R-Tipping of Non-Linear Oscillator

09.02.2022

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Alexander Robinson
Jorge Álvarez-Solas

I. Motivation

II. Nonlinear Oscillator Design

III. “Strange” R-Tipping Behaviors

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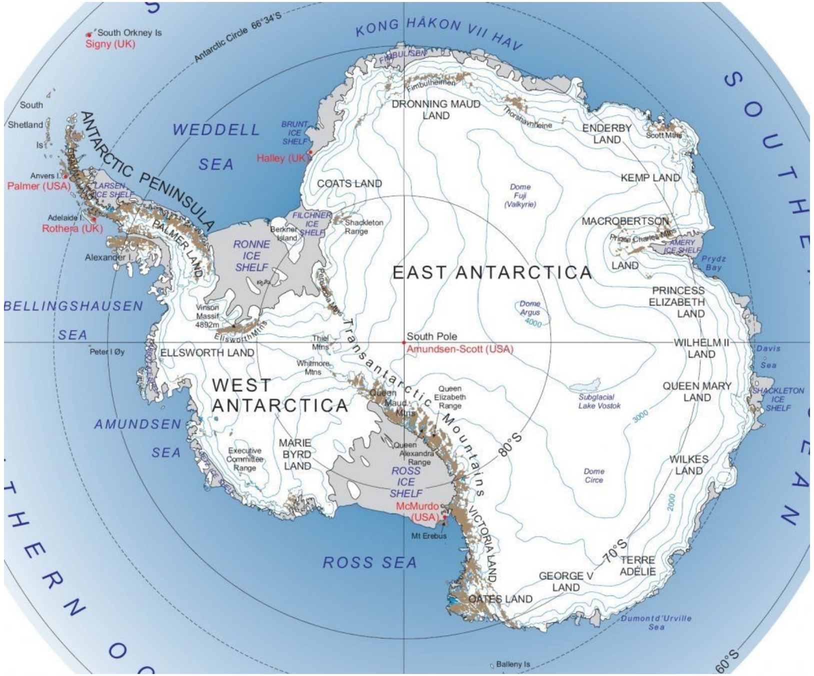
- Studying difference between B-Tipping and R-tipping of the West-Antarctic Ice Sheet (WAIS).



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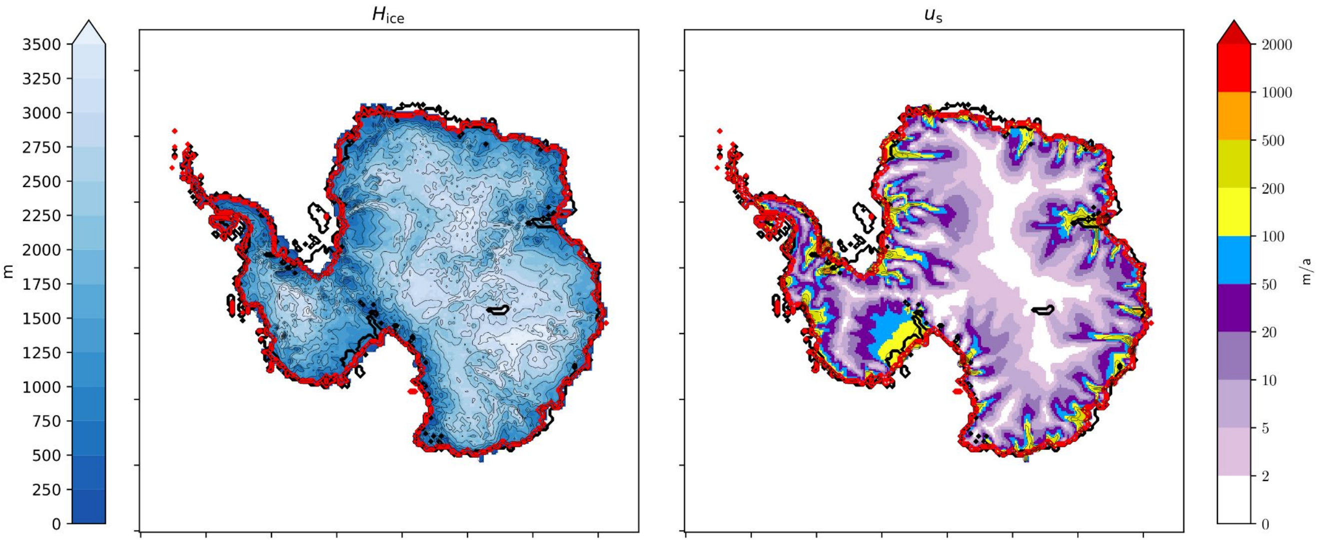
→ Perform experiments on Yelmo (comparable to an ISMIP6 model w.r.t. complexity).



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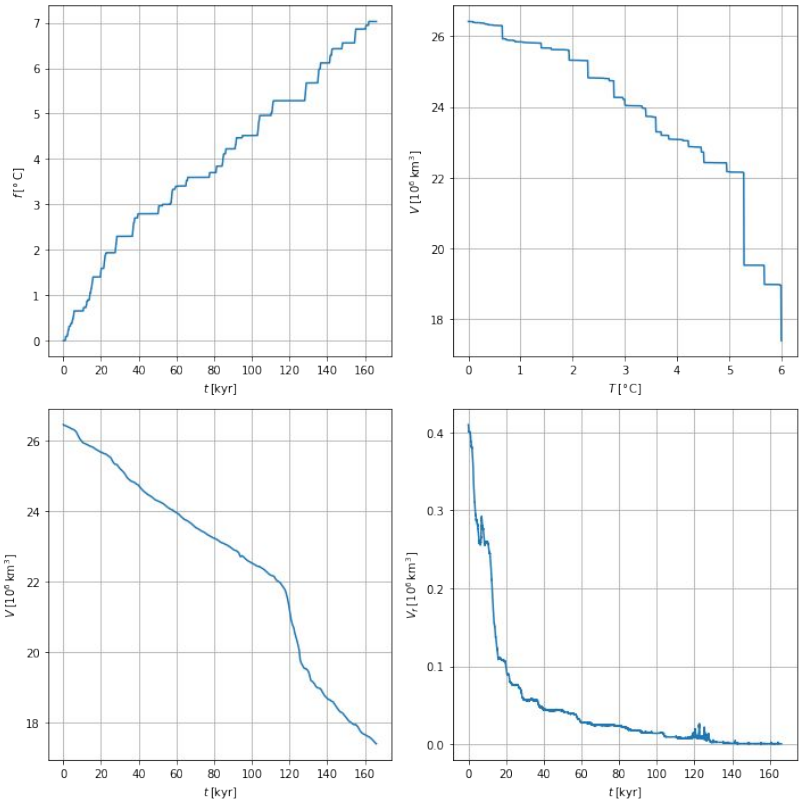

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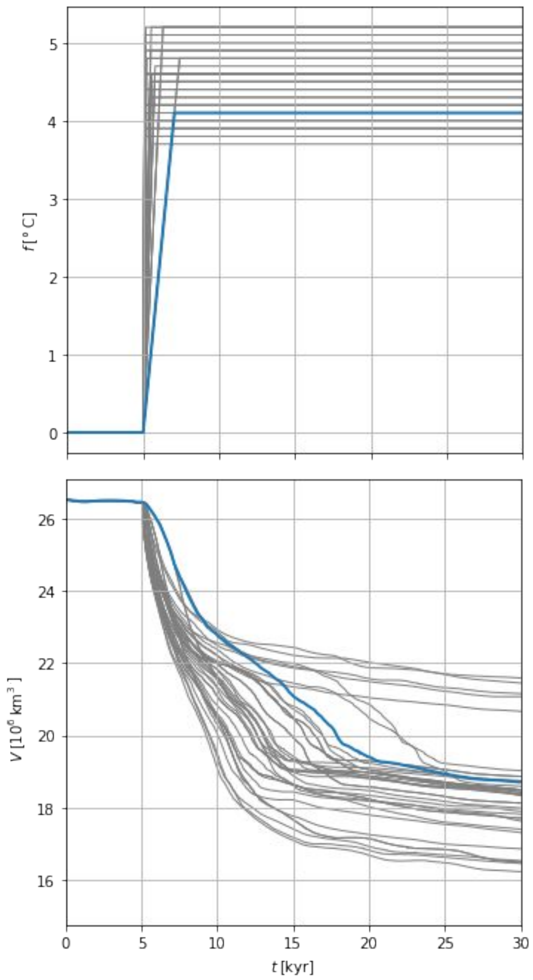

