Universität Bonn

Notes for the lecture

Topology II

held by

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Corrections and improvements

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[07.04.2025, Lecture 1]

1.1 Last Term

In last term, we discussed

- CW-complexes
- higher homotopy groups
- Whitehead theorem
- Singular homology
- cellular homology

In the very end, cohomology was started. Remeber

$$\begin{tikzcd}
 & & & \mathrm{Ab} \\
\mathrm{TOP} \arrow[r, "\mathcal{S}"] \arrow[urrr, bend left, "H_n(X;A)"] \arrow[drrr, bend right, "H^n(X;A)"'] & (\text{simpl.Sets}) \arrow[r, "C(_{,}\mathbb{Z})"] & (\text{Chains}) \arrow[ur, "H_n(_\otimes A)"] \arrow[dr, "H^n(\mathrm{Hom}(_{,}A))"] & \\
 & & & \mathrm{Ab}^{op}
\end{tikzcd}$$

1.2 Cup-product

Let X be a simplicial set, and R[1] a ring.

$$C^n(X, R) = \mathrm{maps}(X_n, R)$$

is an abelian group under pointwise addition. There is a differential

$$d^n \colon C^n(X, R) \to C^{n+1}(X, R)$$

given by

$$d^n(f)(y) = \sum_{i=0}^{n+1} (-1)^i f(d_i^*(y))$$

with $f \colon X_n \to R, y \in X_{n+1}$

[1] A ring is not necessarily commutative, but has a unit

Construction 1.1 (Cup product/Alexander Whitney map)**.** The cup prodcut/Alexander Withney map

$$\cup\colon C^n(X,R)\times C^m(X,R)\to C^{m+n}(X,R)$$

with $n, m \geq 0$ is defined by

$$(f\cup g)(x) \coloneqq f(d^*_{front}(x))\cdot g(d^*_{back}(x))$$

with $f\colon X_n \to R, g\colon X_m \to R$, $x \in X_{n+m}$.

Where we use $[n+m] = \{0, 1, \dots, n+m\}$ and $d_{front}\colon [n] \to [n+m], d_{back}\colon [m] \to [n+m]$ are given by $d_{front}(i) = i$, $d_{back}(i) = n+i$. Note, that d_{front} and d_{back} respectively suppress in their noation n and m.

Theorem 1.2: fundamental properties of cup product

The cup-product satisfies the following properties.

1. *The AW-map is biadditive and satisfies a boundary formula:*

$$d(f\cup g) = (df)\cup g + (-1)^n f\cup(dg) \in C^{m+n+1}(X,R)$$

2. *Associativity: For $h \in C^k(X,R)$, $(f\cup g)\cup h = f\cup(g\cup h) \in C^{n+m+k}(X,R)$.*
   Let $1 \in C^0(X,R)$ be the constant function $1\colon X_0 \to R$ with value 1. Then $1\cup f = f\cup 1 = f$.

3. *Naturality: Let $\alpha\colon Y \to X$ be a morphism of symplicial sets. Then*

$$\alpha^*(f\cup g) = \alpha^*(f)\cup\alpha^*(g), \quad \alpha^*(1) = 1.$$

   where $\alpha^*\colon C^n(X,R)\to C^n(Y,R)$, $\quad f \mapsto f\circ\alpha_n$.

Proof.

1. We check some properties: Let $d_{front}\colon [n] \to [n+m]$, $d_{back}\colon [m] \to [n+m]$ be as in the definition of $\cup$. Then

$$d_i \circ d_{front} = \begin{cases} d_{front}\circ d_i & 0 \leq i \leq n+1 \\ d_{front} & n+1 \leq i \leq n+m+1 \end{cases}$$

   and

$$d_i \circ d_{back} = \begin{cases} d_{back}\circ d_i & 0 \leq i \leq n \\ d_{back}\circ d_{i-n} & n \leq i \leq n+m+1 \end{cases}$$

   Note, that for $n+1$ and n respectively the cases are the same.

Now we calculate

$$
\begin{aligned}
d(f\cup g)(x) &= \sum_{i=0}^{n+m+1}(-1)^i(f\cup g)(d_i^*(x))\\
&= \sum_{i=0}^{n+m+1}(-1)^i\cdot f(d^*_{front}(x))\cdot g(d^*_{back}(d_i^*(x)))\\
&= \sum_{i=0}^{n}(-1)^i\cdot f(d^*_{front}(d_i^*(x)))\cdot g(d^*_{back}(d_i^*(x))) + \sum_{j=1}^{m+1}(-1)^{n+j}\cdot f(d^*_{front}(d^*_{j+n}(x)))\cdot g(d^*_{back}(d^*_{j+n}(x)))\\
&= \sum_{i=0}^{n+1}(-1)^i\cdot f(d_i^*(d^*_{front}(x)))\cdot g(d^*_{back}(x)) + \sum_{j=0}^{m+1}(-1)^{n+j} f(d^*_{front}(x))\cdot g(d_j^*(d^*_{back}(x)))\\
&= d(f)(d^*_{front}(x))\cdot g(d^*_{back}(x)) + (-1)^n\cdot f(d^*_{front}(x))\cdot d(g)(d^*_{back}(x))\\
&= ((df)\cup g)(x) + (-1)^n\cdot(f\cup dg)(x)\\
&= ((df)\cup g + (-1)^n\cdot f\cup(dg))(x)
\end{aligned}
$$

2. For $x\in X_{n+m+k}$ we see

$$
\begin{aligned}
((f\cup g)\cup h)(x) &= (f\cup g)(d^*_{front}(x))\cdot h(d^*_{back}(x))\\
&= f(d^*_{front}(d^*_{front}(x)))\cdot g(d^*_{back}(d^*_{front}(x)))\cdot h(d^*_{back}(x))\\
&= f(d^*_{front}(x))\cdot g(d^*_{middle}(x))\cdot h(d^*_{back}(x))
\end{aligned}
$$

Note that we abuse that d_{front} suppresses the indices for which the map is the front map. We have in the last line

$$d_{front}\colon [n]\to[n+m+k], d_{middle}\colon [m]\to[n+m+k], d_{back}\colon [k]\to[n+m+k]$$

defined by

$$d_{front}(i)=i, d_{middle}(i)=n+i, d_{back}(i)=n+m+i$$

this is obviously associative in the inputs[2]

3. Naturality for $\alpha\colon Y\to X$ we see

$$
\begin{aligned}
(\alpha^*(f\cup g))(y) &= (f\cup g)(\alpha_{n+m}(y))\\
&= f(d^*_{front}(\alpha_{n+m}(y)))\cdot g(d^*_{back}(\alpha_{n+m}(y))) = f(\alpha_n(d^*_{front}(y)))\cdot g(\alpha_m(d^*_{back}(y)))\\
&= \alpha^*(f)(d^*_{front}(y))\cdot\alpha^*(g)(d^*_{back}(y))\\
&= (\alpha^*(f)\cup\alpha^*(g))(y).
\end{aligned}
$$

□

[2]for Schwede at least.

Definition 1.3: Differential graded ring

A differential graded ring (dg-ring) is a cochain-complex $A = \{A^n, d^n\}_{n\in\mathbb{Z}}$ equipped with biadditive maps
$$\cdot\colon A^n \times A^m \to A^{n+m}, \quad n, m \in \mathbb{Z}$$
and a unit $1 \in A^0$, such that;

- $\cdot$ is associative and has 1 as a unit element.
- the Leibniz rule holds:
$$d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$$
with $a \in A^n, b \in A^m$.[1]

[1]The sign is somehow connected to a sign-rule I couldn't follow. The d moved past the a or something.

Example 1.4. Some Differential graded rings are:

- $C^{\cdot}(X, R)$ for a simplicial set X and a ring R.
- De Rham complex of a smooth manifold.

Construction 1.5 (Cup-Product on cohomology)**.** Let $A = (A^n, d, \cdot)$ be a dg-ring. We define a map
$$\cdot\colon H^n(A) \times H^m(A) \to H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$
This is well defined:
$$d(a \cdot b) = \underbrace{(da)}_{=0} \cdot b + (-1)^n @.a \cdot \underbrace{(db)}_{=0} = 0$$
so $a \cdot b$ is a cycle and we can take its homology class. Let $x \in A^{n-1}$.
$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b) = [(a + dx) \cdot b] = [a \cdot b]$$
so it only depends on the cohomology class of a, analogous for b.

The product on cohomology inherits associativity and unity with $1 = [1] \in H^0(A)$. We need to see 1 is a cocycle:
$$d(1) = d(1 \cdot 1) = (d1) \cdot 1 + (-1)^0 1 \cdot (d1) = 2 \cdot d(1)$$
and so $d(1) = 0$.

The cup product on the R-cohomology of a simplicial set X is the product induced by the cup product on $C^*(X, R)$ in $H^*(C(X, R)) = H^*(X, R)$.

Theorem 1.6: Properties of the cup-product on homology

Let X be a simplicial set and R a ring. Then

- *The cup product on $H^*(X, R)$ is associative and unital, with unit the cohomology class of the constant function $1\colon X_0 \to R$.*
- *For a morphism of simplicial sets $\alpha\colon Y \to X$, the relation*
$$\alpha^*([x] \cup [y]) = \alpha^*[X] \cup \alpha^*[y]$$
holds for all $[x] \in H^n(X, R), [y] \in H^m(X, R)$.

Remark 1.7. The cup product generalizes to relative cohomology: For A, B simplicial subsets of X. We have
$$C^n(X, A; R) = \{f: X_n \to R \mid f(A_n) = \{0\}\}$$
The relative cup product is the restriciton of $\cup$ on $C^*(X, R)$ to

$$C^n(X, A; R) \times C^m(X, B; R) \xrightarrow{\cup} C^{n+m}(X, A \cup B; R).$$

Let $x \in (A \cup B)_{n+m}$, then

$$(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x))$$

if $x \in A_{n+m}$ then $f(d^*_{front}(x)) = 0$ and analogous with B_{n+m}, anyways the product is 0. This gives us biadditive well defined maps

$$\cup: H^n(X, A; R) \times H^n(X, B; R) \to H^{n+m}(X, A \cup B; R)$$

In particular for $A = B$ we get

$$\cup: H^n(X, A; R) \times H^n(X, A; R) \to H^{n+m}(X, A; R)$$

which is well defined and associative, but not unital anymore.

1.3 Commutativity of the cup-product

Theorem 1.8: Commutativity of the cup-product

Let X be a simplicial set and R a commutative ring. Then for all $[x] \in H^n(X, R); [y] \in H^m(X, R)$ the realtion
$$[x] \cup [y] = (-1)^{n \cdot m} \cdot [y] \cup [x]$$
holds.

Schwede points out, that the easy way doesn't work. **Warning.** For $f \in C^n(X, R), g \in C^m(Y, R)$, then in general $f \cup g \neq (-1)^{n+m}(g \cup f)$ in $C^{n+m}(X, R)$. The commutativity is a property we only get on homology.

Construction 1.9. The $\cup_1$-product (spoken Cup-one)

$$\cup_1: C^n(X, R) \times C^m(X, R) \to C^{n+m-1}(X, R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-1)\cdot(m+1)} f((d_i^{out})^*(x)) \cdot g((d_i^{inner})^*(x))$$

for $f \in C^n$, $g \in C^m$ and $x \in X_{n+m-1}$.[3] where $d_i^{out}: [n] \to [n+m-1], d_i^{inner}: [m] \to [n+m-1]$ are the unique monotone injective maps with images $\mathrm{Im}(d_i^{out}) = \{0, \ldots, i\} \cup \{i+m, \ldots, n+m-1\}$ and $\mathrm{Im}(d_i^{inn}) = \{i, \ldots, i+m\}$.

[3] There are also $\cup_i$ for $i \in \mathbb{N}$. However, they are quite messy and combinatorical.

Theorem 1.10: $\cup_1$-Product

The $\cup_1$-product satisfies the following formula

$$d(f \cup_1 g) = (df) \cup_1 g + (-1)^n \cdot f \cup_1 (dg) - (-1)^{n+m}(f \cup g) - (-1)^{n+1}m + 1(g \cup f)$$

for $f \in C^n(X, R)$ and $g \in C^m(X, R)$.

Remark 1.11. What we want to see, is that $f \cup g$ and $g \cup f$ are not the same but rather homotopic, and $\cup_1$ wittnesses that homotopy.

Proof. This theorem will not be prooven, because it is quite messy. You should find a lecture-video for that. □

Now suppose that f and g are cocycles, i.e. $df = 0$, $dg = 0$. Then

$$d(f \cup_1 g) = -(-1)^{n+m}(f \cup g) - (-1)^{(n+1)(m+1)}(g \cup f)$$

and we get

$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m}(g \cup f)$$

and as such

$$0 = [(-1)^{n+m-1}] = [f] \cup [g] - (-1)^{n \cdot m}[g] \cup [f]$$

Remark 1.12. Last term we discussed the tensor product of two chain complexes (in an exercise):

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} \cdot x \otimes (dy)$$

Remark 1.13. Reinterpretation of $d(f \cup_1 g)$. The cup product yields a morphism of cochain complexes

$$C^*(X, R) \otimes C^*(X, R) \to C^*(X, R)$$

and we get a diagram

$$\begin{tikzcd} x \otimes y \arrow[d] & C^*(X,R) \otimes C^*(X,R) \arrow[r, "\cup"] \arrow[d] & C^*(X,R) \\ y \otimes x & C^*(X,R) \otimes C^*(X,R) \arrow[ur, "\cup"] & \end{tikzcd}$$

that does not commute, however it does so up to cochain homotopy and $\cup_1$ is exactly a cochain homotopy between the two maps.

[07.04.2025, Lecture 1]

[09.04.2025, Lecture 2]

Only with the definition of the cup-product we cannot calculate a lot yet. Some methods to compute cup-products are:

- directly from the definition
- cellular approximation of the diagonal (whatever that means, he gives a little intuition I failed to record.) (this might be used later)
- Group homology (one exapmle later today, something for AT I)

- Poincaré duality (later this term)
- Analysis on smooth manifolds together with De Rahm Cohomology

The first two methods are not very practical.

Example 1.14. Let X be a discrete space, Then $\mathcal{S}(X)$ is a constant simplicial set. The chain complex has the form

$$\xrightarrow{0} \mathbb{Z}[X] \xrightarrow{=} \mathbb{Z}[X] \xrightarrow{0} \mathbb{Z}[X]$$

And so $H^n(X,R) = 0$ for $n \geq 0$. And only for $n = m = 0$ something nontrivial happens. for $f\colon X_0 \to R, g\colon X_0 \to R$, we have $(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x)) = f(x) \cdot g(x)$ and so the cup product is just pointwise multiplication in dimension 0.

More generally: $H^0(X,R) = \mathrm{maps}(\pi_0(X), R)$ with $\cup$-prodcut pointwise multiplication

Example 1.15. Let G be a group: Define a category $\underline{G}$[4] wit one object $*$ and $\mathrm{Hom}_{\underline{G}}(*,*) = G$. We then define

$$BG = N(\underline{G})$$

Where N is the Nerve-Functor $\mathbf{CAT} \to \mathbf{Sset}$. Then

$$(BG)_n = G^n, \quad d_i^*\colon G^n \to G^{n-1} (g_1,\dots,g_n) \mapsto \begin{cases} (g_2,\dots,g_n) & i = 0 \\ (g_1,\dots,g_i \circ g_{i+1},\dots,g_n) & 1 \leq i \leq n-1 \\ (g_1,\dots,g_{n-1}) & i = n \end{cases}$$

And $s_i(g_1,\dots,g_n) = (g_1,\dots,g_i,1,g_{i+1},\dots,g_n)$.

The general case of this is too hard to calculate. We take $G = (\mathbb{F}_2,+)$ and $R = \mathbb{F}_2$ and we calculate $H^*(B\mathbb{F}_2,\mathbb{F}_2)$. We see

$$\begin{array}{ccccccc} C^0(BG,A) & \xrightarrow{d} & C^1(BG,A) & \xrightarrow{d} & C^2(BG,A) & \longrightarrow & \dots \\ \| & & \| & & \| & & \\ \mathrm{maps}(\{1\},A) & \xrightarrow{0} & \mathrm{maps}(G,A) & \longrightarrow & \mathrm{maps}(G^2,A) & & \\ \wr\| & & & & & & \\ A & & (f\colon G \to A) & \longrightarrow & (df)(g,h) & & \end{array}$$

And the map is defined by

$$f(d_0^*(g,h)) - f(d_1^*(g,h)) + f(d_2^*(g,h)) = f(h) - f(g \cdot h) + f(g)$$

and

$$df = 0 \Leftrightarrow f(g,h) = f(g) + f(h)$$

$\implies$ 1-cocycles are the group homomorphisms from G to A

$$H^1(BG,A) \cong \mathrm{Hom}(G,A)$$

and for $G = (\mathbb{F}_2,+)$, $A = \mathbb{F}_2$

We define

$$0 \neq x := [\mathrm{Id}_{\mathbb{F}_2}] \in H^1(B\mathbb{F}_2,\mathbb{F}_2).$$

[4] via geometric realization, these define interesting spaces, namely some (missed word)-Maclane spaces $M(G,1)$, didn't catch it all

We will show that $x^n = x \cup \cdots \cup x$ (n-times) $\in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is nonzero.

