IBM **Quantum**

GHZ-Game

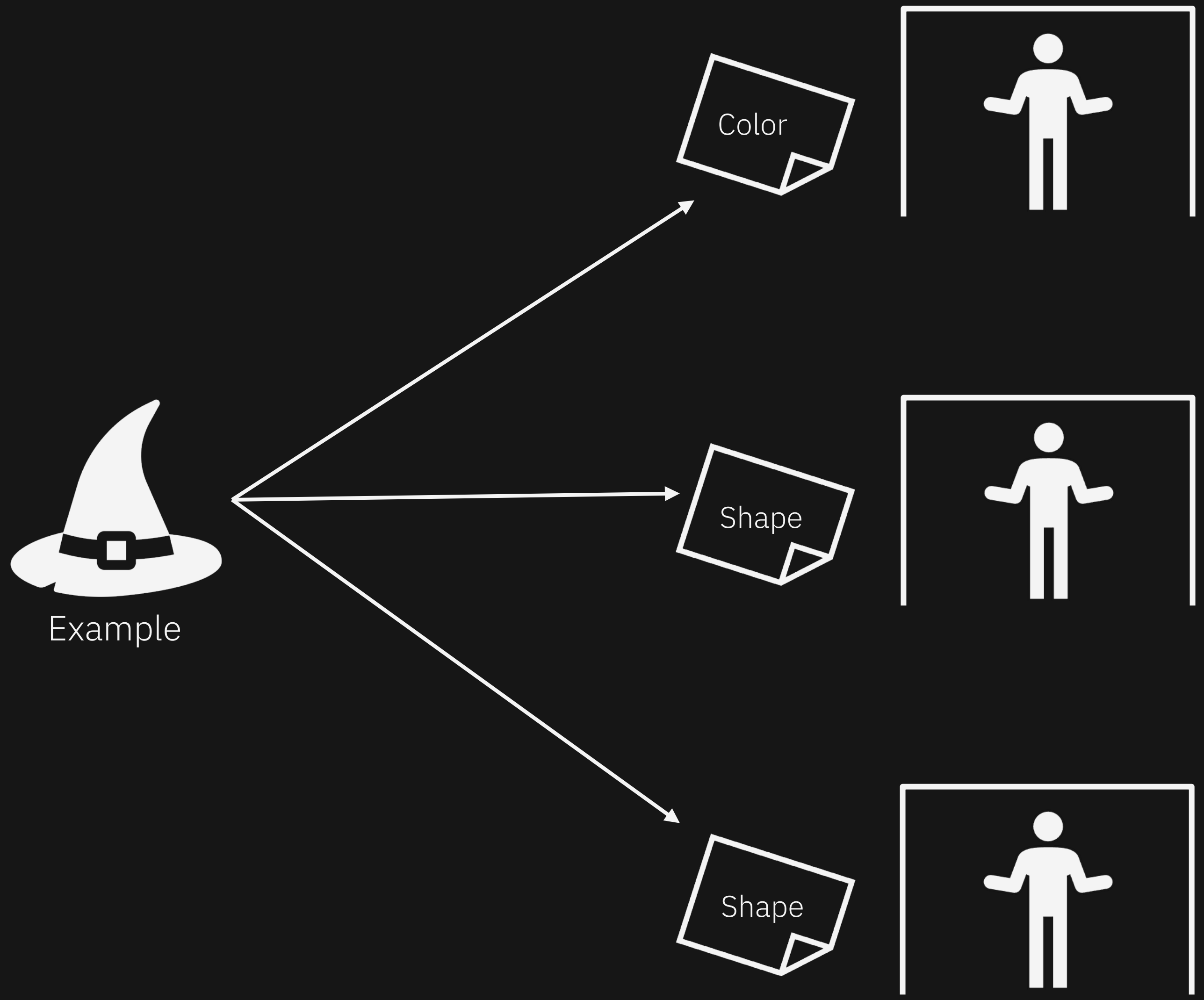
Using Quantum Entanglement to escape from an evil wizard...

