CH 3 – METRIC SPACES

"We can develop a general notion of distance that covers distances between numbers, vectors, sequences, sets and much more. Within this theory, we can develop and formulate and prove results about convergence and continuity."

3.1 DEFINITIONS AND EXAMPLES

Def (3.1.1) A metric space (X, d) consists of $X \neq \emptyset$ and $d : X \times X \to [0, \infty)$, such that:

(1) $\forall x, y \in X : \quad d(x,y) \geq 0 \quad \wedge \quad d(x,y) = 0 \Leftrightarrow x = y$

(2) $\forall x, y \in X : \quad d(x,y) = d(y,x)$

(3) $\forall x, y, z \in X : \quad d(x,y) \leq d(x,z) + d(z,y)$

A function d satisfying these conditions is said to be a metric on X.
When it is clear, we shall use "the metric space X" instead of "the metric space (X,d)".

Examples:

a) $d(x,y) = |x-y|$, $(\mathbb{R}, d)$ is a metric space. (1) and (2) are obvious.
(3) follows from triangle inequality of real numbers:

$$d(x,y) = |x-y| = |x+z-z-y| = |(x-z)+(z-y)| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$$

b) $d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$, then $(\mathbb{R}^n, d)$ is a metric space.
(1) and (2) are again obvious. (3) follows from triangle equality for vectors:

$$d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$$

c) Assume we want to move from a point $x = (x_1, x_2)$ to $y = (y_1, y_2)$ on a plane, but we can only move horizontally/vertically. Then the total distance is:

$$d(x,y) = |y_1 - x_1| + |y_2 - x_2|$$

Horizontally from (x_1, x_2) to (y_1, x_2)

Vertically from (y_1, x_2) to (y_1, y_2)

This is reffered to as the TAXI-CAB metric or MANHATTAN metric.

(1) and (2) are obvious.

To prove (3), let $z=(z_1, z_2)$. We can again use triangle inequality from $\mathbb{R}$ numbers.

$$d(x,y) = |y_1 - x_1| + |y_2 - x_2|$$

$$= |y_1 - z_1 + z_1 - x_1| + |y_2 - z_2 + z_2 - x_2|$$

$$= |(z_1 - x_1) + (y_1 - z_1)| + |(z_2 - x_2) + (y_2 - z_2)|$$

$$\leq |z_1 - x_1| + |y_1 - z_2| + |z_2 - x_2| + |y_2 - z_2|$$

$$= \underline{|z_1 - x_1| + |z_2 - x_2|} + \underline{|y_1 - z_1| + |y_2 - z_2|}$$

$$= d(x,z) + d(z,y)$$

d) the distance between functions: let X be the set of all continuous functions $f: [a,b] \to \mathbb{R}$. Then:

$$d_1(f,g) = \sup \{ |f(x) - g(x)| ; x \in [a,b] \}$$

is a metric on X. It determines the distance between two functions by measuring it at the x-value where the graphs are MOST apart (sup).

$$d_2(f,g) = \int_a^b |f(x) - g(x)| \, dx$$

This metric instead sums up the distance between $f(x)$ and $g(x)$ at all points.

$$d_3(f,g) = \left(\int_a^b |f(x) - d(x)|^2 dx \right)^{1/2}$$

This metric is a generalization of the usual (euclidean) metric in $\mathbb{R}^n$:

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

e) Let $X \neq \emptyset$ and define:

$$d(x,y) = \begin{cases} 0 ; & x = y \\ 1 ; & x \neq y \end{cases}$$

This is usually reffered to as the discrete metric.

f) We can make a new metric on some $X \neq \emptyset$ simply by restricting it to a subset e.g. $A \subseteq X$, such that $d_A(x,y)$ is defined $\forall x,y \in A$.

WHEN ARE TWO METRIC SPACES THE SAME?

Let (X, d_X) and (Y, d_Y) be two metric spaces. In the most "strictest" sense, they are the same when the sets X, Y and the metrics d_X, d_Y are equal.
It's often more appropriate to use a looser definition – we are more interested in the relationship between the sets, rather than what they are.

