Universität Bonn

Notes for the lecture

Algebraic Topology I

held by

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TEXed by

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Corrections and improvements

If you have corrections or improvements, contact me via (`s94jmalm@uni-bonn.de`).

Contents

Lecture

[13.10.2025, Lecture 1]

Organizatorial

For this term we will be doing unstable homotopy theory. Next term we will be doing stable homotopy theory. Note that there were 2 previous courses. Note that all important information is shared on the website `https://www.math.uni-bonn.de/people/schwede/at1-ws2526`. You can sign up for the previous topology courses und see the lecture videos for these courses there.

There are no lecture notes for this lecture specifically, but some similar materials are linked on the webpage.

Exercise sheets will be uploaded fridays and handed in 11 days later via eCampus. Registration for eCampus opens at 4 today.

For exam admission you will have to score 50% of the points on the exercise sheets and have presented 2 exercises in tutorial.

The first exam will be written in the last week of semester.

I fear I will not be able to copy pictures here.

1.1 Blakiers-Massy theorem/Homotopy excision

We start with a reminder on relative homotopy groups.

Definition 1.1: Relative Homotopy Groups

Let (X, A) be a space pair i.e. A is a subspace of a topological space X. We write

$$I = [0,1] \quad I^n = [0,1]^n \text{ the } n\text{-cube}$$

$$\partial(I^n) = \text{ boundary of } I^n$$
$$I^{n-1} \subseteq I^n$$

via Inclusion on the first $n-1$ coordinates.

$$J^{n-1} = I^{n-1} \times \{1\} \cup (\partial I^{n-1}) \times [0,1]$$

He draws a picture for $n = 2$.
For $n \geq 1$ the n-th relative homotopy groups $\pi_n(X, A, x)$ is the set of triple homotopy classes of trible maps $x \in A \subseteq X$

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, \{x\})$$

where a triple map takes each subset on the left into the subset on the right. A triple-homotopy must also conserve these conditions.
For $n \geq 2$ or $n = 1$ and $A = \{x\}$ the set $\pi_n(X, A, \{x\})$ has a group structure by concatenation in the first coordinate. He again draws a picture.
The group structure is commutative if $n \geq 3$ or $n = 2$ and $A = \{x\}$.

Definition 1.2: n-Connectedness

Let $n \geq 0$. A space pair (X, A) is *n-connected*, if the following equivalent conditions hold:

1. For all $0 \leq q \leq n$ every pair map $(I^q, \partial I^q) \to (X, A)$ is homotpic relative $\partial(I^q)$ to a map with image in A
2. For all $a \in A$, $\mathrm{incl}_* \colon \pi_q(A, a) \to \pi_q(X, a)$ is bijective for $q \leq n$ and surjective for $q = n$.
3. $\pi_0(A) \to \pi_0(X)$ is bijective and for all $1 \leq q \leq n$ the relative homotopy group

$$\pi_q(X, A, x) \cong 0$$

Proof. You proof the equivalence using the LES[1] of homotopy groups. □

Let Y be a space, Y_1, Y_2 open subsets of $Y = Y_1 \cup Y_2$, $Y_0 := Y_1 \cap Y_2$.
Excision in homology shows that for all abelian groups B, $i \geq 0$

$$H_i(Y_2; Y_0, B) \to H_i(Y, Y_1; B)$$

is an isomorphism.
Excision does not generally hold for homotopy groups, i.e. for $x \in Y_0$

$$\mathrm{incl}_* \colon \pi_i(Y_2, Y_0; x) \to \pi_i(Y, Y_1; x)$$

[1] Long exact sequence

is **not** generally an isomorphism.

„Blakiers Massey theorem implies that excision holds for homotopy groups in a range.“

Theorem 1.3: Blakiers Massey

Let Y be a space, Y_1, Y_2 open subsets with $Y = Y_1 \cup Y_2, Y_0 := Y_1 \cap Y_2$. Let $p, q \geq 0$, such that for all $y \in Y_0$

$$\pi_i(Y_1, Y_0, y) = 0 \text{ for all } 1 \leq i \leq p$$

and

$$\pi_i(Y_2, Y_0, y) = 0 \text{ for all } 1 \leq i \leq q$$

Then for all $y \in Y_0$, the map

$$\mathrm{incl}_* \colon \pi_i(Y_2, Y_0, y) \to \pi_i(Y, Y_1, y)$$

is an isomorphism for $1 \leq i < p + q$ and surjective for $i = p + q$. He notes how the referenced literature uses different indices. They have proofs in more detail and pictures, however Lücks script contains typos

Proof. Schwede explains he doesn't like the proof, it is to technical and not very enlightening.

We define what cubes are

Cubes in $\mathbb{R}^n$, $n \geq 1$. $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ the „lower left corner of the cube“

$\partial \in \mathbb{R}_{\geq 0}$ „side length of the cube“

$L \subset \{1, \ldots n\}$ „relevant dimensions“

$W = W(a, \delta, L) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_1 \leq a_i + \delta \text{ for all } i \in L,\ x_i = a_i \text{ for all } i \in \{1, \ldots n\} \backslash L\}$

[2] A face W' of W is a subset of the form

$$W' = \{x \in W : x_i = a_i \text{ for all } i \in L_0, x_i = a_i + \delta \text{ for all } i \in L_1\}$$

for some subsets $L_0, L_1 \subseteq L$

Let $1 \leq p \leq n$ we define two subsets of a cube $W =^(a, \delta, L)$.

$$K_p(W) = \{x \in W : x_i < a_i + \delta/2 \text{ for at least} p \text{values of} i \text{ in } L\}$$

We call these „p small coordinates“

$$G_p(W) = \{x \in W : x_i > a_i + \delta/2 \text{ for at least } p \text{ coordinates } i \text{ in } L\}$$

these are „p big coordinates“.

