Universität Bonn

Mitschrift zur Vorlesung

Lineare Algebra II

gehalten von

Stefan Schwede

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Jan Malmström

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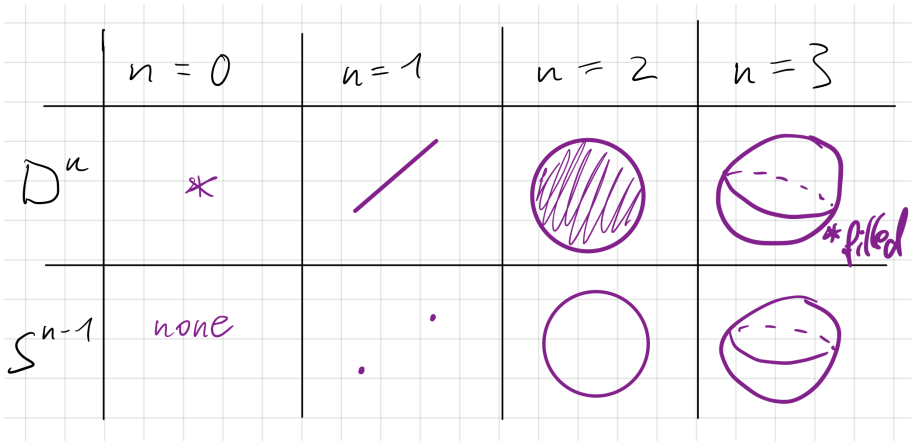

Figure 1: D^n and S^{n-1} for small n

1 CW-Complexes

The name abbreviates compact-Closure-Weak-Topology. They are „nice“ classes of spaces for the purpose of homotopy theory/algebraic topology. They are build by successively attaching cells.

The n-cell is $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. It may also be called n-balls or n-discs. $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : |x| = 1\}$ is the $n-1$-Sphere. See figure 1 for examples.

1.1 Definition

Construction. Let $n \geq 0$, let $f\colon S^{n-1} \to X$ be a continuous map, the *attaching map*. We form the quotient space

$$X \cup_{f,\partial D^n} D^n = X \cup_f D^n = X \cup_{\partial D^n} D^n := X \amalg D^n / \sim$$

where $\sim$ is the equivalencce relation on $X \amalg D^n$ genearted by $\forall\, x \in S^{n-1} : f(x) \sim x$.

Terminology. We say: „$X \cup_f D^n$ is obtained by attaching an n-cell to X along f“.

Example 1.1.

- $X \cup_f D^0 = X \amalg D^0$
- $\{*\} \cup_{S^{n-1}} D^n = D^n / \sim = D^n / S^{n-1} \cong S^n$

  In this example $\sim$ identifies all of S^{n-1} to a point, which then is homeomorphic to S^n[1].
- Remark, that the attaching map matters greatly. See figure 2

$$S^{n-1} \cup_f D^n \cong D^n \quad \text{with } f = \mathrm{Id}\colon S^{n-1} \to S^{n-1}$$

$$S^{n-1} \cup_f D^n \quad \text{with } f\colon S^{n-1} \to S^{n-1} \text{ constant}$$

Simultaneous attachment of several cells

Let J be an indexing[2] set, considered as a discrete space ($J = \emptyset$ is allowed).

[1] supposed as known

[2] „indexing“does not carry mathematical meaning

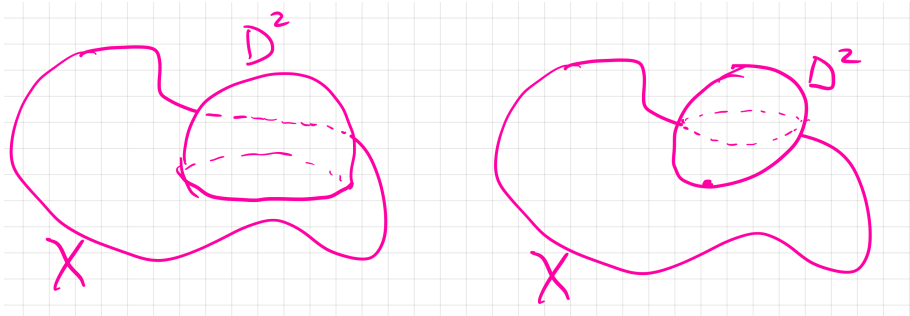

Figure 2: The attaching map influences how D^n is attached.

