

UNIVERSITÄT BONN

Notes for the lecture

Algebraic Topology I

held by

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T_EXed by

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Corrections and improvements

If you have corrections or improvements, contact me via (s94jmalm@uni-bonn.de).

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Lecture

Organizatorial

For this term we will be doing unstable homotopy theory. Next term we will be doing stable homotopy theory. Note that there were 2 previous courses. Note that all important information is shared on the website <https://www.math.uni-bonn.de/people/schwede/at1-ws2526>. You can sign up for the previous topology courses und see the lecture videos for these courses there.

There are no lecture notes for this lecture specifically, but some similar materials are linked on the webpage.

Exercise sheets will be uploaded fridays and handed in 11 days later via eCampus. Registration for eCampus opens at 4 today.

For exam admission you will have to score 50% of the points on the exercise sheets and have presented 2 exercises in tutorial.

The first exam will be written in the last week of semester.

I fear I will not be able to copy pictures here.

1.1 Blakiers-Massy theorem/Homotopy excision

We start with a reminder on relative homotopy groups.

Definition 1.1: Relative Homotopy Groups

Let (X, A) be a space pair i.e. A is a subspace of a topological space X . We write

$$I = [0, 1] \quad I^n = [0, 1]^n \text{ the } n\text{-cube}$$

$$\partial(I^n) = \text{boundary of } I^n$$

$$I^{n-1} \subseteq I^n$$

via Inclusion on the first $n - 1$ coordinates.

$$J^{n-1} = I^{n-1} \times \{1\} \cup (\partial I^{n-1}) \times [0, 1]$$

He draws a picture for $n = 2$.

For $n \geq 1$ the n -th relative homotopy groups $\pi_n(X, A, x)$ is the set of triple homotopy classes of triple maps $x \in A \subseteq X$

$$(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, \{x\})$$

where a triple map takes each subset on the left into the subset on the right. A triple-homotopy must also conserve these conditions.

For $n \geq 2$ or $n = 1$ and $A = \{x\}$ the set $\pi_n(X, A, \{x\})$ has a group structure by concatenation in the first coordinate. He again draws a picture.

The group structure is commutative if $n \geq 3$ or $n = 2$ and $A = \{x\}$.

Definition 1.2: n -Connectedness

Let $n \geq 0$. A space pair (X, A) is n -connected, if the following equivalent conditions hold:

1. For all $0 \leq q \leq n$ every pair map $(I^q, \partial I^q) \rightarrow (X, A)$ is homotpic relative $\partial(I^q)$ to a map with image in A
2. For all $a \in A$, $\text{incl}_*: \pi_q(A, a) \rightarrow \pi_q(X, a)$ is bijective for $q \leq n$ and surjective for $q = n$.
3. $\pi_0(A) \rightarrow \pi_0(X)$ is bijective and for all $1 \leq q \leq n$ the relative homotopy group

$$\pi_q(X, A, x) \cong 0$$

Proof. You proof the equivalence using the LES¹ of homotopy groups. □

Let Y be a space, Y_1, Y_2 open subsets of $Y = Y_1 \cup Y_2$, $Y_0 := Y_1 \cap Y_2$.

Excision in homology shows that for all abelian groups B , $i \geq 0$

$$H_i(Y_2; Y_0, B) \rightarrow H_i(Y, Y_1; B)$$

is an isomorphism.

Excision does not generally hold for homotopy groups, i.e. for $x \in Y_0$

$$\text{incl}_*: \pi_i(Y_2, Y_0; x) \rightarrow \pi_i(Y, Y_1; x)$$

¹Long exact sequence

is **not** generally an isomorphism.

„Blakers Massey theorem implies that excision holds for homotopy groups in a range.“

Theorem 1.3: Blakers Massey

Let Y be a space, Y_1, Y_2 open subsets with $Y = Y_1 \cup Y_2, Y_0 := Y_1 \cap Y_2$. Let $p, q \geq 0$, such that for all $y \in Y_0$

$$\pi_i(Y_1, Y_0, y) = 0 \text{ for all } 1 \leq i \leq p$$

and

$$\pi_i(Y_2, Y_0, y) = 0 \text{ for all } 1 \leq i \leq q$$

Then for all $y \in Y_0$, the map

$$\text{incl}_*: \pi_i(Y_2, Y_0, y) \rightarrow \pi_i(Y, Y_1, y)$$

is an isomorphism for $1 \leq i < p + q$ and surjective for $i = p + q$. He notes how the referenced literature uses different indices. They have proofs in more detail and pictures, however Lück's script contains typos

Proof. Schwede explains he doesn't like the proof, it is too technical and not very enlightening.