For $p > \dim(W), K_p(W) = G_p(W) = \emptyset$ If $p + q \geq \dim(W)$, then $K_p(W) \cap G_q(W) = \emptyset$.

He draws pictures.

[2] W weil Würfel

Lemma 1.4: 1.14

It is Lemma 1.14 in Lücks Script.
Let (Y, A) be a space pair, $W \subseteq \mathbb{R}^n$ a cube, $f\colon W \to Y$ continuous. Suppose that for some $p \leq \dim(W)$, $f^{-1}(A) \cap W' \subseteq K_p(W')$ for all proper[1] faces W' of W.
Then there is a continuous map $g\colon W \to Y$ homotopic to f relative ∂W such that all $g^{-1}(A) \subseteq K_p(W)$

[1] subcube of the boundary

Proof. Wlog: $W = I^n = W(0, 1, \{1, \dots, n\})$

Let I_2^n be the subcube $[0, 1/2]^n$. He draws a picture. $x_4 = (1/4, \dots 1/4) \in I_2^n$. We define a continuous map $h\colon I^n \to I^n$ by radical projection away from x_4. Picture. Let $r(y)$ be the ray from x to y. We map all of $r(y) \cap I^n \setminus I_2^n$ to the intersection point of $r(y)$ and ∂I^n and the rest linearly extends as far as required.[3]

Obviously[4] h is homotopic relative boundary ∂I^n to the identity.

We set $g\colon f \circ h\colon I^n \to Y$, which is then homotopic relative $\partial(I^n)$[5] to f.

It remains to show that $g^{-1}(A) \subseteq K_p(W)$. Consider $z \in I^n$ with $g(z) \in A$.

Case 1 for all $i = 1, \dots, n, z_1 < 1/2$, i.e. $z \in I_2^n$, then $z \in K_n(I^n) \subseteq K_p(I^n)$.

Case 2 There is an $i \in \{1, \dots, n\}$, s.t. $z \geq 1/2$. Then $h(z) \in \partial(I^n)$. Let W' be some proper face of W, with $h(z) \in W'$. Since $f(h(z)) = g(z) \in A$, by hypothesis, $h(z) \in K_p(W')$, so $h(z) < 1/2$ for at least p coordinates. By expansion[6] property of h, also p coordinates of z are small coordinates.

□

Proposition 1.5. [7] Let Y_1, Y_2 be open subsets of $Y, Y_0 := Y_1 \cap Y_2$. Suppose that (Y_1, Y_0) is p-connected, (Y_2, Y_0) is q-connected. Let $f\colon I^n \to Y$ be continuous. Let $\mathcal{W} = \{W\}$ be a subdivision of I^n into subcubes of the same side length s.t. for all $W \in \mathcal{W}$ $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$. Then there is a homotopy $h\colon I^n \times I \to Y$ with $h_0 = f$ such that for all $W \in \mathcal{W}$:

1. If $f(W) \subseteq Y_j, j \in 0, 1, 2$, then $h_t(W) \subseteq Y_j$ for all $t \in [0, 1]$
2. If $f(W) \subseteq Y_0$, then $h_t|_W = f|_W$, i.e. h is constant on W.
3. If $f(W) \subseteq Y_1$, then $h_1^{-1}(Y_1 \setminus Y_0) \subseteq K_{p+1}(W)$.
4. If $f(W) \subseteq Y_2$, then $h_1^{-1}(Y_2 \setminus Y_0) \subseteq G_{q+1}(W)$.

Proof. We let $C^k \subseteq I^n$ be the union of all cubes in $\mathcal{W}$ of dimension at most k. We construct homotopies $h[k]\colon C_k \times I \to Y$, such that for all $W \in \mathcal{W}, W \subseteq C_k$ conditions 1. to 4. hold, and $h[k]$ is constant on $C_{k-1} \times I$. Then the final $h[n]$ does the job.

Note. If $W \in \mathcal{W}$ and $f(W) \subseteq Y_0$ and 2. holds, then also 3. and 4. hold.

$$h^{-1}(Y_1 \setminus Y_0) = h_1^{-1}(Y_2 \setminus Y_0) = \emptyset$$

If $W \in \mathcal{W}$, is such that $f(W) \subseteq Y_1$ and $f(W) \subseteq Y_2$, then $f(W) \subseteq Y_1 \cap Y_2 = Y_0$. So each $W \in \mathcal{W}$ is in excactly one of the following cases

- $f(W) \subseteq Y_0$

[3] I hope this description is clear, hard without the picture.
[4] Meaning he's too lazy to come up with formulas for the map.
[5] I am very inconsistent in remembering these parantheses with the boundary operator. Just imagine it always being as here.
[6] no idea if this is the word he wrote
[7] 11.5 in Lücks notes

- $f(W) \subseteq Y_1$ and $f(W) \nsubseteq Y_1$
- $f(W) \subseteq Y_2$ and $f(W) \nsubseteq Y_2$

Inductive construction $k = 0$, i.e. vertexes of the cubes $w \in \mathcal{W}$. If $w \in Y_2$, take $h[0]_t = \text{const}_{w_0}$.

Suppose $f(W_0) \in Y_1$, but $f(w_0) \notin Y_2$. Since Y_1, Y_0 is 0-connected, there is a path $\pi\colon I \to Y_1$ from w_0 to a point in Y_0. We take $h[0]$ as the path on w_0. Analoguos if $f(W_0) \in Y_2 \setminus Y_1$.

Inductive Step Let $W \in \mathcal{W}$ be a cube of exact dimension k. Then $\partial W = W \cap C_{k-1}$. Since $(W, \partial W)$ has the HEP, we can extend the previous homotopy $h[k-1]|_{\partial W}$ to some homotopy on W relative to $f|_W$. Let this be $h'[k]\colon C_k \times I \to Y$: this satisfies conditions 1. and 2. but not yet 3. and 4.