Give $J \times D^n$ the product topology, then

$$J \times D^n \cong \coprod_{j \in J} \{j\} \times D^{n3}$$

as a topological space. The $\coprod$ represents the disjoint union topology.

It follows, that

$$\begin{array}{ccc} \{\text{continuous maps } f \colon J \times D^n \to X\} & & f \\ \text{\rotatebox{90}{$\cong$}} & & \big\downarrow \\ \{\text{J-indexed families of continuous maps } \{f_j \colon D^n \to X\}_{j \in J}\} & & f_j = f(j, \_) \end{array}$$

We will identify them from now on.

Definition 1.2

Let $f \colon J \times \partial D^n \to X$ be a continuous map, the *attaching map*.

$$X \cup_{f, J \times \partial D^n} J \times D^n = X \cup_f J \times D^n = X \cup_{J \times \partial D^n} J \times D^n := X \amalg J \times D^n / \sim$$

where $\sim$ is the equivalence relation generated by $f(x) \sim x$ for all $x \in J \times \partial D^n$.

Remark. Write

$$p \colon X \amalg J \times D^n \to X \cup_f J \times D^n$$

for the quotient map. From the universal property of the qoutient map follows: Given maps $g \colon X \to Y$ and $\Psi_j \colon D^n \to Y$ such that $g(f_j(x)) = \psi_j(x)$ for all $j \in J, x \in \partial D^n$ there is a unique map $\psi \colon X \cup_f J \times D^n \to Y$, such that

$$\psi \circ p = g + \coprod_{j \in J} \psi_j \colon X \amalg (J \times D^n) \to Y$$

and ψ is continuous iff g and all f_j are continuous.

Remeber the quotient-topology: A subset O in $X \cup_f J \times D^n$ is open iff $p^{-1}(O)$ is open in $X \amalg J \times D^n$. This is equivalent to $p^{-1}(O) \cap X$ is open in X and for all $j \in J$ $p^{-1}(O) \cap j \times D^n$ is open in D^n.

X is a closed subspace of $X \cup_f J \times D^n$ $J \times \mathring{D}^n$ is an open subset of $X \cup_f J \times D^n$ $X \cup_f J \times D^n$ is as a set (not as a space) the disjoint union of X and $J \times \mathring{D}^n$. We elaborate

Proposition 1.3. *1. The composition*

$$X \longrightarrow X \amalg (J \times D^n) \xrightarrow{p} X \cup_f J \times D^n$$

is a closed embedding (i.e. a closed injective map).

2. The composition

$$J \times \mathring{D}^n \xhookrightarrow{incl} J \times D^n \longrightarrow X \amalg J \times D^n \xrightarrow{p} X \cup_f J \times D^n$$

is an open embedding (i.e. injective and open)

3. The underlying set of $X \cup_f J \times D^n$ is the disjoint union of the image of X and $J \times \mathring{D}^n$.

Proof. Suppose $M \subseteq X \amalg J \times D^n$ is saturated, i.e. $M = p^{-1}(p(M))$. If M is saturated and open, then $p(M)$ is open in $X \cup_f J \times D^n$.

1. $n = 0$ $X \cup J \times D^0 = X \amalg J \times D^0$ is obvious.

   $n \geq 1$ let $r\colon D^n \to S^{n-1}$ be a map, such that $r(x) = x$ for all $x \in S^{n-1}$.This <u>cannot</u> be done continuously. Define $X \amalg J \times D^n \to X$ by $x \mapsto x, (j, y) \mapsto r(y)$. This is compatible with the equivalence relation, so it descends to a (noncontinuous) map $X \cup_f J \times D^n \to X$. This prooves injectivity. To show this is a closed map, we consider a closed subset $A \subseteq X$. Then $p^{-1}(p(A)) = A \amalg f^{-1}(A) \subseteq X \amalg J \times D^n$ $\subset J \times \partial D^n \subset J \times D^n$ is closed in $X \amalg J \times D^n$. So $p(A)$ is closed in $X \cup_f J \times D^n$.

2. All points in $J \times \mathring{D}^n$ are their own equivalence classes, so the map is injective. To show that the map of 2. is open, we let B be an open subset of $J \times \mathring{D}^n$. This is then also open in $J \times D^n$. $p^{-1}(p(B)) = \emptyset \amalg B \subset X \amalg J \times D^n$ open, so $p(B)$ is open in $X \cup_f J \times D^n$.

3. I think this was prooven with a picture I didn't draw.

□

Exercise. Let V_j be an open subset of D^n for every $j \in J$, such that $V_j \supset \partial D^n$. Show, that the set $V = X \cup \bigcup_{j \in J} V_j$ is open in $X \cup_f J \times D^n$.

