Universität Bonn

Notes for the lecture

Topology II

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Corrections and improvements

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[07.04.2025, Lecture 1]

1.1 Last Term

In last term, we discussed

- CW-complexes
- higher homotopy groups
- Whitehead theorem
- Singular homology
- cellular homology

In the very end, cohomology was started. Remeber

$$\begin{tikzcd}
 & & & \mathrm{Ab} \\
\mathrm{TOP} \arrow[r, "\mathcal{S}"] \arrow[urrr, bend left, "H_n(X;A)"] \arrow[drrr, bend right, "H^n(X;A)"'] & (\text{simpl.Sets}) \arrow[r, "C(_{,}\mathbb{Z})"] & (\text{Chains}) \arrow[ur, "H_n(_\otimes A)"] \arrow[dr, "H^n(\mathrm{Hom}(_{,}A))"] & \\
 & & & \mathrm{Ab}^{op}
\end{tikzcd}$$

1.2 Cup-product

Let X be a simplicial set, and R[1] a ring.

$$C^n(X,R) = \mathrm{maps}(X_n, R)$$

is an abelian group under pointwise addition. There is a differential

$$d^n\colon C^n(X,R) \to C^{n+1}(X,R)$$

given by

$$d^n(f)(y) = \sum_{i=0}^{n+1} (-1)^i f(d_i^*(y))$$

with $f\colon X_n \to R, y \in X_{n+1}$

[1] A ring is not necessarily commutative, but has a unit

Construction 1.1 (Cup product/Alexander Whitney map)**.** The cup prodcut/Alexander Withney map

$$\cup\colon C^n(X,R)\times C^m(X,R)\to C^{m+n}(X,R)$$

with $n,m\geq 0$ is defined by

$$(f\cup g)(x) := f(d^*_{front}(x))\cdot g(d^*_{back}(x))$$

with $f\colon X_n\to R, g\colon X_m\to R$, $x\in X_{n+m}$.

Where we use $[n+m]=\{0,1,\dots,n+m\}$ and $d_{front}\colon [n]\to[n+m], d_{back}\colon [m]\to[n+m]$ are given by $d_{front}(i)=i$, $d_{back}(i)=n+i$. Note, that d_{front} and d_{back} respectively suppress in their noation n and m.

Satz 1.2: fundamental properties of cup product

The cup-product satisfies the following properties.

1. *The AW-map is biadditive and satisfies a boundary formula:*

$$d(f\cup g)=(df)\cup g+(-1)^n f\cup(dg)\in C^{m+n+1}(X,R)$$

2. *Associativity: For $h\in C^k(X,R)$, $(f\cup g)\cup h=f\cup(g\cup h)\in C^{n+m+k}(X,R)$.*

   Let $1\in C^0(X,R)$ be the constant function $1\colon X_0\to R$ with value 1. Then $1\cup f=f\cup 1=f$.

3. *Naturality: Let $\alpha\colon Y\to X$ be a morphism of symplicial sets. Then*

$$\alpha^*(f\cup g)=\alpha^*(f)\cup\alpha^*(g),\quad \alpha^*(1)=1.$$

   where $\alpha^*\colon C^n(X,R)\to C^n(Y,R)$, $\quad f\mapsto f\circ\alpha_n$.

Proof.

- Let $d_{front}\colon[n]\to[n+m]$, $d_{back}\colon[m]\to[n+m]$ be as in the definition of $\cup$. Then

$$d_i\circ d_{front}=\begin{cases} d_{front}\circ d_i & 0\leq i\leq n+1\\ d_{front} & n+1\leq i\leq n+m+1\end{cases}$$

  and

$$d_i\circ d_{back}=\begin{cases} d_{back}\circ d_i & 0\leq i\leq n\\ d_{back}\circ d_{i-n} & n\leq i\leq n+m+1\end{cases}$$

  Note, that for $n+1$ and n respectively the cases are the same.