We produce another homotopy $h[k]''$ and set $h[k] = h[k]' * h[k]''$.

Consider a cube $W \in \mathcal{W}$ of dimension k.

If $f(W) \subseteq Y_0$ set $h[k]''$ as the constant homotopy on W.

If $h[k]'_1(W) \subseteq Y_1$, but $h[k]'_1(W) \nsubseteq Y_2$ there is a homotopy relative ∂W from $h[k]'_1$ to a map $f_1(W) \subseteq Y_0$.

If $k = \dim(W) > p$ the we use the lemma 1.4 for $f = h[k]'_1|_W$ and the resulting homotopy is $h[k]''|_W$.

If $h[k]'_1(W) \subseteq Y_2$ but $h[k]'_1(W) \nsubseteq Y_1$, use the complement case of the lemma[8] □

[13.10.2025, Lecture 1]

[15.10.2025, Lecture 2]

Now for the actual proof of Blakiers Massey Let $F(Y_1, Y, Y_2) = Y_1 \times_Y Y^{[0,1]} \times_Y Y_2$.

Let $F(Y_1, Y_1, Y_0)$ be the subspace of those $w \in F(Y_1, Y, Y_2)$ such that $w([0,1]) \subseteq Y_1$.

Proposition 1.6. Assume that (Y_1, Y_0) is p-connected, (Y_2, Y_0) is q-connected. Then the pair $(F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$ is $(p + q - 1)$-connected.

Proof. We consider a map of pairs

$$\phi\colon (I^n \partial I^n) \to (F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$$

for $1 \leq n \leq p+q+1$. We want to homotop ϕ relative ∂I^n to a map with image in $F(Y_1, Y_1, Y_0)$. We use the adjoint

$$\text{maps}(X \times [0,1], Z) \cong \text{maps}(X, Z^{[0,1]})$$

We let $\Phi\colon I^n \times I \to Y$ be the adjoint of ϕ, this is *admissable*, in the sense that

1. $\Phi(x, 0) \in Y_1$ for all $x \in I^n$
2. $\Phi(x, 1) \in Y_2$ for all $x \in I^n$
3. $\Phi(x, t) \in Y_1$ for all $x \in \partial I^n$, $t \in [0,1]$

We want a homotopy of Φ through admissable maps to a map $\Phi'\colon I^n \times I \to Y$ such that $\text{Im}(\Phi') \subseteq Y_1$.

Apply Proposition 11.5 to Φ (with $n+1$ instead of n). This gives a homotopy through admissable maps to $\Psi = g$. Let $h\colon I^n \times I \times I \to Y$ be a homotopy witnessing this. $h_0 = \Phi, h_1 F \Psi$.

Consider the projection $\text{pr}\colon I^n \times I \to I^n$ away from the last coordinate.

Claim. The image under pr of $\Psi^{-1}(Y \setminus Y_1)$ and $\Psi^{-1}(Y \setminus Y_2)$ are disjoint.

Suppose there is $y \in I^n$ in the intersection of the preimages, so $z_1 \in \Psi^{-1}(Y \setminus Y_1)$, $z_2 \in \Psi^{-1}(Y \setminus Y_2)$ s.t $\text{pr}(x_1) = y = \text{pr}(z_2)$. Let $W = I^{n+1}$ be a subcube of the subdivision of Proposition 11.5 such

[8] rest of the proof next lecture. I am very sure some words won't make sense, as they were unreadable on the board.

that $z_1 \in W$. Since $z_1 \in \Psi^{-1}(Y \setminus Y_1)$, $z_1 \in K_{p+1}(W)$, so $y = \mathrm{pr}(z_1) \in K_p(\mathrm{Im}(W))$. Analogous $y = \mathrm{pr}(z_2) \in G_q(\mathrm{Im}(W))$. Since $p + q > n$, this is a contradiction and the claim is proven.

Claim. The intersection of $\mathrm{pr}(\Psi^{-1}(Y \setminus Y_1))$ with ∂I^n is empty since Ψ is admissable, and thus $\Psi(\partial I^n) \subseteq Y_1$. So $\mathrm{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\mathrm{pr}(\Psi^{-1}(Y \setminus Y_2)) \cup (\partial I^n)$ are two disjoint closed subsets of the compact space I^n.

So there is a continuous separating function $\tau\colon I^n \to [0,1]$, s,t. $\tau \equiv 0$ is $\mathrm{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\tau \equiv 1$ is $\partial I^n \cup \mathrm{pr}(Y \setminus Y_2)$

We define another homotopy starting with Ψ by

$$h\colon (I^n \times I) \times I \to Y$$

by

$$((x,t),s) \mapsto \Psi(x, (1-s)t + s \cdot t\tau(x))$$

This is

- Homotopy through admissable maps
- $h(x,t,0) = \Psi(x,t)$

Claim. $h(\_,\_,1)$ has image in Y_1.

$$h(x,t,1) = \Psi(x, t \cdot \tau(x))$$

- if $x \in \Psi^{-1}(Y \setminus Y_1)$, then $\tau(x) = 0$, $h(x,t,1) = \Psi(x,0) \in Y_1$.
- if $x \in \Psi^{-1}(Y_1)$, then by admissability $\Psi(x, t \cdot \tau(x)) \in Y$.

$\square$

For the actual proof (Y_1, Y_0) p-connected (Y_2, Y_0) q-connected

some diagram about something being Serre fibration

We compare wo Serre fibrations

$$\begin{tikzcd}
F(\{y_0\}, Y_1, Y_0) & F(\{y_0\}, Y, Y_2) \\
F(Y_1, Y_1, Y_0) \arrow[r] & F(Y_1, Y, Y_2) \\
Y_1 & Y_1
\end{tikzcd}$$

partial 5-lemma shows that also the pair

$$(F(\{y_0\}, Y, Y_2), F(\{y_0\}, Y_1, Y_0))$$

is $(p+q-1)$-connected.

