Notes for the lecture

Topology II

held by

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TEXed by

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Corrections and improvements

If you have corrections or improvements, contact me via (`s94jmalm@uni-bonn.de`).

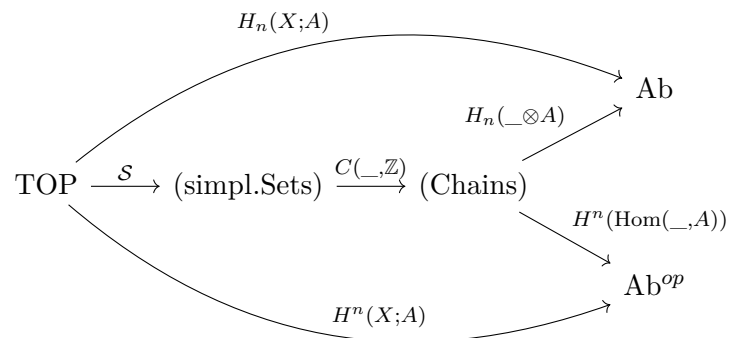# Contents

Lecture

Chapter 1

Cohomology

1.1 Last Term

In last term, we discussed

- CW-complexes
- higher homotopy groups
- Whitehead theorem
- Singular homology
- cellular homology

In the very end, cohomology was started. Remeber

$$
\begin{array}{ccc}
& H_n(X;A) & \\
& & \text{Ab} \\
& H_n(\_\otimes A) & \\
\text{TOP} \xrightarrow{\mathcal{S}} (\text{simpl.Sets}) \xrightarrow{C(\_,\mathbb{Z})} (\text{Chains}) & & \\
& H^n(\text{Hom}(\_,A)) & \\
& & \text{Ab}^{op} \\
& H^n(X;A) &
\end{array}
$$

1.2 Cup-product

Let X be a simplicial set, and R[1] a ring.

$$
C^n(X,R) = \text{maps}(X_n, R)
$$

is an abelian group under pointwise addition. There is a differential

$$
d^n \colon C^n(X,R) \to C^{n+1}(X,R)
$$

given by

$$
d^n(f)(y) = \sum_{i=0}^{n+1} (-1)^i f(d_i^*(y))
$$

with $f \colon X_n \to R, y \in X_{n+1}$

---

[1] A ring is not necessarily commutative, but has a unit

Construction 1.1 (Cup product/Alexander Whitney map)**.** The cup prodcut/Alexander Withney map

$$\cup\colon C^n(X,R)\times C^m(X,R)\to C^{m+n}(X,R)$$

with $n,m\geq 0$ is defined by

$$(f\cup g)(x):=f(d^*_{front}(x))\cdot g(d^*_{back}(x))$$

with $f\colon X_n\to R, g\colon X_m\to R, x\in X_{n+m}$.

Where we use $[n+m]=\{0,1,\ldots,n+m\}$ and $d_{front}\colon [n]\to [n+m], d_{back}\colon [m]\to [n+m]$ are given by $d_{front}(i)=i, d_{back}(i)=n+i$. Note, that d_{front} and d_{back} respectively suppress in their noation n and m.

Satz 1.2: fundamental properties of cup product

The cup-product satisfies the following properties.

1. *The AW-map is biadditive and satisfies a boundary formula:*

$$d(f\cup g)=(df)\cup g+(-1)^n f\cup(dg)\in C^{m+n+1}(X,R)$$

2. *Associativity: For $h\in C^k(X,R)$, $(f\cup g)\cup h=f\cup(g\cup h)\in C^{n+m+k}(X,R)$.*

   Let $1\in C^0(X,R)$ be the constant function $1\colon X_0\to R$ with value 1. Then $1\cup f=f\cup 1=f$.

3. *Naturality: Let $\alpha\colon Y\to X$ be a morphism of symplicial sets. Then*

$$\alpha^*(f\cup g)=\alpha^*(f)\cup\alpha^*(g),\quad \alpha^*(1)=1.$$

   where $\alpha^*\colon C^n(X,R)\to C^n(Y,R),\quad f\mapsto f\circ\alpha_n.$

Proof.