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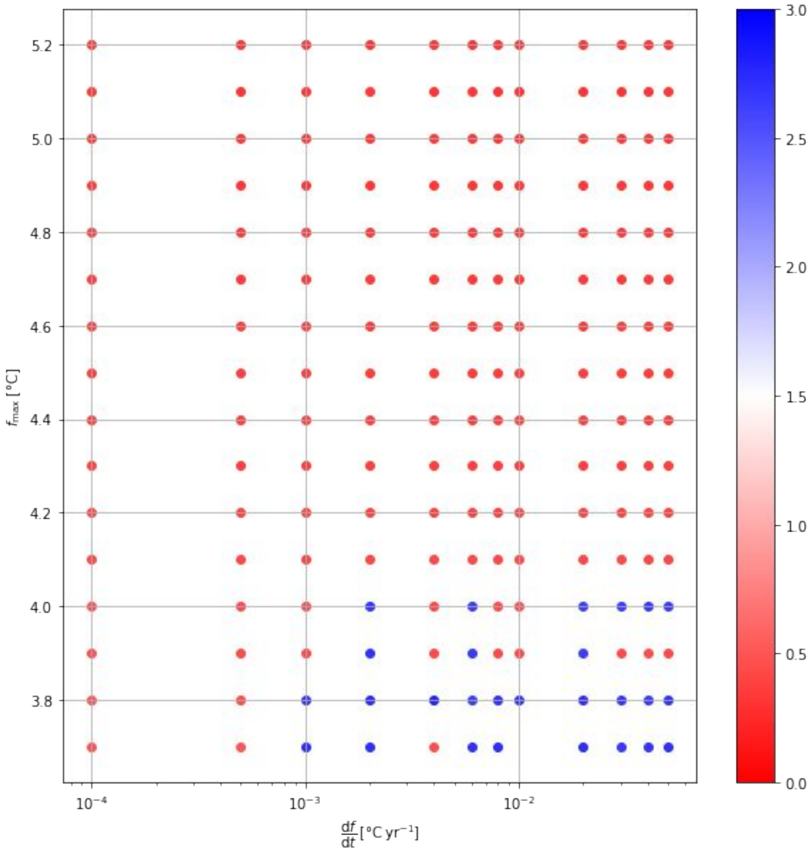
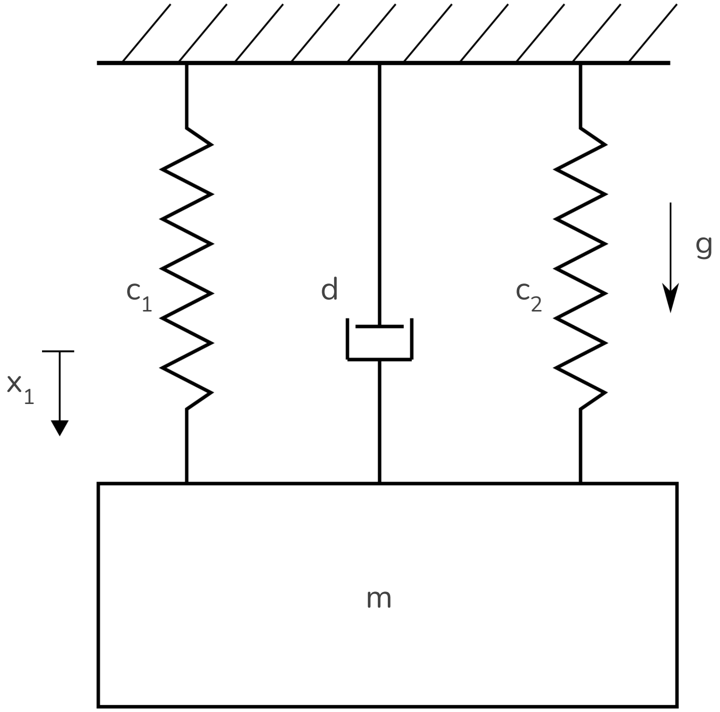
- B-Tipping threshold obtained from adaptive quasi-equilibrium forcing.
- R-Tipping experiment design as discussed in the Bus in Denmark (without forcing back).
- Scatter plot displays unexpected behavior. I.e. separation is not monotonous function of ramp slope.
- Noise → strange placement of some points?
- But maybe not for all of them?



II. Nonlinear Oscillator Design

II. A Simple Nonlinear Oscillator



Notice: m is point mass and there are no DoF in rotation!

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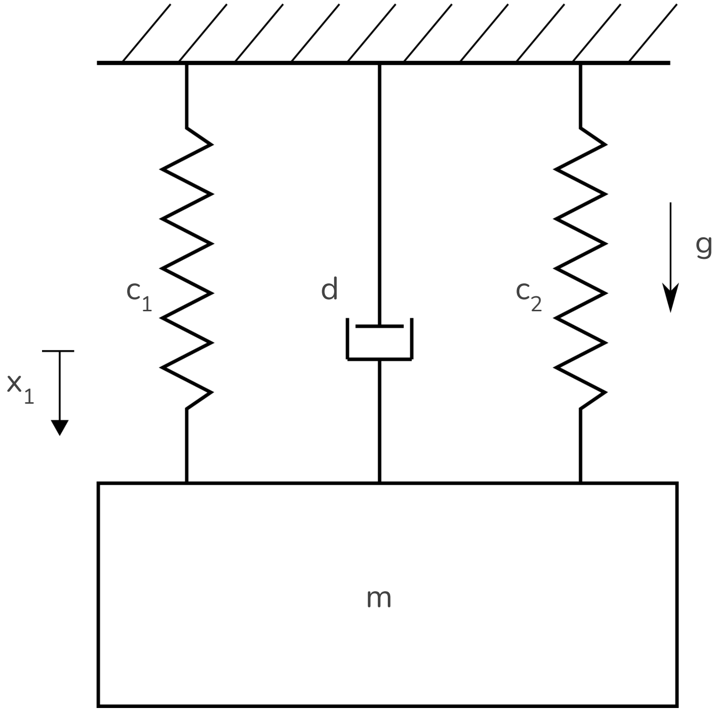
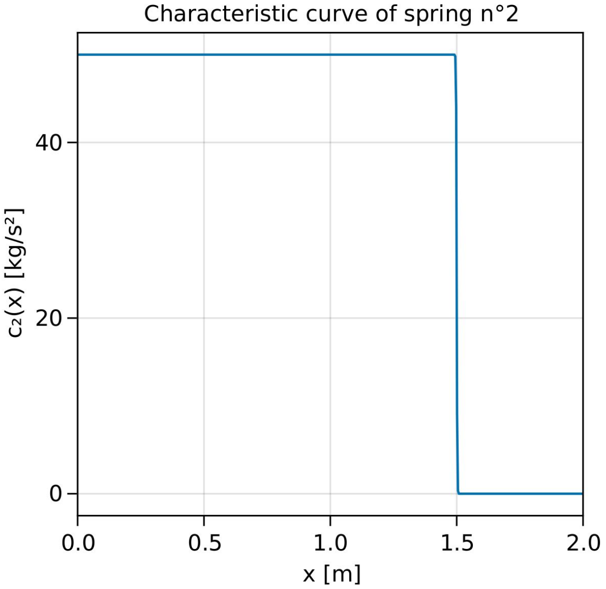


Spring n°2 breaks at x_t=1.5 m ⇒ Before: $c = c_1 + c_2$, After: $c = c_1$

$$c_2(x) = \frac{k_1}{1 + \exp\left(k_2\left(x - x_{\text{tip}}\right)\right)}$$



II. Governing Equations, Equilibria & Parameters

$$m\ddot{x} \quad = \quad -\left(c_1 + c_2(x)\right)x - d\dot{x} + mg + F(t)$$

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Static equilibria:

$$\tilde{x} = \begin{cases} \dfrac{F + mg}{c_1 + c_2}, & \text{if system did not tip} \\[2ex] \dfrac{F + mg}{c_1}, & \text{if system tipped} \end{cases}$$

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Param	Value
m	10 kg
c_1	50 kg/s²
d	20 kg/s
k_1	50 kg/s²
k_2	1000 1/m
g	10 m/s²
x_t	1.5 m
t_1	1 s
F_t + mg	150 N