The following square commutes for all $n \geq 1$

$$\begin{tikzcd}
\pi_{n-1}(F(\{y_0\}, Y_1, Y_0), \mathrm{const}_{y_0}) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(F(\{y_0\}, Y, Y_1), x) \arrow[d, "\cong"] \\
\pi_n(Y_1, Y_0, y_0) \arrow[r, "\mathrm{id}_*"] & \pi_n(Y, Y_1, y_0)
\end{tikzcd}$$

$\square$

1.2 Feudenthal suspension theorem

Section 6.4 in tom Deicks book

Definition 1.7

Let X be a based space. The *unreduced suspension* is $X^\diamond = X \times [-1,1]/\sim$ where $\sim$ identifies all points with second variable -1 to S and all points with second variable 1 to N. We use S as the base point of $X^\diamond$.[1]

[1]south and north pole

Note. If X is well based, i.e. $\{x_0\} \hookrightarrow X$ has the HEP, then the quotient map $X^\diamond \sigma X = \frac{X\times[0,1]}{X\times\{0,1\}\cup\{x_0\}\times[0,1]}$ is a homotopy equivalence.

The suspension homomorphism $S\colon \pi_k(X,x_0) \to \pi_{k+1}(X^\diamond, S)$ for $k \geq 1$ is $S[f\colon S^k \to X] := [f^\diamond\colon S^{k+1} = (S^k)^\diamond \to X^\diamond]$

Theorem 1.8: Freudenthal suspension theorem

Let X be an $(n-1)$-connected space, $n \geq 1$. Then the suspension homomorphism is

- *bijective foor $i \leq 2n-2$*
- *surjective for $i = 2n-1$*

Proof. We cover $X^\diamond$ by $Y_1 = X^\diamond \setminus \{N\}$, $Y_2 F X^\diamond \setminus \{S\}$. We use repeatedly that Y_1 and $Y_2$0are contractible. Then $X \to Y_1 \cap Y_2 = Y_0$ is a homotopy equivalence. We claim without proof that the following diagram commutes:

I couldn't keep up.

So we may show that $\pi_{i+1}(Y_2, Y_0, x_0) \to \pi_{i+1}(Y, Y_1, y_0)$ is bijective for $i \leq 2n-2$ and surjective for $i \leq 2n-1$.

Because X is $(n-1)$-connected

$$\pi_{k+1}(Y_1, Y_0, x_0) \xrightarrow[\partial]{\cong} \pi_k(Y_0, x_0)$$

So $\pi_i(Y_1, Y_0, x_0) = 0$ for $i \leq n$, i.e. (Y_1, Y_0) is n-connected. Also (Y_2, Y_0) is n-connected.

BM gives $\pi_k(Y_2, Y_0, y) \to \pi_k(X^\diamond, Y_1, y)$ is bjective for $k < 2n-1$ and surjective for $k = 2n$.

Setting $i = k-1$ ends the proof. □

Take $X = S^n$

Corollary 1.9. The suspension homomorphism

$$S\colon \pi_i(S^n, *) \to \pi_{i+1}(S^{n+1}, *)$$

is bijective for $i \leq 2n-2$ and surjective for $i = 2n-1$.

Corollary 1.10. For all $n \geq 1$, $\pi_n(S^n, *) \cong \mathbb{Z}$, generated by the class of id_{S^n}. Moreover $\deg\colon \pi_n(S^n, *) \to \mathbb{Z}$ is an isomorphism.

Proof. Induction on n. For $n = 1$ we have this by covering theory.

$n \geq 1$ By Freudenthal, the suspension homomorphism

$$S\colon \pi_n(S^n, *), \to \pi_{n+1}(S^{n+1}, *)$$

is surjective. The composite

$$\pi_n(S^n,*) \xrightarrow{S} \pi_{n+1}(S^{n+1},*) \xrightarrow{\deg} \mathbb{Z}$$

is bijective by induction. So S is also injective and deg in one dimension higher is also an isomorphism. □

Recall. $\pi_3(S^2,*) \cong \mathbb{Z}\{\eta\}$, where $\eta\colon S^3 \to S^2$ is the Hopf map.

Proposition 1.11. For $n \geq 3$, $\pi_{n+1}(S^n,*) = \mathbb{Z}/2\{S^{n-2}(\eta)\}$ cycles of order two.

Proof. Only partial. For $n \geq 2$, $S\colon \pi_{n+1}(S^n,q) \mapsto \pi_{n+2}(S^{n+1},S^{n+1},*)$ is surjective.

$$\mathbb{Z}\{\eta\} = \pi_3(S^2,*) \twoheadrightarrow \pi_4(S^3,*) \underset{\cong}{\xrightarrow{S}} \pi_5(S^4,*)\dots$$

Claim. $S(2\eta) = 0$ in $\pi_4(S^3,*)$. Which gives $\pi_{n+1}(S^n,*)$ is either trivial or order 2.

Consider the commutative square

I did not manage to copy. Something complex conjugation

$\implies$ $[\eta] = [\eta \circ \text{complex conjugation}] = [d \circ \eta]$ in $\pi_3(S^2,* \cong \mathbb{Z})$.

If $f\colon S^n \to S^n$ has degree k, then precomposition of $\pi_n(X,x) \to \pi_n(X,x)$ is multiplication by k.

Let $c\colon S^1 \to S^1$ be any map of degree -1. Then $c \wedge S^2$, $S^1 \wedge d$: bothe have degree -1. So $c \wedge S^2 \sim S^1 \wedge d$

two diagrams I did not copy.

