3.2 – CONVERGENCE AND CONTINUITY

Def. (3.2.1) Let (X,d) be a metric space. A sequence $\{x_n\} \subseteq X$ converges to $a \in X$ if and only if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N: \quad d(x_n, a) < \varepsilon$$

$$\Leftrightarrow \quad \lim_{n \to \infty} x_n = a$$

Lemma (3.2.2) A sequence $\{x_n\}$ in a m.s. (X,d) converges to $a \in X$, if and only if:

$$\lim_{n \to \infty} d(x_n, a) = 0$$

Proposition (3.2.3) A sequence in a metric space cannot converge to more than one point.

Proof: We must show that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$ only if $a = b$ (they're equal). If we take the triangle inequality to write:

$$d(a,b) \leq d(x_n, a) + d(x_n, b)$$

and take their limits:

$$\lim_{n\to\infty} d(a,b) \leq \lim_{n\to\infty} d(x_n, a) + \lim_{n\to\infty} d(x_n, b)$$

this is a constant

$\Downarrow$

$$d(a,b) \leq \lim_{n\to\infty} d(x_n, a) + \lim_{n\to\infty} d(x_n, b)$$

from lemma (3.2.3) this is equal to 0

$\Downarrow$

$$d(a,b) \leq 0 + 0$$

$$d(a,b) \leq 0$$

since d is a metric, it cannot be less than 0, leaving us with the equality:

$$d(a,b) = 0$$

and we know, again since d is a metric, that this is possible only when $a = b$. □

$\downarrow$

A SEQUENCE IN A METRIC SPACE CAN THEREFORE CONVERGE ONLY TO ONE POINT

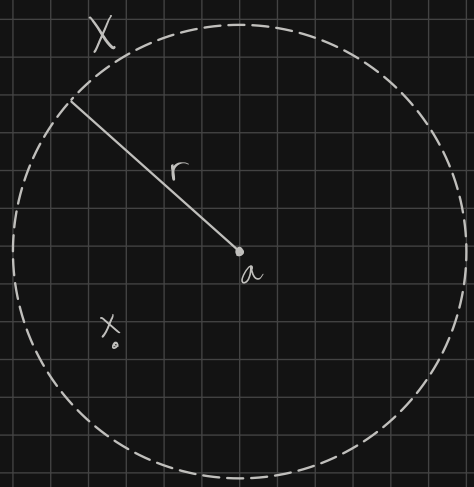
Observe how we used definitions from 3.1.1. As the theory develops, we'll have more and more tools to work with.

We can also phrase the notion of convergence in geometric terms:

If a is an element of a metric space X, and $r > 0$, then the (open) ball centered at a with radius r is the set:

$$B_r(a) = \{x \in X ; d(x,a) < r\}$$

So saying $x \in B_r(a)$ is equivalent to saying $d(x,a) < r$.



The def. of convergence can now be rephrased by saying that $\{x_n\}$ converges to a if the elements in the sequence eventually end up inside ANY ball $B_\varepsilon(a)$ centered at a:

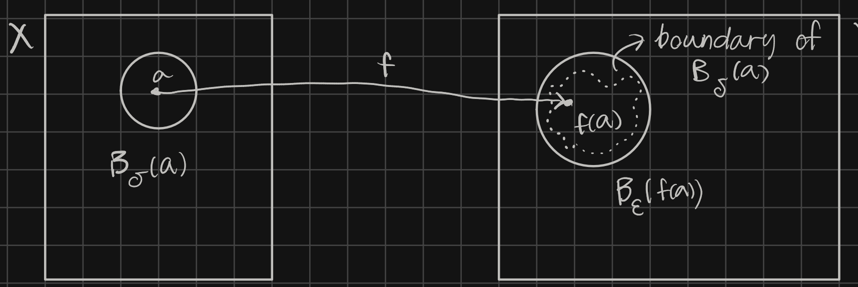
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : x_n \in B_\varepsilon(a) \Leftrightarrow d(x_n, a) < \varepsilon \Leftrightarrow \lim_{n\to\infty} d(x_n, a) = 0 \Leftrightarrow \lim_{n\to\infty} x_n = a$$

Next we define the continuity of functions from one metric space to another, by following the same principles as for $\mathbb{R}^m$ (section 2.1).