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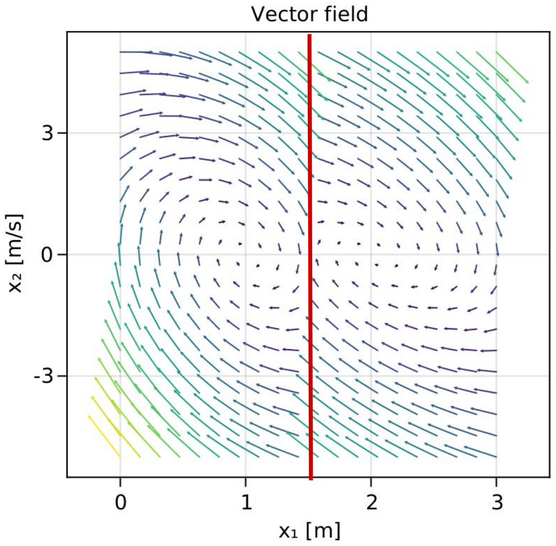

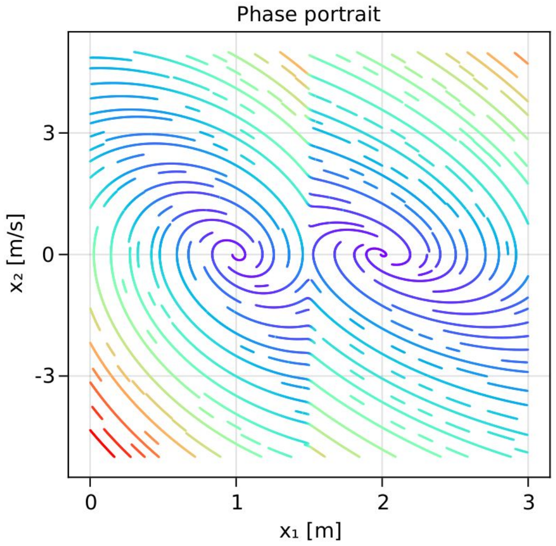

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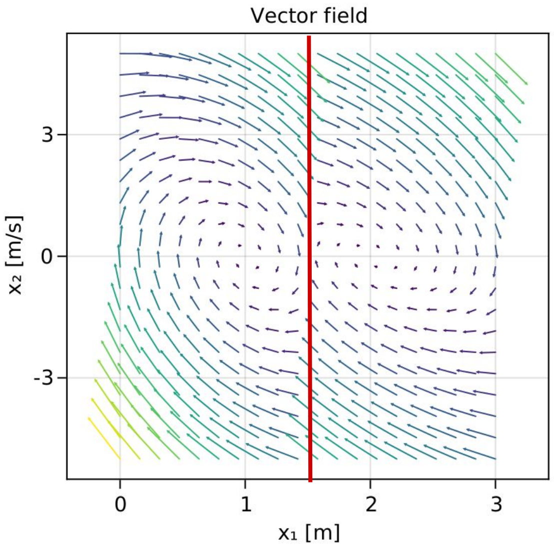
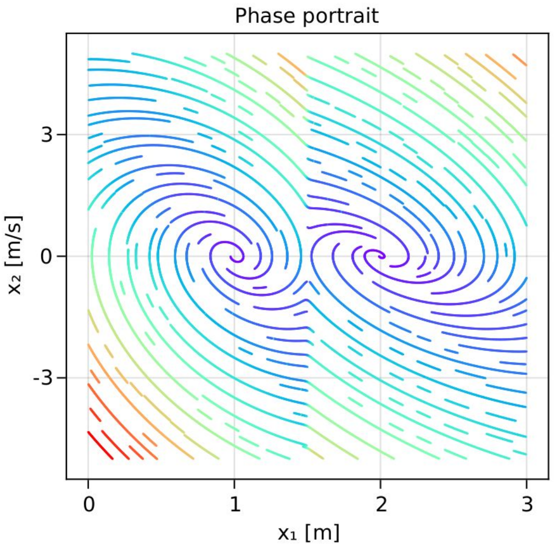
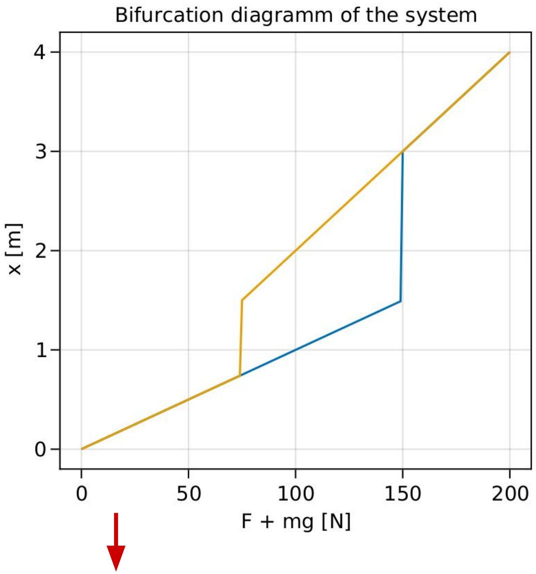


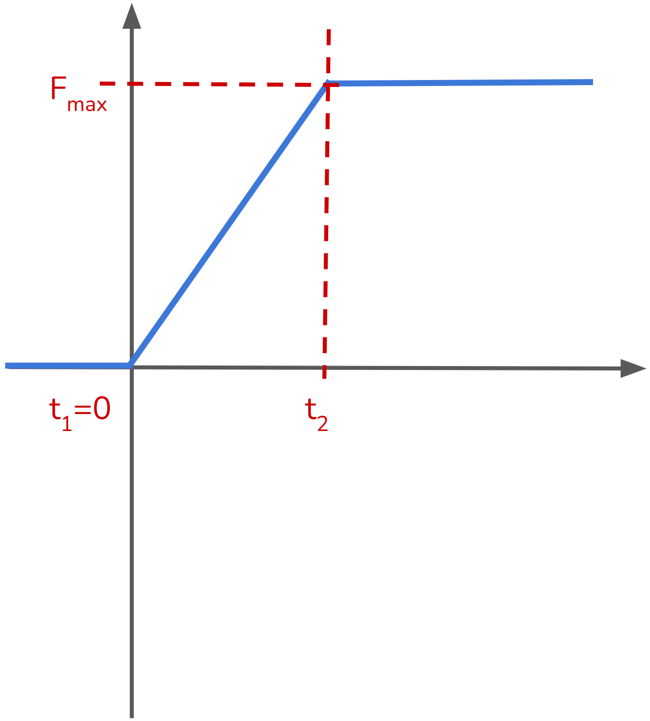




Orange curve does not really correspond to the physical properties of the system, but we are rather interested in the blue one!

III. “Strange” R-Tipping Behaviors

III. Forcing Design



Recall hypothesis:

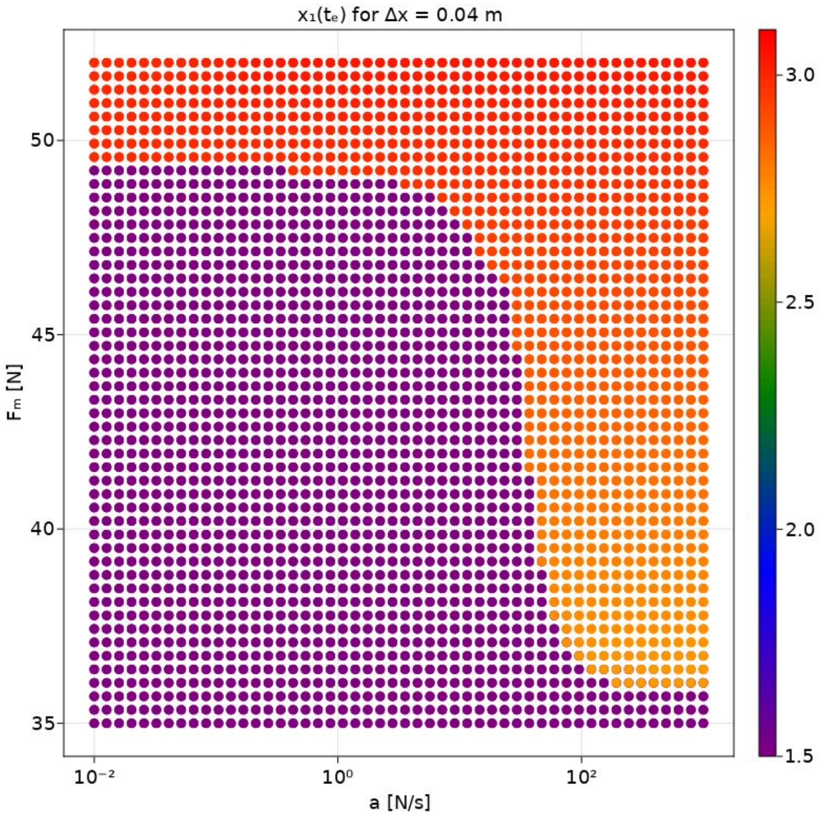
- Unexpected separation between tipped and non-tipped WAIS due to different time scales involved

