

UNIVERSITÄT BONN

Notes for the lecture

Algebraic Topology I

held by

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T_EXed by

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Corrections and improvements

If you have corrections or improvements, contact me via (s94jmalm@uni-bonn.de).

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Lecture

Organizatorial

For this term we will be doing unstable homotopy theory. Next term we will be doing stable homotopy theory. Note that there were 2 previous courses. Note that all important information is shared on the website <https://www.math.uni-bonn.de/people/schwede/at1-ws2526>. You can sign up for the previous topology courses and see the lecture videos for these courses there.

There are no lecture notes for this lecture specifically, but some similar materials are linked on the webpage.

Exercise sheets will be uploaded fridays and handed in 11 days later via eCampus. Registration for eCampus opens at 4 today.

For exam admission you will have to score 50% of the points on the exercise sheets and have presented 2 exercises in tutorial.

The first exam will be written in the last week of semester.

I fear I will not be able to copy pictures here.

1.1 Blakiers-Massy theorem/Homotopy excision

We start with a reminder on relative homotopy groups.

Definition 1.1: Relative Homotopy Groups

Let (X, A) be a space pair i.e. A is a subspace of a topological space X . We write

$$I = [0, 1] \quad I^n = [0, 1]^n \text{ the } n\text{-cube}$$

$$\partial(I^n) = \text{boundary of } I^n$$

$$I^{n-1} \subseteq I^n$$

via Inclusion on the first $n - 1$ coordinates.

$$J^{n-1} = I^{n-1} \times \{1\} \cup (\partial I^{n-1}) \times [0, 1]$$

He draws a picture for $n = 2$.

For $n \geq 1$ the n -th relative homotopy groups $\pi_n(X, A, x)$ is the set of triple homotopy classes of triple maps $x \in A \subseteq X$

$$(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, \{x\})$$

where a triple map takes each subset on the left into the subset on the right. A triple-homotopy must also conserve these conditions.

For $n \geq 2$ or $n = 1$ and $A = \{x\}$ the set $\pi_n(X, A, \{x\})$ has a group structure by concatenation in the first coordinate. He again draws a picture.

The group structure is commutative if $n \geq 3$ or $n = 2$ and $A = \{x\}$.

Definition 1.2: n-Connectedness

Let $n \geq 0$. A space pair (X, A) is *n-connected*, if the following equivalent conditions hold:

1. For all $0 \leq q \leq n$ every pair map $(I^q, \partial I^q) \rightarrow (X, A)$ is homotpic relative $\partial(I^q)$ to a map with image in A
2. For all $a \in A$, $\text{incl}_*: \pi_q(A, a) \rightarrow \pi_q(X, a)$ is bijective for $q \leq n$ and surjective for $q = n$.
3. $\pi_0(A) \rightarrow \pi_0(X)$ is bijective and for all $1 \leq q \leq n$ the relative homotopy group

$$\pi_q(X, A, x) \cong 0$$

Proof. You proof the equivalence using the LES¹ of homotopy groups. □

Let Y be a space, Y_1, Y_2 open subsets of $Y = Y_1 \cup Y_2$, $Y_0 := Y_1 \cap Y_2$.

Excision in homology shows that for all abelian groups B , $i \geq 0$

$$H_i(Y_2; Y_0, B) \rightarrow H_i(Y, Y_1; B)$$

is an isomorphism.

Excision does not generally hold for homotopy groups, i.e. for $x \in Y_0$

$$\text{incl}_*: \pi_i(Y_2, Y_0; x) \rightarrow \pi_i(Y, Y_1; x)$$

¹Long exact sequence

is **not** generally an isomorphism.