We define what cubes are

Cubes in \mathbb{R}^n , $n \geq 1$. $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ the „lower left corner of the cube“

$\partial \in \mathbb{R}_{\geq 0}$ „side length of the cube“

$L \subset \{1, \dots, n\}$ „relevant dimensions“

$W = W(a, \delta, L) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq a_i + \delta \text{ for all } i \in L, x_i = a_i \text{ for all } i \in \{1, \dots, n\} \setminus L\}$

² A face W' of W is a subset of the form

$$W' = \{x \in W : x_i = a_i \text{ for all } i \in L_0, x_i = a_i + \delta \text{ for all } i \in L_1\}$$

for some subsets $L_0, L_1 \subseteq L$

Let $1 \leq p \leq n$ we define two subsets of a cube $W = (a, \delta, L)$.

$$K_p(W) = \{x \in W : x_i < a_i + \delta/2 \text{ for at least } p \text{ values of } i \text{ in } L\}$$

We call these „ p small coordinates“

$$G_p(W) = \{x \in W : x_i > a_i + \delta/2 \text{ for at least } p \text{ coordinates } i \text{ in } L\}$$

these are „ p -big coordinates“.

For $p > \dim(W)$, $K_p(W) = G_p(W) = \emptyset$. If $p + q \geq \dim(W)$, then $K_p(W) \cap G_q(W) = \emptyset$.

He draws pictures.

²W weil Würfel

Lemma 1.4: 1.14

It is Lemma 1.14 in Lück's Script.

Let (Y, A) be a space pair, $W \subseteq \mathbb{R}^n$ a cube, $f: W \rightarrow Y$ continuous. Suppose that for some $p \leq \dim(W)$, $f^{-1}(A) \cap W' \subseteq K_p(W')$ for all proper¹ faces W' of W .

Then there is a continuous map $g: W \rightarrow Y$ homotopic to f relative ∂W such that all $g^{-1}(A) \subseteq K_p(W)$

¹subcube of the boundary

Proof. Wlog: $W = I^n = W(0, 1, \{1, \dots, n\})$

Let I_2^n be the subcube $[0, 1/2]^n$. He draws a picture. $x_4 = (1/4, \dots, 1/4) \in I_2^n$. We define a continuous map $h: I^n \rightarrow I^n$ by radical projection away from x_4 . Picture. Let $r(y)$ be the ray from x to y . We map all of $r(y) \cap I^n \setminus I_2^n$ to the intersection point of $r(y)$ and ∂I^n and the rest linearly extends as far as required.³

Obviously⁴ h is homotopic relative boundary ∂I^n to the identity.

We set $g: f \circ h: I^n \rightarrow Y$, which is then homotopic relative $\partial(I^n)$ ⁵ to f .

It remains to show that $g^{-1}(A) \subseteq K_p(W)$. Consider $z \in I^n$ with $g(z) \in A$.

Case 1 for all $i = 1, \dots, n, z_i < 1/2$, i.e. $z \in I_2^n$, then $z \in K_n(I^n) \subseteq K_p(I^n)$.

Case 2 There is an $i \in \{1, \dots, n\}$, s.t. $z \geq 1/2$. Then $h(z) \in \partial(I^n)$. Let W' be some proper face of W , with $h(z) \in W'$. Since $f(h(z)) = g(z) \in A$, by hypothesis, $h(z) \in K_p(W')$, so $h(z) < 1/2$ for at least p coordinates. By expansion⁶ property of h , also p coordinates of z are small coordinates.