Proposition. $x^n \in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is represented by

$$f_n \colon (\mathbb{F}_2)^n \to \mathbb{F}_2, f_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdot \dots \cdot \lambda_n = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_n = 1 \\ 0 & \text{else} \end{cases}$$

Proof. By induction on n. We checked for $n = 1$. For $n \geq 2$ we have

$$\begin{aligned} x^n &= x^{n-1} \cup x = [f_{n-1}] \cup [\mathrm{Id}_{\mathbb{F}_2}] \\ &= [f_{n-1} \cup \mathrm{Id}] \end{aligned}$$

Then

$$\begin{aligned} (f_{n-1} \cup \mathrm{Id})(\lambda_1, \dots, \lambda_n) &= f_{n-1}(d^*_{front}(\lambda_1, \dots, \lambda_n)) \cdot \mathrm{Id}_(d^*_{back}(\lambda_1, \dots, \lambda_n)) \\ &= f_{n-1}(\lambda_1, \dots, \lambda n - 1) \cdot \mathrm{Id}(\lambda_n) \\ &= (\lambda_1 \cdot \dots \cdot \lambda_{n-1}) \cdot \lambda_n \end{aligned}$$

□

Claim: $x^n \neq 0$. In the UCT for cohomology we used the evaluation pair

$$\Phi \colon H^n(X, A) \to \mathrm{Hom}(H_n(X; \mathbb{Z}); A), \quad [f_n \colon X_n \to A] \mapsto \left\{ [\sum b_i x_i] \mapsto \sum b_i f(x_i) \right\}$$

for $b_i \in \mathbb{Z}, x_i \in X_n$. We can slightly variate that for ring coefficients:

$$\Phi \colon H^n(X, R) \to \mathrm{Hom}(H_n(X, R), R)$$

and $[f \colon X_n \to R] \mapsto \{[\sum r_i \cdot x_i] \mapsto \sum r_i \cdot f(x_i)\}$ with $r_i \in R, x_i \in X_n$.
With $X = B\mathbb{F}_2, R = \mathbb{F}_2$, we consider

$$y \coloneqq \sum_{(\lambda_1 \dots, \lambda_n) \in (\mathbb{F}_2)^n} 1(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2[(\mathbb{F}_2)^n] = \mathbb{F}_2[(B\mathbb{F}_2)_n]$$

Claim: y is an n-cycle in $C_*(B\mathbb{F}_2, \mathbb{F}_2)$.

$$\begin{aligned} dy &= \sum_{i=0,\dots n} (-1)^i \cdot d^*_i (\sum_1 \cdot (\lambda_1, \dots, \lambda_n)) \\ &= \sum_{i=0,\dots,n} \underbrace{\sum_{(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2^n} (-1)^i \cdot d^*_i(\lambda_1, \dots, \lambda_n)}_{\text{cancel in pairs}} \\ &= 0 \end{aligned}$$

Now

$$d^*_0(0, \lambda_2, \dots, \lambda_n) = (\lambda_2, \dots \lambda_n) = d^*_0(1, \lambda_2, \dots, \lambda_n)$$

So

$$\Phi(x^n) \colon H_n(B\mathbb{F}_2, \mathbb{F}_2) \to \mathbb{F}_2$$

$$\Phi(x^n)[y] = \Phi[f_n][\sum_{(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2^n} (\lambda_1, \dots, \lambda_n)] = \sum_{(\lambda_1, \dots, \lambda_n)} f_n(\lambda_1, \dots, \lambda_n) = \sum_{(\lambda_1, \dots, \lambda_n)} \lambda_1, \cdot \dots \cdot \lambda_n = 1 \neq 0$$

and $[y] \neq 0$ in $H_n(B\mathbb{F}_2, \mathbb{F}_2)$.
We will later see, that in fact $H^*(B\mathbb{F}_2; \mathbb{F}_2) = \mathbb{F}_2[X]$.

Remark. Let p be an odd prime. $H^*(B\mathbb{F}_p, \mathbb{F}_p) = ?$.

$$0 \neq x = [\mathrm{Id}_{\mathbb{F}_p} \in H^1(B\mathbb{F}_p; \mathbb{F}_p)]$$

still makes sense, but now there are more scalars and

$$x^n = 0$$

for $n \geq 2$. The graded commutativity says:

$$x \cup x = (-1)^{1 \cdot 1} x \cup x = -x \cup x$$

so if R is commutative, $x \in H^n(X, R)$ and n is odd, then $2 \cdot (x \cup x) = 0$ in $H^{2n}(X, R)$. And then $2 \cdot x^2 = 0 \Rightarrow x^2 = 0$.

Define $h\colon \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ by

$$h(i,j) = \begin{cases} 0 & \text{if } i+j < p \\ 1 & \text{if } i+j \geq p \end{cases}$$

where we write $\mathbb{F}_p = \{0, \ldots, p-1\}$. Now $h \in C^2(B\mathbb{F}_p, \mathbb{F}_p)$. Fact: $dh = 0$ and $0 \neq y \coloneqq [h] \in H^2(B\mathbb{F}_p, \mathbb{F}_p)$.

We then get (but do not proove)

$$H^*(B\mathbb{F}_p, \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$$

and

$$H^{2n}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{y^n\}, \quad H^{2n+1}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{xy^n\}$$

1.4 Künneth theorem

The Künneth theorem is an algebraic relationship between $H^*_*(X, R), H^*_*(Y, R)$ and $H^*_*(X \times Y, R)$[5].

Here is a simplest version in homology with field coefficients:

Theorem 1.16: Künneth, simple version

Let X and Y be spaces and k a field. Then

$$H_n(X \times Y, k)$$

is natural isomorphic to

$$\bigoplus_{p+q=n} H_p(X,k) \otimes_k H_q(Y,k)$$

1.4.1 The Eilenberg-Zilber-theorem

Let A, B be simplicial abelian groups. Then we get two natural chain homotopy equivalences

$$\begin{tikzcd} C_*(A) \otimes C_*(B) \arrow[r, bend left, "\text{Eilenberg-Zilber-Map}"] & C_*(A \otimes B) \arrow[l, bend left, "\text{Alexander Whitney map}"] \end{tikzcd}$$

[5] H^*_* denotes, that Schwede was too lazy to write the statement for homology and cohomology separately

Definition 1.17: Simplicial abelian group

A *simplicial abelian group* is a functor $A\colon \Delta^{Op} \to \mathbf{Ab.Groups}$.

Remark 1.18. Equivalently a simplicial abelian group is a colleciton of abelian groups A_n, and homomorphisms $\alpha^*\colon A_m \to A_n$ for all $\alpha\colon [n] \to [m]$ in Δ, s.t. $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$.

Equivalently a simplicial abelian group is a simplical set endorsed with abelian group structure on the sets of n-simplices, such that all α^* are homomorphisms.

Example 1.19. Let X be a simplicial set and A an abelian group. Then the composite

$$\begin{tikzcd} \Delta^{op} \arrow[r, "X"] \arrow[rr, bend left, "A[X]"] & (\mathbf{Sets}) \arrow[r, "A[\_]"] & (\mathbf{ab.grps}) \end{tikzcd}$$

is a simplicial abelian group.

Construction 1.20. Let $A\colon \Delta^{op} \to (\mathbf{ab.grps})$ be a simplicial abelian groups. Its *chain complex* $C_*(A)$ is the chain complex with $C_n(A) = A_n$ with differential

$$d\colon C_n(A) = A_n \to A_{n-1} = C_{n-1}(A), \quad d(a) = \sum_{i=0,\dots,n} (-1)^i d_i^*(a)$$

And one can easily check $d \circ d = 0$.

Note. The following commutes

$$\begin{tikzcd} (\mathbf{Ssets}) \arrow[rr, "X \mapsto C_*(X,A)"] \arrow[dr, "A[\_]"'] & & (\mathbf{Chains}) \\ & (\mathbf{s.ab.grps}) \arrow[ur, "C_*"'] & \end{tikzcd}$$

Remark 1.21. The tensor product of chain complexes C, D is

$$(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential

$$d(x \otimes y) = (dx \otimes y) + (-1)^p x \otimes (dy)$$

for $x \in C_p, y \in D_q$.

We can also form the tensor product of simplical abelian groups:

Definition 1.22: Tensor product of simplicial abelian groups

$A, B\colon \Delta^{op} \to (\mathbf{ab.grps})$ by

$$(A \otimes B)_n = A_n \otimes B_n, \quad \alpha^*\colon (A \otimes B)_n \to (A \otimes B)_m$$

for $\alpha\colon [m] \to [n]$ is defined as $\alpha^*(a \otimes b) = \alpha^*(a) \otimes \alpha^*(b)$ and we write $\alpha^*_{A \otimes B} := \alpha^*_A \otimes \alpha^*_B$. This can be equally described as the composite

$$\Delta^{op} \xrightarrow{(A,B)} (\mathbf{ab.grps}) \times (\mathbf{ab.grps}) \xrightarrow{\otimes} (\mathbf{ab.grps})$$

Warning. For $A, B \in (\mathbf{SAB})$ = simplicial abelian groups

$$C_*(A \otimes B) \neq C_*(A) \otimes C_*(B)$$

Also he did this in dimension n, but I lacked time to copy.

The Eilenberg-Zilber theorem is a natural pair of chain homotopy equivalences between these two.

Construction 1.23. Let A, B be simplicial chain groups. The *Alexander-Whitney map* is the chain map

$$AW\colon C_*(A \otimes B) \to C_*(A) \otimes C_*(B)$$

defined by

$$\begin{array}{ccc} C_n(A \otimes B) & \longrightarrow & \bigoplus_{p+q=n, p,q \geq 0} A_p \otimes B_q \\ \parallel & & \parallel \\ A_n \otimes B_n & & C_*(A) \otimes C_*(B) \end{array}$$

$$AW_n(a \otimes b) = \sum_{p+q=n} d^*_{front}(a) \otimes d^*_{back}(b)$$

Where $[p] \xrightarrow{d_{front}} [p+q] = [n] \xleftarrow{d_{back}} [q]$.

You may check for yourself, that this is a chain map, however Schwede didn't do that.

[09.04.2025, Lecture 2]

[14.04.2025, Lecture 3]

Remark. An example for a simplicial abelian group, that is not of the form

$$\Delta^{op} \xrightarrow{X} \mathbf{sets} \xrightarrow{A[\_]} (\mathbf{ab.grps.})$$

is for any abelian group G the simplicial set BG, that also admits structure of a simplicial abelian group.

Remark 1.24 (Relation between AW-map and cup-product)**.** For a simplicial set X and ring R,

$$C^*(X, R) = \mathrm{Hom}(C_*(X, \mathbb{Z}), R) = \mathrm{Hom}(C_*(\mathbb{Z}[X]), R)$$

and $C^n(X, R) = \mathrm{Hom}(C_n(X, \mathbb{Z}), R)$. If $\psi \in C^n(X, R)$ is a cocycle, i.e. $d(\psi) = 0$, then it extends to a chain map

$$\tilde{\psi}\colon C_*(\mathbb{Z}[X]) \to R[n]$$

where $R[n]$ is the complex with R in dimension n and 0 otherwise. and $\tilde{\psi}$ is ψ in dimension n and 0 otherwise.

For $f \in C^n(X, R), g \in C^m(X, R)$ cocycles, we have $f \cup g \in C^{n+m}(X, R)$. Then $f \,\tilde{\cup}\, g$ is the following composite

$$\begin{tikzcd} & C_*(\mathbb{Z}[X]) \arrow[rr, "C_*(\mathbb{Z}[\text{diagonal}])"] & & C_*(\mathbb{Z}[X \times X]) \arrow[r, phantom, "\cong"] & C_*(\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \arrow[dllll, "\text{AW}"] \\ C_*(\mathbb{Z}[X]) \otimes C_*(\mathbb{Z}[X]) \arrow[r, "\tilde{f} \otimes \tilde{g}"] & R[n] \otimes R[m] \arrow[r, "\text{mult}"] & R[n+m] & & \end{tikzcd}$$

Definition 1.25: (p,q)-shuffle

A (p,q)-shuffle for $p,q \geq 0$ is a permutation σ of $\{0,1,\dots,p+q-1\}$, such that the restriction of σ to $\{0,1,\dots,p-1\}$ is monotone, and the restriction of σ to $\{p,\dots,p+q-1\}$ is monotone.

Remark. „Shuffles leave the first p elements in order and the last q elements in order."

Example 1.26. The only $(p,0)$-shuffle or $(0,q)$-shuffles are the identity.

There are precisely two $(1,1)$-shuffles, namely both permutations of $\{0,1\}$.

$\sigma \in S_3$ given by $\sigma(0) = 0\sigma(1) = 2, \sigma(2) = 1$ is not a $(2,1)$-shuffle, but it is a $(1,2)$-shuffle.

Remark 1.27. (p,q)-shuffles biject with p-element subsets of $\{0,1,\dots,p+q-1\}$ by $\sigma \mapsto \{\sigma(0),\dots,\sigma(p)\}$ and also wit q-element subsets of $\{0,1,\dots,p+q-1\}$ by $\sigma \mapsto \{\sigma(p),\dots,\sigma(p+q-1)\}$.

This means $|(p,q)\text{-shuffles}| = \binom{p+q}{p} = \binom{p+q}{q}$.

Notation 1.28. Let σ be a (p,q)-shuffle. We write $\mu_i ::= \sigma(i-1)$ for $1 \leq 1 \leq p$ and $\nu_i := \sigma(p+i-1)$ for $1 \leq i \leq q$.

This means $0 \leq \mu_1 \leq \dots \leq \mu_p$ and $0 \leq \nu_1 \leq \dots \leq \nu_q \leq p+q-1$.

Definition 1.29: Eilenberg-Zilber map

Let A, B be simplicial abelian groups. The Eilenberg-Zilber map /shuffle map is

$$EZ\colon C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$$

is the direct sum of the homomorphisms

$$\nabla_{p,q}\colon A_p \otimes B_q \to A_{p+q} \otimes B_{p+q}$$

given by

$$a \otimes b \mapsto a\nabla b := \sum_{\sigma\colon (p,q)\text{-shuffle}} \operatorname{sgn}(\sigma) \cdot (s_{\nu_i} \circ \dots \circ s_{\nu_q})^*(a) \otimes (s_{\mu_1} \circ \dots \circ s_{\mu_p})^*(b)$$

Example 1.30. There is only one $(p,0)$-shuffle, the identity of $\{0,\dots,p-1\}$. Then $\mu_i = i-1$.

$$\nabla_{p,0}\colon A_p \otimes B_0 \to A_p \otimes B_p$$

is defined by

$$a \otimes b \mapsto a\nabla b = a \otimes (s_0 \circ \dots \circ s_{p-1})^*(b).$$

For $p = q = 1$ i didn't have the time to copy.

Schwede claims, that the Eilenberg-Zilber map is a chain map and he can't believe he actually did those calculations 4 years ago. He will not torture us, but you may watch the videos.

Theorem 1.31: Shuffle maps form a chain map

The shuffle maps $\nabla_{p,q}$ for varying $p,q \geq 0$ assemble into a chain map. Furthermore, for $a \in A_p, b \in B_q$

$$d(a\nabla b) = (da)\nabla b + (-1)^p a\nabla(db)$$

He specifies, that the calculation takes up 8 pages of his notes.

Theorem 1.32: Eilenberg-Zilber

Let A, B be simplicial abelian groups. Then the morphisms

$$\begin{tikzcd} C_*(A) \otimes C_*(B) \arrow[rr, bend left, "\text{Eilenberg-Zilber}"] & & C_*(A \otimes B) \arrow[ll, bend left, "AW"] \end{tikzcd}$$

are mutually inverse natural chain homotopy equivalences.

Proof. A first method of proof would be explicit formulas for the chain homotopies AW$\circ EZ \sim$ Id and $EZ \circ AW \sim$ Id. That is however infinitely annoying and we will not do this.

For the special case, where $A = \mathbb{Z}[X], B = \mathbb{Z}[Y]$ for simplicial sets X, Y we proove this via acyclic models. For that we need some category-theory:

Proposition 1.33 (Yoneda lemma)**.** Let $\mathcal{C}$ be a category and c an object of $\mathcal{C}$. Let $F\colon \mathcal{C} \to$ (**sets**) be a functor: Then the evaluation map

$$\mathrm{Nat}_{\mathcal{C}\to\mathbf{sets}}(\mathcal{C}(c, \_), F) \to F(c)$$

given by

$$(\tau\colon \mathcal{C}(c, \_) \to F) \mapsto (\tau_c\colon \mathcal{C}(c, c) \to F(c))(\mathrm{id}_c)$$

is bijective.

Equally: for every $x \in F(c)$, there is a unique natural transformation $\tau\colon (\mathcal{C}(c, \_) \to F)$, such that $\tau_c(\mathrm{id}_c) = x$.

Remark. A special case of this is

$$\mathrm{Hom}_{\mathbf{sset}}(\Delta^n, X) \cong X_n, \quad (f\colon \Delta^n \to X) \mapsto f_n(\mathrm{id}_{[n]}).$$

where $\Delta^n = \Delta(\_, [n])$.

Proof. We show injectivity and surjectivity.

Injectivity Let $\tau\colon \mathcal{C}(c, \_) \to F$ be any natural transformation. Let d be another object of $\mathcal{C}$, $f\colon c \to d$ any morphism. Then we have

$$\tau_d\colon \mathcal{C}(c, d) \to F(d)$$

and

$$\tau_d(f\colon c \to d) = \tau_d(\mathcal{C}(c, f)(\mathrm{id}_c)) = F(f)(\tau_c(\mathrm{id}_c))$$

where we use naturality of τ:

$$\begin{tikzcd} \mathcal{C}(c, d) \arrow[r, "\tau_d"] \arrow[d, "\mathcal{C}(c,g)"] & F(d) \arrow[d, "F(g)"] \\ \mathcal{C}(c, e) \arrow[r, "\tau_e"] & F(e) \end{tikzcd}$$

which implies the value of τ at $d, f\colon c \to d$ is determined by its value of (c, id_c) and the functorality of F.

