University of Bonn

notes for the lecture

Topology I

held by

Stefan Schwede

TeXed by

Jan Malmström

WiSe 2024/25

Contents

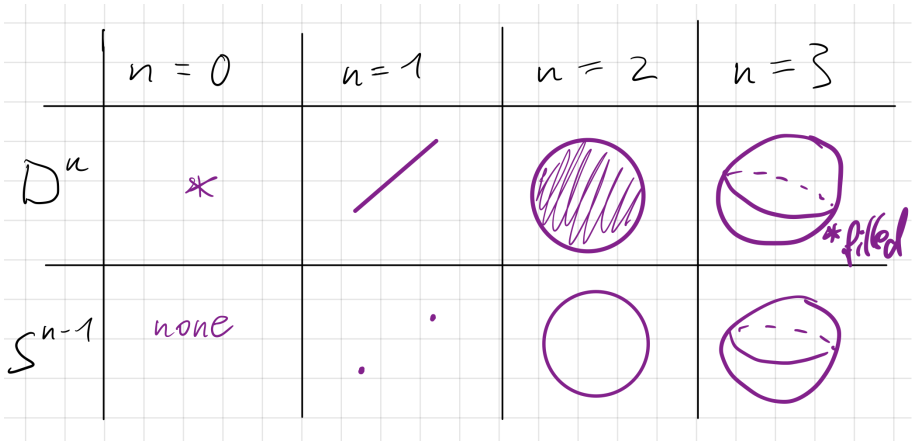

Figure 1: D^n and S^{n-1} for small n

1 CW-Complexes

The name abbreviates compact-Closure-Weak-Topology. They are „nice“ classes of spaces for the purpose of homotopy theory/algebraic topology. They are build by successively attaching cells.

The n-cell is $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. It may also be called n-balls or n-discs. $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : |x| = 1\}$ is the $n-1$-Sphere. See figure 1 for examples.

1.1 Definition

Construction. Let $n \geq 0$, let $f\colon S^{n-1} \to X$ be a continuous map, the *attaching map*. We form the quotient space

$$X \cup_{f,\partial D^n} D^n = X \cup_f D^n = X \cup_{\partial D^n} D^n := X \amalg D^n / \sim$$

where $\sim$ is the equivalencce relation on $X \amalg D^n$ genearted by $\forall\, x \in S^{n-1} : f(x) \sim x$.

Terminology. We say: „$X \cup_f D^n$ is obtained by attaching an n-cell to X along f“.

Example 1.1.

- $X \cup_f D^0 = X \amalg D^0$
- $\{*\} \cup_{S^{n-1}} D^n = D^n/\sim = D^n/S^{n-1} \cong S^n$

  In this example $\sim$ identifies all of S^{n-1} to a point, which then is homeomorphic to S^n[1].
- Remark, that the attaching map matters greatly. See figure 2

$$S^{n-1} \cup_f D^n \cong D^n \quad \text{with } f = \mathrm{Id}\colon S^{n-1} \to S^{n-1}$$

$$S^{n-1} \cup_f D^n \quad \text{with } f\colon S^{n-1} \to S^{n-1} \text{ constant}$$

Simultaneous attachment of several cells

Let J be an indexing[2] set, considered as a discrete space ($J = \emptyset$ is allowed).

[1] supposed as known

[2] „indexing“does not carry mathematical meaning

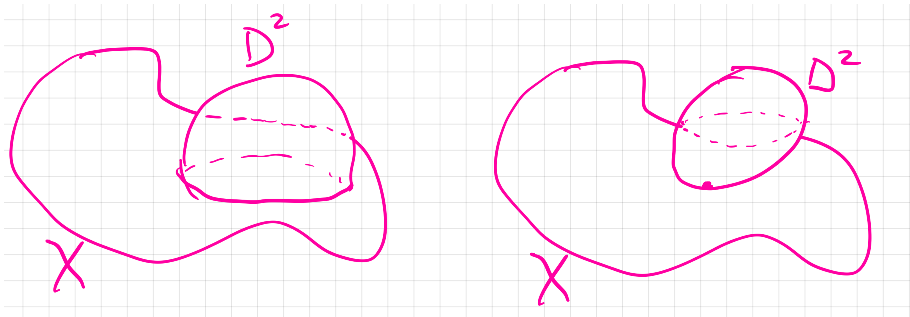

Figure 2: The attaching map influences how D^n is attached.

Give $J \times D^n$ the product topology, then

$$J \times D^n \cong \coprod_{j \in J} \{j\} \times D^{n3}$$

as a topological space. The $\coprod$ represents the disjoint union topology.

It follows, that

$$\begin{array}{ccc} \{\text{continuous maps } f\colon J \times D^n \to X\} & & f \\ \wr\!\| & & \big\downarrow \\ \{\text{J-indexed families of continuous maps } \{f_j\colon D^n \to X\}_{j \in J}\} & & f_j = f(j, \_) \end{array}$$

We will identify them from now on.

Definition 1.2

Let $f\colon J \times \partial D^n \to X$ be a continuous map, the *attaching map*.

$$X \cup_{f, J \times \partial D^n} J \times D^n = X \cup_f J \times D^n = X \cup_{J \times \partial D^n} J \times D^n := X \amalg J \times D^n / \sim$$

where $\sim$ is the equivalence relation generated by $f(x) \sim x$ for all $x \in J \times \partial D^n$.

Remark. Write

$$p\colon X \amalg J \times D^n \to X \cup_f J \times D^n$$

for the quotient map. From the universal property of the qoutient map follows: Given maps $g\colon X \to Y$ and $\Psi_j\colon D^n \to Y$ such that $g(f_j(x)) = \psi_j(x)$ for all $j \in J, x \in \partial D^n$ there is a unique map $\psi\colon X \cup_f J \times D^n \to Y$, such that

$$\psi \circ p = g + \coprod_{j \in J} \psi_j\colon X \amalg (J \times D^n) \to Y$$

and ψ is continuous iff g and all f_j are continuous.

Remeber the quotient-topology: A subset O in $X \cup_f J \times D^n$ is open iff $p^{-1}(O)$ is open in $X \amalg J \times D^n$. This is equivalent to $p^{-1}(O) \cap X$ is open in X and for all $j \in J$ $p^{-1}(O) \cap j \times D^n$ is open in D^n.

X is a closed subspace of $X \cup_f J \times D^n$ $J \times \mathring{D}^n$ is an open subset of $X \cup_f J \times D^n$ $X \cup_f J \times D^n$ is as a set (not as a space) the disjoint union of X and $J \times \mathring{D}^n$. We elaborate

Proposition 1.3. *1. The composition*

$$X \longrightarrow X \amalg (J \times D^n) \xrightarrow{p} X \cup_f J \times D^n$$

is a closed embedding (i.e. a closed injective map).

2. The composition

$$J \times \mathring{D}^n \xhookrightarrow{incl} J \times D^n \longrightarrow X \amalg J \times D^n \xrightarrow{p} X \cup_f J \times D^n$$

is an open embedding (i.e. injective and open)

3. The underlying set of $X \cup_f J \times D^n$ is the disjoint union of the image of X and $J \times \mathring{D}^n$.

Proof. Suppose $M \subseteq X \amalg J \times D^n$ is saturated, i.e. $M = p^{-1}(p(M))$. If M is saturated and open, then $p(M)$ is open in $X \cup_f J \times D^n$.

1. $n = 0$ $X \cup J \times D^0 = X \amalg J \times D^0$ is obvious.

   $n \geq 1$ let $r\colon D^n \to S^{n-1}$ be a map, such that $r(x) = x$ for all $x \in S^{n-1}$.This <u>cannot</u> be done continuously. Define $X \amalg J \times D^n \to X$ by $x \mapsto x, (j, y) \mapsto r(y)$. This is compatible with the equivalence relation, so it descends to a (noncontinuous) map $X \cup_f J \times D^n \to X$. This prooves injectivity. To show this is a closed map, we consider a closed subset $A \subseteq X$. Then $p^{-1}(p(A)) = A \amalg f^{-1}(A) \subseteq X \amalg J \times D^n$ $\subset J \times \partial D^n \subset J \times D^n$ is closed in $X \amalg J \times D^n$. So $p(A)$ is closed in $X \cup_f J \times D^n$.

2. All points in $J \times \mathring{D}^n$ are their own equivalence classes, so the map is injective. To show that the map of 2. is open, we let B be an open subset of $J \times \mathring{D}^n$. This is then also open in $J \times D^n$. $p^{-1}(p(B)) = \emptyset \amalg B \subset X \amalg J \times D^n$ open, so $p(B)$ is open in $X \cup_f J \times D^n$.

3. I think this was prooven with a picture I didn't draw.

$\square$

Exercise. Let V_j be an open subset of D^n for every $j \in J$, such that $V_j \supset \partial D^n$. Show, that the set $V = X \cup \bigcup_{j \in J} V_j$ is open in $X \cup_f J \times D^n$.