- Let $d_{front}\colon [n]\to [n+m], d_{back}\colon [m]\to [n+m]$ be as in the definition of $\cup$. Then

$$d_i\circ d_{front}=\begin{cases} d_{front}\circ d_i & 0\leq i\leq n+1 \\ d_{front} & n+1\leq i\leq n+m+1\end{cases}$$

  and

$$d_i\circ d_{back}=\begin{cases} d_{back}\circ d_i & 0\leq i\leq n \\ d_{back}\circ d_{i-n} & n\leq i\leq n+m+1\end{cases}$$

  Note, that for $n+1$ and n respectively the cases are the same.

now

$$d(f \cup g)(x) = \sum_{i=0}^{n+m+1} (-1)^i (f \cup g)(d_i^*(x))$$

$$= \sum_{i=0}^{n+m+1} (-1)^i \cdot f(d_{front}^*(x)) \cdot g(d_{back}^*(d_i^*(x)))$$

$$= \sum_{i=0}^{n} (-1)^i \cdot f(d_{front}^*(d_i^*(x))) \cdot g(d_{back}^*(d_i^*(x))) + \sum_{j=1}^{m+1} (-1)^{n+j} \cdot f(d_{front}^*(d_{j+n}^*(x))) \cdot g(d_{back}^*(d_{j+n}^*(x)))$$

$$= \sum_{i=0}^{n+1} (-1)^i \cdot f(d_i^*(d_{front}^*(x))) \cdot g(d_{back}^*(x)) + \sum_{j=0}^{m+1} (-1)^{n+j} f(d_{front}^*(x)) \cdot g(d_j^*(d_{back}^*(x)))$$

$$= d(f)(d_{front}^*(x)) \cdot g(d_{back}^*(x)) + (-1)^n \cdot f(d_{front}^*(x)) \cdot d(g)(d_{back}^*(x))$$

$$= ((df) \cup g)(x) + (-1)^n \cdot (f \cup dg)(x)$$

$$= ((df) \cup g + (-1)^n \cdot f \cup (dg))(x)$$

- For $x \in X_{n+m+k}$ we see

$$((f \cup g) \cup h)(x) = (f \cup g)(d_{front}^*(x)) \cdot h(d_{back}^*(x))$$

$$= f(d_{front}^*(d_{front}^*(x))) \cdot g(d_{back}^*(d_{front}^*(x))) \cdot h(d_{back}^*(x))$$

$$= f(d_{front}^*(x)) \cdot g(d_{middle}^*(x)) \cdot h(d_{back}^*(x))$$

Note that we abuse that d_{front} suppresses the indices for which the map is the front map. We have in the last line

$$d_{front} \colon [n] \to [n+m+k], d_{middle} \colon [m] \to [n+m+k], d_{back} \colon [k] \to [n+m+k]$$

defined by

$$d_{front}(i) = i, d_{middle}(i) = n+i, d_{back}(i) = n+m+i$$

this is obviously associative in the inputs[2]

- Naturality for $\alpha \colon Y \to X$ we see

$$(\alpha^*(f \cup g))(y) = (f \cup g)(\alpha_{n+m}(y))$$

$$= f(d_{front}^*(\alpha_{n+m}(y))) \cdot g(d_{back}^*(\alpha_{n+m}(y))) = f(\alpha_n(d_{front}^*(y))) \cdot g(\alpha_m(d_{back}^*(y)))$$

$$= \alpha^*(f)(d_{front}^*(y)) \cdot \alpha^*(g)(d_{back}^*(y))$$

$$= (\alpha^*(f) \cup \alpha^*(g))(y).$$

$\square$

---

[2]for Schwede at least.

Definition 1.3: Differential graded ring

A differential graded ring (dg-ring) is a cochain-complex $A = \{A^n, d^n\}_{n \in \mathbb{Z}}$ equipped with biadditive maps
$$\cdot : A^n \times A^m \to A^{n+m}, \quad n, m \in \mathbb{Z}$$
and a unit $1 \in A^0$, such that;

- $\cdot$ is associative and has 1 as a unit element.

- the Leibniz rule holds:

$$d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$$

with $a \in A^n, b \in A^m$.[1]

---

[1] The sign is somehow connected to a sign-rule I couldn't follow. The d moved past the a or something.