„Blakers Massey theorem implies that excision holds for homotopy groups in a range.“

Theorem 1.3: Blakers Massey

Let Y be a space, Y_1, Y_2 open subsets with $Y = Y_1 \cup Y_2$, $Y_0 := Y_1 \cap Y_2$. Let $p, q \geq 0$, such that for all $y \in Y_0$

$$\pi_i(Y_1, Y_0, y) = 0 \text{ for all } 1 \leq i \leq p$$

and

$$\pi_i(Y_2, Y_0, y) = 0 \text{ for all } 1 \leq i \leq q$$

Then for all $y \in Y_0$, the map

$$\text{incl}_*: \pi_i(Y_2, Y_0, y) \rightarrow \pi_i(Y, Y_1, y)$$

is an isomorphism for $1 \leq i < p + q$ and surjective for $i = p + q$. He notes how the referenced literature uses different indices. They have proofs in more detail and pictures, however Lück's script contains typos

Proof. Schwede explains he doesn't like the proof, it is too technical and not very enlightening.

We define what cubes are

Cubes in \mathbb{R}^n , $n \geq 1$. $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ the „lower left corner of the cube“

$\partial \in \mathbb{R}_{\geq 0}$ „side length of the cube“

$L \subset \{1, \dots, n\}$ „relevant dimensions“

$W = W(a, \delta, L) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq a_i + \delta \text{ for all } i \in L, x_i = a_i \text{ for all } i \in \{1, \dots, n\} \setminus L\}$

² A face W' of W is a subset of the form

$$W' = \{x \in W : x_i = a_i \text{ for all } i \in L_0, x_i = a_i + \delta \text{ for all } i \in L_1\}$$

for some subsets $L_0, L_1 \subseteq L$

Let $1 \leq p \leq n$ we define two subsets of a cube $W = (a, \delta, L)$.

$$K_p(W) = \{x \in W : x_i < a_i + \delta/2 \text{ for at least } p \text{ values of } i \text{ in } L\}$$

We call these „ p small coordinates“

$$G_p(W) = \{x \in W : x_i > a_i + \delta/2 \text{ for at least } p \text{ coordinates } i \text{ in } L\}$$

these are „ p big coordinates“.

For $p > \dim(W)$, $K_p(W) = G_p(W) = \emptyset$. If $p + q \geq \dim(W)$, then $K_p(W) \cap G_q(W) = \emptyset$.

He draws pictures.

²W weil Würfel

Lemma 1.4: 1.14

It is Lemma 1.14 in Lück's Script.

Let (Y, A) be a space pair, $W \subseteq \mathbb{R}^n$ a cube, $f: W \rightarrow Y$ continuous. Suppose that for some $p \leq \dim(W)$, $f^{-1}(A) \cap W' \subseteq K_p(W')$ for all proper¹ faces W' of W .

Then there is a continuous map $g: W \rightarrow Y$ homotopic to f relative ∂W such that all $g^{-1}(A) \subseteq K_p(W)$

¹subcube of the boundary

Proof. Wlog: $W = I^n = W(0, 1, \{1, \dots, n\})$

Let I_2^n be the subcube $[0, 1/2]^n$. He draws a picture. $x_4 = (1/4, \dots, 1/4) \in I_2^n$. We define a continuous map $h: I^n \rightarrow I^n$ by radical projection away from x_4 . Picture. Let $r(y)$ be the ray from x to y . We map all of $r(y) \cap I^n \setminus I_2^n$ to the intersection point of $r(y)$ and ∂I^n and the rest linearly extends as far as required.³

Obviously⁴ h is homotopic relative boundary ∂I^n to the identity.

We set $g: f \circ h: I^n \rightarrow Y$, which is then homotopic relative $\partial(I^n)$ ⁵ to f .

It remains to show that $g^{-1}(A) \subseteq K_p(W)$. Consider $z \in I^n$ with $g(z) \in A$.

Case 1 for all $i = 1, \dots, n, z_i < 1/2$, i.e. $z \in I_2^n$, then $z \in K_n(I^n) \subseteq K_p(I^n)$.

