Analiza III - Funkcije več spremenljivk, študijsko leto 2019/2020

4. vaje - 29.10.2019

METRIKA

1. Izračunajte limito zaporedja $(a_n)_{n=1}^{\infty}$, če je

   (a) $a_n = \left(\sqrt{n^2+n} - \sqrt{n^2-n}, \left(\frac{2n+3}{2n+1}\right)^{n-1}, \frac{3n+2}{\sqrt{4n^2-1}}\right)$

   (b) $a_n = \left(\frac{\sin 2n}{\sqrt{n}}, \frac{e^n}{1+2e^n}, n^3 e^{-2n}\right)$

   (c) $a_n = \left(\left(\frac{n+3}{n+2}\right)^n, \frac{(\ln n)^7}{n^2}, \frac{2^n+4^n}{3^n+6^n}\right)$

2. Poiščite vsa stekališča zaporedja $(x_n)_{n=1}^{\infty}$ v $(\mathbb{R}^2, d_2)$, za $x_n = \left(\frac{(-1)^n}{n}, \cos\frac{n\pi}{2}\right)$.

3. Naj bo $\varphi_1 > \varphi_2 > \varphi_3 > \dots$ in $\lim\limits_{n\to\infty} \varphi_n = 0$.

   Ali je zaporedje točk $(x_n)_{n=1}^{\infty}$ z $x_n = (\cos\varphi_n, \sin\varphi_n)$ v evklidski metriki konvergentno? Ali je Cauchyjevo?

4. Dano je zaporedje funkcij $f_n(x) = \frac{nx}{2nx+1}$. Ali je to zaporedje konvergentno/Cauchyjevo

   (a) v prostoru zveznih funkcij na $\left[\frac{1}{2}, 1\right]$ opremljenim z maximum metriko: $d_\infty(f,g) = \max\limits_{a\le x\le b}\{|f(x)-g(x)|\}$?

   (b) v prostoru zveznih funkcij na $\left[\frac{1}{2}, 1\right]$ opremljenim z integralsko metriko: $d_1(f,g) = \int_a^b |f(x)-g(x)|\, dx$?

ZAPOREDJA V METRIČNIH PROSTORIH

Naj bo (M, d) met. prostor in $(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \dots)$ zaporedje z $a_i \in M$.
a_i – člen zaporedja

LASTNOSTI:

1. Zaporedje $(a_n)_{n=1}^{\infty}$ je v (M, d) OMEJENO, če $\exists r > 0$ in $\exists a \in M$, da $\forall n \in \mathbb{N}$ velja $a_n \in K_r(a)$.

2. Zaporedje $(a_n)_{n=1}^{\infty}$ je v (M, d) KONVERGENTNO, če $\exists\, a \in M$, za katero velja: $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$; da $\forall n > n_0$ velja $a_n \in K_\varepsilon(a)$

   Tedaj je $a \in M$ LIMITA ZAPOREDJA: $\lim\limits_{n \to \infty} a_n = a$.

   * Vsako konvergentno zaporedje ima v (M, d) natanko 1 limito
   * Vsako – " – je omejeno
     (obratno v splošnem ne velja)

3.) Zaporedje $(a_n)_{n=1}^{\infty}$ je v (M, d) CAUCHYJEVO, če $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$, da $\forall m, n > n_0$ velja $d(a_m, a_n) < \varepsilon$.
("dva poljubna "pozna" člena sta si dovolj blizu")

   * Vsako Cauchyjevo zaporedje je omejeno
   * Vsako konvergentno zap. je Cauchyjevo (obr. v spl. ne velja)

4.) Metrični prostor (M, d) je POLN, če je v njem vsako Cauchyjevo zaporedje konvergentno.