From now on we often identify X with its image in $X \cup_f J \times D^n$ and $J \times \mathring{D}^n$ with its image in $X \cup_f J \times D^n$

Definition 1.4: Compactness

A space X is *compact*, if it is Hausdorff (any two points can be separated by two disjoint open sets) and *quasicompact* (any open cover has a finite subcover).

Remark. Some literature defines compactness equivalent to quasicompactness. This lecture uses the definition that was given.

Theorem 1.5: L

t $f\colon J \times \partial D^n \to X$ be a continuous attacing map.

- *If X is Hausdorff, then so is $X \cup_f J \times D^n$.*
- *If X is compact and J is finite, then $X \cup_f J \times D^n$ is compact.*
- *Let K be a quasicompact subset of $X \cup_f J \times D^n$. Then $K \cap (\{j\} \times \mathring{D}^n) = \emptyset$ for almost all[a] $j \in J$.*

[a]mathematical term for all but finitely many.

Lemma 1.6: T

ere exists an open neighborhood V of X in $X \cup_f J \times D^n$ and a continuous map $r\colon V \to X$ that is the identity on X. (X is a neighborhood retract inside $X \cup_f J \times D^n$).

Proof. picture. We take $V = X \cup_{J\times\partial D^n} J \times (D^n \setminus 0)$is open in $X \cup_f J \times D^n$. We define $r\colon V \to X$ by $X \mapsto x, (j,z) \mapsto f(j, z/|z|)$ □

Proof. of the theorem

1. **Case 1** $x, y \in J \times \mathring{D}^n$. Since $\mathring{D}^n$ is Hausdorff, so is $J \times \mathring{D}^n$, so we can separate x and y by open disjoint subsets in $J \times \mathring{D}^n$, Since $J \times \mathring{D}^n$ is open in $X \cup_f J \times D^n$, theses subsets are also open in $X \cup_f J \times D^n$.

   Case 2 $x \in X, y \in \{j\} \times \mathring{D}^n$. We choose an $y \in O_y \subset j \times D^n$ open $j \times \partial D^n \subseteq V_j \subseteq j \times D^n$ s.t. $O_j \cap V_j = \emptyset$. Then $V := X \cup V_j \cup \bigcup_{k\in J\setminus\{j\}} D^n$ is open[4] in $X \cup_f J \times D^n$. $V \cap O_j = \emptyset$, $x \in V, y \in O_j$.

   Case 3 $x, y \in X$. Since X is Hausdorff, there are open subsets O_x, O_y of X with $x \in O_x$, $y \in O_y$, $O_x \cap O_y = \emptyset$. We let V be an open subset of $X \cup_f J \times D^n$ with a continuous retraction $r\colon V \to X$, $r|_X = \mathrm{Id}_X$. Then $x \in r^{-1}(O_x), y \in r^{-1}(O_y)$, $r^{-1}(O_y), r^{-1}(O_y)$ are open, and disjoint.
2. If X is compact and J is finite, so is $X \amalg J \times D^n = X \amalg \coprod_{j\in J}\{j\} \times D^n$ compact hence also the quotient space $X \cup_f J \times D^n$ is quasi-compact. Hausdorff is inherted by 1..
3. Let K be a quasicompact subset of $X \cup_{J\times D^n} J \times D^n$. We define subsets $\mathring{V}_j$ of D^n for all $j \in J$ as follows: If $K \cap (j \times \mathring{D}^n) = \emptyset$, we set $V_j = D^n$. If $K \cap (j \times \mathring{D}^n) \neq \emptyset$, we choose a V_j, that doen't contain at least one point of K, is open, and contains ∂D^n. Now $(X \cup_{j\in J} V_j) \cup \bigcup_{j\in J}\{j\} \times \mathring{D}^n$ is an open cover of X. Since K is quasicompact, there is a finite subset L of J such that $K \subset (X \cup_{j\in J} V_j) \cup \bigcup_{j\in L}\{j\} \times \mathring{D}^n$ much

□

Example 1.7. Hawaiian Earrings

$$H = H_1 \cup H_2 \cup H_3 \cup \cdots = \bigcup_{i\geq 1} H_j$$

where $H_i = circlein\mathbb{R}^2 ofradius 1/iandcenter(0, 1/i)$

picture. with the subspace topology of $\mathbb{R}^2$. Is H obtained from $\{(0,0)\}$ by attaching countably many 1-cells?