Case 2 There is an $i \in \{1, \dots, n\}$, s.t. $z \geq 1/2$. Then $h(z) \in \partial(I^n)$. Let W' be some proper face of W , with $h(z) \in W'$. Since $f(h(z)) = g(z) \in A$, by hypothesis, $h(z) \in K_p(W')$, so $h(z) < 1/2$ for at least p coordinates. By expansion⁶ property of h , also p coordinates of z are small coordinates.

□

Proposition 1.5. ⁷ Let Y_1, Y_2 be open subsets of $Y, Y_0 := Y_1 \cap Y_2$. Suppose that (Y_1, Y_0) is p -connected, (Y_2, Y_0) is q -connected. Let $f: I^n \rightarrow Y$ be continuous. Let $\mathcal{W} = \{W\}$ be a subdivision of I^n into subcubes of the same side length s.t. for all $W \in \mathcal{W}$ $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$. Then there is a homotopy $h: I^n \times I \rightarrow Y$ with $h_0 = f$ such that for all $W \in \mathcal{W}$:

1. If $f(W) \subseteq Y_j, j \in 0, 1, 2$, then $h_t(W) \subseteq Y_j$ for all $t \in [0, 1]$
2. If $f(W) \subseteq Y_0$, then $h_t|_W = f|_W$, i.e. h is constant on W .
3. If $f(W) \subseteq Y_1$, then $h_1^{-1}(Y_1 \setminus Y_0) \subseteq K_{p+1}(W)$.
4. If $f(W) \subseteq Y_2$, then $h_1^{-1}(Y_2 \setminus Y_0) \subseteq G_{q+1}(W)$.

Proof. We let $C^k \subseteq I^n$ be the union of all cubes in \mathcal{W} of dimension at most k . We construct homotopies $h[k]: C_k \times I \rightarrow Y$, such that for all $W \in \mathcal{W}, W \subseteq C_k$ conditions 1. to 4. hold, and $h[k]$ is constant on $C_{k-1} \times I$. Then the final $h[n]$ does the job.

Note. If $W \in \mathcal{W}$ and $f(W) \subseteq Y_0$ and 2. holds, then also 3. and 4. hold.

$$h^{-1}(Y_1 \setminus Y_0) = h_1^{-1}(Y_2 \setminus Y_0) = \emptyset$$

If $W \in \mathcal{W}$, is such that $f(W) \subseteq Y_1$ and $f(W) \subseteq Y_2$, then $f(W) \subseteq Y_1 \cap Y_2 = Y_0$. So each $W \in \mathcal{W}$ is in exactly one of the following cases

- $f(W) \subseteq Y_0$

³I hope this description is clear, hard without the picture.

⁴Meaning he's too lazy to come up with formulas for the map.

⁵I am very inconsistent in remembering these parantheses with the boundary operator. Just imagine it always being as here.

⁶no idea if this is the word he wrote

⁷11.5 in Lück's notes

- $f(W) \subseteq Y_1$ and $f(W) \not\subseteq Y_1$
- $f(W) \subseteq Y_2$ and $f(W) \not\subseteq Y_2$

Inductive construction $k = 0$, i.e. vertexes of the cubes $w \in \mathcal{W}$. If $w \in Y_2$, take $h[0]_t = \text{const}_{w_0}$.

Suppose $f(W_0) \in Y_1$, but $f(w_0) \notin Y_2$. Since Y_1, Y_0 is 0-connected, there is a path $\pi: I \rightarrow Y_1$ from w_0 to a point in Y_0 . We take $h[0]$ as the path on w_0 . Analogous if $f(W_0) \in Y_2 \setminus Y_1$.

Inductive Step Let $W \in \mathcal{W}$ be a cube of exact dimension k . Then $\partial W = W \cap C_{k-1}$. Since $(W, \partial W)$ has the HEP, we can extend the previous homotopy $h[k-1]|_{\partial W}$ to some homotopy on W relative to $f|_W$. Let this be $h'[k]: C_k \times I \rightarrow Y$: this satisfies conditions 1. and 2. but not yet 3. and 4.