1.) Izračunajte limito zaporedja $(a_n)_{n=1}^{\infty}$, če je

a) $a_n = \left(\sqrt{n^2+n} - \sqrt{n^2-n},\ \left(\frac{2n+3}{2n+1}\right)^{n-1},\ \frac{3n+2}{\sqrt{4n^2-1}}\right)$

$a_1 = \left(\sqrt{2} - \sqrt{0},\ \left(\frac{5}{3}\right)^0,\ \frac{5}{\sqrt{3}}\right) = \left(\underset{1,4}{\sqrt{2}},\ 1,\ \underset{2,89}{\frac{5}{\sqrt{3}}}\right)$

$a_2 = \left(\underset{1,03}{\sqrt{6} - \sqrt{2}},\ \underset{1,4}{\left(\frac{7}{5}\right)^1},\ \underset{2,07}{\frac{8}{\sqrt{15}}}\right)$

$\vdots$

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2-n}\right)$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - \sqrt{n^2-n}\right)\left(\sqrt{n^2+n} + \sqrt{n^2-n}\right)}{\left(\sqrt{n^2+n} + \sqrt{n^2-n}\right)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n - \cancel{n^2}+n}{\sqrt{n^2+n} + \sqrt{n^2-n}} = \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+n} + \sqrt{n^2-n}} \quad :n \quad :n$$

$$= \lim_{n\to\infty} \frac{\frac{2\cancel{n}}{\cancel{n}}}{\sqrt{\underbrace{\frac{n^2}{n^2}}_{1} + \underbrace{\frac{\cancel{n}}{n^{\cancel{2}}}}_{\to 0}} + \sqrt{\underbrace{\frac{n^2}{n^2}}_{1} - \underbrace{\frac{\cancel{n}}{n^{\cancel{2}}}}_{\to 0}}} = \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1$$

$$\lim_{n\to\infty} \left(\frac{2n+3}{2n+1}\right)^{n-1} \qquad \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$= \lim_{n\to\infty} \left(\frac{2n+1+2}{2n+1}\right)^{n-1} = \lim_{n\to\infty} \left(1 + \frac{2}{2n+1}\right)^{n-1}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{\frac{2n}{2}+\frac{1}{2}}\right)^{n-1} = \lim_{n\to\infty} \left(1 + \frac{1}{n+\frac{1}{2}}\right)^{n-1}$$

$t = n + \frac{1}{2} \Rightarrow n = t - \frac{1}{2}$

$$= \lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^{t-\frac{1}{2}-1} = \lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^{t-\frac{3}{2}}$$

$$= \lim_{t\to\infty} \left(\left(1+\frac{1}{t}\right)^t \cdot \left(1+\cancel{\frac{1}{t}}\right)^{-\frac{3}{2}}\right) = e \cdot 1 = e$$

$$\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}} \quad :n \quad :n = \lim_{n\to\infty} \frac{\frac{3\cancel{n}}{\cancel{n}} + \cancel{\frac{2}{n}}^{\to 0}}{\sqrt{\frac{4\cancel{n^2}}{\cancel{n^2}} - \cancel{\frac{1}{n^2}}_{\to 0}}} = \frac{3}{\sqrt{4}} = \frac{3}{2}$$

$$\lim_{n\to\infty} a_n = \left(1, e, \frac{3}{2}\right)$$

b) $a_n = \left(\frac{\sin 2n}{\sqrt{n}}, \frac{e^n}{1+2e^n}, n^3 e^{-2n} \right)$

$$\lim_{n\to\infty} \frac{\sin 2n}{\sqrt{n}} \quad \in [-1,1] \quad \Rightarrow \frac{-1}{\sqrt{n}} \leq \frac{\sin 2n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\lim_{n\to\infty} \frac{e^n}{1+2e^n} \quad \begin{array}{l} :e^n \\ :e^n \end{array} = \lim_{n\to\infty} \frac{\frac{e^n}{e^n}}{\underbrace{\frac{1}{e^n}}_{0}+\frac{2e^n}{e^n}} = \frac{1}{0+2} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{n^3}{e^{2n}} \quad \left(= \frac{\infty}{\infty} \Rightarrow \text{L'Hopital} \right)$$

$$= \lim_{n\to\infty} \frac{3n^2}{2e^{2n}} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{6n}{4e^{2n}} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{6}{8e^{2n}} = 0$$

$$\lim_{n\to\infty} a_n = \left(0, \frac{1}{2}, 0\right)$$

c) $a_n = \left(\left(\frac{n+3}{n+2}\right)^n, \frac{(\ln n)^7}{n^2}, \frac{2^n+4^n}{3^n+6^n} \right)$

$$\lim_{n\to\infty} \left(\frac{n+3}{n+2}\right)^n = \lim_{n\to\infty} \left(\frac{n+2+1}{n+2}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n+2}\right)^n$$

