

UNIVERSITÄT BONN

Notes for the lecture

# **Algebraic Topology I**

held by

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TExed by

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**Corrections and improvements**

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# **Lecture**

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[13.10.2025, Lecture 1]

## Organizational

For this term we will be doing unstable homotopy theory. Next term we will be doing stable homotopy theory. Note that there were 2 previous courses. Note that all important information is shared on the website <https://www.math.uni-bonn.de/people/schwede/at1-ws2526>. You can sign up for the previous topology courses und see the lecture videos for these courses there.

There are no lecture notes for this lecture specifically, but some similar materials are linked on the webpage.

Exercise sheets will be uploaded fridays and handed in 11 days later via eCampus. Registration for eCampus opens at 4 today.

For exam admission you will have to score 50% of the points on the exercise sheets and have presented 2 exercises in tutorial.

The first exam will be written in the last week of semester.

I fear I will not be able to copy pictures here.

## 1.1 Blakers-Massey theorem/Homotopy excision

We start with a reminder on relative homotopy groups.

### Definition 1.1: Relative Homotopy Groups

Let  $(X, A)$  be a space pair i.e.  $A$  is a subspace of a topological space  $X$ . We write

$$\begin{aligned} I &= [0, 1] \quad I^n = [0, 1]^n \text{ the } n\text{-cube} \\ \partial(I^n) &= \text{ boundary of } I^n \\ I^{n-1} &\subseteq I^n \end{aligned}$$

via Inclusion on the first  $n - 1$  coordinates.

$$J^{n-1} = I^{n-1} \times \{1\} \cup (\partial I^{n-1}) \times [0, 1]$$

He draws a picture for  $n = 2$ .

For  $n \geq 1$  the  $n$ -th relative homotopy groups  $\pi_n(X, A, x)$  is the set of triple homotopy classes of trile maps  $x \in A \subseteq X$

$$(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, \{x\})$$

where a triple map takes each subset on the left into the subset on the right. A triple-homotopy must also conserve these conditions.

For  $n \geq 2$  or  $n = 1$  and  $A = \{x\}$  the set  $\pi_n(X, A, \{x\})$  has a group structure by concatenation in the first coordinate. He again draws a picture.

The group structure is commutative if  $n \geq 3$  or  $n = 2$  and  $A = \{x\}$ .

### Definition 1.2: n-Connectedness

Let  $n \geq 0$ . A space pair  $(X, A)$  is  $n$ -connected, if the following equivalent conditions hold:

1. For all  $0 \leq q \leq n$  every pair map  $(I^q, \partial I^q) \rightarrow (X, A)$  is homotopic relative  $\partial(I^q)$  to a map with image in  $A$
2. For all  $a \in A$ ,  $\text{incl}_*: \pi_q(A, a) \rightarrow \pi_q(X, a)$  is bijective for  $q \leq n$  and surjective for  $q = n$ .
3.  $\pi_0(A) \rightarrow \pi_0(X)$  is bijective and for all  $1 \leq q \leq n$  the relative homotopy group

$$\pi_q(X, A, x) \cong 0$$

*Proof.* You proof the equivalence using the LES<sup>1</sup> of homotopy groups. □

Let  $Y$  be a space,  $Y_1, Y_2$  open subsets of  $Y = Y_1 \cup Y_2$ ,  $Y_0 := Y_1 \cap Y_2$ .

Excision in homology shows that for all abelian groups  $B$ ,  $i \geq 0$

$$H_i(Y_2; Y_0, B) \rightarrow H_i(Y, Y_1; B)$$

is an isomorphism.

Excision does not generally hold for homotopy groups, i.e. for  $x \in Y_0$

$$\text{incl}_*: \pi_i(Y_2, Y_0; x) \rightarrow \pi_i(Y, Y_1; x)$$

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<sup>1</sup>Long exact sequence

is **not** generally an isomorphism.

"Blakers Massey theorem implies that excision holds for homotopy groups in a range."<sup>2</sup>

### Theorem 1.3: Blakers Massey

Let  $Y$  be a space,  $Y_1, Y_2$  open subsets with  $Y = Y_1 \cup Y_2$ ,  $Y_0 := Y_1 \cap Y_2$ . Let  $p, q \geq 0$ , such that for all  $y \in Y_0$

$$\pi_i(Y_1, Y_0, y) = 0 \text{ for all } 1 \leq i \leq p$$

and

$$\pi_i(Y_2, Y_0, y) = 0 \text{ for all } 1 \leq i \leq q$$

Then for all  $y \in Y_0$ , the map

$$\text{incl}_*: \pi_i(Y_2, Y_0, y) \rightarrow \pi_i(Y, Y_1, y)$$

is an isomorphism for  $1 \leq i < p + q$  and surjective for  $i = p + q$ . He notes how the referenced literature uses different indices. They have proofs in more detail and pictures, however Lücks script contains typos

*Proof.* Schwede explains he doesn't like the proof, it is too technical and not very enlightening.

We define what cubes are

Cubes in  $\mathbb{R}^n$ ,  $n \geq 1$ .  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$  the "lower left corner of the cube"

$\partial \in \mathbb{R}_{\geq 0}$  "side length of the cube"

$L \subset \{1, \dots, n\}$  "relevant dimensions"

$$W = W(a, \delta, L) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_i \leq x_1 \leq a_i + \delta \text{ for all } i \in L, x_i = a_i \text{ for all } i \in \{1, \dots, n\} \setminus L\}$$

<sup>2</sup> A face  $W'$  of  $W$  is a subset of the form

$$W' = \{x \in W : x_i = a_i \text{ for all } i \in L_0, x_i = a_i + \delta \text{ for all } i \in L_1\}$$

for some subsets  $L_0, L_1 \subseteq L$

Let  $1 \leq p \leq n$  we define two subsets of a cube  $W = (a, \delta, L)$ .

$$K_p(W) = \{x \in W : x_i < a_i + \delta/2 \text{ for at least } p \text{ values of } i \text{ in } L\}$$

We call these " $p$  small coordinates"

$$G_p(W) = \{x \in W : x_i > a_i + \delta/2 \text{ for at least } p \text{ coordinates } i \text{ in } L\}$$

these are " $p$  big coordinates".

For  $p > \dim(W)$ ,  $K_p(W) = G_p(W) = \emptyset$ . If  $p + q \geq \dim(W)$ , then  $K_p(W) \cap G_q(W) = \emptyset$ .

He draws pictures.

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<sup>2</sup>W weil Würfel

**Lemma 1.4: 1.14**

*It is Lemma 1.14 in Lücks Script.*

Let  $(Y, A)$  be a space pair,  $W \subseteq \mathbb{R}^n$  a cube,  $f: W \rightarrow Y$  continuous. Suppose that for some  $p \leq \dim(W)$ ,  $f^{-1}(A) \cap W' \subseteq K_p(W')$  for all proper<sup>1</sup> faces  $W'$  of  $W$ .

Then there is a continuous map  $g: W \rightarrow Y$  homotopic to  $f$  relative  $\partial W$  such that all  $g^{-1}(A) \subseteq K_p(W)$

---

<sup>1</sup>subcube of the boundary

*Proof.* Wlog:  $W = I^n = W(0, 1, \{1, \dots, n\})$

Let  $I_2^n$  be the subcube  $[0, 1/2]^n$ . He draws a picture.  $x_4 = (1/4, \dots, 1/4) \in I_2^n$ . We define a continuous map  $h: I^n \rightarrow I^n$  by radical projection away from  $x_4$ . Picture. Let  $r(y)$  be the ray from  $x$  to  $y$ . We map all of  $r(y) \cap I^n \setminus I_2^n$  to the intersection point of  $r(y)$  and  $\partial I^n$  and the rest linearly extends as far as required.<sup>3</sup>

Obviously<sup>4</sup>  $h$  is homotopic relative boundary  $\partial I^n$  to the identity.

We set  $g: f \circ h: I^n \rightarrow Y$ , which is then homotopic relative  $\partial(I^n)$ <sup>5</sup> to  $f$ .

It remains to show that  $g^{-1}(A) \subseteq K_p(W)$ . Consider  $z \in I^n$  with  $g(z) \in A$ .

**Case 1** for all  $i = 1, \dots, n$ ,  $z_i < 1/2$ , i.e.  $z \in I_2^n$ , then  $z \in K_n(I^n) \subseteq K_p(I^n)$ .

**Case 2** There is an  $i \in \{1, \dots, n\}$ , s.t.  $z \geq 1/2$ . Then  $h(z) \in \partial(I^n)$ . Let  $W'$  be some proper face of  $W$ , with  $h(z) \in W'$ . Since  $f(h(z)) = g(z) \in A$ , by hypothesis,  $h(z) \in K_p(W')$ , so  $h(z) < 1/2$  for at least  $p$  coordinates. By expansion<sup>6</sup> property of  $h$ , also  $p$  coordinates of  $z$  are small coordinates.

□

**Proposition 1.5.** <sup>7</sup> Let  $Y_1, Y_2$  be open subsets of  $Y$ ,  $Y_0 := Y_1 \cap Y_2$ . Suppose that  $(Y_1, Y_0)$  is  $p$ -connected,  $(Y_2, Y_0)$  is  $q$ -connected. Let  $f: I^n \rightarrow Y$  be continuous. Let  $\mathcal{W} = \{W\}$  be a subdivision of  $I^n$  into subcubes of the same side length s.t. for all  $W \in \mathcal{W}$   $f(W) \subseteq Y_1$  or  $f(W) \subseteq Y_2$ . Then there is a homotopy  $h: I^n \times I \rightarrow Y$  with  $h_0 = f$  such that for all  $W \in \mathcal{W}$ :

1. If  $f(W) \subseteq Y_j$ ,  $j \in 0, 1, 2$ , then  $h_t(W) \subseteq Y_j$  for all  $t \in [0, 1]$
2. If  $f(W) \subseteq Y_0$ , then  $h_t|_W = f|_W$ , i.e.  $h$  is constant on  $W$ .
3. If  $f(W) \subseteq Y_1$ , then  $h_1^{-1}(Y_1 \setminus Y_0) \subseteq K_{p+1}(W)$ .
4. If  $f(W) \subseteq Y_2$ , then  $h_1^{-1}(Y_2 \setminus Y_0) \subseteq G_{q+1}(W)$ .

*Proof.* We let  $C^k \subseteq I^n$  be the union of all cubes in  $\mathcal{W}$  of dimension at most  $k$ . We construct homotopies  $h[k]: C_k \times I \rightarrow Y$ , such that for all  $W \in \mathcal{W}$ ,  $W \subseteq C_k$  conditions 1. to 4. hold, and  $h[k]$  is constant on  $C_{k-1} \times I$ . Then the final  $h[n]$  does the job.

**Note.** If  $W \in \mathcal{W}$  and  $f(W) \subseteq Y_0$  and 2. holds, then also 3. and 4. hold.

$$h^{-1}(Y_1 \setminus Y_0) = h_1^{-1}(Y_2 \setminus Y_0) = \emptyset$$

If  $W \in \mathcal{W}$ , is such that  $f(W) \subseteq Y_1$  and  $f(W) \subseteq Y_2$ , then  $f(W) \subseteq Y_1 \cap Y_2 = Y_0$ . So each  $W \in \mathcal{W}$  is in exactly one of the following cases

- $f(W) \subseteq Y_0$

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<sup>3</sup>I hope this description is clear, hard without the picture.

<sup>4</sup>Meaning he's too lazy to come up with formulas for the map.

<sup>5</sup>I am very inconsistent in remembering these parentheses with the boundary operator. Just imagine it always being as here.

<sup>6</sup>no idea if this is the word he wrote

<sup>7</sup>11.5 in Lücks notes

- $f(W) \subseteq Y_1$  and  $f(W) \not\subseteq Y_1$
- $f(W) \subseteq Y_2$  and  $f(W) \not\subseteq Y_2$

Inductive construction  $k = 0$ , i.e. vertexes of the cubes  $w \in \mathcal{W}$ . If  $w \in Y_2$ , take  $h[0]_t = \text{const}_{w_0}$ .

Suppose  $f(W_0) \in Y_1$ , but  $f(w_0) \notin Y_2$ . Since  $Y_1, Y_0$  is 0-connected, there is a path  $\pi: I \rightarrow Y_1$  from  $w_0$  to a point in  $Y_0$ . We take  $h[0]$  as the path on  $w_0$ . Analogous if  $f(W_0) \in Y_2 \setminus Y_1$ .

**Inductive Step** Let  $W \in \mathcal{W}$  be a cube of exact dimension  $k$ . Then  $\partial W = W \cap C_{k-1}$ . Since  $(W, \partial W)$  has the HEP, we can extend the previous homotopy  $h[k-1]|_{\partial W}$  to some homotopy on  $W$  relative to  $f|_W$ . Let this be  $h'[k]: C_k \times I \rightarrow Y$ : this satisfies conditions 1. and 2. but not yet 3. and 4.

We produce another homotopy  $h[k]''$  and set  $h[k] = h[k]' * h[k]''$ .

Consider a cube  $W \in \mathcal{W}$  of dimension  $k$ .

If  $f(W) \subseteq Y_0$  set  $h[k]''$  as the constant homotopy on  $W$ .

If  $h[k]'_1(W) \subseteq Y_1$ , but  $h[k]'_1(W) \not\subseteq Y_2$  there is a homotopy relative  $\partial W$  from  $h[k]'_1$  to a map  $f_1(W) \subseteq Y_0$ .

If  $k = \dim(W) > p$  the we use the lemma 1.4 for  $f = h[k]'_1|_W$  and the resulting homotopy is  $h[k]''|_W$ .

If  $h[k]'_1(W) \subseteq Y_2$  but  $h[k]'_1(W) \not\subseteq Y_1$ , use the complement case of the lemma<sup>8</sup> □

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[13.10.2025, Lecture 1]
[15.10.2025, Lecture 2]

Now for the actual proof of Blakers Massey Let  $F(Y_1, Y, Y_2) = Y_1 \times_Y Y^{[0,1]} \times_Y Y_2$ .

Let  $F(Y_1, Y_1, Y_0)$  be the subspace of those  $w \in F(Y_1, Y, Y_2)$  such that  $w([0, 1]) \subseteq Y_1$ .

**Proposition 1.6.** Assume that  $(Y_1, Y_0)$  is  $p$ -connected,  $(Y_2, Y_0)$  is  $q$ -connected. Then the pair  $(F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$  is  $(p + q - 1)$ -connected.

*Proof.* We consider a map of pairs

$$\phi: (I^n \partial I^n) \rightarrow (F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$$

for  $1 \leq n \leq p + q + 1$ . We want to homotop  $\phi$  relative  $\partial I^n$  to a map with image in  $F(Y_1, Y_1, Y_0)$ . We use the adjoint

$$\text{maps}(X \times [0, 1], Z) \cong \text{maps}(X, Z^{[0,1]})$$

We let  $\Phi: I^n \times I \rightarrow Y$  be the adjoint of  $\phi$ , this is *admissible*, in the sense that

1.  $\Phi(x, 0) \in Y_1$  for all  $x \in I^n$
2.  $\Phi(x, 1) \in Y_2$  for all  $x \in I^n$
3.  $\Phi(x, t) \in Y_1$  for all  $x \in \partial I^n$ ,  $t \in [0, 1]$

We want a homotopy of  $\Phi$  through admissible maps to a map  $\Phi': I^n \times I \rightarrow Y$  such that  $\text{Im}(\Phi') \subseteq Y_1$ .

Apply Proposition 11.5 to  $\Phi$  (with  $n+1$  instead of  $n$ ). This gives a homotopy through admissible maps to  $\Psi = g$ . Let  $h: I^n \times I \times I \rightarrow Y$  be a homotopy witnessing this.  $h_0 = \Phi, h_1 F\Psi$ .

Consider the projection  $\text{pr}: I^n \times I \rightarrow I^n$  away from the last coordinate.

**Claim.** The image under  $\text{pr}$  of  $\Psi^{-1}(Y \setminus Y_1)$  and  $\Psi^{-1}(Y \setminus Y_2)$  are disjoint.

Suppose there is  $y \in I^n$  in the intersection of the preimages, so  $z_1 \in \Psi^{-1}(Y \setminus Y_1)$ ,  $z_2 \in \Psi^{-1}(Y \setminus Y_2)$  s.t  $\text{pr}(x_1) = y = \text{pr}(z_2)$ . Let  $W = I^{n+1}$  be a subcube of the subdivision of Proposition 11.5 such

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<sup>8</sup>rest of the proof next lecture. I am very sure some words won't make sense, as they were unreadable on the board.

that  $z_1 \in W$ . Since  $z_1 \in \Psi^{-1}(Y \setminus Y_1)$ ,  $z_1 \in K_{p+1}(W)$ , so  $y = \text{pr}(z_1) \in K_p(\text{Im}(W))$ . Analogous  $y = \text{pr}(z_2) \in G_q(\text{Im}(W))$ . Since  $p + q > n$ , this is a contradiction and the claim is proven.

**Claim.** The intersection of  $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$  with  $\partial I^n$  is empty since  $\Psi$  is admissible, and thus  $\Psi(\partial I^n) \subseteq Y_1$ . So  $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$  and  $\text{pr}(\Psi^{-1}(Y \setminus Y_2)) \cup (\partial I^n)$  are two disjoint closed subsets of the compact space  $I^n$ .

So there is a continuous separating function  $\tau: I^n \rightarrow [0, 1]$ , s.t.  $\tau \equiv 0$  is  $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$  and  $\tau \equiv 1$  is  $\partial I^n \cup \text{pr}(\Psi^{-1}(Y \setminus Y_2))$

We define another homotopy starting with  $\Psi$  by

$$h: (I^n \times I) \times I \rightarrow Y$$

by

$$((x, t), s) \mapsto \Psi(x, (1 - s)t + s \cdot t\tau(x))$$

This is

- Homotopy through admissible maps
- $h(x, t, 0) = \Psi(x, t)$

**Claim.**  $h(\_, \_, 1)$  has image in  $Y_1$ .

$$h(x, t, 1) = \Psi(x, t \cdot \tau(x))$$

- if  $x \in \Psi^{-1}(Y \setminus Y_1)$ , then  $\tau(x) = 0$ ,  $h(x, t, 1) = \Psi(x, 0) \in Y_1$ .
- if  $x \in \Psi^{-1}(Y_1)$ , then by admissability  $\Psi(x, t \cdot \tau(x)) \in Y$ .

□

For the actual proof  $(Y_1, Y_0)$   $p$ -connected  $(Y_2, Y_0)$   $q$ -connected

some diagram about something being Serre fibration

We compare two Serre fibrations

$$F(\{y_0\}, Y_1, Y_0) \quad F(\{y_0\}, Y, Y_2)$$

$$F(Y_1, Y_1, Y_0) \longrightarrow F(Y_1, Y, Y_2)$$

$$Y_1$$

$$Y_1$$

partial 5-lemma shows that also the pair

$$(F(\{y_0\}, Y, Y_2), F(\{y_0\}, Y_1, Y_0))$$

is  $(p + q - 1)$ -connected.

The following square commutes for all  $n \geq 1$

$$\begin{array}{ccc} \pi_{n-1}(F(\{y_0\}, Y_1, Y_0), \text{const}_{y_0}) & \longrightarrow & \pi_{n-1}(F(\{y_0\}, Y, Y_1), x) \\ \downarrow \cong & & \downarrow \cong \\ \pi_n(Y_1, Y_0, y_0) & \xrightarrow{\text{id}_*} & \pi_n(Y, Y_1, y_0) \end{array}$$

□

## 1.2 Freudenthal suspension theorem

Section 6.4 in tom Deicks book

### Definition 1.7

Let  $X$  be a based space. The *unreduced suspension* is  $X^\diamond = X \times [-1, 1] / \sim$  where  $\sim$  identifies all points with second variable  $-1$  to  $S$  and all points with second variable  $1$  to  $N$ . We use  $S$  as the base point of  $X^\diamond$ .<sup>1</sup>

<sup>1</sup>south and north pole

**Note.** If  $X$  is well based, i.e.  $\{x_0\} \hookrightarrow X$  has the HEP, then the quotient map

$$X^\diamond \sigma X = \frac{X \times [0, 1]}{X \times \{0, 1\} \cup \{x_0\} \times [0, 1]}$$

is a homotopy equivalence.