We produce another homotopy $h[k]''$ and set $h[k] = h[k]' * h[k]''$.

Consider a cube $W \in \mathcal{W}$ of dimension k .

If $f(W) \subseteq Y_0$ set $h[k]''$ as the constant homotopy on W .

If $h[k]'_1(W) \subseteq Y_1$, but $h[k]'_1(W) \not\subseteq Y_2$ there is a homotopy relative ∂W from $h[k]'_1$ to a map $f_1(W) \subseteq Y_0$.

If $k = \dim(W) > p$ then we use the lemma 1.4 for $f = h[k]'_1|_W$ and the resulting homotopy is $h[k]''|_W$.

If $h[k]'_1(W) \subseteq Y_2$ but $h[k]'_1(W) \not\subseteq Y_1$, use the complement case of the lemma⁸ □

[13.10.2025, Lecture 1]
[15.10.2025, Lecture 2]

Now for the actual proof of Blakers Massey Let $F(Y_1, Y, Y_2) = Y_1 \times_Y Y^{[0,1]} \times_Y Y_2$.

Let $F(Y_1, Y_1, Y_0)$ be the subspace of those $w \in F(Y_1, Y, Y_2)$ such that $w([0, 1]) \subseteq Y_1$.

Proposition 1.6. Assume that (Y_1, Y_0) is p -connected, (Y_2, Y_0) is q -connected. Then the pair $(F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$ is $(p + q - 1)$ -connected.

Proof. We consider a map of pairs

$$\phi: (I^n \partial I^n) \rightarrow (F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$$

for $1 \leq n \leq p + q + 1$. We want to homotop ϕ relative ∂I^n to a map with image in $F(Y_1, Y_1, Y_0)$. We use the adjoint

$$\text{maps}(X \times [0, 1], Z) \cong \text{maps}(X, Z^{[0,1]})$$

We let $\Phi: I^n \times I \rightarrow Y$ be the adjoint of ϕ , this is *admissable*, in the sense that

1. $\Phi(x, 0) \in Y_1$ for all $x \in I^n$
2. $\Phi(x, 1) \in Y_2$ for all $x \in I^n$
3. $\Phi(x, t) \in Y_1$ for all $x \in \partial I^n, t \in [0, 1]$

We want a homotopy of Φ through admissable maps to a map $\Phi': I^n \times I \rightarrow Y$ such that $\text{Im}(\Phi') \subseteq Y_1$.

Apply Proposition 11.5 to Φ (with $n+1$ instead of n). This gives a homotopy through admissable maps to $\Psi = g$. Let $h: I^n \times I \times I \rightarrow Y$ be a homotopy witnessing this. $h_0 = \Phi, h_1 F\Psi$.

Consider the projection $\text{pr}: I^n \times I \rightarrow I^n$ away from the last coordinate.

Claim. The image under pr of $\Psi^{-1}(Y \setminus Y_1)$ and $\Psi^{-1}(Y \setminus Y_2)$ are disjoint.

Suppose there is $y \in I^n$ in the intersection of the preimages, so $z_1 \in \Psi^{-1}(Y \setminus Y_1), z_2 \in \Psi^{-1}(Y \setminus Y_2)$ s.t $\text{pr}(z_1) = y = \text{pr}(z_2)$. Let $W = I^{n+1}$ be a subcube of the subdivision of Proposition 11.5 such

⁸rest of the proof next lecture. I am very sure some words won't make sense, as they were unreadable on the board.

that $z_1 \in W$. Since $z_1 \in \Psi^{-1}(Y \setminus Y_1)$, $z_1 \in K_{p+1}(W)$, so $y = \text{pr}(z_1) \in K_p(\text{Im}(W))$. Analogous $y = \text{pr}(z_2) \in G_q(\text{Im}(W))$. Since $p + q > n$, this is a contradiction and the claim is proven.

