 0$

$\Updownarrow$ (if $x^\star$ regular point)

$$\exists y = \begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \vdots \\ -\nu_i \\ \vdots \end{bmatrix} \in \mathbb{R}^{2n_h + |\mathcal{A}(x^\star)|} \quad \left.\right\} i \in \mathcal{A}(\bar{x}) \qquad y \geq 0 \quad \rightarrow A(x^\star)^T y = \nabla f(x^\star)$$

Let's write down $A^T y = \nabla f$

$$\nabla h(x^\star)(\hat{\lambda}_1 - \hat{\lambda}_2) - \sum_{i \in \mathcal{A}(x^\star)} \nabla g_i(x^\star)\nu_i = \nabla f(x^\star), \quad \hat{\lambda}_1, \hat{\lambda}_2, \nu_i \geq 0 : \text{ But } (\hat{\lambda}_1 - \hat{\lambda}_2) \gtreqless 0$$

Equivalently: $\lambda \coloneqq -(\hat{\lambda}_1 - \hat{\lambda}_2) \in \mathbb{R}^{n_h}$, sign undefined

$$\exists \lambda \in \mathbb{R}^{n_h}, \quad \nu \in \mathbb{R}^{n_g}, \quad \nu \geq 0, \quad \nu_i = 0, \quad i \notin \mathcal{A}(x^\star)$$

$$\text{s.t. } \nabla f(x^\star) + \nabla h(x^\star)\lambda + \nabla g(x^\star)\nu = 0$$

$$\text{with } \nabla g(x^\star) \coloneqq \begin{bmatrix} \nabla g_1(x^\star) & \nabla g_2(x^\star) & \cdots & \nabla g_{n_g}(x^\star) \end{bmatrix}$$

$$\text{and } \nabla h(x^\star) \coloneqq \begin{bmatrix} \nabla h_1(x^\star) & \nabla h_2(x^\star) & \cdots & \nabla h_{n_h}(x^\star) \end{bmatrix}$$

We are now ready for a fully algebraic characterization.

Definition 1.9 (Karash-Kuhn-Tucker (KKT) points)

A triplet of vectors $(\bar{x}, \bar{\lambda}, \bar{\nu}) \in \mathbb{R}^n \times \mathbb{R}^{n_h} \times \mathbb{R}^{n_g}$

$\bar{x}$: opt. variable, $\quad \bar{\lambda}$: multiplier equality constraints,

$\bar{\nu}$: multiplier inequality constraints

is a KKT point if

1. $\nabla f(\bar{x}) + \nabla h(\bar{x})\bar{\lambda} + \nabla g(\bar{x})\bar{\nu} = 0$
2. $g(\bar{x}) \leq 0$
3. $h(\bar{x}) = 0$
4. $\bar{\nu} \geq 0$
5. $\bar{\nu}^T g(\bar{x}) = 0$

Lemma 1.13 (KKT necessary condition for local minimizer)

If $x^\star$ is a local minimizer **and** a regular point.
Then $\exists \lambda^\star, \nu^\star$ s.t. ($x^\star, \lambda^\star, \nu^\star$ is a KKT point)

Proof: Corollary of previous discussion

1. $\iff \exists \lambda \in \mathbb{R}^{n_h}$, $\nu \in \mathbb{R}^{n_g}$ s.t. $\nabla f(x^\star) + \nabla h(x^\star)\lambda + \nabla g(x^\star)\nu = 0$ It can be written

   equivalently as
$$\nabla_x \mathcal{L}(x, \lambda, \nu)|_{x=x^\star, \lambda=\lambda^\star, \nu=\nu^\star} = 0$$
   where $\mathcal{L}(x, \lambda, \nu) \coloneqq f(x) + \lambda^T h(x) + \nu^T g(x)$
2. $\iff x^\star \in \mathcal{D}$
3. $\iff x^\star \in \mathcal{D}$
4. $\iff$ non-negativity of "y" from Farkas Lemma
5. $\iff \nu_i = 0,\ i \notin \mathcal{A}(x^\star)$

   $\nu^T g(x^\star) = \sum_i \nu_i g_i(x^\star) = 0$ Complementary slackness

$$g_i(x^\star) = \begin{cases} = 0, & i \in \mathcal{A}(x^\star) \\ < 0, & i \notin \mathcal{A}(x^\star) \end{cases} \quad \text{because } x^\star \in \mathcal{D}$$

   $\nu_i \geq 0,\ \forall i$ because of Farkas' lemma.

   Then $\sum_i \nu_i g_i(x^\star)$ automatically sets $\nu_i = 0$ when $i \notin \mathcal{A}(x^\star)$ or $g_i(x^\star) < 0$. □

Intrestingly, if NLP is convex, KKT conditions are sufficient for global optimality:

Lemma 1.14 (KKT sufficient conditions for global minimizer)

Suppose $f, g_i \quad (i = 1, \dots, n_g)$ are convex functions and
$h_j \quad (j = 1, \dots, n_h)$ are affine functions.
If $(x^\star, \lambda^\star, \nu^\star)$ is a KKT point, then $x^\star$ is a global minimizer.

Proof: For $(\lambda^\star, \nu^\star)$ KKT points:

$$b(x) \coloneqq \mathcal{L}(x, \lambda^\star, \nu^\star) = f(x) + \sum_{i=1}^{n_g} \nu_i^\star g_i(x) + \sum_{j=1}^{n_h} \lambda_j^\star g_j(x) \qquad \otimes$$

f, g_i, h_j are convex functions

Linear combination of cvx functions with non-negative coefficients is a convex function

$\Rightarrow b(x)$ convex

1. $b(x)$ convex
2. $\nabla b(x^\star) = 0$ because of 1., $(x^\star, \lambda^\star, \nu^\star)$ is a KKT point $b(x) \geq b(x^\star) \quad \forall x \in \mathbb{R}^n$

$\Updownarrow$ (if $x^\star$ regular point)

$$f(x) - f(x^\star) \geq \underbrace{- \sum_{i \in \mathcal{A}(x^\star)} \nu_i^\star g_i(x)}_{\leq 0} \underbrace{- \sum_{j=1}^{n_h} \lambda_i^\star h_i(x)}_{=0} \geq 0, \qquad x \in \mathcal{D}$$

because $g(x) \leq 0$, $\nu^\star \geq 0$ because $h(x) = 0$

$\rightarrow x^\star$ is a global minimizer. □

<u>Second-order conditions</u>

Similary to the unconstrained case, we can use the Hessian.

$$\nabla^2_{xx} \mathcal{L}$$

We need to check positive semi-definiteness of the Hessian only along feasible directions:

Precisel, we are interestes in this property along

$$\text{Critical Directions} = \{p \mid \underbrace{p \in \mathcal{T}_\mathcal{D}(x^\star)}_{\text{feasible directions}}, \quad \underbrace{\nabla f(x^\star)^T p = 0}_{\substack{\text{directions that cannot} \\ \text{be excluded based on} \\ \text{on first order arguments}}} \}$$

$$\begin{cases} \nabla f(x^\star)^T p < 0 & \rightarrow \quad p \text{ descent direction: } {}^{\substack{\text{already excluded by necessary} \\ \text{condition of order 1}}}, \\ \nabla f(x^\star)^T p > 0 & \rightarrow \quad p \text{ ascent direction: “not harmful”}, \\ \nabla f(x^\star)^T p = 0 & \rightarrow \quad \text{this is what is “new” compared to first order.} \end{cases}$$

Lemma 1.15 (Second order necessary condition)

- $f, g, h \in \mathcal{C}^2$ at $x^\star$
- $x^\star$ local minimizer and regular point
- $(x^\star, \lambda^\star, \nu^\star)$ KKT point (which exists by Lemma 1.13)

Then

$$p^T \nabla^2_{xx} \mathcal{L}(x^\star, \lambda^\star, \nu^\star) p \geq 0 \quad \text{(curvature non-negative along critical directions)}$$

$\forall p \neq 0$ with

- $\nabla h(x^\star)^T p = 0$
- $\nabla g_i(x^\star)^T p \leq 0 \quad \forall i \in \mathcal{A}(x^\star)$ with $\nu_i^\star = 0 \mid p \in \mathcal{T}_{\mathcal{D}}(x^\star)$
- $\nabla g_i(x^\star)^T p = 0 \quad \forall i \in \mathcal{A}(x^\star)$ with $\nu_i^\star > 0 \mid \nabla f(x^\star)^T p = 0$

Lemma 1.16 (Second order suffiecient conditions for local minimizer)

If $(x^\star, \lambda^\star, \nu^\star)$ is a KKT point with

$$p^T \nabla_{xx} \mathcal{L}(x^\star, \lambda^\star, \nu^\star) p > 0$$

for same p as in Lemma 1.15.
Then $x^\star$ is a strict local minimizer.

2 Calculus of Variations

Goal in OC: Find a function that maximizes a functional (function of function) subject to dynamic constraints

In Chapter 1 we characterized solutions to optimization problems over vectors ($\mathbb{R}^n$)

$$\min f(x) \text{ s.t. } x \in \mathcal{D} \subseteq \mathbb{R}^n, \qquad \text{Static problem}$$

- We should introduce "time" or "stages" in the problem

$$\min_{x_1,\ldots,x_N} \sum_{k=1}^{N} f(k, x_k, x_{k-1}) \qquad \text{N coupled stages}$$

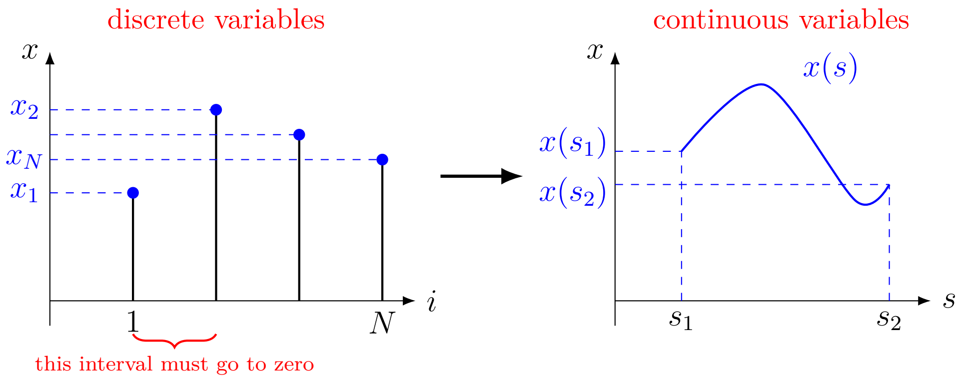
$$\text{s.t. } x_k \in \mathcal{D}_k \subseteq \mathbb{R}^n, \quad k = 1, \dots, N, \quad x_0 \text{ given}$$

equivalent to: (loses structure)

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \to \min_z \; p(z) \qquad z \in \mathcal{Z}, \quad z \in \mathbb{R}^{n \times N}$$

- Continuous-time description of dynamics

  From N stages to continuous time by taking ∞ many stages:



$$\min_{x(\cdot)} \int_{s_1}^{s_2} f\Big(s, x(s), \dot{x}(s)\Big)\, ds$$

$$\text{s.t.} \quad \begin{aligned} & x(s) \in \mathcal{X} \subseteq \mathbb{R}^n, \quad s \in [s_1, s_2], \\ & x(s_1) = x_1, \end{aligned} \quad \left\{ \begin{array}{l} \text{prototypical CV problem} \\ \bullet \text{ no ODE yet} \\ \bullet \text{ opt. variable lives in a function space} \end{array} \right.$$

2.1 Introduction to CV (Calculus of Variations)

Function CLASSES & NORMS

$$\big(\underbrace{V}_{\text{vector space}}, \underbrace{\|\cdot\|}_{\text{norm}}\big) \quad \text{normed vector space}$$

V is the set of vector functions

$$x(s), \quad s \in [s_1, s_2] \text{ taking values in } \mathbb{R}^n, \quad [s_1, s_2] \subseteq \mathbb{R}$$

Two classes:

- $V = \mathcal{C}^1([s_1, s_2], \mathbb{R}^n)$: continuously differentiable functions $x : [s_1, s_2] \to \mathbb{R}^n$
- $\hat{V} = \hat{\mathcal{C}}^1([s_1, s_2], \mathbb{R}^n)$: piecewise continuously differentiable functions

$$x : [s_1, s_2] \to \mathbb{R}^n.$$

Definition 2.1 (Piecewise continuously differentiable functions)

$x : [s_1, s_2] \to \mathbb{R}^n$ is piecewise continuously differentiable (PCD) if

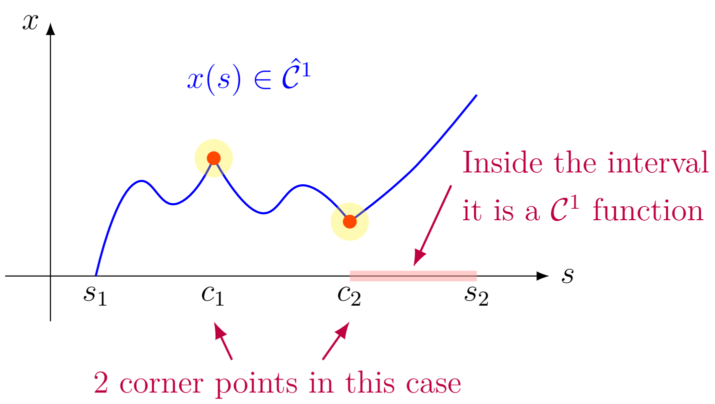
- $x \in \mathcal{C}$ on $[s_1, s_2]$,
- $\exists$ a finite partition $\{c_k\}_{k=0}^{N+1}$ with

$$s_1 = c_0 < c_1 < \cdots < c_{N+1} = s_2,$$

such that $x : [c_k, c_{k+1}] \to \mathbb{R}^n$ is $\mathcal{C}^1$.

That is $x \in \mathcal{C}^1([c_k, c_{k+1}], \mathbb{R}^n), \quad \forall k = 0, 1, \ldots, N.$

Example:



Norms

Definition 2.2 (Strong and weak norms)

<u>Case $V = \mathcal{C}^1$:</u>

- Strong norm (or ∞-norm)

$$\|x\|_\infty := \max_{s_1 \le s \le s_2} \|x(s)\|_{\leftarrow \text{ any norm in } \mathbb{R}^n}$$

- Weak norm (or 1-norm)

$$\|x\|_1 := \|x\|_\infty + \max_{s_1 \le s \le s_2} \|\dot{x}(s)\|_{\leftarrow \text{ any norm in } \mathbb{R}^n}$$

$$(\mathcal{C}^1([s_1, s_2]), \|\cdot\|) \text{ full notation}$$

Note $\forall x \in V, \|x\|_1 \ge \|x\|_\infty$

<u>Case $V = \hat{\mathcal{C}}^1$:</u>

- Strong norm $\to$ same as for $\mathcal{C}^1$
- weak norm

$$\|x\|_1 := \|x\|_\infty + \sup_{s \in \bigcup_{k=0}^{N} (c_k, c_{k+1})} \|\dot{x}(s)\|$$

CV problem

$$\min_{x \in V} \underbrace{J}_{\hookrightarrow \text{ functional } J: V \to \mathbb{R}}(x)$$

$$\text{s.t. } x \in \underbrace{\mathcal{D}}_{\hookrightarrow \text{ admissible set}}$$

$\bar{x} \in \mathcal{D}$ admissible curve for trajectory.

3 forms of J:

- Langrangian form

$$J(x) := \int_{s_1}^{s_2} \underbrace{l}_{\hookrightarrow \text{ running cost, } L: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}}(s, x(s), \dot{x}(s)) ds$$

- Bolza form

$$J(x) \coloneqq \varphi(s_2, x(s_2)) + \int_{s_1}^{s_2} l(s, x(s), \dot{x}(s))ds \qquad \varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

- Mayer form

$$J(x) \coloneqq \varphi(s_2, x(s_2))$$

These 3 forms are (gernerally) interchangeble, e.g. $L \Rightarrow B,\ B \Rightarrow L$.

We will see them again in Chapter 3.

Admissible sets:

- Free problems $\to$ only endpoints are constrained. Example:

$$\mathcal{D} = \{x \in V \mid x(s_1) = x_1,\ x(s_2) = x_2\}, \qquad x_1, x_2 \text{ fixed vectors}$$

- Isoperimetric constraints $\to$ level sets

$$\mathcal{D} = \bigcap_{i=1}^{n_g} \Lambda_i(k_i)$$

$$\Lambda_i(k_i) \coloneqq \{x \in V \mid \int_{s_1}^{s_2} g_i(s, x(s), \dot{x}(s))ds = k_i\}$$

Example:

$$\min_x J(x)$$

$$\text{s.t.} \quad \left.\begin{aligned} \int_{s_1}^{s_2} x^2(s)\, ds = 1 \\ \int_{s_1}^{s_2} x(s)\, ds = 0 \end{aligned}\right\} \qquad \begin{aligned} &n_g = 2, \\ &g_1(s) = x^2(s), \\ &g_2(s) = x(s) \end{aligned}$$

Minimizers for CV problems

Definition 2.3 (Global and local minimizers)

$x^\star \in \mathcal{D}$ is a global minimizer of [CV] if

$$J(x) \geq J(x^\star), \qquad x \in \mathcal{D}$$

$x^\star \in \mathcal{D}$ is a strong local minimizer of [CV] if

$$\exists \varepsilon > 0 \text{ s.t. } J(x) \geq J(x^\star), \quad \forall x \in B_\varepsilon^\infty(x^\star) \cap \mathcal{D}$$

$$B_\varepsilon^\infty := \{y \in V \mid \|x - y\|_\infty \leq \varepsilon\} \quad \text{(strong ball)}$$

$x^\star \in \mathcal{D}$ is a weak local minimizer of [CV] if

$$\exists \varepsilon > 0 \text{ s.t. } J(x) \geq J(x^\star), \quad \forall x \in B_\varepsilon^1(x^\star) \cap \mathcal{D}$$

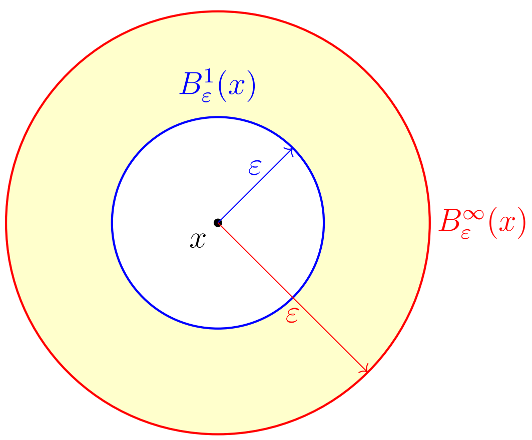
$$B_\varepsilon^1 := \{y \in V \mid \|x - y\|_1 \leq \varepsilon\} \quad \text{(weak ball)}$$

We will often call strong/weak minimizers (without local).

Note: Every strong minimizer is a weak minimizer. In general, the converse is not true.

Why?

$\forall x \in V, \ \forall \varepsilon > 0, \ B_\varepsilon^1(x) \subseteq_\varepsilon^\infty (x)$



$$\|x\|_1 \geq \|x\|_\infty$$

Implication for CV is that a curve x that is better than all elements in $B^1_\varepsilon(x)$ is not necessarily better than all elements in $B^\infty_\varepsilon(x)$.

Example: $s_1 = 0,\ s_2 = 1$

$$J(x) = \int_0^1 \dot{x}^2(s) - \dot{x}^4(s)ds$$
$$\mathcal{D} = \{x \in \hat{\mathcal{C}}^1([0,1]) \mid x(0) = x(1) = 0\}$$
$$\bar{x}(s) = 0, \quad J(\bar{x}) = 0, \quad \bar{x} \text{ is a weak minimum but not strong}$$

Weak minimum

$B^1_\varepsilon(0)$, take $0 < \varepsilon \leq 1$

$$\forall x \in B^1_\varepsilon(0),\ \|\dot{x}(s)\| \leq \varepsilon \qquad \forall s \in [0,1]$$

because x is in the weak ball with radius ε.

$$J(x) = \int_0^1 \underbrace{\dot{x}^2(s)}_{\geq 0}(\underbrace{1 - \dot{x}^2(s)}_{\geq 0 \text{ see above, } \varepsilon \leq 1})ds \geq 0, \quad \forall x \in B^1_\varepsilon$$

$J(\bar{x}) = 0 \rightarrow \bar{x}$ is a weak minimum, but not strong. Try to see why? $\rightarrow$ Find counterexamples.

Existence of solutions for problems of CV:

Weierstrass theorem still holds but compactness $\neq$ closed and bounded in function vector spaces.

A subspace $\mathcal{D}$ of a metric space V is compact, if "every sequence in $\mathcal{D}$ has a subsequence converging to some point in $\mathcal{D}$".

Example:
$$B^\infty_1(0) = \{x \in \mathcal{C}([0,1]) \mid \|x\|_\infty \leq 1\}$$

closed and bounded, not a compact set.

Bottom line: checking Weierstrass in CV can be overly restrictive because our common sets in $\mathcal{D}$ are not compact.

Existence of solutions difficult to guarantee a-priory.

Convexity is a special case where this is possible.

Variations: (extension of perturbations to the cost seen in NLP)

Definition 2.4 (First-variation of a functional)

First variation (Gateaux derivative) of J at $x \in V$ in direction $\xi \in V$ is

$$\delta J(x, \xi) \coloneqq \lim_{\eta \to 0} \frac{J(x+\eta\xi) - J(x)}{\eta} = \left.\frac{\partial J(x+\eta\xi)}{\partial \eta}\right|_{\eta=0}$$

$$J : V \to \mathbb{R}$$

δJ can be interpreted as follows

$$J(x+\eta\xi) = J(x) + \eta\delta J(x;\xi) + \underline{o(\eta)}_{\text{second order term } \lim_{\eta\to 0} \frac{0(\eta)}{\eta} = 0}$$

δJ is functional associated with J and a point x mapping a perturbation ξ into a scalar, representing the variation of J in that direction $\approx$ "directional derivative" for CV.

Example:

$$J(x) = \int_{s_1}^{s_2} x^2(s) ds, \quad V = \mathcal{C}^1$$

δJ? Apply defintion:

$$\frac{J(x+\eta\xi) - J(x)}{\eta} = \frac{1}{\eta}\left[\int_{s_1}^{s_2} (x(s) + \eta\xi(s))^2 - x^2(s) ds\right]$$

$$= 2\int_{s_1}^{s_2} x(s)\xi(s) ds + \eta \int_{s_1}^{s_2} \xi^2(s) ds$$

$$\lim_{\eta \to 0} \quad \longrightarrow \quad \delta J(x;\xi) = 2\int_{s_1}^{s_2} x(s)\xi(s) ds$$

From the defintion, we can see that δJ is a linear operator on V.

$$\delta(J_1 + J_2)(x;\xi) = \delta J_1(x;\xi) + \delta J_2(x;\xi).$$

Moreover, it is a homogeneous operator: $\forall \alpha \in \mathbb{R}$ it holds

$$\delta J(x; \alpha\xi) = \alpha \delta J(x;\xi).$$

Definition 2.5 (Second variation of a functional)

$$\delta^2 J(x;\xi) \coloneqq \left.\frac{\partial^2}{\partial \eta^2} J(x+\eta\xi)\right|_{\eta=0}$$

Interpretation $J(x + \eta\xi) = J(x) + \eta\delta J(x;\xi) + \eta^2\delta^2 J(x;\xi) + o(\eta^2)$

Fundamental Lemma in CV.

Definition 2.6 (Descent direction in CV)

Given V, $J : V \to \mathbb{R}$ Gateaux-differentiable ($\delta J(x;\xi)$ exists) at $\bar{x} \in V$, we call $\xi \in V$ a descent direction for J at $\bar{x}$ if

$$\delta J(\bar{x};\xi) < 0$$

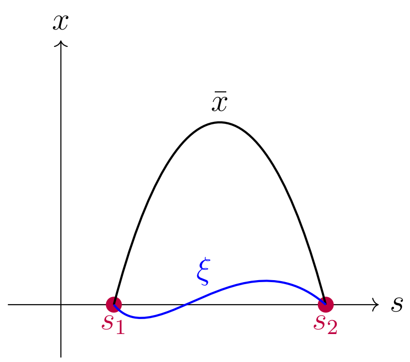
There is a close connection with "descent direction" from NLP.

Definition 2.7 ($\mathcal{D}$-admissible direction)

Given V, $\mathcal{D} \subseteq V$ and $J : V \to \mathbb{R}$.
$\xi \in V$, $\xi \neq 0$ is $\mathcal{D}$-admissible for J at $\bar{x} \in \mathcal{D}$ if

- $\delta J(\bar{x};\xi)$ exists
- $\exists \beta > 0$ s.t. $\bar{x} + \eta\xi \in \mathcal{D}, \quad \forall \eta \in (-\beta, \beta)$

Example:



$\mathcal{D}:\ \bar{x}(s_1) = \bar{x}(s_2) = 0$

ξ is D-admissible

Lemma 2.1 (Negative result for minimizers in unconstrained CV)

$(V, \|\cdot\|)$, J.
Suppose at $\bar{x}\exists$ descent direction $\bar{\xi} \in V$.
Then $\bar{x}$ cannot be a local minimizer for J (neither strong or weak).

Proos: Use definition of 1^{st} variation.

If $\delta J(\bar{x}_i\bar{\xi}) < 0$ then

$$J(\bar{x} + \eta\bar{\xi}) < J(\bar{x}) \quad \forall \eta \in (0, \beta)$$

This comes from

$$J(x + \eta\xi) = J(x) + \eta\xi J(x;\eta) + o(\eta)$$

Thus $\bar{x}$ can't be a local minimizer.

Lemma 2.2 (Geometric necessary condition for a local minimum, Fundamental Lemma of CV)

$(V, \|\cdot\|),\ \mathcal{D} \subseteq V,\ J \to \mathbb{R}$.
Suppose $x^\star \in \mathcal{D}$ is a local minimizer for J on $\mathcal{D}$, then

$$\boxed{\delta J(x^\star;\xi) = 0} \qquad \forall \mathcal{D}\text{-admissible directions at } x^\star$$

Proof: By contradiction:

Case 1: $\delta J(x^\star;\xi) < 0$

By Lemma 2.1 $x^\star$ can't be a local minimizer $\to$ contradiction

Case 2: $\delta J(x^\star;\xi) > 0$

By definition of the $\mathcal{D}$-admissible direction, if ξ is $\mathcal{D}$-admissible.

Then $-\xi$ is also $\mathcal{D}$-admissible.

$$\delta J(x^\star;-\xi) = -\delta J(x^\star;\xi) < 0 \quad \text{(because of linearity)}$$

$\to x^\star$ cannot be a local minimizer (because $\delta J(x^\star;-\xi) < 0$ and $-\xi$ is $\mathcal{D}$-admissible). Thus $\delta J(x^\star;\xi) = 0$.

All the results in the rest of this chapter are $\underbrace{\text{“merely corollaries”}}_{\substack{\text{they turn such an abstract}\\ \text{requirenment into algebraic tests}}}$ of this Lemma.

2.2 Free problems of CV

2.2.1 $V = \mathcal{C}^1$

$$\min_{x(\cdot)} \int_{s_1}^{s_2} l(s, x(s), \dot{x}(s))ds \qquad [CV - P1]$$

$$\text{s.t. } x \in \{y \in \mathcal{C}^1([s_1, s_2]) \mid \underbrace{y(s_1) = x_1,\ y(s_2) = x_2}_{\substack{\text{special case of “free problems”}\\ \text{we fix the endpoint to } x_2}}\}$$

Lemma 2.3

Suppose $x^\star \in \mathcal{C}^1([s_1, s_2])$ is a weak minimum of $[CV - P1]$.
Then

$$\frac{d}{ds} l_{\dot{x}_i}(s, x^\star(s), \dot{x}^\star(s)) = l_{x_i}(s, x^\star(s), \dot{x}^\star(s)), \quad \forall s \in [s_1, s_2],\ i = 1, \dots n$$

with $l_{\dot{x}_i} = \frac{\partial l}{\partial \dot{x}_i}$ and $l_{x_i} = \frac{\partial l}{\partial x_i}$ (Euler equations).

