Universität Bonn

Notes for the lecture

Topology II

held by

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Corrections and improvements

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[07.04.2025, Lecture 1]

1.1 Last Term

In last term, we discussed

- CW-complexes
- higher homotopy groups
- Whitehead theorem
- Singular homology
- cellular homology

In the very end, cohomology was started. Remeber

$$\begin{tikzcd}
 & & & \mathrm{Ab} \\
\mathrm{TOP} \arrow[r, "\mathcal{S}"] \arrow[urrr, bend left, "H_n(X;A)"] \arrow[drrr, bend right, "H^n(X;A)"'] & (\text{simpl.Sets}) \arrow[r, "C(_{,}\mathbb{Z})"] & (\text{Chains}) \arrow[ur, "H_n(_\otimes A)"] \arrow[dr, "H^n(\mathrm{Hom}(_{,}A))"] & \\
 & & & \mathrm{Ab}^{op}
\end{tikzcd}$$

1.2 Cup-product

Let X be a simplicial set, and R[1] a ring.

$$C^n(X, R) = \mathrm{maps}(X_n, R)$$

is an abelian group under pointwise addition. There is a differential

$$d^n \colon C^n(X, R) \to C^{n+1}(X, R)$$

given by

$$d^n(f)(y) = \sum_{i=0}^{n+1} (-1)^i f(d_i^*(y))$$

with $f \colon X_n \to R, y \in X_{n+1}$

[1] A ring is not necessarily commutative, but has a unit

Construction 1.1 (Cup product/Alexander Whitney map)**.** The cup prodcut/Alexander Withney map

$$\cup\colon C^n(X,R)\times C^m(X,R)\to C^{m+n}(X,R)$$

with $n,m\geq 0$ is defined by

$$(f\cup g)(x) := f(d^*_{front}(x))\cdot g(d^*_{back}(x))$$

with $f\colon X_n\to R, g\colon X_m\to R$, $x\in X_{n+m}$.

Where we use $[n+m]=\{0,1,\dots,n+m\}$ and $d_{front}\colon[n]\to[n+m], d_{back}\colon[m]\to[n+m]$ are given by $d_{front}(i)=i$, $d_{back}(i)=n+i$. Note, that d_{front} and d_{back} respectively suppress in their noation n and m.

Satz 1.2: fundamental properties of cup product

The cup-product satisfies the following properties.

1. *The AW-map is biadditive and satisfies a boundary formula:*

$$d(f\cup g)=(df)\cup g+(-1)^n f\cup(dg)\in C^{m+n+1}(X,R)$$

2. *Associativity: For $h\in C^k(X,R)$, $(f\cup g)\cup h=f\cup(g\cup h)\in C^{n+m+k}(X,R)$.*
   Let $1\in C^0(X,R)$ be the constant function $1\colon X_0\to R$ with value 1. Then $1\cup f=f\cup 1=f$.

3. *Naturality: Let $\alpha\colon Y\to X$ be a morphism of symplicial sets. Then*

$$\alpha^*(f\cup g)=\alpha^*(f)\cup\alpha^*(g),\quad \alpha^*(1)=1.$$

   where $\alpha^*\colon C^n(X,R)\to C^n(Y,R),\quad f\mapsto f\circ\alpha_n$.

Proof.

1. We check some properties: Let $d_{front}\colon[n]\to[n+m]$, $d_{back}\colon[m]\to[n+m]$ be as in the definition of $\cup$. Then

$$d_i\circ d_{front}=\begin{cases} d_{front}\circ d_i & 0\leq i\leq n+1\\ d_{front} & n+1\leq i\leq n+m+1\end{cases}$$

   and

$$d_i\circ d_{back}=\begin{cases} d_{back}\circ d_i & 0\leq i\leq n\\ d_{back}\circ d_{i-n} & n\leq i\leq n+m+1\end{cases}$$

   Note, that for $n+1$ and n respectively the cases are the same.

Now we calculate

$$\begin{aligned}
d(f \cup g)(x) &= \sum_{i=0}^{n+m+1} (-1)^i (f \cup g)(d_i^*(x)) \\
&= \sum_{i=0}^{n+m+1} (-1)^i \cdot f(d_{front}^*(x)) \cdot g(d_{back}^*(d_i^*(x)))
\end{aligned}$$

$$= \sum_{i=0}^{n} (-1)^i \cdot f(d_{front}^*(d_i^*(x))) \cdot g(d_{back}^*(d_i^*(x))) + \sum_{j=1}^{m+1} (-1)^{n+j} \cdot f(d_{front}^*(d_{j+n}^*(x))) \cdot g(d_{back}^*(d_{j+n}^*(x)))$$

$$\begin{aligned}
&= \sum_{i=0}^{n+1} (-1)^i \cdot f(d_i^*(d_{front}^*(x))) \cdot g(d_{back}^*(x)) + \sum_{j=0}^{m+1} (-1)^{n+j} f(d_{front}^*(x)) \cdot g(d_j^*(d_{back}^*(x))) \\
&= d(f)(d_{front}^*(x)) \cdot g(d_{back}^*(x)) + (-1)^n \cdot f(d_{front}^*(x)) \cdot d(g)(d_{back}^*(x)) \\
&= ((df) \cup g)(x) + (-1)^n \cdot (f \cup dg)(x) \\
&= ((df) \cup g + (-1)^n \cdot f \cup (dg))(x)
\end{aligned}$$

