Analiza III - Funkcije več spremenljivk, študijsko leto 2017/2018

2. vaje - 11.10.2017

METRIKA

1. Na $\mathbb{R}^2$ je dana matrika

$$d_p\left((x_1,y_1),(x_2,y_2)\right)=\sqrt[p]{|x_1-x_2|^p+|y_1-y_2|^p},\ p\geq 1.$$

   (a) Narišite enotske krogle za $p=1$, $p=2$ in $p=\infty$.

   (b) Narišite elipso v d_1 metriki.

2. Naj bo $g:\mathbb{R}\to\mathbb{R}$,

$$g(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases},\qquad d(x,y)=\begin{cases}|x-y|, & g(x)=g(y)\\ |x-y|+1, & g(x)\neq g(y)\end{cases}.$$

   (a) Dokažite, da je d metrika.

   (b) Skicirajte korglo $K_{\frac{3}{2}}(0)$.

3. Naj bo (M,d) metrični prostor,

$$d(m_1,m_2)=\begin{cases}1; & m_1\neq m_2\\ 0; & m_1=m_2\end{cases}.$$

   in $m\in M$ poljuben. Določite naslednje množice:

   (a) $K_{\frac{1}{2}}(m)$, $K_1(m)$, $K_2(m)$,

   (b) $\overline{K}_{\frac{1}{2}}(m)$, $\overline{K}_1(m)$, $\overline{K}_2(m)$,

   (c) $S_{\frac{1}{2}}(m)$, $S_1(m)$, $S_2(m)$.

4. Ugotovite, ali so naslednje množice (v običajni evklidski metriki) odprte, zaprte ali nič od tega:

   (a) $A=\left\{\frac{1}{n};\, n\in\mathbb{N}\right\}\subset\mathbb{R}$

   (b) $\mathbb{Z}\subset\mathbb{R}$

   (c) $\mathbb{Q}\cap(0,1)\subset\mathbb{R}$

   (d) $\mathbb{R}\times\{0\}\subset\mathbb{R}^2$

   (e) $\left\{(x,y);\,(x-1)^2+(y-1)^2<1\right\}\subset\mathbb{R}^2$

   (f) $\left\{(x,y);\,(x-1)^2+(y-1)^2\leq 1\right\}\subset\mathbb{R}^2$

   (g) $\left\{(x,y);\,(x-1)^2+(y-1)^2\leq 1,\ y<1\right\}\subset\mathbb{R}^2$

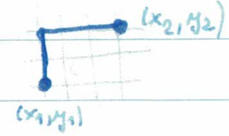
Metrike ("oddaljenosti") v $\mathbb{R}^2$:

(euclidean metrics)

$$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \rightarrow \text{evklidska metrika}$$

$$d_p((x_1,y_1),(x_2,y_2)) = \sqrt[p]{|x_1-x_2|^p+|y_1-y_2|^p} \quad (\text{metrika za } p \geq 1)$$

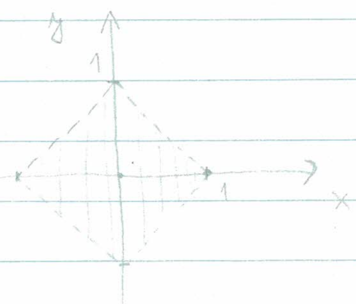
Primeri: Enotske krogle za $p=1$, $p=2$ in $p=\infty$



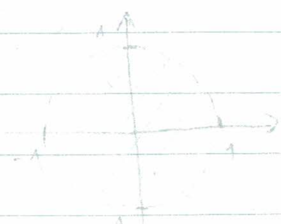
$p=1$: $d_1((x_1,y_1),(x_2,y_2)) = |x_1-x_2|+|y_1-y_2|$

$$K_1((0,0)) = \{(x,y)\in\mathbb{R}^2 \mid d((x,y),(0,0))<1\}$$

$$d((x,y),(0,0)) = |x-0|+|y-0| = |x|+|y| < 1$$

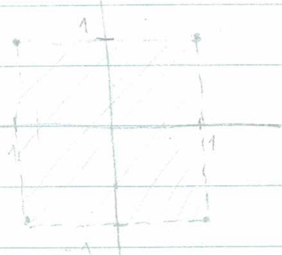


$p=2$: $d(T_1,T_2) = \sqrt{|x_1-x_2|^2+|y_1-y_2|^2}$



$\leftarrow$ $K_1((0,0))$: $d((x,y),(0,0)) = \sqrt{x^2+y^2} < 1$

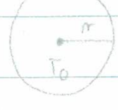
$p\to\infty$: $d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$