Claim. The intersection of $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$ with ∂I^n is empty since Ψ is admissible, and thus $\Psi(\partial I^n) \subseteq Y_1$. So $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\text{pr}(\Psi^{-1}(Y \setminus Y_2)) \cup (\partial I^n)$ are two disjoint closed subsets of the compact space I^n .

So there is a continuous separating function $\tau: I^n \rightarrow [0, 1]$, s.t. $\tau \equiv 0$ is $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\tau \equiv 1$ is $\partial I^n \cup \text{pr}(Y \setminus Y_2)$

We define another homotopy starting with Ψ by

$$h: (I^n \times I) \times I \rightarrow Y$$

by

$$((x, t), s) \mapsto \Psi(x, (1 - s)t + s \cdot t\tau(x))$$

This is

- Homotopy through admissible maps
- $h(x, t, 0) = \Psi(x, t)$

Claim. $h(_, _, 1)$ has image in Y_1 .

$$h(x, t, 1) = \Psi(x, t \cdot \tau(x))$$

- if $x \in \Psi^{-1}(Y \setminus Y_1)$, then $\tau(x) = 0$, $h(x, t, 1) = \Psi(x, 0) \in Y_1$.
- if $x \in \Psi^{-1}(Y_1)$, then by admissibility $\Psi(x, t \cdot \tau(x)) \in Y$.

□

For the actual proof (Y_1, Y_0) p -connected (Y_2, Y_0) q -connected
some diagram about something being Serre fibration

We compare two Serre fibrations

$$F(\{y_0\}, Y_1, Y_0) \quad F(\{y_0\}, Y, Y_2)$$

$$F(Y_1, Y_1, Y_0) \longrightarrow F(Y_1, Y, Y_2)$$

$$Y_1$$

$$Y_1$$

partial 5-lemma shows that also the pair

$$(F(\{y_0\}, Y, Y_2), F(\{y_0\}, Y_1, Y_0))$$

is $(p + q - 1)$ -connected.

The following square commutes for all $n \geq 1$

$$\begin{array}{ccc} \pi_{n-1}(F(\{y_0\}, Y_1, Y_0), \text{const}_{y_0}) & \longrightarrow & \pi_{n-1}(F(\{y_0\}, Y, Y_1), x) \\ \downarrow \cong & & \downarrow \cong \\ \pi_n(Y_1, Y_0, y_0) & \xrightarrow{\text{id}_*} & \pi_n(Y, Y_1, y_0) \end{array}$$

□

1.2 Freudenthal suspension theorem

Section 6.4 in tom Dieck's book

Definition 1.7

Let X be a based space. The *unreduced suspension* is $X^\diamond = X \times [-1, 1] / \sim$ where \sim identifies all points with second variable -1 to S and all points with second variable 1 to N . We use S as the base point of X^\diamond .¹

¹south and north pole

Note. If X is well based, i.e. $\{x_0\} \hookrightarrow X$ has the HEP, then the quotient map $X^\diamond \rightarrow X$ is a homotopy equivalence. $\frac{X \times [0, 1]}{X \times \{0, 1\} \cup \{x_0\} \times [0, 1]}$

The suspension homomorphism $S: \pi_k(X, x_0) \rightarrow \pi_{k+1}(X^\diamond, S)$ for $k \geq 1$ is $S[f: S^k \rightarrow X] := [f^\diamond: S^{k+1} = (S^k)^\diamond \rightarrow X^\diamond]$

Theorem 1.8: Freudenthal suspension theorem

Let X be an $(n-1)$ -connected space, $n \geq 1$. Then the suspension homomorphism is

- bijective for $i \leq 2n-2$
- surjective for $i = 2n-1$

Proof. We cover X^\diamond by $Y_1 = X^\diamond \setminus \{N\}$, $Y_2 = X^\diamond \setminus \{S\}$. We use repeatedly that Y_1 and Y_2 are contractible. Then $X \rightarrow Y_1 \cap Y_2 = Y_0$ is a homotopy equivalence. We claim without proof that the following diagram commutes:

I couldn't keep up.