The suspension homomorphism  $S: \pi_k(X, x_0) \rightarrow \pi_{k+1}(X^\diamond, S)$  for  $k \geq 1$  is  $S[f: S^k \rightarrow X] := [f^\diamond: S^{k+1} = (S^k)^\diamond \rightarrow X^\diamond]$

### Theorem 1.8: Freudenthal suspension theorem

Let  $X$  be an  $(n - 1)$ -connected space,  $n \geq 1$ . Then the suspension homomorphism is

- bijective for  $i \leq 2n - 2$
- surjective for  $i = 2n - 1$

*Proof.* We cover  $X^\diamond$  by  $Y_1 = X^\diamond \setminus \{N\}$ ,  $Y_2 = X^\diamond \setminus \{S\}$ . We use repeatedly that  $Y_1$  and  $Y_2$  are contractible. Then  $X \rightarrow Y_1 \cap Y_2 = Y_0$  is a homotopy equivalence. We claim without proof that the following diagram commutes:

I couldn't keep up.

So we may show that  $\pi_{i+1}(Y_2, Y_0, x_0) \rightarrow \pi_{i+1}(Y, Y_1, y_0)$  is bijective for  $i \leq 2n - 2$  and surjective for  $i \leq 2n - 1$ .

Because  $X$  is  $(n - 1)$ -connected

$$\pi_{k+1}(Y_1, Y_0, x_0) \xrightarrow[\partial]{\cong} \pi_k(Y_0, x_0)$$

So  $\pi_i(Y_1, Y_0, x_0) = 0$  for  $i \leq n$ , i.e.  $(Y_1, Y_0)$  is  $n$ -connected. Also  $(Y_2, Y_0)$  is  $n$ -connected.

BM gives  $\pi_k(Y_2, Y_0, y) \rightarrow \pi_k(X^\diamond, Y_1, y)$  is bijective for  $k < 2n - 1$  and surjective for  $k = 2n$ .

Setting  $i = k - 1$  ends the proof. □

Take  $X = S^n$

**Corollary 1.9.** The suspension homomorphism

$$S: \pi_i(S^n, *) \rightarrow \pi_{i+1}(S^{n+1}, *)$$

is bijective for  $i \leq 2n - 2$  and surjective for  $i = 2n - 1$ .

**Corollary 1.10.** For all  $n \geq 1$ ,  $\pi_n(S^n, *) \cong \mathbb{Z}$ , generated by the class of  $\text{id}_{S^n}$ . Moreover  $\deg: \pi_n(S^n, *) \rightarrow \mathbb{Z}$  is an isomorphism.

*Proof.* Induction on  $n$ . For  $n = 1$  we have this by covering theory.

$n \geq 1$  By Freudenthal, the suspension homomorphism

$$S: \pi_n(S^n, *), \rightarrow \pi_{n+1}(S^{n+1}, *)$$

is surjective. The composite

$$\pi_n(S^n, *) \xrightarrow{S} \pi_{n+1}(S^{n+1}, *) \xrightarrow{\text{deg}} \mathbb{Z}$$

is bijective by induction. So  $S$  is also injective and deg in one dimension higher is also an isomorphism.  $\square$

**Recall.**  $\pi_3(S^2, *) \cong \mathbb{Z}\{\eta\}$ , where  $\eta: S^3 \rightarrow S^2$  is the Hopf map.

**Proposition 1.11.** For  $n \geq 3$ ,  $\pi_{n+1}(S^n, *) = \mathbb{Z}/2\{S^{n-2}(\eta)\}$  cycles of order two.

*Proof.* Only partial. For  $n \geq 2$ ,  $S: \pi_{n+1}(S^n, q) \mapsto \pi_{n+2}(S^{n+1}, *)$  is surjective.

$$\mathbb{Z}\{\eta\} = \pi_3(S^2, *) \rightarrow \pi_4(S^3, *) \xrightarrow[\cong]{S} \pi_5(S^4, *) \dots$$

**Claim.**  $S(2\eta) = 0$  in  $\pi_4(S^3, *)$ . Which gives  $\pi_{n+1}(S^n, *)$  is either trivial or order 2.

Consider the commutative square

I did not manage to copy. Something complex conjugation

$$\Rightarrow [\eta] = [\eta \circ \text{complex conjugation}] = [d \circ \eta] \text{ in } \pi_3(S^2, *) \cong \mathbb{Z}.$$

If  $f: S^n \rightarrow S^n$  has degree  $k$ , then precomposition of  $\pi_n(X, x) \rightarrow \pi_n(X, x)$  is multiplication by  $k$ .

Let  $c: S^1 \rightarrow S^1$  be any map of degree  $-1$ . Then  $c \wedge S^2, S^1 \wedge d$ : both have degree  $-1$ . So  $c \wedge S^2 \sim S^1 \wedge d$

two diagrams I did not copy.

We get  $S(\eta) = S(d \circ \eta) = S(\eta \circ (c \wedge S^1)) = S(-\eta) = -S(\eta)$  and hence  $2 \cdot S(\eta) = 0$ .  $\square$

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[13.10.2025, Lecture 2]
[20.10.2025, Lecture 3]

We make a bit of a preview<sup>9</sup> for stable homotopy theory. Following Lücks notes today rather closely

### Definition 1.12

Let  $X$  be a based space. The  $n$ -th stable homotopy group of  $X$  is the colimit  $\pi_n^S(X, *)$

$$\pi_n(X, *) \xrightarrow{S} \pi_{n+1}(S^1 \wedge X, *) \xrightarrow{S} \pi_{n+2}(S^2 \wedge X, *) \dots$$

Since  $S^k \wedge X$  is  $(k-1)$ -connected, the suspension stabilises (i.e.  $S$  is isomorphism for  $\pi_{n+k}(S^k \wedge X, *)$  onwards).

$\pi_n^S()$  is a functor from based spaces to abelian groups. and it is homotopy invariant. It comes with a natural transformation  $\pi_n(X, *) \rightarrow \pi_n^S(X, *)$ .

**Preview:**  $\{\pi_n^S\}_{n \in \mathbb{Z}}$  form a generalized homology theory on based spaces.

If  $(X, A)$  is a pair of based space,  $x \in A \subseteq X$ , we define

$$\pi_n^S(X, A, *) := \pi_n^S(X \cup_A CA, *)$$

<sup>9</sup>He just spammed random stuff, I don't think I copied enough for it to make sense.

The map collapsing  $X$  is

$$X \cup_A CA \rightarrow \frac{X \cup_A CA}{X} \cong \frac{CA}{A} \cong A^\diamond$$

induces an connecting homomorphism

$$\partial: \pi_n^S(X, A, x) = \pi_n^S(X \cup_A CA, *) \xrightarrow{p_*} \pi_n^S(A^\diamond) \rightarrow \pi_n^S(A \wedge S^1) \xrightarrow{\cong} \pi_{n-1}^S(A)$$

The following sequence will then be exact:

$$\dots \pi_n^S(A, x) \xrightarrow{\text{incl}*} \pi_n^S(X, x, *) \xrightarrow{\text{incl}*} \pi_n^S(X, A, *) \xrightarrow{\partial} \pi_{n-1}^S(A, *) \rightarrow \dots$$

Stable stems are the special case

$$\pi_n^S(S^0) = \text{colim}(\pi_n(S^0, *) \rightarrow \pi_{n+1}(S^1, *) \rightarrow \dots)$$

n	0	1	2	3	4	5	6	7	8
$\pi_n^S = \pi_n^S(S^0)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
Generator	id	$\eta$	$\eta^2$	$\nu, \eta^3 = 12\nu$	-	-	$\nu^2$	$\sigma$	$\eta\sigma, \epsilon$

There is a graded-commutative ring structure on  $\pi_*^S = \{\pi_n^S\}_{n \in \mathbb{Z}}$ .  $\pi_n^S \times \pi_m^S \rightarrow \pi_{n+m}^S$

$$[f: S^{n+k} \rightarrow S^k \times S^k] \times [g: S^{m+l} \rightarrow S^l] = [SWn + m + k + l \rightarrow S^{n+k} \wedge S^{m+l} \xrightarrow{f \wedge g} S^{k+l}]$$

I missed a bit more

Nishidas theorem says: every positive dimensional element of  $\pi_*^3$  is nilpotent.

From Serre spectral sequence we will see: For  $m > n \geq 1$   $\pi_m(S^n, *)$  is finite except  $\pi_{4k-1}(S^{2k}, *) \cong \mathbb{Z} \oplus$  some finite group.

This was exercise 5.2 of last summer term: Hopf invariant  $h: \pi_{2k-1}(S^k, *) \rightarrow \mathbb{Z}$  for  $[f: S^{2k-1} \rightarrow S^k]$

$$H^*(S^k \cup_f D^{2k}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, k, 2k \\ 0 & \text{else} \end{cases}$$

we have  $H^k(Cf, \mathbb{Z}) = \mathbb{Z}\{a\}$   $H^{2k}(Cf, \mathbb{Z}) \cong \mathbb{Z}\{b\}$  and  $a \cup a = h(f) \cdot b$ .

In general we can look at

$t_k: S^{2k-1} \rightarrow S^k \vee S^k$  the attaching map of the  $2k$ -cell in the product minimal CW-structure of  $S^k \times S^k$ . And we get  $h(t_k) = 2$ .

The Hopf invariant 1 theorem tells us when we can realize Hopf invariant 1 and here we see that we always find Hopf invariant 2.

### 1.3 Hurewicz-theorem

This is a "trivial"<sup>10</sup> Corollary of the Blakers-Massey theorem.

<sup>10</sup>meaning 1.5 lectures of intermediate steps

**Definition 1.13**

Let  $X$  be a based space. Choose an orientation of  $S^n$ ,  $n \geq 1$ , i.e.  $[S^n] \in H_n(S^n, \mathbb{Z})$ . The Hurewicz homomorphism

$$h: \pi_n(X, x) \rightarrow H_n(X; \mathbb{Z})$$

is defined by

$$h[f: S^n \rightarrow X] = H_n(f, \mathbb{Z})[S^n].$$

This is a group homomorphism.

**Group Homomorphism:** Let  $\Delta: S^n \rightarrow S^n \vee S^n$  be a pinch map. Group structure on  $\pi_n(X, x)$  is given by  $[f] + [g] = [S^n \xrightarrow{\Delta} S^n \vee S^n \xrightarrow{f+g} X]$ . Then

A diagram is missing here.

$$([f] + [g])_* = [f]_* + [g]_*: H_n(S^n) \rightarrow H_n(X)$$

So Hurewicz is a homomorphism.

**Lemma 1.14**

Let  $w: [0, 1] \rightarrow X$  be a path from  $x = w(0)$  to  $y = w(1)$ . Then the following commutes:

$$\begin{array}{ccc} \pi_n(X, x) & & \\ \downarrow \cong & \searrow h & \\ \pi_n(X, y) & \xrightarrow{h} & H_n(X, \mathbb{Z}) \end{array}$$

*Proof.*  $f$  and  $f * w$  are freely homotopic  $h_n(\_, \mathbb{Z})$  is homotopy invariant □

For  $n = 1$  we have Poincaré:  $X$  based path connected,  $h: \pi_1(X, x) \rightarrow H_1(X, \mathbb{Z})$  is surjective with kernel the commutator subgroup or equivalently

$$\pi_1(X, x)_{ab} \xrightarrow{\cong} H_1(X, \mathbb{Z})$$

**Theorem 1.15: Hurewicz**

Let  $X$  be an  $(n - 1)$ -connected space,  $n \geq 2$ . Then the Hurewicz homomorphism

$$h: \pi_n(X, x) \rightarrow H_n(X, \mathbb{Z})$$

is an isomorphism.

We will need to work a bit for the proof.

**Proposition 1.16** (11.9 in Lücks notes). Let  $m, n \geq 0$ , let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow \bar{i} \\ X & \xrightarrow{\bar{f}} & Y \end{array}$$

be a pushout square of spaces. Suppose  $i: A \rightarrow X$  is a *cofibration*, i.e. a closed embedding with the HEP.

1. If  $f$  is  $n$ -connected, then so is  $\bar{f}$
2. If  $f$  is  $n$ -connected, and  $i$  is  $m$ -connected, then for all  $a \in A$ ,  $\pi_i(f, \bar{f}): \pi_i(X, A, a) \rightarrow \pi_i(Y, B, f(a))$  is bijective for  $1 \leq i < m + n$  and surjective for  $i = m + n$ .

*Proof.* "Basically reduction to Blakers-Massey"

We can replace  $X, B$  and  $Y$  up to homotopy equivalence by appropriate mapping cylinders:

$$\text{cyl}(f) = A \times [0, 1] \cup_{A \times 1, f} B$$

We get a homotopy commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & \text{Cyl}(f) & & \\ \downarrow & & \downarrow & \searrow \sim & \\ \text{Cyl}(i) & \longrightarrow & \text{cyl}(i) \cup_A \text{cyl}(f) & & B \\ & \swarrow \sim & & \searrow \sim & \downarrow \bar{i} \\ & & X & \xrightarrow{\bar{f}} & Y \end{array}$$

This diagram does not commute. The homotopy equivalences shown are not trivial.

We apply Blakers-Massey then to the following open subspace of  $W = \text{cyl}(i) \cup_A \text{cyl}(f)$ .

$$W_2 = [0, 1/2] \times A \cup_{A \times 0} \text{cyl}(f), \quad W_1 = \text{cyl}(i) \cup_{A \times 0} [0, 1/2]. \quad W_0 = W_1 \cap W_2 \sim A$$

Now having  $f$  is  $n$ -connected gives  $(W_2, W_0)$  is  $n$ -connected.  $i$  being  $m$ -connected gives  $(W_1, W_0)$  is  $m$ -connected.

Now BM gives  $\pi(W_1, W_0, a) \rightarrow \pi_i(W, W_2, a)$  is bijective for  $1 \leq i \leq m + n - 1$  and surjective for  $i = m + n$ .

Using the homotopy equivalences, we translate this back. □

**Proposition 1.17** (11.1 in Lücks). Let  $m, n \geq 0$ , let  $c: A \rightarrow X$  be a cofibration. Suppose that  $i$  is  $m$ -connected and  $A$  is  $n$ -connected. Then

$$\pi_k(\text{pr}): \pi_k(X, A, a) \rightarrow \pi_k(X/A, *)$$

for all  $a \in A$  is bijective for  $1 \leq k \leq m + n$  and surjective for  $k = m + n + 1$

*Proof.* We consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{j} & C(A) = A \times [0, 1]/A \times 1 \\ \downarrow i & & \downarrow \\ X & \longrightarrow & X \cup_A CA \xrightarrow{\sim} X/A \end{array}$$

**Remark 1.18.**  $X$   $k$ -connected  $\Leftrightarrow \{*\} \rightarrow X$   $k$ -connected  $\Leftrightarrow X \rightarrow \{*\}$  is  $(n + 1)$ -connected. □

**Proposition 1.19** (11.12 for Lück). Also an exercise (1.2) Let  $X, Y$  be well-pointed spaces. Let  $m, n \geq 1$ . Let  $X$  be  $m$ -connected,  $Y$   $n$ -connected. Then

1. The inclusion  $X \vee Y \rightarrow X \times Y$  induces isomorphisms

$$\pi_k(X \vee Y, *) \rightarrow \pi_k(X \times Y, *)$$

for all  $0 \leq k \leq m + n$ .

2.  $\pi_k(X \times Y, X \vee D, *)$  and  $\pi_k(X \wedge @, *)$  are trivial for all  $0 \leq k \leq m + n + 1$ .  
3. The map  $(pr_*^X, pr_*^Y): \pi_k(X \vee Y, x) \rightarrow \pi_k(X, x_0) \times \pi_k(Y, y_0)$  is an isomorphism for all  $1 \leq k \leq m + n$ .

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[20.10.2025, Lecture 3]
[22.10.2025, Lecture 4]

**Proposition 1.20.** Let  $n \geq 2$ . Let  $\{X_i\}_{i \in I}$  be a family of well-pointed,  $(n - 1)$ -connected spaces. Then the canonical map

$$\bigoplus_{i \in I} \pi_n(X_i, *) \rightarrow \pi_n(\bigvee_{i \in I} X_i, *)$$

is an isomorphism.

*Proof.* **Step 1** If  $I$  is finite, we do Induction on  $|I|$ . Nothing to show if  $I = \emptyset$ ,  $|I| = 1$ . Let  $|I| \geq 2$ . Wlog  $I = \{1, 2, \dots, k\}$ ,  $k \geq 2$ .

By Proposition 1.19,  $\pi_n(\bigvee_{i=1, \dots, k-1} X_i, *) \oplus \pi_n(X_k, *) \xrightarrow{\cong} \pi_n(\bigvee_{i=1, \dots, k} X_i, *)$  and the first part is isomorphic to  $\bigoplus_{i=1, \dots, k} \pi_n(X_i)$  by induction.

**Case 2**  $I$  is infinite: We first show injectivity. The projection

$$p_j: \bigvee_{i \in I} X_i \rightarrow X_j$$

induces an homomorphism

$$(p_j)_*: \pi_n(\bigvee_I X_i, *) \rightarrow \pi_n(X_j, *)$$

For varying  $j \in I$ , these assemble into a homomorphism

$$\bigoplus_{i \in I} \pi_n(X_i, *) \xrightarrow{\text{can}} \pi_n(\bigvee_{i \in I} X_i, *) \xrightarrow{((p_j)_*)} \prod_{j \in I} \pi_n(X_j, *)$$

and the composition is inclusion of sum into product, hence injective. So also the first map is injective.

For surjectivity let  $f: S^n \rightarrow \bigvee_{i \in I} X_i$  be a continuous map that represents a class in  $\pi_n(\bigvee_{i \in I} X_i)$ . Because  $S^n$  is compact, there is a finite subset  $J \subseteq I$  s.t.  $\text{Im}(f) \subseteq \bigvee_{i \in J} X_i$ <sup>11</sup>. This implies  $[f] \subseteq \pi_n(\bigvee_{j \in J} X_j, *) \rightarrow \pi_n(\bigvee_{i \in I} X_i, *)$ .

$$\begin{array}{ccc} \pi_n(\bigvee_{j \in J} X_j, *) & \longrightarrow & \pi_n(\bigvee_{i \in I} X_i, *) \\ \text{Case 1} \uparrow & & \text{can} \uparrow \\ \bigoplus_{j \in J} \pi_n(X_j, *) & \longrightarrow & \bigoplus_{i \in I} \pi_n(X_i, *) \end{array}$$

so  $[f]$  is in the image of the canonical map.

□

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<sup>11</sup>he explains, why this is not easy to see. But it is point-set topology, so we won't do it.

**Theorem 1.21: Hurewicz**

Let  $n \geq 2$ . Let  $X$  be  $(n-1)$ -connected base space. Then the Hurewicz homomorphism  $h: \pi_n(X, x) \rightarrow H_n(X, \mathbb{Z})$ ,  $f \mapsto H_n(f, \mathbb{Z})[S^n]$  is an isomorphism.

*Proof.* • By CW-approximation (see later in this class) there is a CW-complex  $Y$  with one 0-cell and no cells in dimensions  $1 \leq i \leq n-1$  and a weak homotopy equivalence  $Y \xrightarrow{\sim} X$ .  
• Every weak homotopy equivalence induces isomorphisms of  $H_n(\_, A)$  for all  $n \geq 0$ , for all abelian groups  $A$ . This will either be an exercise or proven later on.

We get a commutative diagram

$$\begin{array}{ccc} \pi_n(Y, y) & \xrightarrow[f_*]{\cong} & \pi_n(X, f(*)) \\ \downarrow h & & \downarrow h \\ H_n(Y, \mathbb{Z}) & \xrightarrow[f_*]{\cong} & H_n(X, \mathbb{Z}) \end{array}$$

So wlog we can assume that  $X$  admits a CW-structure with a single 0-cell and no cells of dimensions  $1, \dots, n-1$ . The inclusion of the  $(n+1)$ -skeleton  $X_{n+1} \rightarrow X$  induces isomorphisms

$$\pi_n(X_{n+1}, *) \rightarrow \pi_n(X, *)$$

by cellular approximation. Also

$$H_n(X_{n+1}, \mathbb{Z}) \rightarrow H_n(X, \mathbb{Z})$$

is an isomorphism for example by cellular homology.

It suffices to show the Hurewicz theorem for the  $n+1$ -skeleton, i.e.  $X$  is a CW-complex with a single 0-cell,  $I$  many  $n$ -cells,  $J$  many  $(n+1)$ -cells and no cells in any other dimension.

$$X = (\{x\} \cup_{I \times \partial D^n} D^n) \cup_{J \times \partial D^{n+1}} (J \times D^{n+1}) \cong (\bigvee_{i \in I} S^n) \cup_{J \times D^{n+1}} J \times D^{n+1}$$

We can assume that all attaching maps  $\alpha: \partial D^{n+1} \rightarrow \bigvee_{i \in I} S^n$  are based.