This is a set of nonlinear ordinary time-varying second-order differential equations. Their solutions are candidate local minimizers of $[CV - P1]$.

solution $x(s): \ s \in [s_1, s_2]$

also called "stationary solutions" of the corresponding CV problem. Why?

$$\text{Because } \delta J(x^\star; \xi) = 0 \quad \forall \xi \ \mathcal{D}\text{-admissible}$$

Proof: The idea is to derive algebraic conditions that are sufficient to guarantee $\delta J(x^\star; \xi) = 0$.

First step is to write $\delta J: \ \eta \in \mathbb{R},\ \xi \in \mathcal{C}^1$

$$\frac{\partial}{\partial \eta} J(x^\star + \eta\xi) \underset{\text{Leibniz rule}}{=} \int_{s_1}^{s_2} \frac{\partial}{\partial \eta} l(s, x^\star + \eta\xi, \dot{x}^\star + \eta\dot{\xi})\, ds$$

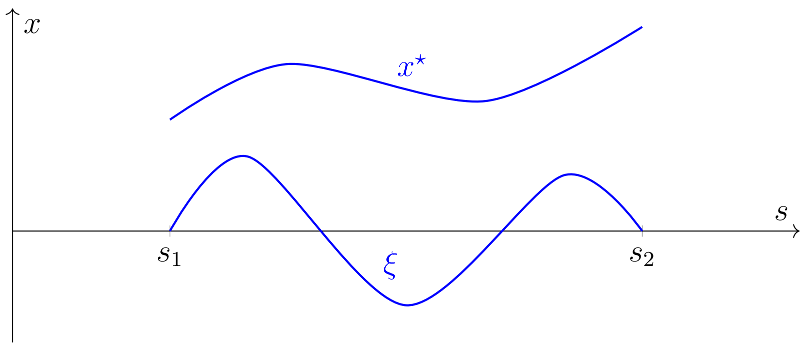
$$= \int_{s_1}^{s_2} l_x \left[x^\star + \eta\xi\right]^T \xi + l_{\dot{x}} \left[x^\star + \eta\xi\right]^T \dot{\xi}\, ds$$

with $l_z[y] := l_z(s, y, \dot{y})$.

Take $\eta \to 0 \quad \delta J(x^\star; \xi) = \int_{s_1}^{s_2} \underbrace{l_x \left[x^\star\right]^T + l_x \left[x^\star\right]^T \dot{\xi}}_{\text{integrand is a continuous function}}\, ds$

$\rightarrow$ first-variation exists $\forall \xi \rightarrow J$ is Gateaux-differentiable

We want to obtain conditions enforcing $\delta J = 0 \ \forall \mathcal{D}$-admissible ξ. This means $\xi(s_1) = \xi(s_2) = 0$ and $\xi \in \mathcal{C}^1$.



To do this, we select n perturbations $\xi^{(i)}(i = 1, \dots n)$ defined as follows

$$\xi^{(i)} = \begin{bmatrix} \xi_1^{(i)} \\ \vdots \\ \xi_n^{(i)} \end{bmatrix}$$

- $\xi_j^{(i)} = 0, \quad j \neq i$
- $\xi_i^{(i)}$ arbitrary but not identically zero with $\xi_i^{(i)}(s_1) = \xi_i^{(i)}(s_2) = 0$.

We replace those n perturbations in the equation $\delta J = 0$

$$\forall i \in \{1, \dots, n\}, \quad 0 = \delta J(x^\star, \xi^{(i)}) = \int_{s_1}^{s_2} \left[l_{x_i}[x^\star]\xi_i + l_{\dot{x}_i}[x^\star]\dot{\xi}_i \right] ds.$$

$$= \int_{s_1}^{s_2} l_{\dot{x}_i}[x^\star]\dot{\xi}_i \, ds + \int_{s_1}^{s_2} \underbrace{\frac{d}{ds}\left[\left(\int_{s_1}^{s} l_{\dot{x}_i}[x^\star] d\sigma\right)\right]}_{v'} \underbrace{\dot{\xi}_i}_{u} \, ds$$

integral by parts $\int_a^b uv' = [uv]_a^b - \int_a^b u'v$

$$= \int_{s_1}^{s_2} l_{\dot{x}_i}[x^\star]\dot{\xi}_i \, ds + \underbrace{\left[\xi_i \int_{s_1}^{s} l_{x_i}[x^\star] d\sigma\right]_{s_1}^{s_2}}_{=0 \text{ because } \mathcal{D}\text{-admissible}} - \int_{s_1}^{s_2} \left[\int_{s_1}^{s} l_{x_i}[x^\star] d\sigma\right] \dot{\xi}_i \, ds$$

$$= \int_{s_1}^{s_2} \left[l_{\dot{x}_i}[x^\star] - \int_{s_1}^{s_2} l_{x_i}[x^\star] d\sigma \right] \dot{\xi}_i \, ds$$

DuBois-Reymond's Lemma if

- $h(s)$ continuous in $[s_1, s_2]$
- $\displaystyle\int_{s_1}^{s_2} h(s)\, \dot{y}(s)\, ds = 0$
- $y(s_1) = y(s_2) = 0$

$\Rightarrow \quad h(s)$ constant in $[s_1, s_2]$

This applies to our problem $\quad y \equiv \dot{\xi}_i$

$$h \equiv L_{\dot{x}_i}[x^\star] - \int_{s_1}^{s} C_{x_i}[x^\star]\, d\sigma$$

$$\Rightarrow \quad l_{\dot{x}_i}[x^\star] - \int_{s_1}^{s} l_{x_i}[x^\star]\, d\sigma = c_i, \qquad \forall s \in [s_1, s_2],\ i = 1, \dots, n,\ c_i: \text{ constants}$$

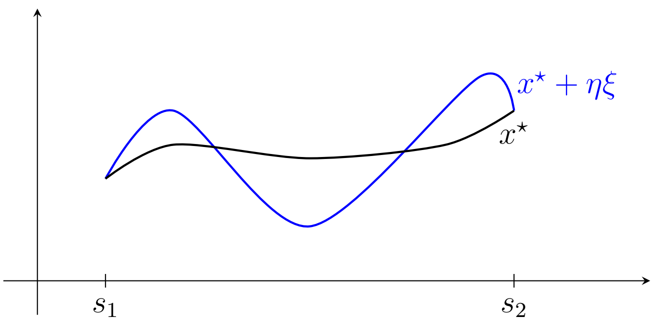
Note that this shows $l_{\dot{x}_i} \in \mathcal{C}^1$

What we obtain are the "integral" Euler equations (EE)

If we take $\frac{d}{ds}$ we obtain the EE.

Remarks

1. Necessary conditions for local minimizers ($\rightarrow$ weak)



as $\eta \rightarrow 0$

$(x^\star + \eta\xi)$ and $x^\star$ differentiable both in magnitude and in derivative.

$\Rightarrow$ "$(x^\star + \eta\xi)$ is inside the weak ball of $x^\star$"

EE $\Rightarrow$ detect weak minimizers

2. To solve ODE we need Boundary conditions (BC).

BC come from the admissible set $x(s_1) = x_1,\ x(s_2) = x_2$

$\rightarrow 2n$ equations for a 2^{nd} order ODE in n unknowns.

Two Point Boundary Value Problem (TPBVP)

3. There exists a reformulation of EE:

$$p(s) := l_{\dot{x}}(s, x, \dot{x}) \quad \text{Momentum associated with a given } x.$$

$$H(s, x, \dot{x}, p) := -l(s, x, \dot{x}) + \dot{x}^T p. \quad \text{Hamiltonian.}$$

EE can be rewritten as:

$$\dot{x} = H_p(s, x, \dot{x}, p).$$

$$\dot{p} = -H_x(s, x, \dot{x}, p).$$

Canonical equation and x, p canonical Variables.

An immediate benefit of this reformulation is that we easily see the following special cases.

[A] $l(x, \dot{x})$. l does not depend on s.

$$\frac{d}{ds} H = -l_x^T \dot{x} - l_{\dot{x}}^T \ddot{x} + \dot{x}^T \dot{p} + \ddot{x}^T p = \dot{x}^T \underbrace{\left(\underbrace{\frac{dl_{\dot{x}}}{ds}}_{\dot{p}} - l_x \right)}_{=0 \text{ because of EE}} = 0$$

$$H = \text{const.} := c_1 \quad \text{on stationary solutions.}$$

[B] $l(s, \dot{x})$ no dependence on x.

$$\frac{d}{ds} p = \dot{p} = 0.$$

$$p = c_2.$$

Lemma 2.4 (Second-order necessary conditions for $(CV - P1)$)

Assume $l \in \mathcal{C}^2$.
If $x^\star$ is a weak minimizer of $[CV - P1]$, then

1. $x^\star$ satisfies EE
2. $\nabla^2_{\dot{x}\dot{x}} l(s, x^\star, \dot{x}^\star) \succeq 0, \quad \forall s \in [s_1, s_2]$.
   Legendre condition

Lemma 2.5 (First-order sufficient condition for global minimizers)

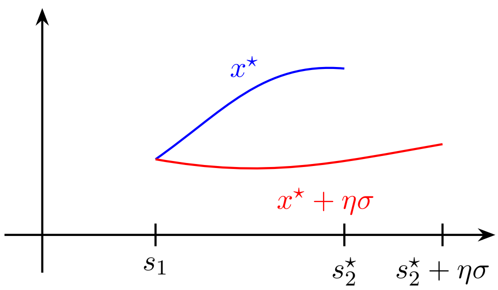
Assume that $l(s, x, \dot{x})$ is jointly convex in x and $\dot{x}$.
If $x^\star \in \mathcal{D}$ satisfies the EE, then $x^\star$ is a global minimizer.
Every global minimizer is a strong local minimizer.

Free end-point problems

$$\mathcal{D} = \{(x, s_2) \in \mathcal{C}^1\left([s_1, \infty) \times (s_1, \infty)\right) \mid x(s_1) = x_1, x(s_2) \text{ free}, s_2 \text{ free}\}$$

First-variation also suitably redefined:

$$\delta J(x, s_2; \xi, \sigma) := \lim_{\eta \to 0} \frac{J(x + \eta\xi, s_2 + \eta\sigma) - J(x, s_2)}{\eta}$$
$$= \frac{\partial}{\partial \eta} J(x + \eta\xi, s_2 + \eta\sigma)\Big|_{\eta=0}$$



[CV-P2] $\quad \min_{x(\cdot), s_2} \varphi(s_2, x(s_2)) + \int_{s_1}^{s_2} l(s, x(s), \dot{x}(s))\, ds$

$$\text{s.t.} \quad x \in \{y \in \mathcal{C}^1([s_1, s_2]) \mid y(s_1) = x_1\}, \quad s_2 \in (s_1, \infty)$$

Lemma 2.6 (First order necessary conditions for local minimizers of $(CV - P2)$)

Suppose $(x^\star, s_2^\star)$ is a weak minimizer of $[CV - P2]$, then

1. $x^\star$ solves EE on $[s_1, s_2^\star]$
2. The transversal conditions

   A) $[l_{\dot{x}} + \varphi_x]\Big|_{x=x^\star, s=s_2^\star} = 0 \quad \leftarrow$ use if $x(s_2)$ free

   B) $[l - \dot{x}^T l_{\dot{x}} + \varphi_s]\Big|_{x=x^\star, s=s_2^\star} = 0 \quad \leftarrow$ use if s_2 free

The transversal conditions "replace" the BC at s_2 because in $[CV - P2]$ we have none.

- 2A is a vector equation with n components $\rightarrow$ it replaces "$x(s_2) = x_2$"
- 2B is a scalar equation $\rightarrow$ it provides an equation to find s_2

"Partially" free end-point problems

Case 1 s_2 free, $x(s_2) = x_2$ given

Same as Lemma 2.6, but we only use 2B

(2A not needed bc the BC on $x(s_2)$ is given)

Case 2 s_2 fixed, $x(s_2)$ free

Same as Lemma 2.6, but we only use 2A

Case 3 $s_2, x(s_2)$ are free but related through $\psi : \mathbb{R} \rightarrow \mathbb{R}^n, \quad x(s_2) = \psi(s_2)$

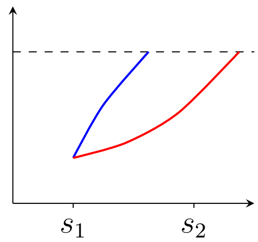
We do not need 2A $\rightarrow$ it is replaced by $x(s_2) = \psi(s_2)$.

We need an extension to 2B.

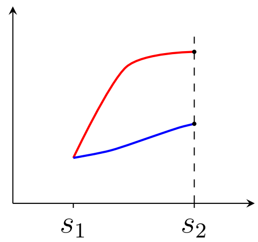
$$2B' : \left[l + l_{\dot{x}}^T(\dot{\psi} - \dot{x}) + \varphi_s + \varphi_x^T \dot{\psi} \right]_{x=x^\star, s=s_2^\star} = 0$$

Note $\psi(s_2) = x_2$

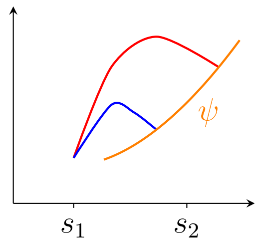
When $\dot{\psi} = 0$, 2B' collapses to 2B.



Case 1



Case 2

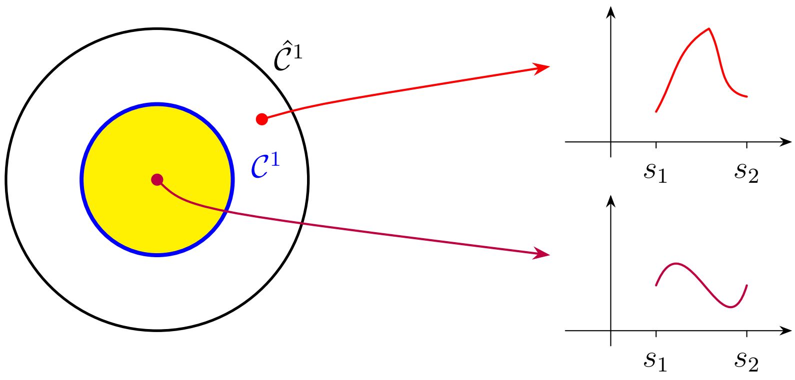


Case 3

2.2.2 $V = \hat{\mathcal{C}}^1$ (Piecewise-continuously differentiable case)

Why?

1. "$\mathcal{C}^1 \subseteq \hat{\mathcal{C}}^1$" by enlarging our search space we can achieve better costs

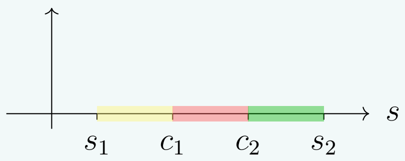


2. We can study conditions that are necessary for strong minimizers (only)

Lemma 2.7 (First-order necessary conditions for strong minimizers)

Consider $[CV - P1]$ with $x \in \hat{\mathcal{C}}^1([s_1, s_2])$.
Suppose $x^\star$ with corner points $\{c_i\}_{i=1}^N$ is a strong minimizer.
Then:

- $x^\star$ solves the EE inside the intervals (c_i, c_{i+1}), $\quad i = 0, 1, \ldots, N$
  $c_0 = s_1,\ c_{N+1} = s_2$



- At every corner point c_i the following continuity condition holds:

$\boxed{A}$ $l_{\dot{x}}(c_i^-) = l_{\dot{x}}(c_i^+)$

with $l(c_i^\pm) := l(c_i, x^\star(c_i), \dot{x}^\star(c_i^\pm))$

$\boxed{B}$ $\left[-l(c_i^-) + \dot{x}^\star(c_i^-)^T l_{\dot{x}}(c_i^-)\right] = \left[-l(c_i^+) + \dot{x}^\star(c_i^+)^T l_{\dot{x}}(c_i^+)\right]$

with $l_{\dot{x}}(c_i^\pm) := l_{\dot{x}}(c_i, x^\star(c_i), \dot{x}^\star(c_i^\pm))$

$\boxed{A}$ and $\boxed{B}$ are also called Weierstrass-Erdmann-conditions (WE).

$\boxed{A}$ is the 1^{st} WE-condition and is also necessary for weak minimizers.

$\boxed{B}$ is the 2^{nd} WE-condition and is <u>only</u> necessary for strong minimizers.

$\boxed{A}$ prescribes continuity of p (momentum)

$\boxed{B}$ prescribes continuity of H (Hamiltonian)

In the $\mathcal{C}^1$ case we need (and have) $2n$ boundary conditions.

In the $\hat{\mathcal{C}}^1$ case we need $2(N+1)n$ boundary conditions.

$$\left.\begin{array}{l} n \text{ at } c_0 = s_1 \\ n \text{ at } c_1^- \\ n \text{ at } c_1^+ \\ n \text{ at } c_2^- \\ \quad \vdots \\ n \text{ at } c_N^+ \\ n \text{ at } c_{N+1} = s_2 \end{array}\right\} 2(N+1)n \text{ in total.}$$

$\rightarrow$ The Problem $[CV-P1]$ still gives you only $2n$ conditions $x(s_1) = x_1,\ x(s_2) = x_2$.

$$\underbrace{2(N+1)n}_{\text{required}} - \underbrace{2n}_{\text{given by the BC}} = 2Nn$$

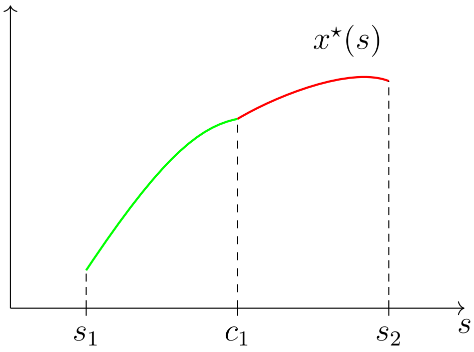
We have N corner points. At each c_i we can enforce:

- continuity of x $\quad$ (n conditions)
- $\boxed{A}$: continuity of p $\quad$ (n conditions)

$\rightarrow$ The problem is closed

Proof:

$x^\star \in \hat{\mathcal{C}}^1$ with 1 corner point $c_1 \in (s_1, s_2)$.



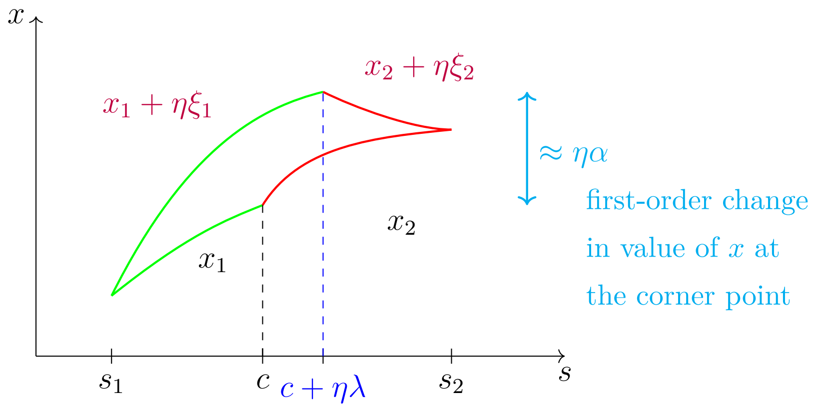
I will drop * from x.

$$x_1 : [s_1, c] \to \mathbb{R}^n \quad \xi_1 \text{ is the perturbation to } x_1$$

$$x_2 : [c, s_2] \to \mathbb{R}^n \quad \xi_2 \text{ is the perturbation to } x_2$$

The location of c can also be perturbed

$$c \to c + \eta\lambda$$



$$\begin{aligned} J(x + \eta\xi; c + \eta\lambda) &= \int_{s_1}^{s_2} l[x + \eta\xi] ds \\ &= \underbrace{\int_{s_1}^{c+\eta\lambda} l[x + \eta\xi] ds}_{J_1} + \underbrace{\int_{c+\eta\lambda}^{s_2} l[x + \eta\xi] ds}_{J_2} \end{aligned}$$

To show that (x, c) is optimal, δJ must be zero.

In fact $\delta J_1 = \delta J_2 = 0 \rightarrow x_1$ and x_2 are optimal in their intervals.

Enforcing $\delta J = 0$ results in this expression. (as well as continuity in the new corners $c + \eta\lambda$)

$$\forall \alpha, \forall \lambda : \underbrace{\left[l_{\dot{x}}(c_i^-) - l_{\dot{x}}(c_i^+) \right]}_{\boxed{A}} \alpha - \underbrace{\left[-l(c_i^-) - \dot{x}^\star(c_i^-)^T l_{\dot{x}}(c_i^-) + l(c_i^+) + \dot{x}^\star(c_i^+)^T l_{\dot{x}}(c_i^+) \right]}_{\boxed{B}} \lambda = 0$$

with α related to change in function value at the corner point and λ as the change of location of the corner points.

1. Because α, λ are independent, this means that $\boxed{A}$, $\boxed{B}$ must hold. The proof shows why only $\boxed{B}$ is necessary for strong minima.

   When c can change the perturbed curve

   $$(x + \eta\xi)$$

   is not in the weak ball of x.

   In the interval $[c, c + \eta\lambda]$ we have that

   $$\|(x + \eta\xi) - x\|_1 \approx \underbrace{\left\| \dot{x}(c^-) - \dot{x}(c^+) \right\|}_{\neq 0 \quad \forall \eta \neq 0}.$$

   This shows that the perturbation is outside of the weak ball around x. It is instead in the strong ball around x because $\|\cdot\|_\infty$ does not look at derivatives of the functions.

   If instead $\boxed{\lambda = 0}$ (no perturbation to c) then $\|(x + \eta\xi) - x\|_1 \approx \eta$.

   $\rightarrow$ The perturbed area is inside the weak ball of x.

2. Condition $\boxed{A}$ is not surprising after all.

   Recall the proof of EE, you can see, that the integral version

   $$l_{\dot{x}_i}[x^\star] - \int_{s_1}^{s} l_{x_i}[x^\star] d\sigma = c_i$$

   This shows already that $l_{\dot{x}}$ is always continuous if $x^\star$ satisfies the EE.

Definition 2.8 (Weierstrass excess function)

$$E(s, x, \dot{x}, w) := l(s, x, w) - \Big[l(s, x, \dot{x}) + (w - \dot{x})^T l_{\dot{x}}(s, x, \dot{x})\Big]$$

interpretation: difference between $l(s, x, w)$ and its first order approximation around $w = \dot{x}$.

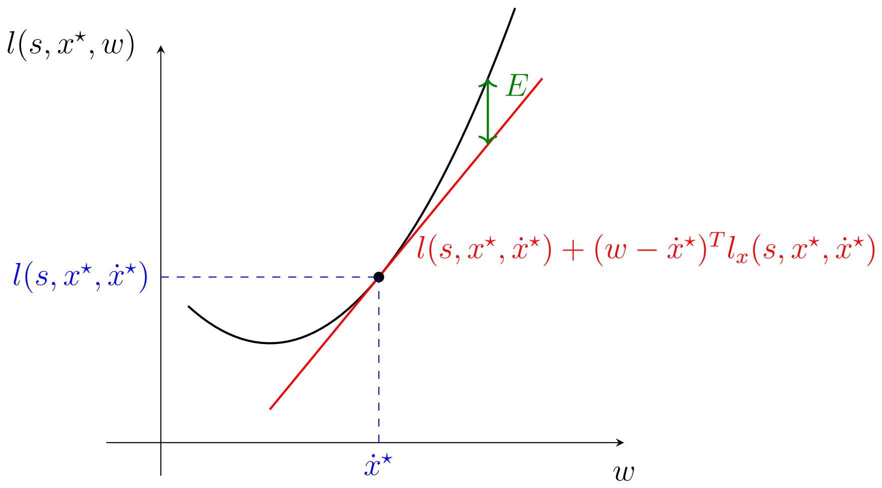
Lemma 2.8 (First-order necessary conditions for strong minimizers - the Weierstrass condition)

Consider $[CV - P1]$ $x \in \hat{\mathcal{C}}^1$.
Suppose $x^\star$ with corner point $\{c_i\}_{i=1}^N$ is a strong minimizer.
Then

- $x^\star$ solves EE inside the intervals $(c_i, c_{i+1}), \quad i = 0, 1, \ldots, N$
- $E(s, x, \dot{x}, w) \geq 0 \quad \forall w \in \mathbb{R}^n, \quad \forall s \in [s_1, s_2]$ except corner points

w condition implies $\boxed{A}$ and $\boxed{B}$ from previous Lemma (it is a stronger condition).

interpretation: for a given solution $x^\star$, we draw for fixed $s \in [s_1, s_2]$ the following curve:



This can be interpreted as a convexity requirement of the function $l(s, x^\star, \cdot)$ for fixed $s, x^\star$.

So this is a weaker requirement than joint convexity of l
$\rightarrow$ we only need to check that at $x^\star$

The Weierstrass condition can equivalently be written as a maximization condition on H:

$$\underbrace{E(s, x^\star, \dot{x}^\star, w) \geq 0 \quad \Leftrightarrow}_{\text{substitude definition of } E \text{ in } E \geq 0} \quad H(s, x^\star, \dot{x}^\star, p^\star) - H(s, x^\star, w, p^\star) \geq 0 \qquad \forall w \in \mathbb{R}^n$$

$$\rightarrow H(s, x^\star, \cdot, p^\star) \text{ has a maximum at } w = \dot{x}^\star$$

2.3 Isoperimetric constraints

$[CV - P3]$ $\quad \min_{x(\cdot)} \int_{s_1}^{s_2} l(s, x, \dot{x}) ds$

$$\text{s.t. } x \in \left\{ y \in \mathcal{C}^1 \mid \underbrace{\int_{s_1}^{s_2} g_i(s, y(s), \dot{y}(s)) ds = k_i}_{G_i(y): V \to \mathbb{R}, \quad k_i \in \mathbb{R}}, \quad i = 1, \dots, n_g; \; y(s_1) = x_1; \; y(s_2) = x_2 \right\}$$

Lemma 2.9

Consider $[CV - P3]$ and assume the following regularity condition

$$\text{Det}\left(G(\{\xi_i\}_{i=1}^{n_g})\right) \neq 0$$

$$(G(\{\xi_i\}_{i=1}^{n_g})) := \begin{bmatrix} \delta G_1(x^\star, \xi_1) & \cdots & \delta G_1(x^\star, \xi_{n_g}) \\ \vdots & \ddots & \vdots \\ \delta G_{n_g}(x^\star, \xi_1) & \cdots & \delta G_{n_g}(x^\star, \xi_{n_g}) \end{bmatrix}$$

for n_g independent directions $\{\xi_i\}_{i=1}^{n_g} \in V$.