now

$$\begin{aligned}
d(f\cup g)(x) &= \sum_{i=0}^{n+m+1}(-1)^i (f\cup g)(d_i^*(x))\\
&= \sum_{i=0}^{n+m+1}(-1)^i \cdot f(d^*_{front}(x))\cdot g(d^*_{back}(d_i^*(x)))\\
&= \sum_{i=0}^{n}(-1)^i \cdot f(d^*_{front}(d_i^*(x)))\cdot g(d^*_{back}(d_i^*(x))) + \sum_{j=1}^{m+1}(-1)^{n+j}\cdot f(d^*_{front}(d^*_{j+n}(x)))\cdot g(d^*_{back}(d^*_{j+n}(x)))\\
&= \sum_{i=0}^{n+1}(-1)^i \cdot f(d_i^*(d^*_{front}(x)))\cdot g(d^*_{back}(x)) + \sum_{j=0}^{m+1}(-1)^{n+j} f(d^*_{front}(x))\cdot g(d_j^*(d^*_{back}(x)))\\
&= d(f)(d^*_{front}(x))\cdot g(d^*_{back}(x)) + (-1)^n\cdot f(d^*_{front}(x))\cdot d(g)(d^*_{back}(x))\\
&= ((df)\cup g)(x) + (-1)^n\cdot (f\cup dg)(x)\\
&= ((df)\cup g + (-1)^n \cdot f\cup (dg))(x)
\end{aligned}$$

- For $x \in X_{n+m+k}$ we see

$$\begin{aligned}
((f\cup g)\cup h)(x) &= (f\cup g)(d^*_{front}(x))\cdot h(d^*_{back}(x))\\
&= f(d^*_{front}(d^*_{front}(x)))\cdot g(d^*_{back}(d^*_{front}(x)))\cdot h(d^*_{back}(x))\\
&= f(d^*_{front}(x))\cdot g(d^*_{middle}(x))\cdot h(d^*_{back}(x))
\end{aligned}$$

Note that we abuse that d_{front} suppresses the indices for which the map is the front map. We have in the last line

$$d_{front}\colon [n]\to[n+m+k], d_{middle}\colon [m]\to [n+m+k], d_{back}\colon [k]\to[n+m+k]$$

defined by

$$d_{front}(i) = i, d_{middle}(i) = n+i, d_{back}(i) = n+m+i$$

this is obviously associative in the inputs[2]

- Naturality for $\alpha\colon Y\to X$ we see

$$\begin{aligned}
(\alpha^*(f\cup g))(y) &= (f\cup g)(\alpha_{n+m}(y))\\
&= f(d^*_{front}(\alpha_{n+m}(y)))\cdot g(d^*_{back}(\alpha_{n+m}(y))) = f(\alpha_n(d^*_{front}(y)))\cdot g(\alpha_m(d^*_{back}(y)))\\
&= \alpha^*(f)(d^*_{front}(y))\cdot \alpha^*(g)(d^*_{back}(y))\\
&= (\alpha^*(f)\cup\alpha^*(g))(y).
\end{aligned}$$

□

[2] for Schwede at least.

Definition 1.3: Differential graded ring

A differential graded ring (dg-ring) is a cochain-complex $A = \{A^n, d^n\}_{n\in\mathbb{Z}}$ equipped with biadditive maps
$$\cdot\colon A^n \times A^m \to A^{n+m}, \quad n, m \in \mathbb{Z}$$
and a unit $1 \in A^0$, such that;

- $\cdot$ is associative and has 1 as a unit element.
- the Leibniz rule holds:
$$d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$$
with $a \in A^n, b \in A^m$.[1]

[1]The sign is somehow connected to a sign-rule I couldn't follow. The d moved past the a or something.

Example 1.4. Some Differential graded rings are:

- $C^{\cdot}(X, R)$ for a simplicial set X and a ring R.
- De Rham complex of a smooth manifold.