Example 1.4. Some Differential graded rings are:
- $C^{\cdot}(X, R)$ for a simplicial set X and a ring R.
- De Rham complex of a smooth manifold.

Construction 1.5 (Cup-Product on cohomology)**.** Let $A = (A^n, d, \cdot)$ be a dg-ring. We define a map
$$\cdot : H^n(A) \times H^m(A) \to H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$

This is well defined:
$$d(a \cdot b) = \underset{=0}{(da)} \cdot b + (-1)^n @. a \cdot \underset{=0}{(db)} = 0$$

so $a \cdot b$ is a cycle and we can take its homology class. Let $x \in A^{n-1}$.

$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b) = [(a + dx) \cdot b] = [a \cdot b]$$

so it only depends on the cohomology class of a, analogous for b.

The product on cohomology inherits associativity and unity with $1 = [1] \in H^0(A)$. We need to see 1 is a cocycle:
$$d(1) = d(1 \cdot 1) = (d1) \cdot 1 + (-1)^0 1 \cdot (d1) = 2 \cdot d(1)$$

and so $d(1) = 0$.

The cup product on the R-cohomology of a simplicial set X is the product induced by the cup product on $C^*(X, R)$ in $H^*(C(X, R)) = H^*(X, R)$.

Satz 1.6: Properties of the cup-product on homology

Let X be a simplicial set and R a ring. Then

- *The cup product on $H^*(X, R)$ is associative and unital, with unit the cohomology class of the constant function $1 \colon X_0 \to R$.*

- *For a morphism of simplicial sets $\alpha \colon Y \to X$, the relation*

$$\alpha^*([x] \cup [y]) = \alpha^*[X] \cup \alpha^*[y]$$

holds for all $[x] \in H^n(X, R), [y] \in H^m(X, R)$.

Remark 1.7. The cup product generalizes to relative cohomology: For A, B simplicial subsets of X. We have

$$C^n(X, A; R) = \{f\colon X_n \to R \mid f(A_n) = \{0\}\}$$

The relative cup product is the restriciton of $\cup$ on $C^*(X, R)$ to

$$C^n(X, A; R) \times C^m(X, B; R) \xrightarrow{u} C^{n+m}(X, A \cup B; R).$$

Let $x \in (A \cup B)_{n+m}$, then

$$(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x))$$

if $x \in A_{n+m}$ then $f(d^*_{front}(x)) = 0$ and analogous with B_{n+m}, anyways the product is 0.

This gives us biadditive well defined maps

$$\cup\colon H^n(X, A; R) \times H^n(X, B; R) \to H^{n+m}(X, A \cup B; R)$$

In particular for $A = B$ we get

$$\cup\colon H^n(X, A; R) \times H^n(X, A; R) \to H^{n+m}(X, A; R)$$

which is well defined and associative, but not unital anymore.

1.3 Commutativity of the cup-product

Satz 1.8: Commutativity of the cup-product

Let X be a simplicial set and R a commutative ring. Then for all $[x] \in H^n(X, R); [y] \in H^m(X, R)$ the realtion

$$[x] \cup [y] = (-1)^{n \cdot m} \cdot [y] \cup [x]$$

holds.

Schwede points out, that the easy way doesn't work. **Warning.** For $f \in C^n(X, R), g \in C^m(Y, R)$, then in general $f \cup g \neq (-1)^{n+m}(g \cup f)$ in $C^{n+m}(X, R)$. The commutativity is a property we only get on homology.

Construction 1.9. The $\cup_1$-product (spoken Cup-one)

$$\cup_1\colon C^n(X, R) \times C^m(X, R) \to C^{n+m-1}(X, R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-1)\cdot(m+1)} f((d^{out}_i)^*(x)) \cdot g((d^{inner}_i)^*(x))$$

for $f \in C^n$, $g \in C^m$ and $x \in X_{n+m-1}$.[3] where $d^{out}_i\colon [n] \to [n+m-1], d^{inner}_i\colon [m] \to [n+m-1]$ are the unique monotone injective maps with images $\operatorname{Im}(d^{out}_i) = \{0, \ldots, i\} \cup \{i+m, \ldots, n+m-1\}$ and $\operatorname{Im}(d^{inn}_i) = \{i, \ldots, i+m\}$.