From now on we often identify X with its image in $X \cup_f J \times D^n$ and $J \times \mathring{D}^n$ with its image in $X \cup_f J \times D^n$

Definition 1.4: Compactness

A space X is *compact*, if it is Hausdorff (any two points can be separated by two disjoint open sets) and *quasicompact* (any open cover has a finite subcover).

Remark. Some literature defines compactness equivalent to quasicompactness. This lecture uses the definition that was given.

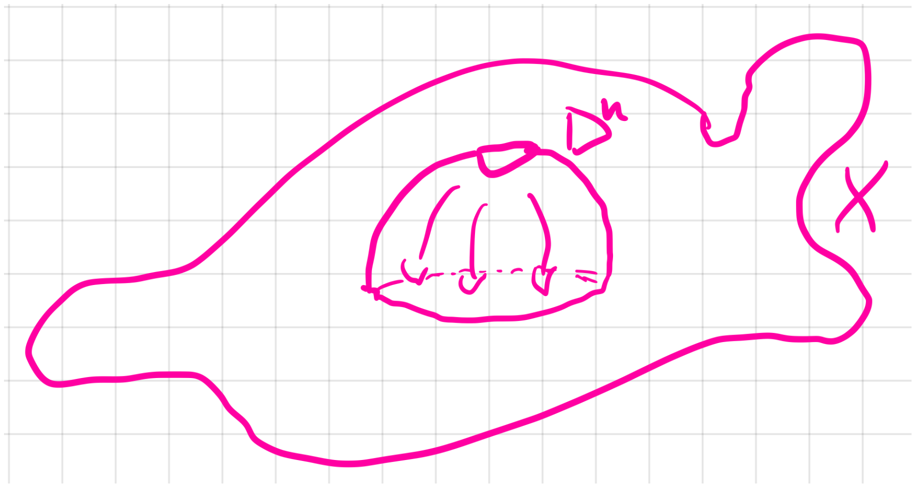

Figure 3: If a point in D^n is missing, it can be continuously retracted.

Theorem 1.5

Let $f\colon J \times \partial D^n \to X$ be a continuous attaching map.

- *If X is Hausdorff, then so is $X \cup_f J \times D^n$.*
- *If X is compact and J is finite, then $X \cup_f J \times D^n$ is compact.*
- *Let K be a quasicompact subset of $X \cup_f J \times D^n$. Then $K \cap (\{j\} \times \mathring{D}^n) = \emptyset$ for almost all[a] $j \in J$.*

[a]mathematical term for all but finitely many.

Lemma 1.6

There exists an open neighborhood V of X in $X \cup_f J \times D^n$ and a continuous map $r\colon V \to X$ that is the identity on X. (X is a neighborhood retract inside $X \cup_f J \times D^n$).

Proof. See figure 3. We take $V = X \cup_{J \times \partial D^n} J \times (D^n \setminus 0)$. This is open in $X \cup_f J \times D^n$. We define $r\colon V \to X$ by $x \mapsto x, (j, z) \mapsto f(j, z/|z|)$. □

Proof of theorem 1.5.

1. **Case 1** $x, y \in J \times \mathring{D}^n$. Since $\mathring{D}^n$ is Hausdorff, so is $J \times \mathring{D}^n$, so we can separate x and y by open disjoint subsets in $J \times \mathring{D}^n$, Since $J \times \mathring{D}^n$ is open in $X \cup_f J \times D^n$, theses subsets are also open in $X \cup_f J \times D^n$.

   Case 2 $x \in X, y \in \{j\} \times \mathring{D}^n$. We choose an $y \in O_y \subset j \times D^n$ open $j \times \partial D^n \subseteq V_j \subseteq j \times D^n$ s.t. $O_j \cap V_j = \emptyset$. Then $V := X \cup V_j \cup \bigcup_{k \in J \setminus \{j\}} D^n$ is open[4] in $X \cup_f J \times D^n$. $V \cap O_j = \emptyset$, $x \in V, y \in O_j$.

   Case 3 $x, y \in X$. Since X is Hausdorff, there are open subsets O_x, O_y of X with $x \in O_x$, $y \in O_y$, $O_x \cap O_y = \emptyset$. We let V be an open subset of $X \cup_f J \times D^n$ with a continuous retraction $r\colon V \to X$, $r|_X = \mathrm{Id}_X$. Then $x \in r^{-1}(O_x), y \in r^{-1}(O_y)$, $r^{-1}(O_y), r^{-1}(O_y)$ are open, and disjoint.

[4]by an exercise.

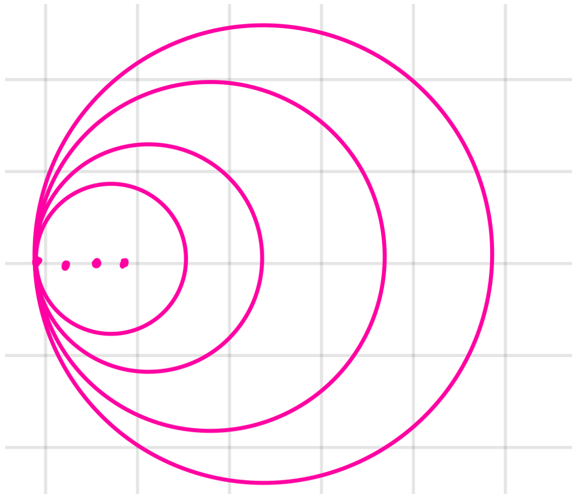

Figure 4: Hawaiian earrings

2. If X is compact and J is finite, then $X \amalg J \times D^n = X \amalg \coprod_{j \in J} \{j\} \times D^n$ is compact hence also the quotient space $X \cup_f J \times D^n$ is quasi-compact. Hausdorff is inherited by 1..
3. Let K be a quasicompact subset of $X \cup_{J \times \mathring{D}^n} J \times D^n$. We define subsets V_j of D^n for all $j \in J$ as follows: If $K \cap (j \times \mathring{D}^n) = \emptyset$, we set $V_j = D^n$. If $K \cap (j \times \mathring{D}^n) \neq \emptyset$, we choose a V_j, that doen't contain at least one point of K, is open, and contains ∂D^n. Now
$$(X \bigcup_{j \in J} V_j) \cup \bigcup_{j \in J} \{j\} \times \mathring{D}^n$$
is an open cover of $X \cup_f J \times D^n$. Since K is quasicompact, there is a finite subset L of J such that
$$K \subset (X \cup_{j \in J} V_j) \cup \bigcup_{j \in L} \{j\} \times \mathring{D}^n.$$

□

Example 1.7 (Hawaiian Earrings)**.** The set
$$H = H_1 \cup H_2 \cup H_3 \cup \dots = \bigcup_{i \geq 1} H_i$$
wherein H_i is the circle in $\mathbb{R}^2$ with radius $1/i$ and center $(1/i, 0)$, equipped with the subspace topology of $\mathbb{R}^2$ is called the Hawaiin earrings (see figure 4).

Is H obtained from $\{(0,0)\}$ by attaching countably many 1-cells? It is not.

Consider a continuous map $\psi_j \colon D^l = [-1,1]$ such that it is a surjective, and $[-1,1]/{-1} \sim 1$ onto $H_j \subset H$ is a homeomorphism.
$$\{(0,0)\} \amalg \mathbb{N} \times D^1 \to H, \quad (j,x) \mapsto \psi_j(x)$$
is a continuous surjection. Then
$$\{(0,0)\} \cup_{\mathbb{N} \times \partial D^1} \mathbb{N} \times D^1 \to H$$
is a continuous bijection. However, it is not a homeomorphism.

Consider $V = \{(0,0)\} \cup_{\mathbb{N}\times\partial D^1} \mathbb{N} \times ([-1,0) \cup (0,1])$. This is open in $\{(0,0)\} \cup_{\mathbb{N}\times\partial D^1} \mathbb{N} \times D^1$. Its complement is closed, but the image of that complement, $(1/n, 0)_{n\in\mathbb{N}}$ is not closed in H.

Definition 1.8: CW-Complex

A relative *CW-complex* is a space X equipped with a sequence of closed subspaces

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \ldots$$

such that

1. For every $n \geq 0$ X_n can be obtained from X_{n-1} by attaching n-cells.
2. $X = \bigcup_{n\geq 0} X_n$ and X has the weak topology with respect to the sequences.

precisely:

1. There exists an index set J, a continuous map $f\colon J \times \partial D^n \to X_{n-1}$ and a homeomorphism $\psi\colon X_{n-1} \cup_f J \times D^n \to X_n$ that is the identity on X_{n-1}.
2. A subset O of X is open in X iff $O \cap X_n$ is open in X_n for all $n \geq 0$.

Remark. 2. is equivalent to: a subset C of X is closed in X iff $C \cap X_n$ is closed in X_n for all $n \geq 0$.

2. implies, that a map $f\colon X \to Y$ is already continuous if $f|_{X_n}\colon X_n \to Y$ is continuous for all $n \geq 0$.

Notation.