[4]by an exercise.

It is not. Consider a continuous map $\psi_j \colon D^l = [-1,1]$ such that is surjective and a homeomorphism of $[-1,1]/-1 \sim 1$ onto $H_j \subset H$ $\{(0,0)\} \amalg \mathbb{N} \times D^1 \to H$ is a continuous surjection$\{(0,0)\} \cup_{\mathbb{N}\times\partial D^1} \mathbb{N} \times D^1$ is a continuous bijection. This is not a homeomorphism.

Proof: Consider $V = \{(0,0)\} \cup_{\mathbb{N}\times\partial D^1} \mathbb{N} \times [-1,0) \cup (0,1]$ open in

Complement of that is not closed.

Definition 1.8

A relative CW-complex is a space X equipped with a sequence of closed subspaces

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq \dots$$

such that

1. For every $n \geq 0$ X_n can be obtained from X_{n-1} by attaching n-cells.
2. $X = \bigcup_{n\geq 0} X_n$ and X has the weak topology with respect to the sequences.

more precise: There exists an index set J, a continuous map $f \colon J \times \partial D^n \to X_{n-1}$ and a homeomorphism $\psi \colon X_{n-1} \cup_f J \times D^n \to X_n$ that is the identity on X_{n-1}. a subset O of X is open in X is open in X iff $O \cap X_n$ is open in X_n for all $n \geq 0 \Leftrightarrow$ a subset C of X is closed in X iff $C \cap X_n$ is closed in X_n for all $n \geq 0$. $\implies$ A map $f \colon X \to Y$ is already continuous if $f|_{X_n} \colon X_n \to Y$ is continuous for all $n \geq 0$.

Notation. We sometimes say (X,A) is a relative $CW-complex$ and leave the X_n implicit. For $A = \emptyset$ X is called a absolute CW-complex, or just a CW-complex.

The subspace X_n in a CW-complex is the n-skeleton.

A relative CW-complex (X,A) is finite-dimensional if $X_n = X$ for some $n \geq 0$.

A relative CW-complex (X,A) is finite, if there are only finitely many cells altogether.

Once chosen a homeomorphism ψ as above, then the characteristic map of the j-th n-cell is the composite $D^n \to X_{n-1} \cup_{J\times\partial D^n} J \times D^n \to X_n \hookrightarrow X$ erste abb $j, \_,$ zweite $\psi_n, cong$.

$X_j|_{\mathring{D}^n} \mapsto X_j(\mathring{D}^n)$ is a homeomorphism ... , which is one path component of $X_n \setminus X_{n-1}$. The restriction $f_j \colon X_j|_{\partial D^n} \to X_{n-1}$ is called the attacing map as before.

Comment: The space $X_n \setminus X_{n-1}$ is a disjoint union of open cells $\mathring{D}^n$. So the indexing set could be taken as $\pi_0(X_n \setminus X_{n-1})$.

For every path-component of $X_n \setminus X_{n-1}$ there exists a homeomorphism $f \colon \mathring{D}^n \to pathcomponent$, that extends to a continuous map $\bar{f} \colon D^n \to X_n$.

example. Any discrete space is an absolute 0-dimensional CW-complex.

Let $z \in S^n$ be any point. Then the minimal CW-structure on S^n is $X_{-1} = \emptyset, X_0 = \{z\} = X_1 = \dots = X_{n-1}$ $X_n = X_{n+1} = \dots = S^n$. It consists of 1 0-cell and 1 n-cell.

$S^n \cong D^n/\partial D^{n-1}$ $z \leftarrow \partial D^{n-1}$

Example $X = S^n$ $n \geq 2$ Another CW-structure:

picture

$X_{-1} = \emptyset, X_0 = X_1 = \dots = X_{n-2} = \{(1,0,\dots,0)\}$ $X_{n-1} = equator = \{(x,0) : x \in S^{n-1}\}$ $X_n = X_{n+1} = \dots = S^n$ 1 0 cell 1 $n-1$-cell 2 n-cells $S^n \cong D^n \cup_{S^{n-1}} D^n$

Example: S^2 2 1 cell 2 2 cell 2 0 cell picture

Analog for S^n is a CW-complex with 2 i-cells for $i = 0, \dots, n$.

On S^1 pick any finite subset $A \subseteq S^1$. Then S^1 has a CW-structure with $X_{-1} = \emptyset, X_1 = A, X_2 = S^1$. n 0 cells n 1 cells.

Any non-discrete space, that admits an absolute CW-structure admits uncountably many different CW-structures.

