

UNIVERSITÄT BONN

Notes for the lecture

Topology II

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Corrections and improvements

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Lecture

Chapter 1

Cohomology

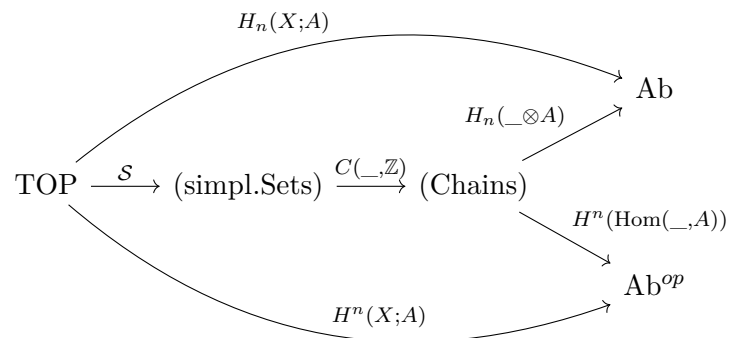
[07.04.2025, Lecture 1]

1.1 Last Term

In last term, we discussed

- CW-complexes
- higher homotopy groups
- Whitehead theorem
- Singular homology
- cellular homology

In the very end, cohomology was started. Remember



1.2 Cup-product

Let X be a simplicial set, and R^1 a ring.

$$C^n(X, R) = \text{maps}(X_n, R)$$

is an abelian group under pointwise addition. There is a differential

$$d^n : C^n(X, R) \rightarrow C^{n+1}(X, R)$$

given by

$$d^n(f)(y) = \sum_{i=0}^{n+1} (-1)^i f(d_i^*(y))$$

with $f : X_n \rightarrow R, y \in X_{n+1}$

¹A ring is not necessarily commutative, but has a unit

Construction 1.1 (Cup product/Alexander Withney map). The cup product/Alexander Withney map

$$\cup: C^n(X, R) \times C^m(X, R) \rightarrow C^{m+n}(X, R)$$

with $n, m \geq 0$ is defined by

$$(f \cup g)(x) := f(d_{front}^*(x)) \cdot g(d_{back}^*(x))$$

with $f: X_n \rightarrow R, g: X_m \rightarrow R, x \in X_{n+m}$.

Where we use $[n+m] = \{0, 1, \dots, n+m\}$ and $d_{front}: [n] \rightarrow [n+m], d_{back}: [m] \rightarrow [n+m]$ are given by $d_{front}(i) = i, d_{back}(i) = n+i$. Note, that d_{front} and d_{back} respectively suppress in their notation n and m .

Satz 1.2: fundamental properties of cup product

The cup-product satisfies the following properties.

1. The AW-map is biadditive and satisfies a boundary formula:

$$d(f \cup g) = (df) \cup g + (-1)^n f \cup (dg) \in C^{m+n+1}(X, R)$$

2. Associativity: For $h \in C^k(X, R), (f \cup g) \cup h = f \cup (g \cup h) \in C^{n+m+k}(X, R)$.

Let $1 \in C^0(X, R)$ be the constant function $1: X_0 \rightarrow R$ with value 1. Then $1 \cup f = f \cup 1 = f$.

3. Naturality: Let $\alpha: Y \rightarrow X$ be a morphism of simplicial sets. Then

$$\alpha^*(f \cup g) = \alpha^*(f) \cup \alpha^*(g), \quad \alpha^*(1) = 1.$$

where $\alpha^*: C^n(X, R) \rightarrow C^n(Y, R), f \mapsto f \circ \alpha_n$.

Proof.

- Let $d_{front}: [n] \rightarrow [n+m], d_{back}: [m] \rightarrow [n+m]$ be as in the definition of \cup . Then

$$d_i \circ d_{front} = \begin{cases} d_{front} \circ d_i & 0 \leq i \leq n+1 \\ d_{front} & n+1 \leq i \leq n+m+1 \end{cases}$$

and

$$d_i \circ d_{back} = \begin{cases} d_{back} \circ d_i & 0 \leq i \leq n \\ d_{back} \circ d_{i-n} & n \leq i \leq n+m+1 \end{cases}$$

Note, that for $n+1$ and n respectively the cases are the same.

