3.2 – CONVERGENCE AND CONTINUITY

Def. (3.2.1) Let (X,d) be a metric space. A sequence $\{x_n\} \subseteq X$ converges to $a \in X$ if and only if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N: \quad d(x_n, a) < \varepsilon$$

$$\Leftrightarrow \quad \lim_{n \to \infty} x_n = a$$

Lemma (3.2.2) A sequence $\{x_n\}$ in a m.s. (X,d) converges to $a \in X$, if and only if:

$$\lim_{n \to \infty} d(x_n, a) = 0$$

Proposition (3.2.3) A sequence in a metric space cannot converge to more than one point.

Proof: We must show that $\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} x_n = b$ only if $a = b$ (they're equal). If we take the triangle inequality to write:

$$d(a,b) \leq d(x_n, a) + d(x_n, b)$$

and take their limits:

$$\lim_{n \to \infty} d(a,b) \leq \lim_{n \to \infty} d(x_n, a) + \lim_{n \to \infty} d(x_n, b)$$

this is a constant

$\Downarrow$

$$d(a,b) \leq \lim_{n \to \infty} d(x_n, a) + \lim_{n \to \infty} d(x_n, b)$$

from lemma (3.2.3) this is equal to 0

$\Downarrow$

$$d(a,b) \leq 0 + 0$$

$$d(a,b) \leq 0$$

since d is a metric, it cannot be less than 0, leaving us with the equality:

$$d(a,b) = 0$$

and we know, again since d is a metric, that this is possible only when $a = b$. $\square$

$\Downarrow$

A SEQUENCE IN A METRIC SPACE CAN THEREFORE CONVERGE ONLY TO ONE POINT

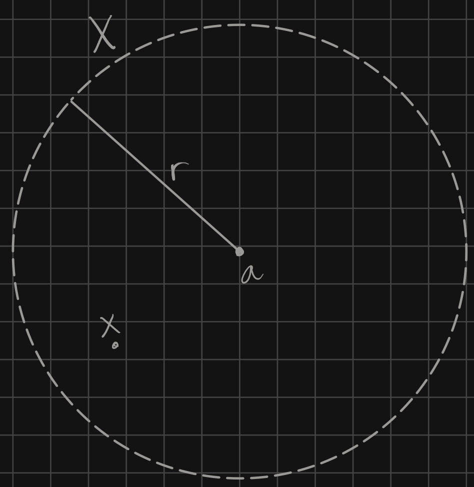
Observe how we used definitions from 3.1.1. As the theory develops, we'll have more and more tools to work with.

We can also phrase the notion of convergence in geometric terms:

If a is an element of a metric space X, and $r>0$, then the (open) ball centered at a with radius r is the set:

$$B_r(a) = \{x \in X ; d(x,a) < r\}$$

So saying $x \in B_r(a)$ is equivalent to saying $d(x,a) < r$.



The def. of convergence can now be rephrased by saying that $\{x_n\}$ converges to a if the elements in the sequence eventually end up inside ANY ball $B_\varepsilon(a)$ centered at a:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : x_n \in B_\varepsilon(a) \Leftrightarrow d(x_n, a) < \varepsilon \Leftrightarrow \lim_{n\to\infty} d(x_n, a) = 0 \Leftrightarrow \lim_{n\to\infty} x_n = a$$

Next we define the continuity of functions from one metric space to another, by following the same principles as for $\mathbb{R}^m$ (section 2.1).