Preview: The Euler characteristic of a finite absolute CW-comples is $\chi(X) = \sum_{n \geq 0} (-1)^n \# n$-cells does not depend on the CW-structure. We will eventually show this using singular homology.

Then: Let (X, A) be a relative CW-complex.

1. If A is Hausdorff, then so is X.
2. If A is compact and (X, A) is finite, then X is also compact.

Proof. Because $X_{-1} = A$ is Hausdorff and X_n can be obtained from X_{n-1}, by attaching cells, inductively X_n is Hausdorff for all $n \geq 0$. Claim: Let O_n, P_n be open disjoint subsets of X_n. Then there exist disjoint open subsets O_{n+1}, P_{n+1} of X_{n+1}, such that $O_n = O_{n+1} \cap X_n, P_n = P_{n+1} \cap X_n$.

Proof. Since X_{n+1} can be obtained from X_n by attaching $(n+1)$-cells X_n is a neighborhood retract in X_{n+1}, i.e. there are open neighborhood V of X_n in X_{n+1} and a continuous retraction $r \colon V \to X_n$ with $r|_{X_n} = \mathrm{Id}$. We set $O_{n+1} = r^{-1}(O_n), P_{n+1} = r^{-1}(P_n)$.

Proof of the Hausdorff property: Let $x, y \in X$ be disjoint points. Since $X = \bigcup_{n \in \mathbb{N}} X_n$. then for some $n \geq 0$, $x, y \in X_n$. Since X_n is Hausdorff, there are open, disjoint subsets O_n, P_n of X_n with $x \in O_n, y \in P_n$. Inductiveleuse the claim to find open disjoint subsets O_m, P_m of X_m for all $m \geq n$, such that $O_{m+1} \cap X_m = O_m, P_{m+1} \cap X_m = O_m$ for all $m \geq n$. Then set $O = \bigcup_{m \geq n} O_m, P = \bigcup_{m \geq n} PM$ disjoint subsets of X and open in X by the weak topology, as $O \cap X_m = O_m$ open in X_m. □

Induction of n such that X_n is compact because X_n is obtained from X_{n-1} by attaching finitely many cells. Also $X = X_n$for sufficently large n. So X is compact. □

Note: Suppose that X admits a CW-structure. Then the following are equivalent: X admits a finite CW-structure $\Leftrightarrow$ X is compact.

From now on standing assumption: the base A in a relative CW-complex X, A is Hausdorff.Then X is also Hausdorff.

Thus: Let X, A be a relative CW-complex.

1. The closure of every open n-cell (= path component of $X_n \setminus X_{n-1}$) is compact.
2. Let $\chi \colon D^n \to X$ be a characteristic map for some n-cell, then the image $\chi(D^n)$ is the closure of the open cell $\chi(\mathring{D}^n)$
3. Let U be a subset of X s.t. $A \subseteq U$. Suppose that the intersection of U with the closure of every cell is closed. Then U is closed in X.

Warning: the closure of a cell is not necessary a closed cell:

minimal CW-tructure on S^2 open 2-cell $S^2 \setminus \{z\}$ closure $= S^2 \neq D^2$.

Proof. 1. By definition every open n-cells admits a characteristic map $\chi \colon D^n \to X_n$ continuous s.t. $\chi|_{\mathring{D}^n}$ is a homeomorphis onto the open cells. Then $\chi(D^n) \subseteq closureofopencell \chi(\mathring{D}^n)$ so they are the same.

2. Let $U \subseteq X$ be as in 2. It suffices to show that $U \cap X_n$ is closed in X_n for all $n \geq 0$ (weak topology). We argue by induction on n. $n = -1$ $U \cap X_{-1} = U \cap A = A$ closed in $A = X_{-1}$. $n \geq 0$ We choose a homeomorphism $\psi\colon X_n \cong X_{n-1} \cup_{J \times \partial D^n} J \times D^n$ that is the in?? on X_{n-1}. We let $p\colon X_{n-1} \amalg J \times D^n \to X_{n-1} \cup_{J \times \partial D^n} J \times D^n \psi \cong\to X_n$ be the ??

   $p^{-1}(U \cap X_n = (U \cap X_{n-1}) \amalg \coprod_{j \in J} p^{-1}(U \cap closureofj - thn - cell))$ closed by hypothesis $\subseteq X_{n-1} \amalg J \times D^n \implies U \cap X_n$ is closed in X_n

□

2 Higher homotopy groups

3 singular homology groups

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