Construction 1.5 (Cup-Product on cohomology)**.** Let $A = (A^n, d, \cdot)$ be a dg-ring. We define a map
$$\cdot\colon H^n(A) \times H^m(A) \to H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$
This is well defined:
$$d(a \cdot b) = \underbrace{(da)}_{=0} \cdot b + (-1)^n @.a \cdot \underbrace{(db)}_{=0} = 0$$
so $a \cdot b$ is a cycle and we can take its homology class. Let $x \in A^{n-1}$.
$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b) = [(a + dx) \cdot b] = [a \cdot b]$$
so it only depends on the cohomology class of a, analogous for b.

The product on cohomology inherits associativity and unity with $1 = [1] \in H^0(A)$. We need to see 1 is a cocycle:
$$d(1) = d(1 \cdot 1) = (d1) \cdot 1 + (-1)^0 1 \cdot (d1) = 2 \cdot d(1)$$
and so $d(1) = 0$.

The cup product on the R-cohomology of a simplicial set X is the product induced by the cup product on $C^*(X, R)$ in $H^*(C(X, R)) = H^*(X, R)$.

Satz 1.6: Properties of the cup-product on homology

Let X be a simplicial set and R a ring. Then

- *The cup product on $H^*(X, R)$ is associative and unital, with unit the cohomology class of the constant function $1\colon X_0 \to R$.*
- *For a morphism of simplicial sets $\alpha\colon Y \to X$, the relation*
$$\alpha^*([x] \cup [y]) = \alpha^*[X] \cup \alpha^*[y]$$
holds for all $[x] \in H^n(X, R), [y] \in H^m(X, R)$.

Remark 1.7. The cup product generalizes to relative cohomology: For A, B simplicial subsets of X. We have

$$C^n(X, A; R) = \{f : X_n \to R \mid f(A_n) = \{0\}\}$$

The relative cup product is the restriciton of $\cup$ on $C^*(X, R)$ to

$$C^n(X, A; R) \times C^m(X, B; R) \xrightarrow{u} C^{n+m}(X, A \cup B; R).$$

Let $x \in (A \cup B)_{n+m}$, then

$$(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x))$$

if $x \in A_{n+m}$ then $f(d^*_{front}(x)) = 0$ and analogous with B_{n+m}, anyways the product is 0.
This gives us biadditive well defined maps

$$\cup : H^n(X, A; R) \times H^n(X, B; R) \to H^{n+m}(X, A \cup B; R)$$

In particular for $A = B$ we get

$$\cup : H^n(X, A; R) \times H^n(X, A; R) \to H^{n+m}(X, A; R)$$

which is well defined and associative, but not unital anymore.

1.3 Commutativity of the cup-product

Satz 1.8: Commutativity of the cup-product

Let X be a simplicial set and R a commutative ring. Then for all $[x] \in H^n(X, R); [y] \in H^m(X, R)$ the realtion

$$[x] \cup [y] = (-1)^{n \cdot m} \cdot [y] \cup [x]$$

holds.

Schwede points out, that the easy way doesn't work. **Warning.** For $f \in C^n(X, R), g \in C^m(Y, R)$, then in general $f \cup g \neq (-1)^{n+m}(g \cup f)$ in $C^{n+m}(X, R)$. The commutativity is a property we only get on homology.

Construction 1.9. The $\cup_1$-product (spoken Cup-one)

$$\cup_1 : C^n(X, R) \times C^m(X, R) \to C^{n+m-1}(X, R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-1)\cdot(m+1)} f((d_i^{out})^*(x)) \cdot g((d_i^{inner})^*(x))$$

for $f \in C^n$, $g \in C^m$ and $x \in X_{n+m-1}$.[3] where $d_i^{out} : [n] \to [n+m-1], d_i^{inner} : [m] \to [n+m-1]$ are the unique monotone injective maps with images $\mathrm{Im}(d_i^{out}) = \{0, \ldots, i\} \cup \{i+m, \ldots, n+m-1\}$ and $\mathrm{Im}(d_i^{inn}) = \{i, \ldots, i+m\}$.

[3] There are also $\cup_i$ for $i \in \mathbb{N}$. However, they are quite messy and combinatorical.