2. For $x \in X_{n+m+k}$ we see

$$\begin{aligned}
((f \cup g) \cup h)(x) &= (f \cup g)(d_{front}^*(x)) \cdot h(d_{back}^*(x)) \\
&= f(d_{front}^*(d_{front}^*(x))) \cdot g(d_{back}^*(d_{front}^*(x))) \cdot h(d_{back}^*(x)) \\
&= f(d_{front}^*(x)) \cdot g(d_{middle}^*(x)) \cdot h(d_{back}^*(x))
\end{aligned}$$

Note that we abuse that d_{front} suppresses the indices for which the map is the front map. We have in the last line

$$d_{front} \colon [n] \to [n+m+k], d_{middle} \colon [m] \to [n+m+k], d_{back} \colon [k] \to [n+m+k]$$

defined by

$$d_{front}(i) = i, d_{middle}(i) = n+i, d_{back}(i) = n+m+i$$

this is obviously associative in the inputs[2]

3. Naturality for $\alpha \colon Y \to X$ we see

$$\begin{aligned}
(\alpha^*(f \cup g))(y) &= (f \cup g)(\alpha_{n+m}(y)) \\
&= f(d_{front}^*(\alpha_{n+m}(y))) \cdot g(d_{back}^*(\alpha_{n+m}(y))) = f(\alpha_n(d_{front}^*(y))) \cdot g(\alpha_m(d_{back}^*(y))) \\
&= \alpha^*(f)(d_{front}^*(y)) \cdot \alpha^*(g)(d_{back}^*(y)) \\
&= (\alpha^*(f) \cup \alpha^*(g))(y).
\end{aligned}$$

□

[2]for Schwede at least.

Definition 1.3: Differential graded ring

A differential graded ring (dg-ring) is a cochain-complex $A = \{A^n, d^n\}_{n\in\mathbb{Z}}$ equipped with biadditive maps
$$\cdot\colon A^n \times A^m \to A^{n+m}, \quad n, m \in \mathbb{Z}$$
and a unit $1 \in A^0$, such that;

- $\cdot$ is associative and has 1 as a unit element.
- the Leibniz rule holds:
$$d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$$
with $a \in A^n, b \in A^m$.[1]

[1]The sign is somehow connected to a sign-rule I couldn't follow. The d moved past the a or something.

Example 1.4. Some Differential graded rings are:

- $C^{\cdot}(X, R)$ for a simplicial set X and a ring R.
- De Rham complex of a smooth manifold.

Construction 1.5 (Cup-Product on cohomology)**.** Let $A = (A^n, d, \cdot)$ be a dg-ring. We define a map
$$\cdot\colon H^n(A) \times H^m(A) \to H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$
This is well defined:
$$d(a \cdot b) = \underbrace{(da)}_{=0} \cdot b + (-1)^n @.a \cdot \underbrace{(db)}_{=0} = 0$$
so $a \cdot b$ is a cycle and we can take its homology class. Let $x \in A^{n-1}$.
$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b) = [(a + dx) \cdot b] = [a \cdot b]$$
so it only depends on the cohomology class of a, analogous for b.

The product on cohomology inherits associativity and unity with $1 = [1] \in H^0(A)$. We need to see 1 is a cocycle:
$$d(1) = d(1 \cdot 1) = (d1) \cdot 1 + (-1)^0 1 \cdot (d1) = 2 \cdot d(1)$$
and so $d(1) = 0$.

The cup product on the R-cohomology of a simplicial set X is the product induced by the cup product on $C^*(X, R)$ in $H^*(C(X, R)) = H^*(X, R)$.

Satz 1.6: Properties of the cup-product on homology

Let X be a simplicial set and R a ring. Then

- *The cup product on $H^*(X, R)$ is associative and unital, with unit the cohomology class of the constant function $1\colon X_0 \to R$.*
- *For a morphism of simplicial sets $\alpha\colon Y \to X$, the relation*
$$\alpha^*([x] \cup [y]) = \alpha^*[X] \cup \alpha^*[y]$$
holds for all $[x] \in H^n(X, R), [y] \in H^m(X, R)$.

Remark 1.7. The cup product generalizes to relative cohomology: For A, B simplicial subsets of X. We have

$$C^n(X, A; R) = \{f \colon X_n \to R \mid f(A_n) = \{0\}\}$$

The relative cup product is the restriciton of $\cup$ on $C^*(X, R)$ to

$$C^n(X, A; R) \times C^m(X, B; R) \xrightarrow{u} C^{n+m}(X, A \cup B; R).$$

Let $x \in (A \cup B)_{n+m}$, then

$$(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x))$$

if $x \in A_{n+m}$ then $f(d^*_{front}(x)) = 0$ and analogous with B_{n+m}, anyways the product is 0. This gives us biadditive well defined maps

$$\cup \colon H^n(X, A; R) \times H^n(X, B; R) \to H^{n+m}(X, A \cup B; R)$$

In particular for $A = B$ we get

$$\cup \colon H^n(X, A; R) \times H^n(X, A; R) \to H^{n+m}(X, A; R)$$

which is well defined and associative, but not unital anymore.