We get $S(\eta) = S(d \circ \eta) = S(\eta \circ (c \wedge S^1)) = S(-\eta) = -S(\eta)$ and hence $2 \cdot S(\eta) = 0$. □

[13.10.2025, Lecture 2]

[20.10.2025, Lecture 3]

We make a bit of a preview[9] for stable homotopy theory. Following Lücks notes today rather closely

Definition 1.12

Let X be a based space. The n-th stable homotopy group of X is the colimit $\pi_n^S(X,*)$

$$\pi_n(X,*) \xrightarrow{S} \pi_{n+1}(S^1 \wedge X,*) \xrightarrow{S} \pi_{n+2}(S^2 \wedge X,*)\dots$$

Since $S^k \wedge X$ is $(k-1)$-connected, the suspension stabilises (i.e. S is isomorphism for $\pi_{n+k}(S^k \wedge X,*)$ onwards).

$\pi_n^S()$ is a functor from based spaces to abelian groups. and it is homotopy invariant. It comes with a natural transformation $\pi_n(X,*) \to \pi_n^S(X,*)$.

Preview: $\{\pi_n^S\}_{n\in\mathbb{Z}}$ form a generalized homology theory on based spaces.

If (X,A) is a pair of based space, $x \in A \subseteq X$, we define

$$\pi_n^S(X,A,*) \coloneqq \pi_n^S(X \cup_A CA,*)$$

The map collapsing X is

$$X \cup_A CA \to \frac{X \cup_A CA}{X} \cong \frac{CA}{A} \cong A^\diamond$$

[9]He just spammed random stuff, I don't think I copied enough for it to make sense.

induces an connecting homomorphism

$$\partial\colon \pi_n^S(X,A,x) = \pi_n^S(X \cup_A CA, *) \xrightarrow{p_*} \pi_n^S(A^\diamond) \to \pi_n^S(A \wedge S^1) \xrightarrow{\cong} \pi_{n-1}^S(A)$$

The following sequence will then be exact:

$$\dots \pi_n^S(A,x) \xrightarrow{\text{incl}_*} \pi_n^S(X,x,) \xrightarrow{\text{incl}_*} \pi_n^S(X,A,*) \xrightarrow{\partial} \pi_{n-1}^S(A,*) \to \dots$$

Stable stems are the special case

$$\pi_n^S(S^0) = \operatorname{colim}(\pi_n(S^0,*) \to \pi_{n+1}(S^1,*) \to \dots)$$

n	0	1	2	3	4	5	6	7	8
$\pi_n^S = \pi_n^S(S^0)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
Generator	id	η	η^2	ν, $\eta^3 = 12\nu$	-	-	ν^2	σ	$\eta\sigma, \epsilon$

There is a graded-commutative ring structure on $\pi_*^S = \{\pi_n^S\}_{n\in\mathbb{Z}}$. $\pi_n^S \times \pi_m^S \to \pi_{n+m}^S$

$$[f\colon S^{n+k} \to S^k \times S^k] \times [g\colon S^{m+l} \to S^l] = [SWn+m+k+l \to S^{n+k} \wedge S^{m+l} \xrightarrow{f\wedge g} S^{k+l}]$$

I missed a bit more

Nishidas theorem says: every pohitive dimensional element of π_*^3 is nilpotent.

From Serre spectral sequence we will see: For $m > n \geq 1$ $\pi_m(S^n,*)$ is finite except $\pi_{4k-1}(S^{2k},*) \cong \mathbb{Z}\oplus$ some finite group.

This was exercise 5.2 of last summer term: *Hopf invariant* $h\colon \pi_{2k-1}(S^k,*) \to \mathbb{Z}$ for $[f\colon S^{2k-1} \to S^k]$

$$H^*(S^k \cup_f D^{2k}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0,k,2k \\ 0 & \text{else} \end{cases}$$

we have $H^k(Cf,\mathbb{Z}) = \mathbb{Z}\{a\}$ $H^{2k}(Cf,\mathbb{Z}) \cong \mathbb{Z}\{b\}$ and $a \cup a = h(f)\cdot b$.

In general we can look at

$t_k\colon S^{2k-1} \to S^k \vee S^k$ the attaching map of the $2k$-cell in the product minimal CW-structure of $S^k \times S^k$. And we get $h(t_k) = 2$.

The Hopf invariant 1 theorem tells us when we can realize Hopf invariant 1 and here we see that we always find Hopf invariant 2.

1.3 Hurewicz-theorem

This is a „trivial“[10] Corollary of the Blakiers-Massey theorem.

Definition 1.13

Let X be a based space. Choose an orientation of S^n, $n \geq 1$, i.e. $[S^n] \in H_n(S^n,\mathbb{Z})$. The Hurewicz homomorphism

$$h\colon \pi_n(X,x) \to H_n(X;\mathbb{Z})$$

is defined by

$$h[f\colon S^n \to X] = H_n(f,\mathbb{Z})[S^n].$$

This is a group homomorphism.

[10] meaning 1.5 lectures of intermediate steps

Group Homomorphism: Let $\Delta\colon S^n \to S^n \vee S^n$ be a pinch map. Group structure on $\pi_n(X,x)$ is given by $[f]+[g] = [S^n \xrightarrow{\Delta} S^n \vee S^n \xrightarrow{f+g} X]$. Then

$$\begin{tikzcd} \pi_n(X,x) \arrow[d, "\cong"'] \arrow[dr, "h"] & \\ \pi_n(X,y) \arrow[r, "h"] & H_n(X,\mathbb{Z}) \end{tikzcd}$$

$$([f]+[g])_* = [f]_* + [g]_* \colon H_n(S^n) \to H_n(X)$$

So Hurewicz is a homomorphism.