Satz 1.10: $\cup_1$-Product

The $\cup_1$-product satisfies the following formula

$$d(f \cup_1 g) = (df) \cup_1 g + (-1)^n \cdot f \cup_1 (dg) - (-1)^{n+m}(f \cup g) - (-1)^{n+1}m + 1(g \cup f)$$

for $f \in C^n(X, R)$ and $g \in C^m(X, R)$.

Remark 1.11. What we want to see, is that $f \cup g$ and $g \cup f$ are not the same but rather homotopic, and $\cup_1$ wittnesses that homotopy.

Proof. This theorem will not be prooven, because it is quite messy. You should find a lecture-video for that. □

Now suppose that f and g are cocycles, i.e. $df = 0$, $dg = 0$. Then

$$d(f \cup_1 g) = -(-1)^{n+m}(f \cup g) - (-1)^{(n+1)(m+1)}(g \cup f)$$

and we get

$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m}(g \cup f)$$

and as such

$$0 = [(-1)^{n+m-1}] = [f] \cup [g] - (-1)^{n \cdot m}[g] \cup [f]$$

Remark 1.12. Last term we discussed the tensor product of two chain complexes (in an exercise):

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} \cdot x \otimes (dy)$$

Remark 1.13. Reinterpretation of $d(f \cup_1 g)$. The cup product yields a morphism of cochain complexes

$$C^*(X, R) \otimes C^*(X, R) \to C^*(X, R)$$

and we get a diagram

$$\begin{tikzcd} x \otimes y \arrow[d] & C^*(X,R) \otimes C^*(X,R) \arrow[r, "\cup"] \arrow[d] & C^*(X,R) \\ y \otimes x & C^*(X,R) \otimes C^*(X,R) \arrow[ur, "\cup"] & \end{tikzcd}$$

that does not commute, however it does so up to cochain homotopy and $\cup_1$ is exactly a cochain homotopy between the two maps.

[07.04.2025, Lecture 1]

[09.04.2025, Lecture 2]

Only with the definition of the cup-product we cannot calculate a lot yet. Some methods to compute cup-products are:

- directly from the definition
- cellular approximation of the diagonal (whatever that means, he gives a little intuition I failed to record.) (this might be used later)
- Group homology (one exapmle later today, something for AT I)

- Poincaré duality (later this term)
- Analysis on smooth manifolds together with De Rahm Cohomology

The first two methods are not very practical.

Example 1.14. Let X be a discrete space, Then $\mathcal{S}(X)$ is a constant simplicial set. The chain complex has the form

$$\xrightarrow{0} \mathbb{Z}[X] \xrightarrow{=} \mathbb{Z}[X] \xrightarrow{0} \mathbb{Z}[X]$$

And so $H^n(X,R) = 0$ for $n \geq 0$. And only for $n = m = 0$ something nontrivial happens. for $f\colon X_0 \to R, g\colon X_0 \to R$, we have $(f \cup g)(x) = f(d^*_{front}(x)) \cdot g(d^*_{back}(x)) = f(x) \cdot g(x)$ and so the cup product is just pointwise multiplication in dimension 0.

More generally: $H^0(X,R) = \text{maps}(\pi_0(X), R)$ with $\cup$-prodcut pointwise multiplication

Example 1.15. Let G be a group: Define a category $\underline{G}$[4] wit one object $*$ and $\text{Hom}_{\underline{G}}(*,*) = G$. We then define

$$BG = N(\underline{G})$$

Where N is the Nerve-Functor **CAT** $\to$ **Sset**. Then

$$(BG)_n = G^n, \quad d_i^*\colon G^n \to G^{n-1}(g_1,\dots,g_n) \mapsto \begin{cases} (g_2,\dots,g_n) & i = 0 \\ (g_1,\dots,g_i \circ g_{i+1},\dots,g_n) & 1 \leq i \leq n-1 \\ (g_1,\dots,g_{n-1}) & i = n \end{cases}$$

And $s_i(g_1,\dots,g_n) = (g_1,\dots,g_i,1,g_{i+1},\dots,g_n)$.