1.3 Commutativity of the cup-product

Satz 1.8: Commutativity of the cup-product

Let X be a simplicial set and R a commutative ring. Then for all $[x] \in H^n(X, R); [y] \in H^m(X, R)$ the realtion

$$[x] \cup [y] = (-1)^{n \cdot m} \cdot [y] \cup [x]$$

holds.

Schwede points out, that the easy way doesn't work. **Warning.** For $f \in C^n(X, R), g \in C^m(Y, R)$, then in general $f \cup g \neq (-1)^{n+m}(g \cup f)$ in $C^{n+m}(X, R)$. The commutativity is a property we only get on homology.

Construction 1.9. The $\cup_1$-product (spoken Cup-one)

$$\cup_1 \colon C^n(X, R) \times C^m(X, R) \to C^{n+m-1}(X, R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-1)\cdot(m+1)} f((d_i^{out})^*(x)) \cdot g((d_i^{inner})^*(x))$$

for $f \in C^n$, $g \in C^m$ and $x \in X_{n+m-1}$.[3] where $d_i^{out} \colon [n] \to [n+m-1], d_i^{inner} \colon [m] \to [n+m-1]$ are the unique monotone injective maps with images $\mathrm{Im}(d_i^{out}) = \{0, \ldots, i\} \cup \{i+m, \ldots, n+m-1\}$ and $\mathrm{Im}(d_i^{inn}) = \{i, \ldots, i+m\}$.

[3]There are also $\cup_i$ for $i \in \mathbb{N}$. However, they are quite messy and combinatorical.

Satz 1.10: $\cup_1$-Product

The $\cup_1$-product satisfies the following formula

$$d(f \cup_1 g) = (df) \cup_1 g + (-1)^n \cdot f \cup_1 (dg) - (-1)^{n+m}(f \cup g) - (-1)^{n+1}m + 1(g \cup f)$$

for $f \in C^n(X,R)$ and $g \in C^m(X,R)$.

Remark 1.11. What we want to see, is that $f \cup g$ and $g \cup f$ are not the same but rather homotopic, and $\cup_1$ wittnesses that homotopy.

Proof. This theorem will not be prooven, because it is quite messy. You should find a lecture-video for that. □

Now suppose that f and g are cocycles, i.e. $df = 0$, $dg = 0$. Then

$$d(f \cup_1 g) = -(-1)^{n+m}(f \cup g) - (-1)^{(n+1)(m+1)}(g \cup f)$$

and we get

$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m}(g \cup f)$$

and as such

$$0 = [(-1)^{n+m-1}] = [f] \cup [g] - (-1)^{n \cdot m}[g] \cup [f]$$

Remark 1.12. Last term we discussed the tensor product of two chain complexes (in an exercise):

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} \cdot x \otimes (dy)$$

Remark 1.13. Reinterpretation of $d(f \cup_1 g)$. The cup product yields a morphism of cochain complexes

$$C^*(X,R) \otimes C^*(X,R) \to C^*(X,R)$$

and we get a diagram

$$\begin{tikzcd} x \otimes y \arrow[d] & C^*(X,R) \otimes C^*(X,R) \arrow[r, "\cup"] \arrow[d] & C^*(X,R) \\ y \otimes x & C^*(X,R) \otimes C^*(X,R) \arrow[ur, "\cup"'] & \end{tikzcd}$$

that does not commute, however it does so up to cochain homotopy and $\cup_1$ is exactly a cochain homotopy between the two maps.

[07.04.2025, Lecture 1]

[09.04.2025, Lecture 2]

Only with the definition of the cup-product we cannot calculate a lot yet. Some methods to compute cup-products are:

- directly from the definition
- cellular approximation of the diagonal (whatever that means, he gives a little intuition I failed to record.) (this might be used later)
- Group homology (one exapmle later today, something for AT I)

- Poincaré duality (later this term)
- Analysis on smooth manifolds together with De Rahm Cohomology

The first two methods are not very practical.

Example 1.14. Let X be a discrete space, Then $\mathcal{S}(X)$ is a constant simplicial set. The chain complex has the form

$$\xrightarrow{0} \mathbb{Z}[X] \xrightarrow{=} \mathbb{Z}[X] \xrightarrow{0} \mathbb{Z}[X]$$

And so $H^n(X,R) = 0$ for $n \geq 0$. And only for $n = m = 0$ something nontrivial happens. for $f\colon X_0 \to R, g\colon X_0 \to R$, we have $(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x)) = f(x) \cdot g(x)$ and so the cup product is just pointwise multiplication in dimension 0.

