Universität Bonn

Notes for the lecture

Topology II

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Corrections and improvements

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Chapter 1

Cohomology

[07.04.2025, Lecture 1]

1.1 Last Term

In last term, we discussed

- CW-complexes
- higher homotopy groups
- Whitehead theorem
- Singular homology
- cellular homology

In the very end, cohomology was started.

$$\begin{tikzcd}
 & & & \mathrm{Ab} \\
\mathrm{TOP} \arrow[r, "\mathcal{S}"] \arrow[urrr, "H_n(X;A)"] \arrow[drrr, "H^n(X;A)"'] & (\text{simpl.Sets}) \arrow[r, "C(_{,}\mathbb{Z})"] & (\text{Chains}) \arrow[ur, "H_n(_\otimes A)"] \arrow[dr, "H^n(\mathrm{Hom}(_{,}A))"] & \\
 & & & \mathrm{Ab}^{op}
\end{tikzcd}$$

1.2 Cup product on cohomology

Let X be a simplicial set, and R[1] a ring.

$$C^n(X,R) = \mathrm{maps}(X_n, R)$$

is an abelian group under pointwise addition. There is a differential

$$d^n \colon C^n(X,R) \to C^{n+1}(X,R)$$

given by

$$d^n(f)(y) = \sum_{i=0}^{n+1} (-1)^i f(d_i^*(y))$$

with $f\colon X_n \to R, y \in X_{n+1}$

construction 1.1 (Cup product/Alexander Withney map)**.** The cup prodcut/Alexander Withney map

$$\cup\colon C^n(X,R) \times C^m(X,R) \to C^{m+n}(X,R)$$

[1] A ring is not necessarily commutative, but has a unit

with $n, m \geq 0$ is defined by

$$(f \cup g)(x) := f(d^*_{front}(x)) \cdot g(d^*_{back}(x))$$

with $f\colon X_n \to R, g\colon X_m \to R$, $x \in X_{n+m}$.

Where we use $[n+m] = \{0, 1, \ldots, n+m\}$ and $d_{front}\colon [n] \to [n+m], d_{back}\colon [m] \to [n+m]$ are given by $d_{front}(i) = i$, $d_{back}(i) = n+i$. Note, that d_{front} and d_{back} respectively suppress in their noation n and m.

Satz 1.2: fundamental properties of cup product

The cup-product satisfies the following properties.

1. *The AW-map is biadditive and satisfies a boundary formula:*

$$d(f \cup g) = (df) \cup g + (-1)^n f \cup (dg) \in C^{m+n+1}(X, R)$$

2. *For $h \in C^k(X, R)$, $(f \cup g) \cup h = f \cup (g \cup h) \in C^{n+m+k}(X, R)$. let $1 \in C^0(X, R)$ be the constant function $1\colon X_0 \to R$ with value 1. Then $1 \cup f = f \cup 1 = f$.*

3. *Naturality: let $\alpha\colon Y \to X$ be a morphism of symplicial sets. Then*

$$\alpha^*(f \cup g) = \alpha^*(f) \cup \alpha^*(g), \quad \alpha^*(1) = 1.$$

where $\alpha^*\colon C^n(X, R) \to C^n(Y, R), \quad f \mapsto f \circ \alpha_n$.

Proof.

- Let $d_{front}\colon [n] \to [n+m]$, $d_{back}\colon [m] \to [n+m]$ be as in the definition of $\cup$. Then

$$d_i \circ d_{front} = \begin{cases} d_{front} \circ d_i & 0 \leq i \leq n+1 \\ d_{front} & n+1 \leq i \leq n+m+1 \end{cases}$$

and

$$d_i \circ d_{back} = \begin{cases} d_{back} \circ d_i & 0 \leq i \leq n \\ d_{back} \circ d_{i-n} & n \leq i \leq n+m+1 \end{cases}$$

Note, that for $n+1$ and n respectively the cases are the same.

