1 Introduction

R-Tipping was proved to arise in some nonlinear dynamic system, such as a 5-box model of the AMOC. As this effect has not been studied for ice-sheet models, we proposed to do so on yelmo. However, some unexpected pattern arose: sometimes, for a fixed target value, a lower rate led to tipping whereas a higher one did not. This has not been observed yet and we supposed that it might be due to the second-order nature of the system (AMOC model is first-order). In this case, a frequency dependence can arise and we propose an approach to the problem following this idea.

Studying this on the WAIS is pretty tough, due to the large amount of variables. Therefore we want to find a system of lower complexity that might display such a behavior. For this we design a simple oscillator with two springs and a dampener.

2 Model Design

We model the fact that one of the springs can break by defining a characteristic curve that displays a constant value k_1 until it suddenly drops to 0 at x_t. This can be achieved by using a steep logistic function requiring $k_2 >> 1$. The governing equations can be written as:

$$
\begin{aligned}
m\ddot{x} &= -\left(c_1 + c_2(x)\right)x - d\dot{x} + mg + F(t) && (2.1)\\
c_2(x) &= \frac{k_1}{1 + \exp\left(k_2\left(x - x_{\text{tip}}\right)\right)} && (2.2)
\end{aligned}
$$

Rewriting this as a first order model, we obtain the belowed written equation, which is suited for numerical integration.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\dfrac{c_1 + c_2}{m} & -\dfrac{d}{m}x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u(t) && (2.3)\\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} && (2.4)
\end{aligned}
$$

The static force associated with the tipping boundary can be easily computed by:

$$
F_{\text{tip}} = (c_1 + c_2)x_t - mg \tag{2.5}
$$

Given a force F and depending on the spring state, the equilibrium point $\tilde{x}$ is given by:

$$
\tilde{x} = \begin{cases} \dfrac{F + mg}{c_1 + c_2}, & \text{if system did not tip} \\[2ex] \dfrac{F + mg}{c_1}, & \text{if system tipped} \end{cases} \tag{2.6}
$$

Notice that the system displays the big advantage to be linear on both sides of the transitions, which allow to define a Bode diagramm and thus study the influence of the frequency easily. The system displays a bifurcation diagramm that is similar to the one admitted for the WAIS (fold bifurcation) and essentially has two equilibria (spring alive or dead) corresponding to WAIS alive or dead.

Recall that for such an oscillator, the eigenfrequency ω_0, the damping ratio D and the forcing frequency ration η are given by:

$$\omega_0 = \sqrt{\frac{c}{m}}, \qquad D = \frac{d}{2\sqrt{mc}}, \qquad \eta = \frac{\omega}{\omega_0}. \tag{2.7}$$

We choose D such that the system is not over-damped. Hence we allow some overshooting in a step response but keep it low to correspond to the AIS (ok, we do not really know here, important assumption made). Just as in the yelmo R-tipping experiments, we apply a saturated ramp starting at $t_1 = 0$, ending at t_2, with target value $F_{\max}$ and therefore with slope $a = F_{\max} t_2^{-1}$. So far, the system has only one time scale - the ramp response. As we believe the strange tipping patterns of the yelmo experiments to be arising due to the interplay of different time scales, we introduce an initial perturbation Δx from the equilibrium position $\tilde{x}$ and therefore another decay time:

$$x_1(t = 0) = \tilde{x} - \Delta x \tag{2.8}$$

For a wide range of Δx values, we also observe a nonmonotonous tipping pattern, wow that's nice! As the system is linear in each half of the state space, we can apply a Laplace transform to ease the study of the dynamics.

3 Transformed System

We can compute the Laplace transform $Y(s)$ of the solution $y(t) = x_1(t)$ as well as the transfer function $G(s)$ of the system as written below.

$$\begin{aligned}
m\ddot{y} &= -cy - d\dot{y} + u(t) && (3.1)\\
s^2Y(s) - sy_0 - \dot{y}_0 &= -\omega_0^2 Y(s) - 2D\omega_0\left(sY(s) - y_0\right) + \frac{1}{m}U(s) && (3.2)\\
Y(s)\left(s^2 + 2D\omega_0 s + \omega_0^2\right) &= y_0 s + \dot{y}_0 + 2D\omega_0 y_0 + KU(s) && (3.3)\\
Y(s) &= \frac{K}{s^2 + 2D\omega_0 s + \omega_0^2}U(s) + \frac{y_0 s + \dot{y}_0 + 2D\omega_0 y_0}{s^2 + 2D\omega_0 s + \omega_0^2} && (3.4)\\
Y(s) &= G_1(s)U(s) + Y_0(s, y_0, \dot{y}_0) && (3.5)
\end{aligned}$$