now

$$\begin{aligned}
d(f \cup g)(x) &= \sum_{i=0}^{n+m+1} (-1)^i (f \cup g)(d_i^*(x)) \\
&= \sum_{i=0}^{n+m+1} (-1)^i \cdot f(d_{front}^*(x)) \cdot g(d_{back}^*(d_i^*(x))) \\
&= \sum_{i=0}^n (-1)^i \cdot f(d_{front}^*(d_i^*(x))) \cdot g(d_{back}^*(d_i^*(x))) + \sum_{j=1}^{m+1} (-1)^{n+j} \cdot f(d_{front}^*(d_{j+n}^*(x))) \cdot g(d_{back}^*(d_{j+n}^*(x))) \\
&= \sum_{i=0}^{n+1} (-1)^i \cdot f(d_i^*(d_{front}^*(x))) \cdot g(d_{back}^*(x)) + \sum_{j=0}^{m+1} (-1)^{n+j} f(d_{front}^*(x)) \cdot g(d_j^*(d_{back}^*(x))) \\
&= d(f)(d_{front}^*(x)) \cdot g(d_{back}^*(x)) + (-1)^n \cdot f(d_{front}^*(x)) \cdot d(g)(d_{back}^*(x)) \\
&= ((df) \cup g)(x) + (-1)^n \cdot (f \cup dg)(x) \\
&= ((df) \cup g + (-1)^n \cdot f \cup (dg))(x)
\end{aligned}$$

- For $x \in X_{n+m+k}$ we see

$$\begin{aligned}
((f \cup g) \cup h)(x) &= (f \cup g)(d_{front}^*(x)) \cdot h(d_{back}^*(x)) \\
&= f(d_{front}^*(d_{front}^*(x))) \cdot g(d_{back}^*(d_{front}^*(x))) \cdot h(d_{back}^*(x)) \\
&= f(d_{front}^*(x)) \cdot g(d_{middle}^*(x)) \cdot h(d_{back}^*(x))
\end{aligned}$$

Note that we abuse that d_{front} suppresses the indices for which the map is the front map. We have in the last line

$$d_{front}: [n] \rightarrow [n+m+k], d_{middle}: [m] \rightarrow [n+m+k], d_{back}: [k] \rightarrow [n+m+k]$$

defined by

$$d_{front}(i) = i, d_{middle}(i) = n+i, d_{back}(i) = n+m+i$$

this is obviously associative in the inputs²

- Naturality for $\alpha: Y \rightarrow X$ we see

$$\begin{aligned}
(\alpha^*(f \cup g))(y) &= (f \cup g)(\alpha_{n+m}(y)) \\
&= f(d_{front}^*(\alpha_{n+m}(y))) \cdot g(d_{back}^*(\alpha_{n+m}(y))) = f(\alpha_n(d_{front}^*(y))) \cdot g(\alpha_m(d_{back}^*(y))) \\
&= \alpha^*(f)(d_{front}^*(y)) \cdot \alpha^*(g)(d_{back}^*(y)) \\
&= (\alpha^*(f) \cup \alpha^*(g))(y).
\end{aligned}$$

□

²for Schwede at least.

Definition 1.3: Differential graded ring

A differential graded ring (dg-ring) is a cochain-complex $A = \{A^n, d^n\}_{n \in \mathbb{Z}}$ equipped with biadditive maps

$$\cdot : A^n \times A^m \rightarrow A^{n+m}, \quad n, m \in \mathbb{Z}$$

and a unit $1 \in A^0$, such that;

- \cdot is associative and has 1 as a unit element.
- the Leibniz rule holds:

$$d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$$

with $a \in A^n, b \in A^m$.¹

¹The sign is somehow connected to a sign-rule I couldn't follow. The d moved past the a or something.

Example 1.4. Some Differential graded rings are:

- $C^*(X, R)$ for a simplicial set X and a ring R .
- De Rham complex of a smooth manifold.

Construction 1.5 (Cup-Product on cohomology). Let $A = (A^n, d, \cdot)$ be a dg-ring. We define a map

$$\cdot : H^n(A) \times H^m(A) \rightarrow H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$

This is well defined:

$$d(a \cdot b) = \underset{=0}{(da)} \cdot b + (-1)^n \underset{=0}{a} \cdot (db) = 0$$

so $a \cdot b$ is a cycle and we can take its homology class. Let $x \in A^{n-1}$.

$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b) = [(a + dx) \cdot b] = [a \cdot b]$$

so it only depends on the cohomology class of a , analogous for b .

The product on cohomology inherits associativity and unity with $1 = [1] \in H^0(A)$. We need to see 1 is a cocycle:

$$d(1) = d(1 \cdot 1) = (d1) \cdot 1 + (-1)^0 1 \cdot (d1) = 2 \cdot d(1)$$

and so $d(1) = 0$.

The cup product on the R -cohomology of a simplicial set X is the product induced by the cup product on $C^*(X, R)$ in $H^*(C(X, R)) = H^*(X, R)$.

Satz 1.6: Properties of the cup-product on homology

Let X be a simplicial set and R a ring. Then

- The cup product on $H^*(X, R)$ is associative and unital, with unit the cohomology class of the constant function $1: X_0 \rightarrow R$.
- For a morphism of simplicial sets $\alpha: Y \rightarrow X$, the relation

$$\alpha^*([x] \cup [y]) = \alpha^*[X] \cup \alpha^*[y]$$

holds for all $[x] \in H^n(X, R), [y] \in H^m(X, R)$.