More generally: $H^0(X,R) = \mathrm{maps}(\pi_0(X), R)$ with $\cup$-prodcut pointwise multiplication

Example 1.15. Let G be a group: Define a category $\underline{G}$[4] wit one object $*$ and $\mathrm{Hom}_{\underline{G}}(*,*) = G$. We then define

$$BG = N(\underline{G})$$

Where N is the Nerve-Functor $\mathbf{CAT} \to \mathbf{Sset}$. Then

$$(BG)_n = G^n, \quad d_i^*\colon G^n \to G^{n-1}(g_1,\dots,g_n) \mapsto \begin{cases} (g_2,\dots,g_n) & i = 0 \\ (g_1,\dots,g_i \circ g_{i+1},\dots,g_n) & 1 \leq i \leq n-1 \\ (g_1,\dots,g_{n-1}) & i = n \end{cases}$$

And $s_i(g_1,\dots,g_n) = (g_1,\dots,g_i,1,g_{i+1},\dots,g_n)$.

The general case of this is too hard to calculate. We take $G = (\mathbb{F}_2,+)$ and $R = \mathbb{F}_2$ and we calculate $H^*(B\mathbb{F}_2,\mathbb{F}_2)$. We see

$$\begin{array}{ccccccc} C^0(BG,A) & \xrightarrow{d} & C^1(BG,A) & \xrightarrow{d} & C^2(BG,A) & \longrightarrow & \dots \\ \| & & \| & & \| & & \\ \mathrm{maps}(\{1\},A) & \xrightarrow{0} & \mathrm{maps}(G,A) & \longrightarrow & \mathrm{maps}(G^2,A) & & \\ \wr\| & & & & & & \\ A & & (f\colon G \to A) & \longrightarrow & (df)(g,h) & & \end{array}$$

And the map is defined by

$$f(d_0^*(g,h)) - f(d_1^*(g,h)) + f(d_2^*(g,h)) = f(h) - f(g \cdot h) + f(g)$$

and

$$df = 0 \Leftrightarrow f(g,h) = f(g) + f(h)$$

$\implies$ 1-cocycles are the group homomorphisms from G to A

$$H^1(BG,A) \cong \mathrm{Hom}(G,A)$$

and for $G = (\mathbb{F}_2,+)$, $A = \mathbb{F}_2$

We define

$$0 \neq x := [\mathrm{Id}_{\mathbb{F}_2}] \in H^1(B\mathbb{F}_2,\mathbb{F}_2).$$

[4]via geometric realization, these define interesting spaces, namely some (missed word)-Maclane spaces $M(G,1)$, didn't catch it all

We will show that $x^n = x \cup \cdots \cup x$ (n-times) $\in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is nonzero.

Proposition. $x^n \in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is represented by

$$f_n\colon (\mathbb{F}_2)^n \to \mathbb{F}_2, f_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdot \cdots \cdot \lambda_n = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_n = 1 \\ 0 & \text{else} \end{cases}$$

Proof. By induction on n. We checked for $n = 1$. For $n \geq 2$ we have

$$\begin{aligned} x^n &= x^{n-1} \cup x = [f_{n-1}] \cup [\mathrm{Id}_{\mathbb{F}_2}] \\ &= [f_{n-1} \cup \mathrm{Id}] \end{aligned}$$

Then

$$\begin{aligned} (f_{n-1} \cup \mathrm{Id})(\lambda_1, \dots, \lambda_n) &= f_{n-1}(d^*_{front}(\lambda_1, \dots, \lambda_n)) \cdot \mathrm{Id}_(\, d^*_{back}(\lambda_1, \dots, \lambda_n)) \\ &= f_{n-1}(\lambda_1, \dots, \lambda n - 1) \cdot \mathrm{Id}(\lambda_n) \\ &= (\lambda_1 \cdot \cdots \cdot \lambda_{n-1}) \cdot \lambda_n \end{aligned}$$

□

Claim: $x^n \neq 0$. In the UCT for cohomology we used the evaluation pair

$$\Phi\colon H^n(X, A) \to \mathrm{Hom}(H_n(X; \mathbb{Z}); A), \quad [f_n\colon X_n \to A] \mapsto \left\{ [\sum b_i x_i] \mapsto \sum b_i f(x_i) \right\}$$

for $b_i \in \mathbb{Z}, x_i \in X_n$. We can slightly variate that for ring coefficients:

$$\Phi\colon H^n(X, R) \to \mathrm{Hom}(H_n(X, R), R)$$

and $[f\colon X_n \to R] \mapsto \{[\sum r_i \cdot x_i] \mapsto \sum r_i \cdot f(x_i)\}$ with $r_i \in R, x_i \in X_n$.
With $X = B\mathbb{F}_2, R = \mathbb{F}_2$, we consider

$$y \coloneqq \sum_{(\lambda_1 \dots, \lambda_n) \in (\mathbb{F}_2)^n} 1(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2[(\mathbb{F}_2)^n] = \mathbb{F}_2[(B\mathbb{F}_2)_n]$$

Claim: y is an n-cycle in $C_*(B\mathbb{F}_2, \mathbb{F}_2)$.

$$\begin{aligned} dy &= \sum_{i=0,\dots n} (-1)^i \cdot d^*_i(\sum_1 \cdot (\lambda_1, \dots, \lambda_n)) \\ &= \sum_{i=0,\dots,n} \underbrace{\sum_{(\lambda_1,\dots,\lambda_n) \in \mathbb{F}_2^n} (-1)^i \cdot d^*_i(\lambda_1, \dots, \lambda_n)}_{\text{cancel in pairs}} \\ &= 0 \end{aligned}$$