Def (3.1.2) An isometry between two metric spaces is a bijection that preserves the distance between points (what is important for metric spaces).

Assume (X, d_X) and (Y, d_Y) are metric spaces. An isometry between them is a bijection $i : X \to Y$, such that:

$$d_X(x, y) = d_Y(i(x), i(y)) \quad ; \ \forall x, y \in X$$

(X, d_X) and (Y, d_Y) are isometric if there exists an isometry from (X, d_X) to (Y, d_Y).

The inverse i^{-1} is an isometry from (Y, d_Y) to (X, d_X), hence being isometric is a symetric relation.

(3.1.4) (Inverse triangle inequality) $\forall x, y, z \in (X, d)$, we have:

$$|d(x,y) - d(x,z)| \leq d(y,z)$$

Proof: Pokazati moramo, da veljata dve neenačbi:

i) $d(x,y) - d(x,z) \leq d(y,z)$

Začni s (3): $d(x,y) \leq d(x,z) + d(z,y)$

$$d(x,y) - d(x,z) \leq d(z,y) = d(y,z)$$

$\blacksquare$

ii) $-(d(x,y) - d(x,z)) \leq d(y,z)$

$d(x,z) - d(x,y) \leq d(y,z)$

Spet začni s (3): $d(x,y) \leq d(x,z) + d(z,y)$

$$d(x,y) - d(x,z) \leq d(z,y)$$

$$d(x,z) - d(x,y) \leq d(y,z)$$

$\blacksquare$

$\square$

EXCERCISES FOR SECTION 3.1

1. Pokaži, da je (X,d) v Example 4 (Hamming - metric) metrika.

- Naj bo L jezik z N simboli
- Želimo pošiljati sporočila; vsa so dolžine K
- Naj bo X množica vseh sporočil (i.e. zaporedij simbolov iz L dolžine K)
- Če sta $x = (x_1, \dots, x_K)$, $y = (y_1, \dots, y_K)$ dve sporočili, potem je

$d(x,y)$ = vsi indeksi n, tako da velja: $x_K \neq y_K$ $(0 \leq n \leq K)$

npr. $d((\underline{1},2,3,4), (\underline{1},1,2,2)) = |\{\dots, 2,3,4\}| = |\{2,3,4\}| = 3!$

Def (3.1.1) A metric space (X,d) consists of $X \neq \emptyset$ and $d: X \times X \to [0,\infty)$, such that:

(1) $\forall x,y \in X: \quad d(x,y) \geq 0 \quad \wedge \quad d(x,y) = 0 \Leftrightarrow x = y$

(2) $\forall x,y \in X: \quad d(x,y) = d(y,x)$

(3) $\forall x,y,z \in X: \quad d(x,y) \leq d(x,z) + d(z,y)$

(1) $d(x,y) \geq 0$: res je $0 \leq n \leq K$. $d(x,y) = 0 \Leftrightarrow x = y$: res, npr. $x = (1,1)$, $y = (1,1)$

$d(x,y) = d((1,1),(1,1))$
$\Rightarrow 1 = 1$ in $1 = 1$
$= 0$

(2) $d(x,y) = d(y,x)$: res... očitno

(3) $d(x,y) \leq d(x,z) + d(z,y) \Leftrightarrow d(x,y) \leq d(z,x) + d(z,y)$

Naj bo $n \in \{1,2,\dots,K\}$. Če velja $x_n \neq z_n$, mora veljati tudi nekaj za y_n:

$x_n \neq z_n \wedge z_n = y_n \Rightarrow \underline{y_n = x_n}$ ✗

$x_n \neq z_n \wedge z_n = y_n \Rightarrow x_n \neq y_n$

ali $z_n \neq y_n$

Naj bodo: $I_{xz} = \{n;\ x_n \neq z_n\}$ $\qquad d(x,z) = |I_{xz}|$

$I_{yz} = \{n;\ y_n \neq z_n\}$ $\Rightarrow$ $d(y,z) = |I_{yz}|$

$I_{xy} = \{n;\ x_n \neq y_n\}$ $\qquad d(x,y) = |I_{yz}|$

Izjava "$x_n \neq z_n \Rightarrow x_n \neq y_n \vee y_n \neq z_n$" pomeni, da je $n \in I_{xz} \Rightarrow n \in I_{xy} \vee n \in I_{yz}$:

$n \in I_{xz} \Rightarrow n \in I_{xy} \cup I_{yz}$

$I_{xz} \subseteq I_{xy} \cup I_{yz}$

Število elementov podmnožice: $|I_{xz}| \leq \underline{|I_{xy} \cup I_{yz}|}$

To lahko razčlenimo po pravilu: $|A \cup B| = |A| + |B| - |A \cap B|$

$$|I_{xy} \cup I_{yz}| = |I_{xy}| + |I_{yz}| - |I_{xy} \cap I_{yz}|$$

ker $n \in I_{xy}$ ali $n \in I_{yz}$ $\Rightarrow$ lahko je v enem, lahko je v obeh

presek $\neq \emptyset$ presek $= \emptyset$

$$|I_{xy} \cap I_{yz}| \geq 0$$

dodamo na obe strani:

$$|I_{xy}| + |I_{yz}| + |I_{xy} \cap I_{yz}| \geq |I_{xy}| + |I_{yz}|$$

Če je presek neprazen, potem imata množici skupen element (vsaj en), zato to moč unije manjša (ali enaka) vsoti moči množic!

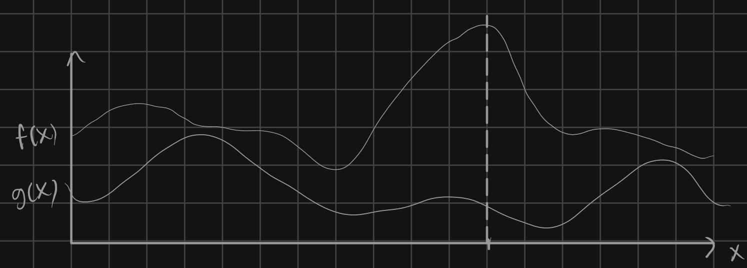
$$|I_{xy} \cup I_{yz}| \leq |I_{xy}| + |I_{yz}| \longrightarrow |I_{xz}| \leq |I_{xy} \cup I_{yz}| \leq |I_{xy}| + |I_{yz}|$$

$$d(x,z) \leq d(x,y) + d(y,z)$$

□

2. Pokaži, da je (X, d_1) v Example 5 metrika

$d_1(f,g) = \sup\{|f(x) - g(x)| ;\ x \in [a,b]\}$ --→ vrne največjo razliko med funkcijama, npr.:



tu je največja razlika

(1) $d(x,y) \geq 0$; $d(x,y) = 0 \Leftrightarrow x = y$

večje od 0 je, ker je absolutna vrednost razlike ■

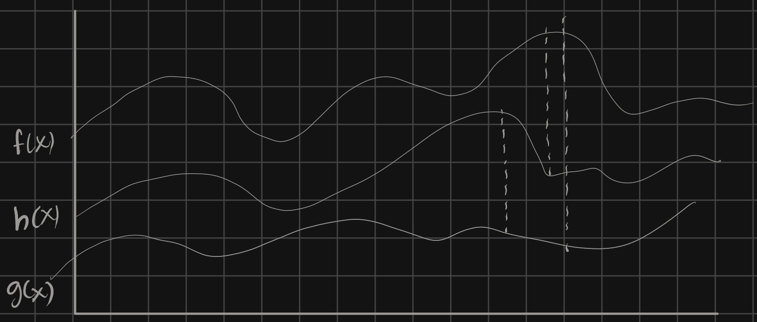
če je $x = y$, potem $f = g$ in zato

$$d(x,x) = \sup\{|f(x) - f(x)| = |0| = 0;\ \forall x \in [a,b]\} = \sup\{0\} = 0$$