We see this since  $\bigvee_{i \in I} S^n$  is path connected,  $\alpha$  is homotopic by HEP to a based map  $\alpha'$ . Since homotopic attaching maps yield homotopy equivalent glueings.

So  $X$  can be written as a pushout

$$\begin{array}{ccc} \bigvee_{j \in J} S^n & \xrightarrow{f} & \bigvee_{i \in I} S^n & = & X_n \\ \downarrow & & \downarrow & & \\ \bigvee_{j \in J} D^{n+1} & \longrightarrow & X & = & X_{n+1} \end{array}$$

Since  $\bigvee_I S^n$  and  $\bigvee_J S^n$  are  $(n-1)$ -connected,  $f: \bigvee_J S^n \rightarrow \bigvee_I S^n$  is  $(n-1)$ -connected. Also  $\bigvee_J S^n \rightarrow \bigvee_I D^{n+1}$  is  $n$ -connected, as  $\bigvee_I D^{n+1}$  is contractible.

By Proposition ??  $\pi_k(\bigvee_J D^{n+1}, \bigvee_J S^n) \rightarrow \pi_k(X, X_n) = \pi_k(X, \bigvee_I S^n)$  is isomorphic for  $1 \leq k \leq 2n-2$  and surjective for  $k = 2n-1$ . In particular  $\pi_{n+1}(\bigvee_J D^{n+1}, \bigvee_J S^n) \rightarrow \pi_{n+1}(X, X_n)$  is

surjective. We compare the long exact homotopy sequences of the vertical pairs:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{n+1}(\vee_J D^{n+1}, \vee_J S^n) & \xrightarrow[\cong]{\partial} & \pi_n(\vee_J S^n) & \longrightarrow & 0 \\ & & \downarrow g_* & & \downarrow f_* & & \\ & & \longrightarrow & \pi_{n+1}(X, X_n) & \xrightarrow{\partial} & \pi_n(X_n) & \xrightarrow[\text{surjective by cell.}]{} \pi_n(X, *) \xrightarrow[\text{approx.}]{} \end{array}$$

So the upper row is exact:

$$\begin{array}{ccccccc} \pi_n(\vee_J S^n) & \xrightarrow{f_*} & \pi_n(\vee_I S^n) & \xrightarrow{\text{incl}_*} & \pi_n(X) & \longrightarrow & 0 \\ \downarrow h & & \downarrow h & & \downarrow h & & \\ H_n(\vee_J S^n, \mathbb{Z}) & \xrightarrow{f_*} & H_n(\vee_I S^n, \mathbb{Z}) & \longrightarrow & H_n(X, \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

We claim the first 2  $h$  are isomorphisms. We have a long exact sequence by excision. For all sets  $I$

$$\begin{array}{ccc} \bigoplus_{i \in I} \pi_n(S^n, *) & \xrightarrow{\cong} & \pi_n(\vee_{i \in I} S^n, *) \\ \downarrow \bigoplus h & & \downarrow h \\ \bigoplus_{i \in I} H_n(S^n, \mathbb{Z}) & \xrightarrow{\cong} & H_n(\vee_I S^n, \mathbb{Z}) \end{array}$$

so  $h: \pi_n(X, x) \rightarrow H_n(X, \mathbb{Z})$  is an isomorphism by 5-lemma.  $\square$

Some applications

**Recall.** If  $X$  is simply connected, then  $H_1(X, \mathbb{Z}) \cong \pi_1(X, x)_{\text{ab}} = 0$ . But if  $X$  is path connected and  $H_1(X, \mathbb{Z}) = 0$ ,  $X$  need not be simply connected, because  $\pi_1(X, x)$  could be non-trivial and perfect (= abelianization is trivial) (E.g.  $A_5$ ).

**Proposition 1.22** (12.7(i) for Lück). Let  $X$  be simply connected,  $n \geq 1$ . Then the following are equivalent

1.  $X$  is  $n$ -connected.
2.  $H_i(X, \mathbb{Z}) = 0$  for all  $2 \leq i \leq n$ .

*Proof.* By induction on  $n$ . Nothing to show for  $n = 1$ . The induction step is the Hurewicz theorem  $\pi_n(X, x) \cong H_n(X, \mathbb{Z})$ .  $\square$

**Proposition 1.23** (12.7 (ii) for Lück). Let  $X$  be simply connected. Then the following are equivalent:

1.  $X$  is weakly contractible<sup>12</sup>
2.  $H_i(X, \mathbb{Z}) = 0$  for all  $i \geq 2$ .

**Warning.** There exist acyclic spaces, i.e. non-contractible CW-complexes, path connected with  $H_i(X, \mathbb{Z}) = 0$  for all  $i \geq 1$ .

**Remark 1.24.** There is a slightly better version of the Hurewicz theorem: If  $X$  is  $n - 1$ -connected,  $n \geq 2$ , then  $h: \pi_n(X, x) \rightarrow H_n(X, \mathbb{Z})$  is isomorphism and  $h: \pi_{n+1}(X, x) \rightarrow H_{n+1}(X, \mathbb{Z})$  is surjective.

<sup>12</sup>All homotopy groups vanish

### 1.3.1 Relative Hurewicz theorem

#### Definition 1.25: Relative Hurewicz map

Let  $(X, A)$  be a space pair,  $a \in A$ . Choose a generator  $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1}, \mathbb{Z})$ . The relative Hurewicz homomorphism is

$$h: \pi_n(X, A, a) \rightarrow H_n(X, A, \mathbb{Z})$$

is defined by

$$[f: (D^n, S^{n-1}, z) \rightarrow (X, A, \{a\})] \mapsto h[f] := H_n(f, \mathbb{Z})[D^n, S^{n-1}]$$

The following diagram commutes:

$$\begin{array}{ccccc} \pi_n(X, a) & \longrightarrow & \pi_n(X, A, a) & \xrightarrow{\partial} & \pi_{n-1}(A, a) \\ \downarrow h & & \downarrow h & & \downarrow h \\ H_n(X, \mathbb{Z}) & \longrightarrow & H_n(X, A, \mathbb{Z}) & \xrightarrow{\partial} & H_{n-1}(A, \mathbb{Z}) \end{array}$$

#### Theorem 1.26: relative Hurewicz, (simply connected case)

Let  $n \geq 2$ . Let  $(X, A)$  be a space pair, such that  $X$  and  $A$  are simply connected and  $(X, A)$  is  $(n - 1)$ -connected. Then

1. The Hurewicz homomorphism  $h: \pi_n(X, A, a) \rightarrow H_n(X, A, \mathbb{Z})$  is an isomorphism, and
2. The group  $H_i(X, A, \mathbb{Z}) = 0$  for all  $0 \leq i \leq n - 1$ .

*Proof.* We will deduce this from the absolute version and some other things we already did.

By replacing  $X$  by the mapping cylinder  $A \times [0, 1] \cup_{A \times 1} X$  and replacing  $A$  by  $A \times 0$ , we can assume wlog that the inclusion  $i: A \rightarrow X$  is a cofibration. Since  $A$  is 1-connected, and  $(X, A)$  is  $(n - 1)$ -connected,

$$pr: \pi_k(X, A, a) \rightarrow \pi_k(X/A, *)$$

is bijective for  $1 \leq k \leq n$  and surjective for  $k = n + 1$ .

$X/A$  is simply connected by the van Kampen theorem.

Since  $\pi_k(X, A, *) = 0$  for  $k \leq n - 1$  by hypothesis, we get that  $\pi_k(X/A, *) = 0$  for  $k \leq n - 1$ . So  $X/A$  is  $(n - 1)$ -connected. By the absolute Hurewicz theorem for  $X/A$ ,  $H_k(X/A, \mathbb{Z}) = 0$  for  $1 \leq k \leq n - 1$  and

$$\begin{array}{ccc} \pi_n(X, A, a) & \xrightarrow{\cong_{pr_*}} & \pi_n(X/A, *) \\ \downarrow & & \downarrow h \cong \\ H_n(X, A, \mathbb{Z}) & \xrightarrow[\text{excision}]{} & H_n(X/A, \mathbb{Z}) \end{array}$$

□

[22.10.2025, Lecture 4]
[27.10.2025, Lecture 5]

**Proposition 1.27.** Let  $f: X \rightarrow Y$  be a map of simply connected spaces. Let  $n \geq 1$ . Then TFAE:

1.  $f$  is  $n$ -connected
2.  $f_*: H_i(X, \mathbb{Z}) \rightarrow H_i(Y, \mathbb{Z})$  is bijective for  $2 \leq i \leq n-1$  and surjective for  $i = n$ .

*Proof.* By replacing  $Y$  by the mapping cylinder of  $f: X \rightarrow Y$ , we may assume  $f$  is the inclusion  $i: A \hookrightarrow X$  of a closed subspace  $(X, A)$  is  $n$ -connected is equivalent to  $\pi_k(X, A, a) = 0$  for all  $k \leq n$  which is then by Hurewicz equivalent to  $H_k(X, A, \mathbb{Z}) = 0$  for all  $k \leq n$ . But this is equivalent to 2.  $\square$

### Theorem 1.28: Whitehead theorem

Let  $f: X \rightarrow Y$  be a continuous map between simply connected CW-complexes. Then the following are equivalent:

1.  $f$  is a homotopy equivalence.
2.  $f$  is a weak homotopy equivalence.
3.  $f_*: H_i(X, \mathbb{Z}) \rightarrow H_i(Y, \mathbb{Z})$  is an isomorphism for all  $i \geq 0$ .

**Note.**  $1 \Leftrightarrow 2$  without simply connectedness is what we previously called the Whitehead theorem.

### Theorem 1.29

Let  $f: X \rightarrow Y$  be a continuous map between path connected CW-complexes. Suppose that for some (hence any)  $x \in X$   $f_*: \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$  is an isomorphism. Let  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  be a lift to universal covers.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow q & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

Then TFAE:

1.  $f$  is a homotopy equivalence
2.  $\tilde{f}_*: H_i(\tilde{X}, \mathbb{Z}) \rightarrow H_i(\tilde{Y}, \mathbb{Z})$  is an isomorphism for all  $i \geq 0$ .

*Proof.*  $p_*: \pi_i(\tilde{X}, \tilde{x}) \rightarrow \pi_i(X, x)$  is an isomorphism for  $i \geq 2$ . So  $f$  is a weak homotopy equivalence iff  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  is a weak homotopy equivalence. and now this is equivalent too  $H_*(\tilde{f}, \mathbb{Z})$  is an isomorphism for all  $* \geq 0$ .  $\square$

### Theorem 1.30: 12.16 for Lück

Let  $X$  be a path connected CW-complex,  $n \geq 2$ . Then the following are equivalent:

1.  $X$  is homotopy equivalent to  $S^n$
2.  $X$  is simply connected and  $H_i(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases}$

*Proof.* 1.  $\implies$  2.

2.  $\implies$  1. By the Hurewicz Theorem  $h: \pi_n(X, x) \rightarrow H_n(X, \mathbb{Z}) \cong \mathbb{Z}$  is an isomorphism. Let  $f: S^n \rightarrow X$  represent a generator of  $\pi_n(X, x)$ . Then  $f$  induces an isomorphism of all integral homology groups. Since  $S^n$  and  $X$  are simply connected CW-complexes,  $f$  is a homotopy equivalence.

□

We will not proof a even more general relative Hurewicz theorem:

Let  $(X, A)$  be a space pair,  $a \in A$ . Recall that  $\pi_1(A, a)$  acts on  $\pi_n(X, A, a)$  for  $n \geq 1$ :

With  $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, a)$  and  $w: ([0, 1], \{0, 1\}) \rightarrow (A, a)$  with  $w * f$ , which he explains by a picture. Note  $w * f$  is pair homotopic to  $f$  (but **Not** triple homotopic).

If  $[I^n, \partial I^n] \in H_n(I^n, \partial I^n, \mathbb{Z}) \cong \mathbb{Z}$  is a generator, then

$$f_*[I^n, \partial I^n] = (w * f)_*[I^n, \partial I^n]$$

so  $h: \pi_n(X, A, a) \rightarrow H_n(X, A, \mathbb{Z})$  satisfies  $h[f] = h[w * f]$ .

### Definition 1.31

Let  $(X, A, a)$  be a space triple,  $n \geq 2$ . Set

$$\pi_n(X, A, a)^\dagger := \text{quotient of } \pi_n(X, A, a) \text{ by the normal subgroup generated by } [w * f] \cdot [f]^{-1}$$

**Note.** For  $n \geq 3$ , the group  $\pi_n(X, A, a)$  is abelian, hence so is  $\pi_n(X, A, a)^\dagger$ . For  $n = 2$   $\pi_2(X, A, a)$  need not be abelian, but  $\pi_2(X, A, a)^\dagger$  is.

Let  $f, g: (I^2, \partial I^2, J^1) \rightarrow (X, A, a)$ , set  $w = g|_{[0,1]}: ([0, 1], \{0, 1\}) \rightarrow (A, a)$ . The rest of the proof is pictures, so good luck understanding without them. I'm sorry. We have  $[g]^{-1} \cdot [f] \cdot [g] = [w * f]$ . See this in Tom Dieck Prop. 6.2.6.

So in particular  $[g]^{-1} \cdot [f] \cdot [g] \equiv [f]$  in  $\pi_2(X, A, a)^\dagger$ , so it is abelian.

### Theorem 1.32: Relative Hurewicz with $\pi_1$

Let  $(X, A)$  be a path connected space pair. Suppose for all  $a \in A$ ,  $\text{incl}_*: \pi_1(A, a) \rightarrow \pi_1(X, a)$  is an isomorphism. Let  $n \geq 2$  be such that  $\pi_i(X, A, a) = 0$  for all  $1 \leq i \leq n - 1$ . Then  $H_i(X, A, \mathbb{Z}) = 0$  for  $0 \leq i \leq n - 1$  and the modified Hurewicz map

$$h^\dagger: \pi_n(X, A, a)^\dagger \rightarrow H_n(X, A, \mathbb{Z})$$

is an isomorphism.

**Warning.** The hypothesis is on  $\pi_i(X, A, a)$ , but the conclusion on  $\pi_n(X, A, a)^\dagger$ !

## 1.4 CW-Approximation

"Every topological space can be approximated by a CW-complex"<sup>13</sup>. More detailed, you can find a CW-complex with a weak homotopy equivalence to your space.

we state a relative refined version of CW-approximation and proof that.

<sup>13</sup>See 6 in Lücks notes

**Definition 1.33**

Let  $(Y, A)$  be a space pair,  $n \geq 0$ . A  $n$ -CW-model for  $(Y, A)$  is a relative CW-complex  $(Z, A)$  and a certain map  $f: Z \rightarrow Y$  such that

- $f|_A =$  inclusion  $A \hookrightarrow Y$
- $(Z, A)$  is  $n$ -connected,
- The map  $f_*: \pi_i(Z, z) \rightarrow \pi_i(Y, f(z))$  is injective for  $i = n$  and bijective for  $i > n$  for all  $z \in Z$ .

We have

$$A \xrightarrow[n\text{-connected}]{\text{relative CW}} Z \xrightarrow[n\text{-coconnected}]{ } Y$$

**Theorem 1.34**

Let  $(Y, A)$  be a space pair,  $A \neq \emptyset$ , such that  $A$  is Hausdorff,  $n \geq 0$ . Then there is a  $n$ -CW-model  $(A, Z, f)$ , s.t.  $(Z, A)$  has no relative cells of dimension  $\leq n$ .

**Addendum.** If  $A$  comes with a CW-structure, then  $Z$  can be chosen as a CW-complex that contains  $A$  as a subcomplex.

**Special Case.**  $Y$  is  $n$ -connected,  $n \geq 0$ ,  $A = \{y_0\}$ . Let  $(Z, \{y_0\})$  be a  $n$ -CW-model with  $f: Z \rightarrow Y$ , without relative  $i$ -cells for  $0 \leq i \leq n$ . Then both  $Y$  and  $Z$  are  $n$ -connected,  $f: \pi_i(Z, z) \rightarrow \pi_i(Y, z)$  is an isomorphism for  $i > n$ , hence  $f$  is a weak homotopy equivalence

*Proof.* We will inductively construct

$$A = Z_n \subseteq Z_{n+1} \subseteq \dots$$

and  $f_i: Z_i \rightarrow Y$ ,  $i \geq n$ , such that

- for  $i > n$ ,  $Z_i$  is obtained from  $Z_{i-1}$  by attaching  $i$ -cells.
- $f_i|_{Z_{i-1}} = f_{i-1}$ ,  $f_n = \text{incl}: A \hookrightarrow Y$
- For all  $z \in Z_i$ ,  $\pi_j(f, z): \pi_j(Z_i, z) \rightarrow \pi_j(Y, f_i(z))$  is
  - injective for  $j = n$
  - bijective for  $n < j < i$
  - surjective for  $j = i$

Given this, we take  $Z = \bigcup_{i \geq n} Z_i$  then  $(Z, A)$  is a relative CW-complex with cells of dimensions  $\geq n + 1$ , and  $f = \bigcup_{i \geq n} f_i$  has the desired property.

$(Z, A)$  is  $n$ -connected by cellular approximation

$$\begin{array}{ccc} f_*: \pi_j(Z, z) & \longrightarrow & \pi_j(Y, f(z)) \\ \uparrow & \nearrow \cong & \\ \pi_j(Z_{j+1}, z) & & (f_{j+1})_* \end{array}$$

where the left up map is an isomorphism by cellular approximation.

For the inductive Step A: make  $\pi_i$  injective

Step B: make  $\pi_{i+1}$  surjective.

Suppose  $i \geq n$  and  $Z_i \xrightarrow{f_i} Y$  have been constructed with the desired properties. For each path component  $C$  of  $A$  choose a basepoint  $x_c \in A$  in that component. For each element in the kernel of  $(f_i)_*: \pi_i(Z_i, x_c) \rightarrow \pi_i(Y, f_i(x_c))$  choose a based continuous map  $q_{c,u}: S^i \rightarrow Z_i$ , s.t.  $[g_{c,u}] = u$ . Define  $Z'_{i+1}$  as the pushout

$$\begin{array}{ccc} \coprod_{C,u \in \text{Ker}(f_{i*})} S^i & \xrightarrow{\coprod_{q_{c,u}}} & Z_i \xrightarrow{f_i} Y \\ \downarrow & & \downarrow f_{i+1} \\ \coprod_{C,u} D^{i+1} & \longrightarrow & Z'_{i+1} Q \end{array}$$

For  $j \leq i+1$  this is bijective for  $j < i$ , surjective for  $j = i$  by cellular approximation. I missed a bit here.

This was step A, now comes step B.

For each  $C \in \pi_0(A)$  and each element  $v$  of  $\pi_{i+1}(Y, f(x_c))$  choose a representation  $q_{c,v}: S^{i+1} \rightarrow Y$  s.t.  $[q_{c,v}] = v$ . Define  $Z_{i+1} = Z'_{i+1} \vee \coprod_{\substack{C \in \pi_0 A \\ v \in \pi_{i+1}(Y, f(x_c))}} S^{i+1} \xrightarrow{f_{i+1} = f'_{i+1} \vee q_{c,v}} Y$ .

This now has all required properties, missed the diagramm with explaining everything.  $\square$

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[27.10.2025, Lecture 5]
[29.10.2025, Lecture 6]

Was ill, might add in later.

This is from Tiens (thank him for it) lecture notes

### Theorem 1.35: Lück Thm. 6.8

Let  $Y$  be any space.

1. There is a CW-approximation, i.e., a pair  $(X, f)$  consisting of a CW-complex  $X$  and a weak homotopy equivalence  $f: X \cong Y$ .
2. Let  $(X, f)$  and  $(X', f')$  be two CW-approximations of  $Y$ . Then there is a continuous map  $g: X \rightarrow X'$  such that

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \searrow \sim & & \swarrow \sim \\ f & & f' \\ Y & & \end{array}$$

commutes up to homotopy. Moreover,  $g$  is unique up to homotopy and a homotopy equivalence.