By inserting $j\omega$ for s we directly obtain the Fourier transform for positive frequencies:

$$\begin{aligned}
Y(\omega) &= \frac{K}{(j\omega)^2 + 2D\omega_0 j\omega + \omega_0^2}U(j\omega) + \frac{y_0 j\omega + \dot{y}_0 + 2D\omega_0 y_0}{(j\omega)^2 + 2D\omega_0 j\omega + \omega_0^2} && (3.6)\\
Y(\omega) &= G_1(\omega)U(\omega) + Y_0(\omega, y_0, \dot{y}_0) && (3.7)
\end{aligned}$$

The Laplace transform of the saturated ramp $u(t)$, beginning at t_1, ending at t_2 and with slope a can be computed by the combination of Laplace transforms of step functions $H(t)$:

$$\begin{aligned}
u(t) &= a \cdot t \cdot (H(t-t_1) - H(t-t_2)) && (3.8)\\
U(s) &= \frac{a}{s^2}(\exp(-t_1 s) - \exp(-t_2 s)) && (3.9)\\
U(\omega) &= \frac{a}{(j\omega)^2}(\exp(-t_1 j\omega) - \exp(-t_2 j\omega)) && (3.10)
\end{aligned}$$

Again, replacing s by $j\omega$ directly yields the results in the frequency domain. Based on the previously derived transfer function, it is possible to compute its amplitude and phase response, respectively named $|G_1|$ and φ:

$$\begin{aligned}
|G_1(\eta)| &= \frac{1}{c\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}} && (3.11)\\
\varphi(\eta) &= \arctan\frac{2D\eta}{1-\eta^2} && (3.12)
\end{aligned}$$

We can also obtain these functions by numerical estimation and therefore obtain a good way to cross-validate our transformations. Now that the system was derived in the Laplace and (upto a scaling?) in the Fourier domain, we believe that finding a signal for the tipping detection should be possible without performing an integration (or backtransformation). The next chapters are dedicated to different ideas allowing such an EWS.

4 Resonance Frequency

For $D < 1$, notice that the denominator of the amplitude is not monotonously increasing. The first intuition might be to connect this nonmonotnous behaviour with the one of the R-tipping pattern. In particular, notice that for $D = 0$ and $\eta = 1$, we obtain an infinite amplitude, corresponding to the extreme case of resonance. Notice that for non extreme cases - i.e. $0 < D < 1$ and $\eta \simeq 1$ - resonance could still play an important role in driving the system beyond its tipping boundary. We can compute the resonance frequency by studying the optimum of $g(\eta)$, the term under the denominator square root. As the non-zero solution has to be positive, we obtain:

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}\eta} &= 2\eta(4D^2 - 2(1-\eta^2)) && (4.1)\\
\eta &= \sqrt{1-2D^2} && (4.2)\\
\hat{\omega} &= \omega_0\sqrt{1-2D^2} && (4.3)
\end{aligned}$$

This gives a way to cross-validate our numerically computed results. More generally, we can compute the resonance region by simple commands, e.g. by detecting any amplitude response increase of 10% compared to the low-frequency asymptotic. Because of the simple shape of the amplitude response, we therefore get two frequencies ω_1 and ω_2 between which the system is particularly sensitve to forcing. We define a scalar γ_1 measuring the intensity of the forcing response in the resonance region:

$$\gamma_1 = \int_{\omega_1}^{\omega_2} |Y(\omega)| d\omega \tag{4.4}$$

This does not give any information on the tipping so far. However, if its pattern corresponds to the nonmonotonous peak of R-tipping, it provides a way to visualise the fact that resonance is at work! Of course, it presents the drawback to be a relative measure but not an absolute one. Hence we cannot get a clear statement "the system will tip" vs. "the system won't tip".