The general case of this is too hard to calculate. We take $G = (\mathbb{F}_2,+)$ and $R = \mathbb{F}_2$ and we calculate $H^*(B\mathbb{F}_2,\mathbb{F}_2)$. We see

$$\begin{array}{ccccccc} C^0(BG,A) & \xrightarrow{d} & C^1(BG,A) & \xrightarrow{d} & C^2(BG,A) & \longrightarrow & \dots \\ \| & & \| & & \| & & \\ \text{maps}(\{1\},A) & \xrightarrow{0} & \text{maps}(G,A) & \longrightarrow & \text{maps}(G^2,A) & & \\ \wr\| & & & & & & \\ A & & (f\colon G \to A) & \longrightarrow & (df)(g,h) & & \end{array}$$

And the map is defined by

$$f(d_0^*(g,h)) - f(d_1^*(g,h)) + f(d_2^*(g,h)) = f(h) - f(g \cdot h) + f(g)$$

and

$$df = 0 \Leftrightarrow f(g,h) = f(g) + f(h)$$

$\implies$ 1-cocycles are the group homomorphisms from G to A

$$H^1(BG,A) \cong \text{Hom}(G,A)$$

and for $G = (\mathbb{F}_2,+)$, $A = \mathbb{F}_2$

We define

$$0 \neq x := [\text{Id}_{\mathbb{F}_2}] \in H^1(B\mathbb{F}_2,\mathbb{F}_2).$$

[4]via geometric realization, these define interesting spaces, namely some (missed word)-Maclane spaces $M(G,1)$, didn't catch it all

We will show that $x^n = x \cup \dots \cup x$ (n-times) $\in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is nonzero.

Proposition. $x^n \in H^n(B\mathbb{F}_2, \mathbb{F}_2)$ is represented by

$$f_n \colon (\mathbb{F}_2)^n \to \mathbb{F}_2, f_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdot \dots \cdot \lambda_n = \begin{cases} 1 & \text{if } \lambda_1 = \lambda_2 = \dots = \lambda_n = 1 \\ 0 & \text{else} \end{cases}$$

Proof. By induction on n. We checked for $n = 1$. For $n \geq 2$ we have

$$\begin{aligned} x^n &= x^{n-1} \cup x = [f_{n-1}] \cup [\mathrm{Id}_{\mathbb{F}_2}] \\ &= [f_{n-1} \cup \mathrm{Id}] \end{aligned}$$

Then

$$\begin{aligned} (f_{n-1} \cup \mathrm{Id})(\lambda_1, \dots, \lambda_n) &= f_{n-1}(d^*_{front}(\lambda_1, \dots, \lambda_n)) \cdot \mathrm{Id}_(\, d^*_{back}(\lambda_1, \dots, \lambda_n)) \\ &= f_{n-1}(\lambda_1, \dots, \lambda n - 1) \cdot \mathrm{Id}(\lambda_n) \\ &= (\lambda_1 \cdot \dots \cdot \lambda_{n-1}) \cdot \lambda_n \end{aligned}$$

$\square$

Claim: $x^n \neq 0$. In the UCT for cohomology we used the evaluation pair

$$\Phi \colon H^n(X, A) \to \mathrm{Hom}(H_n(X; \mathbb{Z}); A), \quad [f_n \colon X_n \to A] \mapsto \left\{ [\sum b_i x_i] \mapsto \sum b_i f(x_i) \right\}$$

for $b_i \in \mathbb{Z}, x_i \in X_n$. We can slightly variate that for ring coefficients:

$$\Phi \colon H^n(X, R) \to \mathrm{Hom}(H_n(X, R), R)$$

and $[f \colon X_n \to R] \mapsto \{[\sum r_i \cdot x_i] \mapsto \sum r_i \cdot f(x_i)\}$ with $r_i \in R, x_i \in X_n$.
With $X = B\mathbb{F}_2, R = \mathbb{F}_2$, we consider

$$y \coloneqq \sum_{(\lambda_1 \dots, \lambda_n) \in (\mathbb{F}_2)^n} 1(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2[(\mathbb{F}_2)^n] = \mathbb{F}_2[(B\mathbb{F}_2)_n]$$