□

Proposition 1.5. Let Y_1, Y_2 be open subsets of $Y, Y_0 := Y_1 \cap Y_2$. Suppose that (Y_1, Y_0) is p -connected, (Y_2, Y_0) is q -connected. Let $f: I^n \rightarrow Y$ be continuous. Let $\mathcal{W} = \{W\}$ be a subdivision of I^n into subcubes of the same side length s.t. for all $W \in \mathcal{W}$ $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$. Then there is a homotopy $h: I^n \times I \rightarrow Y$ with $h_0 = f$ such that for all $W \in \mathcal{W}$:

1. If $f(W) \subseteq Y_j, j \in 0, 1, 2$, then $h_t(W) \subseteq Y_j$ for all $t \in [0, 1]$
2. If $f(W) \subseteq Y_0$, then $h_t|_W = f|_W$, i.e. h is constant on W .
3. If $f(W) \subseteq Y_1$, then $h_1^{-1}(Y_1 \setminus Y_0) \subseteq K_{p+1}(W)$.
4. If $f(W) \subseteq Y_2$, then $h_1^{-1}(Y_2 \setminus Y_0) \subseteq G_{q+1}(W)$.

Proof. We let $C^k \subseteq I^n$ be the union of all cubes in \mathcal{W} of dimension at most k . We construct homotopies $h[k]: C_k \times I \rightarrow Y$, such that for all $W \in \mathcal{W}, W \subseteq C_k$ conditions 1. to 4. hold, and $h[k]$ is constant on $C_{k-1} \times I$. Then the final $h[n]$ does the job.

Note. If $W \in \mathcal{W}$ and $f(W) \subseteq Y_0$ and 2. holds, then also 3. and 4. hold.

$$h^{-1}(Y_1 \setminus Y_0) = h_1^{-1}(Y_2 \setminus Y_0) = \emptyset$$

If $W \in \mathcal{W}$, is such that $f(W) \subseteq Y_1$ and $f(W) \subseteq Y_2$, then $f(W) \subseteq Y_1 \cap Y_2 = Y_0$. So each $W \in \mathcal{W}$ is in exactly one of the following cases

- $f(W) \subseteq Y_0$

³I hope this description is clear, hard without the picture.

⁴Meaning he's too lazy to come up with formulas for the map.

⁵I am very inconsistent in remembering these parantheses with the boundary operator. Just imagine it always being as here.

⁶no idea if this is the word he wrote

- $f(W) \subseteq Y_1$ and $f(W) \not\subseteq Y_1$
- $f(W) \subseteq Y_2$ and $f(W) \not\subseteq Y_2$

Inductive construction $k = 0$, i.e. vertexes of the cubes $w \in \mathcal{W}$. If $w \in Y_2$, take $h[0]_t = \text{const}_{w_0}$.

Suppose $f(W_0) \in Y_1$, but $f(w_0) \notin Y_2$. Since Y_1, Y_0 is 0-connected, there is a path $\pi: I \rightarrow Y_1$ from w_0 to a point in Y_0 . We take $h[0]$ as the path on w_0 . Analogous if $f(W_0) \in Y_2 \setminus Y_1$.

Inductive Step Let $W \in \mathcal{W}$ be a cube of exact dimension k . Then $\partial W = W \cap C_{k-1}$. Since $(W, \partial W)$ has the HEP, we can extend the previous homotopy $h[k-1]|_{\partial W}$ to some homotopy on W relative to $f|_W$. Let this be $h'[k]: C_k \times I \rightarrow Y$: this satisfies conditions 1. and 2. but not yet 3. and 4.

We produce another homotopy $h[k]''$ and set $h[k] = h[k]' * h[k]''$.

Consider a cube $W \in \mathcal{W}$ of dimension k .

If $f(W) \subseteq Y_0$ set $h[k]''$ as the constant homotopy on W .

If $h[k]'_1(W) \subseteq Y_1$, but $h[k]'_1(W) \not\subseteq Y_2$ there is a homotopy relative ∂W from $h[k]'_1$ to a map $f_1(W) \subseteq Y_0$.

If $k = \dim(W) > p$ then we use the lemma 1.4 for $f = h[k]'_1|_W$ and the resulting homotopy is $h[k]''|_W$.

If $h[k]'_1(W) \subseteq Y_2$ but $h[k]'_1(W) \not\subseteq Y_1$, use the complement case of the lemma⁷

□

□

⁷rest of the proof next lecture. I am very sure some words won't make sense, as they were unreadable on the board.

Appendix

List of definitions

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List of statements

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