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

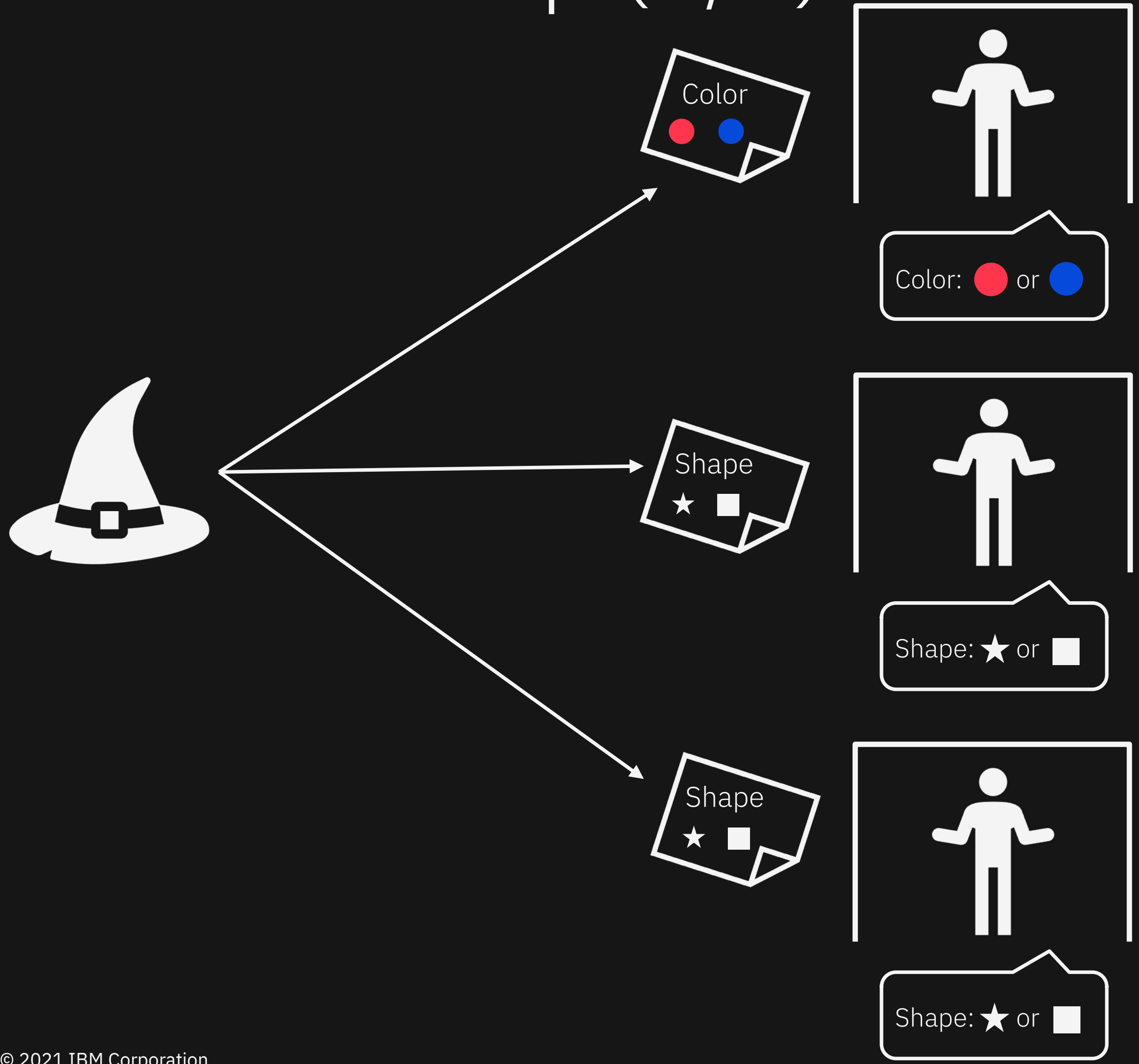
Rules – Setup (1/3)

The three of you are getting locked in different rooms by a wizard.

He gives each of you a piece of paper asking either for a color or for a shape.



Rules – Setup (2/3)