Surjectivity Let $y \in F(c)$ be given. For an object d of $\mathcal{C}$ and morphism $f\colon c \to d$, we define

$$\tau_d\colon \mathcal{C}(c,d) \to F(d) \quad \tau_d(f) \coloneqq F(f)(y).$$

We check $\tau_c(\mathrm{id}_c) = F(\mathrm{id}_c)(y) = y$. We need to check for naturality. Let $g\colon d \to e$ be another morphism. Then

$$\begin{aligned} F(g)(\tau_d(f)) &= F(g)(F(f)(y)) = F(g \circ f)(y) \\ &= \tau_e(g \circ f) = \tau_e(\mathcal{C}(c,g)(f)) \end{aligned}$$

□

Let $\mathcal{C}$ be a category, c an object of $\mathcal{C}$. We define the functor $\mathbb{Z}[\mathcal{C}(c,\_)]\colon \mathcal{C} \to (\mathbf{ab.grps.})$ as the composite

$$\mathcal{C} \xrightarrow{\mathcal{C}(c,\_)} (\mathbf{sets}) \xrightarrow{\mathbb{Z}[\_]} (\mathbf{ab.grps.}).$$

In particular, $\mathbb{Z}[\mathcal{C}(c,\_)](d) = \mathbb{Z}[\mathcal{C}(c,d)]$.

Proposition (Additive Yoneda lemma)**.** Let $c \in ob(\mathcal{C})$, $F\colon \mathcal{C} \to (\mathbf{ab.grps.})$ any functor. Then the evaluation map

$$\mathrm{Nat}_{\mathcal{C}\to(\mathbf{ab.grps.})}(\mathbb{Z}[\mathcal{C}(c,\_)], F) \to F(c)$$

is bijective. $(\tau\colon \mathbb{Z}[\mathcal{C}(c,\_)] \to F) \mapsto \tau_c(1 \cdot \mathrm{id}_c)$.

Proof. For varying objects d of $\mathcal{C}$, the bijections

$$\mathrm{Hom}_{AB}(\mathbb{Z}[\mathcal{C}(c,d)], F(d)) \cong \mathrm{Hom}_{\mathbf{sets}}(\mathcal{C}(c,d), F(d))$$

assemble into a bijection[6]

$$\mathrm{Nat}_{\mathcal{C}\to\mathbf{Ab}}(\mathbb{Z}[\mathcal{C}(c,\_)], F) \cong \mathrm{Nat}_{\mathcal{C}\to\mathbf{sets}}(\mathcal{C}(c,\_), F) \overset{\text{Yoneda}}{\cong} F(c)$$

□

Definition 1.34: Representable functor

A functor $F\colon \mathcal{C} \to \mathbf{Ab}$ is representable if there is an object $c \in \mathcal{C}$ and a natural isomorphism $F \cong \mathbb{Z}[\mathcal{C}(c,\_)]$

Note. Any isomorphism $F \cong \mathbb{Z}[\mathcal{C}(c,\_)]$ is determined by the „universal element“ in $F(c)$.

Example 1.35. Let $\mathcal{C} = (\mathbf{ssets}) \times (\mathbf{ssets})$ be the product of two copies of the category of simplicial sets. Define $f\colon (\mathbf{ssets}) \times (\mathbf{ssets}) \to \mathbf{Ab}$ given by $F(X,Y) = \mathbb{Z}[X_p \times Y_q]$ for some $p, q \geq 0$. **Claim.** This functor is representable by (Δ^p, Δ^q) with natural isomorphisms.

$$(\mathbf{ssets} \times \mathbf{ssets})((\Delta^p, \Delta^q), (X,Y)) = \mathbf{sets}(\Delta^p, X) \times \mathbf{sets}(\Delta^q, Y) \cong X_p \times Y_q$$

Apply free abelian groups to get

$$\mathbb{Z}[(\mathbf{ssets} \times \mathbf{ssets})((\Delta^p, \Delta^q)(X,Y))] \cong \mathbb{Z}[X_P \times Y_q]$$

Notation 1.36. For $F\colon \mathcal{C} \to \mathbf{Chains}$ we write $F_n = (\_)_n \circ F\colon \mathcal{C} \to \mathbf{Ab}$ as the composite.

$$\mathcal{C} \xrightarrow{F} \mathbf{Chains} \xrightarrow{(\_)_n} \mathbf{Ab}$$

[6] I don't know why though.

and the second map sends $C = C(n, d_n)_{n\in\mathbb{Z}} \mapsto C_n$.

Theorem 1.37: Acyclic models

*Let $\mathcal{C}$ be a category, $F, G\colon \mathcal{C} \to$ **Chains**$_+$ = non-negative grade chain complexes. Let $\psi\colon F \to G$ be a natural transformation of functors. Suppose;*

1. *The transformation $\psi_0\colon F_0 \to G_0\colon \mathcal{C} \to$ **Ab** is the zero natural transformation*
2. *For every $n \geq 1$, the functor $F_n\colon \mathcal{C} \to$ **Ab** is isomorphic to a direct sum of representable functors, $\bigoplus_{i\in I} IZ[\mathcal{C}(c_i, \_)]$ for some family $\{c_i\}_{i\in I}$ of $\mathcal{C}$-objects such that $H_n(G(c_i)) = 0$.*

Then ψ is naturally chain nullhomotopic.

[14.04.2025, Lecture 3]
[16.04.2025, Lecture 4]

Proof. For $n \geq 0$, we will construct natural transformations

$$s_n\colon F_n \to G_{n+1}$$

of functors $\mathcal{C} \to$ **Ab**, such that

$$d_{n+1} \circ s_n + s_{n-1} \circ d_n = \psi_n \tag{*}$$

as natural transformations (i.e. they have the chain homotopy property).

The construction is by induction on n. We begin with $s_0 = 0$ and $s_{-1} = 0$. Suppose $n \geq 1$ and that $s_0, \dots, s_{n-1}$ have been constructed satisfying (*). Then

$$d_n^G \circ (\psi_n - s_{n-1} \circ d_n^F) = d_n^G \circ \psi_n - d_n^G \circ s_{n-1} \circ d_n^F$$

as ψ is a chain map,

$$= \psi_{n-1} \circ d_n^F - d_n^G \circ s_{n-1} \circ d_n^F = (\psi_{n-1} - d_n^G \circ s_{n-1}) \circ d_n^F \overset{(*)}{=} s_{n-2} \circ d_{n-1}^F \circ d_n^F = 0.$$

So $\psi_n - s_{n-1} \circ d_n^F\colon F_n \to G_n$ takes values in cycles. By 2.,

$$f_n = \bigoplus_{i\in I} \mathbb{Z}[\mathcal{C}(c, \_)]$$

for some set $\{c_i\}_{i\in I}$ of $\mathcal{C}$-objects, such that $H_n(G(c_i)) = 0$. Let $j \in I$, write

$$x_j \in F(c_j) = \bigoplus_{i\in I} \mathbb{Z}[\mathcal{C}(c_i, c_j)]$$

be the element $1 \cdot \mathrm{id}_j$ in the j-th summand. Then

$$\psi_n^{c_j}(x_j) - s_{n-1}^{c_j}(d_n^{F,c_i}(x_j)) \in G_n(c_j)$$

is a cycle. Since $H_n(G(c_j)) = 0$, the class is a boundary in the complex $G(c_j)$.

Let $y_j \in G(c_j)_{n+1}$ be a element such that

$$d_{n+1}^{c_j}(y_j) = \psi_n^{c_j}(x_j) - s_{n-1}^{c_j}(d_n^{F,c_j}(x_j))$$

The additive Yoneda lemma provides a unique natural transformation

$$s_{n,j}\colon \mathbb{Z}[\mathcal{C}(c_j, \_)] \to G_{n+1}$$

such that $s_{n,j}(x_j) = s_{n,j}^{c_j}(1 \cdot \mathrm{id}_{c_j}) = y_j \in G_{n+1}(c_j)$.

We define the natural transformation

$$s_n \colon F_n = \bigoplus_{i \in I} \mathbb{Z}[\mathcal{C}(c_i, \_)] \to G_{n+1}$$

as $s_n = \bigoplus_{j \in I} s_{n,j}$.

It suffices now to show, that (*) holds on each summand $\mathbb{Z}[\mathcal{C}(c_j, \_)]$. By the additive Yoneda lemma, there it suffices to check the relation on $1 \cdot \mathrm{id}_{c_j}$, which holds by definition. $\square$

Remark. We only prooved „half“ of the acyclic models theorem. The other half states:

Let $\mathcal{C}$ and $F, G \colon \mathcal{C} \to \mathbf{Chains}_+$ be as before, satisfying 2.. Then any natural transformation $\psi_0 \colon F_0 \to G_0$ can be extended to a natural transformation $\psi \colon F \to G$.

Now to actually proove the Eilenberg-Zilber-Theorem **??** (at least in a special case.) Let A, B be simplicial abelian groups. We assume $A = \mathbb{Z}[X]$, $B = \mathbb{Z}[Y]$ for some simplicial sets X, Y. We write $C_*(X), C_*(Y)$. For sets S, T,

$$\begin{tikzcd}
\mathbb{Z}[S] \otimes \mathbb{Z}[T] \arrow[rr, bend left] & & \mathbb{Z}[S \times T] \arrow[ll, bend left] \\
s \otimes t \arrow[rr] & & (s,t)
\end{tikzcd}$$

is naturally isomorphic. Dimensionwise this gives $\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \cong \mathbb{Z}[X \times Y]$.

We want to move this further to $C_*(X) \otimes C_*(Y) \cong C_*(X \times Y)$.

Proposition 1.38.

1. For all $p \geq 0$, the simplicial set Δ^q is simplicially contractible.
2. For all $p \geq 0$, the complex $C_*(\Delta^p)$ is chain homotopy equivalent to the complex $\mathbb{Z}[0]$, the complex consisting of $\mathbb{Z}$ in dimension 0.
3. For $p, q \geq 0$, the chain complex $C_*(\Delta^p) \otimes C_*(\Delta^q)$ is chain homotopy equivalent to $\mathbb{Z}[0]$. In particular,
$$H_n(C_*(\Delta^p) \otimes C_*(\Delta^q)) = 0$$
for $n > 0$.

Proof.

1. We define a morphism of simplicial sets $H \colon \Delta^p \times \Delta^1 \to \Delta^p$ that contracts Δ^p to the last vertex.[7] In dimension n,
$$H_n \colon \Delta([n],[p]) \times \Delta([n],[1]) \to \Delta([n],[p])$$
is given by
$$H_n(\alpha, \beta)(i) = \begin{cases} \alpha(i) & \text{if } \beta(i) = 0 \\ p & \text{if } \beta(i) = 1 \end{cases}$$
for $0 \leq i \leq n$. Let $\gamma \colon [m] \to [n]$ be any morphism in Δ. Then
$$H_m(\gamma^*(\alpha,\beta))(j) = H_m(\alpha \circ \gamma, \beta \circ \gamma)(j) = \begin{cases} \alpha(\gamma(j)) & \text{if } \beta(\gamma(j)) = 0 \\ p & \text{if } \beta(\gamma(j)) = 1 \end{cases} = H_n(\alpha,\beta)(\gamma(j)) = \gamma^*(H_n(\alpha,\beta)(j))$$

[7] remember, that Homotopy is not symmetric in Simplicial sets. This is such an example.

This means H is a homotopy from Id_{Δ^p} to the composite

$$\Delta^p \to \Delta^0 \xrightarrow{p\text{-th vertex}} \Delta^p$$

2. $C_*\colon$ **ssets** $\to$ **chains** takes simplicial homotopies to chain homotopies. So we know $C_*(\Delta^p)$ is chain homotopy equivalent to

$$C_*(\Delta^0) = (\dots \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z})$$

which is chain homotopy equivalent to

$$(\dots 0 \to 0 \to 0 \to 0 \to \mathbb{Z}) = \mathbb{Z}[0]$$

3. The tensor product of chain complexes preserves chain homotopy equivalences in each variable separatedly. So

$$C_*(\Delta^p) \otimes C_*(\Delta^q) \sim \mathbb{Z}[0] \otimes C_*(\Delta^1) \sim \mathbb{Z}[0] \otimes \mathbb{Z}[0] \cong \mathbb{Z}[0].$$

□

We now must produce natural chain homotopies from

$$\mathrm{AW} \circ \mathrm{EZ}\colon C_*(X) \otimes C_*(Y) \to C_*(X) \otimes C_*(Y)$$

and

$$\mathrm{EZ} \circ \mathrm{AW} : C_*(X \times Y) \to C_*(X \times Y)$$

to the respective identities.

Claim. $\mathbf{AW} \circ \mathbf{EZ} - \mathrm{Id}_{C_*(X) \otimes C_*(Y)}\colon C_*(X) \otimes C_*(Y) \to C_*(X) \otimes C_*(Y)$ satisfies the hypothesis of acyclic models.

Proof.

$$\begin{tikzcd}
C_0(X) \otimes C_0(Y) \arrow[r, phantom, "\cong"] \arrow[d, equal] & \mathbb{Z}[X_0] \otimes \mathbb{Z}[Y_0] \arrow[r, bend left, "\cong"] & \mathbb{Z}[X_0 \times Y_0] \arrow[l, bend left, "\cong"] \arrow[d, equal] \\
(C_*(X) \otimes C_*(Y))_0 & & C_0(X \times Y)
\end{tikzcd}$$

Which means $(\mathbf{AW} \circ \mathbf{EZ})_0 = \mathrm{Id}$ and $(\mathbf{EZ} \circ \mathbf{AW})_0 = \mathrm{Id}$. which means $\psi_0 =$ zero natural transformation.

$$(C_*(X) \otimes C_*(Y))_n = \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y) = \bigoplus_{p+q=n} \mathbb{Z}[X_p] \otimes \mathbb{Z}[Y_q] \cong \bigoplus_{p+q=n} \mathbb{Z}[X_p \times Y_q]$$

which is represented by (Δ^p, Δ^q). Then $H_n(C_*(\Delta^p \otimes \Delta^q)) = 0$ (I think, he erased before I could copy.)

We consider ϕ: $\mathbf{EZ} \circ \mathbf{AW} - \mathrm{Id}_{C_*(X \times Y)}\colon C_*(X \times Y) \to C_*(X \times Y)$. We know, $\phi_0 = 0$. We need to show, that ϕ satisfies the hypothesis of acyclic models.

$$C_n(X \times Y) = \mathbb{Z}[X_n \times Y_n]$$

is representable by (Δ^n, Δ^n).

$$H_n(C_*(\Delta^n \times \Delta^n)) \cong H_n(\Delta^0 \times \Delta^0) = H_n(\Delta^0) = 0$$

for $n > 0$, where we used $\Delta^n \sim \Delta^0$ and so $\Delta^n \times \Delta^n \sim \Delta^0 \times \Delta^0$. So acyclic models produces a natural chain nullhomotopy of ϕ. $\square$

This concludes the proof of the Künneth theorem. $\square$

1.4.2 Commutativity of the cup-product revisited

The symmetry isomorphism of chain complexes C, D is the morphism.

$$\tau_{C,D}\colon C \otimes D \xrightarrow{\cong} D \otimes C$$

is given by

$$\begin{array}{cccc} \tau_{C,D_n} & \colon (C \otimes D)_n & & (D \otimes C)_n \\ & \bigoplus_{p+q=n} C_p \otimes D_q & & \bigoplus_{q+p=n} D_q \otimes C_p \\ & c \otimes d & & (-1)^{pq} \cdot d \otimes c \end{array}$$

Fact.

$$\begin{tikzcd} C_*(X) \otimes C_*(Y) \arrow[r, "\mathbf{EZ}"] \arrow[d, "\tau"] & C_*(X, Y) \arrow[d, "C_*(flip)"] \\ C_*(Y) \otimes C_*(X) \arrow[r, "\mathbf{EZ}"] & C_*(Y \otimes X) \end{tikzcd}$$

commutes. where $flip\colon X \times Y \to Y \times X,\ (x, y) \mapsto (y, x)$. Hence, „The Eilenberg-Zilber map is symmetric“.

But however for AW the same diagram does NOT commute.

Howeveer it does so up to natural chain homotopy by applying the acyclic models to the differenc of the two composites. He explains, why we can apply acyclic models.

Let X be a simplicial set. The diagonal $\Delta\colon X \to X \times X$ is flip-invariant, i.e.

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & X \times X \arrow[d, "flip"] \\ & X \times X \end{tikzcd}$$

We draw a diagram:

$$\begin{tikzcd} C_*(X) \arrow[r, "C_*(\Delta)"] \arrow[dr, "C_*(\Delta)"] & C_*(X \times X) \arrow[r, "\mathbf{AW}"] \arrow[d, "C_*(flip)"] & C_*(X) \otimes C_*(X) \arrow[d, "\tau"] \\ & C_*(X \times X) \arrow[r, "\mathbf{AW}"] & C_*(X) \otimes C_*(X) \end{tikzcd}$$

that commutes up to homotopy. We apply the functor $\mathrm{Hom}(\_, R)$ to get a new diagram and my speed at copying was not capable of keeping up. You may want to have a look at the videos for this.

[14.04.2025, Lecture 4]

[23.04.2025, Lecture 5]

The Plan for today is to show the Künneth theorem for homology. The rough approximation is, that product of spaces goes to Tensorproducts of abelian groups.

If X, Y are simplicial sets, then by EZ we have $H_*(X \times Y; R) = H_*(C_*(X \times Y; R)) \cong H_*((C_*(X, R)) \otimes_R C_*(Y; R))$ and we want to see how that relates to $H_*(X, R) \otimes_R H_*(Y; R)$.

In the following R is a commutative ring (have integers and fields in mind).