■

(2) $d(x,y) = d(y,x)$ ... očitno, saj $|f(x) - g(x)| = |-(g(x) - f(x))| = |g(x) - f(x)|$ ■

(3) $d(x,y) \leq d(x,z) + d(z,y)$ oz. $d(f,g) \leq d(f,h) + d(h,g)$



$$\sup\{|f(x)-g(x)|;\ \forall x\} \leq \sup\{|f(x)-h(x)|;\ \forall x\} + \sup\{|h(x)-g(x)|;\ \forall x\}$$

Naj bo $M_{fg} = d(f,g)$ in $M_{gh} = d(g,h)$. Za $\forall x \in [a,b]$ po definiciji supremuma velja:

$$|f(x) - g(x)| \leq M_{fg}$$

$$|g(x) - h(x)| \leq M_{gh}$$

Kaj pa $|f(x) - h(x)|$? Če dodamo 0: $|f(x)-g(x)+g(x)-h(x)| \leq |f(x)-g(x)| + |g(x)-h(x)|$ in za to velja:

$$|f(x) - g(x)| \leq M_{fg}$$

$$|f(x)-g(x)| + |g(x)-h(x)| \leq \underbrace{M_{fg} + M_{gh}}_{\text{neko realno število}}$$

To zaporedje je navzgor omejeno s $M_{fg} + M_{gh}$ oz. to je zgornja meja tega zaporedja. Po definiciji supremuma velja, da je vsaka zgornja meja večja ali enaka supremumu množice za $\forall$ el. v mn.:

$$\sup\{|f(x) - g(x)| + |g(x)-h(x)|;\ \forall x \in [a,b]\} \leq M_{fg} + M_{gh}$$

Rekli smo, da je:

$$|f(x) - h(x)| \leq |f(x)-g(x)| + |g(x)+h(x)| \leq M_{fg} + M_{gh}$$

$$\sup\{|f(x) - h(x)|\} \leq \ldots \leq \sup\{M_{fg} + M_{gh}\}$$

$$\sup\{|f(x) - h(x)|\} \leq M_{fg} + M_{gh}$$

$$d(f,h) \leq M_{fg} + M_{gh}$$

$$d(f,h) \leq d(f,g) + d(g,h)$$

$\square$

4. Pokaži, da je diskretna metrika (example 6.) metrika.

$$d(x,y) = \begin{cases} 1 ; & x \neq y \\ 0 ; & x = y \end{cases}$$

(1) $d(x,y) \geq 0$: očitno po predpisu (df.)

(2) $d(x,y) = 0 \Leftrightarrow x = y$: očitno...

(3) $d(x,y) \leq d(x,z) + d(z,y)$

i) $x = y$:

$d(x,y) = 0$

a) $z = x$: $d(z,x) = d(x,x) = 0$
$\Rightarrow z = y$: $d(z,y) = d(y,y) = d(x,x) = 0$ $\Rightarrow$ $d(x,x) \leq d(x,x) + d(x,x)$ ▩

b) $z \neq x$: $d(z,x) = 1$
$\Rightarrow z \neq y$: $d(z,y) = 1$ $\Rightarrow$ $d(x,y) = d(x,x) \leq d(z,y) + d(z,x)$

$0 \leq 1 + 1$

$0 \leq 2$ ▩

ii) $x \neq y$:

$d(x,y) = 1$

a) $z = x$: $d(z,x) = d(x,x) = 0$ $\Rightarrow$ $d(x,y) \leq d(x,z) + d(z,y)$
$\Rightarrow z \neq y$: $d(z,y) = 1$