If $x^\star$ is is a local minimizer of J, then $\exists \lambda \in \mathbb{R}^{n_g}$ such that

$$\frac{d}{ds} \mathcal{L}_{\dot{x}_i}(s, x^\star(s), \dot{x}^\star(s)) = \mathcal{L}_{x_i}(s, x^\star(s), \dot{x}^\star(s)), \quad \forall s \in [s_1, s_2], \quad i = 1, \dots, n$$

$\mathcal{L}(s, x, \dot{x}) := l(s, x, \dot{x}) + \lambda^T g(s, x, \dot{x}),$ Lagrangian

$$g := \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n_g} \end{bmatrix}^T$$

3 The CV approach to optimal control

3.1 Intro to CV problems

$$[OC] \qquad \begin{aligned} &\min_{u\in V} \ J(u) \\ &\text{s.t. } \dot{x} = f(t,x,u), \quad x(t_0) = x_0, \quad t \in [t_0, t_f] \\ &(t_f, x(t_f)) \in S \subseteq (t_0, \infty) \times \mathbb{R}^{n_x} \end{aligned}$$

Remarks:

- t_0, t_f: initial and final time: $\begin{array}{l} t_0 \in (-\infty, \infty) \\ t_f \in (t_0, \infty) \text{ finite horizon problem} \\ t_f = +\infty \text{ infinite horizon problem} \end{array}$

  t_0 always given

  t_f can be given or free variable (like s_2 in CV)

- $f : [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$

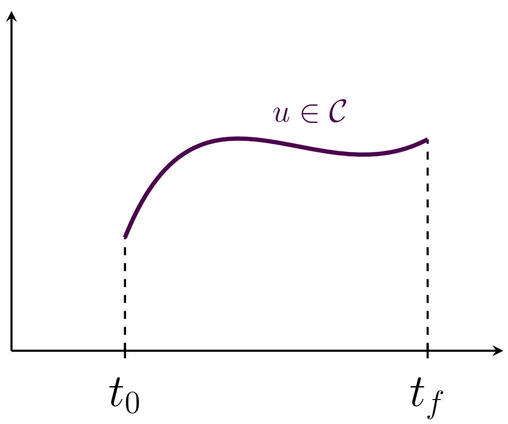
- $(V, \|\cdot\|)$ normed vector space where u belongs

$$u : [t_0, t_f] \to \mathcal{U} \subseteq \mathbb{R}^{n_u}, \qquad \mathcal{U} \subset \mathbb{R}^{n_u} \text{ if we have input contraints}$$

2 classes of functions:

- $V = \mathcal{C}([t_0, t_f], \mathcal{U})$ continuous

- $V = \hat{\mathcal{C}}([t_0, t_f], \mathcal{U})$ piecewise continuous

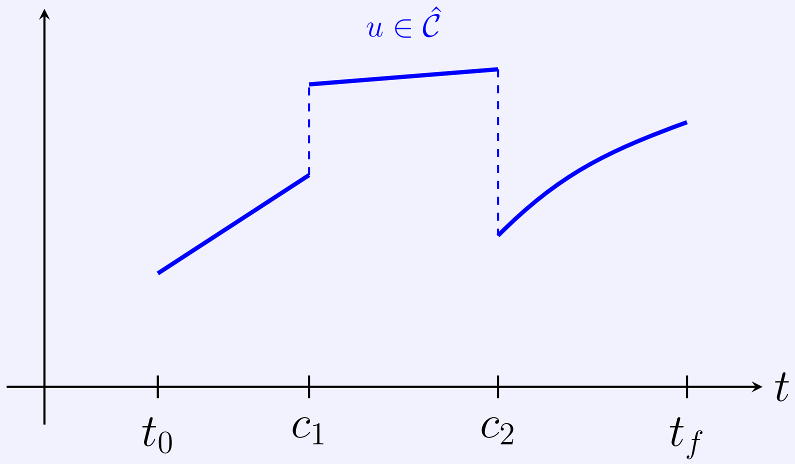


Definition 3.1 (Piecewise continuous function $\hat{\mathcal{C}}$)

$u \in \hat{\mathcal{C}}$ if there is a finite partition $\{c_k\}_{k=0}^{N+1}$ with $t_0 = c_0 < c_1 < \cdots < c_N < c_{N+1} = t_f$ such that

$$u : [c_k, c_{k+1}] \to \mathbb{R}^{n_u} \text{ is continuous}$$

$$\{c_k\}_{k=1}^{N} : \text{ corner points}$$



- $J : V \to \mathbb{R}$
  - Lagrangian form

$$J(u) = \int_{t_0}^{t_f} l(t,x,u)\,dt, \qquad l\text{: running cost}$$

  - Bolza form

$$J(u) = \varphi(t_f, x(t_f)) + \int_{t_0}^{t_f} l(t,x,u)\,dt, \qquad \varphi\text{: terminal cost}$$

  - Mayer form

$$J(u) = \varphi(t_f, x(t_f))$$

  These terms are fully interchangeable: We can go from one to the others by reformulating the problem.

  $L \Rightarrow M$

- Introduce fictitious states: x_l evolving according to $\begin{array}{l} \dot{x}_l = l(t,x,u) \\ x_l(t_0) = 0 \end{array}$

Our new system $\tilde{x} = \begin{bmatrix} x \\ x_l \end{bmatrix}$

$$J(u) = \int_{t_0}^{t_f} l(t, x, u)\, dt = \varphi(t_f, \tilde{x}(t_f)) = x_l(t_f)$$

$M \Rightarrow L$?

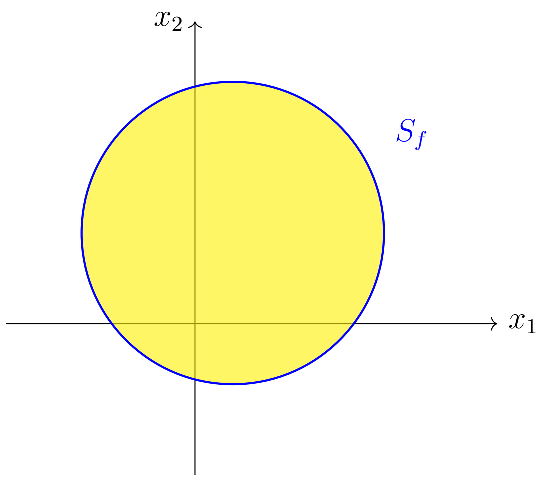
- S is the target set. Examples:
  - free-time free-end point case:

$$S = \underbrace{(t_0, \infty)}_{t_f} \times \underbrace{\mathbb{R}^{n_x}}_{x(t_f)}$$

  - free-time constrained endpoint case:

$$S = (t_0, \infty) \times S_f, \qquad S_f \subseteq \mathbb{R}^{n_x}$$

  E.g. $S_f = \{x_f\}$



  - fixed-time fixed-end point case:

$$S = \{t_f\} \times \{x_f\}$$

- Standing assumptions:
  - l is $\mathcal{C}$ in (t, x, u) and $\mathcal{C}^1$ in x
  - f is $\mathcal{C}$ in (t, x, u) and $\mathcal{C}^1$ in x

In the first part of section 3 we also assume f, l are $\mathcal{C}^1$ in u.

Definition 3.2 (Strong and weak norms)

$V = \mathcal{C}([t_0, t_f], U)$

- $\|\cdot\|_\infty$ strong norm

$$\|u\|_\infty := \max_{t_0 \le t \le t_f} \|u(t)\|$$

- $\|\cdot\|_1$ weak norm

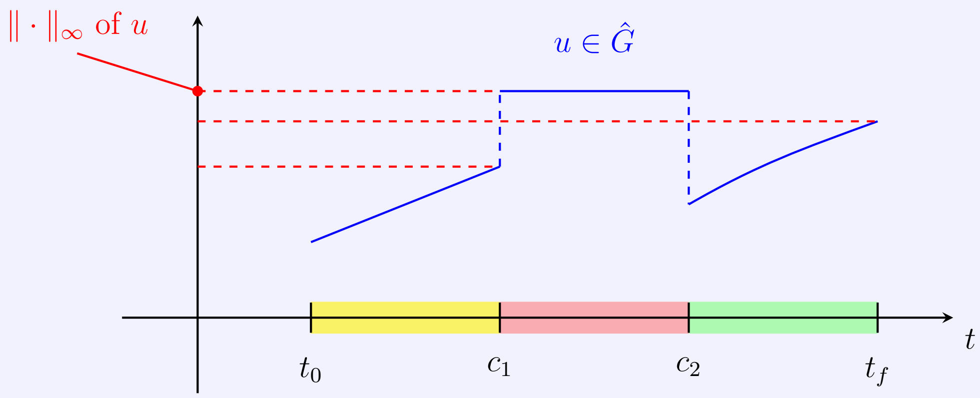
$$\|u\|_1 := \begin{cases} \|u\|_\infty + \max\limits_{t_0 \le t \le t_f} \|\dot{u}(t)\| & \text{if } u \in \mathcal{C}^1 \\ \|u\|_\infty + \sup\limits_{t \in \bigcup_{k=0}^{N} (\hat{c}_k, \hat{c}_{k+1})} \|\dot{u}(t)\| & \text{if } u \in \hat{\mathcal{C}}^1 \end{cases}$$

where $\{\hat{c}_k\}$ are corner points of $\dot{u}$.

$V = \hat{\mathcal{C}}([t_0, t_f], U)$

- $\|\cdot\|_\infty$ strong norm

$$\|u\|_\infty := \sup_{t \in \bigcup_{k=0}^{N} (c_k, c_{k+1})} \|u(t)\|$$



- $\|\cdot\|_1$ weak norm: same rationale

Constraints

- point constraints: Constraints on x/u on specific time points

$$\Psi_1(t, x(t), u(t)) \begin{cases} = 0 \\ \ge 0 \end{cases} \text{at } t = \bar{t}$$

- path constraints:

$$\Psi_2(t, x(t), u(t)) \begin{cases} = 0 \\ \geq 0 \end{cases} \quad \forall t \in [t_1, t_2]$$

- isoperimetric constraints:

$$\int_{t_0}^{t_f} g(t, x, u)\, dt \leq G$$

Ensuring constraint satisfaction in OC problems is hard.

In this course:

- point constraints only at $t = t_f$
- path constraints only on $u \to u : [t_0, t_f] \to \mathcal{U}, \qquad \mathcal{U} = \{u : \Psi_2(t, u) = 0\}$
- isoperimetric constraints

If u satisfies constraints it is called "admissible control" and is denoted by $u \in \mathcal{D}$

Definition 3.3 (Global and local minima)

Admissible control $u^\star(\cdot)$ is a

global minimizer of $[OC]$ if

$$J(u) \geq J(u^\star), \qquad \forall u \in \mathcal{D}$$

strong local minimizer if

$$\exists \epsilon > 0 \text{ s.t. } J(u) \geq J(u^\star), \qquad \forall u \in \mathcal{D} \cap B_\epsilon^\infty(u^\star)$$

is a weak local minimizer if

$$\exists \epsilon > 0 \text{ s.t. } J(u) \geq J(u^\star), \qquad \forall u \in \mathcal{D} \cap B_\epsilon^1(u^\star)$$

Reflection: $u^\star$ is an open-loop controller. We see this from how we defined the OC problem here ("optimizer over $u(\cdot) : [t_0, t_f] \to U$") which are just functions of time. In section 4 we change viewport and study closed-loop optimal controller $u(\underset{t}{\cdot}, \underset{x}{\cdot})$ with Dynamic Programming.

3.2 Unconstrained problems and weak minima

$[OC - P1]$ $$\min_{u(\cdot)\in V} \int_{t_0}^{t_f} l(t,x,u)\,dt$$

$$\dot{x} = f(t,x,u), \quad x(t_0) = x_0, \quad S = \{t_f\} \times \mathbb{R}^{n_x}$$

3.2.1 $V = \mathcal{C}([t_0, t_f])$

Note on regularity of solution of $\dot{x} = f(t,x,u)$:

if $u \in \mathcal{C}, f \in \mathcal{C}$, then $\dot{x} \in \mathcal{C} \Leftrightarrow x \in \mathcal{C}^1$

if $u \in \hat{\mathcal{C}}, f \in \mathcal{C}$, then $\dot{x} \in \hat{\mathcal{C}} \Leftrightarrow x \in \hat{\mathcal{C}}^1$

Lemma 3.1 (First-Order Necessary Conditions for unconstrained OC)

Suppose $u^\star$ is a weak minimum of $[OC - P1]$ and $x^\star \in \mathcal{C}^1$ the associated response. Then $\exists \lambda^\star \in \mathcal{C}^1$ (called adjoint or costate) such that $(x^\star, u^\star, \lambda^\star)$ satisfies:

$$\left.\begin{aligned} &\dot{x}^\star = f(t,x^\star,u^\star), \quad x^\star(t_0) = x_0 \\ &\dot{\lambda}^\star = l_x(t,x^\star,u^\star) - f_x(t,x^\star,u^\star)\lambda^\star, \quad \underset{\text{not if } x(t_f) \text{ given}}{\underset{\uparrow}{\lambda^\star(t_f) = 0}} \\ &0 = -l_u(t,x^\star,u^\star) + f_u(t,x^\star,u^\star)\lambda^\star, \quad \forall t \in [t_0, t_f] \end{aligned}\right\} \text{Euler Lagrange equations (ELE)}$$

where $[f_x]_{i,j} = \frac{\partial f_j}{\partial x_i}, \quad [f_u]_{i,j} = \frac{\partial f_j}{\partial u_i}$.

$\underset{x}{n_x} + \underset{\lambda}{n_x} + \underset{u}{n_u}$ unknowns and $2n_x ODE + n_u$ algebraic equations $\Rightarrow$ Problem is closed

Proof: Notation: I will drop * from x, u, λ.

Rationale: Cast problem $[OC - P1]$ as a CV problem.

Then use the fundamental Lemma of CV.

What is the difference between OC and CV? $\dot{x} = f(t,x,u)$

Conceptional step: given u (candidate), x is fixed. In other words, think of x as $x(t,u)$.

$\rightarrow$ Rewrite J of $[OC - P1]$ as follows:

$$J(u) = \int_{t_0}^{t_f} l(t, x(t,u), u(t))\,dt$$

$$= \int_{t_0}^{t_f} l(t, x(t,u), u(t)) + \underbrace{\lambda^T(t)}_{\mathcal{C}^1 \text{ function}} \Big(\underbrace{\dot{x}(t,u) - f(t, x(t,u), u(t))}_{=0} \Big) \, dt$$

This functional is equivalent to the one in $[OC - P1]$ and it is only a function of u. "There's no more x".

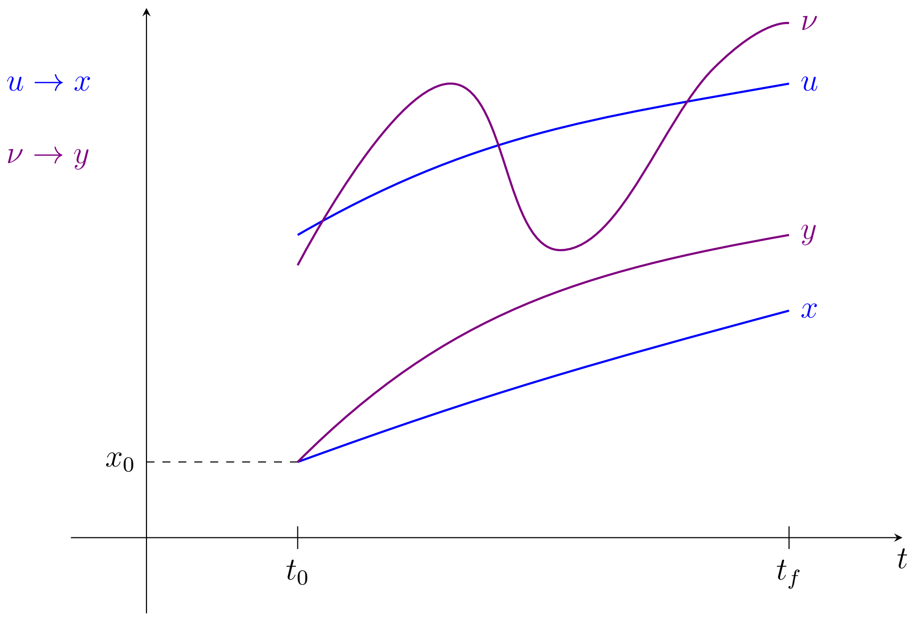
Thus we now have the following CV problem

$$\min_{u(\cdot)} \int_{t_0}^{t_f} l(t, x(t,u), u) + \lambda^T(t) \ldots \, dt, \quad \text{There is only } u \text{ inside } J(u)$$

Step 2: enforce $\delta J = 0$ for this CV problem:

Perturb candidate u:

$$\nu(t, \eta) \coloneqq u(t) + \eta \omega(t), \quad \omega(t) \in \mathcal{C}([t_0, t_f])$$



$y(t, \eta) \in \mathcal{C}^1$ is the state response of the system under ν.
$x(t) \in \mathcal{C}^1$ is the state response of the system under u.

$y_\eta(t; \eta) \coloneqq \frac{\partial y}{\partial \eta}$: $y(t, 0) = x(t), \quad \forall t \in [t_0, t_f]$ by definition

Write J for the perturbed u:

$$J(\nu(\cdot; \eta)) = \int_{t_0}^{t_f} l\Big(t, y(t;\eta), \nu(t;\eta)\Big) + \lambda^T(t)[\dot{y}(t;\eta) - f\Big(t, y(t;\eta), \nu(t;\eta)\Big)] \, dt$$

$$= \int_{t_0}^{t_f} \Big(l(t,y,\nu) - \dot{\lambda}^T y - \lambda^T f(t,y,\nu)\Big)\, dt + \lambda^T(t_f)y(t_f;\eta) - \lambda^T(t_0)y(t_0;\eta)$$

↳integral by parts applied to $\lambda^T \dot{y}$

Now we write the derivative $\frac{\partial J}{\partial \eta}$ and set $\eta \to 0$.

$$\frac{\partial J}{\partial \eta}(\nu) = \int_{t_0}^{t_f} \underline{[l_u(t,y,\nu) - f_u(t,y,\nu)\lambda]^T \omega} + \underline{[l_x(t,y,\nu) - f_x(t,y,\nu)\lambda - \dot{\lambda}]^T y_\eta}\, dt$$

$$+\underline{\lambda(t_f)^T y_\eta(t_f;\eta)} - \underbrace{\lambda(t_0)^T y_\eta(t_0;\eta)}_{=0 \text{ because } y(t_0;\eta)=x_0\ \forall \eta}$$

Take $\eta \to 0$:

$$\delta J(u;\omega) = \int_{t_0}^{t_f} \underline{[l_u(t,x,u) - f_u(t,x,u)\lambda]^T \omega} + \underline{[l_x(t,x,u) - f_x(t,x,u)^T\lambda - \dot{\lambda}]^T y_\eta}\, dt$$

$$+\underline{\lambda^T(t_f) y_\eta(t_f;0)}$$

$\delta J = 0 \quad \forall \omega\ \forall \lambda$ for u to be a candidate minimizer.

We are free to choose any ω, λ we want:

1. We can choose λ as follows:

$$\dot{\lambda} = -l_x + f_x^T \lambda \quad \text{with BC } \lambda(t_f) = 0 \quad \text{2nd ELE + BC}$$

This eliminates the green and blue terms.

2. To set the red term to 0, we can choose n_u "special" perturbations $\omega^{(i)}$ defined as follows: $\begin{cases} \omega_i^{(i)} = l_{u_i} - f_{u_i}^T \lambda & \\ \omega_j^{(i)} = 0, & \forall j \neq i \end{cases}$ where $\omega^{(i)} = \begin{bmatrix} \omega_1^{(i)} \\ \vdots \\ \omega_{n_u}^{(i)} \end{bmatrix}, \quad i = 1, \dots, n_u$

This yields:

$$0 = \int_{t_0}^{t_f} \left[l_{u_i} - f_{u_i}^T \lambda\right]^2 dt, \quad \forall i = 1, \dots, n_u$$

Which is only possible if:

$$l_{u_i} - f_{u_i}^T \lambda = 0, \quad i = 1, 2, \dots, n_u \quad \text{3rd ELE eq.}$$

1^{st} ELE is just the dynamic equation for x. $\square$

Remark:

- we always have $\begin{array}{l} n_x \text{ BC at } t_0 \\ n_x \text{ BC at } t_f \end{array} \rightarrow$ This is a TPBVP
- Typically we extract from the 3^{rd} equation a relationship between u and x, λ:

$$\rightarrow u(x, \lambda).$$

  In this case we can replace $u(x, \lambda)$ in the ODEs. These ODEs then are a system of equations in x, λ. We solve for x, λ and we find $u(x, \lambda)$.

- What happens to the ELE when $f(t, x, u) = u \rightarrow \dot{x} = u$

$$\begin{cases} \dot{x} = u & x(t_0) = x_0 \\ \dot{\lambda} = l_x(t, x, \dot{x}) & \lambda(t_f) = 0 \\ \lambda = l_u(t, x, \dot{x}) \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{x} = u & x(t_0) = x_0 \\ \frac{d}{dt} l_u(t, x, \dot{x}) = l_x(t, x, \dot{x}) & [l_u(t, x, \dot{x})]|_{t=t_f} = 0 \\ \lambda = l_u(t, x, \dot{x}) \end{cases}$$

  These are the EE!

  $\frac{dl_u}{dt} = l_x, \quad [l_u]|_{t=t_f} = 0$ is the transversality condition for the case s_2 fixed and $x(s_2)$ free in the CV problem. **Surprising?**

  If we write $[OC - P1]$ for $\dot{x} = u$ we get

$$\min_{x(\cdot)} \int_{t_0}^{t_f} l(t, x, \underbrace{\dot{x}}_{=u})\, dt$$

$$\text{s.t. } x(t_0) = x_0, \quad t_f \text{ given}, \quad x(t_f) \text{ free}$$

  This problem is equivalent to finding a curve $x(s),\ s \in [s_1, s_2]$ where $s_1 = t_0,\ s_2 = t_f$.

  In this case: $\lambda(t) = l_u(t, x, \dot{x})|_{x(t)} = p(t)$

  Therefore we can think of the adjoint as the momentum for the OC problem.

- We can introduce the Hamiltonian for OC problems:

In CV $\rightarrow \mathcal{H}(s,x,\dot{x},p) = -l(s,x,\dot{x}) + \dot{x}^T p$

In OC $\rightarrow \mathcal{H}(t,x,u,\lambda) = -l(t,x,u) + f^T(t,x,u)\lambda$

Once we have defined $\mathcal{H}$, we can rewrite ELE compactly:

H	
A	D
M	Y
I	N
L	A
T	M
O	I
N	C
I	S
A	
N	

$$\begin{aligned}
\dot{x}^\star &= \mathcal{H}_\lambda(t,x^\star,u^\star,\lambda^\star), && x^\star(t_0) = x_0 \\
\dot{\lambda}^\star &= -\mathcal{H}_x(t,x^\star,u^\star,\lambda^\star), && \lambda^\star(t_0) = 0 \\
0 &= \mathcal{H}_u(t,x^\star,u^\star,\lambda^\star), && \forall t \in [t_0,t_f]
\end{aligned}$$

Candidate optimal controllers
are stationary points for $\mathcal{H}$

- $\left.\frac{d\mathcal{H}}{dt}\right|_{\substack{\text{evaluated at} \\ x^\star,\lambda^\star,u^\star \text{ solutions} \\ \text{of ELE}}} = \mathcal{H}_t + \mathcal{H}_x^T \dot{x} + \mathcal{H}_u^T \dot{u} + f^T \dot{\lambda}$

$$= \mathcal{H}_t + \underbrace{\mathcal{H}_u^T \dot{u}}_{=0} + f^T(\underbrace{\mathcal{H}_x + \dot{\lambda}}_{=0}) = \mathcal{H}_t = \frac{\partial \mathcal{H}(t,x,u,\lambda)}{\partial t}$$

$\mathcal{H}_t \neq 0$ only if

$f(t,x,u)$ depends on time and/or

$l(t,x,u)$ depends on time

Time-invariant problems (dynamics do not depend on t and running cost neither) have $\mathcal{H}_t = 0$.

Thus: $\mathcal{H}$ is constant over ELE solutions in time-invariant problems.

- Consider the case where $S = \{t_f\} \times \underbrace{\{x_f\}}_{①}$ ① fixed end-point

What changes in Lemma 3.1?

We remove $\lambda^\star(t_f) = 0$ and add $x(t_f) = x_f$. All the rest stay the same.

If we have 2^{nd} order regularity properties on f, l, we can derive 2^{nd} order necessary conditions for weak minima.

Lemma 3.2 (Second order necessary conditions for opitmal control problems)

Consider $[OC - P1]$ with the standing assumptions and $u \in \mathcal{C}$.
Assume also that f and l have continuous second-order derivative in u.

$$(\nabla_{uu} l,\ \nabla_{uu} f \text{ exist})$$

Suppose $u^\star$ is a weak minimum and $x^\star$ its response.
Then $(x^\star, u^\star, \lambda^\star)$ satisfies ELE and also

$$-\nabla_{uu} l(t, x^\star, u^\star) + \nabla_{uu}(f^T(t, x^\star, u^\star)\lambda^\star) \preceq 0 \qquad \forall t \in [t_0, t_f]$$

(Legendre-Clebsch condition).
Equivalently:

$$\mathcal{H}_{uu}(t, x^\star, u^\star, \lambda^\star) \preceq 0 \qquad \forall t \in [t_0, t_f]$$

Proof: skipped: It proceeds similary to the proof of Legendre condition in CV by deriving $\delta^2 J = 0$.

Note: If $S = \{t_f\} \times \{x_f\}$, this Lemma still applies. The cange of S only affects the boundary conditions of the ELE.

What about sufficient conditions?

Lemma 3.3 (First order sufficient conditions for unconstrained OC - Mangasarian conditions)

Consider $[OC - P1]$ with standard assumptions and $u \in \mathcal{C}$.
Assume also that

$$l(t, x, u) \text{ and } f(t, x, u) \text{ are jointly cvx in } x \text{ and } u\ \forall t \in [t_0, t_f]$$

If

- $(u^\star, x^\star, \lambda^\star)$ satisfies ELE,
- $\lambda^\star(t) \leq 0 \quad \forall t \in [t_0, t_f]$. $\leftarrow$ if f linear not needed

Then $u^\star$ is a global minimizer.

Proof: The goal is to show that

$$J(u) - J(u^\star) \geq 0 \quad \forall u \in \mathcal{D}$$

$$J(u) - J(u^\star) = \int_{t_0}^{t_f} (l(t,x,u) - l(t,x^\star,u^\star))\, dt \qquad \Big|\ \text{jointly convex } l \text{ wrt } x, u$$

$$\geq \int_{t_0}^{t_f} \underbrace{l_x^\star}_{l_x(t,x^\star,u^\star)} (x - x^\star) + \underbrace{l_u^{\star T}}_{l_u(t,x^\star,u^\star)} (u - u^\star)\, dt \qquad \Big|\ x^\star, u^\star, \lambda^\star \text{ satisfy ELE}$$

$$= \int_{t_0}^{t_f} [f_x^\star \lambda^\star - \dot{\lambda}^\star]^T (x - x^\star) + [f_u^\star \lambda^\star]^T (u - u^\star)\, dt \qquad \Big|\ \text{Integration by parts for } \dot{\lambda}^{\star T}(x - x^\star)$$

$$= \int_{t_0}^{t_f} \lambda^{\star T} \left[f_x^{\star T}(x - x^\star) + f_u^{\star T}(u - u^\star) - \Big(f(t,x,u) - f(t,x^\star,u^\star) \Big) \right] dt$$

$$+ \underbrace{\lambda^{\star T}(t_f)}_{=0 \text{ because BC of ELE}} (x(t_f) - x^\star(t_f)) - \lambda^{\star T}(t_0) \Big(\underbrace{x(t_0) - x^\star(t_0)}_{\substack{=0 \text{ because } x(t_0) = x^\star(t_0) = x_0 \text{ BC of ELE} \\ \text{(all responses must start at } x_0)}} \Big)$$

In summary:

$$J(u) - J(u^\star) = \int_{t_0}^{t_f} \underbrace{\lambda^{\star T}}_{\boxed{A}} \underbrace{\left[f_x^{\star T}(x - x^\star) + f_u^{\star T}(u - u^\star) - \Big(f(t,x,u) - f(t,x^\star,u^\star) \Big) \right]}_{\boxed{B}} dt$$

By joint convexity of f we have

$$f(t,x,u) \geq f(t,x^\star,u^\star) + f_x^{\star T}(x - x^\star) + f_u^{\star T}(u - u^\star) \quad \forall x, u$$

$$\Leftrightarrow \underbrace{f_x^{\star T}(x - x^\star) + f_u^{\star T}(u - u^\star) - \Big(f(t,x,u) - f(t,x^\star,u^\star) \Big)}_{\boxed{B}} \leq 0 \quad \forall x, u$$

$\boxed{B} \leq 0 \rightarrow$ If $\lambda^\star \leq 0 \underset{\text{assumption}}{\rightarrow} \int_{t_0}^{t_f} \boxed{A} \cdot \boxed{B} dt \geq 0$

Therefore: $J(u) - J(u^\star) \geq 0 \quad \forall u \in \mathcal{D}$ $\square$

Remarks:

- When f is linear, can we relax the assumptions of the Lemma?

$$f \text{ linear} \rightarrow \boxed{B} = 0$$

  Thus, we do not need to require $\lambda^\star(t) \leq 0 \quad \forall t$

- What happens if $S = \{t_f\} \times \{x_f\}$?