Idea:

- To introduce a second time scale, let's begin with an initial perturbation of the position:

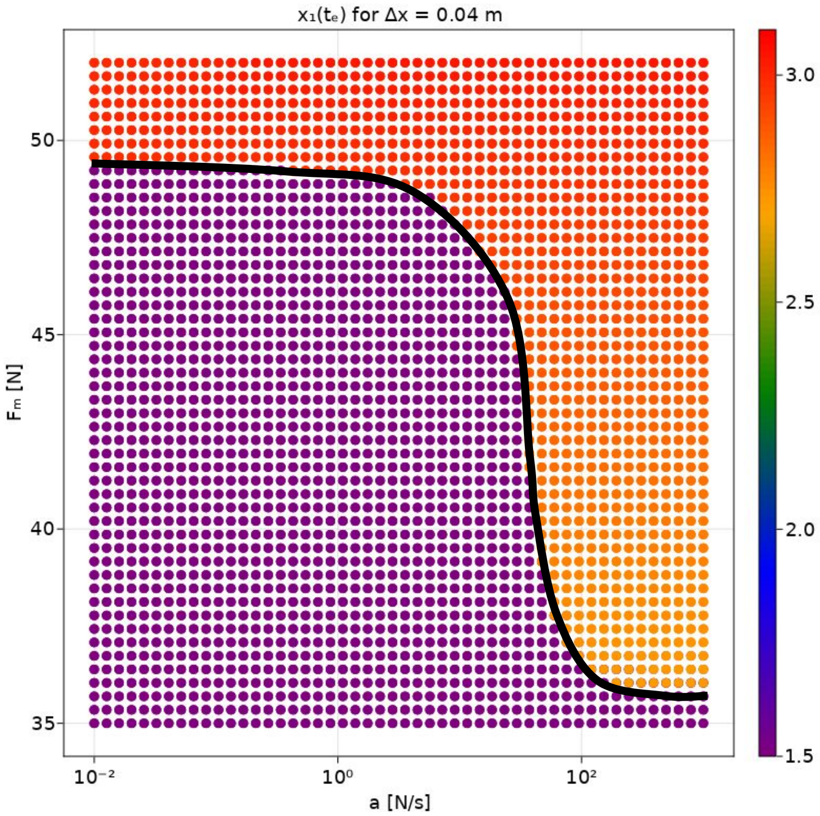
$$x_1(t = 0) = \tilde{x} - \Delta x$$

III. Tipping Pattern n°1



- Notice: perturbation has small magnitude.

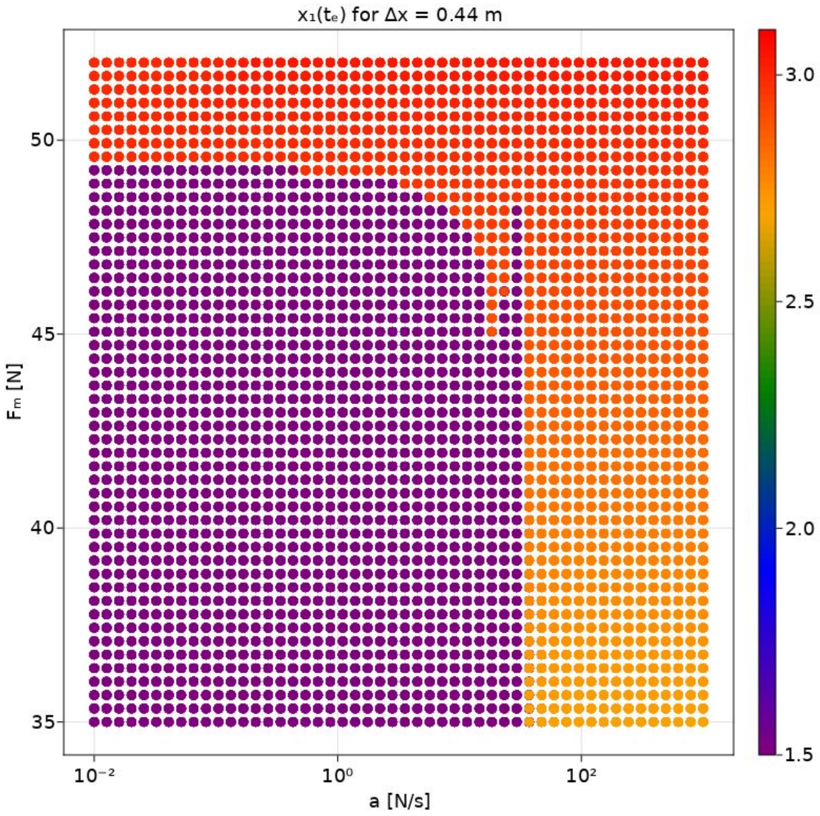
III. Tipping Pattern n°1



- Notice: perturbation has small magnitude.
- The pattern behaves as “expected”.

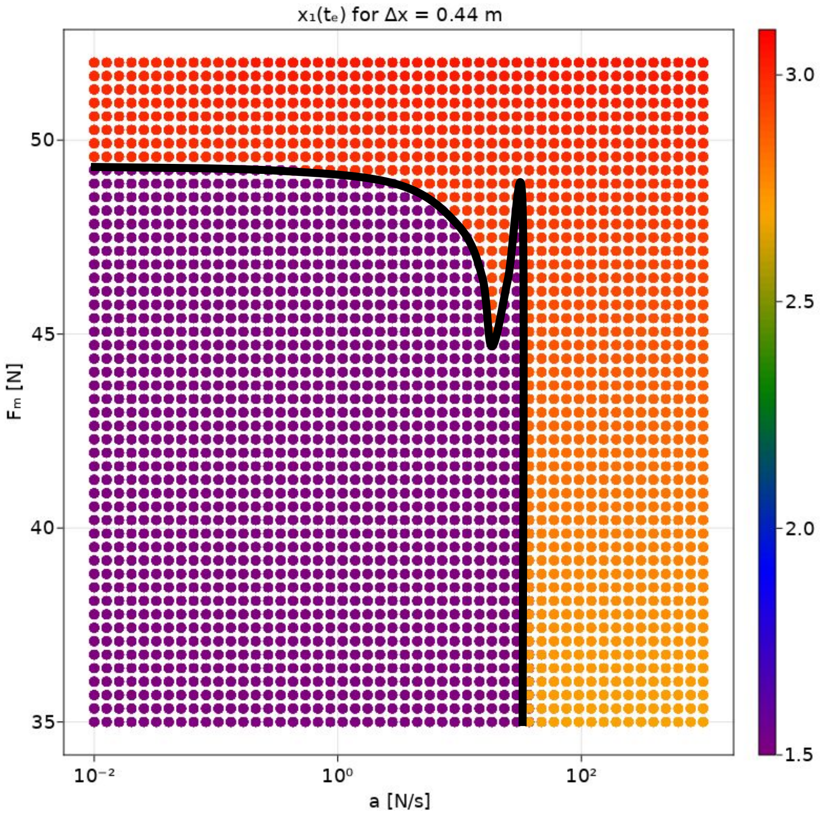
→ Separation function is a monotonously decreasing function of ramp slope.

III. Tipping Pattern n°2



- Notice: perturbation has intermediate magnitude.