$1 \leq 0 + 1$

$1 \leq 1$ ▩

b) $z \neq x$: $d(x,z) = 1$
$\Rightarrow z = y \vee z \neq y$

$d(z,y) = 0$ $d(z,y) = 0$ $\Rightarrow$

1. $d(x,y) \leq d(y,y) + d(z,x)$

$1 \leq 0 + 1$

2. $d(x,y) \leq d(z,y) + d(z,x)$

$1 \leq 1 + 1$ ▩

$\square$

5. Zaporedje realnih števil $\{x_n\}$ je omejeno, če obstaja $M \in \mathbb{R}$, tako da $|x_n| \leq M$, $\forall n \in \mathbb{N}$. Naj bo X mn. vseh omejenih zaporedij. Pokaži, da je d metrika nad X:

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| ; n \in \mathbb{N}\}$$

(1) Naj bo $x = \{x_n\}$ in $y = \{y_n\}$.

$d(x,y) \geq 0$ $\rightarrow$ ne more bit manj kot 0 zaradi absolutne

(2) $d(x,y) = 0 \Leftrightarrow x = y$: Očitno, če je $x_n = y_n$ za $\forall n \in \mathbb{N}$, potem je $|x_n - y_n| = |x_n - x_n| = 0$ in $\sup\{0\} = 0$

Če obstaja vsaj en $n_0 \in \mathbb{N}$, tako da je $x_n \neq y_n$, potem naj bo $s = |x_{n_0} - y_{n_0}|$ in velja

$$\sup\{0, 0, \dots, s, 0, \dots, 0\} = \sup\{0, s\}$$

Ker $s > 0$, je $\sup\{0, s\} = s$ in $s \neq 0$ ⨳

(3) Naj bo $z = \{z_n\}$. Velja $d(x,y) \leq d(x,z) + d(z,y)$.

Naj bo $M_{xz} = d(x,z)$ in $M_{zy} = d(z,y)$ $\Rightarrow$ $M_{xz} = \sup\{|x_n - z_n|;\ n \in \mathbb{N}\}$
$M_{zy} = \sup\{|z_n - y_n|;\ n \in \mathbb{N}\}$

Po def. supremuma za $\forall n \in \mathbb{N}$ velja:

$$|x_n - z_n| \leq M_{xz}$$

$$|z_n - y_n| \leq M_{zy}$$

Enako kot pri nalogi (2.) zdaj obravnavamo

$$|x_n - y_n| = |x_n - z_n + z_n - y_n| \leq |x_n - z_n| + |z_n - y_n| \leq M_{xz} + M_{zy}$$

Po def. supremuma velja, da je manjši/enak vsaki zgornji meji množice:

$$\sup\{|x_n - z_n| + |z_n - y_n|\} \leq M_{xz} + M_{zy}$$

$$\sup\{|x_n - y_n|\} \leq \sup\{\dots\} \leq M_{xz} + M_{zy}$$

$$\Downarrow$$

$$\sup\{|x_n - y_n|\} \leq M_{xz} + M_{zy}\ ;\ \forall n \in \mathbb{N}$$

$$d(\{x_n\}, \{y_n\}) \leq d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$$

$\square$

6. Če je V v.p. nad $\mathbb{R}$ (ali $\mathbb{C}$), je funkcija $\|\cdot\| : V \to \mathbb{R}$ imenovana norma, če izpolnjuje naslednje pogoje:

(1) $\forall x \in V:\ \|x\| \geq 0 \ \wedge\ \|x\| = 0 \Leftrightarrow x = 0$ (x je vektor...?)

(2) $\|\alpha x\| = |\alpha|\,\|x\|$ za $\forall \alpha \in \mathbb{R}$ in $\forall x \in V$

(3) $\|x + y\| \leq \|x\| + \|y\|$; $\forall x, y \in V$

Pokaži, da je v tem primeru $d(x,y) = \|x - y\|$ metrika nad V.

(1) $d(x,y) \geq 0$: $\|x - y\| \geq 0$ ... po def. norme je ta izraz ≥ 0 in $= 0$, ko $\vec{x} = \vec{y}$. $\blacksquare$

(2)

(3) $d(x,y) \leq d(x,z) + d(z,y)$
$\|x - y\| \leq \|x - z\| + \|z - y\|$

Velja:

$$\|x - z\| = \|x + (-z)\| \leq \|x\| + \|\overset{-1}{-}z\| = \|x\| + |-1|\,\|z\| = \|x\| + \|z\|$$

$*$ analogno za $\|z - y\|$ in $\|x - y\|$.