*Proof.*

1. If  $Y = \emptyset$ , take  $X = \emptyset$ . If  $Y \neq \emptyset$ , choose one point  $y_C$  in each path component  $C \in \pi_0(Y)$ . Let  $(Z_C, f_C)$  be a 0-CW-approximation of  $Y_C$ , i.e., the path component of  $C$ , so  $f_C: Z_C \rightarrow Y_C$  is a weak homotopy equivalence. Then set  $X = \coprod_{C \in \pi_0(Y)} Z_C \rightarrow [\coprod f_C]Y$ , which is a CW-approximation.
2. Recall that if  $X$  is a CW-complex and if  $f: Y \rightarrow Z$  is a weak homotopy equivalence, then  $[X, f]: [X, Y] \rightarrow [X, Z]$  is bijective.

Since  $f': X' \rightarrow Y$  is a weak homotopy equivalence and since  $X$  is a CW-complex,  $[X, f']: [X, X'] \cong [X, Y]$  is bijective. So, there is a  $g: X \rightarrow X'$ , unique up to homotopy, such that  $f' \circ g \simeq f$ . Then  $g$  is a weak equivalence because  $f$  and  $f'$  are; hence a homotopy equivalence.  $\square$

### 1.4.1 Killing of homotopy groups

Let  $A$  be a path connected Hausdorff space. Let  $(Z, f)$  be an  $n$ -CW-model of  $(CA, A)$  where  $CA = A \times [0, 1]/A \times 1$  is the cone. Then, for  $n \geq 1$ ,

$$\pi_i(f): \pi_i(Z, z) \rightarrow \pi_i(CA, f(z)) = 0$$

is monic for  $i = n$  and bijective for  $i > n$ . So,  $\pi_i(Z, z) = 0$  for  $i \geq n$ . Also  $(Z, A)$  is a relative CW-complex with relative cells of dimension  $n + 1$  or larger. So,  $\pi_i(A, a) \rightarrow \pi_i(Z, a)$  is an isomorphism for  $i < n$ . This means that  $A \rightarrow Z$  is an isomorphism for  $\pi_{\leq n}$  and  $Z$  has trivial homotopy in  $\pi_{\geq n}$ .

### 1.4.2 Eilenberg-MacLane spaces

- Let  $n \geq 1$ , and let  $A$  be a group, abelian if  $n \geq 2$ . Then there exists a path connected CW-complex  $X = K(X, n)$  such that

$$\pi_i(X, x) \cong \begin{cases} A & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

- $X$  is unique up to homotopy.
- Representability of cohomology: for  $Y$  a CW-complex and  $A$  abelian, we have  $H^n(Y, A) \cong [Y, K(A, n)]$ .

We follow Lück's notes and Schwede's notes (which are in German!).

#### Definition 1.36: Eilenberg-MacLane space

Let  $n \geq 1$ , and let  $A$  be a group, abelian if  $n \geq 2$ . An *Eilenberg-MacLane space* of type  $(A, n)$  is a pair  $(X, \varphi)$  consisting of a based path connected space  $(X, x)$  and an isomorphism  $\varphi: \pi_n(X, x) \cong A$  such that  $\pi_i(X, x) = 0$  for all  $i \geq 1$  and  $i \neq n$ . Shorthand notation: " $X$  is a  $K(A, n)$ ".

(In full generality, Eilenberg-MacLane spaces are very abstract.)

**Example 1.37.**  $S^1$  is a  $K(\mathbb{Z}, 1)$  since  $\pi_1(S^1, z) \cong \mathbb{Z}$  and  $\pi_i(S^1, z) = 0$  for all  $i \geq 2$  by covering space theory, because the universal cover  $\exp: \mathbb{R} \rightarrow S^1$  of  $S^1$  has a contractible total space. More generally, if  $p: \tilde{X} \rightarrow X$  is a universal cover of a path connected space with  $\tilde{X}$  contractible, then  $X$  is a  $K(G, 1)$  with  $G \cong \text{Deck}(p)$ . Hence:

- $\mathbb{R}P^0 = K(\mathbb{Z}/2, 1)$  with universal cover  $S^\infty \simeq *$ .
- $S^1 \times \cdots \times S^1 = K(\mathbb{Z}^m, 1)$  ( $m$  times) with universal cover  $\mathbb{R}^-$ .
- Klein bottle (insert diagram of a square with arrows on the edges that indicate how the Klein bottle is glued) is a  $K(\mathbb{Z} \rtimes \mathbb{Z}, 1)$  with universal cover  $\mathbb{R}^2$ . Here,  $\rtimes$  is a semidirect product with  $\mathbb{Z}$ -action on  $\mathbb{Z}$  by sign.

**Example 1.38.**  $\mathbb{C}P^\infty$  is a  $K(\mathbb{Z}, 2)$  because of the locally trivial fibre bundle  $S(\mathbb{C}^\infty) \rightarrow \mathbb{C}P^\infty$ ,  $x \mapsto \mathbb{C}x$  with fibre  $S^1$ . This is a Serre fibration, so the long exact sequence of homotopy groups gives

$$\{0\} = \pi_i(S(\mathbb{C}^\infty), *) \rightarrow \pi_i(\mathbb{C}P^\infty, *) \cong [\partial] \pi_{i-1}(S^1, x) \rightarrow \pi_{i-1}(S(\mathbb{C}^\infty), *) = \{0\}.$$

Thus,

$$\pi_i(\mathbb{C}P^\infty, *) \cong \begin{cases} \mathbb{Z} & \text{if } i = 2, \\ 0 & \text{if } i \neq 0. \end{cases}$$

(Assume that Eilenberg-MacLane spaces are CW-complexes is not a huge loss of generality because of CW-approximation.)

**Example 1.39.** If  $X$  is a  $K(A, n)$  and  $Y$  is a  $K(B, n)$ , then  $X \times Y$  is a  $K(A \times B, n)$ .

Construction of  $K(A, n)$ 's. We distinguish two cases, whose constructions are very similar.

$K(G, 1)$ 's: let  $G$  be a group. Start with the following 2-dimensional CW-complex

$$CG := \left( \bigvee_{g \in G} S_g^1 \right) \cup \bigcup_{(h, k) \in G^2} D_k^2$$

The 2-cells are attached as follows: let  $i_g: S^1 \rightarrow \bigvee_{g \in G} S_g^1$  denote the inclusion of the  $g$ th For  $(h, k) \in G^2$ , consider the following map  $\alpha_{h,k}: S^1 \rightarrow \bigvee_{g \in G} S_g^1$  which is  $i_h$  on the first third,  $i_k$  on the second third, and  $i_{hk}^{-1}$  on the last third of  $S^1$  (insert image of circle with three equal segments, each segment being named after the respective map). By covering space theory/Seifert-van Kampen, we know that  $\pi_1(\bigvee_{g \in G} S_g^1, *)$  is a free group generated by the elements  $[i_g]$ . By cellular approximation,  $\pi_1(\bigvee_{g \in G} S_g^1, *) \rightarrow \pi_1(CG, *)$  is surjective, so  $\pi_1(CG, *)$  is generated, as a group, by  $[i_g: S^1 \rightarrow \bigvee_{g \in G} S_g^1 \hookrightarrow CG]$ . The two cell indexed by  $(h, k) \in G^2$  witnesses that  $[i_h][i_k][i_{hk}]^{-1}$  map to 1 in  $\pi_1(CG, *)$ , so  $[i_h][i_k] = [i_{hk}]$  in  $\pi_1(CG, *)$ . Thus,  $G \rightarrow \pi_1(CG, *)$ ,  $g \mapsto [S^1 \hookrightarrow [i_g] \vee S_g^1 \hookrightarrow CG]$  is a surjective group homomorphism. By Seifert-van Kampen, attaching a 2-cell to a path connected space precisely kills the normal subgroup generated by the class of the attaching map. Thus,

$$\pi_1(CG, *) = \frac{\text{free group on elements of } G}{[h][k] = [hk]} \cong G.$$

So,  $CG$  is a connected space with correct  $\pi_1$ , so by killing the homotopy group  $\pi_i$  for  $i \geq 2$ , we obtain a  $K(G, 1)$ .

Construction for  $K(A, n)$  for  $n \geq 2$  and  $A$  abelian: choose a presentation of  $A$  as an abelian group:

$$\mathbb{Z}[I] \xrightarrow{[d]} \mathbb{Z}[J] \xrightarrow{[\text{eps}]} A \rightarrow 0$$

is an exact sequence of abelian groups with  $I$  and  $J$  some sets. We set  $X_n = \{x\} \cup_{J \times S^{n-1}} J \times D^n \cong \bigvee_J S^n$  as an  $n$ -dimensional CW-complex with one 0-cell and no cells in dimension  $1 \leq i \leq n-1$ . So,  $X_n$  is  $(n-1)$ -connected by cellular approximation. Then  $\mathbb{Z}[J] \cong \tilde{H}_n(X_n; \mathbb{Z})$ ,  $j \mapsto (i_j)_*[D^n, S^{n-1}]$  where  $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1}; \mathbb{Z})$  is a chosen generator and  $i_j: D^n \rightarrow \{x\} \cup_{J \times S^{n-1}} J \times D^n$  is the characteristic map of the  $j$ th  $n$ -cell. By the Hurewicz theorem,  $h: \pi_n(X_n, *) \rightarrow H_n(X_n; \mathbb{Z}) \cong \mathbb{Z}[J]$  is an isomorphism. For each  $i \in I$ , there is a unique  $[\alpha_i] \in \pi_n(X_n, *)$  where Hurewicz image is  $d(i)$ . Let  $\alpha_i: S^n \rightarrow X_n \cong \bigvee_{j \in J} S^n$  be a representative for this homotopy class. We define  $X_{n+1}$  by attaching  $(n+1)$ -cells along all the  $\alpha_i$ 's:  $X_{n+1} = X_n \cup_{I \times S^n, \alpha_i} I \times D^{n+1}$ . The cellular chain complex of  $X_n$  gives

$$\begin{array}{ccccccc} H_{n+1}(X_{n+1}, X_n; \mathbb{Z}) & \longrightarrow & H_n(X_n; \mathbb{Z}) & \longrightarrow & H_n^{\text{cell}}(X_{n+1}; \mathbb{Z}) & \longrightarrow & 0 \\ \downarrow i \mapsto (\beta_i)_*[D^{n+1}, S^n] & & \downarrow \wr & & \exists! \uparrow \wr & & \\ \mathbb{Z}[I] & \xrightarrow{d} & \mathbb{Z}[J] & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows where  $\beta_i: D^{n+1} \rightarrow X_{n+1}$  is the characteristic map for the  $i$ th  $(n+1)$ -cell. The left square commutes by construction, so we obtain the dashed map. Hence we have an isomorphism  $A \cong H_n(X_{n+1}; \mathbb{Z})$ . Since  $X_{n+1}$  is  $(n-1)$ -connected, the Hurewicz map is an isomorphism  $h: \pi_n(X_{n+1}, *) \cong H_n(X_{n+1}; \mathbb{Z}) \cong A$ . By killing the homotopy groups  $\pi_i$  for  $i \geq n+1$  from  $X_{n+1}$  we obtain a  $K(A, n)$ .

Next aim: for  $X$  an  $(n - 1)$ -connected based CW-complex,

$$\pi_n: [X, K(A, n)]_* \rightarrow \text{hom}(\pi_n(X, *), A), \quad [f: X \rightarrow K(A, n)] \mapsto \varphi \circ \pi_n(f),$$

where  $\varphi: \pi_n(K(A, n), *) \cong A$ , is bijective.

**Lemma 1.40** (Schwede Lem. 49). *Let  $(X, Y)$  be a relative CW-complex and  $Z$  any space. Suppose that for all  $m \geq 1$  such that  $(X, Y)$  has at least one relative  $m$ -cell,  $\pi_{m-1}(Z, z) = 0$  for all  $z \in Z$ .*

*Then any map  $Y \rightarrow Z$  can be extended to a map  $X \rightarrow Z$ .*

*Proof.* By induction on the relative skeleta, we construct continuous maps  $f_m: X_m \rightarrow Z$  such that  $f_{-1} = f$ ,  $f_i|_{X_{i-1}} = f_{i-1}$ . We define  $f_0: X_0 = Y \amalg I \rightarrow Z$  by sending the new 0-cells arbitrarily to  $Z$ . For  $n \geq 1$ , if  $X_i \neq X_{i-1}$ , then there is at least one relative  $i$ -cell. The attaching map  $\alpha_j$  for every relative  $i$ -cell becomes null homotopic after composition with  $f_{i-1}: X_{i-1} \rightarrow Z$  because  $\pi_{i-1}(Z, *) = 0$ . We choose a null homotopy  $h_j: D^{i+1} \rightarrow Z$  of this composite and define  $f_i: X_i \rightarrow Z$  by  $h_j$  on the  $j$ th  $i$ -cell.

$$\begin{array}{ccc} \coprod_J S_j^i & \xrightarrow{\alpha_j} & X_{i-1} \xrightarrow{f_{i-1}} Z \\ \downarrow & & \downarrow \\ \coprod_J D_j^{i+1} & \xrightarrow{\quad} & X_i \xrightarrow{h_j} Z \end{array}$$

(map exist by universal property of the pushout). This completes the induction step. Now, set  $f_\infty = \bigcup_{i \geq 0} f_i$ .  $\square$

[29.10.2025, Lecture 6]  
[3.11.2025, Lecture 7]

We want to show, that for CW-complexes, Eilenberg–MacLane-spaces are unique up to homotopy.

### Theorem 1.41

Let  $n \geq 1$ . Let  $Y$  be an  $(n - 1)$ -connected CW-complex. Let  $Z$  be a based space s.t.  $\pi_m(Z, z) = 0$  for  $m > n$ . Then for every group-homomorphism

$$\Phi: \pi_n(Y, y) \rightarrow \pi_n(Z, z)$$

there is a based continuous map  $f: Y \rightarrow Z$ , s.t.  $\pi_n(f) = \Phi$ . and such an  $f$  is unique up to based homotopy. Equivalently: the map

$$\pi_n: [Y, Z]_* \rightarrow \text{Hom}_{\text{Grp}}(\pi_n(Y, y), \pi_n(Z, z))$$

is bijective.

*Proof.* If  $Y$  is based homotopy equivalent to  $Y'$  and the claim holds for  $Y'$ , then it holds for  $Y$ . Then we can assume wlog, that  $Y$  has one 0-cell and no cells in dimensions  $1 \leq i \leq n - 1$ . So  $Y_n = \bigvee_I D^n / S^{n-1}$ .

**Step 1** we construct a continuous map  $f_n: Y_n \rightarrow Z$  such that the following commutes:

$$\begin{array}{ccc} \pi_n(Y_n, *) & \xrightarrow{(f_n)_*} & \pi_n(Z, z) \\ \searrow \text{incl}_* & & \nearrow \Phi \\ & \pi_n(Y, y) & \end{array}$$

We choose a homeomorphism  $S^n \cong D^n / \partial D^n$  and characteristic maps  $\chi_i: D^n \rightarrow Y$  for all  $i \in I$ . Then the  $S^n \cong D^n / \partial D^n \xrightarrow{\chi_i} Y_n \hookrightarrow Y$  represent a class  $[\chi_i] \in \pi_n(Y, y)$ , so  $\Phi[\chi_i] \in \pi_n(Z, z)$ . Let  $w_i: S^n \rightarrow Z$  be a based map such that  $[w_i] = \Phi[\chi_i]$ . We define

$$f_n: Y_n = \bigvee_{i \in I} D^n / \partial D^n \cong \bigvee_{i \in I} S^n \rightarrow Z$$

as  $w_i$  on the wedge summand indexed by  $i$ . Then the maps commute by construction on the classes  $[\chi_i]$ . Since the class  $[\chi_i]$  for  $i \in I$  generates the group  $\pi_n(Y_n, *)$  (Hurewicz theorem). Since all the maps in the commutative diagram are group homomorphisms and commute on generators, the diagram commutes.

**Step 2** We extend  $f_n$  continuously to  $f_{n+1}: Y_{n+1} \rightarrow Z$ .

Let  $J$  be an index set for the  $(n+1)$ -cells of  $Y$ . For  $j \in J$  let  $\chi_j: D^{n+1} \rightarrow Y$  be a characteristic map for the  $j$ -th  $n+1$ -cell. The associated attaching map is  $\chi_j|_{S^n}: S^n \rightarrow Y_n \cong \bigvee_I D^n / \partial D^n \cong \bigvee_I S^n$ . Since  $Y_n$  is path-connected  $\chi_j|_{S^n}$  is homotopic to a based map. By the HEP for  $(D^{n+1}, S^n)$ , we can extend the homotopy between  $\chi_j|_{S^n}$  and the based map  $\alpha: S^n \rightarrow Y_n$  to the  $(n+1)$ -cell. So

$$[S^n \xrightarrow{\alpha} Y_n \hookrightarrow Y] \in \pi_n(Y, y)$$

is the zero homotopy class, i.e.  $[\alpha] \in \pi_n(Y_n, y)$  lies in the kernel of  $\text{incl}_*$ . By commutativity of the (way above) commutative diagram, we get  $(f_n)_*[a] = 0$  in  $\pi_n(Z, z)$ . So the composite

$$S^n \xrightarrow{\alpha} Y_n \xrightarrow{f_n} Z$$

is nullhomotopic. Hence also  $S^n \xrightarrow{\chi_j|_{S^n}} Y_n \xrightarrow{f_n} Z$  is nullhomotopic. We choose a continuous extension  $\beta_j: D^{n+1} \rightarrow Z$  of  $\chi_j|_{S^n}$  and define  $f_{n+1}: Y_{n+1} = (\bigvee_I S^n) \cup_{J \times S^n} J \times D^{n+1} \rightarrow Z$  by  $f_n$  on  $Y_n$  and by  $\beta_j$  on the  $j$ -th  $(n+1)$ -cell.

**Step 3** We apply the previous lemma to the relative CW-complex  $(Y, Y_{n+1})$  of which all relative cells have dimension  $n \geq 2$ . Since  $\pi_i(Z, z) = 0$  for all  $i \geq n+1$ , the map  $f_{n+1}: Y_{n+1} \rightarrow Z$  extends to a map  $f: Y \rightarrow Z$ .

We have  $\Phi = f_*$  since  $\text{incl}_*$  is surjective and we can cancel it on the left.

We still have to show uniqueness up to based homotopy. Let  $f, f': Y \rightarrow Z$  be based continuous maps, such that  $\pi_n(f) = \pi_n(f'): \pi_n(Y) \rightarrow \pi_n(Z)$ . Since  $Y_{n-1} = *$ , they agree on  $Y_{n-1}$ . Let  $I$  be an index set as before for the  $n$ -cells,  $\chi_i: D^n \rightarrow Y$  characteristic maps for the  $i$ -th  $n$ -cell.  $[S^n \cong D^n / \partial D^n \xrightarrow{\chi_i} Y] \in \pi_n(Y, y)$  which implies  $[f \circ \chi_i] = f_*[\chi_i] = f'_*[\chi_i] = [f' \circ \chi_i]$ . So  $f \circ \chi_i, f' \circ \chi_i: D^n / \partial D^n \cong S^n \rightarrow Z$  are based homotopic. Choose such a homotopy for each  $i \in I$ , and glue them into a homotopy  $H$

$$f|_{Y_n} \sim f'|_{Y_n}: \bigvee_I S^n \rightarrow Z$$

We apply the lemma from last time to the relative CW-complex  $(Y \times [0, 1], Y \times \{0\} \cup Y_n \times [0, 1] \cup Y \times \{1\})$  all of whose relative cells have dimension  $\geq n+2$ . So by the lemma, the map

$$f \cup H \cup f': Y \times \{0\} \cup Y_n \times [0, 1] \cup Y \times \{1\}$$

has a continuous extension  $K: Y \times [0, 1] \rightarrow Z$ . This map is a based homotopy from  $f$  to  $f'$ .  $\square$

**Corollary 1.42.** Let  $n \geq 1$ ,  $A$  a group, abelian if  $n \geq 2$ . Let  $(X, \phi), (Y, \psi)$  be Eilenberg-MacLane spaces of type  $(A, n)$ . If  $X$  is a CW-complex, then there is a based continuous map  $f: X \rightarrow Y$ ,

unique up to based homotopy, such that

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{\cong f_*} & \pi_n(Y, y) \\ & \searrow \phi \cong & \swarrow \psi \cong \\ & A & \end{array}$$

commutes.

Moreover  $f$  is a weak homotopy equivalence, and a homotopy equivalence if  $Y$  is also CW.