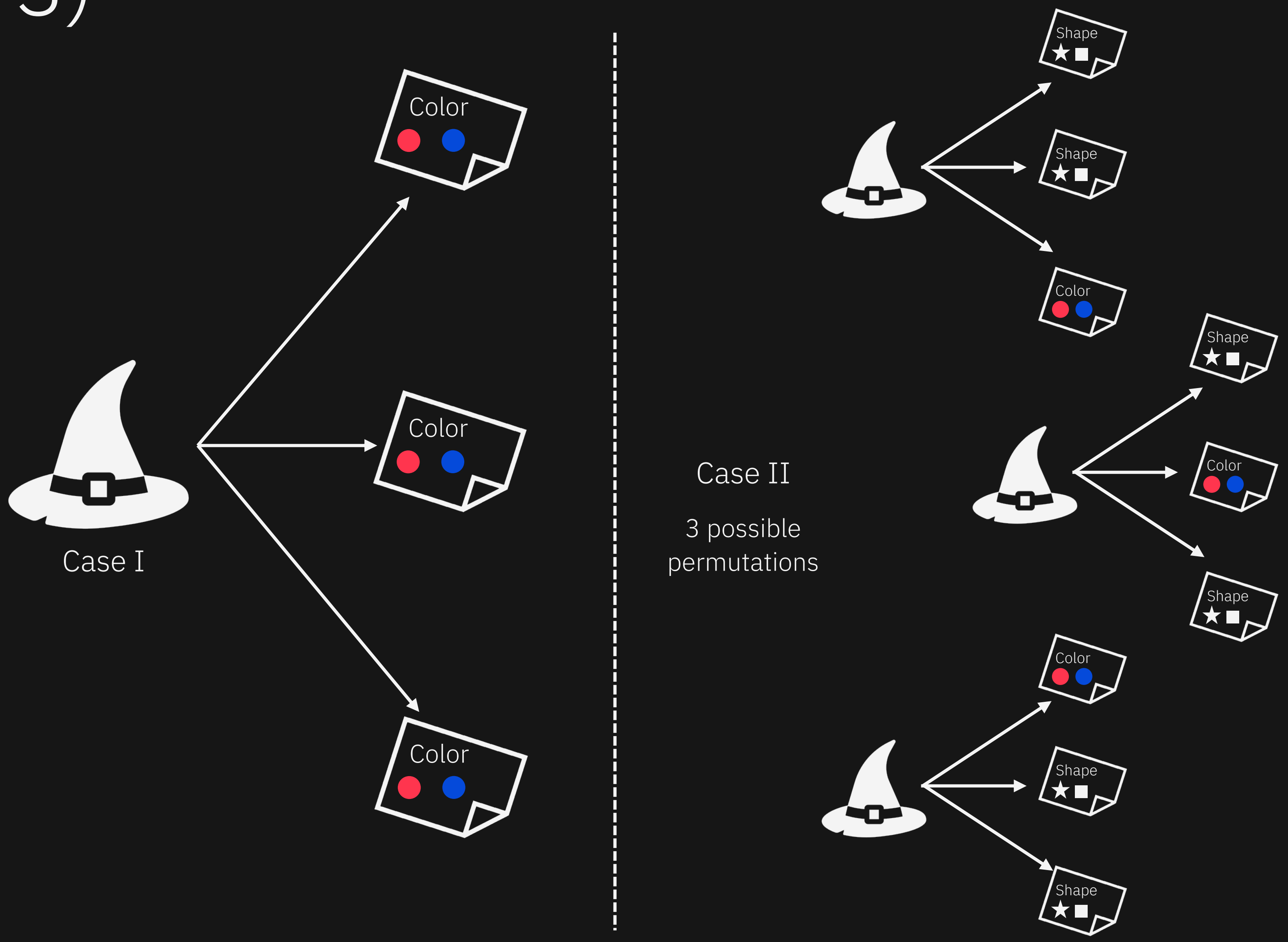
If asked for a color you can write red or blue on the paper, if asked for a shape you can chose either a star or a square.

You have until midnight to chose your answer.

You do not know what your friends were asked and have no possibility to communicate, but you are allowed to agree upon a strategy beforehand.

Rules – Setup (3/3)

The wizard will either ask all of you for a color or 2 for a shape and 1 for a color.

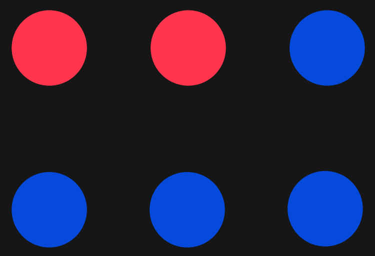


Rules – Winning Conditions

Depending on the question you were asked, you‘ll win against the wizard if one of the following conditions are met.

Case I

a)
b)

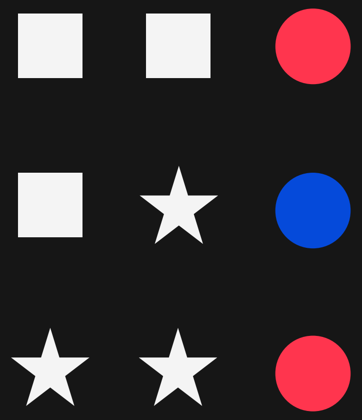


(*)

When asked for 3x color an even number of people needs to answer with red.

Case II

a)
b)
c)