$t = n+2 \Rightarrow n = t-2$

$$= \lim_{t\to\infty} \left(1+\frac{1}{t}\right)^{t-2} = \lim_{t\to\infty} \left(\left(1+\frac{1}{t}\right)^t \cdot \underbrace{\left(1+\cancel{\frac{1}{t}}\right)^{-2}}_{=1} \right) = e$$

$$\lim_{n\to\infty} \frac{(\ln n)^7}{n^2} \stackrel{L'H\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{7\cdot(\ln n)^6 \cdot \frac{1}{n}}{2n} \stackrel{L'H\left(\frac{0}{\infty}\right)}{=} \frac{42\,(\ln n)^5 \cdot \frac{1}{n^2}}{2} = \frac{0}{2} = 0$$

$$\lim_{n\to\infty} \frac{2^n+4^n}{3^n+6^n} \quad \begin{array}{l} :6^n \\ :6^n \end{array} = \lim_{n\to\infty} \frac{\cancel{\frac{2^n}{6^n}}^{0} + \cancel{\frac{4^n}{6^n}}^{0}}{\cancel{\frac{3^n}{6^n}}_{0} + \underbrace{\frac{6^n}{6^n}}_{=1}} = \frac{0+0}{0+1} = 0$$

$$\lim_{n\to\infty} a_n = (e, 0, 0)$$

2.) Poiščite vsa stekališča zaporedja $(x_n)_{n=1}^{\infty}$ v $(\mathbb{R}^2, d_2)$

za $x_n = \left(\frac{(-1)^n}{n}, \cos \frac{n\pi}{2} \right)$

Stekališče $\frac{(-1)^n}{n} \Rightarrow 0$

Stekališče $\cos \frac{n\pi}{2}$:

$\cos \frac{\pi}{2} = 0$

$\cos \frac{2\pi}{2} = \cos \pi = -1$

$\cos \frac{3\pi}{2} = 0$

$\cos \frac{4\pi}{2} = \cos 2\pi = 1$

$\Rightarrow$ Stekališča: $(0,-1), (0,0), (0,1)$

3.) Naj bo $\varphi_1 > \varphi_2 > \varphi_3 > \ldots$ in $\lim\limits_{n\to\infty} \varphi_n = 0$.

Ali je zaporedje točk $(x_n)_{n=1}^{\infty}$ z $x_n = (\cos\varphi_n, \sin\varphi_n)$ v evklidski metriki konvergentno? Ali je Cauchyjevo?

a) konverg.: limita?

$$\lim_{n\to\infty} \cos\varphi_n = \cos 0 = 1$$

$$\lim_{n\to\infty} x_n = (1,0)$$

$$\lim_{n\to\infty} \sin\varphi_n = \sin 0 = 0$$

~~$\Rightarrow x_n$ je konvergentno $\Rightarrow x_n$ je Cauchyjevo~~

b) ~~Cauchyjevo:~~ Izračunajmo: ~~$(\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$, da $\forall m,n > n_0 \Rightarrow d_2(x_m, x_n) < \varepsilon)$~~

$$d_2((\cos\varphi, \sin\varphi), (1,0)) = \sqrt{(\cos\varphi - 1)^2 + \sin^2\varphi}$$

$$= \sqrt{\underbrace{\cos^2\varphi - 2\cos\varphi + 1 + \sin^2\varphi}_{=1}}$$

$\sin\frac{\varphi}{2} = \pm\sqrt{\frac{1}{2}(1-\cos\varphi)}$

$$= \sqrt{2 - 2\cos\varphi} = \sqrt{2(1-\cos\varphi)}$$

$$= 2 \cdot \sqrt{\frac{2(1-\cos\varphi)}{4}} = 2 \cdot \sqrt{\frac{1-\cos\varphi}{2}}$$

$$= 2 \cdot \left|\sin\frac{\varphi}{2}\right| \xrightarrow[\varphi \to 0]{} 0$$

$\Rightarrow x_n$ je konvergentno

$\Rightarrow x_n$ je Cauchyjevo.