Torej:

$$\|x\| \geq \|x - z\| - \|z\| \quad \text{in} \quad \|x\| \geq \|x - y\| - \|y\|$$

$$\|x-y\| = \|x-z+z-y\| = \|(x-z)+(z-y)\| \leq \|x-z\| + \|z-y\|$$

$$\Rightarrow \|x-y\| \leq \|x-z\| + \|z-y\|$$

$$d(x,y) \leq d(x,z) + d(z,y) \quad \square$$

7. Če so $x_1, x_2, \ldots, x_n$ točke v metričnem prostoru (X,d), pokaži da velja:

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

Uh... okej, torej, velja trikotniška neenakost, ker je metrika (predp.):

$$d(x_1, x_n) \leq d(x_1, x_i) + d(x_i, x_n) \quad \text{za nek } i \in \{1, \ldots, n\}$$

Torej, uhm:

$$d(x_1, x_n) \leq d(x_1, x_i) + d(x_i, x_n)$$

$$x_{n-1}$$

(1) $d(x_1, x_n) \leq d(x_1, x_{n-1}) + d(x_{n-1}, x_n)$

(2) $d(x_1, x_{n-1}) \leq d(x_1, x_{n-2}) + d(x_{n-2}, x_{n-1})$

(1) ∧ (2): $d(x_1, x_n) \leq d(x_1, x_{n-1}) + d(x_{n-1}, x_n) \leq d(x_1, x_{n-2}) + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)$

$$\Rightarrow d(x_1, x_n) \leq d(x_1, x_{n-2}) + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)$$

To nadaljujemo, dokler ne dobimo $x_{n-k} = x_2$, za $k \in [1, n-1]$. Takrat dobimo:

$$d(x_1, x_n) \leq d(x_1, x_{n-(n-1)}) + d(x_{n-(n-1)}, x_{n-(n-2)}) + \cdots + d(x_{n-1}, x_n)$$

$$d(x_1, x_n) \leq \underbrace{d(x_1, x_1)}_{=0} + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)$$

$$\Downarrow$$

$$d(x_1, x_n) \leq d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \quad \square$$

8. Naj bosta d_1 in d_2 metriki nad X. Pokaži, da je tudi d metrika nad X:

$$d(x,y) = d_1(x,y) + d_2(x,y)$$

(1) $d(x,y) \geq 0$: $d_1(x,y) \geq 0$ in $d_2(x,y) \geq 0$, zato $d_1(x,y) + d_2(x,y) \geq 0$.

(2) $d(x,y) = 0$: $d_1(x,x) = 0$ in $d_2(x,x) = 0$ takrat ko $x = y$.

(3) $d(x,y) \leq d(x,z) + d(z,y)$

$$d_1(x,y) + d_2(x,y) \leq d_1(x,z) + d_2(x,z) + d_1(z,y) + d_2(z,y)$$

$$d_1(x,y) \leq d_1(x,z) + d_1(z,y) \quad (1)$$

in $\qquad$ ker sta ... metriki

$$d_2(x,y) \leq d_2(x,z) + d_2(z,y) \quad (2)$$

(1) $\wedge$ (2):

$$d_1(x,y) \leq d_1(x,z) + d_1(z,y)$$

$$d_1(x,y) + d_2(x,y) \leq d_1(x,z) + d_1(z,y) + d_2(x,z) + d_2(z,y)$$

$$d(x,y) \qquad d(x,z) \qquad d(z,y)$$

$\square$

9. Naj bosta (X, d_X) in (Y, d_Y) metrična prostora. Definiramo funkcijo:

$$d : (X \times Y) \times (X \times Y) \to \mathbb{R}$$

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

Pokaži, da je d metrika nad $X \times Y$.