Def (3.2.4) Assume (X, d_X) and (Y, d_Y) are two metric spaces. A function $f: X \to Y$ is continuous at $a \in X$ if $\forall \varepsilon > 0$ there $\exists \delta > 0$ such that:

$$d_X(x,a) < \delta \quad \Rightarrow \quad d_Y(f(x), f(a)) < \varepsilon$$

A more geometric formulation: for any open ball $B_\varepsilon(f(a))$ there is an open ball $B_\delta(a)$ such that $f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$.



$$\forall B_\varepsilon(f(a)), \exists B_\delta(a) : f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$$

Proposition (3.2.5) The following are equivalent for a function $f: X \to Y$ between metric spaces:

(i) f is continuous at a point $a \in X$.

(ii) for all sequences $\{x_n\}$ converging to a, the sequence $\{f(a)\}$ converges to $f(a)$.

Proof: Assume that f is continuous at a and that $\{x_n\}$ is a sequence converging to a.
We must show that $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N: \quad d_Y(f(x_n), f(a)) < \varepsilon$

Since f is continuous at a, $\exists \delta > 0: d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$
since $\{x_n\}$ converges to a, there $\exists N \in \mathbb{N}: d_X(x_n, a) < \delta \Rightarrow d_Y(f(x_n), f(a)) < \varepsilon$ when $n \geq N$.

Assume f is not continuous at a. There must exist a sequence $\{x_n\}$ that converges to a such that $\{f(x_n)\}$ does not converge to $f(a)$.

Since $f \neq$ continuous at a: $\exists \varepsilon > 0, \forall \delta > 0: d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) \geq \varepsilon$
For each $n \in \mathbb{N}$ we can find an x_n, such that $d_X(x_n, a) < 1/n \Rightarrow d_Y(f(x_n), f(a)) \geq \varepsilon$.

$\{x_n\}$ converges to a, but $\{f(x_n)\}$ does not converge to $f(a)$.

□

Proposition (3.2.6) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be 3 metric spaces. Assume $f: X \to Y$ and $g: Y \to Z$ are two functions and let $h: X \to Z$ be the composition $h(x) = g(f(x))$.
If f is continuous at $a \in X$ and g is continuous at $b = f(a)$, then h is continuous at a.

Proof: Assume $\{x_n\}$ converges to a. Since f is continuous at a, this means $\{f(x_n)\}$ converges to $f(a)$. Since g is continuous at $b = f(a)$, this means $\{g(f(x_n))\}$ converges to $g(b)$. This also means that $\{h(x_n)\}$, which is just $\{g(f(x_n))\}$, converges to $h(a)$, which is just $g(f(a)) = g(b)$.

$\Rightarrow h$ is continuous at a. □

Def (3.2.7) A function $f: X \to Y$ between 2 metric spaces is continuous if it's continuous at all points $x \in X$.

Def (3.2.8) Assume (X, d_X) and (Y, d_Y) are two metric spaces and that $A \subseteq X$.
A function $f: A \to Y$ is continous at $a \in A$ whenever $x \in A$ and:

$$\forall \varepsilon > 0, \exists \delta > 0: \quad d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

f is continuous if it's continuous at all $a \in A$.

Proposition (3.2.9) Assume (X, d_X) and (Y, d_Y) are two metric spaces and that $A \subseteq X$. The following are equivalent for a function $f: A \to Y$:

(i) f is continuous at a point $a \in A$.

(ii) for all sequences $\{x_n\} \subseteq A$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

EXCERCISES FOR SECTION 3.2

1. Predpostavi, da je (X, d_X) diskretni metrični prostor. Pokaži, da zaporedje $\{x_n\}$ konvergira v a, če in samo če $\exists N \in \mathbb{N}$, da je $x_n = a$ za vsak $n \geq N$.

$$d(x,y) = d_X(x,y) = \begin{cases} 1 ; & x \neq y \\ 0 ; & x = y \end{cases}$$

($\Rightarrow$) Naj $\{x_n\}$ konvergira v $a \in X$. Po definiciji to pomeni:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : \lim_{n\to\infty} x_n = a \Leftrightarrow \lim_{n\to\infty} d(x_n, a) = 0 \Leftrightarrow d(x_n, a) < \varepsilon$$

Na primer, za $\varepsilon = 0.5$ je izraz enak nič le takrat, ko $x_n = a$.

$$\Rightarrow d(x_n, a) = 0 \text{ za } \forall n \geq N \Leftrightarrow x_n = a \text{ za } \forall n \geq N.$$

($\Leftarrow$) Naj $\exists N \in \mathbb{N}$, tako da za $\forall n \geq N$ velja $x_n = a$. To pomeni, da za $\forall n \geq N$ velja $d(x_n, a) = 0$. Če vzamemo limito obeh strani, je to:

$$\lim_{n\to\infty} d(x_n, a) = \lim_{n\to\infty} 0$$

$$\lim_{n\to\infty} d(x_n, a) = 0$$

Kar pa po definiciji pomeni, da $\{x_n\}$ konvergira v a. $\square$

2. Dokaži (3.2.6) brez uporabe (3.2.5) oz. samo z definicijo zveznosti.

Proposition (3.2.5) The following are equivalent for a function $f: X \to Y$ between metric spaces:

(i) f is continuous at a point $a \in X$.