Definition 1.39: Tensor Product of R-chains

Let C, D be chain complexes of R-modules. We define a new complex of R-modules $C \otimes_R D$:

$$(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$

with differential

$$d(x \otimes y) = dx \otimes y + (-1)^{pq} x \otimes dy.$$

Note that $R \otimes \mathbb{Z}[S] \cong R[S]$ for S a simplicial set. And $R[S] \otimes_R R[T] \cong R[S \times T]$ for S, T simplicial sets.

For X, Y simplicial sets, we have

$$R \otimes C_*(X, \mathbb{Z}) \otimes C_*(Y, \mathbb{Z}) \xrightarrow{R \otimes \mathbf{EZ}} R \otimes C_*(X \times Y; \mathbb{Z}) \cong C_*(X \otimes Y; R)$$

and for $R \otimes C_*(X; \mathbb{Z}) \otimes C_*(Y; \mathbb{Z}) \cong (R \otimes C_*(X; Z)) \otimes_R (R \otimes C_*(Y; \mathbb{Z})) = C_*(X, R) \otimes_R C_*(Y, R)$, so we get a Eilenberg-Zilber map

$$C_*(X, R) \otimes_R C_*(Y, R) \xrightarrow{\mathbf{EZ}} C_*(X \times Y; R)$$

Aim. relate $H_*(C \otimes_R D)$ to $H_*(C), H_*(D)$. Our hope is to have a map

$$\bigoplus_{p+q=n} H_p(C) \otimes_R H_q(D) \xrightarrow{???} H_n(C \otimes_R D)$$

For example taking $R = \mathbb{Z}$ and $C = D = (\ldots, \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0)$. Then

$$H_n(C) = H_n(D) = \begin{cases} \mathbb{Z}/2 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

but $C \otimes D = (0 \to 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0)$. And

$$H_1(C \otimes D) = \{(x, -x) \in \mathbb{Z}\} / \{(2y, -2y) \mid y \in \mathbb{Z}\} \cong \mathbb{Z}/2 \neq 0$$

Definition 1.40: Projective R-modules

An R-module P is *projective* if for every epimorphism $\varepsilon \colon M \to N$ of R-modules, the map

$$\mathrm{Hom}(P, \varepsilon) \colon \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$$

is surjective.

$$\begin{tikzcd} & M \arrow[d, two heads, "\varepsilon"] \\ P \arrow[ur, dotted] \arrow[r, "f"] & N \end{tikzcd}$$

Fact. P is projective iff P is a direct summand of a free module iff there exists a R-module Q and a set S, such that

$$P \oplus Q \cong R[S].$$

Proof. Free modulse are projective:

$$\begin{tikzcd} & M \arrow[d, two heads, "\varepsilon"] \\ R[S] \arrow[ur, dotted, "g"] \arrow[r, "f"] & N \end{tikzcd}$$

for every $s \in S$ choose $m_s \in M$ $\varepsilon(m_s) = f(s)$. Then there is a unique homomorphism $g\colon R[S] \to M$ such that $g(s) = m_s$.

Let P be projective and Q a summand of P. For reasons I couldn't copy, then Q is also projective.

Let P be a projective R-modulo. Consider the epimorphism

$$\begin{aligned} R[P] &\to P \\ p &\mapsto p \end{aligned}$$

Then we have

$$\begin{tikzcd} & R[P] \arrow[d, two heads] \\ p \arrow[ur, dotted, "g"] \arrow[r, "\mathrm{id}"] & P \end{tikzcd}$$

So P is a direct summand of $R[P]$.

□

- If R is a field, then all modules are free, hence projective.
- $R = \mathbb{Z}/6$, $P = \mathbb{Z}/2, Q = \mathbb{Z}/3$. Then $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$, so, as $\mathbb{Z}/6$ is free, $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective, but not free.

Proposition 1.41. Let R be a commutative ring, and

$$0 \to I \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

be a short exact sequence of R-modules.

Then for every R-module P, the sequence

$$P \otimes_R I \xrightarrow{P \otimes_R \alpha} P \otimes_R M \xrightarrow{p \otimes_R \beta} P \otimes_R N \to 0$$

is exact. („$P \otimes_R \_$ is right exact“). If moreover P is projective, then it is also exact with a 0 on the left, i.e. $P \otimes_R \alpha$ is injective. („projective modules are flat“).

Proof.

$$p \otimes_R \beta) \circ (p \otimes_R)\alpha = P \otimes_r (\beta \circ \alpha) = P \otimes_R 0 = 0$$

so $\operatorname{Im}(P \otimes_R \alpha) \subseteq \ker(P \otimes_R \beta)$ so we get an induced homomorphism

$$\gamma \frac{P \otimes_R M}{\operatorname{Im}(P \otimes_R \alpha)} \to P \otimes_R N$$

exactness is equivalent to δ being an isomorphism. We define a homomorphism $\delta \colon P \otimes_R N \to \frac{P \otimes_R M}{\mathrm{Im}(P \otimes_R \alpha)}$ given by $(p,n) \in P \otimes N$ choose $\tilde{n} \in M$, such that $\beta(\tilde{n}) = n$.

Claim. $\delta(p \otimes n) = p \otimes \tilde{n} + \mathrm{Im}(P \otimes_R \alpha)$ is independent of choice of $\tilde{n}$

Proof. Let $\tilde{\tilde{n}} \in M$ also satisfy $\beta(\tilde{\tilde{n}}) = n$. Then $\beta(\tilde{\tilde{n}} - \tilde{n}) = 0$, so there is $i \in I$ s.t. $\alpha(i) = \tilde{\tilde{n}} - \tilde{n}$. $p \otimes \tilde{\tilde{n}} - p \otimes \tilde{n} = p \otimes (\tilde{\tilde{n}} - \tilde{n}) = p \otimes \alpha(i) \in \mathrm{Im}(P \otimes_R \alpha)$. □

Claim. The assignment of δ is biadditive and sends (rp, n) and (p, rn) to the same element.

Then this extends to a well defined R-linear map

$$P \otimes_R N \to \frac{P \otimes_R M}{\mathrm{Im}(P \otimes_R \alpha)}$$

which is isomorphic.

Now let P be projective. We show that then $P \otimes_R \alpha$ is injective.

Case 1 $P = R[S]$ free, S some set. Then

$$P \otimes_R M = R[S] \otimes_R M \cong \bigoplus s \in SM$$

we have a natural isomorphism of R-modules in M.

From this we get a commutative square of R-modules:

$$\begin{array}{ccc} P \otimes_R I & \xrightarrow{P \otimes_R \alpha} & P \otimes_R M \\ \wr\Vert & & \wr\Vert \\ \bigoplus_{s \in S} I & \xrightarrow{\bigoplus_{s \in S} \alpha} & \bigoplus_{s \in S} M \end{array}$$

where the bottom map is injective.

General case P projective is a summand of a free module F, i.e. there are homomorphisms

$$P \xrightarrow{\lambda} F \xrightarrow{\mu} P$$

s.t. $\mu \circ \lambda = \mathrm{Id}_P$. We consider thee commutative square

$$\begin{array}{ccc} P \otimes_R I & \xrightarrow{P \otimes_R \alpha} & P \otimes_R N \\ \downarrow{\scriptstyle \lambda \otimes_R I} & & \downarrow{\scriptstyle \lambda \otimes_R N} \\ F \otimes_R I & \xrightarrow{F \otimes_R \alpha} & F \otimes_R N \end{array}$$

where the bottom map is injective by Case 1 and $\lambda \otimes_R I$ is injective, as it admits a retraction.

□

Definition 1.42: Global dimension of rings

A commutative ring R has global dimension ≤ 1 if every submodule of a projective module is projective.

Example 1.43. Some rings with global dimension ≤ 1 are

- fields
- the ring of integers $\mathbb{Z}$ (subgroups of free abelian groups are free).
- every PID[8] is of this form. See for example $k[x]$ for k a field or $\mathbb{Z}[i]$ the gaussian integers
- $\mathbb{Z}_p$ the p-adic integers.

Definition 1.44: Tor of nice rings

Let R be a commutative ring of global dimension ≤ 1. Let M, N be R-modules. Choose an epimorphism $p\colon P \to N$ of R-modules with P projective. Define

$$\mathrm{Tor}^R(M,N) = \mathrm{Ker}(M \otimes_R N \xrightarrow{M\otimes_R incl} M \otimes_R P)$$

Facts. This is independent up to preferred isomorphism of the choice of $p\colon P \to N$.

It is symmetric, i.e. we can resolve M instead of N.

If P is projective, then $\mathrm{Tor}^R(P,N) = 0 = \mathrm{Tor}^R(M,P)$.

Construction 1.45. For R a commutative ring, C, D complexes of R-modules. We define a natural homomorphism

$$\Phi\colon H_p(C) \otimes_R H_q(D) \to H_{p+q}(C \otimes_R D)$$

via $[x] \otimes [y] \mapsto [x \otimes y]$

We can check this is well defined.

Theorem 1.46: Algebraic Künneth theorem

Let R be a commutative ring of global dimension ≤ 1. Let C, D be complexes of projective R-modules. Then the following map is R-linearly split injective

$$\bigoplus \Phi\colon \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(D) \to H_n(C \otimes_R D)$$

Moreover the cokernel is naturally isomorphic to

$$\bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(C), H_q(D)).$$

Equivalently, there is a natural and split short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(D) \xrightarrow{\Phi} H_n(C \otimes D) \to \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(C), H_q(D)) \to 0$$

Proof. We let $Z = \{Z_q\}_{q\in\mathbb{Z}}$ be the comples of R modules with $d = 0$ where $Z_q = \mathrm{Ker}(d\colon D_q \to D_{q-1})$, let $B = \{B_q\}$ be the complex with $d = 0$ where $B_q = \mathrm{Im}(D\colon D_{q+1} \to D_q)$. We have a short exact sequence of complexes of R-modules

$$0 \to Z \xrightarrow{incl} D \xrightarrow{d} B[1] \to 0$$

where $B[1]$ is the complex B shilfted up by 1.

We have $B_q \subseteq Z_q \subseteq D_q$ projective by hypothesis. Since R has global dimension ≤ 1, B_q and Z_q are also projective.

$$0 \to Z_q \to D_q \xrightarrow{d} B_{q-1} \to 0$$

[8]no zero divisors and every ideal is generated by a single element.

is hort exact, B_{q-1} is projective, so the sequence splits.

For every R-module N, the sequence

$$0 \to N \otimes_R Z_p \to N \otimes_R D_q \to N \otimes_R B_{q-1} \to 0$$

is exact.

This means we get a short exact sequence of complexes

$$0 \to C \otimes_R Z \to C \otimes_R D \to C \otimes_R B[1] \to 0$$

This means we get a long exact homology sequence

$$\to H_n(C \otimes_R Z) \xrightarrow{H_n(C \otimes_R incl)} H_n(C \otimes D) \xrightarrow{H_n(C \otimes d)} H_{n-1}(C \otimes_R B) \xrightarrow{\partial} H_{n-1}(C \otimes_R Z) \to \dots$$

Since Z has trivial differential:

$$H_n(C \otimes_R Z) = H_n(\bigoplus_{q \in \mathbb{Z}} C[q] \otimes Z_q) \cong \bigoplus_{q \in \mathbb{Z}} H_n(C[q] \otimes Z_q) \cong \bigoplus_{q \in \mathbb{Z}} H_n(C[q]) \otimes_R Z_q = \bigoplus_{p \in \mathbb{Z}} H_{n-q}(C) \otimes_R Z_q$$

where we use that Z_q is projective.

Similarly $H_n(C \otimes_R B) \cong \bigoplus_{q \in \mathbb{Z}} H_{n-q}(C) \otimes B_q$.

This gives us a long exact sequence

$$\dots \to \bigoplus_{p+q=n} H_p(C) \otimes_R Z_q \to H_n(C \otimes_R D) \to \bigoplus_{p+q=n-1} H_p(C) \otimes B_q \to \bigoplus_{p+q=n-1} H_p(C) \otimes Z_q$$

This splits up into short exact sequences

$$0 \to \bigoplus_{p+q=n} \operatorname{Coker}(H_p(C) \otimes B_q \to H_p(C) \otimes_R Z_p) \to H_n(C \otimes_R D) \to \bigoplus_{p+q=n-1} \operatorname{Ker}(H_p(C) \otimes_R B_q \to H_p(C) \otimes Z_q) \to 0$$

We know $0 \to B_q \to Z_q \to H_q(D)$ is a projective resolution of $H_q(D)$.

This means for all R-modules N,

$$\operatorname{Tor}^R(N; H_q(D)) = \operatorname{Ker}(N \otimes_R B_q \to N \otimes_R Z_q)$$

$$N \otimes_R H_q(D) \cong \operatorname{Coker}(N \otimes_R B_q \to N \otimes_R Z_q)$$

So we get:

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \xrightarrow{\Phi} H_n(C \otimes_R D) \to \bigoplus_{p+q=n-1} \operatorname{Tor}^R(H_p(C), H_q(D)) \to 0$$

for next lecture remains, that Φ has a R-linear retraction!

[23.04.2025, Lecture 5]

[28.04.2025, Lecture 6]

For the R-linear spitting.

Because B_q is projective, the following s.e.s. splits:

$$0 \to Z_q \xrightarrow{\text{incl}} D_q \xrightarrow{d} B_q \to 0$$

and the map Z_q to D_q admits a retraction. We choose a retraction $r_q \colon D_q \to Z_q$ to the inclusion.

Then

$$\begin{tikzcd} D_q+1 \arrow[d, "d"] \arrow[rrdd, bend left, "0"] & & \\ B_q \arrow[rd, "="'] \arrow[rrd, "0"] & & \\ \rotatebox{90}{$\subseteq$} \; D_q \arrow[r, "r_q"] & Z_q \arrow[r] & H_q(D) \end{tikzcd}$$

the retraction $\{r_q\}_{q\in\mathbb{Z}}$ for a morphism of chain complexes

$$r\colon D \to \{H_q(D), d=0\}_q$$

that induces the identity on homology.

$H_q(r) \cong H_q(D) \to H_q(H_*(D), d=0) = H_q(D)$. Similalry, there is a chain map $\rho\colon C \to \{H_p(C), d=0\}$ that is the identity on homology. This gives a chain mpa $\rho \otimes_R r\colon C\otimes_R D \to (H_*(C)\otimes_R H_*(D), d=0)$ which on homology

$$H_n(\rho\otimes_r r)\colon H_n(C\otimes_R D) \to H_n(H_*(C)\otimes RH_*(D), d=0) = \bigoplus_{p+q=n} H_n(C)\otimes_R H_n(D)$$

which is a retraction to

$$\Psi\colon \bigoplus_{p+q=n} H_p(C)\otimes_R H_q(D) \to H_n(C\otimes RD)$$

□

Example 1.47. Let R be a field. Then every module is free, hence projective, and

$$\mathrm{Tor}^R(M,N)=0$$

for all R-modules M,N. For all complexes of R-modules C,D;

$$\psi\colon \bigoplus_{p+q=n} H_p(C)\otimes_R H_q(D) \xrightarrow{\cong} H_n(C\otimes_R D).$$

is an isomorphism.

If $R=\mathbb{Z}$. Let C,D be a complex of free abelian groups. Then there is a split s.e.s.

$$0\to \bigoplus_{p+q=n} H_p(C)\otimes H_q(D) \to H_n(C\otimes D) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C), H_q(D)) \to 0$$

Construction 1.48 (Homology exterior pairing)**.** Let X,Y be simplicial sets. Let R be a commutative ring. We define

$$\times\colon H_p(X,R)\otimes_R H_q(Y,R) \to H_{p+q}(X\times Y, R)$$

as the composite

$$H_p(C_*(X,R))\otimes_R H_q(C_*(Y;R)) \xrightarrow{\Phi} H_{p+q}(C_*(X,R)\otimes C_*(Y,R)) \xrightarrow{H_{p+q}(\mathrm{EZ})} H_{p+q}(C_*(X\times Y,R))$$

For topological spaces A,B we Define

$$\times\colon H_p(A;R)\otimes_R H_q(B,R) \to H_{p+q}(A\times B, R)$$

as the composite

$$H_p(\mathcal{S}(A), R) \otimes_R H_q(\mathcal{S}(B), R) \xrightarrow{\times} H_{p+q}(\mathcal{S}(A) \otimes \mathcal{S}(B), R) \cong H_{p+q}(\mathcal{S}(A \times B); R)$$

where the isomorphism is given by the fact, that simplical complex commutes with products. The isomorphism is the canonical map

$$\mathcal{S}(A) \times \mathcal{S}(B) \xleftarrow{(\mathcal{S}(p_A), \mathcal{S}(p_B))} \mathcal{S}(A \times B)$$

Theorem 1.49: Künneth theorem for homology with field coefficients

Let R be a field. Let X, Y be simplicial sets or spaces. Then the homology external product

$$\times\colon \bigoplus_{p+q=n} H_p(X, R) \otimes_R H_q(X, R) \to H_n(X \times Y; R)$$

is an isomorphism.

Proof. Follows directly from algebraic Künneth + Eilenberg-Zilber □

Theorem 1.50: Künneth theorem for homology

Let X, Y be spaces or simplicial sets. Then there is a natural and split s.e.s.

$$0 \to \bigoplus_{p+q=n} H_p(X, \mathbb{Z}) \otimes H_q(Y, \mathbb{Z}) \to H_n(X \times Y; \mathbb{Z}) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X, \mathbb{Z}), H_q(Y, \mathbb{Z})) \to 0$$

Special Case. Let X, Y be spaces or simplical sets. Suppose that $H_n(X, \mathbb{Z})$ is free for all $n \geq 0$. Then

$$\bigoplus_{p+q=n} H_p(X, \mathbb{Z}) \otimes H_q(Y, \mathbb{Z}) \xrightarrow{\Phi} H_n(X \times Y; \mathbb{Z})$$

is an isomorphism.