- We usually say (X, A) is a relative CW-complex and leave the X_n implicit.
- For $A = \emptyset$, X is called a absolute CW-complex, or just a CW-complex.
- The subspace X_n in a CW-complex is the n-skeleton.
- A relative CW-complex (X, A) is finite-dimensional if $X_n = X$ for some $n \geq 0$.
- A relative CW-complex (X, A) is finite, if there are only finitely many cells altogether.
- Once chosen a homeomorphism ψ as above, then the characteristic map of the j-th n-cell χ_j is the composite

  $$D^n \xrightarrow{(j,\_)} X_{n-1} \cup_{J\times\partial D^n} J \times D^n \xrightarrow[\cong]{\psi} X_n \hookrightarrow X$$

  $\chi_j|_{\mathring{D}^n}\colon \mathring{D}^n \to \chi_j(\mathring{D}^n)$ is a homeomorphism onto its image, which is one path component of $X_n \setminus X_{n-1}$. The restriction

  $$f_j \coloneqq \chi_j|_{\partial D^n}\colon \partial D^n \to X_{n-1}$$

  is called the attaching map as before.

Comment. The space $X_n \setminus X_{n-1}$ is a disjoint union of open cells $\mathring{D}^n$. So the indexing set could be taken as $\pi_0(X_n \setminus X_{n-1})$. Esspecially its cardinality is fixed.

It can be shown, that for every path-component of $X_n \setminus X_{n-1}$ there exists a homeomorphism

$$f\colon \mathring{D}^n \to \text{ that path-component}$$

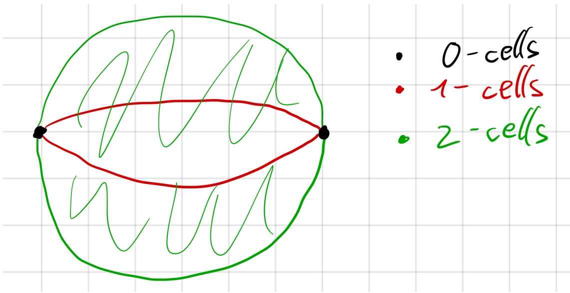

Figure 5: S^2 is built from several cells.

that extends to a continuous map $\bar{f}\colon D^n \to X_n$.

Example 1.9.
- Any discrete space is an absolute 0-dimensional CW-complex.
- Let $z \in S^n$ be any point. Then the minimal CW-structure on S^n is

$$X_{-1} = \emptyset, \quad X_0 = \{z\} = X_1 = \dots = X_{n-1}$$

$$S^n = X_n = X_{n+1} = \dots$$

It consists of one 0-cell and one n-cell. This can be seen, because $S^n \cong D^n/\partial D^{n-1}$ by $\partial D^{n-1} \to \{z\}$.

The CW-strucuture on a given space X is not unique. For example a different CW-structure on S^2 consists of two of each $0, 1$ and 2-cells. See figure 5 for the construction. Analog, S^n is a CW-complex with 2 i-cells for $i = 0, \dots, n$.

Also a CW-structure: For S^1 pick any finite subset $A \subseteq S^1$. Then S^1 has a CW-structure with $X_{-1} = \emptyset, X_1 = A, X_2 = S^1$. n 0 cells n 1 cells.

It can be shown, that any non-discrete space, that admits an absolute CW-structure admits uncountably many different CW-structures.

Preview. The Euler characteristic of a finite absolute CW-complex is

$$\chi(X) = \sum_{n \geq 0} (-1)^n \#n\text{-cells}$$

does not depend on the CW-structure. We will eventually show this using singular homology.

Theorem 1.10

Let (X, A) be a relative CW-complex.

1. *If A is Hausdorff, then so is X.*
2. *If A is compact and (X, A) is finite, then X is also compact.*

Proof. Because $X_{-1} = A$ is Hausdorff and X_n can be obtained from X_{n-1}, by attaching cells, inductively X_n is Hausdorff for all $n \geq 0$.

Claim. Let O_n, P_n be open disjoint subsets of X_n. Then there exist disjoint open subsets O_{n+1}, P_{n+1} of X_{n+1}, such that $O_n = O_{n+1} \cap X_n, P_n = P_{n+1} \cap X_n$.

Proof. Since X_{n+1} can be obtained from X_n by attaching $(n+1)$-cells X_n is a neighborhood retract in X_{n+1}, i.e. there is a open neighborhood V of X_n in X_{n+1} and a continuous retraction $r\colon V \to X_n$ with $r|_{X_n} = \mathrm{Id}$. We set $O_{n+1} = r^{-1}(O_n), P_{n+1} = r^{-1}(P_n)$. □

We proove the Hausdorff property: Let $x, y \in X$ be disjoint points. Since $X = \bigcup_{n\in\mathbb{N}} X_n$. Then for some $n \geq 0$, $x, y \in X_n$. Since X_n is Hausdorff, there are open, disjoint subsets O_n, P_n of X_n with $x \in O_n, y \in P_n$. Inductiveley use the claim to find open disjoint subsets O_m, P_m of X_m for all $m \geq n$, such that $O_{m+1} \cap X_m = O_m, P_{m+1} \cap X_m = O_m$ for all $m \geq n$. Then set $O = \bigcup_{m\geq n} O_m, P = \bigcup_{m\geq n} PM$ disjoint subsets of X and open in X by the weak topology, as $O \cap X_m = O_m$ open in X_m.

For compactness, Induction over n, such that X_n is compact because X_n is obtained from X_{n-1} by attaching finitely many cells. Also $X = X_n$ for sufficently large n. So X is compact. □

Note. Suppose that X admits a CW-structure. Then the following are equivalent: X admits a finite CW-structure $\Leftrightarrow$ X is compact.

From now on we assume, the base A in a relative CW-complex X, A is Hausdorff. Then X is also Hausdorff.

Theorem 1.11

Let X, A be a relative CW-complex.

1. *The closure of every open n-cell (= path component of $X_n \setminus X_{n-1}$) is compact.*
2. *Let $\chi\colon D^n \to X$ be a characteristic map for some n-cell, then the image $\chi(D^n)$ is the closure of the open cell $\chi(\mathring{D}^n)$*
3. *Let U be a subset of X s.t. $A \subseteq U$. Suppose that the intersection of U with the closure of every cell is closed. Then U is closed in X.*

Warning. The closure of a cell is not necessary a closed cell. See for example the minimal CW-tructure on S^2. The closure of the open 2-cell $S^2 \setminus \{z\}$ is $S^2 \neq D^2$.

Proof.

1. By definition every open n-cells admits a characteristic map $\chi\colon D^n \to X_n$ continuous s.t. $\chi|_{\mathring{D}^n}$ is a homeomorphis onto the open cell. Then

   $$\chi(D^n) \subseteq \text{ closure of } \chi(\mathring{D}^n)$$

   and as D^n is compact, and X is Hausdorff, $chi(D^n)$ is closed, so $\chi(D^n) = $ closure of $\chi(\mathring{D}^n)$. As D^n is compact, this is also.
2. Already contained in 1.
3. Let $U \subseteq X$ be as in 2. It suffices to show that $U \cap X_n$ is closed in X_n for all $n \geq 0$ (weak topology). We argue by induction on n.

   $n = -1$ $U \cap X_{-1} = U \cap A = A$ closed in $A = X_{-1}$.

   $n \geq 0$ We choose a homeomorphism $\psi\colon X_n \cong X_{n-1} \cup_{J\times\partial D^n} J \times D^n$ that is the identity on X_{n-1}. We let

   $$p\colon X_{n-1} \amalg J \times D^n \to X_{n-1} \cup_{J\times\partial D^n} J \times D^n \cong X_n$$

be the quotient map. Then

$$p^{-1}(U \cap X_n) = \underbrace{(U \cap X_{n-1})}_{\text{closed by induction}} \amalg \coprod_{j \in J} p^{-1} \underbrace{(U \cap \text{ closure of j-th n-cell})}_{\text{closed by hypothesis}}$$

This is closed as a subspace of $X_{n-1} \amalg J \times D^n$ and hence $U \cap X_n$ is closed in X_n

□

1.2 CW-subcomplexes

Proposition 1.12. *Let A be a Hausdorff-space, $X = A \cup_f J \times D^n$ obtained from A by attaching n-cells. Let $Y \subseteq X$ be a subspace, such that*

- *$Y \cap A$ is closed in A*
- *Y can be obtained from $A \cap Y$ by attaching n-cells.*
- *$Y \cap (J \times \mathring{D}^n)$ is a union of path components of $J \times \mathring{D}^n$.*

Then Y is closed in X.

Proof. **Claim.** If $Y \cap \{j\} \times \mathring{D}^n \neq \emptyset$ ($\Leftrightarrow j \times \mathring{D}^n \subseteq Y$). Then Y contains the closure of $j \times \mathring{D}^n$ in X. ($=$ the closure of this cell).