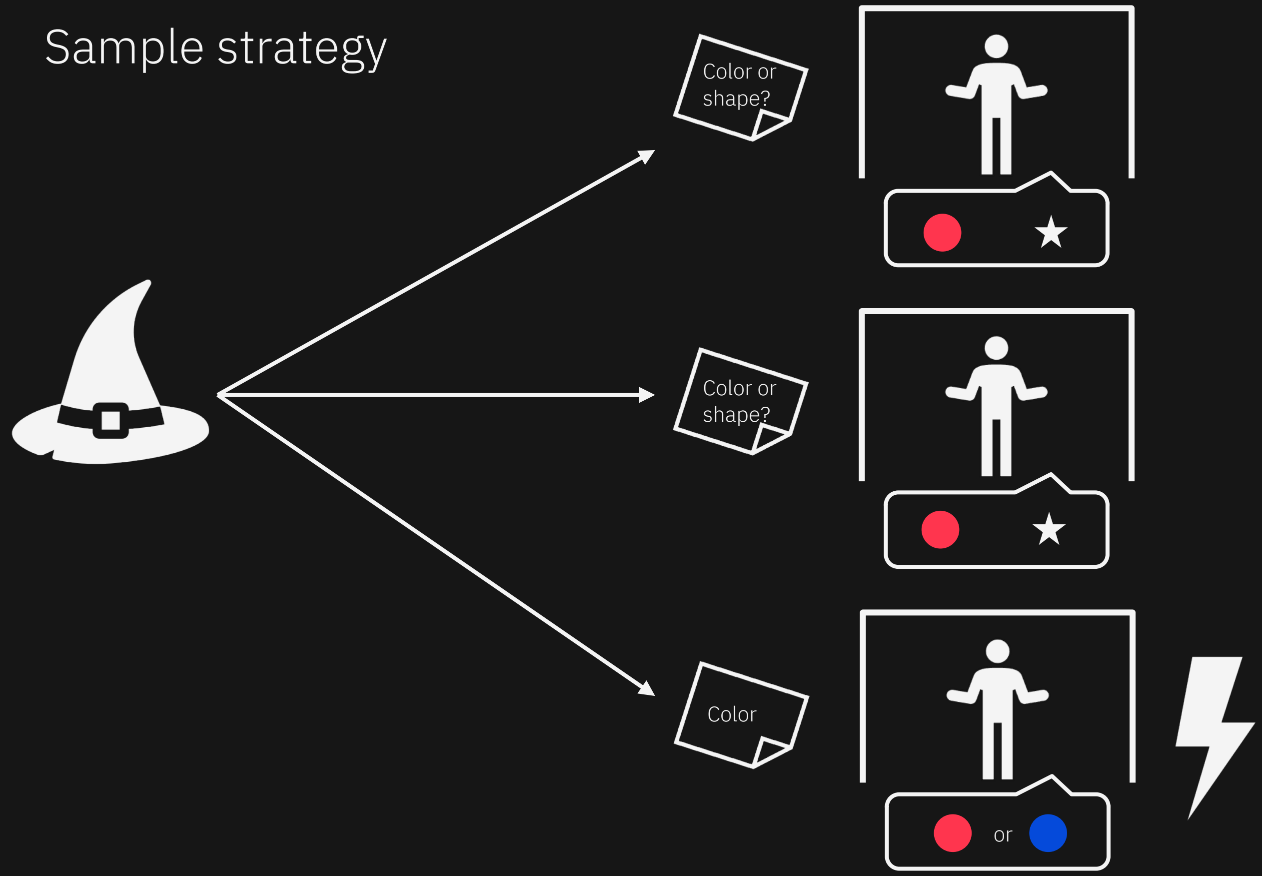


(*)

When asked for 2x shape and 1x color an uneven number of people needs to answer with red or star.

(*)Permutations of the combinations allowed

Is there a pre-agreed winning strategy?



As you want to win the game, the three of you agree with the strategy shown on the left.

But you quickly recognize that the third asked player has the problem that he/she doesn‘t know what the others were asked.

If they were asked color he/she needs to answer blue, but if they were asked shape his/her answer must be red.

Let‘s say you pre-agreed the third player should always answer with red – then you‘d win just in 3/4 cases.

Thus the probability to win with that strategy is 75% and you wonder if there is a chance to find another one with 100% winning probability…

Mathematical approach

to find a pre-agreed 100% winning strategy

A, B, C are the three players:

X is color

Y is shape

Variables: X_A, X_B, X_C, Y_A, Y_B, Y_C

Each can take on the value:

1 or -1

1 Corresponds to red or star

-1 Corresponds to blue or square

(*) Terms were changed to simplify proofing

To satisfy the winning conditions of the GHZ game, the following equations must be true:

- All color, even number of red or star

1. $X_A X_B X_C = -1$

- One player asked for color (X) others for shape (Y), requirement: uneven number of 1, therefore even number of -1

2. $X_A Y_B Y_C = 1$
3. $Y_A X_B Y_C = 1$
4. $Y_A Y_B X_C = 1$

When multiplying equations 2, 3, and 4:

$$X_A (Y_A)^2 X_B (Y_B)^2 X_C (Y_C)^2 = 1$$

since $1^2 = (-1)^2 = 1$

Y^2 is always positive

$$X_A X_B X_C = 1$$

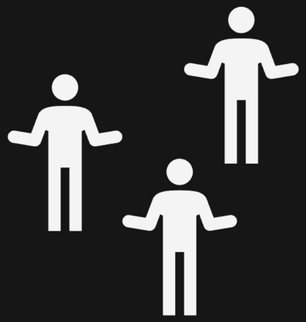
This equation cannot be satisfied if equation 1 is true

Thus it‘ s impossible to win in all cases

As you recognize from the mathematical proof there isn‘ t any chance to find a 100% winning strategy.