Next we want to show the Künneth theorem for cohomology. The strategy:

- EZ provides a chain homotopy equilvalence $C_*(X, \mathbb{Z}) \otimes C_*(Y, \mathbb{Z})$ to $C_*(X \times Y, \mathbb{Z})$.
- $\mathrm{Hom}(\_, R)\colon$ **Chains** $\to$ **coChains**$_R$ preserves chain homotopies, so

$$\mathrm{Hom}(C_*(X, \mathbb{Z}), R) \otimes \mathrm{Hom}(C_*(Y, \mathbb{Z}), R) \cong \mathrm{Hom}((C_*(X \times Y), \mathbb{Z}), R)$$

- in favorable cases we can relate

$$H^*(\mathrm{Hom}(C, R) \otimes_R \mathrm{Hom}(D, R)) \text{ to } H^*(\mathrm{Hom}(C, R)) \otimes_R H^*(\mathrm{Hom}(D, R))$$

- apply the algebraic Künneth theorem.

Step 3 is the hard step.

1.4.3 Relation between Homs and Tensors

Let A be an abelian group and R an commutative ring. We make the set $\mathrm{Hom}(A, R)$ of group homomorphisms into an R module by pointwise addition and skalar multiplication. So $f, g \in \mathrm{Hom}(A, R)$, $r \in R$. then

$$(f + g)(a) = f(a) + g(a), \quad ((r \cdot f)(a) = r \cdot f(a))$$

Let B be another abelian group. Then

$$\bullet\colon \operatorname{Hom}(A,R)\times\operatorname{Hom}(B,R)\to\operatorname{Hom}(A\otimes B,R)$$

by $(f\bullet g)(a\otimes b)=f(a)\cdot g(b)$. This is additive in f and g.

$$(f+f')\bullet g=(f\bullet g)+(f'\bullet g)$$

and

$$(rf)\bullet g=r\cdot(f\bullet g)=f\bullet(r\cdot g)$$

for all $r\in R$. This means this extends to a well-defined R-linear map

$$\operatorname{Hom}(A,R)\otimes_R\operatorname{Hom}(B,R)\to\operatorname{Hom}(A\otimes B,R)$$

Proposition 1.51. Let A,B be abelian groups and R a commutative ring. If A is finitely generated and free, then

$$\operatorname{Hom}(A,R)\otimes_R\operatorname{Hom}(B,R)\to\operatorname{Hom}(A\otimes B,R)$$

is an isomorphism of R-modules.

Proof. For $A=\mathbb{Z}$:

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{Z},R)\otimes_R\operatorname{Hom}(B,R) \arrow[r, "\bullet"] \arrow[d, "\mathrm{ev}\otimes_R\operatorname{Hom}(B,R)"] & \operatorname{Hom}(\mathbb{Z}\otimes B,R) \arrow[d, "\cong\ \operatorname{Hom}(k,R)"] \\
R\otimes_R\operatorname{Hom}(B,R) \arrow[r, "\cong", "r\otimes g\mapsto r\cdot g"'] & \operatorname{Hom}(B,R)
\end{tikzcd}$$

where we have $k\colon B\to\mathbb{Z}\otimes B$ with $b\mapsto 1\otimes b$.

Suppose the claim holds for A and A'. Then it holds for $A\oplus A'$.

$$\begin{tikzcd}
\operatorname{Hom}(A\oplus A*,R)\otimes_R\operatorname{Hom}(B,R) \arrow[r, "\bullet"] \arrow[d, phantom, "\cong"] & \operatorname{Hom}((A\oplus A')\otimes B,R) \arrow[d, phantom, "\cong"] \\
(\operatorname{Hom}(A,R)\oplus\operatorname{Hom}(A',R))\otimes_R\operatorname{Hom}(B,R) \arrow[d, phantom, "\cong"] & \operatorname{Hom}((A\otimes B)\oplus(A'\otimes B),R) \arrow[d, phantom, "\cong"] \\
(\operatorname{Hom}(A,R)\otimes_R\operatorname{Hom}(B,R))\bigoplus(\operatorname{Hom}(A',R)\otimes_R\operatorname{Hom}(B,R)) \arrow[r, "\cong\text{ by assumption}"] & \operatorname{Hom}(A\otimes B,R)\oplus\operatorname{Hom}(A'\otimes B,R)
\end{tikzcd}$$

The claim holfs for $A=\mathbb{Z}^k$, $k\in\mathbb{N}$. any finitely generated free abelian group is isomorphic to $\mathbb{Z}^k$. $\square$

Example 1.52. $R=\mathbb{F}_2$ $A=B=\mathbb{Z}[\mathbb{N}]$. Then $\operatorname{Hom}(\mathbb{Z}[\mathbb{N}],R)\cong\operatorname{maps}(\mathbb{N},R)$ by evaluation of generators. This is R-linear by the R-module structure on $\operatorname{maps}(\mathbb{N},R)$.

$$\begin{tikzcd}
\operatorname{Hom}(A,R)\otimes\operatorname{Hom}(B,R) \arrow[r, "\bullet"] & \operatorname{Hom}(A\otimes B,R) \\
\operatorname{maps}(\mathbb{N},R)\otimes_R\operatorname{maps}(\mathbb{N},R) & \operatorname{Hom}(\mathbb{Z}[\mathbb{N}\times\mathbb{N}],R) \\
 & \operatorname{maps}(\mathbb{N}\times\mathbb{N},R)
\end{tikzcd}$$

This is however not an isomorphism.

$A = B = \mathbb{Z}/2$ and $R = \mathbb{Z}/4$. Then $\mathrm{Hom}(A,R) = \mathrm{Hom}(B,R) = \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/4)$ is cyclif of order two generate by $i\colon \mathbb{Z}/2 \to \mathbb{Z}/4$, $n + 2\mathbb{Z} \mapsto 2n + 4\mathbb{Z}$.

$$\begin{array}{ccc}
\mathrm{Hom}(A,R) \otimes_R \mathrm{Hom}(B,R) & \xrightarrow{\bullet} & \mathrm{Hom}(A \otimes B, R) \\
\| & & \\
\mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/4) \otimes_{\mathbb{Z}/4} \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/4) & & \mathrm{Hom}(\mathbb{Z}/2 \otimes \mathbb{Z}/2, \mathbb{Z}/4) \\
\cong & & \cong \\
\mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2
\end{array}$$

This shows, that both assumptions are strictly necessary.

Now let C, D be complexes of abelian groups. Then $\mathrm{Hom}(C,R), \mathrm{Hom}(D,R)$ are cochian complexes of R-modules.

$$\mathrm{Hom}(C,R)^n = \mathrm{Hom}(C_n, R)$$

and

$$d^n\colon \mathrm{Hom}(C,R)^n \to \mathrm{Hom}(C, R^{n+1}) = \mathrm{Hom}(D_{n+1}, R)$$

The sum of the $\bigoplus$ homomorphism gives a cochain map

$$\bigoplus\colon \mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(D,R) \to \mathrm{Hom}(C \otimes D, R)$$

which is in dimension n:

$$\bigoplus_{p+q=n} \mathrm{Hom}(C_p,R) \otimes_R \mathrm{Hom}(D_q,R) \xrightarrow{\text{sum of } \bigoplus} \mathrm{Hom}(\bigoplus_{p+q=n} C_p \otimes D_q, R)$$

Proposition 1.53. Let C and D be chain complexes of abelian groups, such that $C_n = 0 = D_n$ for $n < 0$ and that C_n is finitely generated and free for all $n \geq 0$. Then $\bigoplus$ is an isomorphism.

$$\bigoplus\colon \mathrm{Hom}(C,R) \otimes \mathrm{Hom}(D,R) \to \mathrm{Hom}(C \otimes D, R)$$

is an isomorphism of cochain complexes.

Proof. The vanishing hypothesis makes the potentially infinte sums

$$\bigoplus_{p+q=n} \mathrm{Hom}(C_p,R) \otimes_R \mathrm{Hom}(D_q,R)$$

finite.

Then $\mathrm{Hom}(\_, R)$ preserves sums. And

$$\mathrm{Hom}(C_p,R) \otimes \mathrm{Hom}(D_q,R) \xrightarrow{\bigoplus} \mathrm{Hom}(C_p \otimes D_q; R)$$

is an isomorphism by the previous proposition. □

This is not yet good enough to apply to topological spaces, as they are very not finitely generated.

Proposition 1.54. Let C be a chain complex of free abelian groups, such that $C_n = 0$ for $n < 0$. Suppose that $H_n(C)$ is finitely generated for all $n > 0$.

Then there is a subcomplex B of C, such that

- B_n is finitely generated and free for all $n \geq 0$.
- The inclusion $B \to C$ is a chain homotopy equivalence.

Proof. We construct subgroups B_n of C_n by induction on $n \geq 0$, such that

- $d(B_n) \subseteq B_{n-1}$
- the inclusions of $0 \to B_n \xrightarrow{d} B_{n-1} \xrightarrow{d} \cdots \to B_0 \to 0$
- into C induces an isomorphism on H_i for all $0 \leq i \leq n-1$ and an epimorphism on H_n.

Induction start: Let $x_1, \ldots, x_m$ be elements of C_0, that generate $H_0(C)$. Select B_0 to be the subgroups of C_0 generated by $x_1, \ldots, x_m$.

Induction step: Suppose $B_0, \ldots, B_{n-1}$ have been constructed fullfilling the conditions. Let $x_1, \ldots, x_m$ be cycles in C_n whose homology classes generate $H_n(C)$, which is possible because $H_n(C)$ is finitely generated. Set

$$Z = \mathrm{Ker}(d\colon B_{n-1} \to B_{n-2}) \cap \mathrm{Im}(d\colon C_n \to C_{n-1})$$

which is finitely generated because B_{n-1} is. Let $z_1, \ldots, z_k$ generate this intersection. Choose $y_1, \ldots, y_k \in C_n$, such that $d(y_i) = z_i$ for $1 \leq i \leq k$.

Let B_n be the subgroup generated by $x_1, \ldots, x_m, y_1, \ldots, y_k$. Then $d(B_n) \subseteq B_{n-1}$.

Let $B_{\leq n}$ and $B_{<n}$ be the subcomplexes of C generated by $B_0, \ldots, B_n$ and $B_0, \ldots, B_{n-1}$

Then $B_{<n} \subseteq B_{\leq n} \subseteq C$ where $B_{<n}$ induces isomorphism on H_i for $0 \leq i \leq n-2$ and epi on H_{n-1}. Similarly $B_{<n} \to B_{\leq n}$ is iso in dimension $\leq n-1$.

Then $B_{\leq n}$ is an Isomorphism on H_i for $0 \leq i \leq n-2$ and surjective on H_n because we include $x_1, \ldots, x_m$ that generate $H_n(C)$.

Let $x \in B_{n-1}$ be any cycle whose class is in the kernel of $H_{n-1}(B_{<n}) \to H_{n-1}(C)$. Then $x \in Z$ so x is a linear combination of the classes $z_1, \ldots, z_k$ and hence a boundary of a linear combination of $y_1, \ldots, y_k$. So $x = d(w)$ for some $w \in B_n$. Then

$$\begin{tikzcd}
& H_{n-1}(B_n) \arrow[dr] & \\
H_{n-1}(B_{<n}) \arrow[ur] \arrow[rr] & & H_{n-1}(C)
\end{tikzcd}$$

the class of x maps to 0 and the map becomes injective and hence an isomorphism.

We let B be the subcomplex of C generated by all B_i for all $i \geq 0$. Then the inclusion $B \to C$ induces an isomorphism on H_i for all $i \geq 0$, so it is a quasi-isomorphism.

By the end of last term we prooved, it is already a chain homotopy equivalence! □

[23.04.2025, Lecture 6]

[30.04.2025, Lecture 7]

Theorem 1.55: Algebraic Künneth theorem, cohomology

Let R be a commutative ring of global dimension ≤ 1. Let C, D be chain complexes of abelian groups such that $C_n = 0 = D_n$ for $n < 0$ and all C_n are free and $H_n(C)$ is finitely generated free.
Then for all $n \geq 0$:

$$\bigoplus_{p+q=n} H^p(\mathrm{Hom}(C,R)) \otimes_R H^q(\mathrm{Hom}(D,R)) \xrightarrow{\Phi} H^n(\mathrm{Hom}(C \otimes D, R))$$

is injective and its cokernel is isomorphic to

$$\bigoplus_{p+q=n+1} \mathrm{Tor}^R(H^p(\mathrm{hom}(C,R)), H^q(\mathrm{Hom}(D,R)))$$

Warning. We do not assume, that there is a splitting.

Proof. „Basically just putting all the hard stuff we've already done together in the right way."

Case 1 Suppose that also C_n is finitely generated for all $n \geq 0$. Then $\bullet\colon \mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(D,R) \to \mathrm{Hom}(C \otimes D, R)$ is an isomorphism of cochain complexes. Applying the homological algebraic Künneth theorem to

$$H^n(\mathrm{Hom}(C \otimes D, R)) \cong H^n(\mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(D,R))$$

since C_n is finitely generated and free, it is isomorphic to $\mathbb{Z}^k$ for some $k \geq 0$, so $\mathrm{Hom}(C,R)^n = \mathrm{Hom}(C_n, R) \cong \mathrm{Hom}(\mathbb{Z}^k, R) = R^k$ which is free hence projective as an R-module for all $n \geq 0$.

Caveat 1. we make cochain complexes into chain complexes, then apply Künneth, then come back. This turns $n-1$ in the $\bigoplus$ for Tor$R$ into $n+1$.

Caveat 2. The proof of the homological Künneth theorem (without the splitting) used only that one complex is dimensionwise projective. Hence it is no problem, that D is not projective.

General case We choose a subcomplex B of C such that B_n is finitely generated for all $n \geq 0$ and $B \hookrightarrow C$ is a chain homotopy equivalence. Then

$$\mathrm{Hom}(i, R)\colon \mathrm{Hom}(B,R) \to \mathrm{Hom}(C,R)$$

is a chain homotopy equivalence of R-module complexes.[9]

Note Additive functors preserve chain homotopy equivalences, however not quasi-isomorphisms. Because of that, quasi-Isomorphisms and chain homotopy equivalences are quite different.

Similarly we see

$$\mathrm{Hom}(i \otimes D, R)\colon \mathrm{Hom}(C \otimes D, R) \to \mathrm{Hom}(B \otimes D, R)$$

is a chain homotopy equivalence.

[9]This is due to the Hom-functor being additive. Unfortunately I don't know what that means.

This gives a commutative square in $\mathbf{coChains}_R$:

$$\begin{tikzcd}
\bigoplus_{p+q=n} H^p(\mathrm{Hom}(C,R)) \otimes H^q(\mathrm{Hom}(D,R)) \arrow[r, hook, "\Phi"] \arrow[d, "\cong"] & H^n(\mathrm{Hom}(C\otimes D,R)) \arrow[d, "\cong"] \\
\bigoplus_{p+q=n} H^p(\mathrm{Hom}(B,R)) \otimes_R H^q(\mathrm{Hom}(D,R)) \arrow[r, hook, "\Phi", "\text{by special case}"'] & H^n(\mathrm{Hom}(B\otimes D,R))
\end{tikzcd}$$

□

Construction 1.56. Let X, Y be spaces or simplicial sets. R an commutative ring. The *exterior cup product*

$$\times \colon H^p(X,R) \times H^q(Y,R) \to H^{p+q}(X \otimes Y, R)$$

is defined by $(x,y) \mapsto p_1^*(x) \cup p_2^*(y)$, where $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$.

Recall. The AW-map is

$$\mathrm{AW} \colon C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

Proposition 1.57. Let X, Y be simplicial sets, R commutative ring. Then the composite

$$H^p(X,R) \otimes_R H^q(Y,R) \xrightarrow[{[f}]{\otimes} [g] \mapsto [f \otimes g]] \Phi H^{p+q}(\mathrm{Hom}(C_*(X),R) \otimes_R \mathrm{Hom}(C_*(Y),R)) \xrightarrow{H^{p+q}(\bullet)} H^{p+q}(\mathrm{Hom}(C_*(X)$$

equals the external cup product.

Proof. In the notes. □

Theorem 1.58: Künneth theorem in cohomology

Let R be a commutative ring of global dimension ≤ 1. Let X, Y be spaces such that $H_n(X,\mathbb{Z})$ is finitely generated for all $n \geq 0$. Then the total exterior cup product map

$$\bigoplus_{p+q=n} H^p(X,R) \otimes_R H^q(Y,R) \to H^n(X \times Y, R)$$

is injective, and its cokernel is naturally isomorphic to

$$\bigoplus_{p+q=n+1} \mathrm{Tor}^R(H^p(X,R), H^q(Y,R))$$

Proof. Similar to the homological one. Use the cohomological algebraic Künneth theorem and the Eilenberg-Zilber theorem. You can read it up somewhere. □

Remark 1.59. Let X be a CW-complex of finite type i.e. such that it has only finitely many cells in every dimension. (ex. $\mathbb{R}P^\infty$). Then

$$C_A^{Cell}(X,\mathbb{Z})$$

is finitely generated free in every dimension, hence $H_n^{cell}(X,\mathbb{Z}) \cong H_n(X,\mathbb{Z})$ if finitely generated, so Künneth theorem applies.

Construction 1.60. Let A, B be graded-commutative[10] rings. Then $A \otimes B$ is another graded-commutative ring by

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes B_q$$

with multiplication for $a \in A_p, b \in B_q, a' \in A_{p'}, b' \in B_{q'}$.

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{p' \cdot q} (aa') \otimes (bb')$$

Check for well-definedness yourself.

Korollar 1.61. *Let R be a field, X, Y spaces and suppose, that $H_n(X, \mathbb{Z})$ is finitely generated for all $n \geq 0$. Then*

$$\times \colon H^*(X, R) \otimes_R H^*(Y, R) \to H^*(X \times Y, R)$$

is an isomorphism of graded-commutative R-algebras.