Now

$$d^*_0(0, \lambda_2, \dots, \lambda_n) = (\lambda_2, \dots \lambda_n) = d^*_0(1, \lambda_2, \dots, \lambda_n)$$

So

$$\Phi(x^n)\colon H_n(B\mathbb{F}_2, \mathbb{F}_2) \to \mathbb{F}_2$$

$$\Phi(x^n)[y] = \Phi[f_n][\sum_{(\lambda_1,\dots,\lambda_n) \in \mathbb{F}_2^n} (\lambda_1, \dots, \lambda_n)] = \sum_{(\lambda_1,\dots,\lambda_n)} f_n(\lambda_1, \dots, \lambda_n) = \sum_{(\lambda_1,\dots,\lambda_n)} \lambda_1, \cdot \cdots \cdot \lambda_n = 1 \neq 0$$

and $[y] \neq 0$ in $H_n(B\mathbb{F}_2, \mathbb{F}_2)$.
We will later see, that in fact $H^*(B\mathbb{F}_2; \mathbb{F}_2) = \mathbb{F}_2[X]$.

Remark. Let p be an odd prime. $H^*(B\mathbb{F}_p, \mathbb{F}_p) =?$.

$$0 \neq x = [\mathrm{Id}_{\mathbb{F}_p} \in H^1(B\mathbb{F}_p; \mathbb{F}_p)]$$

still makes sense, but now there are more scalars and

$$x^n = 0$$

for $n \geq 2$. The graded commutativity says:

$$x \cup x = (-1)^{1\cdot 1} x \cup x = -x \cup x$$

so if R is commutative, $x \in H^n(X, R)$ and n is odd, then $2 \cdot (x \cup x) = 0$ in $H^{2n}(X, R)$. And then $2 \cdot x^2 = 0 \Rightarrow x^2 = 0$.

Define $h\colon \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ by

$$h(i,j) = \begin{cases} 0 & \text{if } i + j < p \\ 1 & \text{if } i + j \geq p \end{cases}$$

where we write $\mathbb{F}_p = \{0, \dots, p-1\}$. Now $h \in C^2(B\mathbb{F}_p, \mathbb{F}_p)$. Fact: $dh = 0$ and $0 \neq y \coloneqq [h] \in H^2(B\mathbb{F}_p, \mathbb{F}_p)$.

We then get (but do not proove)

$$H^*(B\mathbb{F}_p, \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$$

and

$$H^{2n}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{y^n\}, \quad H^{2n+1}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{xy^n\}$$

1.4 Künneth theorem

The Künneth theorem is an algebraic relationship between $H^*_*(X, R), H^*_*(Y, R)$ and $H^*_*(X \times Y, R)$[5].

Here is a simplest version in homology with field coefficients:

Satz 1.16: Künneth, simple version

Let X and Y be spaces and k a field. Then

$$H_n(X \times Y, k)$$

is natural isomorphic to

$$\bigoplus_{p+q=n} H_p(X, k) \otimes_k H_q(Y, k)$$

1.4.1 The Eilenberg-Zilber-theorem

Let A, B be simplicial abelian groups. Then we get two natural chain homotopy equivalences

$$C_*(A) \otimes C_*(B) C_*(A \otimes B)$$

up Eilenberg Zilber map, bottom Alexander Whitney map

[5] H^*_* denotes, that Schwede was too lazy to write the statement for homology and cohomology separately

Definition 1.17: Simplicial abelian group

A *simplicial abelian group* is a functor $A\colon \Delta^{Op} \to \mathbf{Ab.Groups}$.

Remark 1.18. Equivalently a simplicial abelian group is a colleciton of abelian groups A_n, and homomorphisms $\alpha^*\colon A_m \to A_n$ for all $\alpha\colon [n] \to [m]$ in Δ, s.t. $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$.

Equivalently a simplicial abelian group is a simplical set endorsed with abelian group structure on the sets of n-simplices, such that all α^* are homomorphisms.

Example 1.19. Let X be a simplicial set and A an abelian group. Then the composite

$$\begin{tikzcd} \Delta^{op} \arrow[r, "X"] \arrow[rr, bend left, "A[X]"] & (\mathbf{Sets}) \arrow[r, "A[\_]"] & (\mathbf{ab.grps}) \end{tikzcd}$$

is a simplicial abelian group.

Construction 1.20. Let $A\colon \Delta^{op} \to (\mathbf{ab.grps})$ be a simplicial abelian groups. Its *chain complex* $C_*(A)$ is the chain complex with $C_n(A) = A_n$ with differential

$$d\colon C_n(A) = A_n \to A_{n-1} = C_{n-1}(A), \quad d(a) = \sum_{i=0,\ldots,n} (-1)^i d_i^*(a)$$

And one can easily check $d \circ d = 0$.

Note. The following commutes

$$\begin{tikzcd} (\mathbf{Ssets}) \arrow[rr, "X \mapsto C_*(X,A)"] \arrow[dr, "A[\_]"] & & (\mathbf{Chains}) \\ & (\mathbf{s.ab.grps}) \arrow[ur, "C_*"] & \end{tikzcd}$$

Remark 1.21. The tensor product of chain complexes C, D is

$$(C \otimes D)_n \coloneqq \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential

$$d(x \otimes y) = (dx \otimes y) + (-1)^p x \otimes (dy)$$

for $x \in C_p, y \in D_q$.