Claim: y is an n-cycle in $C_*(B\mathbb{F}_2, \mathbb{F}_2)$.

$$\begin{aligned} dy &= \sum_{i=0,\dots n} (-1)^i \cdot d_i^*(\sum_1 \cdot (\lambda_1, \dots, \lambda_n)) \\ &= \sum_{i=0,\dots,n} \underbrace{\sum_{(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2^n} (-1)^i \cdot d_i^*(\lambda_1, \dots, \lambda_n)}_{\text{cancel in pairs}} \\ &= 0 \end{aligned}$$

Now

$$d_0^*(0, \lambda_2, \dots, \lambda_n) = (\lambda_2, \dots \lambda_n) = d_0^*(1, \lambda_2, \dots, \lambda_n)$$

So

$$\Phi(x^n) \colon H_n(B\mathbb{F}_2, \mathbb{F}_2) \to \mathbb{F}_2$$

$$\Phi(x^n)[y] = \Phi[f_n][\sum_{(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2^n} (\lambda_1, \dots, \lambda_n)] = \sum_{(\lambda_1, \dots, \lambda_n)} f_n(\lambda_1, \dots, \lambda_n) = \sum_{(\lambda_1, \dots, \lambda_n)} \lambda_1, \cdot \dots \cdot \lambda_n = 1 \neq 0$$

and $[y] \neq 0$ in $H_n(B\mathbb{F}_2, \mathbb{F}_2)$.
We will later see, that in fact $H^*(B\mathbb{F}_2; \mathbb{F}_2) = \mathbb{F}_2[X]$.

Remark. Let p be an odd prime. $H^*(B\mathbb{F}_p, \mathbb{F}_p) =?$.

$$0 \neq x = [\mathrm{Id}_{\mathbb{F}_p} \in H^1(B\mathbb{F}_p; \mathbb{F}_p)]$$

still makes sense, but now there are more scalars and

$$x^n = 0$$

for $n \geq 2$. The graded commutativity says:

$$x \cup x = (-1)^{1 \cdot 1} x \cup x = -x \cup x$$

so if R is commutative, $x \in H^n(X, R)$ and n is odd, then $2 \cdot (x \cup x) = 0$ in $H^{2n}(X, R)$. And then $2 \cdot x^2 = 0 \Rightarrow x^2 = 0$.

Define $h\colon \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ by

$$h(i,j) = \begin{cases} 0 & \text{if } i + j < p \\ 1 & \text{if } i + j \geq p \end{cases}$$

where we write $\mathbb{F}_p = \{0, \dots, p-1\}$. Now $h \in C^2(B\mathbb{F}_p, \mathbb{F}_p)$. Fact: $dh = 0$ and $0 \neq y \coloneqq [h] \in H^2(B\mathbb{F}_p, \mathbb{F}_p)$.

We then get (but do not proove)

$$H^*(B\mathbb{F}_p, \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$$

and

$$H^{2n}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{y^n\}, \quad H^{2n+1}(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{xy^n\}$$

1.4 Künneth theorem

The Künneth theorem is an algebraic relationship between $H_*^*(X, R), H_*^*(Y, R)$ and $H_*^*(X \times Y, R)$[5].

Here is a simplest version in homology with field coefficients:

Satz 1.16: Künneth, simple version

Let X and Y be spaces and k a field. Then

$$H_n(X \times Y, k)$$

is natural isomorphic to

$$\bigoplus_{p+q=n} H_p(X,k) \otimes_k H_q(Y,k)$$

1.4.1 The Eilenberg-Zilber-theorem

Let A, B be simplicial abelian groups. Then we get two natural chain homotopy equivalences

$$C_*(A) \otimes C_*(B) C_*(A \otimes B)$$

up Eilenberg Zilber map, bottom Alexander Whitney map

[5] H_*^* denotes, that Schwede was too lazy to write the statement for homology and cohomology separately

Definition 1.17: Simplicial abelian group

A *simplicial abelian group* is a functor $A\colon \Delta^{Op} \to \mathbf{Ab.Groups}$.