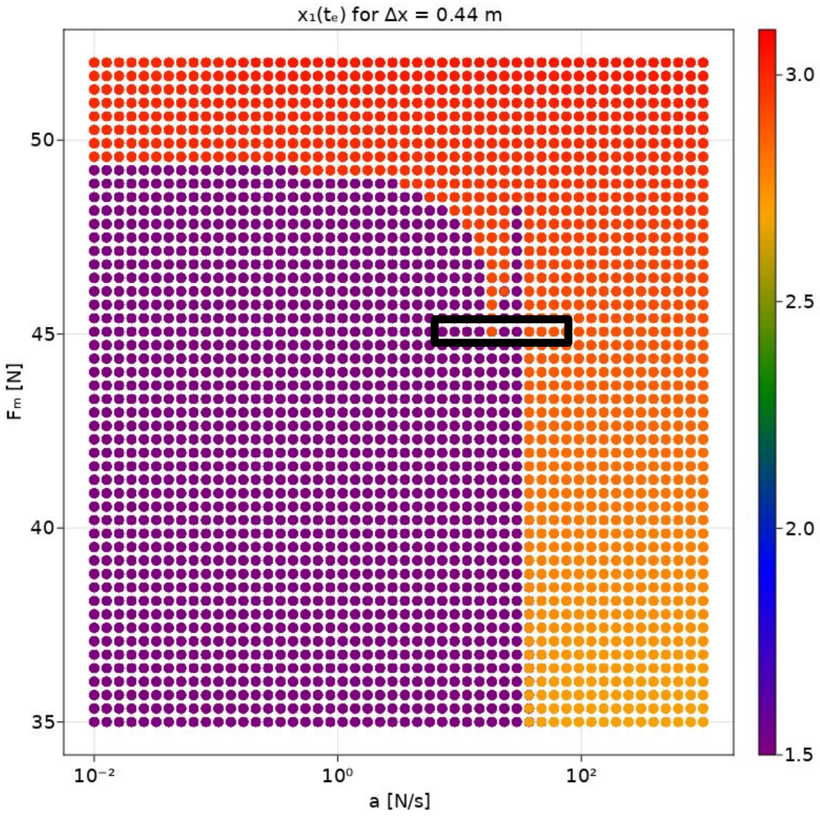
III. Tipping Pattern n°2

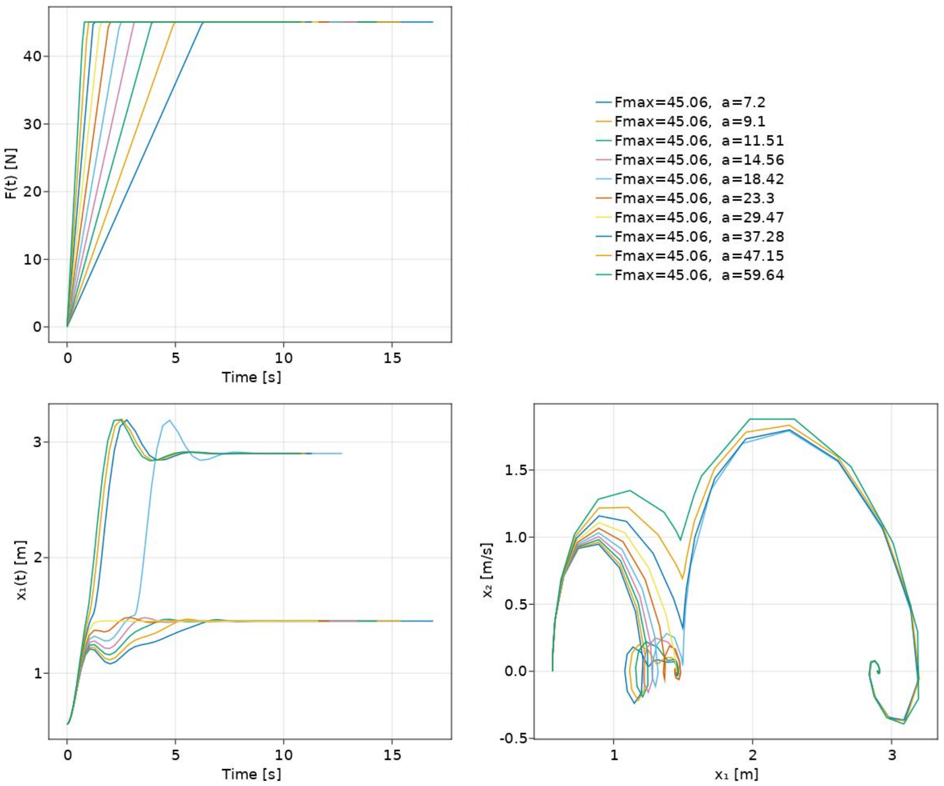


- Notice: perturbation has intermediate magnitude.
- The pattern behaves in unexpected.

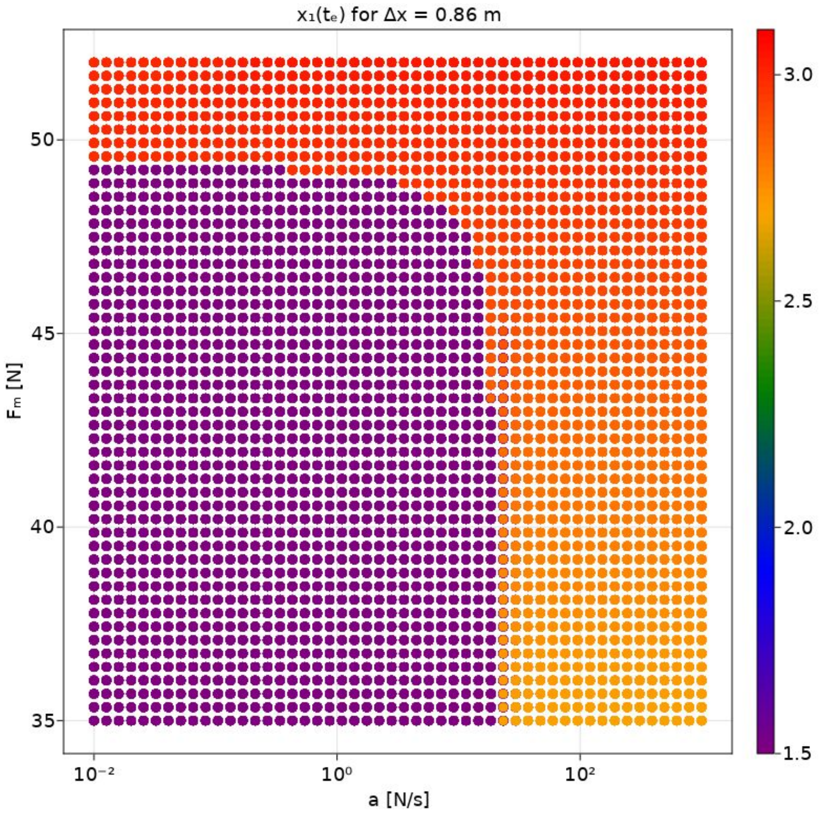
→ Separation function is **not** a monotonously decreasing function of ramp slope.

III. A Slice of Tipping Pattern n°2



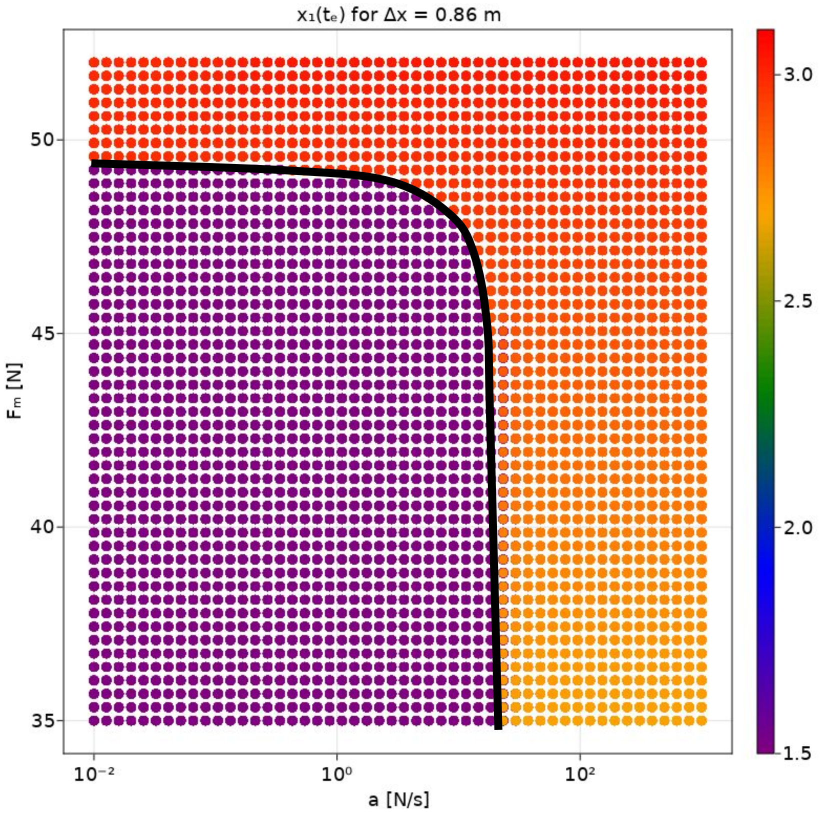


III. Tipping Pattern n°3



- Notice: perturbation has high magnitude.