Remark 1.7. The cup product generalizes to relative cohomology: For A, B simplicial subsets of X . We have

$$C^n(X, A; R) = \{f: X_n \rightarrow R \mid f(A_n) = \{0\}\}$$

The relative cup product is the restriction of \cup on $C^*(X, R)$ to

$$C^n(X, A; R) \times C^m(X, B; R) \xrightarrow{\cup} C^{n+m}(X, A \cup B; R).$$

Let $x \in (A \cup B)_{n+m}$, then

$$(f \cup g)(x) = f(d_{front}^*(x)) \cdot g(d_{back}^*(x))$$

if $x \in A_{n+m}$ then $f(d_{front}^*(x)) = 0$ and analogous with B_{n+m} , anyways the product is 0.

This gives us biadditive well defined maps

$$\cup: H^n(X, A; R) \times H^m(X, B; R) \rightarrow H^{n+m}(X, A \cup B; R)$$

In particular for $A = B$ we get

$$\cup: H^n(X, A; R) \times H^m(X, A; R) \rightarrow H^{n+m}(X, A; R)$$

which is well defined and associative, but not unital anymore.

1.3 Commutativity of the cup-product

Satz 1.8: Commutativity of the cup-product

Let X be a simplicial set and R a commutative ring. Then for all $[x] \in H^n(X, R); [y] \in H^m(X, R)$ the relation

$$[x] \cup [y] = (-1)^{n \cdot m} \cdot [y] \cup [x]$$

holds.

Schwede points out, that the easy way doesn't work. **Warning.** For $f \in C^n(X, R), g \in C^m(Y, R)$, then in general $f \cup g \neq (-1)^{n+m}(g \cup f)$ in $C^{n+m}(X, R)$. The commutativity is a property we only get on homology.

Construction 1.9. The \cup_1 -product (spoken Cup-one)

$$\cup_1: C^n(X, R) \times C^m(X, R) \rightarrow C^{n+m-1}(X, R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-1) \cdot (m+1)} f((d_i^{out})^*(x)) \cdot g((d_i^{inner})^*(x))$$

for $f \in C^n, g \in C^m$ and $x \in X_{n+m-1}$.³ where $d_i^{out}: [n] \rightarrow [n+m-1], d_i^{inner}: [m] \rightarrow [n+m-1]$ are the unique monotone injective maps with images $\text{Im}(d_i^{out}) = \{0, \dots, i\} \cup \{i+m, \dots, n+m-1\}$ and $\text{Im}(d_i^{inner}) = \{i, \dots, i+m\}$.

³There are also \cup_i for $i \in \mathbb{N}$. However, they are quite messy and combinatorical.

Satz 1.10: \cup_1 -Product

The \cup_1 -product satisfies the following formula

$$d(f \cup_1 g) = (df) \cup_1 g + (-1)^n \cdot f \cup_1 (dg) - (-1)^{n+m} (f \cup g) - (-1)^{n+1} m + 1 (g \cup f)$$

for $f \in C^n(X, R)$ and $g \in C^m(X, R)$.

Remark 1.11. What we want to see, is that $f \cup g$ and $g \cup f$ are not the same but rather homotopic, and \cup_1 witnesses that homotopy.

Proof. This theorem will not be proven, because it is quite messy. You should find a lecture-video for that. \square

Now suppose that f and g are cocycles, i.e. $df = 0$, $dg = 0$. Then

$$d(f \cup_1 g) = -(-1)^{n+m} (f \cup g) - (-1)^{(n+1)(m+1)} (g \cup f)$$

and we get

$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m} (g \cup f)$$

and as such

$$0 = [(-1)^{n+m-1}] = [f] \cup [g] - (-1)^{n \cdot m} [g] \cup [f]$$

Remark 1.12. Last term we discussed the tensor product of two chain complexes (in an exercise):

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} \cdot x \otimes (dy)$$

Remark 1.13. Reinterpretation of $d(f \cup_1 g)$. The cup product yields a morphism of cochain complexes

$$C^*(X, R) \otimes C^*(X, R) \rightarrow C^*(X, R)$$

and we get a diagram

$$\begin{array}{ccc} x \otimes y & C^*(X, R) \otimes C^*(X, R) & \xrightarrow{\cup} C^*(X, R) \\ \downarrow & \downarrow & \nearrow \cup \\ y \otimes x & C^*(X, R) \otimes C^*(X, R) & \end{array}$$

that does not commute, however it does so up to cochain homotopy and \cup_1 is exactly a cochain homotopy between the two maps.

Appendix

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