Remark 1.18. Equivalently a simplicial abelian group is a colleciton of abelian groups A_n, and homomorphisms $\alpha^*\colon A_m \to A_n$ for all $\alpha\colon [n] \to [m]$ in Δ, s.t. $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$.

Equivalently a simplicial abelian group is a simplical set endorsed with abelian group structure on the sets of n-simplices, such that all α^* are homomorphisms.

Example 1.19. Let X be a simplicial set and A an abelian group. Then the composite

$$\begin{tikzcd} \Delta^{op} \arrow[r, "X"] \arrow[rr, bend left, "A[X]"] & (\mathbf{Sets}) \arrow[r, "A[\_]"] & (\mathbf{ab.grps}) \end{tikzcd}$$

is a simplicial abelian group.

Construction 1.20. Let $A\colon \Delta^{op} \to (\mathbf{ab.grps})$ be a simplicial abelian groups. Its *chain complex* $C_*(A)$ is the chain complex with $C_n(A) = A_n$ with differential

$$d\colon C_n(A) = A_n \to A_{n-1} = C_{n-1}(A), \quad d(a) = \sum_{i=0,\dots,n} (-1)^i d_i^*(a)$$

And one can easily check $d \circ d = 0$.

Note. The following commutes

$$\begin{tikzcd} (\mathbf{Ssets}) \arrow[rr, "X \mapsto C_*(X{,}A)"] \arrow[dr, "A[]"] & & (\mathbf{Chains}) \\ & (\mathbf{s.ab.grps}) \arrow[ur, "C_*"] & \end{tikzcd}$$

Remark 1.21. The tensor product of chain complexes C, D is

$$(C \otimes D)_n \coloneqq \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential

$$d(x \otimes y) = (dx \otimes y) + (-1)^p x \otimes (dy)$$

for $x \in C_p, y \in D_q$.

We can also form the tensor product of simplical abelian groups: $A, B\colon \Delta^{op} \to (\mathbf{ab.grps})$ by

$$(A \otimes B)_n = A_n \otimes B_n, \quad \alpha^*\colon (A \otimes B)_n \to (A \otimes B)_m$$

for $\alpha\colon [m] \to [n]$ isis defined as $\alpha^*(a \otimes b) = \alpha^*(a) \otimes \alpha^*(b)$ and we write $\alpha^*_{A \otimes B} \coloneqq \alpha^*_A \otimes \alpha^*_B$. Or this can be equally described as

$$\Delta^{op} \xrightarrow{(A,B)} (\mathbf{ab.grps})^2 \xrightarrow{\otimes} (\mathbf{ab.grps})$$

Warning. For $A, B \in (\mathbf{SAB})$ = simplicial abelian groups

$$C_*(A \otimes B) \neq C_*(A) \otimes C_*(B)$$

Also he did this in dimension n, but I lacked time to copy.

The Eilenberg-Zilber theorem is a natural pair of chain homotopy equivalences between these two.

Construction 1.22. Let A, B be simplicial chain groups. The *Alexander-Whitney map* is the chain map

$$AW\colon C_*(A \otimes B) \to C_*(A) \otimes C_*(B)$$

defined by

$$\begin{array}{ccc} C_n(A \otimes B) & \longrightarrow & \bigoplus_{p+q=n, p,q \geq 0} A_p \otimes B_q \\ \| & & \| \\ A_n \otimes B_n & & C_*(A) \otimes C_*(B) \end{array}$$

$$AW_n(a \otimes b) = \sum_{p+q=n} d^*_{front}(a) \otimes d^*_{back}(b)$$

Where $[p] \xrightarrow{d_{front}} [p+q] = [n] \xleftarrow{d_{back}} [q]$.

You may check for yourself, that this is a chain map, however Schwede didn't do that.

Appendix

List of definitions

List of statements

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