Note. We already knew that this is a isomorphism of abelian groups. The new information is, that this is compatible with ring structure.

Proof. We take $x \in H^p(X, R), x' \in H^{p'}(X, R), y \in H^q(Y, R), y' \in H^{q'}(Y, R)$ and then

$$\begin{aligned}
(x \cup x') \times (y \cup y') &= p_1^*(x \cup x') \cup p_2^*(y \cup y') \\
&= (p_1^*(x) \cup p_1^*(x')) \cup (p_2^*(y) \cup p_2^*(y')) \\
&= (-1)^{p' \cdot q} (p_1^*(x) \cup p_2^*(y)) \cup (p_1^*(x') \cup p_2^*(y')) \\
&= (-1)^{p \cdot q'} (x \times y) \cup (x' \times y')
\end{aligned}$$

□

Korollar 1.62. *Let X, Y be spacees such that $H_n(X, \mathbb{Z})$ is finitely genrated and free for all $n \geq 0$. Then*

$$H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z}) \to H^*(X \times Y, \mathbb{Z})$$

is an isomoprhism of graded-commutative rings.

Now we are actually calculating some cohomology rings. Namely for $S^k \times S^l$, $S^1 \times \cdots \times S^1$ and $\mathbb{C}P^2$.

Remember

$$H^n(S^k) = \begin{cases} \mathbb{Z} & n = 0, k \\ 0 & n \neq 0, k \end{cases}$$

and assume $k \geq 1$. For dimensional reasons, the cup product on $H^*(S^k, \mathbb{Z})$ is trivial. $H^*(S^k, \mathbb{Z})$ is dimensionwise finitely generated free, and hence for every space Y the exterior cup product

$$H^*(S^k, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z}) \to H^*(S^k \times Y, \mathbb{Z})$$

is an isomorphism of graded-commutative rings. Take $Y = S^l$ for $l \geq 1$.

Let $e_k \in H^n(S^k; \mathbb{Z})$ be one of the two generators. Then $H^+(S^k, \mathbb{Z}) = \Lambda(e_k)$ where Λ denotes an exterior product. This includes $e_k^2 = 0$. We define $a := p_1^*(e_k) \in H^k(S^k \times S^l; \mathbb{Z})$ and $b := p_2^*(e_l) \in H^l(S^k \times S^l; \mathbb{Z})$. Then

$$H^*(S^k, \mathbb{Z}) \otimes H^*(S^l, \mathbb{Z}) \xrightarrow[x]{\cong} H^*(S^k \times S^l; \mathbb{Z}) = \mathbb{Z}\{1 \times 1, 1 \times e_l, e_k \times 1, e_k \cdot e_l\}$$

where we have $1 \times 1 = 1, 1 \times e_l = b, e_k \times 1 = a, e_k \cdot e_l = a \cup b$.

[10] $a \cdot b = (-1)^{\deg(A) \cdot \deg(B) b \cdot a}$

We look at multiplicative relations:
$$a^2 = 0, b^2 = 0$$
and so
$$a^2 = (p_1^*(e_k))^2 = p_1^*(e_k^2) = p_1^*(0) = 0$$

If k or l is even, then $a \cup b = b \cup a$ and if both are odd, then $a \cup b = -b \cup a$.

We summarize, if k and l are even, then

$$H^*(S^k \times S^l; \mathbb{Z}) = \mathbb{Z}[a,b]/(a^2 = 0, b^2 = 0)$$

and if one is odd
$$H^*(S^k \times S^l; \mathbb{Z}) = \Lambda(a,b)$$

where Λ again denotes exterior products.

We give an inductive description of $H^*(S^1 \times \cdots \times S^1; \mathbb{Z})$ n-times. We use, that

$$\times \colon H^*(S^1;\mathbb{Z}) \otimes H^*(\underbrace{S^1 \times \cdots \times S^1}_{n-1 \text{ times}}) \cong H^*(S^1 \times \cdots \times S^1, \mathbb{Z})$$

we define $a_i = p_i^*(e_1) \in H^1(\underbrace{S^1 \times \cdots \times S^1}_{n}; \mathbb{Z})$, where $p_i \colon (S^1)^n \to S^1$ is projection to the i-th factor for $1 \leq i \leq n$. We get $a_i^2 = 0$ and $a_i \cup a_j = -a_j \cup a_i$ for $i \neq j$. This gives us, that an additive basis of $H^*(S^1)^n; \mathbb{Z}$ is given by

$$a_{i_1} \cup \cdots \cup a_{i_k} \text{ for all tuples } 1 \leq a_i < a_2 < \cdots < a_k \leq n$$

This gives us $\operatorname{rank}(H^*((Sq)^n\mathbb{Z})) = 2^n$. The multiplicative structure is given by $H^*((S^1)^n, \mathbb{Z}) = \Lambda(a_1, \ldots, a_n)$.

Later we will compute $H^*(\mathbb{C}P^n; \mathbb{Z})$ via Poincaré-duality to get $\cong \mathbb{Z}[X]/(X^{n+1})$ for $x \in H^2(\mathbb{C}P^n, 2)$.

We will now use a trick to at least calculate $H^*(\mathbb{C}P^2; \mathbb{Z})$. We know, that

$$H^n(\mathbb{C}P^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0,2,4 \\ 0 & \text{else} \end{cases}$$

we take $x \in H^2(\mathbb{C}P^2; \mathbb{Z})$ a generator. The multiplicative structure is completely defined by which multiple of the generator of $H^4(\mathbb{C}P^2, \mathbb{Z})$ x^2 is.

We use homogenous coordinate noatation for $\mathbb{C}P^2$. For $0 \neq (x,y,z) \in \mathbb{C}^3$ we write $[x,y,z] \coloneqq \mathbb{C} \cdot (x,y,z) \in \mathbb{C}P^2$. We define a continuous map

$$\mu \colon \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$$

given by $([v,w],[x,y]) \mapsto [vx, vy + wx, wy]$. We let $e = [1,0]$ a basepoint in $\mathbb{C}P^1$. Then $\mu(e, \_), \mu(\_, e) \colon \mathbb{C}P^1 \to \mathbb{C}P^2$. are both the „standard inclusions“ $[x,y] \mapsto [x,y,0]$.

Proposition 1.63. The map $\mu^* \colon H^4(\mathbb{C}P^2, \mathbb{Z}) \to H^4(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{Z})$ is injective and its image has index 2.

proof next time.

[30.04.2025, Lecture 7]

[5.05.2025, Lecture 8] Rather sleepy today, quality may be accordingly.

Note. Remember $\mathbb{C}P^2 \cong S^2$.

Proof. We will drop coefficients from the notation. $H^*(X) \coloneqq H^*(X;\mathbb{Z})$. The continuous map $\mathbb{C}^2 \to \mathbb{C}P^2$, $\pi(a,b) = (a^2 - b, 2a, 1)$ is an open embedding and a homoeomorphism onto the open

4-cell $\mathbb{C}P^2 \setminus \mathbb{C}P^1$. That is just the set $[x, y, 1]$ for $(x, y) \in \mathbb{C}^2$. Then

$$(x, y) = (a^2 - b, 2a) \implies a = y/2, b = (a - x = y^2/4 - x)$$

This gives an isomorphism of relative cohomology groups

$$\pi^*\colon H^4(\mathbb{C}P^2 \setminus \mathbb{C}P^1, \mathbb{C}P^2 \setminus (\mathbb{C}P^1 \cup [0,0,1])) \to H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0))$$

Then we hAVE AN EXCISION isomorphism:

$$H^4(\mathbb{C}P^2, \mathbb{C}P^2 \setminus [0,0,1]) \cong H^4(\mathbb{C}P^2 \setminus \mathbb{C}P^1, \mathbb{C}P^2 \setminus \mathbb{C}P^1 \cup [0,0,1])$$

The long exact sequence of the pair gives an isomorphism

$$H^4(\mathbb{C}P^2, \mathbb{C}P^2 \setminus [0,0,1]) \to H^4(\mathbb{C}P^2)$$

We also Define

$$\pi'\colon \mathbb{C}^2 \to \mathbb{C}P^1 \times \mathbb{C}P^1, (a, b) \mapsto ([a + b, 1], [a - b, 1])$$

A similar calulation gives

$$H^4(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus ([0,1],[0,1]))$$

as isomorphic .to $H^4(\mathbb{C}P^1 \times \mathbb{C}P^1)$.

We now also define ν.

$$\nu\colon \mathbb{C}^2 \to \mathbb{C}^2; \quad (a, b) \mapsto (a, b^2)$$

Now a diagram I didn't copy commutes.

The problem nor reduces to show that

$$\nu^*\colon H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0)) \to H^4(\mathbb{C}^2, \mathbb{C}^2 \times 0), 0$$

is multiplication by 2.

A diagramm I didn't copy. He applied Künneth and found out some map is multiplication by 2. □

Proposition 1.64. Let $x \in H^2(\mathbb{C}P^2, Z)$ be an additive generator. Then x^2 is an additive generator of $H^4(\mathbb{C}P^2, \mathbb{Z})$. So $H^*(\mathbb{C}P^2, \mathbb{Z})$ is a truncated polynomial algebra i.e.

$$H^*(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}[X]/(x^3)$$

Outlook. $H^*(\mathbb{C}P^m; \mathbb{Z}) = \mathbb{Z}[X]/(x^{m+1})$ This will be proven later using Poincaré-Duality.

Proof. We write $i\colon \mathbb{C}P^1 \to \mathbb{C}P^2$ for „the inclusion", $i[x, y] = [x, y, 0]$. Then

$$H^*(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}\{1, i^2(x)\}$$

$$\times\colon H^*(\mathbb{C}P^2) \otimes H^*(\mathbb{C}P^1) \cong H^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$$

we write $a := p_1^*(i^*(x)), b := p_2^*(i^*(x))$. Then

$$H^*(\mathbb{C}P^1 \times \mathbb{C}P^1) = \mathbb{Z}\{1, a, b, a \cdot b\}$$

with $a^2 = b^2 = 0, ab = ba$.

Clain. We have

$$\begin{array}{ccc} \mu^*(x) = a + b & \in & H^2(\mathbb{C}p^1 \times \mathbb{C}P^1) \\ & & \downarrow\cong \\ & & H^2(\mathbb{C}P^1 \vee \mathbb{C}P^1) \\ & & \downarrow\cong \\ & & H^2(\mathbb{C}P^1) \times H^2(\mathbb{C}P^1) \end{array}$$

where we use that the wedge is an isomorphis on the 2-skeleton of $\mathbb{C}P^1 \times \mathbb{C}P^1$. The composite map is given by

$$z \mapsto ((e, \_^*(z)), (\_, e)^*(z))$$

We note

$$\begin{aligned} (e, \_)^*(a+b) &= (e, \_)^*(p_1^*(i^*(x))) + (e, \_)^*(p_2^*(i^*(x))) \\ &= (i \circ \underbrace{p_1 \circ (e, \_)}_{\text{constant}})^*(x) + (i \circ \underbrace{p_2 \circ (e, \_)}_{\text{identity}})^*(x) \\ &= i^*(x) \end{aligned}$$

and also

$$(e, \_)^*(\mu^*(x)) = (\underbrace{\mu \circ (e, \_)}_{=1})^* = i^*(x)$$

This gives $\mu^*(x) = a + b$ Now let $y \in H^4(\mathbb{C}P^2)$ be a generator and let $n \in \mathbb{Z}$ be such, that $x^2 = n \cdot y$. Now

$$2ab = (a+b)^2 = (\mu^*(x))^2 = \mu^*(x^2) = \mu^*(ny) = n \cdot \mu^*(y) = n \cdot 2 \cdot ab$$

where the last equality uses degree 2 of μ. This holds in the free abelian group $H^4(\mathbb{C}P^1 \times \mathbb{C}P^1) = \mathbb{Z}\{a, b\}$. This means $2 = 2n$ and hence $n = 1$ and so j$x^2 = y$. □

Application to the Hopf map.

The Hopf map $\eta\colon S^3 \to S^2$ is defined as

$$S^3 = S(\mathbb{C}^2) \to \mathbb{C}P^1 \cong S^2$$

given by $(x, y) \mapsto [x, y]$.

Then $0 \neq [y] \in \pi_3(S^2, *) \cong \mathbb{Z}\{y\}$.

Proposition 1.65. Attaching a 4-cell to $\mathbb{C}P^1$ yields a space homeomorphic to $\mathbb{C}P^2$. Informally: „η is the attahing map of the 4-cell in $\mathbb{C}P^2$.“

Proof. Consider the map $\alpha\colon D(\mathbb{C}^2) \to \mathbb{C}P^2$, $(x, y) \mapsto [x, y, 1 - |x|^2 - |y|^2]$.

This restricts to a homeomorphism from $D(\mathbb{C}^4) \setminus S(\mathbb{C}^2)$ onto $\mathbb{C}P^2 \setminus \mathbb{C}P^1$ and the following commutes:

$$\begin{tikzcd} S(\mathbb{C}^2) \arrow[r, "\eta"] \arrow[d, hook] & \mathbb{C}P^1 \arrow[d, "i"] & {[x, y]} \arrow[d, mapsto] \\ D(\mathbb{C}^2) \arrow[r, "\alpha"] & \mathbb{C}P^2 & {[x, y, 0]} \end{tikzcd}$$

this gives a well-defined continuous map $D(\mathbb{C}^2) \cup_{S(\mathbb{C}^2), \eta} \mathbb{C}P^1 \to \mathbb{C}P^2$, This is a continuous bijection between compact Hausdorff spaces, hence a homeomorphism. □

Theorem 1.66: Hopf map is not constant

The Hopf map η is not hmotopic to a constant map.

Proof. By contradiction. If η was homotopic to the constant map $c\colon S^3 \to S^2$, then $D^4 \cup_{S^3,\eta} \mathbb{C}P^1$ would be homotopy-equivalent to $D^4 \cup_{S^3,\mathrm{const}} \mathbb{C}P^1 = \mathbb{C}P^1 \vee (D^4/S^3) \cong S^2 \vee S^4$.

These spaces have the same additive cohomology. However, their cup-product differs. Namely in $H^*(\mathbb{C}P^1 \vee S^4, \mathbb{Z})$ the square of every 2-dimensional class is 0.

As such, $\mathbb{C}P^1 \vee S^4 \not\simeq \mathbb{C}P^2$. □

Outlook. The Hopf map is sometimes presented as the map

$$S(\mathbb{C}^2) \to \mathbb{C} \cup \{\infty\} = \text{one point compactification of} \mathbb{C} \cong S^2$$

given by $(x, y) \mapsto x/y$. For $\mathbb{H}$ = the quaternions = $\mathbb{R}^4$ with the skew-field multiplication $= \mathbb{R}\{1, i, j, k\}$ and $i^2 = j^2 = k^2 = ijk = -1$. And then we get

$$\nu\colon S^7 = S(\mathbb{H}^2) \mapsto \mathbb{H} \cup \{\infty\} = S^4$$

given by $(x, y) \mapsto x/y = xy^{-1} \vee y^{-1}x$. This map is also called the second Hopf-map. Using that most of linear algebra still applies to skew-fields, we can define $\mathbb{H}P^n$ and see by a similar argument, that ν is not nullhomotopic. Then $[\nu] \in \pi_7(S^4, *) \cong \mathbb{Z}\{\nu\} \oplus \mathbb{Z}/?$ Schwede doesn't remember what exactly π_7 is.

Then we also have $\mathbb{O}$ = Cayley octonians = $\mathbb{R}^8$ with a nonassociative, noncommutative division algebra structure $\cdot \mathbb{O} \times \mathbb{O} \to \mathbb{O}$. Then there is still an $\mathbb{O}P^2$ but no general $\mathbb{O}P^n$.

However this is enough to still calculate that $H^*(\mathbb{O}P^2, \mathbb{Z}) = \mathbb{Z}[w]/w^3$ where $w \in H^8(\mathbb{O}P^2, \mathbb{Z})$. And you can show

$$\sigma\colon S(\mathbb{O}^2) \to \mathbb{O}P^1 = \mathbb{O} \cup \{\infty\}$$

given by $(x, y) \mapsto x/y$ is non zero-homotopic. And $[\sigma] \in \pi_{15}(S^8) = \mathbb{Z} \oplus \mathbb{Z}/120$.

He also talks about a theorem, that these are all the Hopf-Maps that exist. No more in higher dimensions.

[5.05.2025, Lecture 8]

[07.05.2025, Lecture 9]

Chapter 2

Poincaré Duality

The long-time goal is to proove Poincaré duality. For that we first need to study manifolds.

Definition 2.1: Manifold

An m-manifold is a Hausdorff space M such that every point of M has an open neighborhood homeomorphic to $\mathbb{R}^m$.[1]

[1]This is sometimes called a topological manifold to differentiate from smooth ones.

Remark 2.2.

- The empty space is an m-manifold for all $m \geq 0$.
- Let M be a non empty manifold. Then the dimension m is an intrinsic invariant. Let $x \in M$ be a point, let U be an open neighborhood of x homeomorphic to $\mathbb{R}^m$. Let $\varphi\colon \mathbb{R}^m \to U$ be a homeomorphism such that $\varphi(0) = x$. Then

  $$H_i(M, M \setminus \{x\}, \mathbb{Z}) \xleftarrow{\cong} H_i(U, U \setminus \{x\}, \mathbb{Z}) \xleftarrow{\varphi_*} H_i(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}, \mathbb{Z})$$

  where we use excision for the first homeomorphism. Furthermore we see

  $$H_i(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}, \mathbb{Z}) \sim H_i(D^m, S^{m-1}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = m \\ 0 & i \neq m \end{cases}$$

  We call this the local homology of x. From this we can reproduce the dimension of M.
- The Hausdorff condition is important to rule out pathological examples such as the „line with double origin“:

  $$\mathbb{R} \amalg \mathbb{R}/(x, 0) \sim (x, 1) \text{ for all } x \in \mathbb{R} \setminus \{0\}$$

  Can't draw the picture of the space.