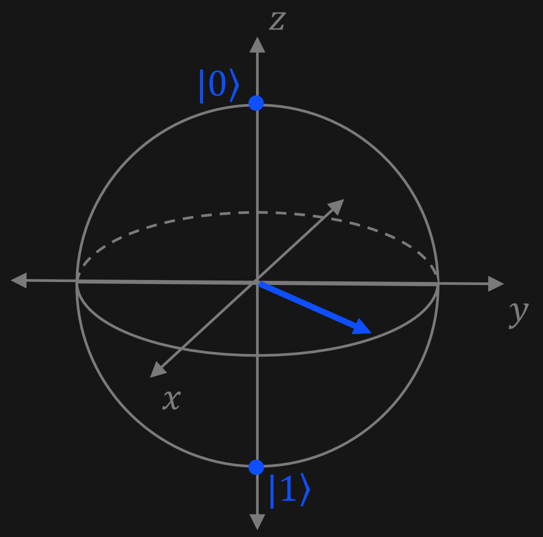
You have already found the best one with just 75% winning probability.

Quantum Solution – Qubits

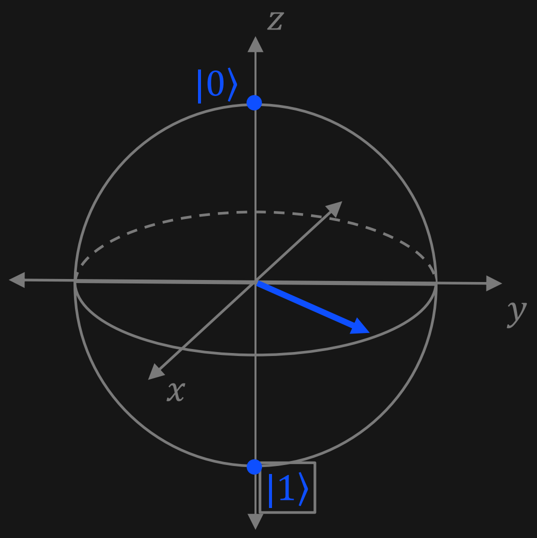


Thinking and discussing about how to win against the wizard, one of you (a quantum physicist :-) finds a surprising solution to that problem which guarantees a correct answer in all cases.

Each of you will get a qubit that has been specifically prepared, which we will learn about later. A qubit can be illustrated as a little sphere (the so called Bloch Sphere) with a vector pointing in one direction, that describes the state of the qubit.

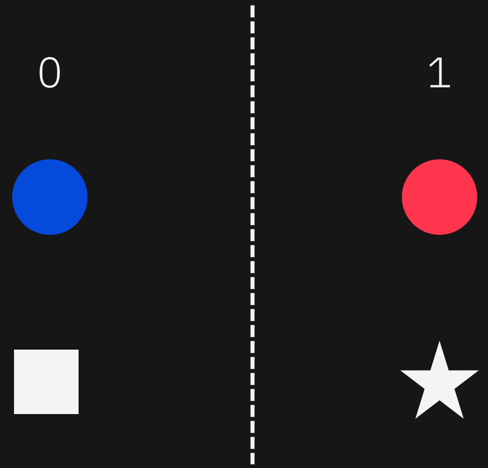


Quantum Solution – Qubits



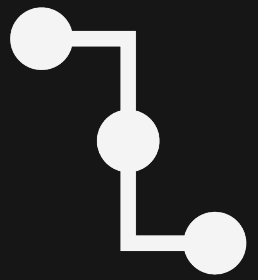
However, you cannot see or measure in which state the qubit is exactly. The result of a measurement is always either 0 or 1, which only reveals partial information about the exact state. It is random which of the results 0 or 1 you get. In this specific case (when the state vector points to the equator of the sphere), the measurement result will be 0 or 1 each with 50% probability.

We encode (translate) the shape and color of the objects in the following way: answer with blue or square whenever the result of the measurement of your qubit is 0, and accordingly with red or star when the result of the measurement is 1.

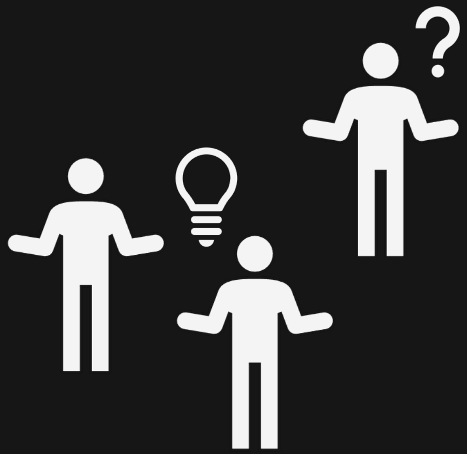


Quantum Solution – Entanglement

To get the correct answers for all cases the qubits need to be entangled, which means they have a strong dependency. Therefore, their states cannot be described individually but only in a correlated manner.