now

$$d(f \cup g)(x) = \sum_{i=0}^{n+m+1} (-1)^i (f \cup g)(d_i^*(x)) = \sum_{i=0}^{n+m+1} (-1)^i \cdot f(d^*_{front}(x)) \cdot g(d^*_{back}(d^*_i(x))) = \sum_{i=0}^{n} (-1)^i \cdot f(d^*_{front}(d^*_i($$

- For $x \in X_{n+m+k}$ we see

$$((f \cup g) \cup h)(x) = (f \cup g)(d^*_{front}(x)) \cdot h(d^*_{back}(x)) = f(d^*_{front}(d^*_{front}(x))) \cdot g(d^*_{back}(d^*_{front}(x))) \cdot h(d^*_{back}(x)) = f(d$$

Note that we abuse that d_{front} suppresses the indices for which the map is the front map. We have in the last line

$$d_{front}\colon [n] \to [n+m+k], d_{middle}$$

this is obviously associative in the inputs[2]

- Naturality for $\alpha\colon Y \to X$ we see

$$(\alpha^*(f \cup g))(y) = (f \cup g)(\alpha_{n+m}(y)) = f(d^*_{front}(\alpha_{n+m}(y))) \cdot g(d^*_{back}(\alpha_{n+m}(y))) = f(\alpha_n(d^*_{front}(y))) \cdot g(\alpha_m(d^*_{back}$$

[2] for Schwede at least.

□

Definition 1.3

A differential graded ring (dg-ring) is a cochain-complex $A = \{A^n, d^n\}_{n\in\mathbb{Z}}$ equipped with biadditive maps

$$\cdot\colon A^n \times A^m \to A^{n+m}, \quad n, m \in \mathbb{Z}$$

and a unit $1 \in A^0$, such that;

- $\cdot$ is associative and has 1 as a unit element.
- the Leibniz rule holds:

$$d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$$

with $a \in A^n, b \in A^m$.[1]

[1] The sign is somehow connected to a sign-rule I couldn't follow. The d moved past the a or something.

example 1.4. $C^{\cdot}(X, R)$ for a simplicial set X and a ring R.

De Rham complex of a ssmooth manifold.

construction 1.5 (Cup-Product on cohomology)**.** Let $A = (A^n, d, \cdot)$ be a dg-ring. We define a map

$$\cdot\colon H^n(A) \times H^m(A) \to H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$

This is well defined:

$$d(a \cdot b) = \underbrace{(da)}_{=0} \cdot b + (-1)^n@.a \cdot \underbrace{(db)}_{=0} = 0$$

so $a \cdot b$ is a cycle and we can take its homology class. Let $x \in A^{n-1}$.

$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b) = [(a + dx) \cdot b] = [a \cdot b]$$

so it only depends on the cohomology class of a, analogous for b.

The product on cohomology inherits associativity and unity with $1 = [1] \in H^0(A)$. We need to see 1 is a cocycle:

$$d(1) = d(1 \cdot 1) = (d1) \cdot 1 + (-1)^0 1 \cdot (d1) = 2 \cdot d(1)$$

and so $d(1) = 0$.

The cup product on the R-cohomology of a simplicial set X is the product induced by the cup product on $C^*(X, R)$ in $H^*(C(X, R)) = H^*(X, R)$.

Satz 1.6: L

t X be a simplicial set and R a ring. Then

- *The cup product on $H^*(X, R)$ is associative and unital, with unit the cohomology class of the constant function $1\colon X_0 \to R$.*
- *For a morphism of simplicial sets $\alpha\colon Y \to X$, the relation*

$$\alpha^*([x] \cup [y]) = \alpha^*[X] \cup \alpha^*[y]$$

holds for all $[x] \in H^n(X, R), [y] \in H^m(X, R)$.

remark 1.7. The cup product generalizes to relative cohomology: A, B simplicial subsets of X. We have

$$C^n(X, A; R) = \{f\colon X_n \to R \mid f(A_n) = \{0\}\}$$

The relative cup product is the restriciton of $\cup$ on $C^*(X, R)$ to

$$C^n(X, A; R) \times C^m(X, B; R) \xrightarrow{u} C^{n+m}(X, A \cup B; R).$$

Let $x \in (A \cup B)_{n+m}$, then

$$(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x))$$

if $x \in A_{n+m}$ then $f(d^*_{front}(x)) = 0$ and analogous with B_{n+m}, anyways the product is 0.
This gives us biadditive well defined maps

$$\cup\colon H^n(X, A; R) \times H^n(X, B; R) \to H^{n+m}(X, A \cup B; R)$$

In particular for $A = B$ we get

$$\cup\colon H^n(X, A; R) \times H^n(X, A; R) \to H^{n+m}(X, A; R)$$

which is well defined and associative, but not unital anymore.