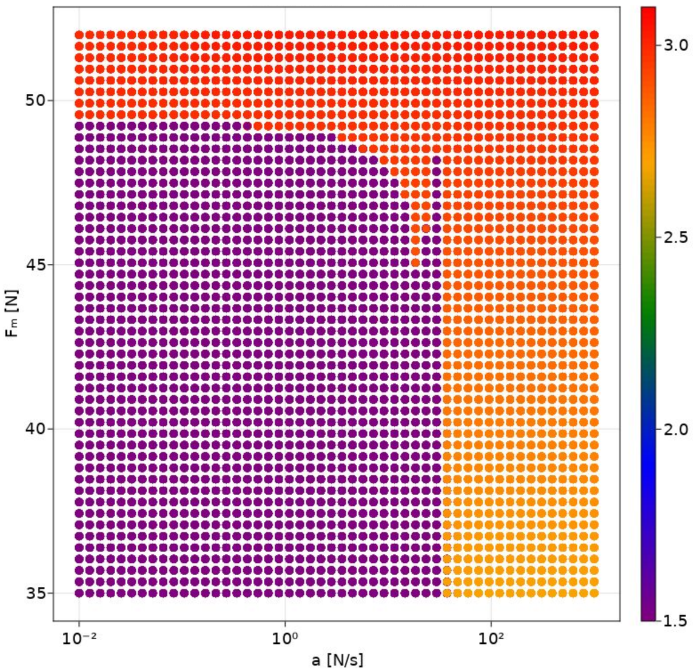
III. Tipping Pattern n°3

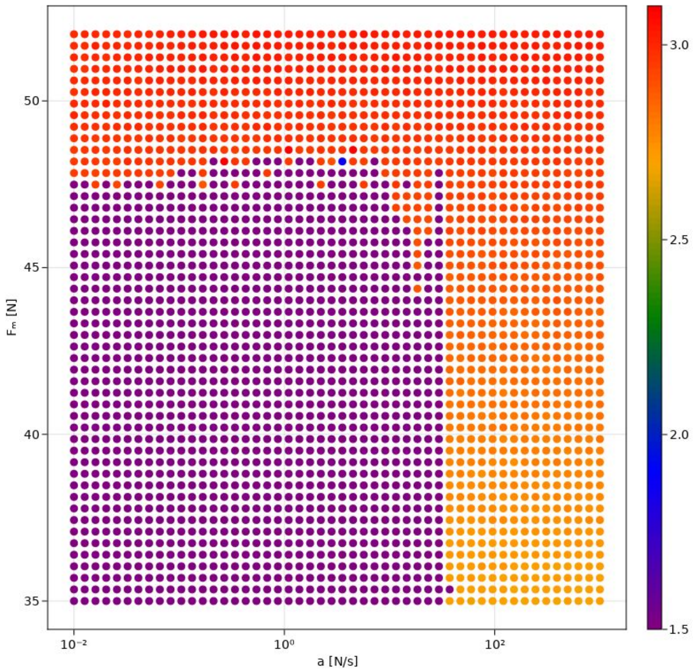


- Notice: perturbation has high magnitude.
- The pattern behaves in expected way again.

Tipping Pattern - Noisy Forcing I

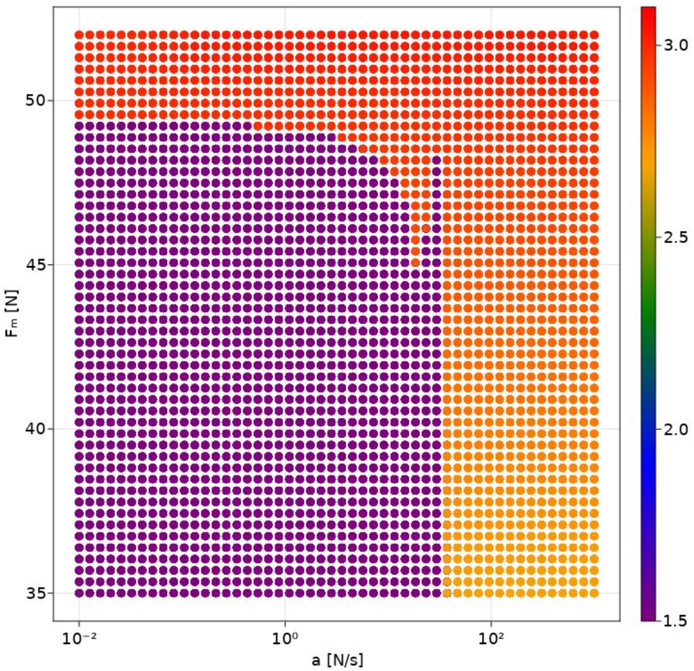
White noise added to forcing with mean = 0 N, **variance = 0.3 N.**

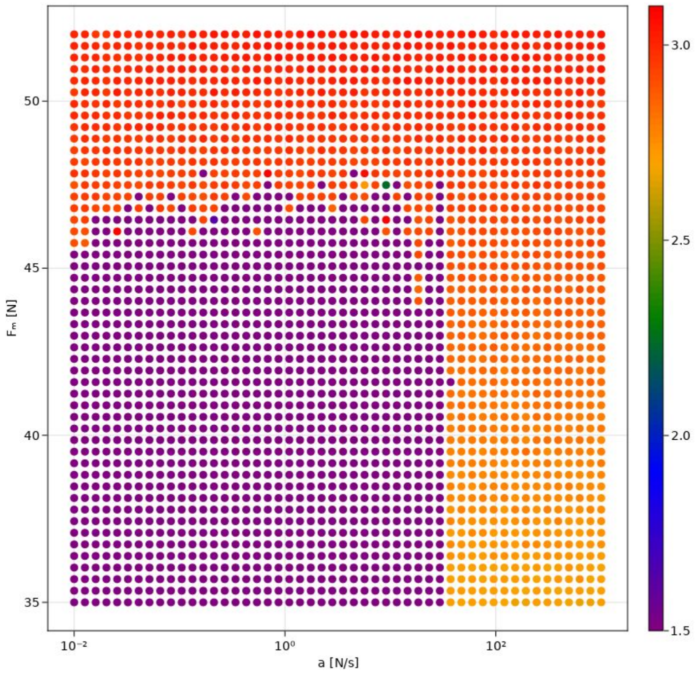




Tipping Pattern - Noisy Forcing II

White noise added to forcing with mean = 0 N, **variance = 0.5 N.**





Tipping Pattern - Summary

- Even on a very simple system, we can observe non-monotonous separation due to different time scales.
- This means: for a given target value F_{max}, **a higher slope does not necessarily mean higher chance to tip.**
- Hypothesis: this arises when the involved time scale have a similar importance (pattern II).
- Hypothesis: if one of the time scales dominates, the separability is “intuitive” (pattern I & III).
- Noisy forcings can give rise to “isolated points” on a finely resolved grid. Even for low variance compared to F_{max}.

Link to glaciology:

- We have a single forcing but feedbacks with different time scales.
- If the shape of the forcing leads these time scales to coincide, “strange” pattern could be observed.
- If one of the time scales dominates, we are in a “nicely” separable part of the ramp-parameter space.
- Noise and small amplitude oscillations can arise within the model and lead to isolated points.

Thank you!