## 1.5 Representativity of cohomology

Here  $A$  is abelian group and  $n \geq 0$ . The Aim of this subsection is a natural transformation of functors in  $Y$

$$[Y, K(A, n)] \rightarrow H^n(Y; A)$$

that is an isomorphism for CW-complexes. We want to construct a group structure on  $[Y, K(A, n)]$ , the "Homotopy group structure on  $K(A, n)$ ".

Our hypothesis are  $n \geq 1$ ,  $A$  abelian group,  $K(A, n)$  some EM-space of type  $(A, n)$  that is a CW-complex. Because  $A$  is abelian, the addition  $A \times A \rightarrow A$ ,  $(a, b) \mapsto a + b$  and inverse  $A \rightarrow A$ ,  $a \mapsto a^{-1}$  are group homomorphisms. We apply the previous theorem to  $Y = K(A, n) \times K(A, n)$  (an EM-space of type  $(A \times A, n)$ ) and  $ZFK(A, n)$  and the momomorphism

$$\pi_n(K(A, n) \times K(A, n)) \cong \pi_n(K(A, n)) \times \pi_n(K(A, n)) \xrightarrow{\cong} A \times A \xrightarrow{+} A \xrightarrow{\cong} \pi_n(K(A, n))$$

So there is a continuous map  $\mu: K(A, n) \times K(A, n) \rightarrow K(A, n)$  that induces the addition on  $\pi_n$ <sup>14</sup>.

Similarly, the composite  $\pi_n(K(A, n)) \cong A \xrightarrow{a \mapsto -a} A \cong \pi_n(K(A, n))$  is realized by a based continuous map  $i: K(A, n) \rightarrow K(A, n)$  unique up to based homotopy.

$\mu$  is associative up to based homotopy:

$$\mu \circ (\mu \times \text{Id}), \mu \circ (\text{id} \times \mu): K(A, n)^3 \rightarrow K(A, n)$$

Bothe induce the same map on  $\pi_n$ , so by the theorem,  $\mu \circ (\mu \times \text{Id}) \sim \mu \circ (\text{id} \times \mu)$ , i.e.  $\mu$  is homotopy associative.

Let  $\tau: K(A, n) \times K(A, n) \rightarrow K(A, n) \times K(A, n)$  be the flip map  $\tau(x, y) = (y, x)$ . Then  $\mu \circ \tau, \mu: K(A, n) \times K(A, n) \rightarrow K(A, n)$  are by applying the theorem homotopic, i.e.  $\mu$  is homotopy commutative.

Since the constant map  $*: K(A, n) \rightarrow K(A, n)$  to the basepoint is 0 on  $\pi_n$ , the theorem shows that  $\mu \circ (\text{id}, i) = *$ . So  $i: K(A, n) \rightarrow K(A, n)$  is a homotopy inverse for  $\mu$ .

**Upshot.**  $K(A, n)$  is a "homotopy abelian group".

**Remark.** It is possible to even take  $K(A, n)$  as a topological abelian group. E.g.  $|\tilde{A}[\Delta^n / \partial\Delta^n]|$ .

**Construction 1.43.** For every (compactly generated) space  $Y$ , the set of homotopy classes of maps  $[Y, K(A, n)]$  becomes an abelian group via

$$+: [Y, K(A, n)] \times [Y, K(A, n)] \rightarrow [Y, K(A, n)]$$

$$([f: Y \rightarrow K(A, n)], [g: Y \rightarrow K(A, n)]) \mapsto [Y \xrightarrow{(f,g)} K(A, n) \times K(A, n)] \xrightarrow{\mu_*} [Y, K(A, n)]$$

<sup>14</sup>some mumbling about uncountable  $A$  and compactly generated topology, retopologizing  $K(A, n) \times K(A, n)$

The homotopy associativity, homotopy commutativity and homotopy inverse properties of  $(\mu, i)$  imply that  $+$  is associative, commutative, and has an inverse  $-[f] = [i \circ f]$ .

[3.11.2025, Lecture 7]
[05.11.2025, Lecture 8]

**Example 1.44.** We have in this class not talked about Vector- and Line-bundles. Schwede encourages us to do so.  $\mathbb{R}P^\infty$  and  $\mathbb{C}P^\infty$  are EM-spaces of type  $(\mathbb{Z}, 1)$  and  $(\mathbb{Z}, 2)$  respectively, so they have "homotopy group structures".

So we get abelian group structures on  $[Y, \mathbb{R}P^\infty]$ , and  $[Y, \mathbb{C}P^\infty]$ .

Also  $\mathbb{R}P^\infty$  and  $\mathbb{C}P^\infty$  classify real/complex line bundles for compact spaces  $Y$ ,

$$[Y, \mathbb{R}P^\infty] \xrightarrow{\cong} \text{Pic}_{\mathbb{R}}(Y) = \text{groups under } \otimes \text{ of isoclasses of line bundles over } Y$$

$$[f: Y \rightarrow \mathbb{R}P^\infty \mapsto f^*(\gamma)]$$

That is an iso of abelian groups  $\mu^*(\gamma) \cong p_1^*(\gamma) \otimes p_2^*(\gamma)$  as line bundles over  $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ .

Similarly  $[Y, \mathbb{C}P^\infty] \cong \text{Pic}_{\mathbb{C}}(Y)$ .

**Example 1.45.** A strictly associative & commutative model for  $K(\mathbb{Z}/2, 1)$  and  $K(\mathbb{Z}/2, 2)$ .

The ?? as polynomial algebra on  $\mathbb{C}[x]$  has no zero-divisors. So multiplication restricts to a commutative and associative operation

- $(\mathbb{C}[x] \setminus \{0\}) \times (\mathbb{C}[x] \setminus \{0\}) \rightarrow \mathbb{C}[x] \setminus \{0\}$  For  $z \in \mathbb{C} \setminus 0$   $z(fg) = (zf)g = f(zg)$  for  $f, g \in \mathbb{C}[x]$   
So this descnds to a well defined map

$$(\mathbb{C}[x] \setminus \{0\})/\mathbb{C}^* \times \mathbb{C}[x] \setminus \{0\}/\mathbb{C}^* \rightarrow (\mathbb{C}[x] \setminus \{0\})/\mathbb{C}^*$$

which then is a multiplication on  $\mathbb{C}P^\infty$ . This structure is associative and commutative, but not invertible.

#### Lemma 1.46: Fundamental class (non-standard-notation)

Let  $(X, \phi)$  be an EM-space of type  $(A, n)$ . Then there is a unique class  $\iota = \iota_{A,n} \in H^n(X, A)$  such that the composite

$$\pi_n(X, *) \xrightarrow{\text{Hurewicz}} H_n(X, \mathbb{Z}) \xrightarrow{\Phi(\iota)} A$$

equals  $\phi: \pi_n(X, x) \xrightarrow{\cong} A$ . Where we have

$$\Phi: H^n(X, A) \rightarrow \text{Hom}(H_n(X, \mathbb{Z}), A)$$

is the map from UCT.

$\iota$  is called the fundamental class.

*Proof.* Since  $X$  is  $(n - 1)$ -connected and  $A$  abelian, the Hurewicz homomorphism  $\pi_n(X, *) \rightarrow H_n(X, \mathbb{Z})$  is an isomorphism. So

$$\phi \circ h^{-1}: H_n(X, \mathbb{Z}) \xrightarrow{\cong} A \in \text{Hom}(H_n(X, \mathbb{Z}), A)$$

Since  $X$  is  $(n - 1)$ -connected, the  $\Phi: H^n(X, A) \xrightarrow{\cong} \text{Hom}(H_n(X, \mathbb{Z}), A)$  is an isomorphism and  $\iota \mapsto \phi \circ h^{-1}$ .  $\square$

**Construction 1.47.** An EM-space of type  $(A, 0)$  is the group  $A$  with the discrete topology and  $\iota_{A,0} \in H^0(A, A)$  is the class represented by  $\text{id}: A \rightarrow A$ .

### Theorem 1.48: Group homomorphism

Let  $n \geq 0$ ,  $A$  any abelian group. Then for all (compactly generated)  $Y$ , the map  $[Y, K(A, n)] \rightarrow H^n(Y, A)$ ,  $[f: Y \rightarrow K(A, n)] \mapsto f^*(\iota_{A,n})$  is an homomorphism of groups.

*Proof.* In 2 steps

**Step 1** The universal example:  $Y = K(A, n) \times K(A, n)$  and  $f = \mu: K(A, n)^2 \rightarrow K(A, n)$ . Let  $p_1, p_2: K(A, n)^2 \rightarrow K(A, n)$  denote the two projections. Then

$$[p_1] + [p_2] = [K(A, n)^2 \xrightarrow[\text{id}]{} K(A, n)^2 \xrightarrow{\mu} K(A, n)]$$

So we neet to show that  $\mu^*(\iota) = p_1^*(\iota) + p_2^*(\iota)$  in  $H^n(K(A, n) \times K(A, n), A)$ . By UCT and Hurewicz

$$\begin{aligned} H^n(K(A, n) \times K(A, n), A) &\xrightarrow{\Phi} \text{Hom}(H_n(K(A, n) \times K(A, n), \mathbb{Z}), A) \\ &\xrightarrow{\text{Hurewicz}} \text{Hom}(\pi_n(K(A, n), *)^2, A) \cong \text{Hom}(A \times A, A) \end{aligned}$$

The composit isomoprism sends  $\mu^*(\iota) \mapsto$  Addition  $A \times A \rightarrow A$  and  $p_1^*(\iota) \mapsto p_1: A \times A \rightarrow A$  and  $p_2^*(\iota) \mapsto p_2: A \times A \rightarrow A$ . In  $\text{Hom}(A \times A, A)$  the relation Addition  $= p_1 + p_2$  holds.

**Step 2** General case: Let  $Y$  be a compactly generated space,  $f, g: Y \rightarrow K(A, n)$ . Then

$$\begin{aligned} ([f] + [g])^*(\iota) &= [\mu \circ (f, g)]^*(\iota) = (f, g)^*(\mu^*(\iota)) \\ &\stackrel{\text{step 1}}{=} (f, g)^*(p_1^*(\iota) + p_2^*(\iota)) \\ &= (f, g)^*(p_1^*(\iota)) + (f, g)^*(p_2^*(\iota)) \\ &= (p_1 \circ (f, g))^*(\iota) + (p_2 \circ (f, g))^*(\iota) = f^*(\iota) + g^*(\iota) \end{aligned}$$

□

### Lemma 1.49

Let  $n \geq 1$ ,  $A$  an abelian group.  $Y$  an based CW-complex. Then the forgetful map

$$[Y, K(A, n)_*] \rightarrow [Y, K(A, n)]$$

is bijective.

*Proof.* Earlier we have done an exercise:  $Y$  is a non-degenerately<sup>15</sup> based space,  $Z$  any based space. Then the forgetful map

$$[Y, Z]_* \rightarrow [Y, Z]$$

is surjective if  $Z$  is path-connected and bijective if  $Z$  is simply connected.

This implies the lemma for  $n \geq 2$  and surjectivity for  $n = 1$ . Injectivity for  $n = 1$ : Let  $f, g: Y \rightarrow K(A, 1)$  be freely homotopic. Then  $f_* = g_*: H_1(Y, \mathbb{Z}) \rightarrow H_1(K(A, 1), \mathbb{Z})$ . Since  $A$  is abelian,  $\pi_1(K(A, 1), *) \xrightarrow{\cong} H_1(K(A, 1), \mathbb{Z})$ . So  $\pi_1(f) = \pi_1(g)$ . So by the earlier theorem 1.41,  $f$  and  $g$  are based homotopic. □

<sup>15</sup>inclusion of base point has HEP

**Theorem 1.50**

Let  $n \geq 0$ ,  $A$  an abelian group. Then for all CW-complexes  $Y$ , the homomorphism

$$[Y, K(A, n)] \mapsto H^n(Y, A), \quad [f] \mapsto f^*(\iota_{A,n})$$

is an isomorphism.

*Proof.* Both  $[_, K(A, n)]$  and  $H^n( _, A)$  takes disjoint unions in  $Y$  to products of abelian groups. Every CW-complex  $Y$  is the disjoint union of its path-components. So we can assume wlog, that  $Y$  is path-connected.

Induction on  $n$ . For  $n = 0$   $[Y, K(A, 0)] = [Y, A^{\text{disc}}] \cong A$  and also  $H^0(Y, A) = \text{Hom}(\pi_0(Y), A) \cong A$ . We do not check, that the map is indeed the identity, which we get between these two.

Now for  $n \geq 1$ . By the lemma we can replace free homotopy classes by based homotopy classes. We first treat a special case:

**Special Case**  $Y$  is  $n - 1$ -connected. Then  $H_{n-1}(Y, \mathbb{Z})$  is trivial for  $n \geq 2$  or free  $n = 1$ , so  $\text{Ext}(H_{n-1}(Y, \mathbb{Z}), A) = 0$ . So the UCT provides an isomorphism

$$\Phi: H^n(Y, A) \rightarrow \text{Hom}(H_n(Y, \mathbb{Z}), A)$$

also the Hurewicz map  $\pi_n(Y, *) \rightarrow H_n(Y, \mathbb{Z})$  is an isomorphism for  $n \geq 2$  or the universal homomorphism into an abelian group for  $n = 1$ . In any case

$$h^*: \text{Hom}(H_n(Y, \mathbb{Z}), A) \xrightarrow{\cong} \text{Hom}(\pi_n(Y, *), A)$$

is bijective. The composite

$$[Y, K(A, n)]_* \xrightarrow{[f] \mapsto f^*(\iota)} H^n(Y, A) \xrightarrow[\cong]{\Phi} \text{Hom}(H_n(Y, \mathbb{Z}), A) \xrightarrow[\cong]{h^*} \text{Hom}(\pi_n(Y, *), A)$$

sends  $f$  to  $\phi \circ \pi_n(f): \pi_n(Y, *) \rightarrow A$ . So again by theorem 1.41, the composite is bijective. Also  $\Phi$  and  $h^*$  are bijective. So the first map is bijective

**General case** We consider  $Y \cup_{Y_{n-1}} C(Y_{n-1}) = Y \cup_{Y_{n-1} < 0} Y_{n-1} \times [0, 1] \cup_{Y_{n-1} \times 1} \{*\}$ . This is  $n - 1$ -connected. We consider the following commutative diagram:

$$\begin{array}{ccccccc} [\Sigma Y_{n-1}, K(A, n)]_* & \xrightarrow{p^*} & [Y \cup_{Y_{n-1}} CY_{n-1}, K(A, n)]_* & \xrightarrow{i^*} & [Y, K(A, n)]_* & \longrightarrow & 0 \\ \downarrow & & \text{(special case)} \downarrow \cong & & \downarrow & & \\ H^n(\Sigma Y_{n-1}, A) & \xrightarrow{p^*} & H^n(Y \cup_{Y_{n-1}} CY_{n-1}, A) & \longrightarrow & H^n(Y, A) & \longrightarrow & 0 \end{array}$$

Here  $i$  and  $p$  are inclusion and projection  $p: Y \cup_{Y_{n-1}} CY_{n-1} \rightarrow \Sigma Y_{n-1}$ .

**Claim.** The upper row is exact:

Surjectivity of  $i^*$ : all relative cells of  $(Y \cup_{Y_{n-1}} CY_{n-1}, Y)$  have relative cells of dimension at most  $n$ . But  $\pi_i(K(A, n), *) = 0$  for all  $i < n$ . By a previous "extension lemma", every continuous map  $Y \rightarrow K(A, n)$  admits a continuous extension to  $Y \cup_{Y_{n-1}} CY_{n-1}$ . Since  $p \circ i$  is nullhomotopic,  $i^* \circ p^* = 0$ . Suppose  $f: Y \cup_{Y_{n-1}} CY_{n-1} \rightarrow K(A, n)$  is based s.t.  $i^*[f] = 0$ . So  $f|_Y: Y \rightarrow K(A, n)$  is based nullhomotopic. The HEP of the pair  $(Y \cup_{Y_{n-1}} CY_{n-1}, Y)$  lets us replace  $f$  by a based homotopic map  $f'$ , s.t.  $f'|_Y = \text{const}_*$ . So  $f' ??$

The lower sequence is also exact by the l.e.s. for a CW pair  $(Y \cup_{Y_{n-1}} CY_{n-1}, Y)$ .

The left map in the diagram is surjective. We let  $\rho: K(A, n - 1) \rightarrow \Omega K(A, n)$  be the unique up to based homotopy map that induces

$$\pi_{n-1}(K(A, n - 1)) \xrightarrow{\rho_*} \pi_{n-1}(\Omega K(A, n)) \cong \pi_n(K(A, n)) \cong A$$

Such that the composition is  $\phi_{n-1}$ .

Let  $\kappa_n: \Sigma K(A, n-1) \rightarrow K(A, n)$  be the adjoint of  $\rho$ .

Then the following commutes: I did not manage to copy the diagram.

So the right vertical map is an isomorphism by the 5-lemma.

□

[05.11.2025, Lecture 8]
[10.11.2025, Lecture 9]

Schwede remarks that when he was a student someone cleaned the board for the professor.

**Example 1.51.** If you have already heard about vector bundles somewhere:

$K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$ . For  $X$  a CW-complex, Also  $\text{Pic}(X) \xleftarrow{\cong} [X, \mathbb{R}P^\infty] H^1(X, \mathbb{Z}/2)$ .  $[\gamma: E \rightarrow X] \mapsto w_1(\gamma)$  is called the first Stiefel-Whitney class. So real line bundles are completely classified by their first Stiefel-Whitney class.

**Example 1.52.**  $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ . Then

$$\text{Pic}^{\mathbb{C}}(X) \xleftarrow{\cong} [X, \mathbb{C}P^\infty] \xrightarrow{\cong} H^2(X, \mathbb{Z})$$

so complex line bundles  $[\gamma: E \rightarrow X] \mapsto c_1(\gamma)$  are classified by their first Chern class.

## 1.6 Cohomology operations and the Steenrod algebra

**Aim.** Define "Steenrod squares", natural homomorphisms

$$\text{Sq}^i: H^n(X, \mathbb{F}_2) \rightarrow H^{n+i}(X, \mathbb{F}_2)$$

together with properties and applications.

Schwede will follow some notes of his own which are not yet finalized and he will eventually publish on his website.

### Definition 1.53: Cohomology operation

Let  $A, B$  be abelian groups. A  $A$  cohomology operation of type  $(A, n, B, m)$  is a natural transformation

$$\tau = \{\tau_X: H^n(X, A) \rightarrow H^m(X, B)\}$$

of set valued functors on topological spaces. Specifically these operations need not be group homomorphisms.

We write

$$\text{Oper}(A, n, B, m) = \text{abelian group under pointwise addition of all such operations.}$$

This is not standard notation.

### Lemma 1.54

The map  $\text{Oper}(A, n, B, m) \rightarrow H^m(K(A, n), B)$  given by  $\tau \mapsto \tau_{K(A, n)}(\iota_{A, n})$  is an isomorphism of groups.

*Proof.* On the homotopy category of CW-complexes, the pair  $(K(A, n), \iota_{A,n})$  represents the functor  $H^n(\_, A)$ . For any other functor  $F: Ho(\text{CW-complexes}) \rightarrow \text{Sets}$  we have

$$\text{Nat}(H^n(\_), A), F) \xrightarrow{\cong} F(K(A, n))$$

by the Yoneda lemma. Apply this to  $F = H^n(\_, B)$  yields a bijection

$$\text{Nat}_{Ho\text{CW} \rightarrow \text{Sets}}(H^n(\_), A), H^m(\_, B)) \cong H^m(K(A, n), B)$$

CW-approximation  $X^{CW} \xrightarrow{\sim} X$  of spaces  $X$  are natural and unique up to homotopy, and we are considering functors on  $\text{Top}$  which are homotopy invariant and take weak equivalences to isomorphisms.

$$\begin{array}{ccc} H^n(X^{CW}, A) & \xrightarrow{\tau_{X^{CW}}} & H^m(X^{CW}, B) \\ \downarrow \cong & & \downarrow \cong \\ H^n(X, A) & \xrightarrow{\tau_X} & H^m(X, B) \end{array}$$

So

$$\begin{aligned} \text{Nat}_{\text{Top}}(H^n(\_), A), H^m(\_, B)) &\xleftarrow{\cong} \text{Nat}_{Ho(\text{Top})}(H_n(\_), A), H^m(\_, B)) \\ &\xrightarrow{\cong} \text{Nat}_{Ho(\text{CW})}(H^n(\_), A), H^m(\_, B)) \end{aligned}$$

using weak homotopy equivalences and CW-approximation  $\square$

**Example 1.55.** 1.  $K(A, n)$  is simply connected, so  $H^0(K(A, n), B) \cong B$  for  $n \geq 1$  and

$$H^i(K(A, n), B) = 0 \text{ for } 1 \leq i \leq n - 1.$$

so the only operations of type  $(A, n, B, 0)$  are the constant functions with image in  $B$ .