We can also form the tensor product of simplical abelian groups:

Definition 1.22: Tensor product of simplicial abelian groups

$A, B\colon \Delta^{op} \to (\mathbf{ab.grps})$ by

$$(A \otimes B)_n = A_n \otimes B_n, \quad \alpha^*\colon (A \otimes B)_n \to (A \otimes B)_m$$

for $\alpha\colon [m] \to [n]$ is defined as $\alpha^*(a \otimes b) = \alpha^*(a) \otimes \alpha^*(b)$ and we write $\alpha^*_{A \otimes B} \coloneqq \alpha^*_A \otimes \alpha^*_B$. This can be equally described as the composite

$$\Delta^{op} \xrightarrow{(A,B)} (\mathbf{ab.grps}) \times (\mathbf{ab.grps}) \xrightarrow{\otimes} (\mathbf{ab.grps})$$

Warning. For $A, B \in (\mathbf{SAB})$ = simplicial abelian groups

$$C_*(A \otimes B) \neq C_*(A) \otimes C_*(B)$$

Also he did this in dimension n, but I lacked time to copy.

The Eilenberg-Zilber theorem is a natural pair of chain homotopy equivalences between these two.

Construction 1.23. Let A, B be simplicial chain groups. The *Alexander-Whitney map* is the chain map

$$AW \colon C_*(A \otimes B) \to C_*(A) \otimes C_*(B)$$

defined by

$$\begin{tikzcd}
C_n(A \otimes B) \arrow[r] \arrow[d, equal] & \bigoplus\limits_{p+q=n, p,q \geq 0} A_p \otimes B_q \arrow[d, equal] \\
A_n \otimes B_n & C_*(A) \otimes C_*(B)
\end{tikzcd}$$

$$AW_n(a \otimes b) = \sum_{p+q=n} d^*_{front}(a) \otimes d^*_{back}(b)$$

Where $[p] \xrightarrow{d_{front}} [p+q] = [n] \xleftarrow{d_{back}} [q]$.

You may check for yourself, that this is a chain map, however Schwede didn't do that.

[09.04.2025, Lecture 2]

[14.04.2025, Lecture 3]

Remark. An example for a simplicial abelian group, that is not of the form

$$\Delta^{op} \xrightarrow{X} \mathbf{sets} \xrightarrow{A[\_]} (\mathbf{ab.grps.})$$

is for any abelian group G the simplicial set BG, that also admits structure of a simplicial abelian group.

Remark 1.24 (Relation between AW-map and cup-product)**.** For a simplicial set X and ring R,

$$C^*(X, R) = \mathrm{Hom}(C_*(X, \mathbb{Z}), R) = \mathrm{Hom}(C_*(\mathbb{Z}[X])R)$$

and $C^n(X, R) = \mathrm{Hom}(C_n(X, R), R)$. If $\psi \in C^n(X, R)$ is a cocycle, i.e. $d(\psi) = 0$, then it extends to a chain map

$$\tilde{\psi} \colon C_*(\mathbb{Z}[X]) \to R[n]$$

where $R[n]$ is the complex with R in dimension n and 0 otherwise. and $\tilde{\psi}$ is ψ in dimension n and 0 otherwise.

For $f \in C^n(X, R), g \in C^m(X, R)$ cocycles, we have $f \cup g \in C^{n+m}(X, R)$. Then $f \tilde{\cup} g$ is the following composite

$$C_*(\mathbb{Z}[X]) \xrightarrow{C_*(\mathbb{Z}[\text{diagonal}])} C_*(\mathbb{Z}[X \times X]) \quad \cong \quad C_*(\mathbb{Z}([X]) \otimes \mathbb{Z}[X]) \xrightarrow{\text{AW}} C_*(\mathbb{Z}[X]) \otimes C_*(\mathbb{Z}[X]) \xrightarrow{\tilde{f} \otimes \tilde{g}} R[n] \otimes R[m] \xrightarrow{\text{m}}$$

Definition 1.25: (p,q)-shuffle

A (p, q)-shuffle for $p, q \geq 0$ is a permutation σ of $\{0, 1, \ldots, p+q-1\}$, such that the restriction of σ to $\{0, 1, \ldots, p-1\}$ is monotone, and the restriction of σ to $\{p, \ldots, p+q-1\}$ is monotone.

Remark. „Shuffles leave the first p elements in order and the last q elements in order.“

Example 1.26. The only $(p,0)$-shuffle or $(0,q)$-shuffles are the identity.

There are precisely two $(1,1)$-shuffles, namely both permutations of $\{0,1\}$.

$\sigma \in S_3$ given by $\sigma(0) = 0\sigma(1) = 2, \sigma(2) = 1$ is not a $(2,1)$-shuffle, but it is a $(1,2)$-shuffle.

