1 Introduction

R-Tipping was proved to arise in some nonlinear dynamic system, such as a 5-box model of the AMOC. As this effect has not been studied for ice-sheet models, we proposed to do so on yelmo. However, some unexpected pattern arose: sometimes, for a fixed target value, a lower rate led to tipping whereas a higher one did not. This has not been observed yet and we supposed that it might be due to the second-order nature of the system (AMOC model is first-order). In this case, a frequency dependence can arise and we propose an approach to the problem following this idea.

Studying this on the WAIS is pretty tough, due to the large amount of variables. Therefore we want to find a system of lower complexity that might display such a behavior. For this we design a simple oscillator with two springs and a dampener.

2 Model Design

We model the fact that one of the springs can break by defining a characteristic curve that displays a constant value k_1 until it suddenly drops to 0 at x_{tip}. This can be achieved by using a steep logistic function requiring $k_2 >> 1$. The governing equations can be written as:

$$m\ddot{x} = -\left(c_1 + c_2(x)\right)x - d\dot{x} + mg + F(t) \tag{2.1}$$

$$c_2(x) = \frac{k_1}{1 + \exp\left(k_2\left(x - x_{\text{tip}}\right)\right)} \tag{2.2}$$

Rewriting this as a first order model, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\dfrac{c_1 + c_2}{m}x_1 - \dfrac{d}{m}x_2 + g \end{bmatrix} \tag{2.3}$$

The static force associated with the tipping boundary can be easily computed by:

$$F_{\text{tip}} = (c_1 + c_2)x_{\text{tip}} - mg \tag{2.4}$$

Given a force F and depending on the spring state, the equilibrium point $\tilde{x}$ is given by:

$$\tilde{x} = \begin{cases} \dfrac{F + mg}{c_1 + c_2}, & \text{if system did not tip} \\[2ex] \dfrac{F + mg}{c_1}, & \text{if system tipped} \end{cases} \tag{2.5}$$

Notice that the system displays the big advantage to be linear on both sides of the transitions, which allow to define a Bode diagramm and thus study the influence of the frequency easily. The system displays a bifurcation diagramm that is similar to the one admitted for the WAIS (fold bifurcation) and essentially has two equilibria (spring alive or dead) corresponding to WAIS alive or dead.

Recall that for such an oscillator, the eigenfrequency ω_0, the damping ratio D and the forcing frequency ration η are given by:

$$\omega_0 = \sqrt{\frac{c}{m}}, \qquad D = \frac{d}{2\sqrt{mc}}, \qquad \eta = \frac{\omega}{\omega_0}. \tag{2.6}$$

We choose D such that the system is not over-damped. Hence we allow some overshooting in a step response but keep it low to correspond to the AIS (ok, we do not really know here, important assumption made). For the chosen parameters, we also observe the strange tipping pattern on R-grid, wow that's nice! Now let's dive into the frequency analysis...

3 Frequency Analysis – Linear System Theory Approach

We can compute the Laplace transform $Y(s)$ of the solution $y(t)$ as well as the transfer function $G(s)$ of the system as written below. By inserting $j\omega$ for s we directly obtain the Fourier transform for positive frequencies:

$$\begin{aligned}
m\ddot{y} &= -cy - d\dot{y} + u(t) && (3.1)\\
s^2Y(s) - sy_0 - \dot{y}_0 &= -\omega_0^2 Y(s) - 2D\omega_0\left(sY(s) - y_0\right) + \frac{1}{m}U(s) && (3.2)\\
Y(s)\left(s^2 + 2D\omega_0 s + \omega_0^2\right) &= y_0 s + \dot{y}_0 + 2D\omega_0 y_0 + KU(s) && (3.3)\\
Y(s) &= \frac{y_0 s + \dot{y}_0 + 2D\omega_0 y_0}{s^2 + 2D\omega_0 s + \omega_0^2} + \frac{K}{s^2 + 2D\omega_0 s + \omega_0^2}U(s) && (3.4)\\
Y(\omega) &= \frac{K}{(j\omega)^2 + 2D\omega_0 j\omega + \omega_0^2}U(j\omega) + \frac{y_0 j\omega + \dot{y}_0 + 2D\omega_0 y_0}{(j\omega)^2 + 2D\omega_0 j\omega + \omega_0^2} && (3.5)\\
Y(\omega) &= G_1(\omega)U(\omega) + G_2(\omega, y_0, \dot{y}_0) && (3.6)
\end{aligned}$$

The Laplace transform of the saturated ramp $u(t)$, beginning at t_1, ending at t_2 and with slope a can be computed by the combination of Laplace transforms of step functions $H(t)$:

$$\begin{aligned}
u(t) &= a \cdot t \cdot (H(t - t_1) - H(t - t_2)) && (3.7)\\
U(s) &= \frac{a}{s^2}(\exp(-t_1 s) - \exp(-t_2 s)) && (3.8)\\
U(\omega) &= \frac{a}{(j\omega)^2}(\exp(-t_1 j\omega) - \exp(-t_2 j\omega)) && (3.9)
\end{aligned}$$

Again, replacing s by $j\omega$ directly yields the results in the frequency domain. Based on the previously derived transfer function, it is possible to compute its amplitude and phase response, respectively named $|G_1|$ and φ:

$$\begin{aligned}
|G_1(\eta)| &= \frac{1}{c\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}} && (3.10)\\
\varphi(\eta) &= \arctan\frac{2D\eta}{1-\eta^2} && (3.11)
\end{aligned}$$

It achieves an optimum whenever the term $g(\eta)$ under the square root achieves an optimum itself. As non-trivial solution has to be positive, we obtain:

$$\frac{\mathrm{d}g}{\mathrm{d}\eta} = 2\eta(4D^2 - 2(1-\eta^2)) \tag{3.12}$$

$$\eta = \sqrt{1-2D^2} \tag{3.13}$$

$$x_1(t=0) = \tilde{x} - \Delta x \tag{3.14}$$