Proof. Y can be obtained from $Y \cap A$ by attaching n-cells and $Y \setminus (Y \cap A)$ is a union of some of the open cells of $X \setminus A = J \times \mathring{D}^n$. Let $\chi\colon D^n \to Y$ be a characteristic map for the attaching of the j-th n-cell to Y. $\chi(\mathring{D}^n) = j \times \mathring{D}^n$. Since D^n is compact, $f(D^n)$ is quasicompact, and hence closed in X since X is Hausdorff. Then

$$\underset{\chi(\mathring{D}^n)}{j \times \mathring{D}^n = \text{closed in}}\ X \subseteq \chi(D^n) \subseteq Y \subseteq X$$

and the closure of $\chi\mathring{D}^n = j \times \mathring{D}^n$ is in $\chi(D^n)$ and hence in Y. □

We let

$$p\colon A \amalg J \times D^n \to A \cup_f J \times D^n \cong X$$

be the quotient map. Then

$$p^{-1}(Y) = (Y \cap A) \amalg \coprod_{\substack{j \in J \\ Y \cap (j \times \mathring{D}^n) \neq \emptyset}} j \times D^n \amalg \coprod_{\substack{j \in J \\ Y \cap (j \times \mathring{D}^n) = \emptyset}} p^{-1}(Y \cap A) \cap (j \times D^n)$$

So Y is closed in X. □

Theorem 1.13

Let (X, A) be a relative CW-complex and Y a closed subspace of X with $A \subseteq Y$. Suppose that for all $n \geq 0$, $Y \cap X_n \setminus X_{n-1}$ is a disjoint union of path components of $X_n \setminus X_{n-1}$. Then (Y, A) is a relative CW-complex with respect to the induced filtration, i.e.

$$A = Y_{-1} \subseteq Y_0 = (X_0 \cap Y) \subseteq Y_1 = X_1 \cap Y \subseteq \dots \subseteq Y_n = X_n \cap Y \subseteq \dots$$

Proof.

1. Y_n can be obtained from Y_{n-1} by attaching n-cells. Let
$$I = \{j \in J \colon Y \cap (j \times \mathring{D}^n) \neq \emptyset\} = \{j \in J : j \times \mathring{D}^n \subseteq Y\}.$$
Let $\chi_j \colon D^n \to X_n \subseteq X$ be a charactreistic map for the j-th n-cell of X. If $j \in I$, then
$$\chi(D^n) = \text{closure of } \chi(\mathring{D}^n)$$
and since Y is closed, this is a closed subspace of Y. So we can (and will) consider χ as a map with target $Y \cap X_n = Y_n$. We get a continuous map
$$\psi \colon Y_{n-1} \cup_{I \times \partial D^n} I \times D^n \to Y_n$$
(induced by $\coprod_{j \in I} \chi_j$), which is bijective because source and target are - as sets - both the disjoint union of Y_{n-1} and $I \times \mathring{D}^n$. We argue, that ψ is a closed map and hence a homeomorphism. See
$$\begin{tikzcd}
Y_{n-1} \amalg I \times D^n \arrow[r, hook] \arrow[d, "q"] & X_{n-1} \amalg J \times D^n \arrow[d, "p"] \\
Y_{n-1} \cup_{I \times \partial D^n} I \times D^n \arrow[r, "\psi"] & Y_n \subseteq X_n
\end{tikzcd}$$
Let $B \subseteq Y_{n-1} \cup_{I \times \partial D^n} I \times D^n$ be a closed subset, where $f_j \colon \partial D^n \to X_{n-1}$ is the attaching map for the j-th n-cell i.e. $f_j = \chi_j|_{\partial D^n}$. Then
$$p^{-1}(\psi(B)) = X_n \underset{q^{-1}(B)}{\amalg I \times D^n} \amalg \coprod_{j \in J \setminus I} j \times f_j^{-1}(B \cap X_{n-1})$$
With $f_j = \chi_j|_{\partial D^n} \colon \partial D^n \to X_{n-1}$. As all these are closed, $p^{-1}(\psi(B))$ is closed. Hence $\psi(B)$ is closed in X_n and also in Y_n.

2. Y has the weak topology with respect to
$$Y = Y \cap X = Y \cap (\bigcup_{n \geq 0} X_n) = \bigcup_{n \geq 0} (Y \cap X_n) = \bigcup_{n \geq 0} Y_n.$$
Let $B \subseteq Y$ be a subset such that for all $n \geq 0$, $B \cap Y_n$ is closed in Y_n. Since Y is closed in X, Y_n is closed in X_n, so $B \cap Y_n$ is closed in X_n. Since X has the weak topology, B is closed in X, hence also in Y.

□

Definition 1.14

A CW-subcomplex of a relative CW-complex (X, A) is a closed subspace Y of X, such that $A \subseteq Y$ and for all $n \geq 0$ $Y \cap (X_n \setminus X_{n-1})$ is a union of path components of $X_n \setminus X_{n-1}$.

Note. Let (Y, A) be a CW-subcomplex of (X, A). Then (Y, A) is a relative CW-complex with respect to the induced filtration.

Theorem 1.15

Let (X, A) be a relative CW-complex.

1. *The closure of every cell is contained in a finite subcomplex.*
2. *Every compact subset of X is contained in a finite subcomplex of X.*

Remark. The Historically first definition of CW-complexes (J.H.C. Whitehead). A CW-complex is a space X equipped with a decomposition $X = \dot{\bigcup}_{n\geq 0, i\in J_n} e_i^n$, such that

1. e_i^n is homeomorphic do $\mathring{D}^n$.
2. The closure of e_i^n is contained in the union of finitely many e_j^m-s („closure finite“).
3. a subset Y of X is closed iff $Y \cap \overline{e_i^n}$ is closed for all e_i^n. then called weak topology.[5]

Proof. Since the closure of every cell is compact, 1 is a special case of 2.

Let K be a compact subset of X. **Claim** There is an $n \geq 0$, such that $K \subseteq X_n$.

Proof by contradiction. If $K \not\subseteq X_n$ for all $n \geq 0$. Then we can choose points in K $x_1, x_2, x_3, \dots \in K$, such that $x_i \in X_{n_i} \setminus X_{n_i - 1}$ for some $n_1 < n_2 < n_3 < \dots$. Set $D \coloneqq \{x_1, x_2, x_3, \dots\}$.

Subclaim. Every subset of D is closed in X. Let $S \subseteq D$ be any subset. Thus for all $n \geq 0$ $S \cap X_n$ is finite, hence closed in X (Hausdorff). In particular, D is Closed in X and contained in K hence compact. But D has discrete topology and D is infinite. Contradiction. ☐

Now we assume that the compact subset K is contained in X_n. We argue by induction over n.

$n = -1$ If K is contained in A, then A, A is a finite CW complex.

$n \geq 0$ We choose a representatoin $X_n \cong X_{n-1} \cup_{J\times\partial D^n} J \times D^n$ We showed earlier, that K only meets finitely many of the n-cells in the interior. Set

$$I = \{j \in J : K \cap (j \times \mathring{D}^n) \neq 0\}$$

a finite subset of J. Set

$$L \coloneqq K \cup \bigcup_{j\in I} \underbrace{(\text{ closure of } j\text{-th } n\text{-cell})}_{\text{compact}}$$

Note that L is compact. Since X_{n-1} is closed in X, $L \cap X_{n-1}$ is closed in X_{n-1}, and hence compact. So by induction, $L \cap X_{n-1}$ is contained in some finite CW-subcomplex of (X_{n-1}, A). Then K is contained in $Y \cup_{I\times\partial D^n} I \times D^n$, another finite subcomplex of (X, A).

☐

1.3 Cellular approximation theorem

We will formulate the cellular approximation theorem and spend some time to prove it.

[5] The equivalence of this definition to ours will be shown later.

Definition 1.16

Let (X, A) and (Y, B) be relative CW-complexes. Let $f\colon X \to Y$ be a continous map, such that $f(A) \subseteq B$. The map f is *cellular* if $f(X_n) \subseteq Y_n$ for all $n \geq 0$.

Theorem 1.17: Cellular approximation

Let (X, A), (Y, B) be relative CW-complexes, and $f\colon X \to Y$ continuous with $f(A) \subseteq B$. Then f is homotopic, relative A, to a cellular map.

Reminder. „relatively homotopic" means, there is a homotopy $H\colon X \times [0,1] \to Y$, such that $f = H(\_, 0)\colon X \to Y$, $H(\_, 1\colon X \to Y)$ is cellular, $H(a,t) = f(a)$ for all $a \in A, t \in [0,1]$.

Example 1.18. Consider a minimal CW-structure on S^n, i.e. one 0-cell and one n-cell. $A = X_{-1} = \{z\} = X_0 = \cdots = X_{n-1} \subseteq X_n = S^n$. Suppose that $m < n$, give S^m a minimal CW-structure. Let $f\colon S^m \to S^n$ be continuous. Take $z := f(x)$

CAT gives f is homotpoic to a constant map!