Remark 1.27. (p,q)-shuffles biject with p-element subsets of $\{0,1,\dots,p+q-1\}$ by $\sigma \mapsto \{\sigma(0),\dots,\sigma(p)\}$ and also wit q-element subsets of $\{0,1,\dots,p+q-1\}$ by $\sigma \mapsto \{\sigma(p),\dots,\sigma(p+q-1)\}$.

This means $|(p,q)\text{-shuffles}| = \binom{p+q}{p} = \binom{p+q}{q}$.

Notation 1.28. Let σ be a (p,q)-shuffle. We write $\mu_i ::= \sigma(i-1)$ for $1 \leq 1 \leq p$ and $\nu_i := \sigma(p+i-1)$ for $1 \leq i \leq q$.

This means $0 \leq \mu_1 \leq \dots \leq \mu_p$ and $0 \leq \nu_1 \leq \dots \leq \nu_q \leq p+q-1$.

Definition 1.29: Eilenberg-Zilber map

Let A, B be simplicial abelian groups. The Eilenberg-Zilber map /shuffle map is

$$EZ\colon C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$$

is the direct sum of the homomorphisms

$$\nabla_{p,q}\colon A_p \otimes B_q \to A_{p+q} \otimes B_{p+q}$$

given by

$$a \otimes b \mapsto a\nabla b := \sum_{\sigma\colon (p,q)\text{-shuffle}} \operatorname{sgn}(\sigma) \cdot (s_{\nu_i} \circ \dots \circ s_{\nu_q})^*(a) \otimes (s_{\mu_1} \circ \dots \circ s_{\mu_p})^*(b)$$

Example 1.30. There is only one $(p,0)$-shuffle, the identity of $\{0,\dots,p-1\}$. Then $\mu_i = i-1$.

$$\nabla_{p,0}\colon A_p \otimes B_0 \to A_p \otimes B_p$$

is defined by

$$a \otimes b \mapsto a\nabla b = a \otimes (s_0 \circ \dots \circ s_{p-1})^*(b).$$

For $p = q = 1$ i didn't have the time to copy.

Schwede claims, that the Eilenberg-Zilber map is a chain map and he can't believe he actually did those calculations 4 years ago. He will not torture us, but you may watch the videos.

Satz 1.31: Shuffle maps form a chain map

The shuffle maps $\nabla_{p,q}$ for varying $p,q \geq 0$ assemble into a chain map. Furthermore, for $a \in A_p, b \in B_q$

$$d(a\nabla b) = (da)\nabla b + (-1)^p a\nabla(db)$$

He specifies, that the calculation takes up 8 pages of his notes.

Satz 1.32: Eilenberg-Zilber

Let A, B be simplicial abelian groups. Then the morphisms

$$\begin{tikzcd} C_*(A) \otimes C_*(B) \arrow[rr, bend left, "\text{Eilenberg-Zilber}"] & & C_*(A \otimes B) \arrow[ll, bend left, "AW"] \end{tikzcd}$$

are mutually inverse natural chain homotopy equivalences.

Proof. A first method of proof would be explicit formulas for the chain homotopies $\mathrm{AW} \circ EZ \sim \mathrm{Id}$ and $EZ \circ AW \sim \mathrm{Id}$. That is however infinitely annoying and we will not do this.

For the special case, where $A = \mathbb{Z}[X], B = \mathbb{Z}[Y]$ for simplicial sets X, Y via acyclic models.

Proposition 1.33 (Yoneda lemma)**.** Let $\mathcal{C}$ be a category and c an object of $\mathcal{C}$. Let $F\colon \mathcal{C} \to$ (**sets**) be a functor: Then the evaluation map

$$\mathrm{Nat}_{\mathcal{C}\to\mathbf{sets}}(\mathcal{C}(c, \_), F) \to F(c)$$

given by

$$(\tau\colon \mathcal{C}(c, \_) \to F) \mapsto (\tau_c\colon \mathcal{C}(c, c) \to F(c))(\mathrm{id}_c)$$

is bijective.

Equally: for every $x \in F(c)$, there is a unique natural transformation $\tau\colon (\mathcal{C}(c, \_) \to F)$, such that $\tau_c(\mathrm{id}_c) = x$.

Remark. A special case of this is

$$\mathrm{Hom}_{\mathbf{sset}}(\Delta^n, X) \cong X_n, \quad (f\colon \Delta^n \to X) \mapsto f_n(\mathrm{id}_[n]).$$

where $\Delta^n = \Delta(\_, [n])$.

Proof. We show injectivity and surjectivity.

Injectivity Let $\tau\colon \mathcal{C}(c, \_) \to F$ be any natural transformation. Let d be another object of $\mathcal{C}$, $f\colon c \to d$ any morphism. Then we have

$$\tau_d\colon \mathcal{C}(c, d) \to F(d)$$

and

$$\tau_d(f\colon c \to d) = \tau_d(\mathcal{C}(c, f)(\mathrm{id}_c)) = F(f)(\tau_c(\mathrm{id}_c))$$

where we use naturality of τ:

$$\begin{tikzcd} \mathcal{C}(c, d) \arrow[r, "\tau_d"] \arrow[d, "\mathcal{C}(c,g)"] & F(d) \arrow[d, "F(g)"] \\ \mathcal{C}(c, e) \arrow[r, "\tau_e"] & F(e) \end{tikzcd}$$

which implies the value of τ at $d, f\colon c \to d$ is determined by its value of (c, id_c) and the functorality of F.