For  $1 \leq i \leq n - 1$ ,  $\text{Oper}(A, n, B, i) = \{b\}$

2. Let  $f: A \rightarrow B$  be a group homomorphism. This induces a natural group homomorphism  
3.

$$f_*: H^n(X, A) \rightarrow H^n(X, B)$$

This is a additive cohomology operation for all  $n \geq 0$ . So we have

$$\begin{aligned} \text{Hom}_{grp}(A, B) &\rightarrow \text{Oper}(H^n(\_), A), H^n(\_, B)) \cong H^n(K(A, n), B) \\ &\xrightarrow[\text{UCT}]{\cong} \text{Hom}(H_n(K(A, n), \mathbb{Z}), B) \cong_{\text{Hurewicz}} \text{Hom}(\pi_n(K(A, n), *), B) \cong_{\phi} \text{Hom}(A, B) \end{aligned}$$

4. Bockstein Homomorphisms associated with a short exact sequence

$$0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$$

of abelian groups are cohomology operations  $\beta: H^n(X, A) \rightarrow H^{n+1}(X, B)$  for all  $n \geq 0$ .

$\beta$  only depends on the equivalence class of the s.e.s. in  $\text{Ext}(A, B)$ . So we get

$$\text{Ext}(A, B) \rightarrow \text{Oper}(A, n, B, n + 1) \xrightarrow{\cong} H^{n+1}(K(A, n), B)$$

where for  $n \geq 2$  the homomorphism from UCT is a isomorphism.

These are all cohomology operations for  $n \geq 2$ .

5. The group  $H_2(K(A, 1), \mathbb{Z})$  is not generally trivial. For a group  $G$  (not necessarily abelian)

$$H^2(K(G, 1), B) = \text{isomorphism classes of centric extensions, i.e.}$$

short exact sequences of groups

$$0 \rightarrow BE \rightarrow G \rightarrow 0$$

such that  $B$  is central in  $E$ . If  $G$  is abelian, then

$$\text{abelian extensions} = \text{Ext}(A, B) \rightarrow H^2(K(A, 1), B) = \text{Central extensions}$$

As an exercise we will see a sort of Bockstein that gives a cohomology operation from degree 1 to degree 2.

So far we have

$$\text{Oper}(A, n, B, i) \cong \begin{cases} B & i = 0 \\ 0 & 1 \leq i \leq n - 1 \\ \text{Hom}(A, B) & i = n \\ \text{Ext}(A, B) & i = n + 1 \geq 3 \end{cases}$$

6. Let  $R$  be a ring,  $k \geq 0$ . Then

$$H^n(X, R) \rightarrow H^{kn}(X, R) \quad x \mapsto x^k = x \cup \dots \cup x$$

is a cohomology operation of type  $(R, n, R, kn)$ . This is typically not additive.

For example

$$H^n(K(\mathbb{F}_2, 1, \mathbb{F}_2)) \cong \text{Oper}(\mathbb{F}_2, 1, \mathbb{F}_2, k) \in \{x \mapsto x^k\}$$

where this is equal to

$$H^n(\mathbb{R}P^\infty, \mathbb{F}_2)$$

and

$$H^*(\mathbb{R}P^\infty, \mathbb{F}_2) = \mathbb{F}_2[\iota_{\mathbb{F}_2, 1}]$$

So  $\text{Oper}(\mathbb{F}_2, 1, \mathbb{F}_2, k) = \{0, x \mapsto x^k\}$ .

We also calculated  $H^*(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}[\iota_{\mathbb{Z}, 2}]$ . So this gives

$$\text{Oper}(\mathbb{Z}, 2, \mathbb{Z}, k) = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z}\{x \mapsto x^m\} & k = 2m \end{cases}$$

7.  $\text{Oper}(\mathbb{Z}, 1, B, n) = ?$  We need to compute the cohomology of  $K(\mathbb{Z}, 1) = S^1$ .

$$H^n(S^1, B) = \begin{cases} B & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

This is now the operations.

This completes the operations we can really calculate.

**Reminder** Let  $R$  be a commutative ring. We discussed

$$\cup_1: C^n(X, R) \otimes C^m(X, R) \rightarrow C^{n+m-1}(X, R)$$

with coboundary formula  $\delta(f \cup_1 g) = (\delta f) \cup_1 g + (-1)^n f \cup_1 (\delta g) - (-1)^{n+m} f \cup g - (-1)^{(n+1)(m+1)} (g \cup f)$ . This was how we proofed commutativity of the cup product. We can use this to produce new cohomology operations.

Suppose  $n$  is even and  $f \in C^n(X, R)$  is a cocycle,  $\delta f = 0$ . Then  $\delta(f \cup_1 f) = 0$ , and the cohomology class  $[f \cup_1 f] \in H^{2n-1}(X, R)$  only depends on the cohomology class of  $f$ . Same for  $n$  odd and  $2 = 0$  in  $R$ .

So we get cohomology operations  $\text{Sq}_1: H^n(X, R) \rightarrow H^{2n-1}(X, R)$  for  $n$  even and

$$\text{Sq}_1: H^n(X, R) \rightarrow H^{2n-1}(X, R/2)$$

for  $n$  odd. They are defined by  $\text{Sq}_1[f] := [f \cup_1 f]$ .

**Preview.** We will define Stable cohomology operations of type  $(A, B, n) = \{\tau_i: H^i(\_, A) \rightarrow H^{n+i}(\_, B)\}$  and some compatibility with a suspensions isomorphism.

### Definition 1.56: Reduced cohomology operation

A reduced cohomology operation of type  $(A, n, B, m)$  is a natural transformation  $\tilde{H}^n(\_, A) \rightarrow \tilde{H}^m(\_, B)$  of set valued functors of based spaces.

Then  $\text{redOper}(A, n, B, m) \cong \tilde{H}^m(K(A, n), B) = H^m(K(A, n), B) = \text{Oper}(A, n, B, m)$  for  $m \geq 1$ . And for  $m = 0$  the constants operations induced by  $B \setminus \{0\}$  vanish.

**Construction 1.57.** Let  $(X, \phi)$  and  $(Y, \psi)$  be EM-spaces of type  $(A, n)$  and  $(A, n+1)$ , respectively for  $n \geq 1$ . By an earlier theorem, there is a unique-up-to-based homotopy map  $\rho: X \rightarrow \Omega Y$ , such that  $\rho_*: \pi_n(X, *) \rightarrow \pi_n(\Omega Y, *)$  agrees with the composite

$$\pi_n(X, *) \xrightarrow{\phi} A \xrightarrow{\psi^{-1}} \pi_{n+1}(Y, *) \cong \pi_n(\Omega Y, *)$$

we have  $S^n \wedge S^1 \cong S^{n+1}$  and by understanding  $S^n = \mathbb{R}^n \cup \{\infty\}$ , this isomorphism is by coordinates.

Further  $\Sigma X = X \wedge S^1$  and  $\Omega X = \text{maps}_*(S^1, X)$

We let  $\varepsilon: \Sigma X \rightarrow Y$  be the adjoint of  $\rho$ .

### Lemma 1.58

Let  $(X, \phi)$ ,  $(Y, \psi)$  be EM-spaces of type  $(A, n)$  and  $(A, n+1)$  respectively.

1. The following commutes:

$$\begin{array}{ccc} \pi_n(X, *) & \xrightarrow{\Sigma} & \pi_{n+1}(\Sigma X, *) & \xrightarrow{\varepsilon} & \pi_{n+1}(Y, *) \\ & & \searrow \phi & & \downarrow \psi \cong \\ & & & & A \end{array}$$

2. The fundamental class  $\iota_{A,n} \in H^n(X, A)$ ,  $\iota_{A,n+1} \in H^{n+1}(Y, A)$  fulfill

$$\varepsilon^*(\iota_{A,n+1}) = \Sigma(\iota_{A,n}) \in H^{n+1}(\Sigma X, A)$$

*Proof.* 1.

$$\begin{array}{ccccc} & \pi_n(X, *) & \xrightarrow{\Sigma} & \pi_{n+1}(\Sigma X, *) & \\ & \downarrow \rho_* & & \downarrow \varepsilon_* & \\ \phi \swarrow & \cong \pi_n(\Omega Y, *) & \xrightarrow{\psi} & \pi_{n+1}(Y, *) & \\ & \searrow & & \nearrow \cong & \\ & A & & & \end{array}$$

Commutes because  $\varepsilon$  is adjoint to  $\rho$ .

2. Equivalently  $\Sigma^{-1}(\varepsilon^*(\iota_{A,n+1})) = \iota_{A,n}$ . We show that  $\Sigma^{-1}(\varepsilon^*(\iota_{A,n+1}))$  has the property that defines  $\iota_{A,n}$ :

$$\pi_n(X, *) \xrightarrow[\{f\mapsto f_*[S^n]\}]{} H_n(X, \mathbb{Z}) \xrightarrow{\Sigma^{-1}(\varepsilon^*(\iota_{A,n+1})) \cap} A$$

equals  $\phi$ . We make our choices  $[S^n] \in H_n(S^n, \mathbb{Z})$  consistent with suspension.

Then

$$\begin{aligned} \Sigma^{-1}(\varepsilon^*(\iota_{A,n+1})) \cap f_*[S^n] &= f^*(\Sigma^{-1}(\varepsilon^*(\iota_{A,n+1}))) \cap [S^n] \\ &= \Sigma^{-1}((\Sigma f)^*(\varepsilon^*(\iota_{A,n+1}))) \cap [S^n] \\ &= (\Sigma f)^*(\varepsilon^*(\iota_{A,n+1})) \cap \Sigma[S^n] \\ &= (\varepsilon \circ (\Sigma f))^*(\iota_{A,n+1}) \cap [S^{n+1}] \\ &= \iota_{A,n+1} \cap (\varepsilon \circ (\Sigma f))_*[S^{n+1}] \\ &= \iota_{A,n+1} \cap \text{Hur}(\varepsilon \circ (\Sigma f)) \\ &= \psi[e \circ (\Sigma f)] = \phi(f) \end{aligned}$$

□

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[10.11.2025, Lecture 9]
[12.11.2025, Lecture 10]

Was not there, unfortunately

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[12.11.2025, Lecture 10]
[17.11.2025, Lecture 11]

We give some examples of stable cohomology operations

### Example 1.59.

- There are no stable operations of negative degrees.
- $\text{Stab}(A, B, 0) = \lim_{\leftarrow} H^n(K(A, n), B) \cong \text{Hom}(A, B)$ .
- $\text{Stab}(A, B, 1) = \lim_{\leftarrow} \text{Oper}(A, i, B, i+1) = \lim_{\leftarrow} \text{Ext}(A, B)$
- If  $R$  is a ring,  $x \mapsto x^n = x \cup \dots \cup x$  is usually not additive, so whenever it is not additive, it does not extend to a stable operation.
- If  $R$  is a  $\mathbb{F}_p$ -algebra,  $p$  prime, then  $x \mapsto x^p$  is an additive operation. We will see that  $p = 2$ ,  $x \mapsto x^2 \in \text{Oper}(\mathbb{F}_2, n\mathbb{F}_2, 2n)$  does extend to a stable operation  $\text{Sq}^n \in \text{Stab}(\mathbb{F}_2, \mathbb{F}_2, n)$

If  $p$  is an odd prime ( $x \mapsto x^p \in \text{Oper}(\mathbb{F}_p, 2k, \mathbb{F}_p, 2k \cdot p)$ ) does extend to a stable operation  $P^k \in \text{Stab}(\mathbb{F}_p, \mathbb{F}_p, 2k(p-1))$ . We will do this only for  $p = 2$  and Schwede will mumble how it is more complicated for other  $p$  but works.

### Definition 1.60: Steenrod-Algebra

The *Steenrod algebra* for  $A$  is the graded ring  $\mathcal{A}(A)$  with

$$\mathcal{A}^n(A) = \text{Stab}(A, A, n)$$

The graded multiplication

$$\circ: \mathcal{A}^n(A) \times \mathcal{A}^m(A) \rightarrow \mathcal{A}^{n+m}(A)$$

is composition of operations.

**Remark.** In practice we will mostly have  $A = \mathbb{F}_p$ .

$\mathcal{A}^*(A)$  acts tautologically on the  $A$ -cohomology on any space, by  $\tau \in \text{Stab}(A, A, n)$ ,  $x \in H^m(X, A)$ .

$$\tau \cdot x := \tau_m(x) \in H^{m+n}(X, A)$$

This makes  $H^*(X, A) = \{H^m(X, A)\}_{m \geq 0}$  into a graded left  $\mathcal{A}^*(A)$ -module.

The suspension isomorphisms  $\Sigma: H^*(X, A) \rightarrow H^{*+1}(\Sigma X, A) =: H^*(\Sigma X, A)[1]$  is an isomorphism of left  $\mathcal{A}^*(A)$ -modules.

### Theorem 1.61

Let  $X$  be an  $n$ -connected based space,  $n \geq 1$ ,  $\varepsilon: \Sigma(\Omega X) \rightarrow X$  counit of the adjunction  $(\Sigma, \Omega)$ . Then for all abelian groups  $B$ , the map

$$\varepsilon^*: H^i(X, B) \rightarrow H^i(\Sigma(\Omega X), B)$$

is an isomorphism for  $0 \leq i \leq 2n$  and injective for  $i = 2n + 1$ .

This gives us that the stable operations are in fact quite the nice inverse limit.

*Proof.* Since  $X$  is  $n$ -connected,  $\Omega X$  is  $(n - 1)$ -connected. Freudenthal suspension theorem says

$$\Sigma: \pi_i(\Omega X, *) \rightarrow \pi_{i+1}(\Sigma(\Omega X), *)$$

is an isomorphism for  $1 \leq i \leq 2n - 2$  and surjective for  $i = 2n - 1$ .

The composite

$$\pi_i(\Omega X, *) \xrightarrow{\Sigma} \pi_{i+1}(\Sigma(\Omega X), *) \xrightarrow{\varepsilon_*} \pi_{i+1}(X, *)$$

is an isomorphism, so the first map is injective. So  $\Sigma: \pi_i(\Omega X, *) \rightarrow \pi_{i+1}(\Sigma(\Omega X), *)$  is bijective for all  $1 \leq i \leq 2n - 1$ . So  $\varepsilon_*: \pi_{i+1}(\Sigma(\Omega X), *) \rightarrow \pi_{i+1}(X, *)$  is bijective for all  $1 \leq i \leq 2n - 1$  and surjective for  $i = 2n$ .

Set  $j = i + 1$ .

$$\varepsilon_*: \pi_j(\Sigma(\Omega X), *) \rightarrow \pi_j(X, *)$$

is bijective for  $1 \leq j \leq 2n$  and surjective for  $j = 2n + 1$ .

Relative CW-approximation provides a relative CW-complex  $(Z, \Sigma\Omega X)$  and a weak equivalence  $f: Z \xrightarrow{\sim} X$  extends  $\varepsilon$  and all relative celss have dimension  $\geq 2n + 2$ .

So  $H^i(Z, \Sigma\Omega X; A) = 0$  for  $i \leq 2n + 1$ . The long exact cohomology sequence of the pair shows that

$$H^i(Z, A) \rightarrow H^i(\Sigma\Omega X, A)$$

is an isomorphism for  $i \leq 2n$ . And the following is exact

$$0 \rightarrow H^{2n+1}(Z, A) \rightarrow H^{2n+1}(\Sigma\Omega X, A) \xrightarrow{\partial} H^{2n+2}(Z, \Sigma\Omega X, A)$$

so the last map is injective. □

**Corollary 1.62.** ( $X = K(A, n+1)$ ) Let  $n \geq 1$ ,  $A, B$  abelian groups,  $\varepsilon: \Sigma K(A, n) \rightarrow K(A, n+1)$ . Then

$$\varepsilon^*: H^i(K(A, n+1), B) \rightarrow H^i(\Sigma K(A, n), B)$$

is bijective for  $i \leq 2n$  and injective for  $i = 2n + 1$ .

### Corollary 1.63.

$$\text{Stab}(A, B, n) \cong \varprojlim$$

$$H^{n+i+1}(K(A, i+1), B) \xrightarrow{\cong} H^{n+i}(K(A, n), B) \xrightarrow{\cong} \dots \xrightarrow{\cong} H^{2n+1}(K(A, n+1), B) \hookrightarrow H^{2n+1}(K(A, n+1), B) \rightarrow \dots$$

So

$$\text{Stab}(A, B, n) \xrightarrow[\text{isomorphism}]{\cong} H^{2n+1}(K(A, n+1), B), \quad \tau \mapsto \tau_{n+1}(\iota_{A, n+1})$$

and

$$\text{Stab}(A, B, n) \xrightarrow{\text{injective}} H^{2n}(K(A, n), B), \quad \tau \mapsto \tau_n(\iota_{A, n})$$

Since  $\tau_n$  is additive operation, the class  $u = \tau_n(\iota_{A, n}) \in H^{2n}(K(A, n), B)$  satisfies  $\mu^*(u) = p_1^*(u) + p_2^*(u)$ .

So  $\text{Stab}(A, B, n) \hookrightarrow \{u \in H^{2n}(K(A, n), B) \mid \mu^*(u) = p_1^*(u) + p_2^*(u)\}$ .

### Theorem 1.64

The map  $\text{Stab}(A, B, n) \rightarrow \{u \in H^{2n}(K(A, n), B) \mid \mu^*(u) = p_1^*(u) + p_2^*(u)\}$  given by  $\tau = \{\tau_i\}_{i \geq 0} \tau_n(\iota_{A, n})$  is an isomorphism

Equivalently we could say the map

$$\text{Stab}(A, B, n) \rightarrow \text{Oper}^{\text{add}}(A, n, B, 2n), \quad \tau \mapsto \tau_n$$

is an isomorphism

To proof this we would need a loooot of simplicial homotopy theory which we did not do. We will use this fact to construct Steenrod squares anyways. Later on, Schwede will show a construction not relying on this fact.

**Remark.** If  $A = B = R$  is a ring, then

$$\times: H^k(X, R) \times H^m(Y, R) \mapsto H^{k+m}(X \times Y, R), \quad x \times y = p_1^*(x) \cup p_2^*(y)$$

Then  $\mu^*(u) = p_1^*(u) + p_2^*(u)$  is equivalent to  $\mu^*(u) = u \times 1 + 1 \times u$ , i.e. it is "primitive".

**Example 1.65.** Take  $A = B = \mathbb{F}_2$ ,  $x \mapsto x^2 \in \text{Oper}(\mathbb{F}_2, n, \mathbb{F}_2, 2n)$  and we have

$$(x + y) \cup (x + y) = x \cup x + x \cup y + y \cup x + y \cup y = x^2 + y^2$$

so it is additive. Hence we have a unique stable operation  $\text{Sq}^n = \{\text{Sq}_i^n\}_{i \geq 0} \in \text{Stab}(\mathbb{F}_2, \mathbb{F}_2, n) = \mathcal{A}^n(\mathbb{F}_2) = \mathcal{A}_2^n$  the mod 2 Steenrod algebra.

This operation is uniquely characterised by the property

$$\text{Sq}^n(x) = x^2$$

for all  $x \in H^n(X, \mathbb{F}_2)$ . We want to see how this operation looks in other degrees. We can only do this for  $\mathbb{F}_2$  coefficients, constructing the Steenrod squares in this way. For general  $\mathbb{F}_p$  we will need to use another strategy.

**Example 1.66.** For  $n = 0$  we have  $\iota_0^2 = \iota_0$  where  $\iota_0 \in H^0(K(\mathbb{F}_2, 0), \mathbb{F}_2) = H^0(\mathbb{F}_2, \mathbb{F}_2)$  and  $\iota_0$  corresponds to  $\text{id}_{\mathbb{F}_2}$ . So we get  $\text{Sq}^0 = \text{id}$ .