  This is not Hausdorff, but locally $\mathbb{R}^1$. we don't want this to be a manifold.

Example 2.3.

- open subsets of $\mathbb{R}^m$ are m-manifolds.
- Let M be a Hausdorff space, such that every point has an open neighborhood that is an m-manifold.Then M is an m-manifold.
- Let M be an m-manifold and N an n-manifold. Then $M \times N$ is an $m + n$-manifold.
- The m-sphere $S^m = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^m \mid x_1^2 + \cdots + x_{m+1}^2 = 1\}$ is an m-manifold.

  Let $x = (x_1, \dots, x_{m+1}) \in S^m$ be a point. Let $V = \{y \in \mathbb{R}^{m+1} \mid \langle y, x\rangle = 0\}$ be the orthogonal complement of x. The stereographic projection is a homeomorphism

  $$x \in S^m \setminus \{-x\} \to V$$

  given by some formula I couldn't copy before it was erased and he also had a nice picture.
- The real projective space $\mathbb{R}P^m \cong S^m/x \sim -x$ is an m-manifold. Let $\{x, -x\}$ be a point in $\mathbb{R}P^m$ for $x \in S^m$. Let x be one of the representatives. Let $\mathbb{R}^m \cong U = \{z \in S^m \mid \langle z, x\rangle \geq 0\}$

„The northern hemisphere with north-pole x“. As $U \subseteq S^m$ we get via projection a map to $\mathbb{R}P^m$. This is an open embedding onto a neighborhood.

- Let $\mathbb{C}P^m = \{l \in \mathbb{C}^{n+1} : L \text{ complex line through } 0\}$. is a $2m$ manifold. Consider first the point $[0, 0, \dots, 0, 1]$.

  Then $\mathbb{R}^{2n} \cong \mathbb{C}^n \to \mathbb{C}P^n$ given by $(z_1, \dots, z_m) \mapsto [z_1, \dots, z_m, 1]$ is an homeomorphism onto a open neighborhood U of $[0, 0, \dots, 0, 1]$.

  Let $l \in \mathbb{C}P^n$ be any point, let $v \in l$ be a nonzero vector in l. Let $A \in Gl_n(\mathbb{C})$ such that $A \cdot (0, \dots, 0, 1) = v$. Then $A \colon \mathbb{C}P^n \to \mathbb{C}P^n$, $L_0 = [0, \dots, 0, 1]$ given by $L \mapsto A \cdot L$ sends $A \cdot L_0 = L$. So we can take $A(U)$ as an open neighborhood of L homeomorphic to $\mathbb{R}^{2n}$.

Now we do some examples that are a little more involved.

Example 2.4 (Stiefel manifold)**.** Let $0 \le k \le n$. The Stiefel manifold $V_{k,n} = \{(v_1, \dots, v_k) \in \mathbb{R}^n\} \mid$ orthonomal set.. We call this the „k-frame“ this means

$$\langle v_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

note, that each v_i is a vector in $\mathbb{R}^n$. We give $V_{k,n}$ the subspace topology of $(\mathbb{R}^n)^k$. This is even a closed subspace of $(S^{n-1})^k$, so $V_{k,n}$ is compact.

For example,

- $V_{0,n} = \{\emptyset\}$ is a point hence a 0-manifold.
- $V_{1,n} = S^{n-1}$.
- $V_{n,n} = O(n)$ the n-th orthogonal group.
- $V_{n-1,n} \xleftarrow{\cong} SO(n)$ given by $(Ae_1, \dots, A \cdot e_{n-1}) \leftarrow\!\shortmid A$ where $e_i = {}^t(0, \dots, 1, \dots, 0)$. This is bijective because it sends orthogonal matrices to the orthogonal vectors that span it. That is not what was written on the board. That was erased before I could copy.

Proposition 2.5. *$V_{k,n}$ is a manifold of dimension $(n-1) + (n-2) + \dots + (n-k) = nk - \frac{k(k+1)}{2}$*

Proof. By induction on k. We have already seen $V_{0,n} = \{\emptyset\}$ as a 0-manifold and $V_{1,n} = S^{n-1}$ a $(n-1)$-manifold.

Now let $k \ge 2$. Let $S^{n-1}_+ = \{(x_1, \dots, x_n) \in S^{n-1} : x_1 \ge 0\}$ be the „northern hemisphere“. We define a continuous map $\psi \colon S^{n-1}_+ \to O(n)$ as the following composite

$$S^{n-1}_* \to GL_n(\mathbb{R}) \xrightarrow{\text{Gram-Schmidt}} O(n) \quad w \mapsto ({}_t w, e_2, \dots, e_n) \mapsto \dots$$

where Gram-Schmidt is a continuous way to orthonormalize a matrix.

We remember the properties:

- ψ is contnuous
- $\psi(e_1) = \psi(1, 0, 0, \dots, 0) = E_n$
- $\psi(w) \cdot e_1 = w$.

Warning. Ther is no continuous map $\tilde{\psi} \colon S^{n-1} \to O(n)$ such that $\tilde{\psi}(w) \cdot e_1 = w$.

We show the manifold condition around $(e_1, \dots, e_k) \in V_{k,n}$. We set $U = \{(v_1, \dots, v_k) \in V_{k,n} : v_1 \in S^{n-1}_+\}$ is open in $V_{k,n}$ around $(e_1, \dots, e_k)$. The map

$$U \to S^{n-1} \times V_{k-1,n-1}, \quad (v_1, \dots, v_k) \mapsto (v_1, (\psi(v_1))^{-1}(v_2), \dots, (\psi(v_1))^{-1}(v_k))$$

where $(\psi(v_1))^{-1}(v_i)$ are in $0 \times \mathbb{R}^{n-1}$. The well-definedness follows from $\psi(v_1)^{-1}$ is an orthogonal matrix such that $\psi(v_1)^{-1}(v_1) = e_1$. This means, that $\psi(v_1)^{-1}(v_2, \dots, v_k)$ will be an orthonormal $k-1$-set that is also orthogonal to e_1, i.e. they sit in $0 \times \mathbb{R}^{n-1}$. He also rambles, as to why this is continuous.

It is a homeomorphism. This shows that around $e_1, e_2, \dots, e_k$ $V_{k,n}$ is locally a manifold of dimension $(n-1) + \dim(V_{k-1,n-1}) = (n-1) + (n-2 + \cdots + n-k)$.

We have a continuous inverse:

$$S^{n-1}_+ \times V_{k-1,n-1} \to U \quad (v, w_1, \dots, w_{k-1}) \mapsto (v, \psi(v)(0, w_1), \dots, \psi(v)(0, w_{k-1}))$$

Now let $(v_1, \dots, v_k) \in V_{k,n}$ be any point. We choose an extension to an orthonormal basis $(v_1, \dots, v_k, v_{k+1}, \dots, v_n)$. Set $A = (v_1, \dots, v_n) \in O(n)$. then

$$A \cdot \_ \colon V_{k,n} \to V_{k,n}$$

is a self homeomorphism that sends $(w_1, \dots, w_k) \mapsto (A \cdot w_1, \dots, A \cdot w_k)$ and specifically $e_1, \dots, e_k$ to $v_1, \dots, v_k$. So the homeomorphism takes the previous neighborhood U homeomorphically onto the neighborhood $A \cdot U$ of $(v_1, \dots, v_k)$ $\square$

Remark 2.6. What we really showed is, that $V_{k,n} \to S^{n-1}$, $(v_1, \dots, v_k) \mapsto v_1$ is a smooth locally trivial fiberbundle with fiber $V_{k-1,n-1}$.

Note. Complex Stiefel Manifold. We can also define

$$V^{\mathbb{C}}_{k,n} = \{(v_1, \dots, v_k) \in \mathbb{C}^n : \langle v_i, v_j \rangle = \delta_{i,j}\}$$

where δ denotes the Kronecker-symbol and we use the hermitian complex bilinear product.

This is a manifold of dimension $(2n-1) + (2n-3) + (2n-5) + \cdots + (2n-2k+1) = 2nk - k^2$. We will see.

$$V^{\mathbb{C}}_{0,n} = \{,\} \quad V^{\mathbb{C}}_{1,n} = S^{2n-1}, \quad V_{n-1,n} \cong SU(n), \quad V_{n,n} \cong U(n)$$

For the quaternions $\mathbb{H} = \mathbb{R}\{1, i, j, k\}$ with $i^2 = j^2 = k^2 = ijk = -1$, we have quaternionic conjugation $\lambda = a + bi + cj + dk \to \bar{\lambda} a - bi - cj - dk$ that is an anti-isomorphism: $\lambda \bar{\cdot} \mu = \bar{\mu} \cdot \bar{\lambda}$. This gives a „Quaternionic skalar product" on $\mathbb{H}^n$ is defined by $[x, y] \coloneqq \bar{x_1} y_1 + \cdots + \bar{x_n} \cdot y_n$ for $x, y \in \mathbb{H}^n$. This is an $\mathbb{H}$-sesquilinear, non degenerate positive definete $\mathbb{R}$-bilinear form.

With the right definitions and being careful, all of this works.

This gives Quaternionic Stiefel manifolds:

$$V^{\mathbb{H}}_{k,n} = \{(v_1, \dots, v_k) \in (\mathbb{H}^n)^k : [v_i, v_j] = \delta_{i,j}\}$$

is a manifold of dimension $(4n-1) + (4n-5) + \cdots + (4n-4k+3) = 4nk - k(2k-1)$. And we see again

$$V^{\mathbb{H}}_{1,n} = S^{4n-1}, \quad V^{\mathbb{H}}_{n,n} = Sp(n) = \{A \in M(n \times n, \mathbb{H}) : A \cdot \bar{A}^t = \bar{A}^t \cdot A = E_n\}$$

Where Sp is the simpletic group. There is no such thing as a special simpletic group, because you would need determinant for that, which then really needs commutativity.

Example 2.7 (Graßmann manifolds)**.** Let $0 \leq k \leq n$ The Graßmann manifold of k-pairs in $\mathbb{R}^n$ is

$$Gr(k, n) = Gr_k(n) = Gr_k(\mathbb{R}^n) = \{L \subseteq \mathbb{R}^n : L \text{ is } k\text{-dimensional } \mathbb{R}\text{-subspace.}\}$$

There is a surjective map

$$\text{span} \colon V_{k,n} \to Gr(k, n) \quad (v_1, \dots, v_k) \mapsto \text{span}(v_1, \dots, v_k).$$

we give $Gr(k,n)$ the quotient topology. Next time we will see $Gr(k,n)$ is a compact manifold of dimension $k \cdot (n-k)$.

The map $Gr(k,n) \mapsto Gr(n-k,n)$ given by $L \mapsto L^\perp$ is a homeomorphism.

[07.05.2025, Lecture 9]

[12.05.2025, Lecture 10]

Example 2.8. We have $Gr(1,n) = \mathbb{R}P^{n-1}$.

Theorem 2.9: Graßmann Manifolds

$Gr(k,n)$ is a compact manifold of dimension $k \cdot (n-k)$.

Proof. We first show compactness. Quasicompactness is clear, as it is a quotient space of a compact space.

We will show Hausdorff by constructing an injection into a Hausdorff space. For $V \in Gr(k,n)$ we consider the orthogonal projection $p_V \colon \mathbb{R}^n \to \mathbb{R}^n$. Let $(v_1, \dots, v_k)$ be an orthonomal basis. then

$$p_V(x) = \langle x, v_1 \rangle \cdot v_1 + \dots + \langle x, v_k \rangle \cdot v_k$$

We will sometimes also write $p_V \colon \mathbb{R}^n \to \mathbb{R}^k$.

The map $Gr(k,n) \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n \to \mathbb{R}^n)$ given by $V \mapsto p_V$ is injective. Claim: this map is continuous.

By the quotient topology, we need to show, that the composite $V_{k,n} \to Gr(k,n) \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous. This map is

$$(v_1, \dots, v_k) \mapsto \sum_{i=1,\dots,k} \langle \_, v_i \rangle \cdot v_i$$

and as a sum of continuous maps it is continuous. Because $Gr(kn)$ admits an injective continuous map to a Hausdorff space, it is Hausdorff.

Manifold property. Let $V \in Gr(k,n)$ be any k-plane. Set $U := \{L \in Gr(k,n) : L \cap V^\perp = \{0\}\}$. Claim: U is an open subset of $Gr(k,n)$. We choose an orthonormal basis $(v_1, \dots, v_k)$ of V.

Claim. $\mathrm{span}{-1}(U) = \{(l_1, \dots, l_k) : \det(\langle l_i, v_j \rangle_{1 \le i,j \le k}) \neq 0\} \subseteq V_{k,n}$.

If we show this, we are done, as $\det \neq 0$ is an open condition.

Note. $V^\perp$ is the kernel of $p_V \colon \mathbb{R}^n \to \mathbb{R}^n$. So $L \cap V^\perp = \{0\} \Leftrightarrow pr|_L \colon L \to V$ is injective.

As $\dim(L) = \dim(V) = k$, this is equal to $pr|_L \colon L \to V$ is bijective. Since $(\langle l_i, v_j \rangle)_{1 \le i,j \le k}$ is the matrix that expresses $(pr)|_L$ in terms of the basis $(l_i)_{1 \le i \le k}$ and $(v_j)_{1 \le j \le k}$, this is equivalent to $\det(\langle l_i, v_j \rangle) \neq 0$.

The map $V_{k,n} \to \mathbb{R}, (l_1, \dots, l_k) \mapsto \det(\langle l_i, v_j \rangle)$ is continuous, so $\mathrm{span}^{-1}(U)$ is open in $V_{k,n}$, hence U is open in $Gr(k,n)$.

Next, we exhibit a homeomorphism

$$\begin{tikzcd} U \arrow[r, bend left, "\Psi"] & \mathrm{Hom}_{\mathbb{R}}(V, V^\perp) \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$

We then use $\dim(V) = k, \dim(V^\perp) = n-k$, so $\mathrm{Hom}_{\mathbb{R}}(V, V^\perp) \cong \mathbb{R}^{k(n-k)}$.

Note that $\Gamma(f) \cap V^\perp = \{v \oplus f(V) : v = 0\} = \{0, 0\}$.

We define $\Gamma \colon \mathrm{Hom}(V, V^\perp) \to U$ using that $\mathbb{R}^n = V \oplus V^\perp$. Then

$$\Gamma(f \colon V \to V^\perp) = \text{Graph of } f = \{v \oplus f(v) : v \in V\}$$

The graph map factors as the composite after choice of orthonormal basis $v_1, \ldots, v_k$ of V as

$$\mathrm{Hom}_{\mathbb{R}}(V, V^{\perp}) \xrightarrow{\text{Gram-Schmidt}} V_{k,n} \xrightarrow{\text{span}} Gr(k,n)$$

so Γ is a continuous map.

We define $\Psi\colon U \to \mathrm{Hom}_{\mathbb{R}}(V, V^{\perp})$ as follows: If $L \in U$, then $p_V|_L\colon L \to V$ is a linear isomorphism.

We define $\Psi(L)$ as the composite $V \xrightarrow{(p_V)|_L^{-1}} L \xrightarrow{(p_{V^{\perp}})|_L} V^{\perp}$.

This is inverse to Γ by go check yourself.

For Continuity of $\Psi\colon U \to \mathrm{Hom}_{\mathbb{R}}(V, V^{\perp})$. Since span: $V_{k,n} \to Gr(k,n)$ is a quotient map, so is its restriction

$$\text{span}\colon \text{span}^{-1}(U) \to U$$

So it suffices to show, that the composite

$$\text{span}^{-1}(U) \to U \xrightarrow{\Psi} \mathrm{Hom}_{\mathbb{R}}(V, V^{\perp})$$

is continuous.

To proove that, we choose orthonormal bases $(v_1, \ldots, v_k)$ of V and $w_1, \ldots, w_{n-k}$ of $V^{\perp}$. Expressing a linear map in the basis is a linear isomorphism

$$\mathrm{Hom}_{\mathbb{R}}(V, V^{\perp}) \cong M(k \times (n-k), \mathbb{R}).$$

So we only need to show that

$$\text{span}^{-1}(U) \xrightarrow{\text{span}} U \xrightarrow{\Gamma} \mathrm{Hom}_{\mathbb{R}}(V, V^{\perp}) \cong M(k \times (n-k), \mathbb{R})$$

is continuous. Did not copy the argument. Something about how we just compose matrices. □

Korollar 2.10. *The map* $Gr(k,n) \to P_{k,n} \coloneqq q \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) : q^2 = q = q^*,\ \mathrm{trace}(q) = k$ *is a homeomorphism.*

Korollar 2.11. *For all* $0 \leq k \leq n$*, the map* $Gr(k,n) \to Gr(n-k,n)$ *given by* $V \mapsto V^{\perp}$ *is an homeomorphism.*

Proof. We need only show continuity.

$$\begin{array}{ccc} Gr(k,n) & \xrightarrow{V \mapsto V^{\perp}} & Gr(n-k,n) \\ \wr\| & & \wr\| \\ P_{k,n} & \xrightarrow{f \mapsto \mathrm{Id} - f} & P_{n-k,n} \end{array}$$

□

We can define the compelx analogue: $Gr^{\mathbb{C}}(k,n) = \{L \subseteq \mathbb{C}^n : /Complexlinearsubspaces\}$ with quotient topology by $V_{k,n}^{\mathbb{C}} \xrightarrow{\text{span}} Gr^{\mathbb{C}}(k,n)$ is a compact manifold of dimension $2k \cdot (n-k)$.