$f_1(x) = \frac{x}{2x+1}$

$f_2(x) = \frac{2x}{4x+1}$

$f_3(x) = \frac{3x}{6x+1}$

$f_{100}(x) = \frac{100x}{200x+1}$

4.) Dano je zaporedje funkcij $f_n(x) = \frac{nx}{2nx+1}$

Ali je to zaporedje konvergentno / Cauchyjevo ?

a) v prostoru zveznih funkcij na $[\frac{1}{2}, 1]$, opremljenim z max metriko?

$\hookrightarrow d_\infty(f,g) = \max_{a \leq x \leq b} |f(x) - g(x)|$

b) v prostoru — " — , opremljenim z integralsko metriko d_1?

$$d_p(f,g) = \left(\int_a^b |f(x)-g(x)|^p dx\right)^{\frac{1}{p}}$$

a) Poiščimo limito danega zaporedja

$$L = \lim_{n\to\infty} \frac{nx}{2nx+1} = \lim_{n\to\infty} \frac{\frac{nx}{n}}{\frac{2nx}{n} + \frac{1}{n}} = \frac{x}{2x} = \frac{1}{2}$$

Izračunajmo:

$$d_\infty(f_n, L) = \max_{\frac{1}{2} \leq x \leq 1} \left| \frac{nx}{2nx+1} - \frac{1}{2} \right| = \max_{\frac{1}{2} \leq x \leq 1} \left| \frac{\cancel{2nx} - \cancel{2nx} - 1}{4nx+2} \right|$$

$$= \max_{\frac{1}{2} \leq x \leq 1} \frac{1}{4nx+2} = \frac{1}{2n+2} \qquad \max \text{ pri } x = \frac{1}{2}$$

$$\xrightarrow[n\to\infty]{} 0$$

Ker ~~gre iz~~ je $\lim_{n\to\infty} \frac{1}{2n+2} = 0$ zaporedje tudi v max metriki konvergira proti L.

$\Rightarrow$ zaporedje je Cauchyjevo

b) Limito že poznamo!

Izračunajmo:

$$d_1(f_n, L) = \int_{1/2}^1 \left| \frac{nx}{2nx+1} - \frac{1}{2} \right| dx = \int_{1/2}^1 \frac{1}{4nx+2} dx$$

$4nx+2 = t$

$dt = 4n\, dx$

$$= \int \frac{1}{t} \frac{dt}{4n} = \frac{1}{4n} \int \frac{1}{t} dt = \frac{1}{4n} \ln t$$

$$= \frac{1}{4n} \ln(4nx+2) \Big|_{1/2}^1 = \frac{1}{4n} \left(\ln(4n+2) - \ln(2n+2) \right)$$

$$= \frac{1}{4n} \ln \frac{4n+2}{2n+2} = \frac{1}{4n} \ln \frac{2n+1}{n+1}$$

Ker je $\lim_{n\to\infty} \frac{1}{4n} \ln \frac{2n+1}{n+1} =$ [illegible] $\Longrightarrow$

Ker je $\lim\limits_{n\to\infty} \frac{1}{4n} \cdot \ln \frac{2n+1}{n+1}$

$$= \lim_{n\to\infty} \frac{\ln \frac{2n+1}{n+1}}{4n} = \frac{\infty}{\infty} \Rightarrow L'H$$

$$= \lim_{n\to\infty} \frac{\frac{n+1}{2n+1} \cdot \left(\frac{2n+1}{n+1}\right)'}{4} = \lim_{n\to\infty} \frac{\frac{\cancel{n+1}}{2n+1} \cdot \frac{2(n+1)-(2n+1)\cdot 1}{(n+1)^{\cancel{2}}}}{4}$$

$$= \lim_{n\to\infty} \frac{\frac{\cancel{2n}+2-\cancel{2n}-1}{(2n+1)(n+1)}}{4} = \lim_{n\to\infty} \frac{\frac{1}{(2n+1)(n+1)}}{4} =$$

$$= \lim_{n\to\infty} \frac{1}{4(2n+1)(n+1)} = 0$$

je zaporedje tudi v tej metriki konvergentno $\Rightarrow$ Cauchijevo