For  $n = 1$ , we have seen earlier in an exercise, for all  $x \in H^1(X, \mathbb{F}_2)$ ,  $v^2 = \beta(x)$  where  $\beta$  is the Bockstein of the short exact sequence

$$0 \rightarrow \mathbb{F}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{F}_2 \rightarrow 0$$

where we have  $\beta: H^n(X, \mathbb{F}_2) \rightarrow H^{n+1}(X, \mathbb{F}_2)$  a stable operation. We get  $\text{Sq}^1 = \beta$ .

### Theorem 1.67: Properties of Steenrod squares

For each  $i \geq 0$ , there is a unique stable mod  $-2$  cohomology operation  $\text{Sq}^i$ , such that  $\text{Sq}^i(x) = x^2$  for all  $x \in H^i(X, \mathbb{F}_2)$ . Moreover, the following properties hold

1.  $\text{Sq}^0 = \text{id}, \text{Sq}^1 = \beta$
2. (Unstability) For all  $x \in H^n(X, \mathbb{F}_2)$  and  $i > n$ ,  $\text{Sq}^i(x) = 0$
3. (Cartan Formula)  $x, y \in H^*(X, \mathbb{F}_2)$  homogenous

$$\text{Sq}^i(x \cup y) = \sum_{a+b=i} \text{Sq}^a(x) \cup \text{Sq}^b(y)$$

*Proof.* 1. Was the above example

2. Consider the iterated suspension isomorphism  $i > n$

$$\Sigma^{i-n}: H^n(X, \mathbb{F}_2) \rightarrow H^i(X, \mathbb{F}_2)$$

$$\Sigma^{i-n}(\text{Sq}^i(x)) = \text{Sq}^i(\underbrace{\Sigma^{i-n}(x)}_{i\text{-dim}}) = (\Sigma^{i-n}(x)) \cup (\Sigma^{i-n}(x)) = 0$$

as cup products vanish in suspensions. Since  $\Sigma^{i-n}$  is an isomorphism,  $\text{Sq}_i(x) = 0$

3. Will be deduced from

**Proposition 1.68** (External Catan Formula). For all spaces  $X, Y$  and all  $x \in H^n(X, \mathbb{F}_2)$ ,  $y \in H^m(Y, \mathbb{F}_2)$ ,  $i \geq 0$ :

$$\text{Sq}^i(x \times y) = \sum_{a+b=i} \text{Sq}^a(x) \times \text{Sq}^b(y)$$

The external catan implies the internal catan by

$$x \cup y = \Delta^*(x/times y), \quad \Delta: X \rightarrow X \times X \text{ diagonal}$$

for  $x, y \in H^*(X)$ . Use external form with  $\Delta^*: H^*(X \times X, \mathbb{F}_2) \rightarrow H^*(X, \mathbb{F}_2)$ .

*Proof.* If  $i > m + n$  both sides of the desired equation are 0 by instability. For  $i = n + m$

$$\text{Sq}^{n+m}(x \times y) = (x \times y) \cup (x \times y) = (x \times x) \cup (y \times y) = x^2 \times y^2 = \text{Sq}^m(x) \times \text{Sq}^n(y) = \sum_{a+b=n+m} \text{Sq}^a(x) \times \text{Sq}^b(y)$$

again using instability on most of the sum-terms. For  $v > n + m$  Induction on  $n + m$ . For  $n + m = 0$ ,  $n = m = 0$ , so  $\text{Sq}^0(x \times y) = x \times y = \text{Sq}^0(x) \times \text{Sq}^0(y)$ .

Now let  $n + m \geq 1$ ,  $i < n + m$ .  $X = K(n) = K(\mathbb{F}_2, n)$ ,  $K(m) = K(\mathbb{F}_2, m) = Y$ ,  $x = \iota_n$ ,  $y = \iota_m$ . For  $p \leq 2n - 1$ :

$$\varepsilon^*: H^p(K(n), \mathbb{F}_2) \rightarrow H^p(\Sigma K(n-1), \mathbb{F}_2)$$

is injective for  $q \leq 2m - 1$

$$\varepsilon^*: H^q(K(m), \mathbb{F}_2) \rightarrow H^q(\Sigma K(m-1), \mathbb{F}_2)$$

is injective and

$$H^k(K(n) \times K(m), \mathbb{F}_2) \xrightarrow[\text{K\"unneth theorem}]{} \bigoplus_{p+q=k} H^p(K(n), \mathbb{F}_2) \otimes H^q(K(m), \mathbb{F}_2)$$

so

$$(\varepsilon^* \otimes 1, 1 \otimes \varepsilon^*) \hookrightarrow \bigoplus_{p+q=k} H^p(\Sigma K(n-1), \mathbb{F}_2) \otimes H^q(K(m), \mathbb{F}_2) \oplus H^p(K(n), \mathbb{F}_2^2) \otimes H^q(\Sigma K(n-1), \mathbb{F}_2)$$

is injective for all  $k \leq 2n + 2m - 1$ .

$\text{Sq}^i(x \times y) \in H^{n+m+i}(K(n) \times K(m), \mathbb{F}_2)$ . To prove the desired relation, it suffices to prove it after applying  $\varepsilon^* \otimes 1$  and  $1 \otimes \varepsilon^*$ .

$$\begin{aligned} (\varepsilon \times 1)^*(\text{Sq}^i(\iota_n \times \iota_m)) &= \text{Sq}^i((\varepsilon \times 1)^*(\iota_n \times \iota_m)) \\ &= \text{Sq}^i(\varepsilon^*(\iota_n) \times \iota_m) \\ &= \text{Sq}^i(\Sigma(\iota_{n-1}) \times \iota_m) \\ &= \Sigma(\text{Sq}^i(\iota_{n-1} \times \iota_m)) \\ &\stackrel{\text{induction}}{=} \Sigma \sum_{a+b=i} \text{Sq}^a(\iota_{n-1}) \times \text{Sq}^b(\iota_m) \\ &= \sum_{a+b=i} \text{Sq}^a(\Sigma(\iota_{n-1}) \times \text{Sq}^b(\iota_m)) \\ &= \sum_{a+b=i} \text{Sq}_a(\varepsilon^*(\iota_n)) \times \text{Sq}^b(\iota_m) \\ &= (\varepsilon \times 1)^*\left(\sum_{a+b=i} \text{Sq}^a(\iota_n) \times \text{Sq}^b(\iota_m)\right). \end{aligned}$$

This proofs external caten in the universal example  $X = K(n), Y = K(m), x = \iota_n, y = \iota_m$ . For CW-complexes it follows by representability. In general by CW-approximation.  $\square$

$\square$

[17.11.2025, Lecture 11]  
[19.11.2025, Lecture 12]

Some standard applications of Steenrod squares:

**Reminder.** Let  $f: X \rightarrow Y$  be a continuous map. We want to understand when  $f$  is essential, i.e. not homotopic to a constant map.

- if  $f$  induces a non-zero map on  $\pi_n, H_n(\_, A), H^m(\_, B)$ , then  $f$  is essential.
- if  $f$  is nullhomotopic, the s.e.s. for

$$C(f) := X \times [0, 1] \cup_f Y, \quad i: Y \rightarrow C(f)$$

yields

$$H^{n+1}(X, A) \rightarrow H^n(Cf, A) \xrightarrow{i^*} H^n(Y, A) \xrightarrow{f^*} H^n(X, A)$$

If  $f^* = 0, H^*(Y, A) \rightarrow H^*(X, A)$  also gives a s.e.s. If  $f$  is nullhomotopic, then by a choice of nullhomotopy  $H: X \times [0, 1] \rightarrow Y$  from a constant map to  $f$  and we get

$$\sigma = H \cup \text{id}_Y: C(Y) \rightarrow Y$$

with  $\sigma \circ i = \text{id}_Y$ . So  $i: Y \rightarrow C(f)$  has a retraction, which induces a map in cohomology.

$$0 \rightarrow H^{k+1}(X, a) \xrightarrow{\partial} H^k(Cf, A) \xrightarrow{i^*} H^k(Y, A) \rightarrow 0$$

where  $\sigma^*$  gives a section to  $i^*$ .

Our strategy here is to endow cohomology with more natural structure to show there is no algebraic map that respects the structure and is a section to  $i^*$ .

The Cup-product:  $f = \eta: S^3 \rightarrow S^2$  the Hopf fibration has  $C(f) \cong \mathbb{C}P^2$ .

$$H^*(C(f), \mathbb{Z}) = H^*(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$$

and

$$\mathbb{Z}[x]/(x^3) = H^*(\mathbb{C}P^2, \mathbb{Z}) \xrightarrow{i^*} H^*(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}[x]/(x^2)$$

has additive sections, but *no* multiplicative sections. Hence there is no continuous retraction  $\iota: \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$ .

And hence we get  $\eta$  is essential.  $\pi_3(S^2, *) \cong \mathbb{Z}\{\eta\}$ .

We can do the same for  $S(\mathbb{H}^2) = \nu: S^7 \rightarrow S^4 = \mathbb{H}P^1$ . Hence  $\nu$  is essential and we had  $\pi_7(S^4) \cong \mathbb{Z}\{\nu\} \oplus$  a finite group Schwede forgot.

And once more the same for  $\sigma: S^{15} = S(\mathbb{O}^2) \rightarrow \mathbb{O} \cup \{\infty\} \cong S^8$ . We define  $C(\sigma) =: \mathbb{O}P^2$  and  $H^*(\mathbb{O}P^2, \mathbb{Z}) \cong \mathbb{Z}[z]/(z^3)$ . We see  $\sigma$  is essential,  $\pi_{15}(S^8) \cong \mathbb{Z}\{\sigma\} \oplus$  some finite group.<sup>16</sup>

- We want to see that  $\eta$  is stably essential, i.e. also suspensions of  $\eta$  are essential. Same for  $\nu, \sigma$ . We saw  $2 \cdot (\Sigma\eta) = 0$  in  $\pi_4(S^3)$ . And similarly also

$$24 \cdot (\Sigma\nu) = 0 \in \pi_8(S^5) \quad 240 \cdot (\Sigma\sigma) \in \pi_{16}(S^9)$$

### Definition 1.69: stably essential maps

The map  $f: X \rightarrow Y$  of based spaces is *stably essential* if  $\Sigma^n f: \Sigma^n X \rightarrow \Sigma^n Y$  is not homotopic to a constant map for all  $n \geq 0$ .

**Note.** Cohomology and cup product cannot show that  $\Sigma^n f$  is essential for  $n \geq 1$  because the cup product on the cohomology of  $\Sigma Y$  is trivial!

$\eta: S^3 \rightarrow S^2$  is stably essential, i.e.  $\Sigma^n \eta \not\simeq *$  for all  $n \geq 0$ . We want to show this using cohomology operations.

**Corollary 1.70.**  $\pi_1^{\text{st}} = \mathbb{Z}_2\{\eta\}$ .

$$\Sigma^n \eta: S^{n+3} \rightarrow S^{n+2} \rightarrow \Sigma^n C(\eta) \rightarrow S^{n+4}$$

where we write  $\text{Cone}(\Sigma\eta) \cong \Sigma^n(\eta)$ .

$$0 \rightarrow \tilde{H}^*(S^{n+4}, \mathbb{F}_2) \rightarrow \tilde{H}^*(\Sigma^n(C(\eta)), \mathbb{F}_2) \rightarrow \tilde{H}^*(S^{n+2}, \mathbb{F}_2) \rightarrow 0$$

Schwede explains how to draw cohomology to see there is no section to  $(\Sigma^{n_i})^*$  compatible with Steenrod operation  $\text{Sq}^2$ .

$H^*(X, \mathbb{F}_2)$  is a graded module over  $\mathcal{A}^*(\mathbb{F}_2)$ . Teh s.e.s. descends split as modules over  $\mathcal{A}^*(\mathbb{F}_2)$ . We have  $C(\eta) \cong \mathbb{C}P^2$ .  $H^*(C(\eta), \mathbb{F}_2) = H^*(\mathbb{C}P^2, \mathbb{F}_2) = \mathbb{F}[x]/(x^3)$  and we have  $\text{Sq}^2(x) = x^2 \neq 0$ .

So as the steenrod squares are stable cohomology operations, they are still non-zero after suspending, while the cup-product is gone.

$$\tilde{H}^k(\Sigma^n C(\eta), \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2\{\Sigma^n x\} & k = n + 2 \\ \mathbb{F}_2\{\Sigma^n(x^2)\} & k = n + 4 \\ 0 & \text{else} \end{cases}$$

and

$$\text{Sq}^2(\Sigma^n(x)) = \Sigma^n(X^2)$$

<sup>16</sup>some remark about how we can't proof that  $\sigma$  has infinite order.

Schwede rambles about how there are secondary cohomology operations and he will not talk about that.

For  $\nu$  we have  $C(\nu) \cong \mathbb{H}P^2$  and  $H^*(\mathbb{H}P^2, \mathbb{F}_2) \cong \mathbb{F}_2[y]/(y^3)$  for  $y \in H^4(\mathbb{H}P^2, \mathbb{F}_2)$ . He again draws the cohomology and concludes  $\Sigma^n(\nu) \not\simeq 0$  for all  $n \geq 0$  using  $\text{Sq}^4$ .

For  $C\sigma \cong \mathbb{O}P^2$  we use  $\text{Sq}^8$  to obtain the same result.

Schwede explains that  $\text{Sq}^n$  is decomposable for  $n \neq 2^i$ , i.e.

$$\text{Sq}^n = \sum_{a+b=n} \text{Sq}^a \circ \text{Sq}^b$$

where  $\circ$  is composition of operations. This is just motivation we will do this later.

We give one more application of the Steenrod squares.<sup>17</sup>

### 1.6.1 Moore Spaces

#### Definition 1.71: Moore Space

The mod  $n$  moore space is

$$M(n) = \text{Cone}(S^1 \xrightarrow{\cdot n} S^1)$$

For example  $M(2) \cong \mathbb{R}P^2$ . And we get

$$\tilde{H}_k(M(n), \mathbb{Z}) = \begin{cases} \mathbb{Z}/n & k = 1 \\ 0 & k \neq 1 \end{cases}$$

**Fact.**  $\times_n: S^1 \wedge M(n) \xrightarrow{\deg(n) \wedge \text{id}_{M(n)}} S^1 \wedge M(n)$ .

If  $p$  is odd then  $\times_p$  is stably nullhomotopy on  $M(p)$ . But

$$\times_2: S^1 \wedge M(2) \rightarrow S^1 \wedge M(2)$$

is stably essential. schwede draws us a picture to proof this. You could read it in his script. I did not copy this part.

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[19.11.2025, Lecture 12]	[24.11.2025, Lecture 13]
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We want to now construct the Steenrod-squares. We want to construct the total power operation:  $p$  prime.  $\mathcal{P}_p: H^n(X, \mathbb{F}_p) \rightarrow H^{np}(X \times L(p), \mathbb{F}_2)$  where  $L(p) =$  infinite dimensional lense space  $= S^\infty/C_p$  as generalizations of  $L(2) = \mathbb{R}P^\infty$ . By Künneth theorem we will have

$$H^*(X \times L(p), \mathbb{F}_p) \cong H^*(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(L(p), \mathbb{F}_p)$$

where the second part is 1-dimensional in every degree.

For  $p = 2$ ,  $H^*(L(2), \mathbb{F}_2) = H^*(\mathbb{R}P^\infty, \mathbb{F}_2) = \mathbb{F}_2[u]$  for  $u \in H^1(\mathbb{R}P^\infty, \mathbb{F}_2)$ .  $\mathcal{P}_2(X) = \sum_{i \geq 0} \text{Sq}^i(x) \times u^{n-i}$ . We check that all degrees match up.

And now in more detail:

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<sup>17</sup>Schwede does something for the people in his seminar and tells the rest of us to ignore.

**Definition 1.72**

We write  $S^\infty = \bigcup_{n \geq 0} S(\mathbb{C}^n)$  for the infinite dimensional sphere, where  $S(\mathbb{C}^n) = \{z \in \mathbb{C}^n \mid |z| = 1\}$ . We write

$$C_p = \{z \in \mathbb{C} \mid z^n = 1\}$$

for the  $p$ -th roots of unity.  $C_p$  acts freely on  $S^\infty$  by scalar multiplication. We set

$$L(p) = S^\infty / C_p = S^\infty / (v \sim \zeta_p v)$$

where  $\zeta_p = e^{2\pi i/p}$  product with  $p$ -th root of 1.

For  $p = 2$ , we have  $C_2 = \{\pm 1\}$ , this is the antipodal action on  $S^\infty$ .

Since the  $C_p$ -action on  $S^\infty$  is free and  $S^\infty$  is contractible, the quotient map

$$S^\infty \rightarrow S^\infty / C_p = L(p)$$

is the universal covering. So  $L(p)$  is an Eilenberg-Maclane space of type  $K(C_p, 1)$ .

$S^\infty$  has a CW-structure with  $S_{2k-1}^\infty = S(\mathbb{C}^k)$  and  $S_{2k}^\infty = \text{join in } S(\mathbb{C}^{k+1})$  of  $S_{2k-1}^\infty = S(\mathbb{C}^k \oplus 0)$  and  $\{(0, \dots, 0, z) \mid z \in C_p\}$ . He draws a picture on how this works.

This is a CW-structure on  $S^\infty$  with  $p$  cells in each dimension;  $C_p$  acts cellularly and free permuting the  $k$ -cells for all  $k$ .

As  $L(p)$  is a  $K(p, 1)$  we get

$$C_p \cong \pi_1(L(p), *) \cong H_1(L(p), \mathbb{Z})$$

By UCT we get

$$H^1(L(p), \mathbb{F}_p) \cong \text{Hom}(H_1(L(p), \mathbb{Z}), \mathbb{F}_p) \cong \text{hom}(C_p, \mathbb{F}_p)$$

mapping some  $x$  to  $\zeta_p \mapsto 1$ . We set  $y := \beta(x) \in H^2(L(p), \mathbb{F}_p)$ . Where  $\beta = \text{Bockstein}$  (for  $0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{F}_p \rightarrow 0$ ).

If  $p = 2$ , then  $y = \beta(x) = x^2$ . If  $p \geq 3$ , then  $x^2 = 0$ .

**Theorem 1.73**

Let  $p$  be an odd prime. Then

$$H^*(L(p), \mathbb{F}_p) = \mathbb{F}_p[x, y]/(x^2) = \mathbb{F}_p[y] \otimes \Lambda(x)$$

*Proof.*

$$H^k(L(p), \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p\{y^{k/2}\} & k \text{ even} \\ \mathbb{F}_p\{x \cdot y^{(k-1)/2}\} & k \text{ odd} \end{cases}$$

The  $C_p$ -action on  $S^\infty$  is cellular for the CW-structure, so the induced action on  $C_*^{\text{cell}}(S^\infty)$  makes theis into a complex of modules over  $\mathbb{Z}[C_p]$ . This makes  $C_k^{\text{cell}}(S^\infty)$  a free  $\mathbb{Z}[C_p]$ -module of rank 1.

choose characteristic maps for all of the  $k$ -cells; chooste the other characeristic maps by following with the action of  $C_p$ . We get  $e_k^0, e_k^1 = \zeta_p \cdot e_k^0, \dots, e_k^i, \dots, e_k^{p-1}$ .

**Claim.**

$$\partial^{\text{cell}}(e_k^0) = \begin{cases} e_{k-1}^0 - e_{k-1}^1 & k \text{ odd} \\ e_k^0 + e_k^1 + \dots + e_k^{p-1} & k \text{ even} \end{cases}$$

We again draw pictures to proof this claim. And some notes so scribbled I didn't follow. Now

$$C_*^{\text{cell}}(L(p)) = C_*^{\text{cell}}(S^\infty / C_p) = C_*^{\text{cell}}(S^\infty) \otimes_{\mathbb{Z}[C_p]} \mathbb{Z}$$

He "hits the picture with that tensor." We get that the connecting maps alternately are 0 and  $p$ . So

$$H_*(L(p), \mathbb{F}_p), H^*(L(p), \mathbb{F}_p)$$

is 1-dimensional in every degree.