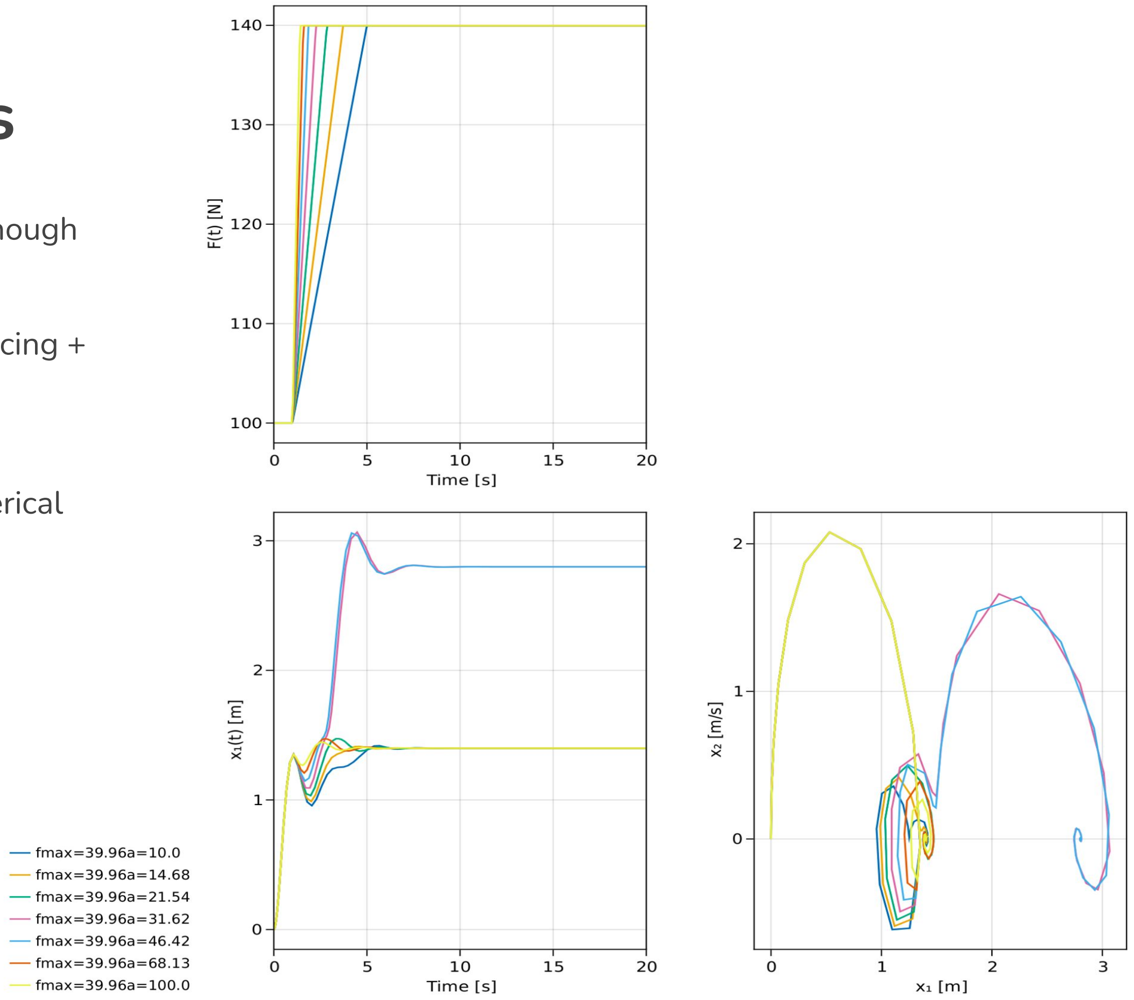
Open Questions

- To what extent is that all known to you?
- Have you observed it on another model?
- Do you know of any theoretical paper about this?
- Do you think it is novel enough to represent a potential future publication?

Appendix

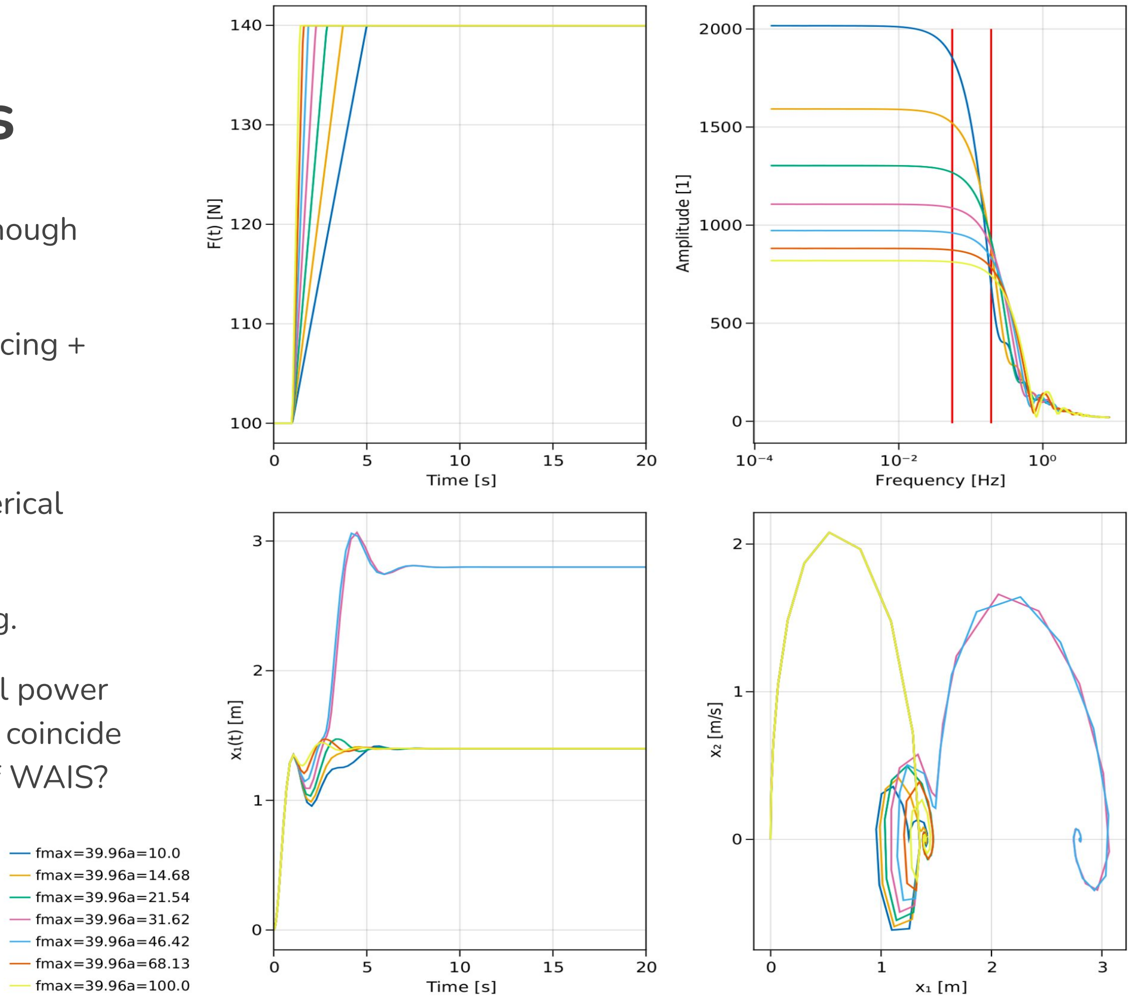
Some Experiments

- Initial perturbation: x(t=0) = 0 although gravitation is there.
- Two time scales introduced by forcing + initial perturbation.
- Important: here we have an unambiguously neglectable numerical noise!



Some Experiments

- Initial perturbation: $x(t=0) = 0$ although gravitation is there.
- Two time scales introduced by forcing + initial perturbation.
- Important: here we have an unambiguously neglectable numerical noise!
- We can perform FFT of the forcing.
- Lower slopes have higher spectral power at low frequencies → maybe they coincide with the non-dampened region of WAIS?



Bonus: Frequency Response

$$m\ddot{x} \quad = \quad -\left(c_1 + c_2(x)\right)x - d\dot{x} + mg + F(t)$$

Equilibria:

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Frequency response:

$$\alpha(\eta) = \frac{1}{\sqrt{\left(1 - \eta^2\right)^2 + 4D^2\eta^2}}$$

$$\eta = \sqrt{1 - 2D^2} \quad \text{with:} \quad D \quad = \quad \frac{d}{2\sqrt{mc}}$$

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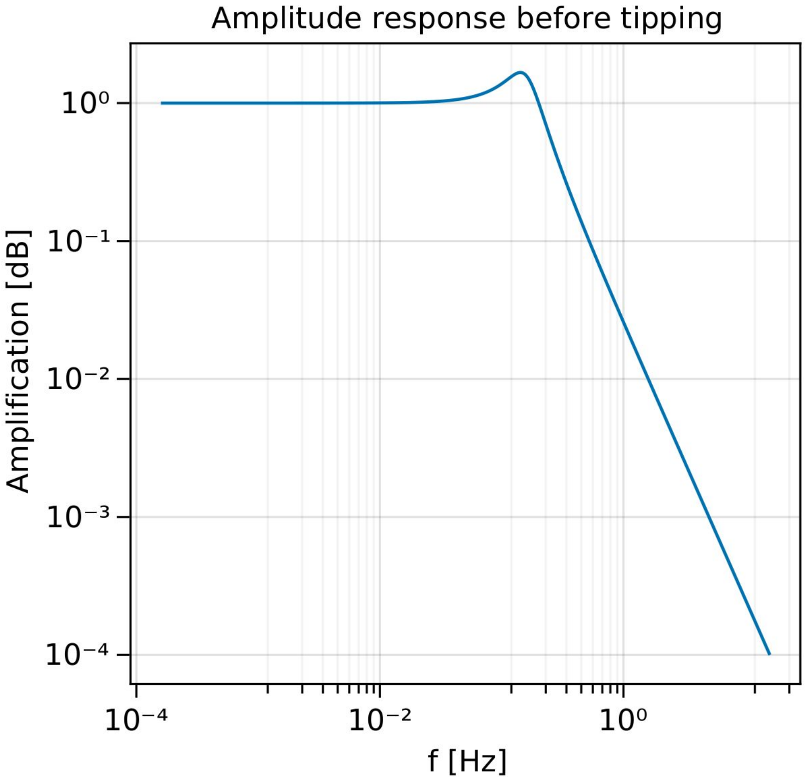
F_t + mg	150 N