(ii) for all sequences $\{x_n\}$ converging to a, the sequence $\{f(a)\}$ converges to $f(a)$.

Proposition (3.2.6) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be 3 metric spaces. Assume $f: X \to Y$ and $g: Y \to Z$ are two functions and let $h: X \to Z$ be the composition $h(x) = g(f(x))$. If f is continuous at $a \in X$ and g is continuous at $b = f(a)$, then h is continuous at a.

Proof: Assume $\{x_n\}$ converges to a. Since f is continuous at a, this means $\{f(x_n)\}$ converges to $f(a)$. Since g is continuous at $b = f(a)$, this means $\{g(f(x_n))\}$ converges to $g(b)$. This also means that $\{h(x_n)\}$, which is just $\{g(f(x_n))\}$, converges to $h(a)$, which is just $g(f(a)) = g(b)$.

$\Rightarrow h$ is continuous at a. $\square$

Def (3.2.4) Assume (X, d_X) and (Y, d_Y) are two metric spaces. A function $f: X \to Y$ is continuous at $a \in X$ if $\forall \varepsilon > 0$ there $\exists \delta > 0$ such that:

$$d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

Predpostavimo, da je $f: X \to Y$ zvezna v $a \in X$ in da je $g: Y \to Z$ zvezna v $b = f(a)$. Hočemo pokazati, da je tudi $h: X \to Z$ zvezna v a. Po definiciji to pomeni:

$$\forall \varepsilon > 0, \exists \delta > 0 : \quad d_X(x,a) < \delta \Rightarrow d_Z(h(x), h(a)) < \varepsilon$$

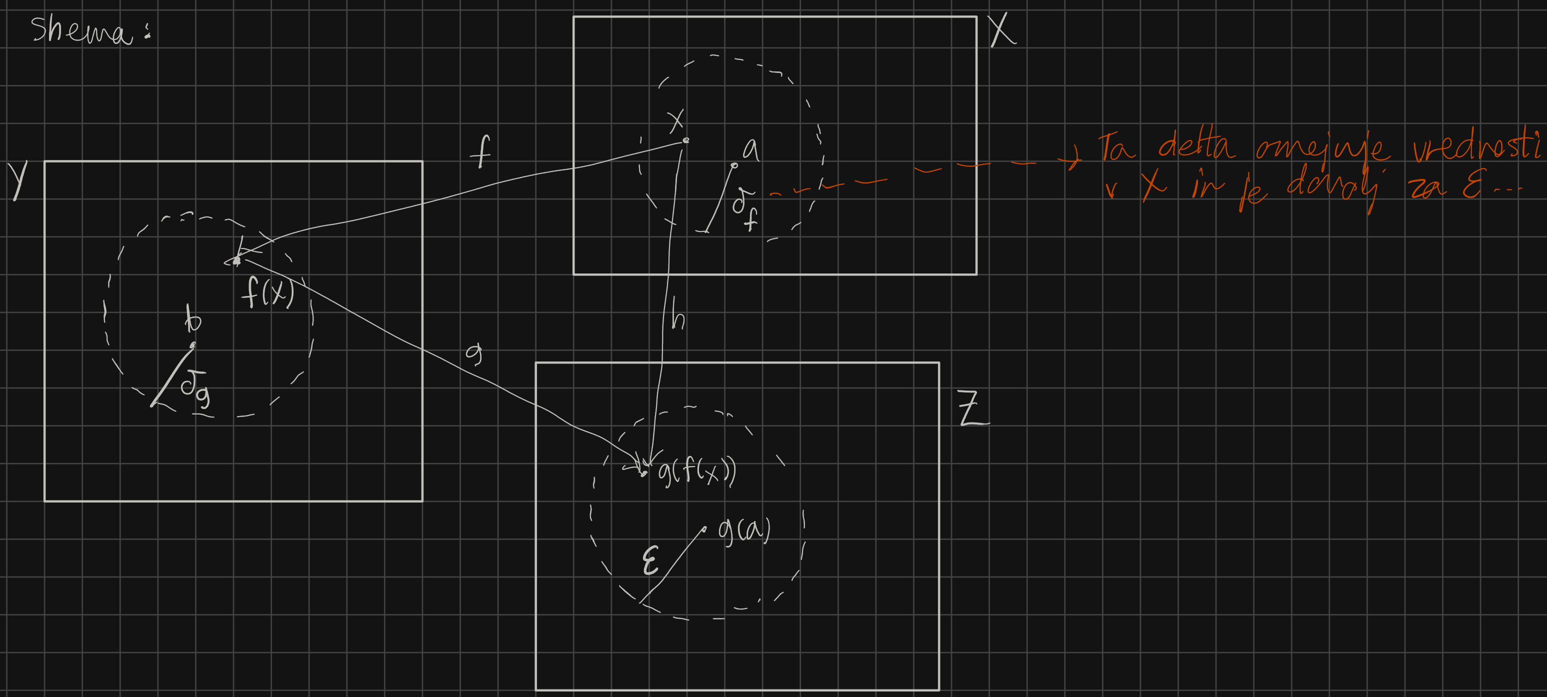
Naj bo $\varepsilon > 0$ poljuben. Ker je g zvezna v točki b, za tak ε obstaja nek $\delta_g > 0$, tako da velja:

$$y \in Y : \quad d_Y(y, b) < \delta_g \Rightarrow d_Z(g(y), g(b)) < \varepsilon \quad (1)$$

Ker je tudi f zvezna v a, za tak $\delta_g > 0$ obstaja nek $\delta_f > 0$, tako da velja:

$$x \in X : \quad d_X(x,a) < \delta_f \Rightarrow d_Y(f(x), f(a)) < \delta_g \quad (2)$$

Shema:



Naj bo $y = f(x)$. Vemo, da je $b = f(a)$. (2) lahko zapišemo kot:

$$d_X(x,a) < \delta_f \Rightarrow d_Y(y, b) < \delta_g$$

Po (1), to pomeni:

$$d_Y(y, b) < \delta_g \Rightarrow d_Z(g(y), g(b)) < \varepsilon$$

$$d_Y(f(x), f(a)) < \delta_g \Rightarrow d_Z(g(f(x)), g(f(a))) < \varepsilon$$

Po definiciji h, to pomeni:

$$d_Z(h(x), h(a)) < \varepsilon$$

Pokazali smo torej, da za katerikoli $\varepsilon > 0$ obstaja tak δ_f oz. δ_x, da če je $x \in X$ in $d_X(x,a) < \delta_x$, potem $d_Z(h(x),h(a)) < \varepsilon$, kar pa je točno definicija zveznosti v točki a.

$\square$

DOKAZ S KROGLAMI

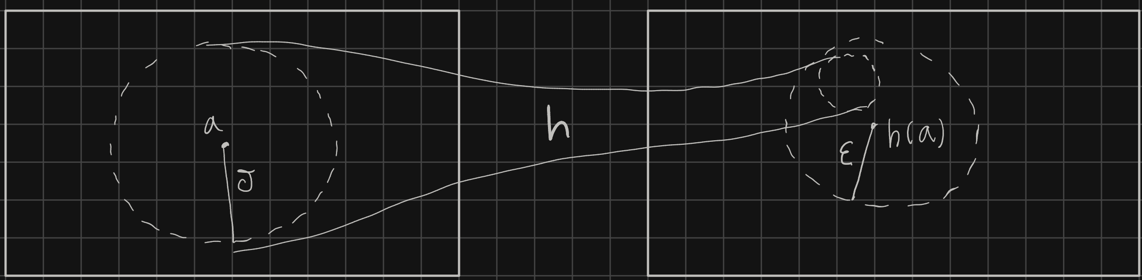
Funkcija $f: X \to Y$ je zvezna v $a \in X$, če za vsako odprto kroglo $K_\varepsilon(f(a))$ obstaja odprta krogla $K_\delta(a)$, tako da velja:

$$f(K_\delta(a)) \subseteq K_\varepsilon(f(a))$$

Želimo dokazat, da je $h: X \to Z$ zvezna v $a \in X$. Po zgornji definiciji to pomeni, da mora za vsako odprto kroglo $K_\varepsilon(h(a))$ obstajati odprta krogla $K_\delta(a)$, tako da:

$$h(K_\delta(a)) \subseteq K_\varepsilon(h(a))$$

Velja, da sta $K_\varepsilon(h(a)) \subseteq Z$ in $K_\delta(a) \subseteq X$.