5 Necessary Condition

The boundary solution denotes the static trajectory arising for $x_1(t=0) = x_t - \varepsilon$, $\varepsilon << 1$, $x_2(t=0) = 0$, $F(t) = cx_t - mg$. This leads to $y(t) = \bar{y}$ and its transformation in the Laplace domain yields:

$$\bar{Y}(s) = \frac{\bar{y}}{s} \tag{5.1}$$

We now make the link between Laplace and time domain by studying the amplitude of the results:

$$\exists\, t : \quad y(t) \quad > \quad \bar{y} \tag{5.2}$$
$$\Rightarrow \exists\, s : |Y(s)| \quad > \quad |\bar{Y}(s)| \tag{5.3}$$

This is a necessary condition but not a sufficient one! Hence it can only tell us that tipping **cannot** occur or **might** occur. It is by no mean a guarantee that tipping **will** occur! We define a scalar γ_2:

$$\Delta Y(s) \quad = \quad \frac{|Y(s)| - |\bar{Y}(s)|}{|\bar{Y}(s)|} \tag{5.4}$$
$$\gamma_2 \quad = \quad \int_{\Delta Y(s) > 0} \Delta Y(s) ds \tag{5.5}$$

If $\gamma_2 < 0$, then we won't tip. Otherwise, we can only suggest that tipping might occur. Notice that it does not only says whether the condition is fulfilled or not, but also "by how much". However, we miss the information "by how much it does not fulfill the condition". A proposition tackling this could be:

$$\gamma_3 \quad = \quad \max_s \Delta Y(s) \tag{5.6}$$

This gives the maximum relative distance from the tipping boundary. The more negative γ_3, the more distant from R-tipping the system will be.

6 Sufficient condition for Tipping

A sufficient condition is given by Parseval's theorem:

$$\int_{-\infty}^{\infty} |y(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, \mathrm{d}\omega \tag{6.1}$$

By evaluating the right-hand side of this equation we get an information on the potential energy of the system over time. If it is superior to the one of the boundary solution, then tipping is sure to occur. Following this idea, we define following measure:

$$\gamma_4 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 - |\bar{Y}(\omega)|^2 \, \mathrm{d}\omega \tag{6.2}$$

If it is positive, tipping will occur. If not tipping is less likely to occur but still could!

The big question is: do we get more information out of this than by computing the static equilibrium? In other words: maybe this measure only gives positive results for obvious cases, e.g. where $F_{\text{max}} > F_{\text{tip}}$. Moreover, the gap between the necessary and the sufficient conditions might quite high - i.e. many points give positive results for the necessary condition but far less yield a positive sufficient condition.

7 Empirical Condition Closing the Gap

However we can try to get a finer measure by following consideration. The results we get out of Parseval's theorem will be dominated by the power at low frequencies because $Y(s)$ has a similar shape to the one of an integrator. However, we know that in the time domain the maximal amplitude will arise roughly at the end of the ramp, that is for times close to t_2. The vicinity of the associated angular frequency $\omega_{t_2} = \dfrac{2\pi}{t_2}$ could give some information on the maximum displacement of the mass. To some extent, what we are doing might be to study whether ω_{t_2} and the resonance frequency $\hat{\omega}$ coincide! We define a further measure:

$$\gamma_5 = \frac{1}{2\pi} \int_{\hat{\omega}/2}^{2\hat{\omega}} |Y(\omega)|^2 - |\bar{Y}(\omega)|^2 \, \mathrm{d}\omega \tag{7.1}$$

$$
\begin{aligned}
\alpha_{+}(t) &= \left(\sqrt{\left(D^{2}-1\right)\omega_{0}^{2}}-D\omega_{0}\right)t && (7.2)\\
\alpha_{-}(t) &= \left(-\sqrt{\left(D^{2}-1\right)\omega_{0}^{2}}-D\omega_{0}\right)t && (7.3)\\
K &= \frac{1}{2\sqrt{\left(D^{2}-1\right)\omega_{0}^{2}}} && (7.4)\\
\beta &= x_{0}\sqrt{\left(D^{2}-1\right)\omega_{0}^{2}} && (7.5)\\
y(t) &= K\left[\left(D\omega_{0}x_{0}+\beta+\dot{x}_{0}\right)\exp\left(\alpha_{+}\left(t\right)\right)+\left(-D\omega_{0}x_{0}+\beta-\dot{x}_{0}\right)\exp\left(\alpha_{-}\left(t\right)\right)\right] && (7.6)
\end{aligned}
$$