For the multiplicative structure We show by induction on  $k$ , that

$$H^*(L(p)_{2k-1}, \mathbb{F}_p) = \mathbb{F}_p[x, y]/(x^2, y^k)$$

We have  $L(p)_{2k-1} = S(\mathbb{C}^k)/C_p$  with  $C_p$ -action free and orientation preserving. So  $L(p)_{2k-1}$  (the Lense space) is an orientable, connected, compact (smooth) manifold. So

$$H^*(L(p)_{2k-1}, \mathbb{F}_p)$$

satisfies Poincare duality. So

$$\cup: H^i(L(p)_{2k-1}, \mathbb{F}_p) \times H^{2k-1-i}(L(p), \mathbb{F}_p) \rightarrow H^{2k-1}(L(p), \mathbb{F}_p)$$

is a perfect pairing. So

$$\cup y: H^{2k-3}(L(p)_{2k-1}, \mathbb{F}_p) \rightarrow H^{2k-1}(L(p)_{2k-1}, \mathbb{F}_p)$$

is an isomorphism. However the first term is also  $H^{2k-3}(L(p)_{2k-3}, \mathbb{F}_p) = \mathbb{F}_p\{x \cdot y^{k-2}\}$ . So  $y^{k-1} \neq 0$ . So  $y^{k-1}$  generates  $H^{2k-2}(L(p)_{2k-1}, \mathbb{F}_p)$   $\square$

#### Definition 1.74: Extended power Construction

the  $p$ -th extended power of a space  $X$  is the space

$$D_p(X) = X^p \times_{C_p} S^\infty = (X^p \times S^\infty)/(x_1, \dots, x_p, v) \sim (x_2, \dots, x_p, x_1, \zeta_p \cdot v)$$

If  $X$  is based, the reduced  $p$ -th extended power is

$$\tilde{D}_p(X) = X^{\wedge p} \wedge_{C_p} S_+^\infty$$

$D_p(X)$  has different notation:  $X^p \times_{C_p} EC_p$  for some  $EC_p$  a free contractible  $C_p$  space.  $(X^p)_{hC_p}$  the homotopy orbit space.

We have  $\tilde{D}_p(X) = \frac{D_p(X)}{(\text{FatWedge}_{C_p}) \times S^\infty}$ . For  $Y$  unbased  $\tilde{D}_p(Y_+) \cong D_p(Y)_+$ .

**Proposition 1.75.** Let  $Y$  be a based  $(n-1)$ -connected CW-complex equipped with a continuous based  $C_p$ -action. Then the space  $Y \wedge_{C_p} S_+^\infty$  is  $(n-1)$ -connected. Moreover, the map

$$j: Y \rightarrow Y \wedge_{C_p} S_+^\infty \quad y \mapsto [y \wedge (1, 0 \dots)]$$

induces an isomorphism  $j_*: H_n(Y, A)/C_p \rightarrow H_n(Y \wedge_{C_p} S_+^\infty, A)$  where the left hand side is a abelian group quotient. And

$$j^*: H^n(Y \wedge_{C_p} S_+^\infty, A) \rightarrow H^n(Y, A)^{C_p}$$

is also an isomorphism, where  $H^n(Y, A)^{C_p}$  denotes fixpoints under the action of  $C_p$ .

*Proof.* The subquotients of the equivariant skeleton filtration on  $S^\infty$  are  $S_k^\infty/S_{k-1}^\infty \cong (C_p)_+ \wedge S^k$  where the isomorphism is  $C_p$ -equivariant.

This induces a filtration on  $Y \wedge_{C_p} S_+^\infty$  by  $\{Y \cap_{C_p} S_k^\infty\}_{k \geq 0}$ . with subquotients

$$\frac{Y \wedge_{C_p} (K_k^\infty)_+}{Y \wedge_{C_p} (S_{k-1}^\infty)_+} \cong Y \wedge_{C_p} ((C_p)_+ \wedge S^k) \cong Y \wedge S^k$$

with the end result being  $(n+k-1)$ -connected. So

$$Y \wedge_{C_p} (S_1^\infty)_+ \rightarrow Y \wedge_{C_p} S_+^\infty$$

induces isomorphisms on  $H_n(\_, A)$  and  $H^n\_, A$ . Hence

$$H_n(Y \wedge_{C_p} S(\mathbb{C})_+, A) \rightarrow H_n(Y \wedge_{C_p} S_+^0, A)$$

is an isomorphism.

We get a cofibre sequence of  $C_p$ -spaces.

$$C_p = \{z \in \mathbb{C} | z^p = 1\} \rightarrow S(\mathbb{C}) \rightarrow (C_p)_+ \wedge S^1$$

and we can apply  $Y \wedge_{C_p} \_\_$  to it. Missed what that looked like. We get

$$H_{n+1}(Y \wedge S^1, A) \rightarrow H_n(Y, A) \rightarrow H_n(Y \wedge_{C_p} S(\mathbb{C})_+, A) \rightarrow 0$$

where the first term is isomorphic to  $H_n(Y, A)$  and we want to understand the first map acting on homology. i missed a bit.

Since  $Y$  is  $n-1$ -connected, UCT gives

$$H^n(Y, A) \xrightarrow{\cong} \text{hom}(H_n(Y, \mathbb{Z}), A)$$

since the map in the UCT is natural, this isomorphism is  $C_p$ -equivariant. So it retracts to an isomorphism of  $C_p$ -fixed points.

$$H^n(Y, A)^{C_p} \xrightarrow{\cong} \text{Hom}(H_n(Y, \mathbb{Z}), A)^{C_p} \cong \text{Hom}(H_n(Y, \mathbb{Z})/C_p, A)$$

Since  $Y \wedge_{C_p} S_+^\infty$  is  $(n-1)$ -connected, so UCT gives

$$H^n(Y \wedge_{C_p} S_+^\infty, A) \xrightarrow{\cong} \text{hom}(H_n(Y \wedge_{C_p} S_+^\infty, \mathbb{Z}), A)$$

These data participate in a commutative diagramm I didn't copy. □

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[24.11.2025, Lecture 13]
[26.11.2025, Lecture 14]

We apply this proposition to  $Y = X^{\wedge p}$ , for  $X$  a based space.

**Proposition 1.76.** Let  $p$  be a prime,  $n \geq 1$ .

1. For every based space  $X$  and every  $x \in \tilde{H}^n(X, \mathbb{F}_p)$  the class

$$x \wedge \cdots \wedge x \in \tilde{H}^{np}(X^{\wedge p}, \mathbb{F}_p)$$

is invariant under the  $C_p$ -action.

2. There is a unique class  $\tilde{\iota}_{n,p} \in \tilde{H}^{np}(\tilde{D}_p(K(\mathbb{F}_p, n), \mathbb{F}_p))$ ,<sup>18</sup> s.t.

$$j^*(\tilde{\iota}_n, p) = \iota \wedge \cdots \wedge \iota \text{ p-times } \in H^{np}(K(\mathbb{F}_p, n)^{\wedge p}, \mathbb{F}_p)$$

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<sup>18</sup>Schwede remarks how this might look random to us right now. Can confirm.

*Proof.* 1. Recall that for based spaces  $X, Y$  and  $x \in \tilde{H}^k(X, \mathbb{F}_p), y \in \tilde{H}^l(Y, \mathbb{F}_p)$  we have

$$x \wedge y = (-1)^{kl} \tau_{x,y}^* y \wedge x$$

where  $\tau_{X,Y}: X \wedge Y \xrightarrow{\cong} Y \wedge X$  is switching around.

Let  $m \geq 2$ ,  $X_1, \dots, X_m$  based spaces

$$c_m: X_1 \wedge \cdots \wedge X_m \xrightarrow{\cong} X_2 \wedge \cdots \wedge X_m \wedge X_1$$

the cyclic permutation of factors,  $x_i \in H^{k_i}(X_i, \mathbb{F}_p)$ .

$$x_1 \wedge \cdots \wedge X_m = (-1)^{k_1 \cdot (k_2 + k_3 + \cdots + k_m)} c_m^*(x_2 \wedge \cdots \wedge x_m \wedge x_1)$$

For  $m = 2$  this is true. For  $m \geq 3$

$$\begin{array}{ccc} X_1 \wedge \cdots \wedge X_m & \xrightarrow{\quad} & X_2 \wedge \cdots \wedge X_m \wedge X_1 \\ X_2 \wedge \cdots \wedge X_{m-1} \wedge X_m & \downarrow c_{m-1}^* \wedge X_m & \nearrow \tau_{X_1, X_m} \\ X_2 \wedge \cdots \wedge X_{m-1} \wedge X_1 \wedge X_m & & \end{array}$$

$$c_m^*(x_2 \wedge \cdots \wedge x_m \wedge x_1) = (c_{m-1} \wedge X_m)^*(X_2 \wedge \cdots \wedge X_{m-1} \wedge \tau_{X_1, X_m})^*(x_2 \wedge \cdots \wedge x_m \wedge x_1) =$$

and this is where he continued with the next part. A lot of this calculation was missing.

Now specializing  $m = p$  prime,  $X_1 = \cdots = X_m = X, x_1 = x_2 = \cdots = x, k_1 = \cdots = k_p = m$  then

$$c_p^*(x \wedge \cdots \wedge x) = (-1)^{n \cdot (n+n+\cdots+n)} x \wedge \cdots \wedge x = (-1)^{n^2 \cdot (p-1)} x \wedge \cdots \wedge x = x \wedge \cdots \wedge x$$

2. We use  $K(\mathbb{F}_p, n)$  is  $n$ -connected. Hence  $K(\mathbb{F}_p, n)^{\wedge p}$  is  $np - 1$ -connected, with  $C_p$ -action by cyclic permutation. So

$$j^*: H^{np}(\tilde{D}_p(K(\mathbb{F}_p, n)), \mathbb{F}_p) = H^{np}(K(\mathbb{F}_p, n)^{\wedge p} \wedge_{C_p} S_+^\infty, \mathbb{F}_p) \rightarrow H^{np}(K(\mathbb{F}_p, n)^{\wedge p}, \mathbb{F}_p)^{C_p}$$

by

$$\tilde{\iota}_{n,p} \mapsto \iota \wedge \cdots \wedge \iota$$

which exists uniquely.

□

We let

$$\pi: D_p(K(\mathbb{F}_p, n)) \rightarrow \tilde{D}_p(K(\mathbb{F}_p, n))$$

be the projection. Set

$$\iota_{n,p} := \pi^*(\tilde{\iota}_{n,p}) \in H^{np}(D_p(K(\mathbb{F}_p, n)), \mathbb{F}_p)$$

The following commutes

$$\begin{array}{ccc} K(\mathbb{F}_p, n)^{\times p} & \xrightarrow{\pi} & K(\mathbb{F}_p, n)^{\wedge p} \\ \downarrow j & & \downarrow j \\ D_p(K(\mathbb{F}_p, n)) & \xrightarrow{\pi} & \tilde{D}_p(K(\mathbb{F}_p, n)) \end{array}$$

Then

$$j^*(\iota_{n,p}) = j^*(\pi^*(\tilde{\iota}_{n,p})) = \pi^*(j^*(\tilde{\iota}_{n,p})) F \pi^*(\iota \wedge \cdots \wedge \iota) = \iota \times \cdots \times \iota$$

Let  $X$  be a CW-complex. The diagonal map  $\Delta: X \rightarrow X^p$  is  $\Delta(x) = (x, \dots, x)$ . This is  $C_p$ -equivariant for the trivial action on the source and the cyclic permutation action on target.

$$\_ \times_{C_p} S^\infty: (X \times_{C_p} S^\infty) \xrightarrow{\Delta \times_{C_p} S^\infty} X^p \times_{C_p} D_p(X)$$

where the first term is also  $X \times L(p)$  and we call this  $\Delta_x$  sending  $x, [v] \mapsto [x, x, \dots, x, v]$

### Definition 1.77: Power operation

The  $p$ -th power operation  $P_p: H^n(X, \mathbb{F}_p) \rightarrow H^{np}(X \times L(p), \mathbb{F}_p)$  is defined by

$$x = f^*(\iota) = \Delta_X^*(D_p(f)^*(\iota_{n,p}))$$

In more detail  $f: X \rightarrow K(\mathbb{F}_p, n)$  is continuous, such that  $f^*(\iota) = x$ , where  $\iota = \iota_{\mathbb{F}_p, n} \in H^n(K(\mathbb{F}_p, n), \mathbb{F}_p)$ .

$$X \times L(p) \longrightarrow D_p(X) \xrightarrow{D_p(f)} D_p(K(\mathbb{F}_p, n))$$

$$P_p(x) \longleftarrow D_p(f)^*(\iota_{n,p}) \longleftarrow \iota_{n,p}$$

$$\text{so } P_p(\iota) = \Delta_{K(\mathbb{F}_p, n)}(\iota_{n,p})$$

Let  $j: X \rightarrow X \times L(p)$  be the map  $j(x) = (x, [1, 0 \dots, 0])$ . This new  $j$  fulfills  $\Delta_X \circ j = j$  for respective  $j$

**Proposition 1.78.** 1. The composite

$$H^n(X, \mathbb{F}_p) \xrightarrow{P_p} H^{np}(X \times L(p), \mathbb{F}_p) \xrightarrow{j^*} H^{np}(X, \mathbb{F}_p)$$

raises a class to its  $p$ -th power.

2. The exterior product interacts with the total power operation in the following way:  $x \in H^n(X, \mathbb{F}_p), y \in H^m(Y, \mathbb{F}_p)$

$$P_p(x \times y) = \Delta^*(P_p(x) \times P_p(y))$$

in  $H^{(n+m)\cdot p}(X \times Y \times L(p), \mathbb{F}_p)$ ,  $\Delta: X \times Y \times L(p) \rightarrow X \times L(p) \times Y \times L(p)$ ,  $(x, y, [v]) \mapsto (x, [v], y, [v])$ .

This is what gives rise to the cartan-formula.

*Proof.* 1. By naturality it suffices to show this relation for the universal example  $X f K(\mathbb{F}_p, n), x = \iota = \iota_{\mathbb{F}_p, n} \in H^n(K(\mathbb{F}_p, n), \mathbb{F}_p)$ . We get a commutative square

$$\begin{array}{ccc} K(\mathbb{F}_p, n) & \xrightarrow{\Delta} & K(\mathbb{F}_p, n)^p \\ \downarrow j & & \downarrow j \\ K(\mathbb{F}_p, n) \times L(p) & \xrightarrow{\Delta_{K(\mathbb{F}_p, n)}} & K(\mathbb{F}_p, n)^p \times_{C_p} S^\infty = D_p(K(\mathbb{F}_p, n)) \end{array}$$

we observe

$$j^*(P_p(i)) = j^*(\Delta_{K(\mathbb{F}_p, n)}^*(\iota_{n,p})) = \Delta^*(j^*(\iota_{n,p})) = \Delta^*(\iota \times \dots \times \iota) = \iota \cup \dots \cup \iota = \iota^p$$

2. By naturality it suffices to show the universal case  $X = K(\mathbb{F}_p, n) = K(n)$  and  $Y = K(\mathbb{F}_p, m) = K(m)$ ,  $x = \iota_n = \iota_{\mathbb{F}_p, n} \in H^n(K(n), \mathbb{F}_p)$ ,  $y = \iota_m \in H^m(K(m), \mathbb{F}_p)$ . We need the reduced diagonal

$$\tilde{\Delta}: \tilde{D}_p(X \wedge Y) \rightarrow \tilde{D}_p(X) \wedge \tilde{D}_p(Y)$$

given by  $[\underline{x} \wedge \underline{y} \wedge v] \mapsto [\underline{x} \wedge v] \wedge [\underline{y} \wedge v]$ . The following commutes

$$\begin{array}{ccc} (X \wedge Y)^p & \xrightarrow[\cong]{\text{shuffle}} & (X^{\wedge p}) \wedge (Y^{\wedge p}) \\ \downarrow j & & \downarrow j \wedge j \\ \tilde{D}_p(X \wedge Y) & \xrightarrow{\tilde{\Delta}} & \tilde{D}_p(X) \wedge \tilde{D}_p(Y) \end{array}$$

We let  $\tilde{c}: K(n) \wedge K(m) \rightarrow K(n+m)$  be the unique up to homotopy based map s.t.

$$\tilde{c}^*(\iota_{n+m}) = \iota_n \wedge \iota_m$$

It induces a based map

$$\tilde{D}_p(K(n) \wedge K(m)) \rightarrow \tilde{D}_p(K(n+m))$$

**Claim.**  $\tilde{\Delta}(\tilde{\iota}_{n,p} \wedge \tilde{\iota}_{m,p}) = \tilde{D}_p(\tilde{c})^*(\tilde{\iota}_{n+m,p})$ . This is in  $H^{(n+m)p}(\tilde{D}_p(K(n) \wedge K(m)), \mathbb{F}_p)$ .

Since  $K(n)$  is  $(n-1)$ -connected,  $K(m)$  is  $(m-1)$ -connected,  $K(n) \wedge K(m)$  is  $(n+m-1)$ -connected. So  $(K(n) \wedge K(m))^{\wedge p}$  is  $(p(n+m)-1)$ -connected. In a previous proposition we had  $j: (K(n) \wedge K(m))^{\wedge p} \rightarrow \tilde{D}_p(K(n) \wedge K(m))$  and  $j^*$  is injective.

$$\begin{aligned} j^*(\tilde{\Delta}^*(\tilde{\iota}_{n,p} \wedge \tilde{\iota}_{m,p})) &= \text{shuffle}^*((j \wedge j)^*(\tilde{\iota}_{n,p} \wedge \tilde{\iota}_{m,p})) \\ &= \text{shuffle}^*(j^*(\tilde{\iota}_{n,p}) \wedge j^*(\tilde{\iota}_{m,p})) \\ &= \text{shuffle}^*((\iota_n \wedge \cdots \wedge \iota_n) \wedge (\iota_m \wedge \cdots \wedge \iota_m)) \\ &= ((\iota_n \wedge \iota_m) \wedge \cdots \wedge (\iota_n \wedge \iota_m)) \\ &= \tilde{c}^*(\iota_{n+m}) \wedge \cdots \wedge \tilde{c}^*(\iota_{n+m}) \\ &= (\tilde{c} \wedge \cdots \wedge \tilde{c})^*(\iota_{n+m} \wedge \cdots \wedge \iota_{n+m}) \\ &= (\tilde{c} \wedge \cdots \wedge \tilde{c})^*(j^*(\tilde{\iota}_{n+m,p})) \\ &= j^*(\tilde{D}_p(\tilde{c})(\tilde{\iota}_{n+m,p})) \end{aligned}$$

Since  $j^*$  is injective, this proves the claim.

We turn the relation into an unreduced form.

$$\Delta: D_p(X \times Y) \rightarrow D_p(X) \times D_p(Y)$$

the diagonal. We define  $c = \tilde{c} \circ \pi: K(n) \times K(m) \xrightarrow{\pi} K(n) \wedge K(m) \xrightarrow{\tilde{c}} K(n+m)$ . We have

$$c^*(\iota_{n+m}) = \pi^*(\tilde{c}^*(\iota_{n+m})) = \pi^*(\iota_n \wedge \iota_m) = \iota_n \times \iota_m$$

If  $X$  and  $Y$  are based, the following commutes:

$$\begin{array}{ccc} X \times Y \times L(p) & \xrightarrow{\Delta} & X \times L(p) \times Y \times L(p) \\ \downarrow \Delta_{X \times Y} & & \downarrow \Delta_X \times \Delta_Y \\ D_p(X \times Y) & \xrightarrow{\Delta} & D_p(X) \times D_p(Y) \\ \downarrow \pi & & \downarrow \pi_0(\pi \times \pi) \\ \tilde{D}_p(X \wedge Y) & \xrightarrow{\tilde{\Delta}} & \tilde{D}_p(X) \wedge \tilde{D}_p(Y) \end{array}$$

$$\begin{aligned}
\Delta^*(\iota_{n,p} \times \iota_{m,p}) &= \Delta^*(\pi^*(\iota_{n,p} \wedge \iota_{m,p})) \\
&= \pi^*(\tilde{\Delta}^*(\iota_{n,p} \wedge \iota_{m,p})) \\
&= \pi^*(\tilde{D}_p(c)^*(\tilde{\iota}_{n+m}, p)) \\
&= D_p(c)^*(\pi^*(\tilde{\iota}_{n+m}, p)) \\
&= D_p(c)^*(\iota_{n+m}, p)
\end{aligned}$$

Thus  $P_p(\iota_n \times \iota_m) = P_p((c \circ \pi)^*(\iota_{n+m})) = \Delta_{K(n) \times K(m)}^*(:_p (c \circ \pi)^*(\iota_{n+m}, p)) = \Delta_{K(n) \times K(m)}(\Delta^*(\iota_{n,p} \times \iota_{m,p})) = \Delta^*((\Delta_{K(n)} \times \Delta_{K(m)})(\iota_{n,p} \times \iota_{m,p})) = \Delta^*(P_p(\iota_n) \times P_p(\iota_m))$

□

# **Appendix**

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