We can say $\pi_m(S^n, z) = \{0\}$ for $m \leq n$

Proof of CAT. We start by prooving a special case:

Theorem 1.19

Let $Y = B \cup_{\partial D^n} D^n$. Then for all $m < n$, every continous map $f\colon D^m \to Y$ with $f(\partial D^m) \subseteq B$,is homotopic relative ∂D^m to a map with image in B.

Proof. By induction on n.

For $n = 1$, $m = 0$, $D^0 = \{x\}$, $\partial D^0 = \emptyset$.

$$f\colon \{x\} \to B \cup_{\partial D^1} D^1$$

is homotpoic to a map with image in B because D^1 is path connected.

Now let $n \geq 2$ and assume the special case for all smaller values of n.

Fact 1 For all $p < n - 1$, every continuous map $S^p \to S^{n-1}$ is homotopic to a constant map.

Proof. By the inductive hypothesis, the composite

$$D^p \to D^p/S^{p-1} \cong S^p \xrightarrow{f} S^{n-1} \cong \{z\} \cup_{\partial D^{n-1}} D^{n-1}$$

with $z := f(\partial D^p)$ is homotopic, relative ∂D^p, to a constant map with value $\{z\}$. Let $H\colon D^p \times [0,1] \to S^{n-1}$ be such a homotopy. This descends to a map

$$\begin{tikzcd}
D^p \times [0,1] \arrow[r, "H"] \arrow[d, "p"] & S^{n-1} \\
D^p/\partial D^p \times [0,1] \cong S^p \times [0,1] \arrow[ur] &
\end{tikzcd}$$

which is again continuous. ∎

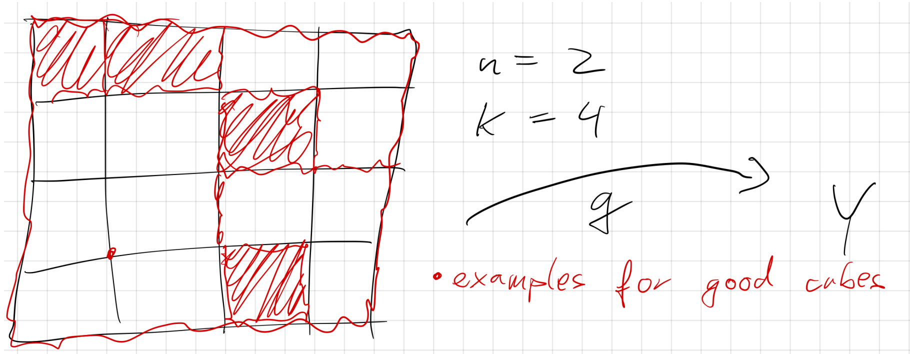

Figure 6: examples for good/bad cubes.

Fact 2 For $p < n-1$, every continuous map

$$h = (h_1, h_2) \colon S^p \to S^{n-1} \times (a,b)$$

with $a < b \in \mathbb{R}$. is homotopic to a constant map.

Proof. Let $H_1 \colon S^p \times [0,1] \to S^{n-1}$ be a homotopy of h_1 to a constant map (Fact 1). Let $H_2 \colon S^p \times [0,1] \to (a,b)$ be a linear homotopy from h_2 to some constant map. Then $H = (H_1, H_2) \colon S^p \times [0,1] \to S^{n-1} \times (a,b)$ is the desired homotopy. □

Fact 3 For $q < n$, every continuous map $h \colon \partial D^q \to S^{n-1} \times (a,b)$ admits a continuous extension to D^q.

Proof. The map $\partial D^q \times [0,1] \to D^q$, $(x,t) \mapsto x \cdot t$ is a quotient map. Let $p = q-1$. $\partial D^q = S^p$, we let $H \colon \partial D^q \to S^{n-1} \times (a,b)$ be a homotopy from a constant map as in Fact 2.

$$\begin{tikzcd}
\partial D^q \times [0,1] \arrow[r, "H"] \arrow[d, "(x,t)\mapsto x\cdot t"'] & S^{n-1} \times (a,b) \\
D^q \arrow[ur, "\overline{H}"] &
\end{tikzcd}$$

So there is a continuous map $\overline{H} \colon D^q \to S^{n-1} \times (a,b)$ with the desired property. □

Inductive Step. Let $m < n$ and $f \colon D^m \to Y = B \cup_{\partial D^n} D^n$, such that $f(\partial D^m) \subseteq B$. We define two open subsets of Y.

$$U = \{x \in D^n : |x| < 2/3\}, \quad V = B \cup_{\partial D^n} \{x \in D^n : |x| > 1/3\}$$

Note that $U \cap V \cong \partial D^n \times (1/3, 2/3)$. Fact 3 gives: Every continuous map $\partial D^q \to U \cap V$ admits a continuous extension to D^q for $q < n$.

We replace the pair $(D^m, \partial D^m)$ by the homeomorphic pair $[0,1]^m, \partial([0,1]^m)$. Let

$$g \colon [0,1]^m \to B \cup_{\partial D^n} D^n = U \cup V, \text{ such that } g(\partial([0,1]^m)) \subseteq B$$

Then $g^{-1}(U), g^{-1}(V)$ is an open cover of the compact metric space $[0,1]^m$, so by Lebeques Lemma there is an $\varepsilon > 0$, such that every ε-ball in $[0,1]^m$ is contained in $g^{-1}(U)$ or in $g^{-1}(V)$. So we can subdivide $[0,1]^m$ into sufficiently small equally sized and equally spaced subcubes, such that each subcube maps wholly U or to V by g. We need to consider all vertices, edges, squares, …, $(m-1)$- cubes and m-cubes. Let W be any such p-cube. We call W

Good if $g(W) \subseteq V$.

Bad if $g(W) \not\subseteq V$

Note, that

- if W is bad, then $g(W) \subseteq U$.
- every face of a good cube is good.
- every cube contained in $\partial([0,1]^m)$ is good.

See figure 6 for an example.

Let Γ be the union of all good cubes of all dimension. $\Gamma \subseteq [0,1]^m$. We define

$$\begin{aligned} K^{-1} &= \Gamma = \text{ all good cubes} \\ K^0 &= K^{-1} \cup \text{ bad 0-cubes} \\ K^1 &= K^0 \cup \text{ bad 1-cubes} \\ &\vdots \\ K^m &= [0,1]^m \end{aligned}$$

By induction on p we will construct continuous maps

$$g_p \colon K^p \to Y = B \cup_{\partial D^n} D^n = U \cup V.$$

such that:

- $g_p|_{K^{p-1}} = g_{p-1}$
- if W is a bad cube, then $g_p(W) \subseteq U \cap V$.

Start: $g_{-1} = g|_\Gamma \colon \Gamma = K^{-1} \to Y$.

Suppose, that $g_{-1}, g_0, \dots, g_{p-1}$ have already been constructed.

Claim. If W is a bad p-cube, then $g_{p-1} \subseteq U \cap V$.

Proof. Let W' be a q-cube in ∂W, so $q < p$. If W' is good, then

$$g_{p-1}(W') = g(W') \subseteq V$$

But also

$$g_{p-1}(W') = g(W') \subseteq g(W) \subseteq U$$

.

If W' is bad, then $g_{p-1}(W') \subseteq U \cap W$ by induction hypothesis. □

Fact 3 implies, that $g_{p-1}|_{\partial W} \colon \partial W \to U \cap V \cong \partial D^n \times (1/3, 2/3)$ admits a continuous extension to W. We choose such a continuous extension for every bad p-cube and then define

$$g_p \colon K^p = K^{p-1} \cup \text{ bad } p\text{-cubes} \to \text{Y} \quad \text{as } g_{p-1} \cup \text{ chosen extensions.}$$

This completes the inductive construction of the maps $g_p \colon K^p \to Y$.

Claim. g_m and g are homotopic relative $\partial[0,1]^m$.

Proof. We show that g and g_m are even homotopic relative to $\Gamma = K^{-1} \supset \partial([0,1]^m)$.

We write C for the union of all bad cubes. Then $[0,1]^m = C \cup \Gamma$. Then $g(C) \subseteq U$ and $g_m(C) \subseteq U \cap V \subseteq U$. So we can consider the restrictions of both g and g_m to C as continuous maps

$$g_m|_C, g|_C \colon C \to U \cong R^n$$

We can use the linear homotopy between g_m and g. This linear homotopy has the additional property, that it is constant on all points, where g and g_m agree. In particular, the homotopy is constant on $C \cap \Gamma$. So the lineare homotopy on C and the constant homotopy on Γ, patch together to a homotopy between g_m and g, that is moreover constant on Γ, hence also constant on $\partial([0,1]^m)$. □

End of the inductive step: We have constructed a homotopy relative to $\partial([0,1]^m)$ from g to g_m, which has image in V. V deformation retracts onto B. Following g_m with such a deformation retraction, is a relative homotopy from g_m to a map with image in B. □

Theorem 1.20

Let (Y,B) be a relative CW-complex, and let $f\colon D^m \to Y$ be a continuous map, such that $f(\partial D^m) \subseteq B$. Then f is homotopic, relative ∂D^m to a map with image in Y_m.