Frequency Response

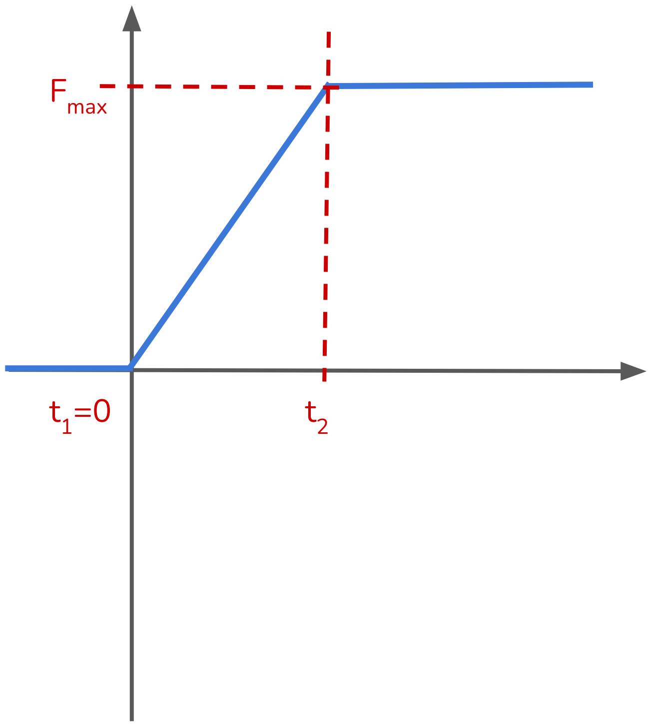


Before tipping:

Resonance interval: **[0.05, 0.19]**

Resonance peak: $\mathbf{f_{res}}$**=0.142**

Forcing Design



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Idea:

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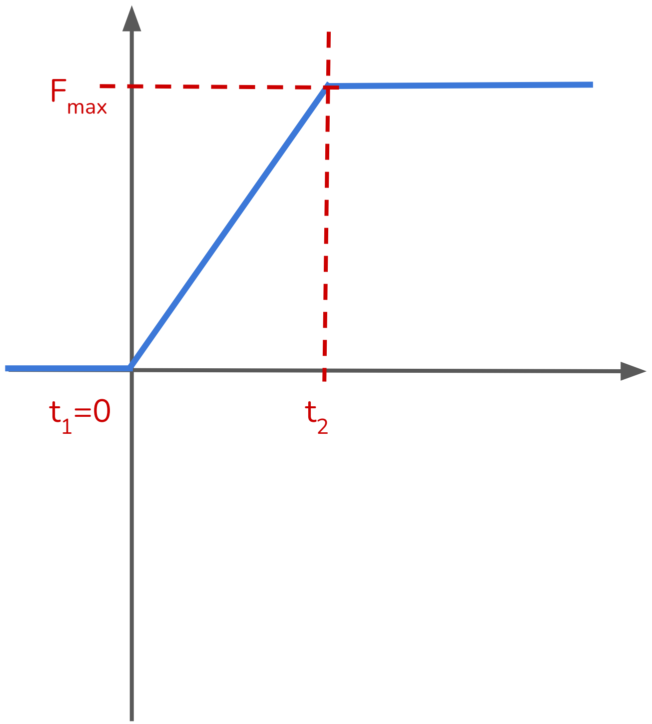
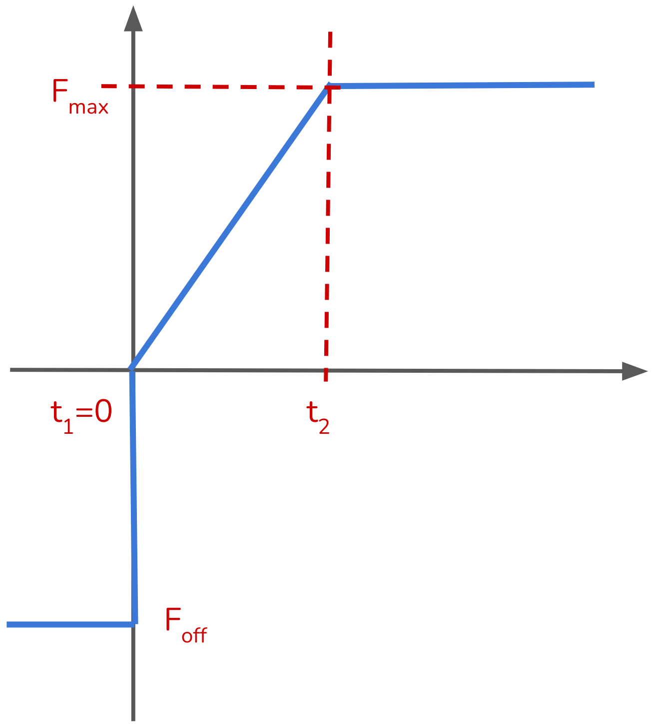
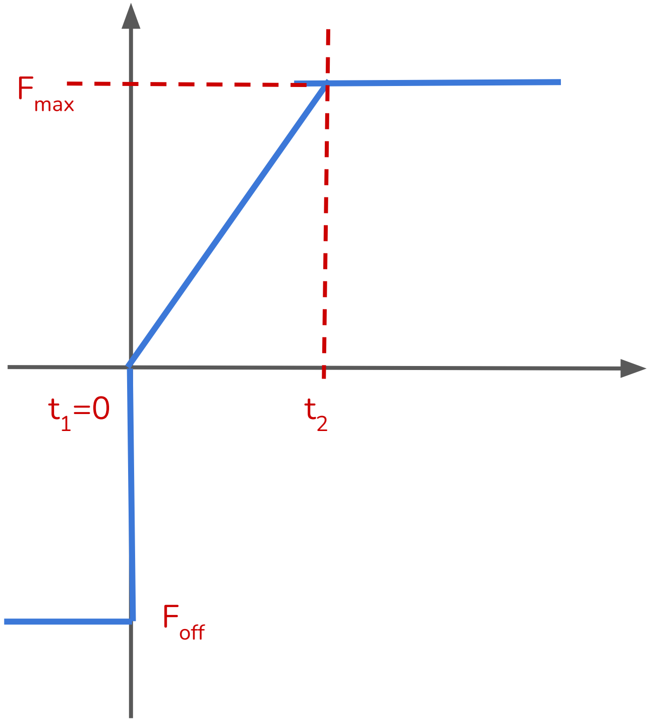
Bonus: Can we include this in the forcing?

Forcing Design





Appendix - Fourier Transform



= Step + Ramp_{t1} + Ramp_{t2}

Use linearity of Fourier Transform!

Appendix - Fourier Transform

$$\begin{aligned}
\mathcal{F}(F(t)) &= \mathcal{F}\left(F_{\max}R\left(\alpha\left(t-t_1\right)\right)-F_{\max}R\left(\alpha\left(t-t_2\right)\right)+F_{\text{off}}u\left(t\right)\right)\\
&= \frac{F_{\max}}{|\alpha|}(\mathcal{F}(R(t-t_1))(\frac{\omega}{\alpha})-\mathcal{F}(R(t-t_2)))(\frac{\omega}{\alpha})+F_{\text{off}}\mathcal{F}(u\left(t\right))\\
&= \frac{F_{\max}}{|\alpha|}\left(\exp(-\frac{j\omega t_1}{\alpha})-\exp(-\frac{j\omega t_2}{\alpha})\right)\mathcal{F}(R(t))+F_{\text{off}}(\pi\delta(\omega)+\frac{1}{j\omega})\\
&= \frac{F_{\max}}{|\alpha|}\left(\exp(-\frac{j\omega t_2}{\alpha})-\exp(-\frac{j\omega t_1}{\alpha})\right)\frac{1}{4\pi^2(\frac{\omega}{\alpha})^2}+F_{\text{off}}(\pi\delta(\omega)+\frac{1}{j\omega})
\end{aligned}$$

Tipping Pattern -