We can even define this for Quarternions:

$$Gr^{\mathbb{H}}(k,n) = \{L \subseteq \mathbb{H}^n : \mathbb{H}\text{- right submoule of dimension } k\}$$

is a compactr manifold of dimension $4 \cdot k \cdot (n-k)$.

Bigger picture. The orthogonal group $O(n)$ acts transitively on $V_{k,n}$. This gives an isomoprhism $O(n)/(1 \times O(n-k)) \to V_{k,n}$, that is even a homeomorphism.

Similarly, we have a transitive action $O(n) \to Gr(k,n)$. Looking at the stabilizor of $\mathbb{R}^k$. We get $O(n)/O(k) \times O(n-k) \xrightarrow{\cong} Gr(k,n)$ an homeomorphism.

This works similarly for complex and quaternionic Stiefel/Graßmann manifolds. This can be summarized as: „Stiefel manifolds and Grassmannians are homogenous spaces“.

Fact. Let G be a liegroup. H a closed subgroup. Then G/H is a (smooth) manifold of dimension $\dim G - \dim H$.

2.1 Orientations

Notation. We will write $H_n(X)$ for $H_n(X, \mathbb{Z})$.

For $Y \subseteq X$, write $H_n(X \mid Y) := H_n(X, X \setminus Y; \mathbb{Z})$, we call the „local homology of X at Y“.

This is because for $Y \subseteq U \subseteq X$, U a neighborhood of Y, then excision gives

$$H_n(U \mid Y) = H_n(U; U \setminus Y; \mathbb{Z}) \xrightarrow{\cong} H_n(X, X \setminus Y; \mathbb{Z}) = H(X \mid Y)$$

If M is an m-manifold, and $x \in M$, then $H_n(X \mid x) = H_n(X \mid \{x\})$. This is $\mathbb{Z}$ iff $m = n$ and else 0.

Definition 2.12: Local orientation

Let M be an m-manifold. A local orientation of M at $x \in M$ is a generator of $H_m(X \mid x)$.

There are exactly two local orientations at every point.

Construction 2.13 (Orientation covering)**.** Let M be an m-manifold. We define the set $\tilde{M} = \{(x, \mu) : x \in M, \mu \text{ is a local orientation at } x\}$. This comes with a map $p\colon \tilde{M} \to M$, $p(x, \mu) = x$. This map is surjective and every point in M has exactly tow preimages.

A subset B of M is a *Local ball* if B is a local subset of M, such that there exists a homeomorphism $\phi\colon \mathbb{R}^n \to M$ onto some open subset, sucht that $\phi(\langle D \rangle^n) = B$.

NOte. IF B is a local ball in M, then $M \setminus B \to M \setminus \{x\}$ is a homotopy-equivalence (here we need the special definition of open ball). This induces a isomorphism $r_x^B\colon H_m(M \mid B) \to H_m(X \mid x) \cong \mathbb{Z}$ for all $x \in B$. If μ is a local orientation at x, i.e. a generator of $H_m(X \mid B)$, we set $U(B, \mu) = \{(x, r_x^B(\mu)) : x \in B\} \subseteq \tilde{M}$.

Theorem 2.14: Orientation covering

Let M be an m-manifold.

1. *As (B, μ) varies over all pairs of local balls B and generators μ of $H_m(M \mid B)$, the subset $U(B, \mu)$ of $\tilde{M}$ are the basis of a topology on $\tilde{M}$.*
2. *In this topology on $\tilde{M}$, the map $p\colon \tilde{M} \to M$, $p(x, \mu) = x$ is a twofold covering, the orientation covering of M.*
3. *$\tilde{M}$ is an m-manifold.*

Proof. 1. We need to show, that for all local balls B, B' and all generators $\mu \in H_m(X \mid B)$, $\mu' \in H_m(X, B')$, the set $U(B, \mu) \cap U(B', \mu')$ is a union of basiss sets. Let $(x, \nu) \in U(B, \mu) \cap U(B', \mu')$. so $x \in B \cap B'$. and $r_x^B(\mu) = r_x^{B'}(\mu') := \nu$.

Choose a smaller local ball, s.t. $x \in B'' \subseteq B \cap B'$. We consider the following diagram of local homology groups:

$$\begin{tikzcd}
H_m(X \mid B) \arrow[d] \arrow[dr, "\cong"] \arrow[drr, "r_x^B", "\cong"'] & & \\
H_m(X \mid B \cap B') \arrow[r] & H_m(X \mid B'') \arrow[r, "r_x^{B''}", "\cong"'] & H_m(X \mid x) \\
H_m(X \mid B') \arrow[u] \arrow[ur, "\cong"] \arrow[urr, "r_x^{B'}", "\cong"'] & &
\end{tikzcd}$$

so μ and μ' map to the same generator of $H_m(X \mid B'')$. Set $\mu'' = \mathrm{incl}_*(\mu) = \mathrm{incl}'_*(\mu')$. Then $(x, \nu) \in U(B'', \mu'') \subseteq U(B, \mu) \cap U(B', \mu)$

2. Because M is a manifold, the local balls form athe basis of a topology of M. So it suffices to establish for all local balls B in M a homeomorphism

$$\begin{tikzcd}
p^{-1}(B) \arrow[d, "p"] & \cong & B \amalg B \arrow[dll, "fold"] \\
B & &
\end{tikzcd}$$

I did not manage to copy the rest of this argument.

3. is a special case of

Proposition 2.15. Let $p \colon N \to M$ be a covering map and M and m-manifold. Then N is an m-manifold.

Proof. Hausdorff is clear.

For $y \in N$ choose an open neighborhood U of $p(y) = x$ in M, such that $U \cong \mathbb{R}^m$ and p is locally trivial over U. Choose a homeomorphism of $p^{-1}(U) \cong U \times F$ for F some discrete space. Then $U \times f$ is again homeomorphic to $\mathbb{R}^n$ and its preimage is an open neighborhood of $y \in N$. □

□

[12.05.2025, Lecture 10]

[19.05.2025, Lecture 11] [1]

Definition 2.16: Orientation

An orientation of an m-manifold is a continuous section $s \colon M \to \tilde{M}$ of p the orientation covering.

Definition 2.17: Orientablity

An manifold is orientable, if it has an orientation.

Remark 2.18. If M is connected and orientable, $s \colon M \to \tilde{M}$ an orientation. Then $\tau \circ s \colon M \to \tilde{M}$ is another orientation where $\tau \colon \tilde{M} \to \tilde{M}, (x, \mu) \mapsto (x, -\mu)$.

Then $M \amalg M \cong \tilde{M}$ is a homeomorphism and $p \colon \tilde{M} \to M$ is the trivial covering.

[1] I recently got a new laptop (including a new keyboard), which may negatively affect my writing speed and subsquently quality of the script for the next few lectures

So for connected M, the following are equivalent: M is orientable, $p\colon \tilde{M} \to M$ has a continuous section, p is trivial, i.e. $\tilde{M} \cong M \amalg M$.

An orientable connected manifold has exactly two orientations

If M is orientable and has n path-components, then M has exactly 2^n orientations.

Korollar 2.19. *Let M be a connected m-manifold such that for some (hence any) $x \in M$, the group $\pi_1(M,x)$ does not have a subgroup of index 2. Then M is orientable.*

Proof. We argue by contradiction. If M was not orientable, then $p\colon \tilde{M} \to M$ is not a product covering. So $\tilde{M}$ is path connected, so for every $\tilde{x} \in \tilde{M}$ the homomorphism on fundamental groups $p_*\colon \pi_1(\tilde{M}, \tilde{x}) \to \pi_1(M, p(\tilde{x}))$ is a monomorphism with image of index 2. So $\mathrm{Im}(p_*)$ is an index 2 subgroup of $\pi_1(M,x)$. □

This gives that in particular every simply connected manifold is orientable.

Example 2.20. The spaces S^n (for $n \geq 2$), $\mathbb{C}P^n, \mathbb{H}P^n$ are orientable manifolds.

He continues to draw, that S^1 is also orientable.

Let M be an m-manifold, that is also a topological group, i.e. there is a continuous map $m\colon M \times M \to M$ that is also a group structure on M and such that $m \mapsto m^{-1}$ is continuous. Then M is orientable.

Proof. choose a local orientation $\mu \in H_m(M \mid 1)$, where $1 \in M$ is the multiplicative unit. For every $m \in M$,

$$m \cdot \_\colon M \to M$$

is a homoeomorphism that takes 1 to m, so $(m \cdot \_)_*\colon H_m(M \mid 1) \to H_m(M \mid m)$ is an isomorphism. Set $\mu_m \coloneqq (m \cdot \_)_*(\mu)$. Then $\{\mu_m\}_{m \in M}$ is an orientation of M. □

Examples for this are $S^1, O(n), U(n), \mathrm{Sp}(n), SO(n), SU(n), \dots$

Proposition 2.21. Let M be any n-manifold.

1. The manifold $\tilde{M}$ is orientable, and the map $\tau\colon \tilde{M} \to \tilde{M}$ given by $\tau(x,\mu) = (x, -\mu)$ is orientation reversing.
2. Let $q\colon N \to M$ be a twofold covering and N be orientable manifold, $\tau\colon N \to N$ the free deck-transformation. If $\tau\colon N \to N$ is orientation reversing, then $q\colon N \to M$ is isomorphic as a covering to $p\colon \tilde{M} \to M$.

Proof.

1. Let $\tilde{x} = (x,\mu) \in \tilde{M}$ be any point in $\tilde{M}$. Since $p\colon \tilde{M} \to M$ is a local homeomorphism. so $p\colon H_n(\tilde{M}, \tilde{x}) \xrightarrow{\cong} H_n(M,x) \ni \mu$. Set $\mu_{\tilde{x}} \coloneqq p_*^{-1}(\mu)$. then $\{\mu_{\tilde{x} \in \tilde{M}}\}$ is an orientation of $\tilde{M}$.

   The map $\tau\colon \tilde{M} \to \tilde{M}$ reverses this orientation.

   $$\tau_*\colon H_n(\tilde{M}, \tilde{x}) \to H_n(\tilde{M}, \tau_*(x)) \quad p_*^{-1}(\mu) \mapsto \tau_*(p_*^{-1}(\mu)) = p_*^{-1}(\mu) = -p_*^{-1}(-\mu) = \mu_{\tau(\tilde{x})}$$

2. We have

$$\begin{tikzcd} N \arrow[dr, "q"] & & \tilde{M} \arrow[dl, "p"'] \\ & M & \end{tikzcd}$$

and we look for a map $N \to \tilde{M}$. Define $f\colon N \to \tilde{M}$ by $f(y) = (q(y), q_*(\mu_y))$. We use $q_*\colon H_n(N \mid y) \xrightarrow{\cong} H_n(M \mid q(y))$. this f is continuous. We will not check this, f commutes with the free involution:

$$f(\tau y) = (q(y), q_*(\mu_{\tau_y})) = (q(y), q_*(-\tau_*(\mu_y))) = (q(y), -q_*(\mu_y)) = \tau(q(y), q_*(\mu_y)) = \tau(f(y))$$

So f is a continuous bijection over M, hence a homeomorphism.

□

2.1.1 Orientability of $\mathbb{R}P^n$

We already know $\mathbb{R}P^1$ and $\mathbb{R}P^3$ are orientable, as $\mathbb{R}P^1 \cong S^1$ and $\mathbb{R}P^3 \cong SO(3)$.

Recall. The antipodal map $A\colon S^n \to S^n$, given by $x \mapsto -x$ has degree $(-1)^{n+1}$.

Let $\mu \in H_n(S^n, \mathbb{Z})$ be any generator, define an orientation on S^n by $x \in S^n : \mu_x \coloneqq \mu_x^{S^n}(\mu) \in H_n(S^n(x))$.

We look at

$$\begin{array}{ccc} H_n(S^n, \mathbb{Z}) & \xrightarrow[r_x^{S^n}]{\cong} & H_n(S^n \mid x) \\ \downarrow A_* & & \downarrow A_* \\ H_n(S^n, \mathbb{Z}) & \xrightarrow{\cong} & H_n(S^n \mid -x) \end{array}$$

Some of this diagram is missing.

this gives if n is even, then $A\colon S^n \to S^n$ is orientation reversing. if n is odd, then $A\colon S^n \to S^n$ is orientation preserving:

So for even n, then $q\colon S^n \to \mathbb{R}P^n$ is twofold covering and flip reverses orientation, so this „is" the orientation covering. As $S^n \ncong \mathbb{R}P^n \amalg \mathbb{R}P^n$ we have no continuous section to $S^n \xrightarrow{q} \mathbb{R}P^n$ and $\mathbb{R}P^n$ is not orientable.

For n odd we have

Proposition 2.22. Let $f\colon N \to N$ be continuous free involution and a connected oriented m-manifold. Then

1. $M \coloneqq N/x \sim f(x)$ is an m-manifold.
2. If N is orientable and f is orientation preserving, then M is orientable.

Proof.

1. We have implicitly already done he's trying to convince me.
2. Choose an orientation $\{\mu_y\}_{y \in N}$ of N. We define an orientation of M as follows: For $x \in M$ choose $y \in N$, such that $p(y) = x$. Then we have

$$p_*\colon H_n(N \mid y) \xrightarrow{\cong} H_n(M \mid x)$$

and set $\mu_x \coloneqq q_*(\mu_y)$. This is independent of the choice of y: the other choice is $f(y)$.

Some diagram I couldn't copy. As f is orientation preserving, the chocie does not matter.

□

Then $\mathbb{R}P^n$ is orientable for n odd.

Next. For a m-manifold and for all $n > m$: $H_n(M, A) = 0$.

„Homology vanishes above the geometric dimension."

In many examples, m-manifolds M have m-dimensional CW-structure, e.g.

$$S^n, \mathbb{R}P^n, \mathbb{H}P^n, \mathbb{C}P^n$$

We could also produce CW-strucutre on the Grassmannians and Stiefel-Manifolds.

Warning. an m-manifold need not admit a CW-structure! Smooth manifolds admit triangulations, hence CW-structures. But there are non-smoothable manifolds, that do not admit CW-structures.

Theorem 2.23: Vanishing homology in high dimensions

Let M be an m-manifold; let K be a compact subset of M. Then

1. *$H_i(M, M \setminus K; A) = 0$ for all $i > n$ and A any abelian group.*

   In particular, if M is compact, then $H_i(M; A) = 0$ for all $i > n$ and all A.

2. *A class in $H_n(M, M \setminus K, A)$ is zero if and only if for all $x \in K$, its image under $r_*^K : H_n(M, M \setminus K, A) \to H_n(M, M \setminus \{x\}; A)$ is zero.*

   In particular, if M is compact, the map $r_x : H_n(M, A) \to H_n(M, M \setminus \{x\}; A)$ are jointly injective.

Proof. In 6 bootstrapping-steps.

Step 1 If $M = \mathbb{R}^n$, and K is a convex compact subset. Choose $R > 0$ such that K is contained in the open ball of radius R around $y \in K$.

$$\{z \in IR^n : |z - y| = R\} =: S_R^{n-1} \subseteq M \setminus K \subseteq M \setminus \{y\}$$

and these are homotopy equivalences.

So $H_i(M \mid K) \cong H_i(M \mid x) = 0$ for $i > n$.

Step 2 K_1, K_2 two compact subsets of M. Suppose the claim holds for K_1, K_2 and $K_1 \cap K_2$. Then it also holds for $K_1 \cup K_2$.

We do this by a Mayer-Vietoris argument for local homology.

$$M \setminus (K_1 \cap K_2) = (M \setminus K_1) \cup (M \setminus K_2)$$

And $M \setminus K_1 \cap M \setminus K_2 = M \setminus (K_1 \cup K_2)$

Remembering the theorem of small simplices we get long fractions of chain complexes, which I did not copy. We get a long exact sequence of homology groups

$$H_{n+1}(M \mid K_1 \cap K_2) \xrightarrow{\partial} H_n(M \mid K_1 \cup K_2) \to H_n(M \mid K_1) \oplus H_n(M \mid K_2) \to H_n(M \mid K_1 \cap K_2)$$

For $i > n$ we have $H_i(M \mid K_1 \cup K_2)$ lies between $H_{i+1}(M \mid K_1 \cap K_2) = 0$ and $H_i(M \mid K_1) \oplus H_i(M \mid K_2) = 0$.

For $i = n$

$$0 = H_{n+1}(M \mid K_1 \cap K_2) \to H_n(M \mid K_1 \cup K_2) \hookrightarrow H_n(M \mid K_1) \oplus H_n(M \mid K_2)$$

Let $Z \in H_n(M \mid K_1 \cup K_2)$ such that $r_x^{K_1 \cup K_2}(z) = 0$ for all $x \in K_1 \cup K_2$.

Claim. $r_{K_1}^{K_1 \cup K_2}(z) = 0$. To see this pick $x \in K_1$. Then

$$\begin{tikzcd}
H_n(M \mid K_1 \cup K_2) \arrow[r, "r_{K_1}^{K_1\cup K_2}"] \arrow[dr] & H_n(M \mid K_1) \arrow[d, "r_X^{K_1}"] \\
& H_n(M \mid x)
\end{tikzcd}$$

For all $x \in K_1$, $r_X^{K_1}(r_{K_1}^{K_1 \cup K_2}) = 0$ and so $r_{K_1}^{K_2 \cap K_1}(z) = 0$ because the claim ii) holds for K_1. Similarly $r_{K_2}^{K_1 \cap K_2}(z) = 0$ and then also $z = 0$ by the injectivity of

$$(r_{K_1}^{K_1 \cap K_2}, r_{K_2}^{K_1 \cup K_2})$$

$\square$

Appendix

List of definitions

List of statements

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