Lemma 1.14

Let $w\colon [0,1] \to X$ be a path from $x = w(0)$ to $y = w(1)$. Then the following commutes:

$$\begin{tikzcd} \pi_n(X,x) \arrow[d, "\cong"] \arrow[dr, "h"] & \\ \pi_n(X,y) \arrow[r, "h"] & H_n(X,\mathbb{Z}) \end{tikzcd}$$

Proof. f and $f * w$ are freely homotopic $h_n(\_,\mathbb{Z})$ is homotopy invariant □

For $n = 1$ we have Poincaré: X based path conneted, $h\colon \pi_1(X,x) \to H_1(X,\mathbb{Z})$ is surjective with kernel the commutator subgroup or equivalently

$$\pi_1(X,x)_{ab} \xrightarrow{\cong} H_1(X,\mathbb{Z})$$

Theorem 1.15: Hurewicz

Let X be an $(n-1)$-connected space, $n \geq 2$. Then the Hurewicz homomorphism

$$h\colon \pi_n(X,x) \to H_n(X,\mathbb{Z})$$

is an isomorphism.

We will need to work a bit for the proof.

Proposition 1.16 (11.9 in Lücks notes)**.** Let $m, n \geq 0$, let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[d, "\bar{i}"] \\ X \arrow[r, "\bar{f}"] & Y \end{tikzcd}$$

be a pushout square of spaces. Suppose $i\colon A \to X$ is a *cofibration*, i.e. a closed embedding with the HEP.

1. If f is n-connected, then so is $\bar{f}$
2. If f is n-connected, and i is m-connected, then for all $a \in A$, $\pi_i(f,\bar{f})\colon \pi_i(X,A,a) \to \pi_i(Y,B,f(a))$ is bijective for $1 \leq i < m+n$ and surjective for $i = m+n$.

Proof. „Basically reduction to Blakiers-Massey“

We can replace X, B and Y up to homotopy equivalence by appropriate mpping cylinders:

$$\mathrm{cyl}(f) = A \times [0,1] \cup_{A \times 1, f} B$$

We get a homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{Cyl}(f) \arrow[d] \arrow[dr, "\sim"] & \\
\mathrm{Cyl}(i) \arrow[r] \arrow[dr, "\sim"] & \mathrm{cyl}(i) \cup_A \mathrm{cyl}(f) \arrow[dr, "\sim"] & B \arrow[d, "\bar{i}"] \\
& X \arrow[r, "\bar{f}"] & Y
\end{tikzcd}$$

This diagram does not commute. The homotopy equivalences shown are not trivial.

We apply Blakiers-Massey then to the following open subspace of $W = \mathrm{cyl}(i) \cup_A \mathrm{cyl}(f)$.

$W_2 = [0, 1/2) \times A \cup_{A \times 0} \mathrm{cyl}(f)$, $W_1 = \mathrm{cyl}(i) \cup_{A \times 0} [0, 1/2)$. $W_0 = W_1 \cap W_2 \sim A$

Now having f is n-connected gives (W_2, W_0) is $n-connected.$ i being m-connected gives (W_1, W_0) is m-connected.

Now BM gives $\pi(W_1, W_0, a) \to \pi_i(W, W_2, a)$ is bijective for $1 \leq i \leq m+n-1$ and surjective for $i = m+n$.

Using the homotopy equivalences, we translate this back. □

Proposition 1.17 (11.1 in Lücks)**.** Let $m, n \geq 0$, let $c\colon A \to X$ be a cofibration. Suppose that i is m-connected and A is n-connected. Then

$$\pi_k(\mathrm{pr})\colon \pi_k(X, A, a) \to \pi_k(X/A, *)$$

for all $a \in A$ is bijective for $1 \leq k \leq m+n$ and surjective for $k = m+n+1$

Proof. We consider the pushout

$$\begin{tikzcd}
A \arrow[r, "j", "(n+1)\text{-con}"'] \arrow[d, "i"] & C(A) = A \times [0,1]/A \times 1 \arrow[d] & \\
X \arrow[r] & X \cup_A CA \arrow[r, "\sim"] & X/A
\end{tikzcd}$$

Remark 1.18. X k-connected $\Leftrightarrow$ $\{*\} \to X$ k-connected $\Leftrightarrow$ $X \to \{*\}$ is $(n+1)$-connected.

□

Proposition 1.19 (11.12 for Lück)**.** Also an exercise (1.2) Let X, Y be well-pointed spaces. Let $m, n \geq 1$. Let X be m-connected, Y n-connected. Then

1. The inclusion $X \vee Y \to X \times Y$ induces isomorphisms

$$\pi_k(X \vee Y, *) \to \pi_k(X \times Y, *)$$

   for all $0 \leq k \leq m+n$.

2. $\pi_k(X \times Y, X \vee D, *)$ and $\pi_k(X \wedge @, *)$ are trivial for all $0 \leq k \leq m+n+1$.

3. The map $(pr_*^X, pr_*^Y)\colon \pi_k(X \vee Y, x) \to \pi_k(X, x_0) \times \pi_k(Y, y_0)$ is an isomorphism for all $1 \leq k \leq m + n$.

[20.10.2025, Lecture 3]

[22.10.2025, Lecture 4]

Proposition 1.20. Let $n \geq 2$. Let $\{X_i\}_{i\in I}$ be a family of well-pointed, $(n-1)$-connected spaces. Then the canonical map

$$\bigoplus_{i\in I} \pi_n(X_i, *) \to \pi_n(\bigvee_{i\in I} X_i, *)$$

is an isomorphism.

Proof. **Step 1** If I is finite, we do Induction on $|I|$. Nothing to show if $I = \emptyset$, $|I| = 1$. Let $|I| \geq 2$. Wlog $I = \{1, 2, \dots k\}$, $k \geq 2$.

By Proposition 1.19, $\pi_n(\bigvee_{i=1,\dots,k-1} X_i, *) \oplus \pi_n(X_k, *) \xrightarrow{\cong} \pi_n(\bigvee_{i=1,\dots,k} X_i, *)$ and the first part is isomorphic to $\bigoplus_{i=1,\dots,k} \pi_n(X_i)$ by induction.