---

[3]There are also $\cup_i$ for $i \in \mathbb{N}$. However, they are quite messy and combinatorical.

> ### Satz 1.10: $\cup_1$-Product
>
> *The $\cup_1$-product satisfies the following formula*
>
> $$d(f \cup_1 g) = (df) \cup_1 g + (-1)^n \cdot f \cup_1 (dg) - (-1)^{n+m}(f \cup g) - (-1)^{n+1}m + 1(g \cup f)$$
>
> *for $f \in C^n(X, R)$ and $g \in C^m(X, R)$.*

Remark 1.11. What we want to see, is that $f \cup g$ and $g \cup f$ are not the same but rather homotopic, and $\cup_1$ wittnesses that homotopy.

Proof. This theorem will not be prooven, because it is quite messy. You should find a lecture-video for that. □

Now suppose that f and g are cocycles, i.e. $df = 0$, $dg = 0$. Then

$$d(f \cup_1 g) = -(-1)^{n+m}(f \cup g) - (-1)^{(n+1)(m+1)}(g \cup f)$$

and we get

$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m}(g \cup f)$$

and as such

$$0 = [(-1)^{n+m-1}] = [f] \cup [g] - (-1)^{n \cdot m}[g] \cup [f]$$

Remark 1.12. Last term we discussed the tensor product of two chain complexes (in an exercise):

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} \cdot x \otimes (dy)$$

Remark 1.13. Reinterpretation of $d(f \cup_1 g)$. The cup product yields a morphism of cochain complexes

$$C^*(X, R) \otimes C^*(X, R) \to C^*(X, R)$$

and we get a diagram

$$
\begin{array}{ccc}
x \otimes y & C^*(X, R) \otimes C^*(X, R) \xrightarrow{\cup} & C^*(X, R) \\
\downarrow & \downarrow \quad \overset{\cup}{\nearrow} & \\
y \otimes x & C^*(X, R) \otimes C^*(X, R) &
\end{array}
$$

that does not commute, however it does so up to cochain homotopy and $\cup_1$ is exactly a cochain homotopy between the two maps.

Only with the definition of the cup-product we cannot calculate a lot yet. Some methods to compute cup-products are:

- directly from the definition

- cellular approximation of the diagonal (whatever that means, he gives a little intuition I failed to record.) (this might be used later)

- Group homology (one exapmle later today, something for AT I)

- Poincaré duality (later this term)

- Analysis on smooth manifolds together with De Rahm Cohomology

The first two methods are not very practical.

Example 1.14. Let X be a discrete space, Then $cS(X)$ is a constant simplicial set. The chain complex has the form

$$\xrightarrow{\ 0\ } \mathbb{Z}[X] \xrightarrow{\ =\ } \mathbb{Z}[X] \xrightarrow{\ 0\ } \mathbb{Z}[X]$$

And so $H^n(X, R) = 0$ for $n \geq 0$. And only for $n = m = 0$ something nontrivial happens. for $f \colon X_0 \to R, g \colon X_0 \to R$, we have $(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x)) = f(x) \cdot g(x)$ and so the cup product is just pointwise multiplication in dimension 0.

More generally: $H^0(X, R) = \mathrm{maps}(\pi_0(X), R)$ with $\cup$-prodcut pointwise multiplication

Example 1.15. Let G be a group: Define a category $\underline{G}^4$ wit one object $*$ and $\mathrm{Hom}_{\underline{G}}(*, *) = G$. We then define

$$BG = N(\underline{G})$$

Where N is the Nerve-Functor $\mathbf{CAT} \to \mathbf{Sset}$. Then

$$(BG)_n = G^n, \quad d^*_i \colon G^n \to G^{n-1}(g_1, \ldots, g_n) \mapsto \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i \circ g_{i+1}, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

And $s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)$.