So we may show that $\pi_{i+1}(Y_2, Y_0, x_0) \rightarrow \pi_{i+1}(Y, Y \cap Y_1, y_0)$ is bijective for $i \leq 2n-2$ and surjective for $i \leq 2n-1$.

Because X is $(n-1)$ -connected

$$\pi_{k+1}(Y_1, Y_0, x_0) \xrightarrow[\partial]{\cong} \pi_k(Y_0, x_0)$$

So $\pi_i(Y_1, Y_0, x_0) = 0$ for $i \leq n$, i.e. (Y_1, Y_0) is n -connected. Also (Y_2, Y_0) is n -connected.

BM gives $\pi_k(Y_2, Y_0, y) \rightarrow \pi_k(X^\diamond, Y_1, y)$ is bijective for $k < 2n-1$ and surjective for $k = 2n$.

Setting $i = k-1$ ends the proof. \square

Take $X = S^n$

Corollary 1.9. The suspension homomorphism

$$S: \pi_i(S^n, *) \rightarrow \pi_{i+1}(S^{n+1}, *)$$

is bijective for $i \leq 2n-2$ and surjective for $i = 2n-1$.

Corollary 1.10. For all $n \geq 1$, $\pi_n(S^n, *) \cong \mathbb{Z}$, generated by the class of id_{S^n} . Moreover $\text{deg}: \pi_n(S^n, *) \rightarrow \mathbb{Z}$ is an isomorphism.

Proof. Induction on n . For $n = 1$ we have this by covering theory.

$n \geq 1$ By Freudenthal, the suspension homomorphism

$$S: \pi_n(S^n, *) \rightarrow \pi_{n+1}(S^{n+1}, *)$$

is surjective. The composite

$$\pi_n(S^n, *) \xrightarrow{S} \pi_{n+1}(S^{n+1}, *) \xrightarrow{\deg} \mathbb{Z}$$

is bijective by induction. So S is also injective and \deg in one dimension higher is also an isomorphism. \square

Recall. $\pi_3(S^2, *) \cong \mathbb{Z}\{\eta\}$, where $\eta: S^3 \rightarrow S^2$ is the Hopf map.

Proposition 1.11. For $n \geq 3$, $\pi_{n+1}(S^n, *) = \mathbb{Z}/2\{S^{n-2}(\eta)\}$ cycles of order two.

Proof. Only partial. For $n \geq 2$, $S: \pi_{n+1}(S^n, q) \mapsto \pi_{n+2}(S^{n+1}, S^{n+1}, *)$ is surjective.

$$\mathbb{Z}\{\eta\} = \pi_3(S^2, *) \rightarrow \pi_4(S^3, *) \xrightarrow{\cong} \pi_5(S^4, *) \dots$$

Claim. $S(2\eta) = 0$ in $\pi_4(S^3, *)$. Which gives $\pi_{n+1}(S^n, *)$ is either trivial or order 2.

Consider the commutative square

I did not manage to copy. Something complex conjugation

$$\implies [\eta] = [\eta \circ \text{complex conjugation}] = [d \circ \eta] \text{ in } \pi_3(S^2, * \cong \mathbb{Z}).$$

If $f: S^n \rightarrow S^n$ has degree k , then precomposition of $\pi_n(X, x) \rightarrow \pi_n(X, x)$ is multiplication by k .

Let $c: S^1 \rightarrow S^1$ be any map of degree -1 . Then $c \wedge S^2, S^1 \wedge d$: both have degree -1 . So $c \wedge S^2 \sim S^1 \wedge d$

two diagrams I did not copy.

We get $S(\eta) = S(d \circ \eta) = S(\eta \circ (c \wedge S^1)) = S(-\eta) = -S(\eta)$ and hence $2 \cdot S(\eta) = 0$. \square

Appendix

List of definitions

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