  The Lemma holds exactly the same. In the proof instead of having $\lambda^\star(t_f) = 0$ we have that $x^\star(t_f) = x(t_f) = x_f$.

- When f is jointly concave ($\leftrightarrow -f$ is convex).

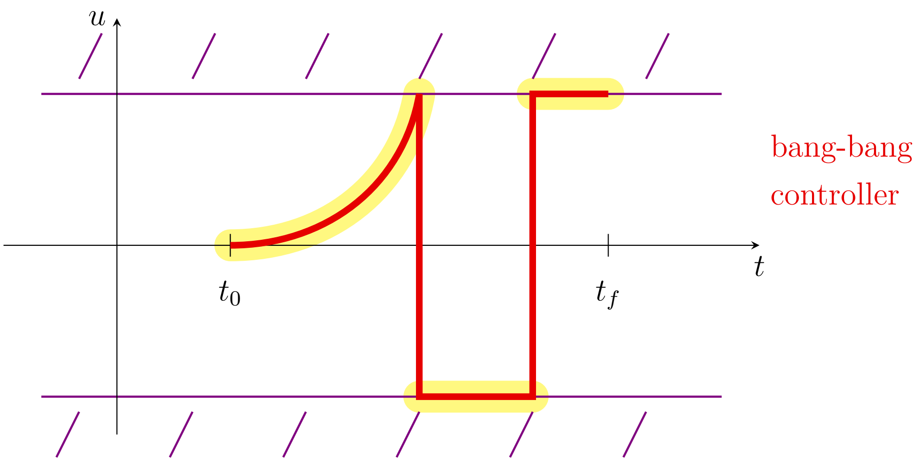
  Then we require $\lambda^\star \geq 0$. The rest ist still valid.

3.2.2 Piecewise continuous control

$$\begin{aligned} u \in \hat{\mathcal{C}}, \quad & S = \{t_f\} \times \mathbb{R}^{n_x} \\ & S = \{t_f\} \times \{x_f\} \end{aligned}$$

Why?

- For some OC problems, a continuous solution to ELE might not exist.
  - This happens very frequently when we have input constraints.



  - Usually u is obtained by solving

$$0 = \mathcal{H}_u(t, x, u, \lambda)$$

$\rightarrow u = u_1(x,\lambda),\ u = u_2(x,\lambda),\ \ldots$ might have more than 1 solution

even without constraints, u might jump from u_1 to u_2.

- To improve the cost J.

  $u \in \hat{\mathcal{C}}$ is a larger class than $u \in \mathcal{C}$.

Lemma 3.4 (First-order necessary conditions for opt $\hat{\mathcal{C}}$ control)

Consider $[OC - P1]$, $V = \hat{\mathcal{C}}([t_0, t_f], \mathbb{R}^{n_u})$.
Suppose $u^\star \in \hat{\mathcal{C}}$ with corner points $\{c_k\}_{k=1}^N$ is a weak minimum with state response $x^\star \in \hat{\mathcal{C}}^1$.
Then $\exists \lambda^\star \in \hat{\mathcal{C}}^1$ such that

1. $(x^\star, u^\star, \lambda^\star)$ solves ELE inside the intervals (c_k, c_{k+1}), $k = 0, \ldots, N$ with $c_0 = t_0,\ c_{N+1} = t_f$.

2. At every corner point of $u^\star$, the following continuity conditions must hold:

   A: $x^\star(c_k^-) = x^\star(c_k^+)$

   $\lambda^\star(c_k^-) = \lambda^\star(c_k^+)$

   B: $\mathcal{H}(c_k^-, x^\star(c_k), u^\star(c_k^-), \lambda^\star(c_k)) = \mathcal{H}(c_k^+, x^\star(c_k), u^\star(c_k^+), \lambda^\star(c_k))$

A is a generalization of the 1^{st} WE conditions (where p had to be cont.).

B is a generalization of the 2^{nd} WE conditions (where $\mathcal{H}$ had to be cont.).

Same result holds for $S = \{t_f\} \times \{x_f\}$.

3.2.3 Continuous control with general target set

$$S = \{(t,x) \in (t_0, \infty) \times \mathbb{R}^{n_x} \mid \psi_k(t,x) = 0,\ k = 1, \ldots, n_\psi\}$$

$$\rightarrow t_f \times x(t_f) \in S \qquad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n_\psi} \end{bmatrix} = 0$$

$$
\begin{aligned}
[OC-P2] \quad \min_{t_f, u(\cdot)} \quad & J(u) := \int_{t_0}^{t_f} l(t,x,u)\,dt + \varphi(t_f, x(t_f)) \\
\text{s.t.} \quad & \dot{x} = f(t,x,u),\ x(t_0) = x_0 \\
& (t_f, x(t_f)) \in S \\
& u \in \mathcal{C}([t_0, t_f])
\end{aligned}
$$

Assumption: $\varphi, \psi \in \mathcal{C}^1$.

$[OC-P1]$ special case of $[OC-P2]$ by defining ψ to recover

$$S = \{t_f\} \times \mathbb{R}^{n_x} \quad \text{or} \quad S = \{t_f\} \times \{x_f\}.$$

Normed vector space $V = \underbrace{\mathcal{C}[t_0, t_f]}_{u} \times \underbrace{(t_0, \infty)}_{t_f}$.

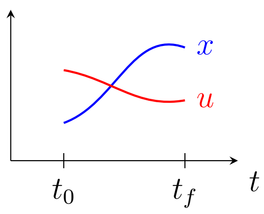
$$\|(u, t_f)\|_\infty = \|u\|_\infty + |t_f|$$

$$J(u, t_f): \text{ cost functional depends on } u, t_f.$$

$$\delta J(u, t_f; \omega, \tau) \quad \begin{cases} \tau : \text{perturbation in } t_f \\ \omega : \text{perturbation in } u \end{cases}$$

Recall what we did in Chapter 2 when s_2 was free. Let's define functionals associated with the eq. constraints $\psi = 0$.

$$P_k(\underline{u}, t_f) := \psi_k(t_f, \underline{x(t_f)}) \qquad k = 1, \ldots, n_\psi$$



$\delta P_k(u, t_f; \omega, \tau)$ first variation of P_k.

We can now state the main result which again consists of:

- ELE

- Boundary conditions taking into account φ and ψ.

For this, we require some regularity conditions as typical in constrained problems.

Lemma 3.5 (First-order necessary cond. for weak minimizers and general terminal problems)

Consider $[OC - P2]$, standing regularity assumptions, $u \in \mathcal{C}$. Suppose $u^\star \in \mathcal{C}$ is a weak minimizer with optimal time $t_f^\star$ and state response $x^\star$. Assume that

$$\text{Det}(P(\{\bar{\omega}_i, \bar{\tau}_i\}_{i=1}^{n_\psi})) \neq 0$$

$$P(\{\bar{\omega}_i, \bar{\tau}_i\}_{i=1}^{n_\psi}) = \begin{bmatrix} \delta P_1(u^\star, t_f^\star; \bar{\omega}_1, \bar{\tau}_1) & \cdots & \delta P_1(u^\star, t_f^\star; \bar{\omega}_{n_\psi}, \bar{\tau}_{n_\psi}) \\ \vdots & \ddots & \vdots \\ \delta P_{n_\psi}(u^\star, t_f^\star; \bar{\omega}_1, \bar{\tau}_1) & \cdots & \delta P_{n_\psi}(u^\star, t_f^\star; \bar{\omega}_{n_\psi}, \bar{\tau}_{n_\psi}) \end{bmatrix}$$

for n_ψ independent perturbations $\{\bar{\omega}_k, \bar{\tau}_k\}_{k=1}^{n_\psi}$. $\rightarrow$ regularity condition on eq. constraints $\psi = 0$ (similar to isoperimetric constraints in CV).
Then $\exists \lambda^\star \in \mathcal{C}^1$ and a vector $\nu^\star \in \mathbb{R}^{n_\psi}$ such that $(u^\star, x^\star, \lambda^\star, \nu^\star, t_f^\star)$ satisfy:

- **Differential-Algebraic eq.** (Hamiltonian dynamics):

$$\begin{aligned} \dot{x} &= \mathcal{H}_\lambda(t, x^\star, u^\star, \lambda^\star) & x^\star(t_0) &= x_0 \\ \dot{\lambda} &= -\mathcal{H}_x(t, x^\star, u^\star, \lambda^\star) & \lambda^\star(t_f) &= -\Phi_x(t_f^\star, x^\star(t_f^\star), \nu^\star) \\ 0 &= \mathcal{H}_u(t, x^\star, u^\star, \lambda^\star) & &\text{(BC on } \lambda \text{ at } t_f) \end{aligned}$$

- **Enforcing terminal set:**

$$0 = \psi(t_f^\star, x^\star(t_f^\star))$$

- **Equation closing the problem by producing an extra scalar equation for t_f:**

$$0 = -\Phi_t(t_f^\star, x^\star(t_f^\star), \nu^\star) + \mathcal{H}(t_f^\star, x^\star(t_f^\star), u^\star(t_f^\star), \lambda^\star(t_f^\star))$$

$$\Phi(t, x, \nu) := \varphi(t, x) + \nu^T \psi(t, x)$$

Remarks

- Reg. conditions must be checked for the result to hold. Typical of constrained problems.

- BC on $\lambda(t_f)$ also called transversal condition.
- Can we interpret this condition?

  1. $\psi_x = \nabla_x \psi = [\nabla\psi_1 \ldots \nabla\psi_{n_\psi}] \in \mathbb{R}^{x \times n_\psi}$

  $$\begin{aligned}\lambda(t_f) &= -\nu^T \psi_x, \quad \nu \in \mathbb{R}^{n_\psi} \\ &= \sum \nu_i \nabla \psi_i\end{aligned}$$

     If $\psi = x(t_f) - x_f = 0 \to \nabla_x \psi$ has full rank. Thus, λ is unconstrained $\to$ we retrieve the case we already knew for $S = \{t_f\} \times \{x_f\}$.

  2. We always have

  $$\begin{array}{lcll} n_x \text{ BC at } t = t_0 & \to & n_x \text{ on } x & x(t_0) = x_0 \\ n_x \text{ BC at } t = t_f & \to & n_\psi \text{ on } x & \text{assuming } t_f \text{ free} \\ & & n_x - n_\psi \text{ on } \lambda & \end{array}$$

     When $\psi \neq 0$ then we have constraints on λ.

What if $S = \{(t, x) \mid \psi(t, x) \leq 0\}$? What changes?

1. Regularity conditions must be checked <u>only</u> on the active inequality constraints.
2. In the conditions of the previous Lemma, replace $\psi = 0$ with

$$\left[\begin{array}{l} \psi(t_f, x(t_f)) \leq 0 \\ \nu \geq 0 \\ \nu^T \psi(t_f, x(t_f)) = 0 \end{array}\right.$$

3.2.4 Time-varying Linear-Quadratic Regulator (LQR)

$$\min_{u(\cdot)} \quad \frac{1}{2} \int_{t_0}^{t_f} x^T Q(t) x + u^T R(t) u \, dt + \frac{1}{2} x(t_f)^T Q_f x(t_f)$$

$$\dot{x} = A(t)x + B(t)u \quad x(t_0) = x_0$$

$$S = \{t_f\} \times \mathbb{R}^{n_x}$$

$$\underbrace{R(t) \succ 0, Q(t) \succeq 0 \quad \forall t \in [t_0, t_f], \quad Q_f \succeq 0}_{\text{running cost and terminal cost are convex}}$$

$$u \in \mathcal{C}$$

Solution: Write the ELE.

$$\mathcal{H} = -\frac{1}{2}[x^T Q x + u^T R u] + \lambda^T (Ax + Bu)$$
$$\dot{x}^\star = \mathcal{H}_\lambda, \quad x^\star(t_0) = x_0$$
$$\dot{\lambda}^\star = -\mathcal{H}_x, \quad \lambda^\star(t_f) = -\nabla\varphi(x(t_f)) = -Q_f x^\star(t_f)$$
$$0 = \mathcal{H}_u, \quad \forall t \in [t_0, t_f]$$

Replacing $\mathcal{H}$ we obtain:

$$\begin{bmatrix} \dot{x}^\star \\ \dot{\lambda}^\star \end{bmatrix} = \begin{bmatrix} A & BR^{-1}B^T \\ Q & -A^T \end{bmatrix} \begin{bmatrix} x^\star \\ \lambda^\star \end{bmatrix}, \quad x^\star(t_0) = x_0,\ \lambda^\star(t_f) = -Q_f x^\star(t_f)$$
$$u^\star = R^{-1}B^T\lambda^\star \quad (0 = \mathcal{H}_u)$$

LQR solution consists of a TPBVP in (x, λ) and plugging its solution in

$$u^\star = R^{-1}B^T\lambda^\star.$$

How do we solve TPBVP? Not easy in general. Computational approaches in Chapter 5.

Ansatz: The TPBVP is homogeneous in x_0 ($\rightarrow$ if $x_0 = 0$ then $x^\star = \lambda^\star = u^\star \equiv 0$ is the solution).

Therefore, we can assume a solution on λ of the form

- $\lambda^\star(t) = -P(t)x^\star(t)$

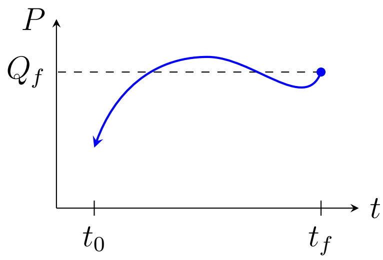
  with $\lambda^\star(t_f) = -Q_f x^\star(t_f) \qquad (P(t_f) = Q_f)$

  $\lambda^\star(t_0) = -P_0 x^\star(t_0)$

We plug this Ansatz into the TPBVP and we get an equation in $P(t)$:

$$\dot{P} = -PA - A^T P + PBR^{-1}B^T P - Q, \quad P(t_f) = Q_f \quad \text{(Riccati differential equation)}$$

ODE with final value condition. Its solution gives us the Optimal Control $u^\star$.



$$
\begin{aligned}
u^\star &= R^{-1}B^T \underbrace{\lambda^\star}_{\to -P(t)x^\star} \\
&= -R^{-1}(t)B(t)^T P(t)x^\star(t)
\end{aligned}
$$

Remarks

- Linear $u^\star$: $u = -K(t)x, \quad K(t) = R^{-1}(t)B(t)^T P(t)$.
- <u>Closed loop controller</u> / control law $u(x)$.

  $\to$ "by chance": combination of lin dynamics + quad cost.
- Why do I keep calling it "optimal controller"? Aren't ELE only necessary for "weak minimizers"?

  Because of the Mangasarian sufficient conditions.

$$
\text{LQR} \quad \begin{array}{l} \nearrow \\ \searrow \end{array} \quad \left.\begin{array}{l} \text{cvx } l \text{ and } \varphi \text{ (in fact quadratic)} \\ \text{cvx } f \text{ (in fact linear)} \end{array}\right\} \quad \text{minimizer is global}
$$

3.3 Pontryagin Maximum Principle (PMP)

3.3.1 Limitations of the variational approach (ELE equations)

① **Constraints on u:**

ELE approach struggles to cope with problems where we require path constraints on $\mathcal{U}$.

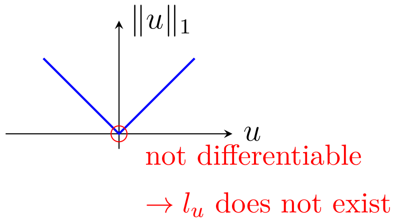
<u>Reason:</u> difficult to build perturbations w s.t. $u + \eta w$ is admissible.

② **Relax regularity:**

- So far we assumed $f_u, l_u \in \mathcal{C}$ ($\Leftrightarrow f, l$ have cont. derivatives w.r.t u).

  e.g. $l = \|u\|_1 \to$ minimum energy control.

$$J = \int_{t_0}^{t_f} \|u\|_1 \, dt$$



- ELE assume $u \in \mathcal{C}$, we have seen corner conditions for $u \in \hat{\mathcal{C}}$.

We would like to have a method that generally works with $u \in \hat{\mathcal{C}}$.

$\to$ We do not want to assume $f_u, l_u \in \mathcal{C}$ if possible.

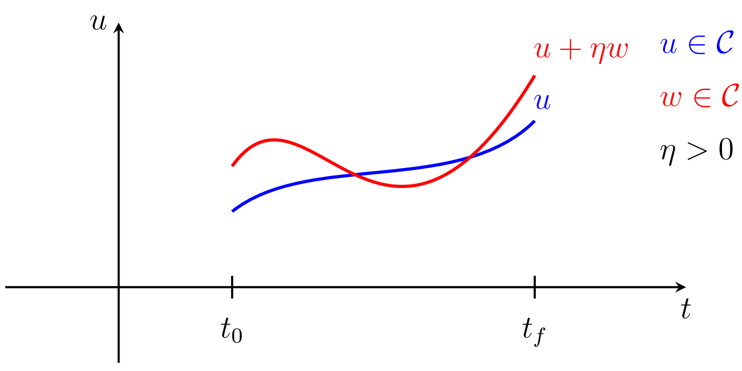
③ **ELE are necessary conditions for weak minimizers.**

How about necessary conditions for strong minimizers?

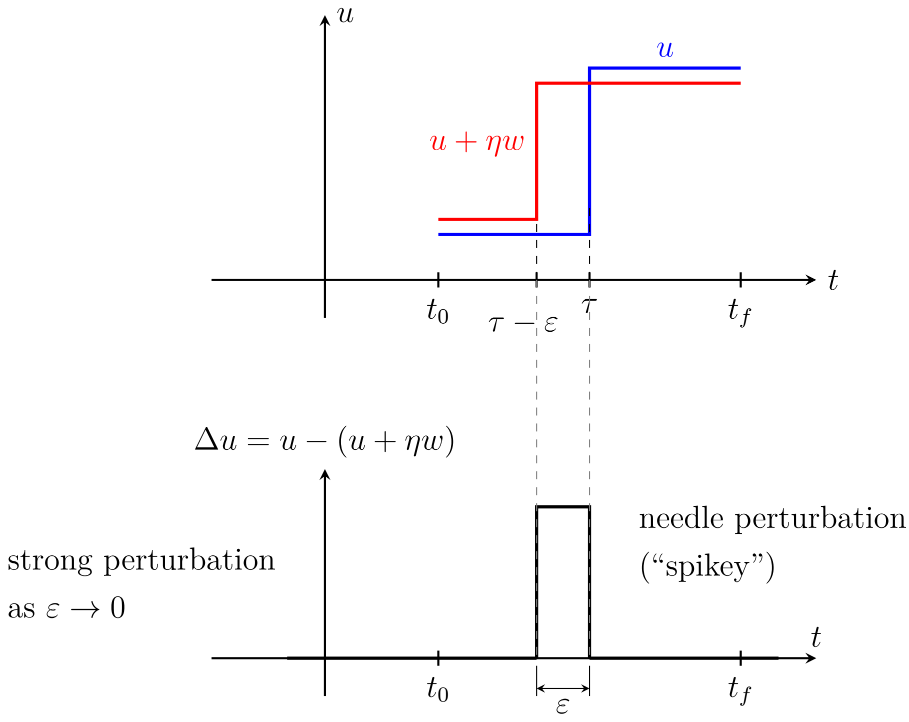
The key-issue is that in ELE we considered this class of perturbations:

$u + \eta w$ is a weak perturbation.

$$\eta \to 0, \quad \|u - (u + \eta w)\| \to 0$$



What type of perturbations would be strong?



We have:

$$\|u - (u + \eta w)\|_1 \neq 0$$

$$\|x - y\|_1 \to 0$$

(y: perturbed x where $(u + \eta w)$ acts).

We want an approach that captures local optimality w.r.t. all perturbations (weak + strong).

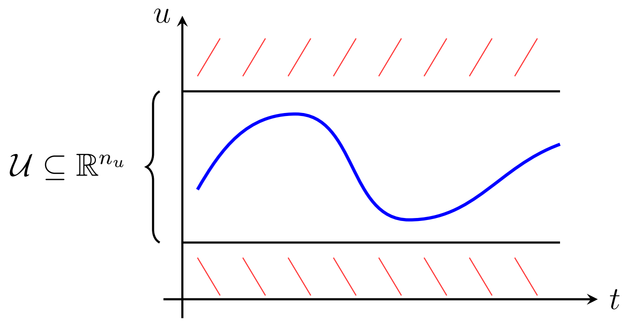
3.3.2 Statements of PMP

$$[OC - P3] \qquad \begin{aligned} &\min_{u(\cdot), t_f} \int_{t_0}^{t_f} l(x,u)\, dt \\ &\text{s.t. } \dot{x} = f(x,u), \quad x(t_0) = x_0 \\ &(t, x(t_f)) \in (t_0, \infty) \times S_f \\ &V = \hat{\mathcal{C}}([t_0, t_f], \mathcal{U}), \quad \mathcal{U} \subseteq \mathbb{R}^{n_u} \end{aligned}$$

Remarks:

- Time-invariant problems (autonomous): l and f depend not on time.

- $\varphi = 0$
- 2 cases for S_f:
  1. $S_f = \{x_f\}$
  2. $S_f = \{x \mid h_i(x) = 0, \forall i = 1, \ldots, n_h\}$, where we assume that all points in S_f are regular (see section 1)
- t_f is free.
- u is constrained to lie inside $\mathcal{U} \subseteq \mathbb{R}^n$.



- $u \in \hat{\mathcal{C}}$ piecewise cont. function.
- Regularity conditions: $l, f \in \mathcal{C}$ and $l_x, f_x \in \mathcal{C}$. No more conditions on l_u, f_u!

Theorem 3.1 (PMP for fixed endpoint)

- Problem $[OC - P3]$
- $S = (t_0, \infty) \times \{x_f\}$

Suppose $u^\star \in \hat{\mathcal{C}}$ is a strong minimizer with associated final time $t_f^\star$ and $x^\star \in \hat{\mathcal{C}}^1$. Then $\exists \lambda \in \hat{\mathcal{C}}^1$ and a constant $\lambda_0^\star \leq 0$ satisfying $(\lambda_0^\star, \lambda^\star(t)) \neq (0, 0)$ $\forall t \in [t_0, t_f^\star]$ s.t.:

1. $(x^\star, \lambda^\star)$ satisfy Hamiltonian dynamics
$$\dot{x}^\star = \mathcal{H}_\lambda(x^\star, u^\star, \lambda^\star, \lambda_0^\star), \quad x^\star(t_f^\star) = x_f, \quad x^\star(t_0) = x_0$$
$$\dot{\lambda}^\star = -\mathcal{H}_x(x^\star, u^\star, \lambda^\star, \lambda_0^\star)$$
where $\mathcal{H}(x, u, \lambda, \lambda_0) = \lambda_0 l(x, u) + f(x, u)^T \lambda$.
2. For every t, the function $\mathcal{H}(x^\star, \cdot, \lambda^\star, \lambda_0^\star)$ has a global maximum at $u = u^\star(t)$ i.e.
$$\mathcal{H}(x^\star(t), u^\star(t), \lambda^\star(t), \lambda_0^\star) \geq \mathcal{H}(x^\star(t), u, \lambda^\star(t), \lambda_0^\star) \quad {}_{\substack{\forall t \in [t_0, t_f^\star] \\ \forall u \in \mathcal{U}}}$$
or $u^\star \in \arg\max_{u \in \mathcal{U}} \mathcal{H}(x^\star(t), u, \lambda^\star(t), \lambda_0^\star)$.
3. $\mathcal{H}(x^\star(t), u^\star(t), \lambda^\star(t), \lambda_0^\star) = 0 \quad \forall t \in [t_0, t_f^\star]$.

Remarks:

- How do we handle λ_0?

  2 cases:

  - $\lambda_0 < 0$: we can divide $\mathcal{H}$ by $|\lambda_0|$ w.l.o.g. We retrieve $\mathcal{H} = -l + \lambda^T f$.
  - $\lambda_0 = 0$: we have $\mathcal{H} = \lambda^T f$.
- $\mathcal{H} = 0$ is typical value of the Hamiltonian for free problems. When t_f is fixed $\to \mathcal{H} = \text{const.} \neq 0$ (because problem is time-invariant). And we do not need 3 because t_f is known.

Proof: Liberzon's book Ch. 4, Sect 4.2 is on the proof of PMP.

Second statement: $S_f = \{x \mid h_i(x) = 0, i = 1, \dots, n_h\}$.

We expect changes on the boundary conditions.

Theorem 3.2 (PMP for variable end point)

Same conditions as Theorem 3.1 except

$$S = (t_0, \infty) \times \{x \mid h_i(x) = 0\} \quad \text{(with regular } x\text{)}.$$

If ...
Then $\exists \lambda \in \hat{\mathcal{C}}^1([t_0, t_f^\star])$ and $\lambda_0 \leq 0$ satisfying $(\lambda_0^\star, \lambda^\star(t)) \neq (0,0)$ $\forall t \in [t_0, t_f^\star]$.
①: $(x^\star, \lambda^\star)$ satisfy the Hamiltonian dynamics with boundary conditions

$$\left.\begin{aligned} x^\star(t_0) &= x_0 \\ x^\star(t_f) &\in S_f \end{aligned}\right\} n_x + n_h \text{ conditions}$$

$$\langle \lambda^\star(t_f^\star), d\rangle = 0 \quad \forall d \in \mathcal{T}_{S_f}(x^\star(t_f^\star)) \quad \Big\} n_x - n_h \text{ conditions}$$

or equivalently $\lambda^\star(t_f^\star) \perp \mathcal{T}_{S_f}(x^\star(t_f^\star))$ ($\mathcal{T}_{S_f}$ tangent cone to S_f at $x^\star(t_f^\star)$, see section 1).
$\to$ in total $n_x + n_x$ BC:

$$\begin{aligned} &n_x \text{ at } t_0 \\ &n_x \text{ at } t_f \end{aligned} \begin{cases} n_h \text{ on } x \\ n_x - n_h \text{ on } \lambda \end{cases}$$

②, ③ stay the same.

BC (or transversal condition) on $\lambda^\star(t_f^\star)$

$$\mathcal{T}_{S_f}(x^\star(t_f^\star)) = \{d \in \mathbb{R}^{n_x} : \langle \nabla h_k(x^\star(t_f^\star)), d\rangle = 0,\ k = 1, \dots, n_h\}$$

The BC prescribes that $\lambda^\star(t_f^\star)$ belongs to the orthogonal complement of $\mathcal{T}_{S_f}$. That is, by definition of $\mathcal{T}_{S_f}$: $\lambda^\star(t_f^\star)$ a linear combination of $\nabla h_k,\ k = 1, \dots, n_h$.

We had a similar interpretation for the transversality condition of ELE. More on this in T5.

$(\lambda_0^\star, \lambda^\star(t)) \neq 0$ $\forall t \in [t_0, t_f^\star]$ why?

Non-triviality condition, because if you have $\lambda_0^\star = \lambda^\star(t) \equiv 0$, then all conditions of PMP hold!

From the proof, one can see that

$$(\lambda_0^\star, \lambda^\star(t)) \neq 0 \;\forall t \iff \exists \bar{t} \text{ s.t. } (\lambda_0^\star, \lambda^\star(\bar{t})) \neq 0.$$

($(\lambda_0, \lambda) \neq 0$ is a condition to rule out the case that $\lambda_0^\star = 0$.)

E.g. $l \neq 0$ running cost. This happens in time-optimal control problems.

$$\min_{t_f} \int_{t_0}^{t_f} 1\, dt$$

$$\begin{aligned} \mathcal{H} &= \lambda^T f + \lambda_0 l = 0, \quad ③ \text{ from PMP} \\ &= \lambda^T f + \lambda_0. \end{aligned}$$

If $\exists t$ s.t. $\lambda(t) = 0$, then we have $\lambda_0 = 0$.

Therefore, for problems where $l(x, u) \neq 0$. We have that $\lambda(t) \neq 0$! Otherwise also $\lambda_0 = 0$ and this is not possible.

3.3.3 Extensions of PMP

1. Fixed terminal time
2. Time-varying problems
3. Non-zero terminal costs

① t_f fixed

We said before that $\mathcal{H} = 0$ in time-free problems. When t_f is fixed then we still have $\mathcal{H} =$ const. but “const.” $\neq 0$.