The general case of this is too hard to calculate. We take $G = (\mathbb{F}_2, +)$ and $R = \mathbb{F}_2$ and we calculate $H^*(B\mathbb{F}_2, \mathbb{F}_2)$. We see

$$C^0(BG, A) \xrightarrow{\ d\ } C^1(BG, A) \xrightarrow{\ d\ } C^2(BG, A) \longrightarrow \ldots$$

$$\| \qquad\qquad\qquad \| \qquad\qquad\qquad \|$$

$$\mathrm{maps}(\{1\}, A) \xrightarrow{\ 0\ } \mathrm{maps}(G, A) \longrightarrow \mathrm{maps}(G^2, A)$$

$$\|\wr$$

$$A \qquad\qquad (f \colon G \to A) \longrightarrow (df)(g, h)$$

And the map is defined by

$$f(d^*_0(g, h)) - f(d^*_1(g, h)) + f(d^*_2(g, h)) = f(h) - f(g \cdot h) + f(g)$$

and

$$df = 0 \Leftrightarrow f(g, h) = f(g) + f(h)$$

$\implies$ 1-cocycles are the group homomorphisms from G to A

$$H^1(BG, A) \cong \mathrm{Hom}(G, A)$$

and for $G = (\mathbb{F}_2, +), A = \mathbb{F}_2$ we have $0 \neq x := [\mathrm{Id}_{\mathbb{F}_2}] \in H^1(B\mathbb{F}_2, \mathbb{F}_2)$.

We will show that $x^n = x \cup \cdots \cup x$ (n-times) $\in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is nonzero.

---

[4]via geometric realization, these define interesting spaces, namely some (missed word)-Maclane spaces $M(G, 1)$, didn't catch it all

Proposition. $x^n \in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is represented by $f_n \colon (\mathbb{F}_2)^n \to \mathbb{F}_2, f_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \cdot \lambda_n = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_n = 1 \\ 0 & \text{else} \end{cases}$

Proof. By induction on n. We checked for $n = 1$. For $n \geq 2$ we have

$$x^n = x^{n-1} \cup x = [f_{n-1}] \cup [\mathrm{Id}_{\mathbb{F}_2}]$$
$$= [f_{n-1} \cup \mathrm{Id}]$$

Then

$$(f_{n-1} \cup \mathrm{Id})(\lambda_1, \ldots, \lambda_n) = f_{n-1}(d^*_{front}(\lambda_1, \ldots, \lambda_n)) \cdot \mathrm{Id}_{(} d^*_{back}(\lambda_1, \ldots, \lambda_n))$$
$$= f_{n-1}(\lambda_1, \ldots, \lambda n - 1) \cdot \mathrm{Id}(\lambda_n)$$
$$= (\lambda_1 \cdot \cdots \cdot \lambda_{n-1}) \cdot \lambda_n$$

$\square$

Claim: $x^n \neq 0$. In the UCT for cohomology we uset the evaluation pair

$$\Psi \colon H^n(X, A) \to \mathrm{Hom}(H_n(X; \mathbb{Z}); A), \quad [f_n \colon X_n \to A] \mapsto \{[\textstyle\sum b_i x_i] \mapsto \textstyle\sum b_i f(x_i)\}$$

for $b_i \in \mathbb{Z}, x_i \in X_n$. We can sligtly variate that for ring coefficients:

$$\Psi \colon H^n(X, R) \to \mathrm{Hom}(H_n(X, R), R)$$

and $[f \colon X_n \to R] \mapsto \{[\sum r_i \cdot x_i] \mapsto \sum r_i \cdot f(x_i)\}$ with $r_i \in R, x_i \in X_n$.

With $X = B\mathbb{F}_2, R = \mathbb{F}_2$, we consider

$$y \coloneqq \sum_{(\lambda_1 \ldots, \lambda_n) \in (\mathbb{F}_2)^n} 1(\lambda_1, \ldots, \lambda_n) \in \mathbb{F}_2[(\mathbb{F}_2)^n] = \mathbb{F}_2[(B\mathbb{F}_2)_n]$$

Claim: y is an n-cycle in $C_*(B\mathbb{F}_2, \mathbb{F}_2)$.