Case 2 I is infinite: We first show injectivity. The projection

$$p_j\colon \bigvee_{i\in I} X_i \to X_j$$

induces an homomorphism

$$(p_j)_*\colon \pi_n(\bigvee_I X_i, *) \to \pi_n(X_j, *)$$

For varying $j \in I$, these assemble into a homomborphism

$$\bigoplus_{i\in I} \pi_n(X_i, *) \xrightarrow{can} \pi_n(\bigvee_{i\in I} X_i, *) \xrightarrow{((p_j)*)} \prod_{j\in I} \pi_n(X_j, *)$$

and the composition is inclusion of sum into product, hence injective. So also the first map is injective.

For surjectivity let $f\colon S^n \to \bigvee_{i\in I} X_i$ be a continuous map that represents a class in $\pi_n(\bigvee_{i\in I} X_i)$. Because S^n is compact, there is a finite subset $J \subseteq I$ s.t. $\mathrm{Im}(f) \subseteq \bigvee_{i\in J} X_i$[11] This implies $[f] \subseteq \pi_n(\bigvee_{j\in J} X_j, *) \to \pi_n(\bigvee_{i\in I} X_i, *)$.

$$\begin{tikzcd}
\pi_n(\bigvee_{j\in J} X_j, *) \arrow[r] & \pi_n(\bigvee_{i\in I} X_i, *) \\
\bigoplus_{j\in J} \pi_n(X_j, *) \arrow[r] \arrow[u, "\text{Case 1}"] & \bigoplus_{i\in I} \pi_n(X_i, *) \arrow[u, "can"']
\end{tikzcd}$$

so $[f]$ is in the image of the canonical map.

□

Theorem 1.21: Hurewicz

Let $n \geq 2$. Let X be $(n-1)$-connected base space. Then the Hurewicz homomorphism $h\colon \pi_n(X, x) \to H_n(X, \mathbb{Z})$, $f \mapsto H_n(f, \mathbb{Z})[S^n]$ is an isomorphism.

[11] he explains, why this is not easy to see. But it is point-set topology, so we won't do it.

Proof.
- By CW-approximation (see later in this class) there is a CW-complex Y with one 0-cell and no cells in dimensions $1 \leq i \leq n-1$ and a weak homotopy equivalence $Y \xrightarrow{\simeq} X$.
- Every weak homotopy equivalence induces isomorphisms of $H_n(\_, A)$ for all $n \geq 0$, for all abelian groups A. This will either be an exercise or proven later on.

We get a commutative diagram

$$\begin{tikzcd}
\pi_n(Y, y) \arrow[r, "f_*", "\cong"'] \arrow[d, "h"] & \pi_n(X, f(*)) \arrow[d, "h"] \\
H_n(Y, \mathbb{Z}) \arrow[r, "f_*", "\cong"'] & H_n(X, \mathbb{Z})
\end{tikzcd}$$

So wlog we can assume that X admits a CW-structure with a single 0-cell and no cells of dimensions $1, \ldots, n-1$. The inclusion of the $(n+1)$-skeleton $X_{n+1} \to X$ induces isomorphisms

$$\pi_n(X_{n+1}, *) \to \pi_n(X, *)$$

by cellular approximation. Also

$$H_n(X_{n+1}, \mathbb{Z}) \to H_n(X, \mathbb{Z})$$

is an isomorphism for example by cellular homology.

It suffices to show the Hurewicz theorem for the $n+1$-skeleton, i.e. X is a CW-complex with a single 0-cell, I many n-cells, J many $(n+1)$-cells and no cellls in any other dimension.

$$X = (\{x\} \cup_{I \times \partial D^n} D^n) \cup_{J \times \partial D^{n+1}} (J \times D^{n+1}) \cong (\bigvee_{i \in I} S^n) \cup_{J \times D^{n+1}} J \times D^{n+1}$$

We can assume that all attaching maps $\alpha \colon \partial D^{n+1} \to \bigvee_{i \in I} S^n$ are based.

We see this since $\bigvee_{i \in I} S^n$ is path connected, α is homotopic by HEP to a based map α'. Since homotopic attaching maps yield homotopy equivalent glueings.

So X can be written as a pushout

$$\begin{tikzcd}
\bigvee_{j \in J} S^n \arrow[r, "f"] \arrow[d, hook] & \bigvee_{i \in I} S^n \arrow[d] & = & X_n \\
\bigvee_{j \in J} D^{n+1} \arrow[r] & X & = & X_{n+1}
\end{tikzcd}$$

Since $\bigvee_I S^n$ and $\bigvee_J S^n$ are $(n-1)$-connected, $f \colon \bigvee_J S^n \to \bigvee_I S^n$ is $(n-1)$-connected. Also $\bigvee_J S^n \to \bigvee_I D^{n+1}$ is n-connected, as $\bigvee_I D^{n+1}$ is contractible.

By Proposition ?? $\pi_k(\bigvee_J D^{n+1}, \bigvee_J S^n) \to \pi_k(X, X_n) = \pi_k(X, \bigvee_I S^n)$ is isomorphic for $1 \leq k \leq 2n-2$ and surjective for $k = 2n-1$. In particular $\pi_{n+1}(\bigvee_J D^{n+1}, \bigvee_J S^n) \to \pi_{n+1}(X, X_n)$ is surjective. We compare the long exact homotopy sequences of the vertical pairs:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(\bigvee_J D^{n+1}, \bigvee_J S^n) \arrow[r, "\partial", "\cong"'] \arrow[d, two heads, "g_*"] & \pi_n(\bigvee_J S^n) \arrow[r] \arrow[d, "f_*"] & 0 \\
\arrow[r] & \pi_{n+1}(X, X_n) \arrow[r, "\partial"] & \pi_n(X_n) \arrow[r, "\text{surjective by cell. approx}"'] & \pi_n(X, *)
\end{tikzcd}$$