Proof. **Special case.** (Y, Y_m) is a finite relative CW-complex. We argue by induction on the number of relative cells of (Y, Y_m).

Start: $Y = Y_m$ is trivial.

Otherwise, choose a cell of Y of top dimension n. Then $m < n$. We choose

$$Y' = B \cup \text{ all cells of } Y \text{ except for the chosen } n\text{-cell}$$

Then (Y', B) is a relative CW-complex. Hence (Y', Y_m) is a relatively finite CW-complex with one cell less than (Y, Y_m). $Y = Y' \cup_{\partial D^n} D^n$. By the previous theorem applied to (Y, Y'), the map f is homotopic relative ∂D^m to a map $g'\colon D^m \to Y$ with image in Y'. By induction g' is homotopic relative ∂D^m to a map $g''\colon D^m \to Y'$ with image in Y_m. g'' is the desired map.

General case $f(D^m)$ is a compact subset of Y, and hence contained in some finite subcomplex $(\bar{Y}, B)$ of (Y, B). Apply the special case to f, considered as a map into $\bar{Y}$. □

Theorem 1.21

Let X be obtained from A by attaching (arbitrarily many) n-cells. Let (Y,B) be a relative CW-complex. Let $f\colon X \to Y$ be a continuous map with $f(A) \subseteq B$. Then f is homotopic, relative A to a map with image in Y_m.

Proof. We may assume $X = A \cup_{J \times \partial D^m} J \times D^m$ for some attaching map $J \times \partial D^m \to A$. For $j \in J$ we define $f_j\colon D^m \to Y$ as the composite

$$\begin{aligned} D^m &\to X = A \cup_{J \times \partial D^m} J \times D^m \xrightarrow{f} Y \\ x &\mapsto (j, x) \end{aligned}$$

This satisfies $f_j(\partial D^m) \subseteq f(A) \subseteq B$. The previous special case provides a homotopy $H_j\colon D^m \times [0,1] \to Y$ relative ∂D^m, from f_j to a map with image in Y_m. We „glue“ the homotopies and the constant homotopy on A to a homotopy on X, i.e.

$$\begin{tikzcd}
A \times [0,1] \amalg J \times D^n \times [0,1] \arrow[r, "(\mathrm{const}\to f|_A)\amalg\coprod_{j\in J} H_j"] \arrow[d, "p\times[0,1]"] & Y \\
X \times [0,1] = (A \cup_{J\times\partial D^m} J \times D^m) \times [0,1] \arrow[ur, "\bar{H}"'] &
\end{tikzcd}$$

where $p\colon A \amalg J \times D^n \to X$ is the quotient map. $\bar{H}$ is continuous by the quotient property of $p \times [0,1]$. $\bar{H}$ is the desired homotopy. That $p \times [0,1]$ is a quotient map will be shown later. $\square$

Definition 1.22: A

ontinuous map $f\colon X \to Y$ is a *quotient map* if it is surjective and $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open

Equivalently: the induced map $X/\sim_f \xrightarrow{\cong} Y$ is a homeomorphism, where $x \sim_f x' \Leftrightarrow f(x) = f(x')$.

In general, if $f\colon X \to Y$ is a quotient map, then $f \times Z\colon X \times Z \to Y \times Z$ is continuous and surjective, but not neccessarily a quotient map!

The next steps will be

- If Z is locally compact, then $\times Z$ preserves quoteint maps.
- Suppose $f\colon X \to Y$ is cellular up to level $m-1$, i.e. $f(X_k) \subseteq Y_k$ for $k = -1, 0, 1, \dots, X_{m-1}$, then apply the previous special case to $f|_{X_m}\colon (X_m, X_{m-1}) \to (Y, Y_{m-1})$ makes $f|_{X_m}$ homotopic to a cellular map.
- Looking at the *Homotopy Extension property*, which some spaces have, allowing to extend a homotopy from a subspace of it to the whole space.
- A limit argument to finish the proof.

Definition 1.23

A space X is *locally compact*, if every neighborhood of any point of X contains a compact neighborhood of that point.

Lemma 1.24

Let X be a space, such that every point has a compact neighborhood. Then X is locally compact. In particular, compact spaces are locally compact.

Example 1.25. $\mathbb{R}^n$ is locally compact, but not compact.

Proof. Let U be a neighborhood of $x \in X$ in X. Then there is a open set U' of X with $x \in U' \subseteq U \subseteq X$. Let K be a compact neighborhood of x in X. Then $K \setminus U'$ and $\{x\}$ are disjoint closed subsets of the compact space K. Compact spaces are normal, so there are relatively open subsetes W_1 and W_2 of K, such that $x \in W_1 \subseteq K$ and $K \setminus U' \subseteq W_2 \subseteq K$ and $W_1 \cap W_2 = \emptyset$.

Then $K \setminus W_2$ is closed in K an hence compact. Since W_1 is a neighborhood of x in K and K is a neighborhood of $x \in X$, W_1 is a neighborhood of x in X. So

$$x \in W_1 \subseteq K \setminus W_2 \subseteq U \subseteq X.$$

□

Lemma 1.26: Slice lemma

Let X and Y be spaces and K a compact subset of Y. Let $x \in X$ and let W be an open subset of $X \times Y$, such that $\{x\} \times K \subseteq W$.Then there is an open subset V of X, such that $x \in V$ and $V \times K \subseteq W$.

This was prooven in GeoTopo.

Theorem 1.27

Let $f\colon X \to Y$ be a quotient map. Then for every locally compact space Z, the map

$$f \times Z\colon X \times Z \to Y \times Z$$

is a quotient map.

Proof. $f \times Z$ is continuous and surjective. We must show: Let $B \subseteq Y \times Z$ such that $f^{-1}(B)$ is open in $X \times Z$, then B is open in $Y \times Z$.

We consider any point $(y, z) \in B$. We choose some $x \in X$, such that $f(x) = y$. Then $(x, z) \in f^{-1}(B)$. We define

$$\begin{aligned} A &:= \{\bar{z} \in Z : (y, \bar{z}) \in B\} = \{\bar{z} \in Z : (x, \bar{z}) \in f^{-1}(B)\} \\ &= \text{preimage of } B \text{ under the continuous map } Z \xrightarrow{(y,\_)} Y \times Z \end{aligned}$$

A is open in Z. Since Z is locally compact, there is a compact neighborhood K of z inside A.

$$z \in K \subseteq A \subseteq Z$$

In particular, $\{y\} \times K \subseteq B$. We define $U := \{\bar{y} \in Y : \{\bar{y} \times K\} \subseteq B\}$. Then $y \in U$.

Claim U is open in Y.

Proof. Since $f\colon X \to Y$ is a quotient map, it suffices to show that

$$f^{-1}(U) = \{\bar{x} \in X : \{\bar{x}\} \times K \subseteq (f \times Z)^{-1}(B)\}$$

is open in X.

Since $\bar{x} \in f^{-1}(U)$ there is an open subset V of $\bar{x}$ in X with $V \times K \subseteq (f \times Z)^{-1}(U)$ (Slice Lemma!). Hence $\bar{x} \in V \subseteq f^{-1}(U)$ so $f^{-1}(U)$ is open in X, hence U is open in Y. □

Consider: Given $(y, z) \in B$ we found $(y, z) \in U \times K \subseteq B$ with U open and K a neighborhood of z. So B is indeed open. □

Corollary 1.28. *Let $X = A \cup_{J \times D^n} J \times D^n$ be obtained from A by attaching n-cells. Then for every locally compact space Z, the map $(A \times Z) \amalg (J \times D^n \times Z) \to (A \cup_{J \times \partial D^n} J \times D^n) \times Z = X \times Z$ is a quotient map.*

Proof. The map f is the composite

$$A \times Z) \amalg (J \times D^n \times Z) \cong (A \amalg J \times D^n) \times Z \to X \times Z$$

Products commutes with disjoint unions. □

Corollary 1.29. *Let (X, A) be a relative CW-complex and Z a locally compact space. Then for any $O \subseteq X \times Z$, the following are equivalent:*

1. *The set O is open in $X \times Z$.*
2. *For every $n \geq -1$, $O \cap (X_n \times Z)$ is open in $X_n \times Z$*
3. *For every finite subcomplex (Y, A) of X, $O \cap (Y \times Z)$ is open in $Y \times Z$.*

Proof.

1. $\implies$ 2., 1. $\implies$ 3. by subspace topology.

2. $\implies$ 1. We define

$$\bar{X} = X_{-1} \amalg X_0 \amalg X_1 \amalg \cdots \amalg X_n \amalg \ldots.$$

Let $\bar{f} \colon \bar{X} \to X$ be the inclusion on all X_m. $\bar{f}$ is a quotient map by the weak topology. By the theorem, $\bar{f} \times Z \colon \bar{X} \times Z \to X \times Z$ is a quotient map. Hence also $\coprod_{n \geq 1}(X_n \times Z) \to X \times Z$ is a quotient map.