1.3 Commutativity of the cup-product

Satz 1.8: Commutativity of the cup-product

Let X be a simplicial set and R a commutative ring. Then for all $[x] \in H^n(X, R); [y] \in H^m(X, R)$ the realtion

$$[x] \cup [y] = (-1)^{n \times m} \cdot [y] \cup [x]$$

holds.

Schwede points out, that the easy way doesn't work. **Warning.** For $f \in C^n(X, R), g \in C^m(Y, R)$, then in general $f \cup g \neq (-1)^{n+m}(g \cup f)$ in $C^{n+m}(X, R)$. The commutativity is a property we only get on homology.

construction 1.9. The $\cup_1$-product (spoken Cup-one)

$$\cup_1\colon C^n(X, R) \times C^m(X, R) \to C^{n+m-1}(X, R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-1)\cdot(m+1)} f((d_i^{out})^*(x)) \cdot g((d_i^{inner})^*(x))$$

for $f \in C^n$, $g \in C^m$ and $x \in X_{n+m-1}$.[3] where $d_i^{out}\colon [n] \to [n+m-1], d_i^{inner}\colon [m] \to [n+m-1]$ are the unique monotone injective maps with images $\mathrm{Im}(d_i^{out}) = \{0, \dots, i\} \cup \{i+m, \dots, n+m-1\}$ and $\mathrm{Im}(d_i^{inn}) = \{i, \dots, i+m\}$.

[3]There are also $\cup_i$ for $i \in \mathbb{N}$. However, they are quite messy and combinatorical.

Satz 1.10

The $\cup_1$-product satisfies the following formula

$$d(f \cup_1 g) = (df) \cup_1 g + (-1)^n \cdot f \cup_1 (dg) - (-1)^{n+m}(f \cup g) - (-1)^{n+1}m + 1(g \cup f)$$

for $f \in C^n(X, R)$ and $g \in C^m(X, R)$.

remark 1.11. What we want to see, is that $f \cup g$ and $g \cup f$ are not the same but rather homotopic, and $\cup_1$ wittnesses that homotopy.

Proof. This theorem will not be prooven, because it is quite messy. You should find a lecture-video for that. □

Now suppose that f and g are cocycles, i.e. $df = 0$, $dg = 0$. Then

$$d(f \cup_1 g) = -(-1)^{n+m}(f \cup g) - (-1)^{(n+1)(m+1)}(g \cup f)$$

and we get

$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m}(g \cup f)$$

and as such

$$0 = [(-1)^{n+m-1}] = [f] \cup [g] - (-1)^{n \cdot m}[g] \cup [f]$$

remark 1.12. Last term we discussed the tensor product of two chain complexes (in an exercise):

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} \cdot x \otimes (dy)$$

remark 1.13. Reinterpretation of $d(f \cup_1 g)$. The cup product yields a morphism of cochain complexes

$$C^*(X, R) \otimes C^*(X, R) \to C^*(X, R)$$

and we get a diagram

$$\begin{tikzcd} x \otimes y \arrow[d] & C^*(X,R) \otimes C^*(X,R) \arrow[r, "\cup"] \arrow[d] & C^*(X,R) \\ y \otimes x & C^*(X,R) \otimes C^*(X,R) \arrow[ur, "\cup"'] & \end{tikzcd}$$

that does not commute, however it does so up to cochain homotopy and $\cup_1$ is exactly a cochain homotopy between the two maps.

Appendix

List of definitions

List of statements

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