In practice, this has little implications. We just ignore the third condition of the PMP (t_f is not unknown here after all).

To further analyze this case, we augment the system with a "state" denoting time:

$$\begin{cases} \dot{x} = f(x, u) & x(t_0) = x_0 \\ \dot{x}_{n+1} = 1 & x_{n+1}(t_0) = t_0 \\ t_f \in (t_0, \infty) & \text{"free final time": final time will be fixed through } x_{n+1} \\ (x, x_{n+1}) \in S_f \times \{t_f\} & \text{target set for the "state"} \end{cases}$$

$\rightarrow$ this problem is in the standard PMP form.

Hamiltonian for our modified system:

$$\overline{\mathcal{H}}(x,u,\lambda,\lambda_0,\lambda_{n+1}) := \underbrace{\lambda_0 l(x,u) + f(x,u)^T\lambda}_{\mathcal{H}(x,u,\lambda,\lambda_0)\text{ (Hamilt. for original system)}} + \underbrace{1}_{\substack{\text{dynamic}\\ \text{of } x_{n+1}}} \cdot \underbrace{\lambda_{n+1}}_{\substack{\text{co-state}\\ \text{of } x_{n+1}}}$$

Because the modified system has free final time, we can write:

$$0 = \overline{\mathcal{H}} = \mathcal{H} + \lambda_{n+1}$$

Thus:

$$\mathcal{H} = -\lambda_{n+1}$$

How do we obtain λ_{n+1} (and thus $\mathcal{H}$ for the original problem)?

Write the Hamiltonian dynamics for the last adjoint:

$$\dot{\lambda}_{n+1} \underset{\substack{\uparrow\\ \text{“}\dot{\lambda}=-\mathcal{H}_x\text{”}}}{=} -\overline{\mathcal{H}}_{x_{n+1}} \underset{\substack{\uparrow\\ \text{because } x_{n+1} \text{ is time}}}{=} -\mathcal{H}_t \underset{\substack{\uparrow\\ \text{because } \mathcal{H}=\text{const}}}{=} 0$$

BC for λ_{n+1}? x_{n+1} is fixed at final time ($= t_f$). Thus $\lambda_{n+1}(t_f)$ is free (because its corresponding state is fixed). Depending on the other variables, $\lambda_{n+1}(t_f)$ will have a certain value.

In summary: for these problems $\mathcal{H} = \text{const}$ but $\neq 0$.

② Time-varying problems

$f(t,x,u)$, $l(t,x,u)$, t_f fixed or free.

Again we augment the dynamic with a fictitious state representing time.

$$\tilde{x} = \begin{bmatrix} x \\ x_{n_x+1} \end{bmatrix} \qquad \begin{array}{ll} \dot{x} = f(x_{n_x+1}, x, u) & x(t_0) = x_0 \\ \dot{x}_{n_x+1} = 1 & x_{n_x+1}(t_0) = t_0 \\ l(x_{n_x+1}, x, u) & \begin{cases} x_{n_x+1}(t_f) = t_f & \text{if } t_f \text{ fixed} \\ x_{n_x+1}(t_f) \in \mathbb{R} & \text{if } t_f \text{ free} \end{cases} \end{array}$$

→ Time-invariant problem. Therefore, “standard PMP” theorem applies.

①-② do not change.

③ $\bar{\mathcal{H}} = 0$ because augmented system, the problem is time-invariant.

$\bar{\mathcal{H}}$: Hamiltonian for the augmented system with state $\tilde{x}$.

$$\bar{\mathcal{H}} = \mathcal{H} + \lambda_{n_x+1} \quad \text{(Like in extension ①)}$$

$\rightarrow \mathcal{H}(t, x, u, \lambda_0, \lambda)$ Hamiltonian of the original problem.

Thus: $\dot{\lambda}_{nx+1} = -\mathcal{H}_t \neq 0$ because $\mathcal{H}$ depends on time.

If t_f is fixed, then we just solve ①+② and keep in mind that $\mathcal{H}$ is now time-varying.

If t_f is free, we need an equation to determine it. We need to obtain a value for $\mathcal{H}$ to have a condition ③.

$$\frac{d}{dt}\mathcal{H} = \mathcal{H}_t, \quad (\mathcal{H}(t_f) = -\lambda_{n_x+1}(t_f))$$

When t_f is free, what is $\lambda_{n_x+1}(t_f)$?

Remember: When $x(t_f)$ is free, $\lambda(t_f) = 0$.

Here: $\underbrace{x_{n_x+1}(t_f)}_{t_f \text{ free}}$ is free, thus $\lambda_{nx+1}(t_f) = 0$.

Therefore we have $\frac{d}{dt}\mathcal{H} = \mathcal{H}_t$ and $\mathcal{H}(t_f) = 0$.

$$\mathcal{H}(t) = -\int_t^{t_f} \mathcal{H}_t(s, x, u, \lambda_0, \lambda)\, ds$$

This term depends on u, x, λ which are obtained from ①-②.

So we have a system of equations in u, x, λ, t_f:

1. Ham. dynamics (x, λ)
2. Ham. maximization (u)
3. $\mathcal{H}(t) = -\int_t^{t_f} \mathcal{H}_t$ BC $\mathcal{H}(t_f) = 0 \quad (t_f)$

③ Terminal cost

W.l.o.g we assume $l = 0$, $\varphi(x_f) \neq 0 \rightarrow J(u) = \varphi(x_f)$.

$t_f, x(t_f)$ are free $\rightarrow \mathcal{H} = \lambda^T f$

The difference to "standard PMP" is that in Theorem 3.1, Theorem 3.2, we consider Lagrangian functionals $J = \int l$. Now we have Mayer functionals $J = \varphi$.

The strategy to solve this problem is then to write our Mayer form in Lagrangian form ($M \Rightarrow L$).

$$\varphi(x_f) = \varphi(x_0) + \int_{t_0}^{t_f} \varphi_x(x)^T f(x,u)\, dt \quad \text{(assuming diff of } \varphi\text{)}$$

$$\rightarrow J(u) = \underbrace{\varphi(x_0)}_{\text{constant, can be dropped}} + \int_{t_0}^{t_f} \underbrace{\varphi_x(x)^T f(x,u)}_{\bar{l}(x,u)}\, dt$$

$\rightarrow$ We have an equivalent problem in L form.

$$\bar{\mathcal{H}} = \bar{\lambda}_0 \bar{l} + \bar{\lambda}^T f = (\bar{\lambda} + \bar{\lambda}_0 \varphi_x)^T f$$

Both the original (M) problem and the reformulated (L) problem are time-invariant and t_f free.

$$\rightarrow \bar{\mathcal{H}} = \mathcal{H} = 0$$

So we can relate the variables of the two problems by setting $\bar{\mathcal{H}} = \mathcal{H}$.

$$(\bar{\lambda} + \bar{\lambda}_0 \varphi_x)^T f = \lambda^T f$$

Thus $\lambda = \bar{\lambda} + \bar{\lambda}_0 \varphi_x$.

Note: the M problem has no terminal cost and free final state. Standard PMP applies.

$$\bar{\lambda}(t_f) = 0$$

$\bar{\lambda}_0 \neq 0$, otherwise we have

$$\underbrace{\begin{pmatrix} \bar{\lambda}(t_f) \\ \bar{\lambda}_0 \end{pmatrix} = 0}_{\text{non-triviality condition would not be satisfied}}$$

Thus we set $\bar{\lambda}_0 = -1$.

$$\forall t : \boxed{\lambda = \bar{\lambda} - \varphi_x} \quad \text{relationship btw} \begin{array}{l} \lambda \rightarrow \text{adjoint of original problem} \\ \bar{\lambda} \rightarrow \text{adjoint in the new problem} \end{array}$$

Specifically at t_f:

$$\lambda(t_f) = \underbrace{\bar{\lambda}(t_f)}_{=0} - \varphi_x(x(t_f)) = -\varphi_x(x(t_f))$$

$\rightarrow$ Summary:

- All stays the same except for the BC on λ.
- Without terminal cost, we would have $\lambda(t_f) = 0$.
- With terminal cost, $\lambda(t_f) = -\varphi_x(x(t_f))$.

Compare this with ELE for general problems $[OC - P2]$. There we also had a similar result on $\lambda(t_f)$.

3.3.4 Time-optimal control problems & bang-bang principle

(Exercises on t-opt in Tutorial and HW.)

Here, we consider LTI systems:

$$\dot{x} = Ax + Bu$$

$$u \in \mathcal{U} \subset \mathbb{R}^{n_u}, \quad \mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid u_i \in [-1, 1], i = 1, \ldots, n_u\}$$

$$-\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \le \begin{bmatrix} u_1 \\ \vdots \\ u_{n_u} \end{bmatrix} \le \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

w.l.o.g. we could have $\underline{u}_i, \bar{u}_i$ instead of $[-1, 1]$.

Goal: steer x from x_0 to given final state x_f in minimal time.

Assumption: controllability of the system, because we need to have at least a feasible controller that steers x from $x_0 \to x_f$.

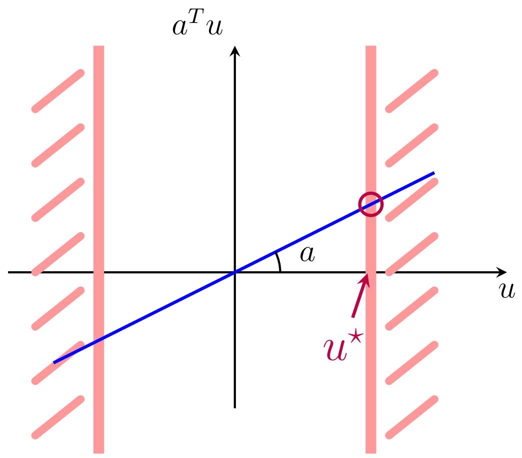
Solution via PMP: $\min\limits_{u,t_f} \int_{t_0}^{t_f} 1\,dt,\ \dot{x} = Ax + Bu,\ x(t_0) = x_0,\ x(t_f) = x_f.$

$$\mathcal{H}(x, u, \lambda, \lambda_0) = \lambda^T(\underbrace{Ax + Bu}_{f(x,u)}) + \underbrace{\lambda_0 \cdot 1}_{l}$$

Condition 2 can be written as:

$$t \in [t_0, t_f^\star]: \quad u^\star(t) \in \arg\max_{u \in \mathcal{U}} \quad \underbrace{\lambda^\star(t)^T}_{\substack{\lambda^\star(x^\star) \text{ solve } 1 \\ \to \text{ Ham. dynamics.}}} \quad (Bu) \qquad | \text{ Linear programme (LP)}$$

When the objective is linear, maximizers are on the boundary of the constraints. The value of the maximizer depends on the slope.



For our problem, this means that $u^\star(t)$ will switch between -1 and 1 depending on the slope $(\lambda^\star(t)^T B)$. $\rightarrow$ as $\lambda^\star(t)$ changes, same will happen to the location of the maximum $u^\star$.

Component-wise, we have:

$$i = 1, \dots, n_u \quad u_i^\star(t) = \begin{cases} 1 & \text{if } \lambda^\star(t)^T b_i > 0 \\ -1 & \text{if } \lambda^\star(t)^T b_i < 0 \\ ? & \text{if } \lambda^\star(t)^T b_i = 0 \end{cases}$$

$$B = [b_1 | b_2 | \dots | b_{n_u}]$$

If ? is satisfied on a non-zero interval $t \in [t_1, t_2]$, then we call this interval a <u>SINGULAR ARC</u>. OC problems with singular arcs are called <u>SINGULAR OPTIMAL CONTROL PROBLEMS</u>. $\rightarrow$ problems in which condition ② of PMP does not provide any information $u^\star$ for a non-zero time-interval.

(Tutorial 6 (next Thursday) is on this topic.)

Let's try to understand when we can rule out that $\lambda^\star(t)^T b_i = 0$ for $t \in [t_1, t_2]$.

For this we need to study $\lambda^\star(t)$ which satisfy

$$\dot{\lambda}^\star = -A^T \lambda^\star \quad (\dot{\lambda} = -\mathcal{H}_x)$$

$$\rightarrow \lambda^\star(t) = e^{A^T(t_f^\star - t)} \lambda^\star(t_f^\star)$$

$$\rightarrow \lambda^\star(t)^T b_i = \underbrace{\lambda^\star(t_f^\star)^T \left(e^{A^T(t_f^\star - t)} b_i\right)}_{\text{when is this 0 on a non-infinitesimal interval?}}$$

This function is an analytic function:

- it is $\mathcal{C}^\infty$ (all derivatives exist)
- it coincides with its Taylor expansion on any neighborhood of its argument.

$$\forall t_0 \in D: \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n \to f(t)$$

Ex: exponentials, poly, logarithm.

Prop: Analytic functions vanish on finite intervals iff they vanish for all t, together with its derivatives.

We can study its derivatives and conditions for which they are zero at a generic t. We pick $t = t_f^\star$.

$$\begin{aligned}
0^{th} \text{ deriv:} \quad & \lambda^\star(t_f^\star)^T b_i \\
1^{th} \text{ deriv:} \quad & \lambda^\star(t_f^\star)^T A b_i \\
& \vdots \\
(n-1)^{th} \text{ deriv:} \quad & \lambda^\star(t_f^\star)^T A^{n-1} b_i
\end{aligned}$$

When are they all zero ($i = 1, \dots, n_u$)?

$$\underbrace{\begin{bmatrix} b_i & Ab_i & \dots & A^{n-1}b_i \end{bmatrix}^T}_{P} \quad \underbrace{\lambda^\star(t_f^\star)}_{\neq 0 \text{ because } x(t_f)=x_f} \quad = 0 \qquad ?$$

$\text{rank}(P) = n_x$ if the system is "normal", that is controllable w.r.t to any input channel u_i.

This is a stronger notion than controllability than the standard one obtained by "replacing $b_i \to B$".

Even if we cannot rule out singular arcs because (A, B) is not "normal", one can still show that for controllable LTI plants there exists an optimal controller switching between its boundary values as a function of $\lambda^\star(t)$.

$$\to u_i^\star = \begin{cases} 1 & \lambda^\star(t)^T b_i > 0 \\ -1 & \lambda^\star(t)^T b_i < 0 \end{cases} \quad \text{bang-bang-controller!}$$

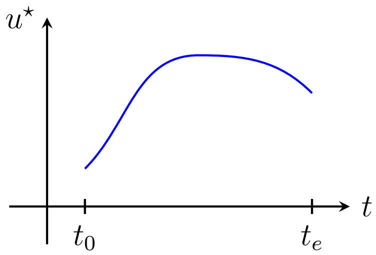
4 Dynamic Programming

In section 3 we learned how to solve **"open-loop optimal control problems"**.

Informally: Given

- f (model of the system)
- Cost function J + constraints
- Initial condition (t_0, x_0)

We find an optimal input trajectory $u^\star : [t_0, t_e] \to \mathbb{R}^{n_u}$.



Here (Chapter 4) we want to solve **"Closed-loop Optimal Control problems"**.

Given $\mathcal{U}, \mathcal{X}$, and the cost, we seek a feedback policy:

$$u = \mu(x, t)$$

$$\mu : \mathcal{X} \times [t_0, t_e] \to \mathcal{U} \subseteq \mathbb{R}^{n_u}$$

4.1 Multi-stage decision problems and the DP recursion

We consider discrete-time stages (k) with finite state and input (or action) spaces.

$$x \in \mathcal{X} := \{x^{(1)}, x^{(2)}, \ldots, x^{(N)}\} \quad \text{(Finite state space)}$$

$$u \in \mathcal{U} := \{u^{(1)}, \ldots, u^{(M)}\} \quad \text{(Finite input space)}$$

Example: $\mathcal{X} = \{1, 0, -1\}$; $N = 3$. Think about games (e.g., chess) as an example of where this scenario applies.

Dynamics:

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, 1, \ldots, T-1$$

Transition function mapping: $X_k \times U_k \to X_{k+1}$.

Problem: Starting from $x_0 \in X$, find the sequence of optimal inputs solving:

$$\min_{\{u_0, u_1, \ldots, u_{T-1}\}} \sum_{k=0}^{T-1} g_k(x_k, u_k) + g_T(x_T)$$

s.t. dynamic constraints.

Optimal solution: Sequence of optimal inputs $\{u_0^\star, u_1^\star, \ldots, u_{T-1}^\star\}$.

Optimal value (or Value Function): $J^\star(x_0)$: optimal value of the cost evaluated at x_0.

How can we approach the solution to this problem?

1. **"Brute force"** Starting from x_0, we can enumerate all possible trajectories going forward from 0 to T, calculate their costs, and select the best.

   The number of operations scales with M^T. This strategy is **computationally expensive** (Curse of Dimensionality). Furthermore, it yields an "open loop" solution for every specific x_0. If x_0 changes (or we have disturbances), we have to recompute everything; no replanning strategy.

2. **Dynamic Programming (DP) algorithm**

   > "Life can only be understood backward, but it must be lived forward."
   > — Kierkegaard

   Let's start from the end time T.

   - At $k = T$: For each $x_T \in X$, we can compute $g_T(x_T)$. This operation is independent of u.

   - At $k = T-1$: For each $x_{T-1} \in X$, we solve the following truncated problem:

     $$\min_{u_{T-1} \in U} [g_{T-1}(x_{T-1}, u_{T-1}) + g_T(x_T)]$$

     subject to $x_T = f_{T-1}(x_{T-1}, u_{T-1})$.

     Since $g_T(x_T)$ is pre-computed (or known), we can find the minimizer $u_{T-1}^\star(x_{T-1})$.

We define the **Cost-to-go** at $T-1$:

$$V_{T-1}(x_{T-1}) = \min_{u_{T-1} \in U} \left[g_{T-1}(x_{T-1}, u_{T-1}) + g_T(f_{T-1}(x_{T-1}, u_{T-1}))\right]$$

- Let's do the same at $k = T-2$:

$$V_{T-2}(x_{T-2}) = \min_{u_{T-2}, u_{T-1}} \left[g_{T-2}(x_{T-2}, u_{T-2}) + g_{T-1}(x_{T-1}, u_{T-1}) + g_T(x_T)\right]$$

Using the dynamic equation and the result from step $T-1$:

$$V_{T-2}(x_{T-2}) = \min_{u_{T-2} \in U} \left[g_{T-2}(x_{T-2}, u_{T-2}) + \underbrace{V_{T-1}(f_{T-2}(x_{T-2}, u_{T-2}))}_{\text{Cost-to-go from } T-1}\right]$$

This holds for all stages and can be written as the **DP recursion**:

For $k = T-1, \ldots, 0$:

$$V_k(x_k) = \min_{u_k \in U} \left[g_k(x_k, u_k) + V_{k+1}(f_k(x_k, u_k))\right]$$

with terminal condition $V_T(x_T) = g_T(x_T)$.

We have a sequence of value functions (Cost-to-go) $V_0(\cdot), V_1(\cdot), \ldots, V_T(\cdot)$. And a sequence of optimal **closed-loop controllers** (policies):

$$\{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{T-1}(\cdot)\}$$

$$\mu_k(x) \in \arg\min_{u \in U} \left[g_k(x, u) + V_{k+1}(f_k(x, u))\right]$$

The optimal control depends on the current state x and the stage k. This is **CLOSED-LOOP**.

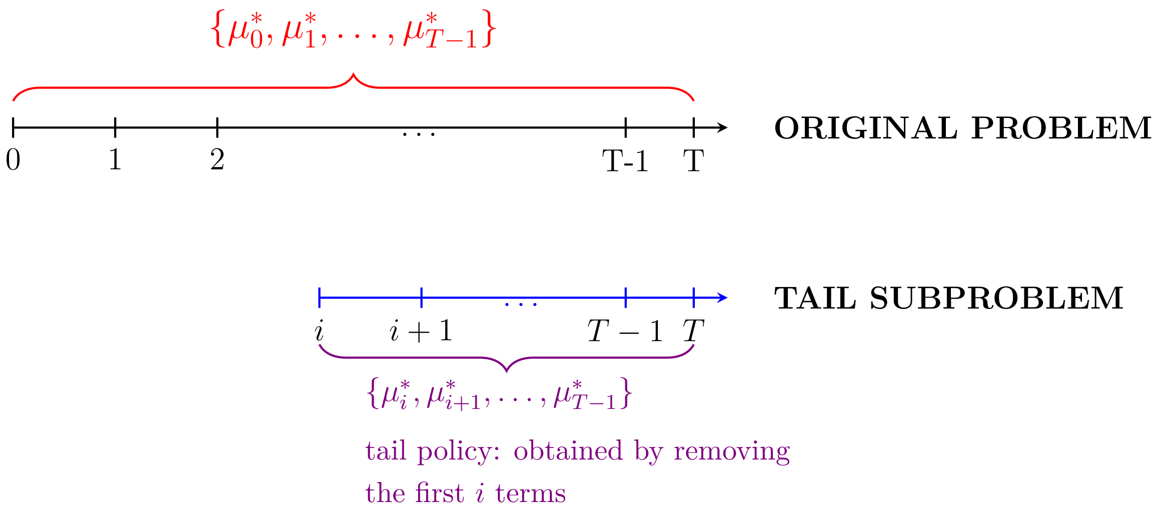
The number of operations required for this procedure scales with $T \cdot N \cdot M$ (Linear in T!). This was enabled by the Principle of Optimality.

Lemma 4.1 (Principle of Optimality - Multi-stage version)

Consider the problem [DP-P1] and assume $\{u_0^\star, u_1^\star, \dots, u_{T-1}^\star\}$ is an optimal policy. Consider the **Tail Subproblem** starting at time step $i \in \{1, \dots, T-1\}$ from state x_i, where we wish to solve:

$$\min_{\{u_i, \dots, u_{T-1}\}} \sum_{k=i}^{T-1} g_k(x_k, u_k) + g_T(x_T)$$

s.t. $x_{k+1} = f_k(x_k, u_k)$, given x_i.
Then the **tail policy** $\{u_i^\star, u_{i+1}^\star, \dots, u_{T-1}^\star\}$ (obtained by removing the first i terms from the original optimal policy) is optimal for the tail subproblem.



This is a central result which basically says that optimization of the future does not depend on what we did in the past. In the DP recursion, this principle guarantees that the paths we discard going back cannot be portions of optimal trajectories.

4.2 Hamilton-Jacobi-Bellman equation

Let us go back to the usual Optimal Control setting (Continuous time, nonlinear dynamics, etc.).

In section 3: $t \in [t_0, t_e]$, $x(t_0)$ fixed.

$$J(t_0, x_0, u) = \int_{t_0}^{t_e} l(t, x(\tau), u(\tau)) d\tau + \varphi(x(t_e))$$

In section 4: We do not want to solve 1 single problem (for t_0, x_0 fixed), but a "family of problems" parametrized by (t, x).

Formally, the problem can be stated as follows:

Find

$$\min_{u[t,t_e]} \int_t^{t_e} l(\tau, x(\tau), u(\tau))d\tau + \varphi(x(t_e))$$

s.t. $\dot{x} = f(\tau, x, u)$, $x(t) = x$ (initial condition for this subproblem).

We seek a closed-loop controller $u^\star(\cdot, \cdot) : \mathbb{R}^{n_x} \times [t_0, t_e] \to U$ which solves the problem $\forall (t, x)$.

Next time we will write the value function for this problem and characterize it through the principle of optimality. This will lead to the **Hamilton-Jacobi-Bellman (HJB) equation**.

$$\begin{aligned} &[DP - P2] \quad \min_{u_{[t,t_f]}} \quad \int_t^{t_f} l(s, x(s), u(s))\, ds + \varphi(x(t_f)) \\ &\dot{x} = f(t, x, u), \quad x(t) = x, \quad u \in \mathcal{U}, \quad (t_f, x(t_f)) \in \mathcal{S} \end{aligned}$$

What is the value function associated with $[DP - P2]$?

$$V(t, x) := \inf_{u_{[t,t_f]}} \{J(t, x, u) \mid \dot{x} = f(t, x, u), u \in \mathcal{U}, (t_f, x(t_f)) \in \mathcal{S}\}$$

$\to$ Extension of the cont. time setting of the value function/cost-to-go introduced in subsection 4.1.

$\Rightarrow$ optimal cost of the problem starting at time $t \in [t_0, t_f]$ from state $x \in \mathbb{R}^{n_x}$.

Lemma 4.2 (Regularity properties of V)

Assume:

- $f(t,x,u)$ bounded $\forall x \in \mathbb{R}^{n_x}, u \in \mathcal{U}$ and Lipschitz cont. in x.
- $l(t,x,u)$ bounded $\forall x \in \mathbb{R}^{n_x}, u \in \mathcal{U}$ and Lipschitz cont. in x.
- $\varphi(x)$ bounded $\forall x \in \mathbb{R}^{n_x}$ and Lipschitz cont.
- $\mathcal{U}$ compact.

Then the value fct V is bounded and uniformly Lipschitz continuous.
$\exists c, c' > 0$ s.t.

- $|V(t,x)| \le c \quad \forall t, \forall x$
- $|V(t_a,x_a) - V(t_b,x_b)| \le c'(|t_a - t_b| + \|x_a - x_b\|) \quad \forall t_a, t_b \in [t_0,t_f], \forall x_a, x_b \in \mathbb{R}^{n_x}$

We want to further char. V in order to compute it. First we know that V must satisfy bound. conditions.

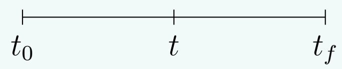
$$V(t_f,x) = \varphi(x) \qquad \forall x \in \mathbb{R}^{n_x},$$

because $\mathcal{S} = \{t_f\} \times \mathbb{R}^{n_x}$ fixed final time, free final state.

Second, we leverage the Principle of Optimality to characterize $V(t,x), t \in [t_0,t_f], \forall x$.

Lemma 4.3 (Principle of Optimality - cont. time)

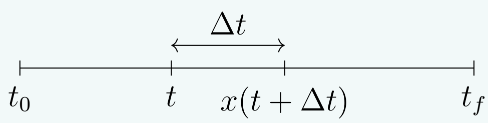
$\forall (t,x) \in [t_0,t_f] \times \mathbb{R}^{n_x}, \forall \Delta t \in (0, t_f - t]$.



The value function V satisfies the relation:

$$V(t,x) = \inf_{u_{[t,t+\Delta t]}} \int_t^{t+\Delta t} l(s,x(s),u(s))\,ds + V(t+\Delta t, x(t+\Delta t))$$

where $x(\cdot)$ is the state traj. starting at $x(t) = x$ and subject to $u_{[t,t+\Delta t]}$.



Interpretation: The optimal cost over any interval $[t,t_f]$ can be split into the "integrated cost" over $[t,t+\Delta t]$ plus the value function at $t+\Delta t$, starting from $x(t+\Delta t)$.

Dynamic equation in V, that appears on the left (at t,x) and on the right (at $t+$

$\Delta t, x(t+\Delta t))$. If we find V that satisfies Lemma 4.3, then we found the value function of our problem.

How to find V that satisfies Lemma? The one above is a functional equation $\to$ difficult to solve. We work on its "infinitesimal version" $\to$ we have Taylor expansions of the terms on the right hand side.

① $x(t+\Delta t) = x + f(t,x,u(t))\Delta t + o(\Delta t), \quad (x = x(t))$

If we assume that $V \in \mathcal{C}^1$ then:

② $V(t+\Delta t, x(t+\Delta t)) = V(t,x) + V_t(t,x)\Delta t + \langle V_x(t,x), f(t,x,u(t))\Delta t\rangle + o(\Delta t)$

③ $\displaystyle\int_t^{t+\Delta t} l(s,x(s),u(s))\,ds = l(t,x,u(t))\Delta t + o(\Delta t)$

Replace 1-2-3 in Lemma

$$V(t,x) = \inf_{u_{[t,t+\Delta t]}} (l(t,x,u(t))\Delta t + V(t,x) + V_t(t,x)\Delta t + \langle V_x(t,x), f(t,x,u(t))\Delta t\rangle + o(\Delta t))$$

$V(t,x)$ cancel out (note that $V(t,x)$ does not depend on u, hence I can take it out of the infimum).

We go infinitesimal by taking the limit $\Delta t \to 0$.