3. $\implies$ 1. Recall from the previous class: Let (X, A) be a relative CW-complex, let $U \subseteq X$, such that

- $U \cap A$ is closed in A
- U intersected with the closure of every cell is closed.

Then U is closed.

Proposition 1.30. *Let (X, A) be a relative CW-complex. Then the tautological map*

$$\coprod_{(Y,A) \text{ finite CW-subcomplex of } (X,A)} Y \to X$$

is a quotient map.

Proof. Every point of X is either contained in A or some open cell of (X, A). Since (A, A) is finite, and the closure of every cell is contained in a finite subcomplex, the map is surjective. Let $U \subseteq X$ be such that $q^{-1}(U)$ is closed. Then $U \cap Y$ is closed in Y for every finite subcomplex (Y, A) of (X, A). This includes (A, A), so $U \cap A$ is closed in A. The closure $\bar{e_j}$ of a cell e_j is contained in some finite subcomplex (Y, A), since $U \cap Y$ is closed in Y, also $U \cap \bar{e_j}$ is cloesd in $\bar{e_j}$. Hence U is cloesd in X. □

Let $O \subseteq X \times Z$ be such that $O \cap (Y \times Z)$ is open in $Y \times Z$ for all finite subcomplexes (Y, A) of X. Then $B = (X \times Z) \setminus O$ has the property that $B \cap (Y \times Z)$ is closed in $Y \times Z$ for every finite subcomplex (Y, A) of (X, A). Since Z is locally compact, product with Z preserves quotient maps, so

$$\begin{tikzcd}
(\coprod_{(Y,A)} Y) \times Z \arrow[dr, "q \times Z"] & \cong & \coprod_{(Y,A)}(Y \times Z) \arrow[dl] \\
& X \times Z &
\end{tikzcd}$$

□

Corollary 1.31. *Let (X, A) be a relative CW-complex, and Z a locally compact space. Let $f\colon X \times Z \to Y$ be any map. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $n \geq -1$, the map $f|_{X_n \times Z}\colon X_n \times Z \to Y$ is continuous.*

Proof. $X \times Z$ has the weak topology of the filtration $\{X_n \times Z\}_{n\geq -1}$ because

$$\coprod_{n\geq 1} X_n \times Z \to X \times Z$$

is a quotient map. □

1.3.1 Homotopy extension property

Definition 1.32

Let X be a space and A a subspace of X. Then (X, A) has the *homotopy extension property*, if the following holds: let $f\colon X \to Y$ be a continuous map and let $H\colon A \times [0, 1] \to Y$ be a homotopy starting with $f|_A$, i.e. for all $a \in A$, $H(a, 0) = f(a)$. Then there is a homotopy

$$\bar{H}\colon X \times [0, 1] \to Y$$

starting with $f$and extending H, i.e.

- for all $x \in X$, $\bar{H}(x, 0) = f(x)$
- for all $(a, t) \in A \times [0, 1]$, $\bar{H}(a, t) = H(a, t)$.

Lemma 1.33

A pair (X, A) has the HEP if and only if for every continuous map $g\colon X \cup_A A \times [0, 1] \to Y$, there is a continuous extension to $X \times [0, 1] \to Y$. Here

$$X \cup_A A \times [0, 1] := (X \amalg A \times [0, 1])/\sim$$

with $a \sim (a, 0)$ for all $a \in A$.

Warning.

$$X \cup_A A \times [0, 1] \to X \times \{0\} \cup A \times [0, 1] \subseteq X \times [0, 1]$$

$x \mapsto (x, 0), (a, t) \mapsto (a, t)$ need not be a homeomorphism.

Proposition 1.34. *The ?? (f, H) of a homotopy extension property is equally defined to a continuous map*

So (X, A) has the HEP iff $f \cup_A H$ extends continuously to $X \times [0, 1]$.

Lemma 1.35: L

t A be a closed subset of X. Then the tautological map

$$\tau\colon X \cup_A A \times [0,1] \to X \times \{0\} \cup A \times [0,1]$$

is a homeomorphism.

Proof. We know that τ is a continuous bijection. We show that τ is also a closed map. Let $B \subseteq X \cup_A A \times [0,1]$ be a closed subset. Let $p\colon X \amalg A \times [0,1] \to X \cup_A A \times [0,1]$ be the quotient map. Then $p^{-1}(B) \cap X$ is closedin X, and $p^{-1}(B) \cap A \times [0,1]$ is closedin $A \times [0,1]$. Since $X \times \{0\}$ is closed in $X \times [0,1]$, $(p^{-1}(B) \cap X) \times \{0\}$ because A is closed in X, hence $A \times [0,1]$ is closed in $X \times [0,1]$. So $\tau(B)$ is the union of two closed subsets in $X \times [0,1]$, and continuous in $X \times \{0\} \cup A \times [0,1]$ and have ?? in $X \times \{0\} \cup A \times [0,1]$. □

Corollary 1.36. *Let A be a closed subspace of X. Then (X, A) has the HEP if and only if the inclusion $X \times \{0\} \cup A \times [0,1]$ into $X \times [0,1]$ has a continuous retraction.*

Proof. $\Rightarrow$ Apply the HEP to $f\colon X \to X \times \{0\} \cup A \times [0,1]$ and $x \mapsto (x,0)$ $H\colon A \times [0,1] \to$So the HEP gives a continuous map $\bar{H}\colon X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$. that extends f& H

$\Leftarrow$ Let $\gamma\colon X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ be a continuous retraction. Let $f\colon X \to Y$, $H\colon A \times [0,1] \to Y$ be a homotopy extension problem. Then

$$X \times [0,1] \to X \times \{0\} \cup A \times [0,1] \to Y$$

Then $\bar{H}$:=is a homotopy extension of f and H.

□

Proposition 1.37. *For every $m \geq 0$, the pair $(\partial D^m, D^m)$ has the HEP.*

We exhibit a retraction $r\colon D^m \times [0,1] \to D^m \times \{0\} \cup \partial D^m \times [0,1]$ to the inclusion. For (x,t) in $D^m \times [0,1]$, the line through (x,t) and $(0,2)$ meets $D^m \times 0 \cup \partial D^m \times [0,1]$ in exactly one point that varies continuously with (x,t), this point defines $r(x,t)$.

Proposition 1.38. *Let X be a space obtained by attaching m-cells to A. Then (X, A) has the HEP.*

Proof. We construct a continuous retraction to $X \times \{0\} \cup A \times [0,1] \to X \times [0,1]$. We let $r\colon D^m \times [0,1] \to D^m \times \{0\} \cup \partial D^m \times [0,1]$ be a continuous retraction to the inclusion. We define the retraction $\rho\colon X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ as follows:

$$X \times [0,1] = (A \cup_{J \times \partial D^m} J \times D^m) \times [0,1] \leftarrow A \times [0,1] \cup_{J \times \partial D^m \times [0,1]} J \times D^m \times [0,1]$$

arrow down $A \times [0,1] \cup X \times \{0\} = A \times [0,1] \cup_{J \times \partial D^m} J \times D^m \cong A \times [0,1] \cup_{J \times \partial D^m \times [0,1]} J \times (D^m \times 0 \cup \partial D^m \times [0,1])$ □

Theorem 1.39

Every relative CW-complex has the HEP.

Proof. Let (X, A) be a relative CW-complex. We construct by induction continuous retractions $r_m \colon X_m \times [0,1] \to X_m \times 0 \cup A \times [0,1]$.

$m = -1$ Nothing to do.

$m \geq 0$ Suppose r_{m-1} has alreadey been constructed. We define r_m as the composite $X_m \times [0,1] \to X_m \times \{0\} \cup X_{m-1} \times [0,1] \to X_m \times \{0\} \cup (X_{m-1} \times \{0\} \cup A \times [0,1]) = X_m \times \{0\} \cup A \times [0,1]$. First arrow any retraction from previous proposition, second $\mathrm{Id} \cup r_{m-1}$.

We now define $r \colon X \times [0,1]$ as the „union“ of the r_ms, i.e. any $(x,t) \in X \times [0,1]$ is contained in $X_m \times [0,1]$ for some $m \geq 0$. We set $r(x,t) \coloneqq r_m(x,t)$. This is independent of m, because $r_{m+1}|_{X_m \times [0,1]} = r_m$. Then $r|_{X_m \times [0,1]} = r_m$ is continuous for all $m \geq 0$. So r is continuous because $X \times [0,1]$ has the weak topology wrt $\{X_m \times [0,1]\}_{m \geq 0}$.

□

non-example Let $X = [-1,0] \cup \{1/n : n \geq 1\}$, $A = [-1,0]$. Claim: (X, A) does not have the HEP.

Let $f \colon X \to X$ be the identity, $H \colon A \times [0,1] \to X$ be $H(a,t) = (1-t) \cdot a - t$ this is contracting $[-1,0]$ onto

$$-1$$

. Suppose there existed a homotopy $\bar{H} \colon X \times [0,1] \to X$ from the identity that extends H. Then $\bar{H}$ would need to be constant on each isolated point $1/n$. By continuity $\bar{H}$ would also have to be the identity on the limit point 0, but H is not.