Naj bo $\varepsilon > 0$ poljuben. Velja $h(x) = g(f(a))$ in $b = f(a)$, zato je:

$$K_\varepsilon(h(a)) = K_\varepsilon(g(f(a))) = K_\varepsilon(g(b))$$

$$g(f(K_\delta(a))) \subseteq K_\varepsilon(g(f(a)))$$

Funkcija g je zvezna v $b = f(a)$. To pomeni, da mora za vsako odprto kroglo $K_\varepsilon(g(b)) \subseteq Z$ obstajati odprta krogla $K_{\delta_y}(b) \in Y$, tako da velja:

$$g(K_{\delta_y}(b)) \subseteq K_\varepsilon(g(b))$$

Funkcija f pa je zvezna v $a \in X$. To pomeni, da $\forall K_{\delta_y}(f(a))$, $\exists K_{\delta_x}(a)$, da velja:

$$f(K_{\delta_x}(a)) \subseteq K_{\delta_y}(f(a))$$

$$\Big\downarrow\ b = f(a)$$

$$f(K_{\delta_x}(a)) \subseteq K_{\delta_y}(b)$$

$$\Big\downarrow\ g(\ldots)$$

$$g(f(K_{\delta_x}(a))) \subseteq g(K_{\delta_y}(b)) \subseteq K_\varepsilon(g(b))$$

$$\Big\downarrow$$

$$g(f(K_{\delta_x}(a))) \subseteq K_\varepsilon(g(f(a)))$$

$$\Big\Vert\ h(\ldots)$$

$$\Downarrow$$

$$h(K_{\delta_x}(a)) \subseteq K_\varepsilon(h(a))$$

Za poljuben $x \in K_{\delta_x}(a)$ velja torej $h(x) \in h(K_{\delta_x}(a))$ in tudi $h(x) \in K_\varepsilon(h(a))$. $\square$

3. Dokaži izrek (3.2.9).

($\Rightarrow$) (i) $\Rightarrow$ (ii)

Predpostavimo, da je f zvezna v $a \in A$ ter vse kar je v proposition (3.2.9). Naj bo $\{x_n\} \subseteq A$ in naj konvergira v a.

$\downarrow$

Po definiciji (za metrične prostore):

$$\forall \varepsilon > 0, \exists \delta > 0: \; d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

Zaporedje $\{x_n\} \subseteq A$ po definiciji konvergira takrat, ko:

$$\forall \delta > 0, \exists N \in \mathbb{N}, \forall n \geq N: \; d(x_n, a) < \delta$$

f je zvezna v $a \in A$, zato velja:

$$\forall \varepsilon > 0, \exists \delta > 0, \exists N \in \mathbb{N}, \forall n \geq N: \; d_X(x_n, a) < \delta \Rightarrow d_Y(f(x_n), f(a)) < \varepsilon$$

$\downarrow$

to pa je definicija konvergence zaporedja $\{f(x_n)\}$ proti $f(a)$. ▨

($\Leftarrow$) (ii) $\Rightarrow$ (i) --- Predpostavimo protislovje: $\neg((ii) \Rightarrow (i)) \equiv (ii) \wedge \neg(i)$

Predpostavimo, da f ni zvezna v $a \in A$. To pomeni, da $\exists \varepsilon > 0$, tako da za $\forall \delta > 0$, $\exists x \in A$:

$$d_X(x,a) < \delta \;\wedge\; d_Y(f(x), f(a)) \geq \varepsilon \qquad (1)$$

Predpostavimo tudi (ii), da za $\forall \{x_n\} \subseteq A$, ki konvergira proti a, $\{f(x_n)\}$ konvergira proti $f(a)$. Naj bo $\delta = 1/n$ za $\forall n \in \mathbb{N}$. Ker f ni zvezna, to pomeni, da $\exists x_n \in A$, tako da:

$$d_X(x_n, a) < 1/n \;\wedge\; d_Y(f(x_n), f(a)) \geq \varepsilon \quad \text{za } \forall n \in \mathbb{N}$$

$$\lim_{n\to\infty} d_X(x_n, a) = 0 \qquad \lim_{n\to\infty} d_Y(f(x_n), f(a)) \geq \varepsilon \neq 0$$

To pomeni, da obstaja zaporedje $\{x_n\}$, ki konvergira proti a, vendar $\{f(x_n)\}$ ne konvergira proti $f(a)$, kar je v protislovju z našo predpostavko.

Očitno $\neg((ii) \Rightarrow (i))$ vodi v protislovje, zato $(ii) \Rightarrow (i)$ drži. ▨

$\square$