If we divide by Δt and then take the limit, we have that

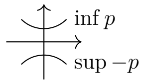
- $\frac{o(\Delta t)}{\Delta t} \to 0$ because $o(\Delta t)$ is second or higher order.
- $\inf\limits_{u[t,t+\Delta t]} \to \inf\limits_{u} \quad \Rightarrow$ infimum over instantaneous value of u.

So we obtain:

$$-V_t(t,x) = \inf_{u\in\mathcal{U}} l(t,x,u) + \langle V_x(t,x), f(t,x,u)\rangle \quad \forall x \in \mathbb{R}^{n_x}, t \in [t_0, t_f]$$

(with bound. condition $V(t_f, x) = \varphi(x)$).

$$\inf p = -\sup(-p)$$



$$\Rightarrow \quad +V_t(t,x) = +\sup_{u\in\mathcal{U}} \mathcal{H}(t,x,u,-V_x(t,x))$$

where $\mathcal{H}(t,x,u,\lambda) = -l(t,x,u) + \langle \lambda(t), f(t,x,u)\rangle\Big|_{\lambda=-V_x(t,x)} = -V_x(t,x)$

$\to$ equivalent reformulation of HJB in terms of $\mathcal{H}$.

What about the optimal controller?

So far, we have not even assumed that it exists, so we use "if" throughout. If $u^\star$ exists, then it satisfies

$$V(t, x^\star(t)) = \int_t^{t+\Delta t} l(s, x^\star(s), u^\star(s))\, ds + V(t + \Delta t, x^\star(t + \Delta t))$$

where $x^\star(t)$ is the optimal state trajectory (state evolution under $u^\star$).

Repeating the same steps we did before, this means $u^\star$ satisfies

$$-V_t(t, x^\star(t)) = l(t, x^\star(t), u^\star(t)) + \langle V_x(t, x), f(t, x^\star(t), u^\star(t))\rangle \quad \forall t$$

$\to$ in other words:

$$u^\star(t, x) \in \arg\min_{u \in \mathcal{U}} l(t, x, u) + \langle V_x(t, x), f(t, x, u)\rangle \qquad \forall t$$

Once we found V, we get $u^\star$ in this way.

HJB + minimization condition of u, solve our problem $[DP - P2]$. HJB is a Nonlinear, time varying PDE with an inf inside $\to$ rarely tractable!

Example 4.1 $\dot{x} = u, \quad x, u \in \mathbb{R}, \quad l(x, u) = x^4 + u^4, \quad J = \int_t^{t_f} l(x, u)\, ds, \quad \varphi = 0$

HJB:

$$-V_t(t, x) = \inf_{u \in \mathbb{R}} \underbrace{x^4 + u^4 + V_x(t, x)u}_{\text{this function is convex}}$$

Generally we want to get rid of the inf. Here, this is simple because rhs. is a convex problem, thus we can solve it by setting the gradient to 0.

$$4u^3 + V_x(t, x) = 0 \quad \Rightarrow \quad u = -\left(\frac{1}{4} V_x(t, x)\right)^{\frac{1}{3}}$$

This step solves the "inf" problem. We replace this u in the HJB and obtain

$$-V_t(t, x) = x^4 - 3\left(\frac{1}{4} V_x(t, x)\right)^{\frac{4}{3}} \qquad \text{standard PDE}$$

$\rightarrow$ Once we have a solution V, the optimal controller is

$$u^\star(t,x) = -\left(\frac{1}{4}V_x(t,x)\right)^{\frac{1}{3}}$$

There is a special case where HJB simplifies:

<u>Infinite horizon</u> ($t_f := +\infty$) and <u>time-invariant problems</u> (f, l do not depend on t).

In this case, V does not depend on t! Value function $V(x), x \in \mathbb{R}^{n_x}$. Because time does not play any role in the optimality.

Let's go back to the previous example and set $t_f = \infty$. Then $V_t = 0$ and HJB:

$$x^4 - 3\left(\frac{1}{4}V_x(x)\right)^{\frac{4}{3}} = 0, \quad x \in \mathbb{R}$$

We can directly extract V_x from this equation and plug it in $u^\star$.

$$\rightarrow u^\star = -\left(\frac{1}{4}V_x(x)\right)^{\frac{1}{3}} = -\left(\frac{1}{3}\right)^{\frac{1}{4}} x$$

Sometimes, the sign of V must be chosen by YOU! If the stage cost is non-negative (like here), then V must also be non-negative.

What we have seen so far is that (assuming $V \in \mathcal{C}^1$)

- if V is a value function (= satisfies the Principle of Optimality) then V must satisfy the HJB.
- if $u^\star$ exists, then it must be given by the minimization condition seen before.

  $\rightarrow$ HJB is necessary condition for V and $u^\star$

Actually, the implication can be proven also in the other direction.

$\rightarrow$ HJB are Nec. and Suf. conditions for optimal controllers and value functions (in finite horizon problems) (see proof in T.9).

Note: For the infinite horizon case, we need an additional condition to make HJB sufficient (see T.9).

PMP vs. HJB Let's compare those 2 approaches to finding optimal controllers.

PMP	HJB
Type of opt. condition	
$u^\star$ satisfies necessary conditions for local (strong) minimizers.	$u^\star$ satisfies necessary and sufficient conditions for global minimizers.
$\rightarrow$ we do not even know if $u^\star$ is opt.	$\rightarrow$ $u^\star$ is the optimal controller.
Open loop vs. closed loop	
"Informally" $\begin{cases} \dot{x} = \mathcal{H}_\lambda \\ \dot{\lambda} = -\mathcal{H}_x \end{cases}$ + B.C.	HJB to determine V.
$u^\star(t) \in \underset{u \in \mathcal{U}}{\operatorname{argmax}}\ \mathcal{H}(t, x(t), u, \lambda(t))$	$u^\star(x,t) \in \underset{u \in \mathcal{U}}{\operatorname{argmax}}\ \mathcal{H}(t, x(t), u, -V_x(t,x))$
$\rightarrow$ $u^\star$ is open-loop because it depends on λ which is "computed offline" (not measured).	$\rightarrow$ $u^\star$ is closed-loop because it depends on t and x.
Computational complexity	
TPBVP (two point boundary val. prob.)	Nasty PDE
ODE + maximization condition	(+ assumption $V \in \mathcal{C}^1$)
$\rightarrow$ very well developed numerical methods to solve this problem (Something in section 5).	$\rightarrow$ still today an open problem for general classes of systems.
Proof technique	
We did not see in lecture (see Liberzon's book $\sim$ 10 pages).	We saw it in the lecture, quite easy to show.

Can we prove PMP with HJB by seeing it as a special case?

Assume that $u^\star, x^\star$ are optimal. Assume time-invariant problem (like in PMP) $\rightarrow$ $f(x,u), l(x,u)$, no time.

Then HJB guarantees that $u^\star, x^\star$ satisfy

$$-V_t(t, x^\star) = l(x^\star, u^\star) + \langle V_x(x^\star), f(x^\star, u^\star) \rangle$$

where V is the value function.

To show PMP, we need to show:

- Hamiltonian dynamics
- Max condition $\rightarrow$ if we define the costate $\lambda^\star(t) := -V_x(t, x^\star)$, then the max condition is automatically satisfied.

So we conjecture that a quantity denoted as $\lambda^\star$ and defined as above must exist.

What is left to show are the Ham. dynamics.

$$\dot{x} = \mathcal{H}_\lambda \rightarrow \text{trivial, } \dot{x} = f.$$

$$\dot{\lambda} = -\mathcal{H}_x + \text{B.C.}$$

BC for our problem $S = \{t_f\} \times \mathbb{R}^{n_x}$ is $\lambda^\star(t_f) = -\varphi_x(x^\star(t_f))$ (see section 3, transversality condition ELE, PMP).

Indeed, from the HJB we use that

$$V(t_f, x) = \varphi(x) \quad \text{BC of HJB}$$

Therefore $\lambda^\star = -V_x = -\varphi_x$

The only thing that is left is to show that if we have $\lambda^\star = -V_x$, V satisfying HJB, then $\dot{\lambda}^\star = -\mathcal{H}_x$.

This is true! We can use the relationship $\lambda = -V_x$ to interpret λ as the sensitivity of the optimal cost wrt x.

Question: Why can't we just use HJB to show PMP?

Underlying HJB there is the assumption that $\underline{V \in \mathcal{C}^1}$. This does not always hold.

Example 4.2:

$$\dot{x} = xu, \quad x \in \mathbb{R}, \quad u \in [-1, 1]$$

$$S = \{t_f\} \times \mathbb{R}^{n_x}$$

$$J(u) = x(t_f), \quad (l = 0, \varphi = x)$$

$u^\star$?

$$\begin{cases} +1, & x_0 < 0 \quad (\dot{x} = x) \\ -1, & x_0 > 0 \quad (\dot{x} = -x) \end{cases}$$

optimal cost $J^\star$?

$$x(t_f) = \begin{cases} e^{(t_f-t_0)}x_0, & x_0 < 0 \\ e^{-(t_f-t_0)}x_0, & x_0 > 0 \\ 0, & x_0 = 0 \quad (x \equiv 0) \end{cases}$$

From this, we can easily obtain the value function

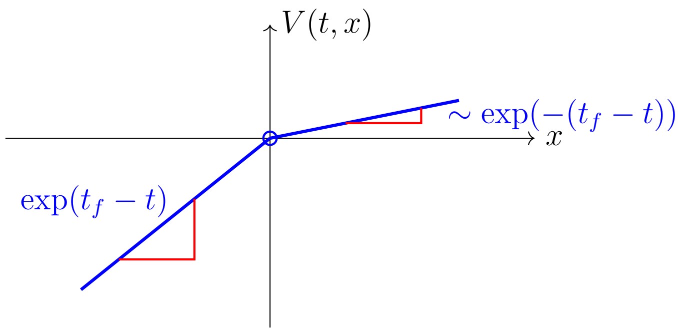
$$V(t,x) = \begin{cases} e^{(t_f-t)}x, & x < 0 \\ e^{-(t_f-t)}x, & x > 0 \\ 0, & x = 0 \end{cases} \qquad x \in \mathbb{R}^{n_x}, \quad t \in [t_0, t_f] \qquad \circledast$$

What is the HJB for this problem?

$$-V_t = \inf_u \{V_x \overbrace{xu}^{f} \underset{l=0}{} \} = -|V_x x|$$

Check that $\circledast$ satisfy the HJB. But only away from "$x = 0$".

At $x = 0$, V is not $\mathcal{C}^1$.



This is not a pathological case. Non-differentiability arises often for problems with constraints and/or terminal costs.

Note: V has some regularity property (see Lemma 4.2). Specifically, it is Lipschitz continuous. From Rademacher theorem, if a function is Lip. cont. then it is "almost everywhere" differentiable. ($\rightarrow$ it can only be non-differentiable at isolated points).

How do we deal with the possibility that $V \notin \mathcal{C}^1$? $\rightarrow$ viscosity solution.

Some concepts from "nonsmooth analysis".

Definition 4.1 (super-differential and sub-differential)

Let $v : \mathbb{R}^{n_x} \to \mathbb{R}$ be a continuous function.
$\zeta \in \mathbb{R}^{n_x}$ is a super-differential of v at x if

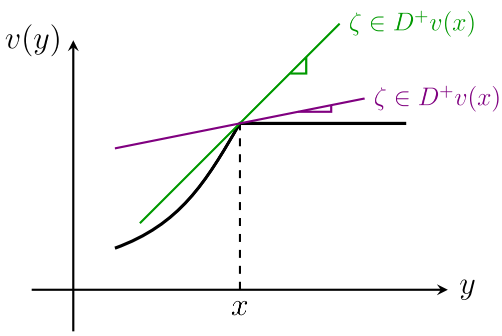
$$v(y) \leq v(x) + \langle \zeta, y - x \rangle + \underbrace{o(\|y - x\|)}_{\text{higher order term } (2^{nd} \text{ or higher}} \qquad \forall y \in \mathbb{R}^n$$

... is a sub-differential of v at x if

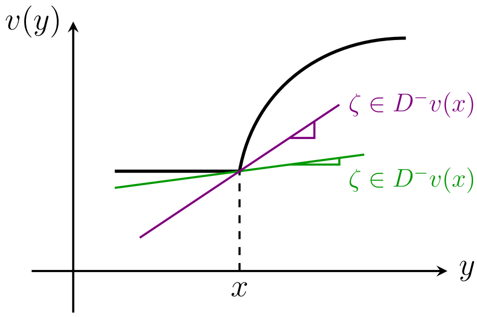
$$v(y) \geq v(x) + \langle \zeta, y - x \rangle + o(\|y - x\|) \quad \forall y \in \mathbb{R}^{n_x}$$

$D^+v(x)$ denotes the set of all super-diff of v at x.
$D^-v(x)$ denotes the set of all sub-diff of v at x.

Geometrically:



The affine function
$y \to v(x) + \langle \zeta, y - x \rangle$
upper bounds $v(y)$
at least locally



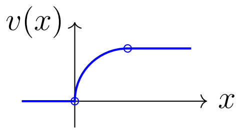
Here the affine function lower bounds $v(y)$

Lemma: Relation with classical differentials $\to$ gradients.

$v \in \mathcal{C}^1$ at x iff $D^+v(x) = D^-v(x) = \{\nabla v(x)\}$.

Example 4.3

$$v(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



Determine $D^+v(x)$ and $D^-v(x)$ for the 3 ranges of x: $\begin{cases} x < 0 \\ 0 \leq x \leq 1 \\ x > 1 \end{cases}$

Viscosity solution of a PDE

PDE in generic form:

$$F(y, v(y), \nabla v(y)) = 0 \quad \blacksquare$$

$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function.

Think about F associated with HJB where $y = \begin{bmatrix} x \\ t \end{bmatrix}$, $v = V(t, x)$, $n = n_x + 1$.

Definition 4.2

Viscosity subsolution of $\blacksquare$ is a function v such that

$$F(y, v(y), \zeta) \leq 0 \quad \forall \zeta \in D^+v(y), \forall y$$

Viscosity supersolution of $\blacksquare$ is a function v such that

$$F(y, v(y), \zeta) \geq 0 \quad \forall \zeta \in D^-v(y), \forall y$$

Viscosity solution of $\blacksquare$ is a function v such that is Both a sub and a supersolution.

$$\forall y \begin{cases} \zeta \in D^+v(y) \Rightarrow F(y, v(y), \zeta) \leq 0 & \text{(subsolution)} \\ \zeta \in D^-v(y) \Rightarrow F(y, v(y), \zeta) \geq 0 & \text{(supersolution)} \end{cases}$$

Let's take now F as our HJB. It turns out that viscosity solutions of the HJB give the (potentially non-differentiable) value function of the optimal control problem.

Note: under the assumptions on l, f, φ in Lemma 4.2, this solution is unique.

$$\underbrace{-V_t(t,x) - \inf_{u \in \mathcal{U}} \left(l(t,x,u) + \langle V_x(t,x), f(t,x,u)\rangle\right)}_{F\left(\left[\begin{smallmatrix} x \\ t \end{smallmatrix}\right], V(t,x), \left[\begin{smallmatrix} V_x \\ V_t \end{smallmatrix}\right]\right)} = 0$$

The viscosity solution <u>always</u> gives us the value function V (no $V \in \mathcal{C}^1$ assumption is required).

This solves the problem in the general case where $V \notin \mathcal{C}^1$ everywhere. You can check by exercise that V in Ex. 4.2 is a Viscosity Solution (you need to check separately the cases $\underbrace{x = 0}_{\substack{\downarrow \\ \text{Sub/super viscosity solutions}}}$ and $\underbrace{x \neq 0}_{\substack{\downarrow \\ \text{classic HJB}}}$).

4.3 DP methods for discrete-time infinite horizon problems

4.3.1 Problem formulation and the Bellman equation

Motivation for the material discussed in 4.3:

- Overcome some of the limitations of HJB (tractability).
- Methods used as basis in many Reinforcement Learning (RL) algorithms.

Major changes compared to Section 4.2:

- Dynamical Systems described in <u>discrete-time</u> and time-invariant:

$$x_{k+1} = f(x_k, u_k), \quad x \in \mathbb{R}^{n_x},\ u \in \mathbb{R}^{n_u}$$

  ($\rightarrow$ Markov Decision Processes, T8)
  (x, u are discrete).

- Infinite horizon (unbounded time-interval).

As in the rest of the chapter, we are interested in closed-loop controllers:

$$\mathcal{M} = \{\mu \mid \mu : x \in X \subseteq \mathbb{R}^{n_x} \to u \in \mathcal{U}(x) \subseteq \mathbb{R}^{n_u}\} \quad \text{input/policy class}$$

The input constraints set $\mathcal{U}(x)$ can be a function of x.

We optimize over the set of policies:

$$\Pi = \{\{\mu_0, \mu_1, \mu_2, \dots\} \mid \mu_k \in \mathcal{M},\ k = 0, 1, 2, \dots\} \quad \text{Set of policies}$$

Goal: Find $\pi^\star \in \Pi$ that solves the OC problem. What is the problem we want to solve (4.3)?

Find $\pi^\star \in \Pi$ solving $\forall x_0 \in X$:

$$\min_{\pi \in \Pi} \ J_\pi(x_0) \coloneqq \lim_{N \to \infty} \sum_{k=0}^{N-1} \alpha^k g(x_k, u_k) \qquad [DP - P3]$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ u_k = \mu_k(x_k),\ u_k \in \mathcal{U}(x_k),\ x_0 \text{ given.}$$

where $\alpha \in (0, 1]$ is the discount factor.

Why α?

(i) Mathematically: it makes it easier to argue existence of limit (convergence of the series).

(ii) Control design: one might want to discount future costs to prioritize "present" costs.

Value function of problem DP-P3:

$$J^\star(x) \coloneqq \inf_{\pi \in \Pi} J_\pi(x), \quad \forall x \in X$$

Not surprisingly, $J^\star$ depends on x, but not on k. ($\to$ because $N \to \infty$ and time-invariant problem, analogous to 4.2).

We want to characterize $J^\star$. Consider:

$$\pi = \{\mu_0, \mu_1, \mu_2, \dots\} \quad \text{generic policy}$$

$$\pi_1 = \{\mu_1, \mu_2, \mu_3, \dots\} \quad \text{shifted policy (first entry dropped)}$$

Then the cost can be written recursively:

$$J_\pi(x) = g(x, \mu_0(x)) + \alpha J_{\pi_1}(f(x, \mu_0(x))), \quad x \in X$$

Why? $J_\pi(x) = g(x, \mu_0(x)) + \lim_{N\to\infty} \sum_{k=1}^{\infty} \alpha^k g(x_k, \mu_k(x_k))$.

Using the Principle of Optimality logic:

$$\begin{aligned}
J^\star(x) &= \inf_\pi J_\pi \\
&= \inf_{\pi=\{\mu_0,\pi_1\}} [g(x, \mu_0(x)) + \alpha J_{\pi_1}(f(x, \mu_0(x)))] \\
&= \inf_{\mu_0 \in \mathcal{M}} \left[g(x, \mu_0(x)) + \alpha \inf_{\pi_1} J_{\pi_1}(f(x, \mu_0(x))) \right] \\
&= \inf_{\mu_0 \in \mathcal{M}} (g(x, \mu_0(x)) + \alpha J^\star(f(x, \mu_0(x))))
\end{aligned}$$

Equivalently:

$$\boxed{J^\star(x) = \inf_{u \in \mathcal{U}(x)} (g(x, u) + \alpha J^\star(f(x, u)))}$$

Bellman Equation $\rightarrow$ the functional equation that the value function of [DP-P3] must satisfy.

In 4.1 we found the "recursion" $V_k(x_k) = \min_{u_k} g_k(x_k, u_k) + V_{k+1}(f_k(x_k, u_k))$.

There is a close connection.

If we have $J^\star$, we can find an optimal stationary policy $\{\mu^\star, \mu^\star, \dots\}$ as:

$$\mu^\star(x) \in \arg \min_{u \in \mathcal{U}(x)} g(x, u) + \alpha J^\star(f(x, u)), \quad x \in X$$

As in HJB, we have 1 equation for the value function and 1 equation for the optimal controller. Is it easier (than HJB) to solve the value function equation? Maybe. We need a small detour in operator theory ($\rightarrow$ maps between functions).

Mathematical preliminaries

Definition 4.3 (Monotone and contractive operators (or mapping))

Let Y be a finite dim. vector-space (e.g. $\mathbb{R}^{n_x}$).
$S(Y)$: vector space of real-valued functions

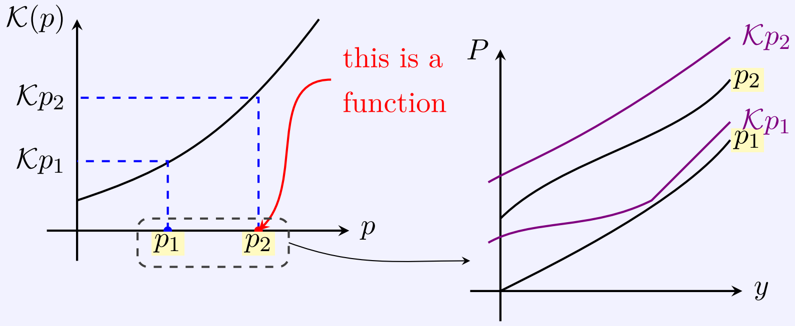
$$S : Y \to \mathbb{R}^n \quad \text{(in our case } n = 1\text{)}$$

(Note: for us, this is the space where $J^\star$ lives $J^\star : X \to \mathbb{R}$).
A mapping $\mathcal{K} : S(Y) \to S(Y)$ ("from functions to functions") is <u>monotone</u> iff:

$$\langle \mathcal{K}p_1 - \mathcal{K}p_2, p_1 - p_2 \rangle \geq 0, \quad \forall p_1, p_2 \in S(Y).$$

The definition depends on the choice of inner product. In our case ($n = 1$), this simplifies to:

$$p_2(y) \geq p_1(y) \implies (\mathcal{K}p_2)(y) \geq (\mathcal{K}p_1)(y), \quad \forall y \in Y$$



$p_2(y) \geq p_1(y), \ \forall y \in Y$

Let $P(Y)$ be the vector space of functions $p : Y \to \mathbb{R}^n$ s.t.

$$\|p\|_\infty = \sup_{y \in Y} |p(y)| < \infty$$

finite sup-norm (or ∞-norm).
$\mathcal{K} : P(Y) \to P(Y)$ is a <u>contraction</u> if $\exists \alpha \in [0, 1)$ s.t.:

$$\|\mathcal{K}p_1 - \mathcal{K}p_2\|_\infty \leq \alpha \|p_1 - p_2\|_\infty, \quad \forall p_1, p_2 \in P(Y).$$

α is the modulus of contraction.

Definition 4.4 (Fixed point of a mapping)

$\mathcal{K} : P(Y) \to P(Y)$.
A fixed point of $\mathcal{K}$ is $x \in P(Y)$ which is mapped onto itself, i.e.,

$$\mathcal{K}x = x.$$

Theorem 4.1 (Banach Fixed Point Theorem (Theorem 4.2))

Let $(P(Y), \|\cdot\|_\infty)$ be a complete normed space ($\to$ Banach space).
Let $\mathcal{K}$ be a contractive mapping.
Then $\mathcal{K}$ has a unique fixed point $x \in P(Y)$.

Very important! In general operators might not have or might have multiple fixed points. We also have a "simple way" of finding the fixed point:

Fixed point iteration: start with x_0 and iterate as follows:

$$x_{n+1} = \mathcal{K}x_n, \quad n = 0, 1, 2, \ldots$$

Using contraction we know that:

$$\|x_{n+1} - x_n\|_\infty \leq \alpha \|x_n - x_{n-1}\|_\infty$$

If we unroll the right-side:

$$\|x_{n+1} - x_n\|_\infty \leq \alpha^n \|x_1 - x_0\|_\infty$$

4.3.2 Bellman operators and properties

We give an operator viewpoint on the relationships found earlier $(J^\star, \mu^\star)$.

DP operators $\to T_\mu, T : S(\mathcal{X}) \to S(\mathcal{X})$ ($s : \mathcal{X} \to \mathbb{R}$, like $J^\star, J_\pi$).

$\boxed{T_\mu}$ For given $\mu \in \mathcal{M}$:

$$(T_\mu J)(x) := g(x, \mu(x)) + \alpha J(f(x, \mu(x))), \quad J \in S(\mathcal{X}), x \in \mathcal{X}$$

$\boxed{T}$ The Bellman Operator:

$$(TJ)(x) := \inf_{u \in \mathcal{U}(x)} \{g(x, u) + \alpha J(f(x, u))\} = \inf_{\mu \in \mathcal{M}} (T_\mu J)(x)$$

The Bellman equation can be written as:

$$\forall x \in X, \quad J^\star(x) = (TJ^\star)(x)$$

① $J^\star = TJ^\star \implies J^\star$ is the fixed point of T.

② The cost of policy μ (J_μ) is the fixed point of T_μ:

$$J_\mu(x) = g(x, \mu(x)) + \alpha J_\mu(f(x, \mu(x))), \quad x \in \mathcal{X}$$

$$\implies J_\mu = T_\mu J_\mu.$$

③ Finally, the condition for policy $\mu^\star$ to be optimal can be written as:

$$T_{\mu^\star} J^\star(x) = TJ^\star(x), \quad x \in \mathcal{X}.$$

Summary:

- ① Bellman equation
- ② Cost of policy
- ③ Optimal policy

They have an operator representation!

Theorem 4.2 (Theorem 4.3: Monotonicity & Contraction of operators in DP)

Given problem [DP-P3]. If:

- $\alpha \in (0, 1)$
- bounded cost $|g(x, u)| < \infty, \ \forall x \in \mathcal{X}, \forall u \in \mathcal{U}$

Then T_μ and T are monotone and contractive.

4.3.3 Algorithmic strategies

1) Limited Lookahead policies Given $\hat{J} \in P(\mathcal{X})$ approximation of $J^\star$. We can obtain a policy $\hat{\mu}$ by "treating $\hat{J}$ as if it was $J^\star$":

$$\hat{\mu}(x) \in \arg\min_{u \in \mathcal{U}(x)} g(x, u) + \alpha \hat{J}(f(x, u)), \quad x \in \mathcal{X}$$

Equivalently: $T_{\hat{\mu}}\hat{J} = T\hat{J}$.

$\hat{\mu}$ is called one-step lookahead policy because it solves an optimal control problem with "one step" and terminal cost $\alpha\hat{J}$.

Thanks to the contraction property of T, T_μ we can bound the suboptimality of $\hat{\mu}$ as a function of the difference between $\hat{J}$ and $J^\star$.

Lemma 4.4 (Lemma 4.4)

Assume T, T_μ contractive, $\hat{\mu}$ is a one-step look-ahead policy associated with $\hat{J}$. Then

$$\| \underbrace{J_{\hat{\mu}}}_{\substack{\text{Cost achieved} \\ \text{by } \hat{\mu}}} - \underbrace{J^\star}_{\text{Value fcn}} \| \leq \frac{2\alpha}{1-\alpha} \| \underbrace{\hat{J}}_{\substack{\text{approximate value function} \\ \text{might be diff to compute if we don't know } J^\star}} - J^\star \|$$

Alternative bound:

$$\|J_{\hat{\mu}} - J^\star\| \leq \frac{2}{1-\alpha}\|\hat{J} - T\hat{J}\|$$

if $\hat{J} = J^\star$ then $\hat{J} = T\hat{J}$ because $J^\star(=\hat{J})$ is the fixed point of T.

What happens if you use "multisteps"? We are given an approximate J_m (like $\hat{J}$ before) and we compute sequences.

1) Value sequence:

$$\left\{ J_i(x) = \min_{u \in \mathcal{U}(x)} g(x,u) + \alpha J_{i+1}(f(x,u)) \right\}_{i=0,\dots,m-1}$$

$$\begin{aligned} i = m-1: &\quad J_{m-1} = \dots J_m \\ i = m-2: &\quad J_{m-2} = \dots J_{m-1} \\ &\quad \vdots \\ &\quad \{J_0, J_1, \dots, J_{m-1}\} \end{aligned}$$

2) Policy sequence:

$$\mu_i(x) \in \operatorname*{argmin}_{u \in \mathcal{U}(x)} g(x,u) + \alpha J_{i+1}(f(x,u))$$

$$\{\mu_0, \mu_1, \dots, \mu_{m-1}\}$$

Any improvement wrt One-step?