Surjectivity Let $y \in F(c)$ be given. For an object d of $\mathcal{C}$ and morphism $f\colon c \to d$, we define

$$\tau_d\colon \mathcal{C}(c, d) \to F(d) \quad \tau_d(f) := F(f)(y).$$

We check $\tau_c(\mathrm{id}_c) = F(\mathrm{id}_c)(y) = y$. We need to check for naturality. Let $g\colon d \to e$ be another morphism. Then

$$\begin{aligned} F(g)(\tau_d(f)) &= F(g)(F(f)(y)) = F(g \circ f)(y) \\ &= \tau_e(g \circ f) = \tau_e(\mathcal{C}(c, g)(f)) \end{aligned}$$

□

Let $\mathcal{C}$ be a category, c an object of $\mathcal{C}$. We define the functor $\mathbb{Z}[\mathcal{C}(c, \_)]\colon \mathcal{C} \to (\mathbf{ab.grps.})$ as the composite

$$\mathcal{C} \xrightarrow{\mathcal{C}(c,\_)} (\mathbf{sets}) \xrightarrow{\mathbb{Z}[\_]} (\mathbf{ab.grps.}).$$

In particular, $\mathbb{Z}[\mathcal{C}(c, \_)](d) = \mathbb{Z}[\mathcal{C}(c, d)]$.

Proposition (Additive Yoneda lemma)**.** Let $c \in ob(\mathcal{C})$, $F\colon \mathcal{C} \to (\mathbf{ab.grps.})$ any functor. Then the evaluation map

$$\mathrm{Nat}_{\mathcal{C}\to(\mathbf{ab.grps.})}(\mathbb{Z}[\mathcal{C}(c, \_)], F) \to F(c)$$

is bijective. $(\tau\colon \mathbb{Z}[\mathcal{C}(c, \_)] \to F) \mapsto \tau_c(1 \cdot \mathrm{id}_c)$.

Proof. For varying objects d of $\mathcal{C}$, the bijections

$$\mathrm{Hom}_{AB}(\mathbb{Z}[\mathcal{C}(c, d)], F(d)) \cong \mathrm{Hom}_{\mathbf{sets}}(\mathcal{C}(c, d), F(d))$$

assemble into a bijection[6]

$$\mathrm{Nat}_{\mathcal{C}\to\mathbf{Ab}}(\mathbb{Z}[\mathcal{C}(c, \_)], F) \cong \mathrm{Nat}_{\mathcal{C}\to\mathbf{sets}}(\mathcal{C}(c, \_), F) \overset{\text{Yoneda}}{\cong} F(c)$$

□

Definition 1.34: Representable functor

A functor $F\colon \mathcal{C} \to \mathbf{Ab}$ is representable if there is an object $c \in \mathcal{C}$ and a natural isomorphism $F \cong \mathbb{Z}[\mathcal{C}(c, \_)]$

Note. Any isomorphism $F \cong \mathbb{Z}[\mathcal{C}(c, \_)]$ is determined by the „universal element“ in $F(c)$.

Example 1.35. Let $\mathcal{C} = (\mathbf{ssets}) \times (\mathbf{ssets})$ be the product of two copies of the category of simplicial sets. Define $f\colon (\mathbf{ssets}) \times (\mathbf{ssets}) \to \mathbf{Ab}$ given by $F(X, Y) = \mathbb{Z}[X_p \times Y_q]$ for some $p, q \geq 0$. **Claim.** This functor is representable by (Δ^p, Δ^q) with natural isomorphisms.

$$(\mathbf{ssets} \times \mathbf{ssets})((\Delta^p, \Delta^q), (X, Y)) = \mathbf{sets}(\Delta^p, X) \times \mathbf{sets}(\Delta^q, Y) \cong X_p \times Y_q$$

Apply free abelian groups to get

$$\mathbb{Z}[(\mathbf{ssets} \times \mathbf{ssets})((\Delta^p, \Delta^q)(X, Y))] \cong \mathbb{Z}[X_P \times Y_q]$$

[6] not very clear, you might want to think, why those are bijections.

Satz 1.36: Acyclic models

Let $\mathcal{C}$ be a category, $F, G\colon \mathcal{C} \to \mathbf{Chains}_+$ = non-negative grade chain complexes. Let $\psi\colon F \to G$ be a natural transformation of functors. Suppose;

1. *The transformation $\psi_0\colon F_0 \to G_0\colon \mathcal{C} \to \mathbf{Ab}$ is the zero natural transformation*
2. *For $n \geq 1$, the functor $F_n\colon \mathcal{C} \to \mathbf{Ab}$ is isomorphic to a direct sum of representable functors, $\mathbb{Z}[\mathcal{C}(c, \_)]$ for some family $\{c_i\}_{i\in I}$ of $\mathcal{C}$-objects such that $H_n(G(c)) = 0$.*

Then ψ is naturally chain nullhomotopic.

$\square$

Appendix

List of definitions

List of statements

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