The fascinating thing is, each qubit when observed individually, behaves perfectly random. But when considering the whole system a dependency becomes obvious.



Thinking about that, one of you wonder if that is against the rule to not communicate with each other after the question was asked.

But quantum mechanics tell us: It isn‘t – because the qubits do not communicate, they are just correlated in such a way that they cannot be seen as independent.

Quantum Solution – GHZ State

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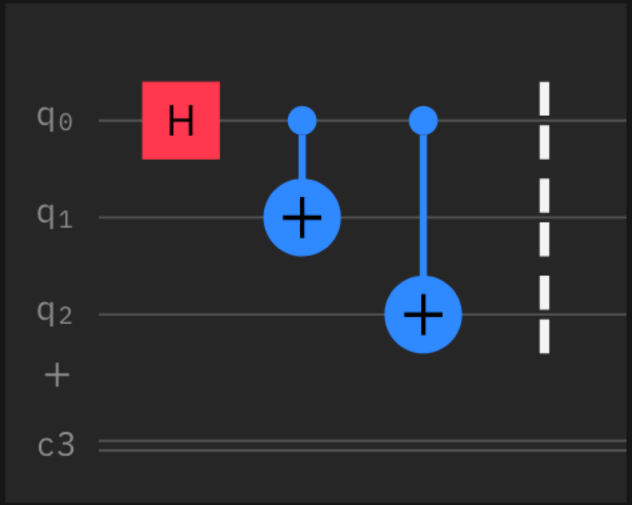
Before you start you need to bring all your three qubits into the so-called GHZ-State. This is an entangled state that creates dependencies between the qubits.

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$



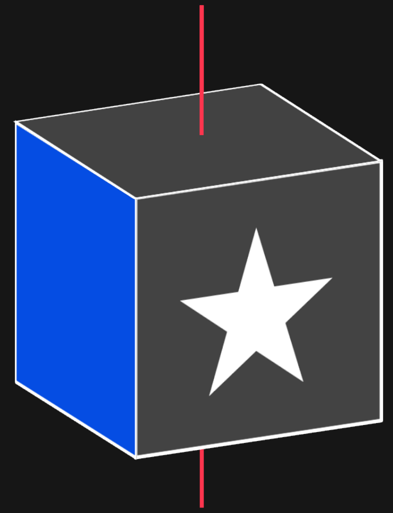
To create the GHZ state, you can apply the Quantum circuit on the left.

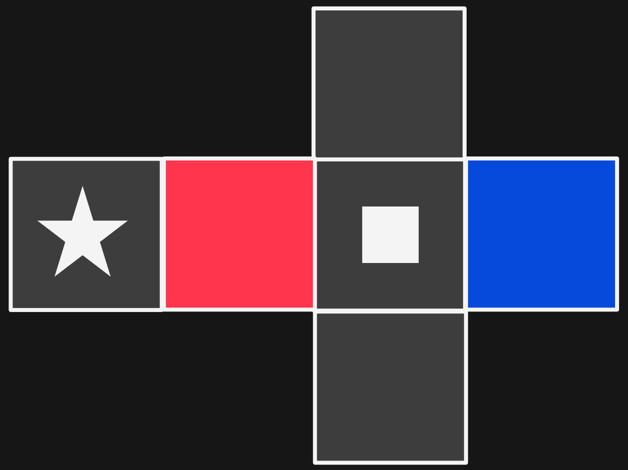
The first gate is the H-gate which brings the qubit into a Superposition. When measured it has a 50% probability of beeing 0 or 50% probability of beeing 1.

The other gates are CNOT-gates operating on 2 qubits. The qubit with the dot is the control qubit and the other one with the cross sign is the target qubit. Basically this means, if the control qubit is 1 an X-gate is applied to the target qubit. Otherwise nothing happens.

Quantum Solution – Measuring in different basis

As the wizard asked for two different properties (aspects), we can imagine these to correspond to different basis’ of the qubit, which in turn basically correspond to measurements along a different axis. The X axis (or basis) stands for the color and the Y axis (or basis) for the shape.





Imagine a dice to simplify. A quantum computer typically measures in the Z-axis, i.e. the red line through dice. However we need the information about what is on the sides of the dice.