Lemma 4.5

Assume that DP operators are contractive. And we apply

$$\pi = \{\mu_0, \mu_1, \dots, \mu_{m-1}, \mu_0, \mu_1, \dots, \mu_{m-1}, \dots\}$$

Then:

$$\|J_\pi - J^\star\| \leq \frac{2\alpha^m}{1-\alpha^m}\|J_m - J^\star\|$$

- if $m = 1$, we recover the previous Lemma.
- if $m > 1$ then $\frac{2\alpha^m}{1-\alpha^m} < \frac{2\alpha}{1-\alpha}$.

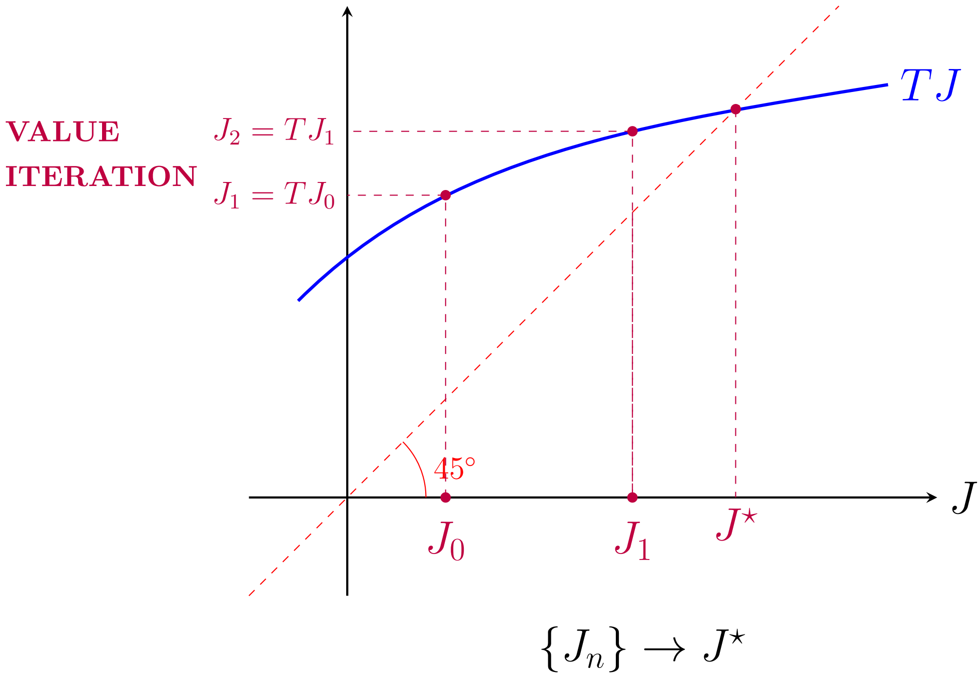
For the same approximation error, by increasing m, we can decrease the suboptimality gap.

2) Value Iteration (VI) straightforward application of Banach fixed point theorem.

Given an initial estimate $J_0 \in P(\mathcal{X})$, we apply fixed point iteration to the Bellman operator

$$J_{n+1} = TJ_n = \underbrace{T^{n+1}}_{T^n(TJ)} J_0, \quad n = 0, 1, 2 \dots$$

We can graphically represent operators acting on function spaces.



Lemma 4.6 (Properties of VI)

Assume contraction of the DP operators, $J_0 \in \mathcal{P}(X)$. Then

- VI converges to $J^\star$
- rate of convergence I

$$\|T^n J_0 - J^\star\| \leq \alpha^n \|J_0 - J^\star\|, \quad n = 0, 1, \ldots$$

- rate of convergence II

$$\|T^n J_0 - J^\star\| \leq \frac{\alpha}{1-\alpha} \underbrace{\|T^{n+1} J_0 - T^n J_0\|}_{\text{always computable}}, \quad n = 0, 1, \ldots$$

Note: $T^n J_0 = J_n$, $T^{n+1} J_0 = J_{n+1}$.

Typically VI is run for a number of steps until we obtain an approximation $\hat{J}$, with which we compute an approximate policy:

$$\hat{\mu}(x) \in \arg\min_{u \in \mathcal{U}(x)} g(x, u) + \alpha \hat{J}(f(x, u))$$

By combining Lemma 4.6 and one-step lookahead policy ($\hookrightarrow$ Lemma 4.4), we can bound suboptimality of $\hat{\mu}$:

$$\|J_{\hat{\mu}} - J^\star\| \leq \frac{2\alpha}{1-\alpha} \underbrace{\|\hat{J} - J^\star\|}_{\substack{\text{estimated with Lemma 4.6,} \\ \text{because } \hat{J} \text{ comes from VI}}}$$

3) Policy Iteration (PI)

In VI, the policy μ does not play any role in the algorithm. It is only estimated at the end to approximate the desired optimal policy. In PI, we instead construct a sequence of policies:

$$\{\mu^k\} \leftarrow \text{iteration counter is in the superscript}$$

PI has 2 steps. We start with μ_0 admissible.

$\boxed{1}$ **Policy evaluation** (input $\mu_k \to J_{\mu^n}$)

$$J_{\mu^n} = T_{\mu^n} J_{\mu^n}$$

given a policy μ^n, we determine "its value" i.e. the cost associated with it.

$\boxed{2}$ **Policy improvement** (input: J_{μ^n}, output: μ^{n+1})

$$T_{\mu^{n+1}} J_{\mu^n} = T J_{\mu^n}$$

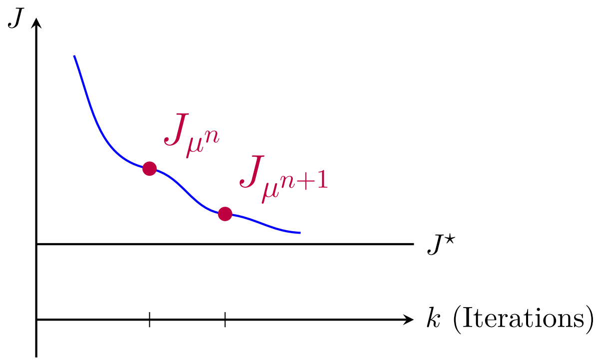
we pretend that J_{μ^n} is $J^\star$ and compute $\mu^{n+1} \in \arg\min \ldots J_{\mu^n}$.

Lemma 4.7 (Properties of PI)

Assume monotonicity and contraction of DP operators. Let $\{\mu^k\}$ sequence generated by PI.
Then

- $\lim_{k\to\infty} \|J_{\mu^k} - J^\star\| = 0$
- and precisely $\|J_{\mu^{k+1}} - J^\star\| \leq \alpha \|J_{\mu^k} - J^\star\|$

That is $J_{\mu^{n+1}} \leq J_{\mu^n}$, with equality (=) only when $J_{\mu^n} = J^\star$.



- If $\mathcal{M}$ is finite (set of policies is finite; subsection 4.1, MDP etc.) then $\exists \bar{n}$ s.t. $J_{\mu^{\bar{n}}} = J^\star$. $\to$ PI converges in finite steps.

VI and PI are examples of ADP (Approximate DP) techniques. Are widely used in RL and applications (robotics) $\to$ many open questions & research directions.

4.3.4 Closing example: infinite horizon LQR

$$\min_{\pi\in\Pi} J_\pi(x_0) := \lim_{N\to\infty} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k, \quad Q \succeq 0, R \succ 0$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \quad u_k = \mu_k(x_k), \; x_0 \text{ given}$$

$$u_k \in \mathbb{R}^{n_u}$$

$\boxed{\text{LQR}}$ specializes $\boxed{\text{DP-P3}}$ to the case of linear dynamics, quadratic convex costs. $\alpha = 1$ because we have that (A, B) stabilizable and this guarantees existence of limit.

We make the Ansatz $J^\star = x^T P^\star x, \quad P^\star \succeq 0$.

- Bellman eq.
- VI
- PI

What is the Bellman equation for our problem?

$$J^\star(x) = \inf_{u \in \mathbb{R}^{n_u}} x^T Q x + u^T R u + J^\star(Ax + Bu), \quad \forall x$$

Now we replace our guess (Ansatz) for $J^\star$:

$$x^T P^\star x = \inf_u \underbrace{x^T Q x + u^T R u + (Ax + Bu)^T P^\star (Ax + Bu)}_{f(u)}, \quad \forall x$$

Find $\inf_u f(u)$**?**

$$f(u) = x^T Q x + x^T A^T P^\star A x + 2u^T B^T P^\star A x + u^T (R + B^T P^\star B) u$$

$$\nabla_u f = 2(R + B^T P^\star B) \succ 0 \qquad \text{strongly convex, unique minimizer}$$

To find $u^\star$ it is enough to set $\nabla f = 0$.

$$\nabla f(u^\star) = 0 \quad \Leftrightarrow \quad 2(R + B^T P^\star B) u^\star + 2B^T P^\star A x = 0$$

$$u^\star = -\underbrace{(R + B^T P^\star B)^{-1} B^T P^\star A}_{K^\star} x = K^\star x$$

$(R + B^T P^\star B)$ has an inverse because $(\cdot) \succ 0$.

How do we find $P^\star$: BF (Bellman Function). We replace the expression of $u^\star$ and remove the inf.

$$x^T P x = x^T Q x + (K^\star x)^T R K^\star x + x^T \left[(A + BK^\star)^T P^\star (A + BK^\star) \right] x \quad , \forall x$$

$$x^T P^\star x = x^T [\dots] x \quad , \forall x$$

$$P^\star = [\dots] = Q + A^T P^\star A - A^T P^\star B (R + B^T P^\star B)^{-1} B^T P^\star A$$

Discrete Algebraic Riccati Equation (DARE)

Under standard assumptions (A, B) stabilizable, $Q \succeq 0$, $R \succ 0$, DARE has unique solution. This shows that our ansatz was correct $\rightarrow J^\star$ is quadratic, $u^\star$ is linear in x.

We can use VI and PI to find $P^\star$ (or approximate versions) at cheaper comp. cost.

Value Iteration (VI) $\{P_n\}$

$$J_{n+1} = TJ_n \qquad n = 0, 1, 2$$

$$x^T P_{n+1} x = \inf_{u \in \mathbb{R}^{n_u}} x^T Q x + u^T R u + (Ax + Bu)^T P_n (Ax + Bu)$$

$\rightarrow$ same as before, except that we have P_n instead of $P^\star$. We know the solution.

At each n, VI consists of:

$$P_{n+1} = Q + A^T P_n A - A^T P_n B(R + B^T P_n B)^{-1} B^T P_n A$$

initialized with $P_0 \succeq 0$.

It can be shown that $\lim_{n\to\infty} P_n = P^\star$. (Note: T here is NOT contractive).

Policy iteration (PI) $\{\mu_n\} \Leftrightarrow \{K^n\}$ Sequence of gains K.

① **Policy evaluation:** J_{μ^n} (given μ^n).

$$J_{\mu^n} = T_{\mu^n} J_{\mu^n}$$

in LQR: $\mu^n = K^n x$. This step computes the cost of μ^n.

$$x^T P_n x = x^T Q x + (K^n x)^T R (K^n x) + x^T (A + BK^n)^T P_n (A + BK^n) x$$

$$\downarrow$$

$$P_n = Q + K^{nT} R K^n + (A + BK^n)^T P_n (A + BK^n) \quad \text{Lyap. equation}$$

$J^n(x) = x^T P_n x$: Cost of gain K^n.

② **Policy improvement:** $J^n \to \mu^{n+1}$

$$K^{n+1} = -(R + B^T P_n B)^{-1} B^T P_n A$$

Same structure of $K^\star$ but with P_n instead of $P^\star$.

PI must be initialized with stabilizing gain $K^0 \to \rho(A + BK^0) < 1$. In this case, it holds that $\lim_{n\to\infty} K^n = K^\star$.

5 Numerical Methods for Optimal Control

Methods to compute solutions to open-loop optimal controllers.

2 categories:

- **Indirect methods:** find solutions to necessary (and sufficient) conditions discussed in section 3. "Optimize and then discretize".
- **Direct methods:** "Discretize and then optimize".

5.1 Indirect methods

They apply to ELE, PMP in all their forms. Here we focus on 1 instance of PMP:

$$\min_{u(\cdot)\in\hat{\mathcal{C}}[t_0,t_f]} \int_{t_0}^{t_f} l(x,u)\,dt + \varphi(x(t_f)) \qquad \text{(time-invariant)}$$

$$\dot{x} = f(x,u), \quad x(t_0) = x_0 \qquad t_0 = 0$$

$$t_f \text{ fixed }, \quad x(t_f) \text{ free} \qquad t_f = T$$

Recap: Necessary conditions are the existence of $(x^\star, \lambda^\star, u^\star)$ s.t.

$$\dot{x}^\star = \mathcal{H}_\lambda(x^\star, u^\star, \lambda^\star), \qquad x^\star(0) = x_0$$

$$\dot{\lambda}^\star = -\mathcal{H}_x(x^\star, u^\star, \lambda^\star), \qquad \lambda^\star(T) = -\varphi_x(x^\star(T))$$

$$u^\star(t) \in \operatorname*{argmax}_{u\in\mathbb{R}^{n_u}} \mathcal{H}(x^\star(t), u, \lambda^\star(t))$$

$$\text{where } \mathcal{H}(x,u,\lambda) \coloneqq -l(x,u) + f(x,u)^T\lambda$$

Note

- $\lambda_0 \neq 0$ in free end-point problems, so we set $\lambda_0 = -1$.
- $\mathcal{H} = \text{const.} \neq 0$ because final time is fixed.

These conditions consist of:

- $2n_x$ ODEs with B.C. at $t = 0, t = T$ (Two point boundary value problem).
- n_u algebraic equations (maximization condition).

The first step is typically to replace "$\max \mathcal{H}$" with "$\mathcal{H}_u (= \nabla_u \mathcal{H}) = 0$".
Note: $\mathcal{H}_u = 0$ is only a necessary condition for maximization of $\mathcal{H}$.

In this way we have:

$$\left[\begin{array}{l} \dot{x} = \mathcal{H}_\lambda \\ \dot{\lambda} = -\mathcal{H}_x + \text{B.C.} \\ 0 = \mathcal{H}_u \end{array}\right. \left.\vphantom{\begin{array}{l} a \\ b \\ c \end{array}}\right\} \begin{array}{l} \text{Differential} \\ \text{Algebraic} \\ \text{Equations (DAE)} \end{array}$$

Challenge ①: solving DAE.
Challenge ②: TPBVP.

For challenge ① we make the assumption that we have an explicit relationship $u(x, \lambda)$ such that $0 = \mathcal{H}_u(x, u(x, \lambda), \lambda)$. In other words:

$$0 = \mathcal{H}_u \leftrightarrow u = u(x, \lambda)$$

(We have this function).

This allows us to replace u in the first 2 equations by $u(x, \lambda)$. This decouples the solution strategy:

- Step 1: Solve TPBVP in (x, λ) only.
- Step 2: Replace (x, λ) from 1 into $u(x, \lambda)$ and find u.

We will focus here on "challenge 2" only $\rightarrow$ solve TPBVP.

u has disappeared from our unknowns and we are left with $2n_x$ ODEs.

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix}}_{\dot{y}} = \underbrace{\begin{bmatrix} \mathcal{H}_\lambda(x, u(x,\lambda), \lambda) \\ -\mathcal{H}_x(x, u(x,\lambda), \lambda) \end{bmatrix}}_{g(y)}, \quad t \in [0, T], \quad y := \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$\underbrace{x(0) - x_0}_{r_0(y(0), x_0)} = 0, \quad \underbrace{\lambda(T) + \varphi_x(x(T))}_{r_T(y(T))} = 0$$

$$b(y(0), y(T), x_0) := \begin{bmatrix} r_0(y(0), x_0) \\ r_T(y(T)) \end{bmatrix}$$

We want to solve the following system of equations:

$$\dot{y}(t) = g(y(t)), \quad t \in [0, T]$$

$$b(y(0), y(T), x_0) = 0$$

5.1.1 Single shooting

Idea: Given x_0, if we guess the initial value $\lambda_0 := \lambda(0)$, we can use a standard numerical integration routine (Euler, R-K) to obtain $y(t, \lambda_0), t \in [0, T]$ by simulating $\dot{y} = g(y)$. ($\rightarrow$ the response depends on λ_0).

In general this will give $r_0 = 0$, but $r_T \neq 0$ because we do not enforce $\lambda(T) + \varphi_x(x(T)) = 0$.

The goal is to update λ_0 iteratively so that the following residual is zero:

$$R_{ss}(\lambda_0) := r_T(y(T, \lambda_0)) \in \mathbb{R}^{n_x}$$

How do we find the "right" λ_0 s.t. $R_{ss}(\lambda_0) = 0$?

Root finding problem $\rightarrow$ Newton's method.

Short recap:

$$h : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\text{Find } z \text{ s.t. } h(z) = 0$$

Newton's method builds a sequence $\{z_k\}$ by setting to 0 the 1^{st} order Taylor expansion at the current iterate:

$$h(z_{k+1}) = h(z_k) + \frac{\partial h}{\partial z}(z_k)(z_{k+1} - z_k) = 0$$

$$\Leftrightarrow z_{k+1} = z_k - \left(\frac{\partial h}{\partial z}(z_k)\right)^{-1} h(z_k)$$

assuming differentiability of h and invertability of $\left(\frac{\partial h}{\partial z}\right)$.

Single shooting = Newton's method applied to R_{ss}.

$$\lambda_0^{i+1} = \lambda_0^i + \gamma_i \left(\nabla R_{ss}(\lambda_0^i)\right)^{-1} R_{ss}, \quad i = 0, 1, \ldots$$

$$\gamma_i \in (0, 1] \text{ step-size}$$

At each iteration of Single Shooting (SS), we require the computation of the residual:

$$R_{SS}(\lambda_0^i) \to \text{can be done by forward simulation of } \dot{y} = g(y) \to y(T, \lambda_0^i)$$

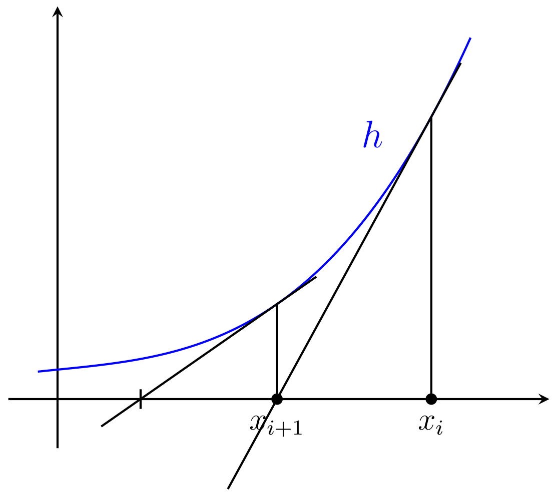
Note: Forward simulation of Hamiltonian dynamics is numerically ill-posed
(λ might increase very fast during simulation).

We also need the gradient:

$$\nabla R_{SS} : \text{ ODE sensitivity } \to \nabla_{\lambda_0} y(T, \lambda_0)$$

There are by now efficient tools to compute both.

However, the reason why SS often fails is that Newton methods fail on highly nonlinear functions, and $R_{SS}(\lambda_0)$ is generally very nonlinear.



Newton method works well when:

- the function is "mildly nonlinear"
- we start close to the zero of the function

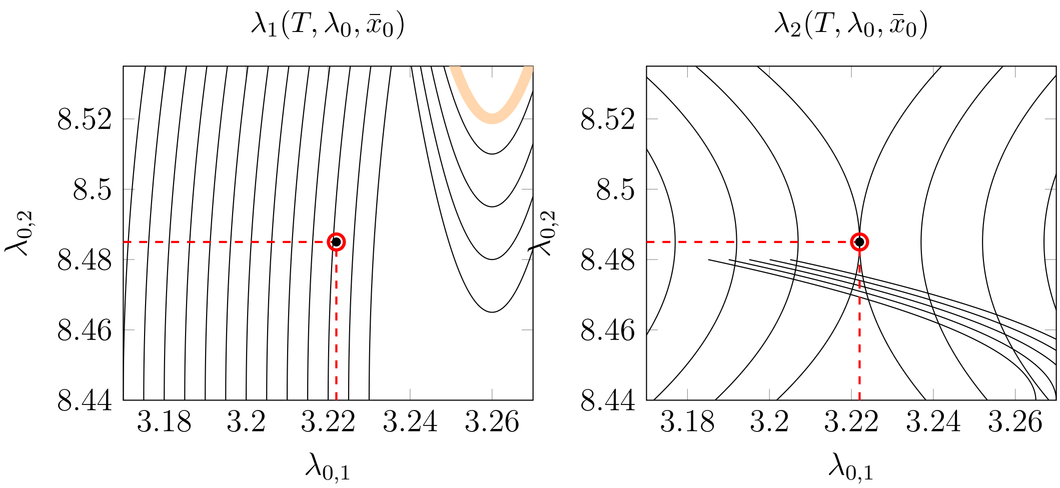
What about R_{SS}? Let's look at a concrete example.

$$P : \begin{cases} \dot{x}_1 = x_1 x_2 + u & x_1(0) = 0 \\ \dot{x}_2 = x_1 & x_2(0) = 1 \end{cases}$$

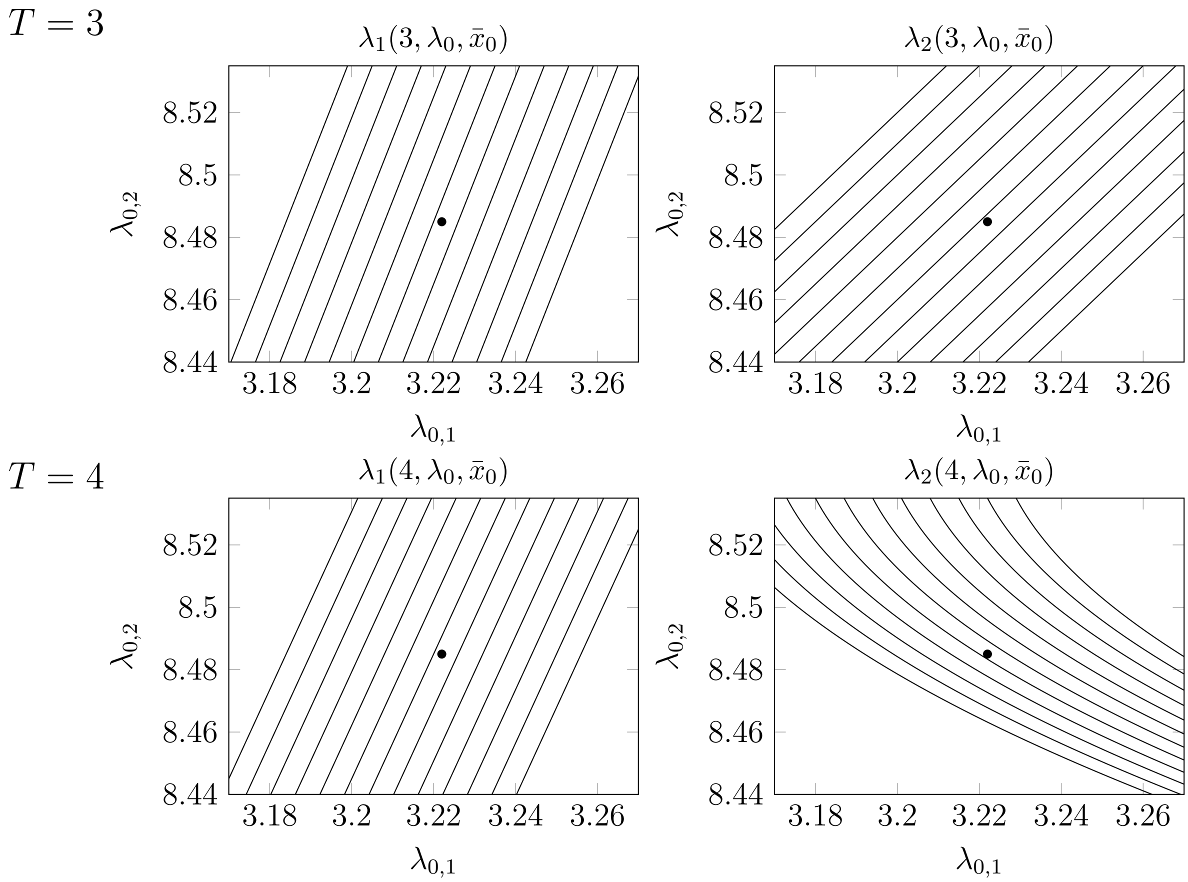
$$l(x,u) = x_1^2 + 10x_2^2 + u^2, \quad \varphi(x) = 0, \quad T = 5$$

$$\rightarrow \lambda(T) = 0$$

R_{SS} for this problem.



Question: How would the level sets of R_{SS} look like if we plot it for smaller values of T?



$\to$ more linear.

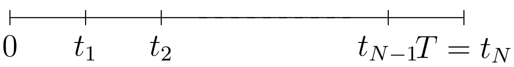
Take-away message: To linearize R_{SS} we should not integrate the solution to $\dot{y} = g$ over long horizons. This idea inspires the multiple shooting method.

5.1.2 Multiple Shooting

Idea: Break down the time interval $[0, T]$ in N shooting intervals.

$$[t_k, t_{k+1}] \subset [0, T], \quad k = 0, \ldots, N-1$$

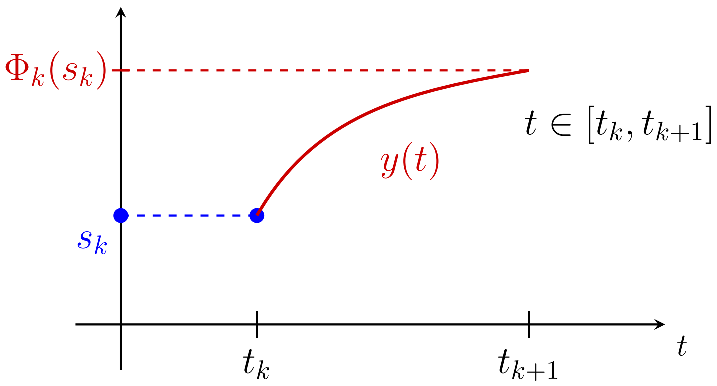
For example $t_k = k\frac{T}{N},\ k = 0, 1, \ldots, N \quad (t_0 = 0, t_N = T)$.



For a generic interval, we define

$$\Phi_k(s_k) := y(t_{k+1})$$

where $\dot{y}(t) = g(y(t)),\ y(t_k) = s_k$.



Integrators: examples Euler / Runge-Kutta; Matlab: `odexy`.

The multiple shooting (MS) method enforces the following conditions:

- $\Phi_k(s_k) - s_{k+1} = 0, \quad k = 0, 1, \dots, N-1$
  $\rightarrow$ Continuity across shooting intervals
- $b(s_0, s_N, x_0) = 0$
  $\rightarrow$ Boundary conditions

(For $N = 1$ we recover SS).

These conditions can be compactly written as this root finding problem:

$$R_{MS}(s) := \begin{bmatrix} \Phi_0(s_0) - s_1 \\ \Phi_1(s_1) - s_2 \\ \vdots \\ b(s_0, s_N, x_0) \end{bmatrix} = 0, \quad s = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_N \end{bmatrix}, \quad R_{MS}(s) \in \mathbb{R}^{2n_x(N+1)}$$

$$s_0 = \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

We apply Newton method to $R_{MS}(s)$:

$$s^{i+1} = s^i - \gamma_i(\nabla R_{MS}(s^i))^{-1} R_{MS}(s^i), \quad i = 0, 1, 2, \dots$$

Advantage: The functions "$\Phi_k(s_k) - s_{k+1}$" are less nonlinear.

$$\rightarrow \text{higher } N \Leftrightarrow \text{less nonlinear}$$

Newton method works even for bad initialization.