Remember 1.17.

We will inductively construct the following data: for $m \geq -1$:

- a continuous map $f_m \colon X \to Y$
- Homotopy $H_m \colon X \times [0,1] \to Y$

such that f_m is „cellular up to level m“, i.e. $f_m(X_k) \subseteq Y_k$ for all $k = -1, 0, \ldots, m$. H_m is a hhomotopy from f_{m-1} to f_m relative to X_{m-1}.

We begin with $f_{-1} = f$. For $m \geq 0$ suppose the previous data has been constructed. By a previous special case of CAT applied to (X_m, X_{m-1}), (Y, Y_{m-1}) and $f_{m-1}|_{X_m} \colon X_m \to Y$ we obtain a homotopy

$$H \colon X_m \times [0, 1 \to Y]$$

relative X_{m-1} from $f_{m-1}|_{X_m}$ to to some map $H(\_, 1) \colon X_m \to Y$ such that $H(X_m \times \{1\}) \subseteq Y_m$. The HEP for the pair (X, X_m) applied to $f_{m-1} \colon X \to Y$ and H yields a homotopy

$$H_m \colon X \times [0,1] \to Y$$

form f_{m-1} that extends H. Then we set $f_m \coloneqq H_m(\_, 1) \colon X \to Y$. This has the desired properties.

If X was a finite-dimensional CW-complex we would be done. We now define a homotopy

$H\colon X \times [0,1] \to Y$ by „running through the homotopies H_m faster and faster."

$$H(x,t) = \begin{cases} H_0(x, 2t) & 0 \leq t \leq 1/2 \\ H_1(x, 6 \cdot (t - 1/2)) & 1/2 \leq t \leq 2/3 \\ \vdots & \\ H_m(x, (m+1)(m+2) \cdot (t - m/(m+1))) & \text{for } m/(m+1) \leq t \leq (m+1)/(m+2) \\ H_m(x, 1) & \text{for } t = 1,\ x \in X_m \end{cases}$$

This map is continuous on $X \times [0,1]$ by the weak topology because it is continuous on $X_m \times [0,1]$ for all $m \geq -1$. □

„The product of two CW-complexes „is" a CW-complex (often)"

Cells multiply: There is a homeomorphism $D^m \times D^n \cong D^{m+n}$ that such $(\partial D^m) \times D^n \cup D^m \times (\partial D^n)$ homeomorphic onto $partial(D^{m+n})$. picture square = circle

Let X and Y be CW-complexes. The conadidate CW-structure on $X \times Y$ is the *product CW-structure* with skeleta $(X \times Y)_n = \bigcup_{k=0,\ldots,n} X_k \times Y_{n-k}$.

Proposition 1.40 (CW-recognition theorem)**.** *Let X be a Hausdorff space, J_k a set for all $k \geq 0$, and $q\colon \coprod_{k\geq 0} J_k \times D^k \to X$ a continuous map. Suppose that:*

1. *For every $n \geq 0$, the restriction of q to $J_n \times \mathring{D}^n$ is injective, and the ... set of X is the disjoint union of $q(J_n \times \mathring{D}^n)$ for $n \geq 0$*
2. *For all $k \geq 0$ and $j \in J_k$, the set $q(j \times \partial D^k)$ is contained in a finite union of sets of the form $q(i \times D^j)$ for some $j < k$, $i \in J_j$.*
3. *A subset $A \subseteq X$ is closed in X if and only if $A \cap q(j \times D^k)$ is closed in $q(j \times D^k)$ for all $k \geq 0, j \in J_k$.*

Then setting $X_n \coloneqq \bigcup_{0\leq k\leq n} q(J_k \times D^k)$ defines a CW-structure on X.

Proof. Convenient notation: $e^k_j \coloneqq q(j \times \mathring{D}^k)$ for $k \geq 0, j \in J_k$ is the „j-th open k-cell". $\bar{e}^k_j$ = closure of $e^k_j = q(j \times D^k)$ „j-th closed cell".

We show by induction on n, that X_n is closed in X and X_n can be obtained from X_{n-1} by attaching n-cells indexed by J_n.

We write $\alpha J_n \times \partial D^n \to X_{n-1}$ for the restriction of q.

$$X_{n-1} \amalg J_n \times D^n \to X$$

arrow down P arrow up f $X_{n-1} \cup_\alpha J \times D^n$ arrow up is continuous and injective with image X_n.

Claim. f is a closed map. Let $A \subseteq X_{n-1} \cup_\alpha J \times D^n$ be a closed subset. We want to show, that $f(A)$ is closed in X. We use 3. and check that $f(A) \cap \bar{e}^k_j$ is closed in $\bar{e}^k_j$ for all $k \geq 0$, $j \in J_k$.

Case 1 $k < n$. Then $\bar{e}^k_j \subseteq X_{n-1}$. Because A is closed, $p^{-1}(A)$ is closed, so $A \cap X_{n-1}$, in X_{n-1} This is closedin X by induction. So $f(A) \cap \bar{e}^k_j$ is closed

Case 2 $k = n$ $p^{-1}(A) \cap (j \times D^n)$ is closed in $j \times D^n$, which is compact. So $f(A) \cap \bar{e}^n_j$ is the continuous image of a compact set hence compact in X, hence closed in X, and in $\bar{e}^n_j$.

Case 3 $k > n$. Because $f(A) \subseteq X_n$, $f(A) \cap \bar{e}_j^k \subseteq q(j \times \partial D^n) \subseteq$ finite union of cells of smaller dimension, each of which are closed in the set by induction. So $f(A) \cap \bar{e}_j^k$ is closed.

X has the weak topology: Let $A \subseteq X$ be such that $A \cap X_n$ is closed in X_n for all $n \geq 0$. Then $A \cap \bar{e}_j^k$ is closed in $\bar{e}_j^k$ for all $k \geq 0$, $j \in J_k$ because $\bar{e}_j^k \subseteq X_k$. By 3. A is closed in X. $\square$

non-example. $D^2 = \bigcup_{j \in \partial D^2} \{j\} \cup \mathring{D}^2$ is a union of uncountably many open 0-cells, and one 2-cell.

$q\colon (\partial D^2)_{\text{discret}} \amalg D^2 \to D^2$ the tautological map. This does not define a CW-structure on D^2. The ffiniteness in 2 fails. Because ∂D^2 is not contained in a finite union of cells of dimension ≤ 1.

Theorem 1.41

Let X, Y be CW-complexes such that Y is locally compact. Then $(X \times Y)_n := \bigcup_{k \leq n} X_k \times Y_{n-k}$ defines a CW-structure on $X \times Y$.
The n-cells of this product CW-structure *biject with pairs of*

$$\bigcup_{k=0,\dots,n} (k\text{-cells of } X) \times ((n-k)\text{-cells of } Y)$$

Proof. We choose indexing sets and characteristic maps for the given CW-structure on X and Y. This yields two quotient maps

$$q\colon \coprod_{k \geq 0} J_k \times D^k \to X \quad q'\colon \coprod_{l \geq 0} J'_l \times D^l \to Y$$

The product yields a continuous map

$$\coprod_{k,l \geq 0} J_k \times J'_l \times D^{k+l} \cong (\coprod_{k \geq 0} J_k \times D^n) \times (\coprod_{l \geq 0} J'_l \times D^l) \xrightarrow{q \times q'} X \times Y$$

The composite satisfies condition 1 and 2 of the previous „recognition theorem“ for CW-structures.

Claim. $q \times q'$ is a quotient map.

Proof.

$$(\coprod_{k \geq 0} J_k \times D^n) \times (\coprod_{l \geq 0} J'_l \times D^l) \xrightarrow{\mathrm{Id} \times q'} (\coprod_{k \geq 0} J_k \times D^k) \times Y \xrightarrow{q \times Y} X \times Y$$

first: quoteint maps because $\coprod_{k \geq 0} J_k \times D^k$ is disjoint union of compact spaces. second: Quoteint map because Y is locally compact. $\square$

Condition 3 of recognitnion theorem: Let $A \subseteq X \times Y$ be a subset such that $A \cap e_j^k \times e_j'^l = A \times (\bar{e}_j^k \times \bar{e}_{j'}^l)$ is closed in $\bar{e}_j^k \times \bar{e}_{j'}^l$ for all $k \geq 0$, $l \geq 0$, $j \in J_k$, $j' \in J_l$. Then $(q \times g')^{-1}(A) \cap ((j, j') \cap D^k \times D^l) = (q \times q')^{-1}|_{(j,j') \times D^k \times D^l}(A \cap (\bar{e}_j^k \times \bar{e}_{j'}^l))$ is closed. Since $(q \times q')^{-1}(A)$ is closed and $q \times q'$ is a quotient map, A is indeed closed in $X \times Y$. $\square$

2 Higher homotopy groups

3 singular homology groups

Index