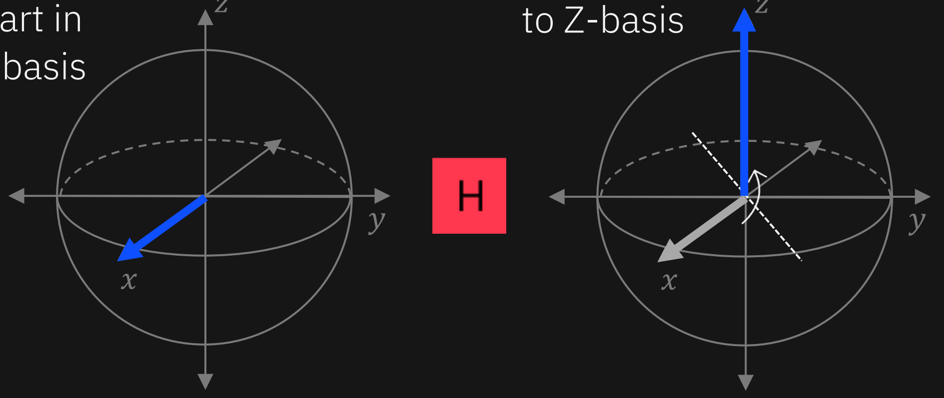
Instead of measuring in the X or Y axis (which quantum computers do not allow directly), we turn the dice with specific gates and then measure in the Z axis.

Quantum Solution – Measuring in different basis

To measure in the X basis (i.e. the color), we first apply an H-gate to rotate the X axis to the Z axis and then apply a standard Z measurement.

Start in X-basis

Transform to Z-basis



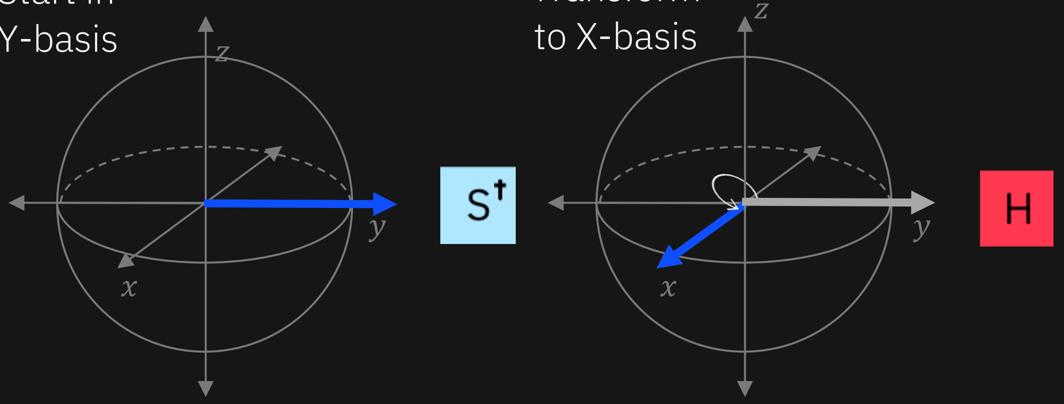
H - Gate

rotate 180° around the X+Z „diagonal“ axis

To measure in the Y basis (i.e. the shape), an sdg-gate followed by an H-gate will be applied. This combination of gates rotates the Y axis to the Z axis, and thus enables us to read out the Y-value.

Start in Y-basis

Transform to X-basis



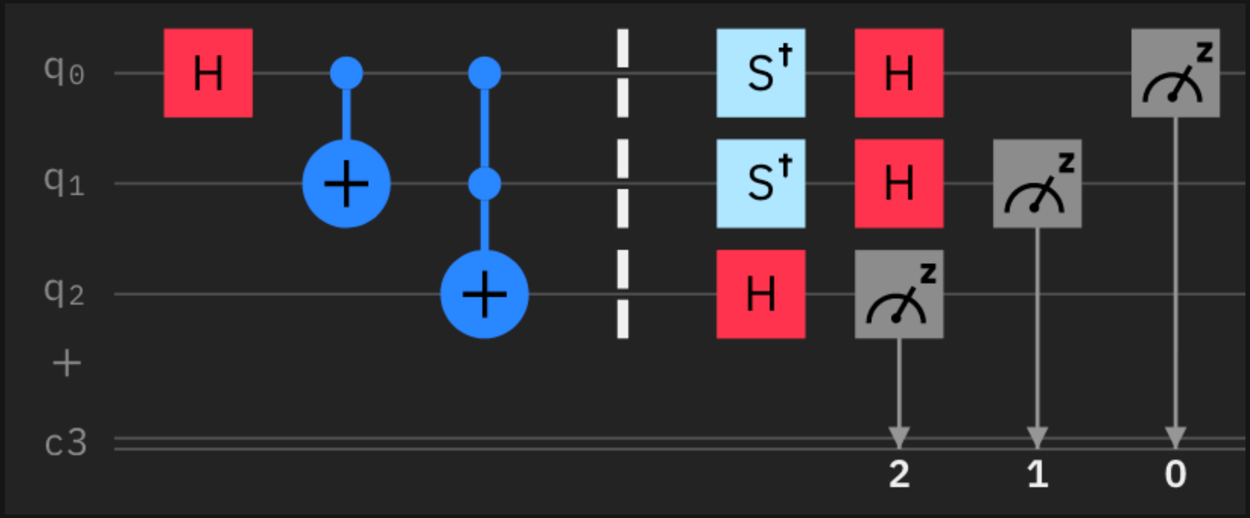
S† - Gate:

rotate 270° around the Z-axis;

Then apply an H-Gate to transform to Z-basis as above.