Disadvantage: Higher computational cost because s is bigger than λ_0.

However, the problem has structure that can be exploited in the computation of the sensitivities ∇R_{MS}. With so-called "condensing techniques", the per-iteration cost of Newton method is comparable to SS.

The only true disadvantage of the MS is the ill-posed integration of Hamiltonian dynamics.

5.1.3 Collocation methods

Overview of "orthogonal collocation method" to solve ODEs.

$$\dot{x} = f(x,t), \quad x(0) = x_0$$

- Collocation intervals $[t_k, t_{k+1}] \subset [0, T], \quad k = 0, \dots, N-1$.
- Inside each collocation interval, we approximate the solution $x(t)$ with a polynomial $p_k(t, v_k) \approx x(t), \quad t \in [t_k, t_{k+1}]$ of degree d.
- $v_k \in \mathbb{R}^{n_x(d+1)}$ vector of coefficients to be determined.

Typical choice: Lagrange polynomials: For each interval $[t_k, t_{k+1}]$, give $d+1$ collocation times $t_{k,0}, t_{k,1}, \dots, t_{k,d}$:

$$p_k(t, v_k) = \sum_{i=0}^{d} v_{k,i} l_{k,i}(t)$$

$$\mathbb{R} \ni l_{k,i}(t) := \prod_{j=0, j\neq i}^{d} \frac{t - t_{k,j}}{t_{k,i} - t_{k,j}}, \quad \text{basis polynomials}$$

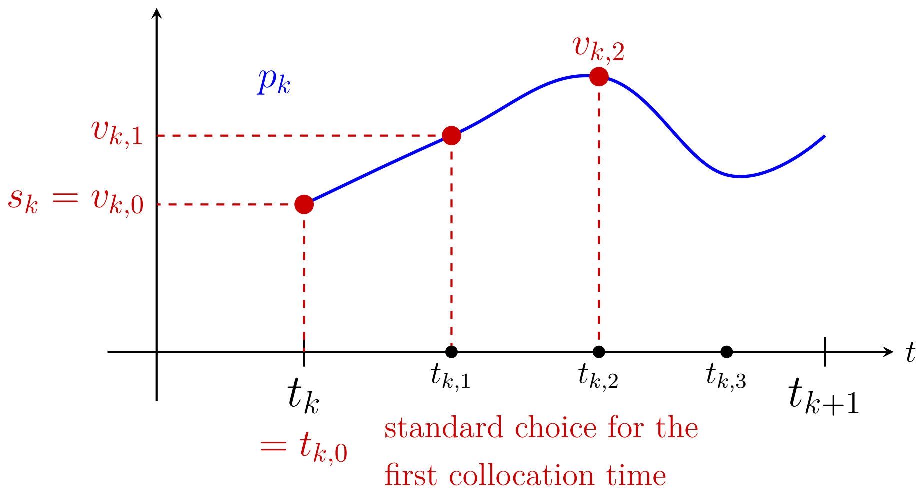
$$v_k = \begin{bmatrix} v_{k,0} \\ v_{k,1} \\ \vdots \\ v_{k,d} \end{bmatrix}. \quad \text{Why such a funny choice of } l_{k,i}?$$

$$l_{k,i}(t) = \frac{t - t_{k,0}}{t_{k,i} - t_{k,0}} \cdot \frac{t - t_{k,1}}{t_{k,i} - t_{k,1}} \cdots$$

$$\rightarrow l_{k,i}(t_{k,j}) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Therefore:

$$p_k(t_{k,i}, v_k) = v_{k,i}, \quad \forall i = 0, 1, 2, \dots, d$$



Note: For each class of polynomials, there are ways of choosing collocation times "$t_{k,i}$" (NOT equispaced in general).

Collocation methods (C) recast the integration of an ODE as the solution of a system of equations. Given s_k at time t_k, the collocation equation enforces these conditions:

$$p_k(t_k, v_k) = s_k \quad \text{initial condition}$$

$$t_k = t_{k,0} \dots t_{k+1}$$

In this case, this means that $\boxed{v_{k,0} = s_k}$.

- $\dot{p}_k(t_{k,i}, v_k) = f(p_k(t_{k,i}, v_k), t_{k,i})$

$$= f(v_{k,i}, t_{k,i}), \quad i = 1, 2, \dots, d$$

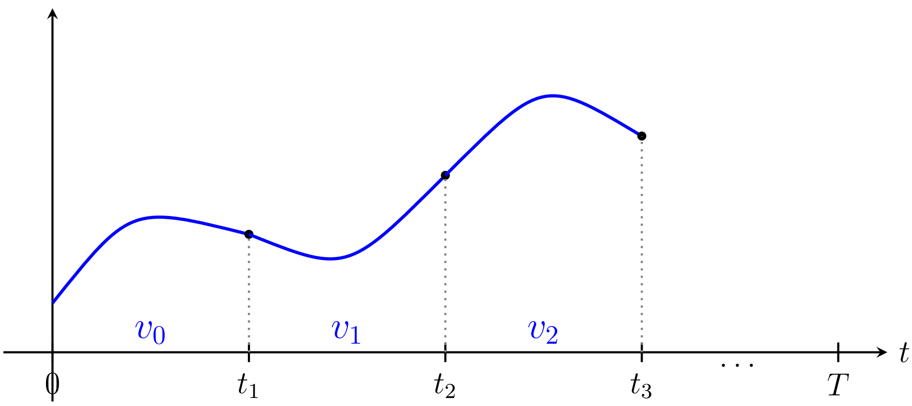
Note: $\dot{p}_k(t, v_k) = \sum_{i=0}^{d} v_{k,i} \dot{l}_{k,i}(t)$.

This can be written in compact form over one collocation interval:

$$C_k(v_k, t_{k,i}, s_k) = \begin{bmatrix} v_{k,0} - s_k \\ \dot{p}_k(t_{k,1}, v_k) - f(v_{k,1}, t_{k,1}) \\ \vdots \\ \dot{p}_k(t_{k,d}, v_k) - f(v_{k,d}, t_{k,d}) \end{bmatrix} = 0$$

Given s_k we solve for $v_k \to p_k(t, v_k) \approx x(t)$ in $t \in [t_k, t_{k+1}]$.

To find a solution over the whole interval $[0, T]$, we need to “glue” these solutions v_k.



Let’s go now to our TPBVP:

$$\dot{y} = g(y),\ \ t \in [0, T]$$

$$b(y(0), y(T), x_0) = 0$$

The corresponding equations to implement the collocation method are:

$$\begin{aligned} v_{k,0} - s_k &= 0, & k &= 0, 1, \dots, N-1 & &\text{internal init. condition} \\ p_k(t_{k+1}, v_k) - s_{k+1} &= 0, & k &= 0, 1, \dots, N-1 & &\text{continuity across intervals} \\ \dot{p}_k(t_{k,i}, v_k) - g(v_{k,i}, t_{k,i}) &= 0, & k &= 0, 1, \dots, N-1 & &\text{"integration"} \\ b(s_0, s_N, x_0) &= 0, & &\text{boundary conditions} \end{aligned}$$

$$v_k \in \mathbb{R}^{2n_x(d+1)}, \quad s_k \in \mathbb{R}^{2n_x}$$

$$\left(\begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \text{degree polynomial} \to v = \begin{bmatrix} v_0 \\ \vdots \\ v_{N-1} \end{bmatrix} \in \mathbb{R}^{N2n_x(d+1)}\right).$$

Clearly, there is a relationship between them:

- $s_k = v_{k,0}, \quad k = 1, \ldots, N-1$
- $s_N = p_{N-1}(t_N, v_{N-1})$

So only v is an independent variable. Once we determine v, we have our solution $y(t)$.

Collocation equations are a set of nonlinear equations. We can use again Newton method, applied to the residual $R_c(v)$.

$$R_c(v) = \begin{bmatrix} R_k(t_{k,1}, v_k, v_{k+1,0}) \\ \vdots \\ \dot{P}_k(t_{z,i}, v_k) - g(v_{k,i}, t_{k,i}) \\ \vdots \\ b(v_{0,0}, P_{N-1}(t_N, v_{N-1}, x_0)) \end{bmatrix} \begin{array}{l} \rightarrow \text{continuity} \\ \\ \rightarrow \text{integration} \\ = 0 \\ \\ \rightarrow \text{boundary condition} \end{array}$$

Newton method: $v^{j+1} = v^j - \gamma_j (\nabla R_c(v^j))^{-1} R_c(v^j), \quad j = 0, 1, 2, \ldots$ iteration counter

Advantages:

- it avoids direct integration of Hamiltonian dynamics.

Disadvantages:

- large-scale problems (factor d compared to Multiple Shooting). There are smart techniques that leverage the structure of the problem to reduce complexity.
- discretization due to collocation (dynamics are not simulated exactly). This can be controlled by increasing N and d.

Main drawbacks of indirect methods:

① It must be possible to "elimnate" u from Ham. dynamics. What if a function $u(x, \lambda)$ satisfying the maximization condition does not exist?

② Even if it exists, $u(x, \lambda)$ can be discontinuous, e.g. bang-bang.
$\rightarrow$ This leads to discontinuous dynamics $\dot{y} = g(y)$.

③ Ham. dynamics sometimes difficult to simulate forward in time (like it is done in Single Shooting, Multiple Shooting).

④ Sensitivity to initializations (because Newton's method is solved to find the solution).

Big advantage:

- They find the "exact solution" of the OC problem because they follow the paradigm "optimize then discretize".

5.2 Direct methods

Basic idea: Assume a fixed shape for u (e.g. piecewise constant, poly, sinusoidal) with some free parameters.

By using integrators, the OC problem becomes an NLP in the space of parameters parametrizing the input u. It is not anymore an optimization problem in function space because the expression of u is fixed, but in finite dim. space.

Advantages: more flexible in the

- problems we can consider (e.g. constraints).
- solution strategies $\rightarrow$ wide variety of NLP solvers.

Drawback: In general, we only get suboptimal solutions (with no quantifiable suboptimality) because of fixing the control input shape.

"Discretize, then optimize":

- time discretization
- the shape of u: $u(t) \approx \sum_{i=1}^{N} q_i u_i(t)$

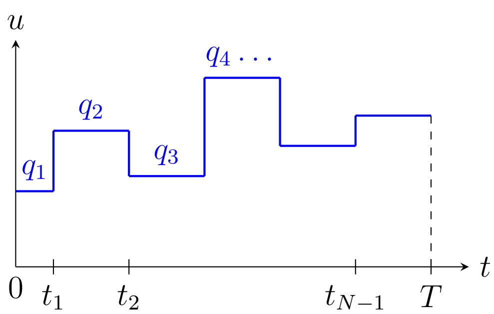
General problem statement (common to the 3 methods we present).

$$\begin{aligned} \min_{u(\cdot)\in\hat{\mathcal{C}}[0,T]} \quad & \int_0^T l(x,u)dt + \varphi(x(T)) \\ \text{s.t.} \quad & \dot{x} = f(x,u),\ x(0) = x_0,\ T \text{ fixed},\ x(T) \text{ free}. \\ & h(x(t),u(t)) \leq 0 \quad \forall t \quad \text{path constraints} \\ & r(x(T)) \leq 0 \quad \text{terminal constraint} \end{aligned}$$

The first step of any direct method is the parametrization of $u(\cdot)$.

$$u(t) \approx u(t,q), \quad q \in \mathbb{R}^{n_q}$$

The most common is piecewise constant controls:



Time-grid:

- $0 = t_0 < t_1 < \cdots < t_N = T$
- N intervals $[t_k, t_{k+1}]$, $\quad k = 0, 1, \ldots, N-1$

- N vectors $q_k \in \mathbb{R}^{n_u}$

$$u(t,q) = q_k, \;\; t \in [t_k, t_{k+1}]$$

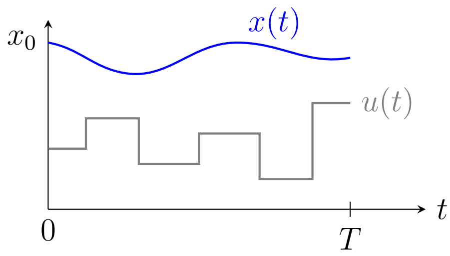
$$q = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{N-1} \end{bmatrix} \in \mathbb{R}^{Nn_u = n_q}$$

The 3 methods (SS, MS, C) only differ in the methodology on how the OC problem is discretized into an NLP.

5.2.1 Single shooting

We see the state trajectory $x(t), t \in [0,T]$ as a dependent variable of u (that is of q). We use an integrator (only ONE) that provides the map

$$x_0, q \to x(t), \;\; t \in [0,T]$$



Let's denote $x(t)$ by $x(t,q)$ to emphasize that $x(t)$ depends on the choice of q. The OC problem becomes:

$$\begin{aligned} \min_{q \in \mathbb{R}^{n_q}} \quad & \int_0^T l(x(t,q), u(t,q)) + \varphi(x(T,q)) \\ \text{s.t.} \quad & h(x(t_k,q), u(t_k,q)) \leq 0 \quad k = 0, 1, \dots, N-1 \\ & r(x(T,q)) \leq 0 \end{aligned}$$

The associated NLP looks as follows:

$$\begin{aligned} \min_{q} \quad & J^{SS}(q) \\ \text{s.t.} \quad & h^{SS}(q) \leq 0 \end{aligned}$$

where

$$J^{SS}(q) := \sum_{k=0}^{N-1} \underbrace{l_k(q)}_{\substack{\downarrow \\ \text{integrated cost in} \\ \text{each interval}}} + \varphi(x(T,q))$$

$$\Lsh \int_{t_k}^{t_{k+1}} l(x(t,q),u(t,q))dt$$

$$h^{SS}(q) := \begin{bmatrix} h_0(q) \\ \vdots \\ h_{N-1}(q) \\ r(x(T,q)) \end{bmatrix} \leq 0, \quad \to h_k(q) := h(x(t_k,q),u(t_k,q))$$

$$\begin{array}{l} x_0 \longrightarrow \\ q \longrightarrow \end{array} \boxed{\begin{array}{l} \dot{x} = f(x,u) \\ u = u(t,q) \\ x(0) = x_0 \end{array}} \longrightarrow x(t,q), \quad t \in (0,T]$$

This block is inside the NLP, NLP solvers typically seek solutions to KKT conditions.
$\to$ Gradients of J^{SS}, h^{SS}.

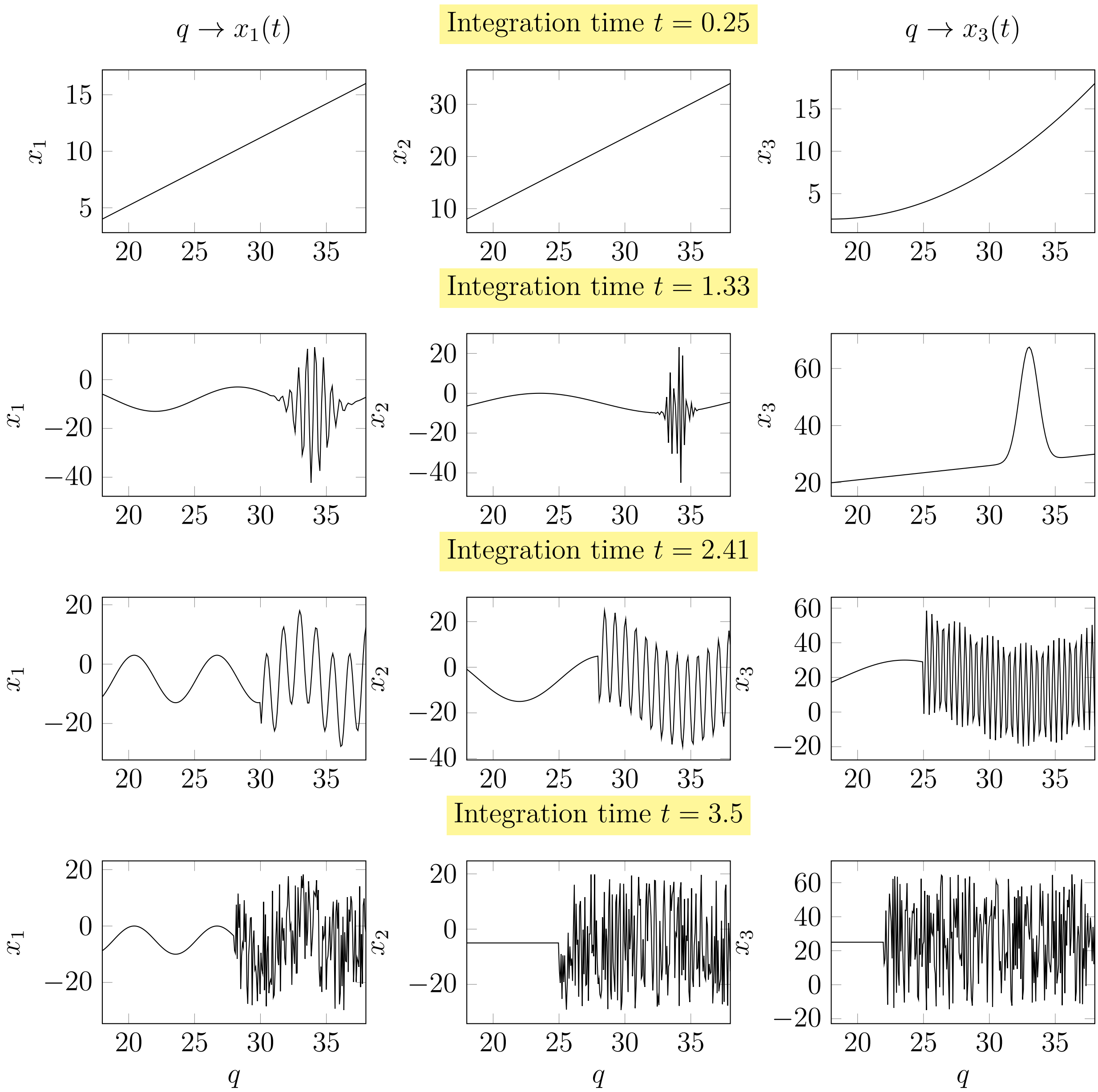
Sensitivities: how obj and constraints change when I modify q (input). Through automatic differentiation (AD), see e.g. CASADI.

Second issue is that the map $(x_0,q) \to (x,q)$ is highly nonlinear.

Example

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1) \\ \dot{x}_2 &= x_1(u - x_3) - x_2 \qquad x(0) = 0 \\ \dot{x}_3 &= x_2 x_1 - 3x_3 \end{aligned}$$

$$u \in \mathbb{R},\ u(t,q) = q,\ \forall t,\ \text{Constant input}$$

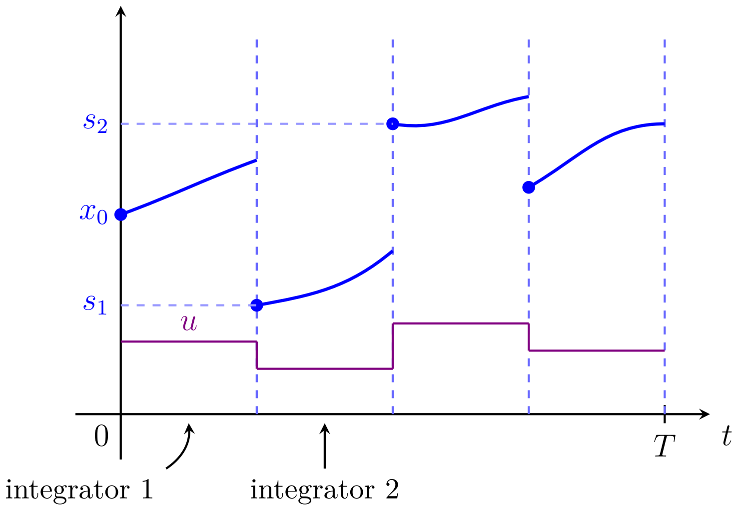


The function $x(t, q)$ is highly NL in general for large t. This makes the SS (direct) not very competitive.

5.2.2 Multiple shooting

Integration is limited to short intervals $[t_k, t_{k+1}]$.

Key difference: we use a bank of integrators simulating the dynamics in parallel in each interval starting from arbitrary initial conditions s_k.



The NLP will take care of enforcing that the initial conditions used by each integrator equals the state at the last interval.

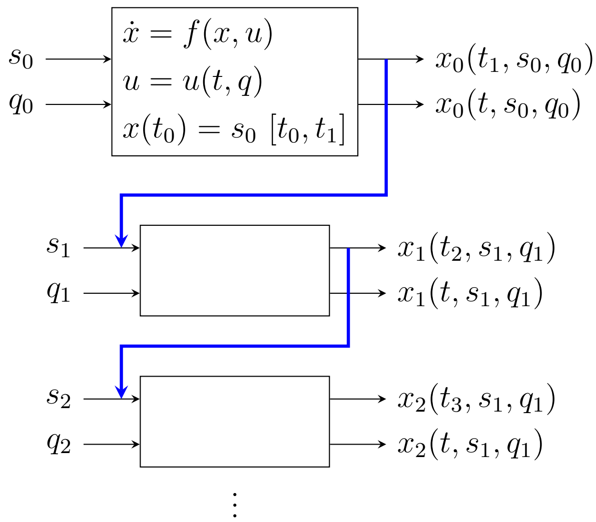
Each integrator solves the following problem:

$$\dot{x}_k(t; s_k, q_k) = f(x_k(t; s_k, q_k), q_k), \quad t \in [t_k, t_{k+1}]$$

$$x_k(t_k; s_k, q_k) = s_k \qquad \text{initial condition}$$

The NLP will have to enforce:

$$\underbrace{s_{k+1} = x_k(t_{k+1}, s_k, q_k)}_{\text{variables of the NLP}}$$



$$(*)\begin{cases} \min\limits_{u(\cdot)\in\hat{C}[0,T]} & \displaystyle\int_0^T l(x,u)\,dt + \varphi(x(T)) \\ & \dot{x} = f(x,u), \quad x(0) = x_0, \quad T \text{ fixed}, \ x(T) \text{ free} \\ & h(x(t),u(t)) \leq 0 \quad \forall t \quad \text{(path constraints)} \\ & r(x(T)) \leq 0 \end{cases}$$

The first step of any direct method is the parametrization of u:

$$u = u(t,q)$$

depends on $q \in \mathbb{R}^{n_q}$ that we will determine by solving an NLP associated with the discretization of $(*)$.

NLP associated with MS

$$s = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \end{bmatrix} \in \mathbb{R}^{N\cdot n_x}, \quad q \in \mathbb{R}^{n_q}$$

$$\begin{aligned} \min_{q,s} \quad & J^{MS}(q,s) \\ \text{s.t.} \quad & x_0 = s_0 \\ & s_{k+1} = x_n(t_{k+1}, q_k, s_k) \quad k = 0,1,2,\dots,N-2 \\ & h^{MS}(q,s) \leq 0 \end{aligned}$$

J^{MS}, h^{MS} conceptually the same as SS.

$$J^{MS}(q,s) = \sum_{k=0}^{N-1} l_k(q,s) + \varphi(x(T,q,s))$$

$$h^{MS}(q,s) = \begin{bmatrix} h_0(q,s) \\ h_1(q,s) \\ \vdots \\ h_{N-1}(q,s) \\ r(x(T,q,s)) \end{bmatrix} \qquad h_k(q,s) \coloneqq h(x(t_k,q,s),u(t_k,q,s))$$

The problem has larger dimension ($n_q + N \cdot n_x$ variables) compared to SS (n_q variables), but the functions in the NLP are given by the integrators $(s,q) \to x(t)$ are much less nonlinear. The problem is numerically better conditioned.

5.2.3 Collocation Methods

- We have N collocation intervals $[t_k, t_{k+1}],\ k = 0, \dots, N-1$.
- We approximate $x(t)$ on each interval with a given polynomial $p_k(t, v_k),\ v_k \in \mathbb{R}^{n_x(d+1)}$ of order d.
- For each interval we have the collocation equations:

$$c_k(v_k, s_k, q_k) \coloneqq \begin{bmatrix} v_{k,0} - s_k \\ \dot{p}_k(t_{k,1}, v_k) - f(v_{k,1}, t_{k,1}, q_k) \\ \vdots \\ \dot{p}_k(t_{k,d}, v_k) - f(v_{k,d}, t_{k,d}, q_k) \end{bmatrix} = 0$$

Same equations we saw in the indirect, here we do not just “simulate” the dynamics. We want to optimize over u, that is q. So we use q inside the NLP.

$$\begin{aligned} \min_{q,s,v} \quad & J^C(q,s,v) \\ \text{s.t.} \quad & x_0 = s_0 \\ & p_k(t_{k+1}, v_k) - s_{k+1} = 0, \quad k = 0,1,2,\dots,N-2 \quad \text{(CONTINUITY)} \\ & c_k(v_k, s_k, q_k) = 0, \quad k = 0,1,2,\dots,N-2 \quad \text{(COLLOCATION EQUATIONS)} \\ & h^C(q,s,V) \le 0 \end{aligned}$$

J^C, h^C contains the integrated cost and evaluated constraints at t_k or at collocation times (finer grid).

\# variables:

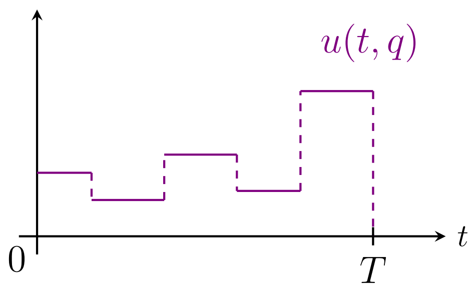
- $q \in \mathbb{R}^{n_q}$
- $\big(s \in \mathbb{R}^{n_x \times N} \quad (\to \text{can be written as a function of } v)\big)$
- $V \in \mathbb{R}^{n_u(d+1)N}$

Advantages

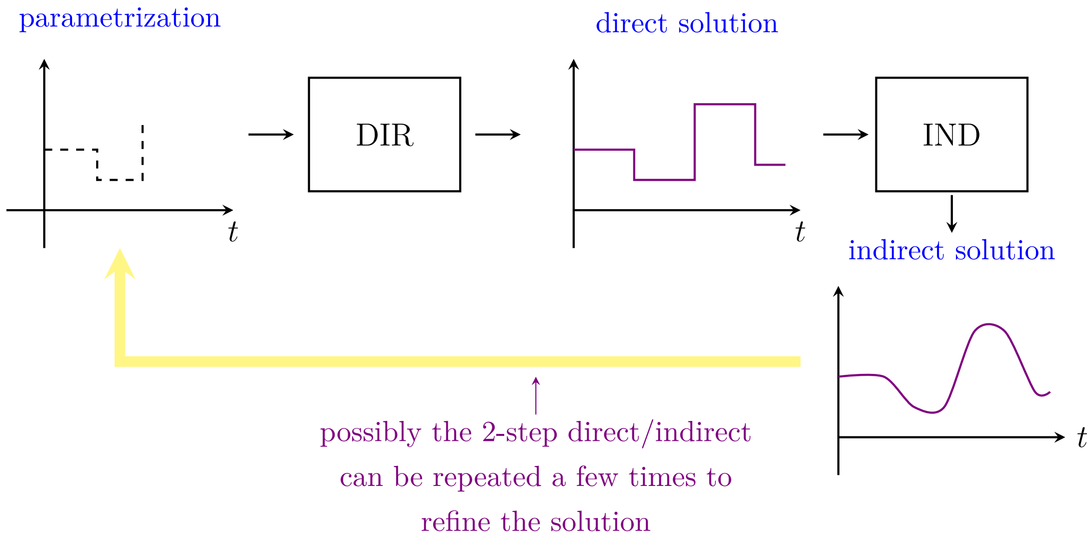
- Flexibility in the problem you can solve (constraints, ...)
- Less sensitive to initializations

Disadvantage

- No guarantee of optimality of u because the shape of u must be guessed.



In general, what many people do in challenging problems is to combine them. For example, one can first apply the direct method (because it is not so sensitive to initialization), and then initialize with its solution the indirect method (because it will work better if we warm-start it with a good solution).


parametrization
t
DIR
direct solution
t
IND
indirect solution
t
possibly the 2-step direct/indirect
can be repeated a few times to
refine the solution