$$dy = \sum_{i=0,\ldots n} (-1)^i \cdot d^*_i (\sum_1 \cdot (\lambda_1, \ldots, \lambda_n))$$
$$= \sum_{i=0,\ldots,n} \underbrace{\sum_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{F}_2^n} (-1)^i \cdot d^*_i(\lambda_1, \ldots, \lambda_n)}_{\text{cancel in pairs}}$$
$$= 0$$

Now

$$d^*_0(0, \lambda_2, \ldots, \lambda_n) = (\lambda_2, \ldots \lambda_n) = d^*_0(1, \lambda_2, \ldots, \lambda_n)$$

So

$$\Psi(x^n) \colon H_n(B\mathbb{F}_2, \mathbb{F}_2) \to \mathbb{F}_2$$

$$\Psi(x^n)[y] = \Psi[f_n][\sum_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{F}_2^n} (\lambda_1, \ldots, \lambda_n)] = \sum_{(\lambda_1, \ldots, \lambda_n)} f_n(\lambda_1, \ldots, \lambda_n) = \sum_{(\lambda_1, \ldots, \lambda_n)} \lambda_1 \cdot \cdots \cdot \lambda_n = 1 \neq 0$$

and $[y] \neq 0$ in $H_n(B\mathbb{F}_2, \mathbb{F}_2)$.

An outlook: $H^*(B\mathbb{F}_2; \mathbb{F}_2) = \mathbb{F}_2[X]$.

Remark. Let p be an odd prime. $H^*(B\mathbb{F}_p, \mathbb{F}_p) = ?$.

$$0 \neq x = [\mathrm{Id}_{\mathbb{F}_p} \in H^1(B\mathbb{F}_p; \mathbb{F}_p)]$$

still makes sense, but now there are more scalars and

$$x^n = 0$$

for $n \geq 2$. The graded commutativity says:

$$x \cup x = (-1)^{1 \cdot 1} x \cup x = -x \cup x$$

so if R is commutative, $x \in H^n(X, R)$ and n is odd, then $2 \cdot (x \cup x) = 0$ in $H^{2n}(X, R)$. And then $2 \cdot x^2 = 0 \Rightarrow x^2 = 0$.

Define $h \colon \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ by

$$h(i, j) = \begin{cases} 0 & \text{if } i + j < p \\ 1 & \text{if } i + j \geq p \end{cases}$$

where we write $\mathbb{F}_p = \{0, \dots, p - 1\}$. Now $h \in C^2(B\mathbb{F}_p, \mathbb{F}_p)$. Fact: $dh = 0$ and $0 \neq y := [h] \in H^2(B\mathbb{F}_p, \mathbb{F}_p)$.

We then get (but do not proove)

$$H^*(B\mathbb{F}_p, \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$$

and

$$H^{2n}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{y^n\}, \quad H^{2n+1}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{xy^n\}$$

1.4 Künneth theorem

The Künneth theorem is an algebraic relationship between $H_*^*(X, R), H_*^*(Y, R)$ and $H_*^*(X \times Y, R)$[5].

Here is a simplest version in homology with field coefficients:

Satz 1.16: Künneth, simple version

Let X and Y be spac3es and k a field. Then

$$H_n(X \times Y, k)$$

is natural isomorphic to

$$\bigoplus_{p+q=n} H_p(X, k) \otimes_k H_q(Y, k)$$

1.4.1 The Eilenberg Zilber Map theorem

Let A, B be simplicial abelian groups. Then we get two natural chain homotopy equivalences

$$C_*(A) \otimes C_*(B) C_*(A \otimes B)$$

up Eilenberg Zilber map, bottom Alexander Whitney map

---

[5] H_*^* denotes, that Schwede was too lazy to write the statement for homology and cohomology separately

---

> ## Definition 1.17: Simplicial abelian group
>
> A *simplicial abelian group* is a functor $A: \Delta^{Op} \to \textbf{Ab.Groups}$.
> Equivalently a colleciton of abelian groups A_n, and homomorphisma $\alpha^*: A_m \to A_n$ for all $\alpha: [n] \to [m]$ in Δ, s.t. $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$.
> Equivalently a simplical set endorsed with abelian group stuctur on the sets of n-simplices, such that all α^* are homomorphisms.

Example 1.18. Let X be a simplicial set and A an abelian group. Then the composite

Appendix

List of definitions

List of statements

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