Quantum Solution – Example Circuit and Solution

You want to try out this strategy and the wizard asks you for color, shape and shape.



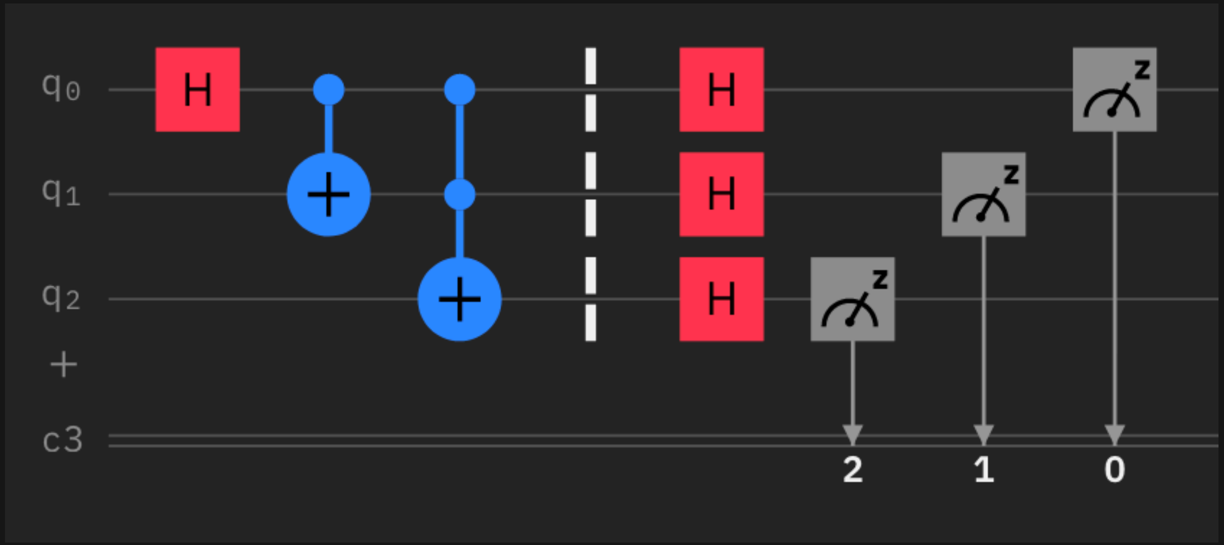
$$H\,S^\dagger \otimes H\,S^\dagger \otimes H\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right) = \underbrace{\frac{1}{2}\left(|001\rangle+|010\rangle+|100\rangle+|111\rangle\right)}$$

When measuring this state, only one of the four winning combinations on the right can occur as a result, but no other combination of colors and shapes.

As you can see – with all combinations you‘ll win against the wizard.

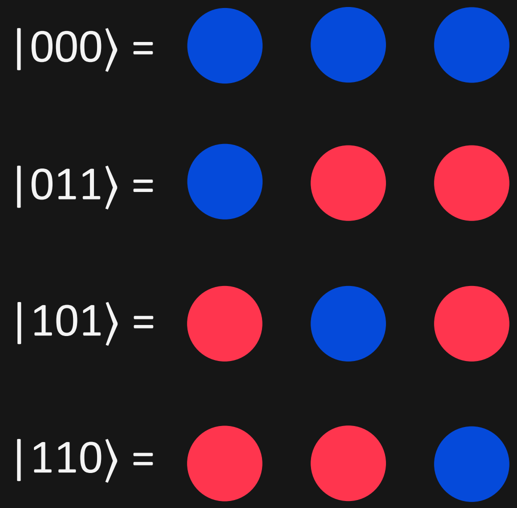
Quantum Solution – Example Circuit and Solution

Remark: When asked color, color color the following holds.



$$H \otimes H \otimes H \left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right) = \frac{1}{2}\underbrace{(|000\rangle+|011\rangle+|101\rangle+|110\rangle)}$$

When measuring this state, only one of the four winning combinations on the right can occur as a result, but no other combination of colors.



As you can see – with all combinations you‘ll win against the wizard.

End of the game

The results for each player seems to be perfectly random, but all three results in fact are strongly correlated and not independent. This is caused by the entanglement.

So using Quantum Mechanics and an entangled GHZ state, you were able to win against the evil wizard – Congrats!