So the upper row is exact:

$$\begin{tikzcd}
\pi_n(\bigvee_J S^n) \arrow[r, "f_*"] \arrow[d, "h"] & \pi_n(\bigvee_I S^n) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "h"] & \pi_n(X) \arrow[r] \arrow[d, "h"] & 0 \\
H_n(\bigvee_J S^n, \mathbb{Z}) \arrow[r, "f_*"] & H_n(\bigvee_I S^n, \mathbb{Z}) \arrow[r] & H_n(X, \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

We claim the first 2 h are isomorphisms We have a long exact sequence by excision. For all sets I

$$\begin{tikzcd}
\bigoplus_{i\in I} \pi_n(S^n, *) \arrow[r, "\cong"] \arrow[d, "\bigoplus h"] & \pi_n(\bigvee_{i\in I} S^n, *) \arrow[d, "h"] \\
\bigoplus_{i\in I} H_n(S^n, \mathbb{Z}) \arrow[r, "\cong"] & H_n(\bigvee_I S^n, \mathbb{Z})
\end{tikzcd}$$

so $h\colon \pi_n(X,x) \to H_n(X,\mathbb{Z})$ is an isomorphism by 5-lemma. □

Some applications

Recall. If X is simply connected, tehn $H_1(X,\mathbb{Z}) \cong \pi_1(X,x)_{\mathrm{ab}} = 0$. But if X is path connceed and $H_1(X,\mathbb{Z}) = 0$, X need not be simply connected, because $\pi_1(X,x)$ could be non-trivial and perfect (= abelianization is trivial) (E.g. A_5).

Proposition 1.22 (12.7(i) for Lück)**.** Let X be simply connected, $n \geq 1$. then the following are equivalent

1. X is n-connected.
2. $H_i(X,\mathbb{Z}) = 0$ for all $2 \leq i \leq n$.

Proof. By induction on n. Nothing to show for $n = 1$. The induction step is the Hurewicz theorem $\pi_n(X,x) \cong H_n(X,\mathbb{Z})$. □

Proposition 1.23 (12.7 (ii) for Lück)**.** Let X be simply connected. Then the following are equivalent:

1. X is weakly contractible[12]
2. $H_i(X,\mathbb{Z}) = 0$ for all $i \geq 2$.

Warning. There exist acyclic spaces, i.e. non-contractible CW-complexes, path connected with $H_i(X,\mathbb{Z}) = 0$ for all $i \geq 1$.

Remark 1.24. There is a slightly better version of the Hurewicz theorem: If X is $n-1$-connected, $n \geq 2$, then $h\colon \pi_n(X,x) \to H_n(X,\mathbb{Z})$ is isomorphism and $h\colon \pi_{n+1}(X,x) \to H_{n+1}(X,\mathbb{Z})$ is surjective.

[12] All homotopy groups vanish

1.3.1 Relative Hurewicz theorem

Definition 1.25: Relative Hurewicz map

Let (X, A) be a space pair, $a \in A$. Choose a generator $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1}, \mathbb{Z})$. The relative Hurewicz homomorphism is

$$h\colon \pi_n(X, A, a) \to H_n(X, A, \mathbb{Z})$$

is defined by

$$[f\colon (D^n, S^{n-1}, z) \to (X, A, \{a\})] \mapsto h[f] \coloneqq H_n(f, \mathbb{Z})[D^n, S^{n-1}]$$

The following diagram commutes:

$$\begin{tikzcd}
\pi_n(X,a) \arrow[r] \arrow[d, "h"] & \pi_n(X,A,a) \arrow[r, "\partial"] \arrow[d, "h"] & \pi_{n-1}(A,a) \arrow[d, "h"] \\
H_n(X,\mathbb{Z}) \arrow[r] & H_n(X,A,\mathbb{Z}) \arrow[r, "\partial"] & H_{n-1}(A,\mathbb{Z})
\end{tikzcd}$$

Theorem 1.26: relative Hurewicz, (simply connected case)

Let $n \geq 2$. Let (X, A) be a space pair, such that X and A are simply connected and (X, A) is $(n-1)$-connected. Then

1. *The Hurewicz homomorphism $h\colon \pi_n(X, A, a) \to H_n(X, A, \mathbb{Z})$ is an isomorphism, and*
2. *The group $H_i(X, A, \mathbb{Z}) = 0$ for all $0 \leq i \leq n-1$.*

Proof. We will deduce this from the absolute version and some other things we already did.

By replacing X by the mapping cylinder $A \times [0,1] \cup_{A\times 1} X$ and replacing A by $A \times 0$, we can assume wlog that the inclusion $i\colon A \to X$ is a cofibration. Since A is 1-connected, and (X, A) is $(n-1)$-connected,

$$pr\colon \pi_k(X, A, a) \to \pi_k(X/A, *)$$

is bijective for $1 \leq k \leq n$ and surjective for $k = n+1$.

X/A is simply connected by the van Kampen theorem.

Since $\pi_k(X, A, *) = 0$ for $k \leq n-1$ by hypothesis, we get that $\pi_k(X/A, *) = 0$ for $k \leq n-1$. So X/A is $(n-1)$-connected. By the absolute Hurewicz theorem for X/A, $H_k(X/A, \mathbb{Z}) = 0$ for $1 \leq k \leq n-1$ and

$$\begin{tikzcd}
\pi_n(X,A,a) \arrow[r, "\cong", "pr_*"'] \arrow[d] & \pi_n(X/A,*) \arrow[d, "h"', "\cong"] \\
H_n(X,A,\mathbb{Z}) \arrow[r, "\cong", "\text{excision}"'] & H_n(X/A,\mathbb{Z})
\end{tikzcd}$$

□

Appendix

List of definitions

List of statements