$K_1((0,0))$: $d((x,y),(0,0)) = \max\{|x_1|,|y_1|\} < 1$

Krogle: d_1:

d_2:



d_∞:

Naloga: $g: \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \notin \mathbb{Q} \end{cases}$, $d(x,y) = \begin{cases} |x-y| & ; g(x)=g(y) \\ |x-y|+1 & ; g(x) \neq g(y) \end{cases}$

a) Dokaži, da je d metrika. (DN!)

b) Skiciraj kroglo $K_{\frac{3}{2}}(0)$.

a) M1) $d(x,y) \geq 0$ ✓

$\underline{d(x,y)=0 \Leftrightarrow x=y}$

$(\Rightarrow)$ $d(x,y)=0 \underset{|x-y|+1 \neq 0}{\Rightarrow} d(x,y)=|x-y|=0 \Rightarrow |x-y|=0 \Rightarrow x=y$

$(\Leftarrow)$ $x=y \Rightarrow g(x)=g(y) \Rightarrow d(x,y) = \underbrace{|x-y|}_{x=y} = 0$

M2) $\underline{d(x,y) \stackrel{?}{=} d(y,x)}$

1.) $g(x)=g(y)$: $d(x,y)=|x-y|=|y-x|=d(y,x)$

2.) $g(x) \neq g(y)$: $d(x,y)=|x-y|+1=|y-x|+1=d(y,x)$

M3) $\underline{d(x,y) \leq d(x,z)+d(z,y)}$

1.) $g(x)=g(y)$

$d(x,y)=|x-y|$

$d(x,z)=\begin{cases} |x-z| \\ |x-z|+1 \end{cases}$

$d(z,y)=\begin{cases} |z-y| \\ |z-y|+1 \end{cases}$

$\Big\}$ $\underset{\text{običajna trikotniška neenakost}}{|x-y| \leq} |x-z|+|y-z| <$ ostale kombinacije

2.) $g(x) \neq g(y)$

$d(x,y)=|x-y|+1 \leq |x-z|+|z-y|+1$ ← na desni se pojavi vsaj en člen z 1

če $d(x,z)=|x-z| \wedge d(z,y)=|z-y| \Rightarrow g(x)=g(z) \wedge g(z)=g(y)$

$\Rightarrow g(x)=g(y)$ ✱ ker tega nimamo je sigurno 1 na koncu

b) $K_{\frac{3}{2}}(0)=\{x \mid d(x,0)<\frac{3}{2}\}$, $d(x,0)=\begin{cases} |x| & ; g(0)=g(x) \Leftrightarrow x \in \mathbb{Q} \\ |x|+1 & ; 1 \neq g(x) \Leftrightarrow x \notin \mathbb{Q} \end{cases}$

1) $x \in \mathbb{Q}$: $d(x,0)=|x|<\frac{3}{2}$

2) $x \notin \mathbb{Q}$: $d(x,0)=|x|+1<\frac{3}{2} \Rightarrow |x|<\frac{1}{2}$

$A_\alpha,\ \alpha \in J$ odprta v (M,d) $\Rightarrow$ $\bigcup\limits_{\alpha \in J} A_\alpha$ odprta v (M,d)

$A_1, \dots, A_n$ odprte v (M,d) $\Rightarrow$ $\bigcap\limits_{k=1}^{n} A_k$ odprta v (M,d)

$A_1, \dots, A_n$ zaprte v (M,d) $\Rightarrow$ $\bigcup\limits_{k=1}^{n} A_k$ zaprta v (M,d)

$A_\alpha,\ \alpha \in J$ zaprte v (M,d) $\Rightarrow$ $\bigcap\limits_{\alpha \in J} A_\alpha$ zaprta v (M,d)

Množica A je v (M,d) OMEJENA $\Leftrightarrow$ $\exists\, R>0$ in $a \in M \ni: A \subseteq K_R(a)$

(ko je vsebovana v neki krogli)

-||- NEOMEJENA $\Leftrightarrow$ sicer

$S_r(a) = \{x \in M \mid d(x,a) = r\}$ ... sfera s središčem v a in polmerom r.

Npr: (M,d) metrični prostor, $d(m_1,m_2) = \begin{cases} 1 & ; m_1 \neq m_2 \\ 0 & ; m_1 = m_2 \end{cases}$, $m \in M$ poljuben.

a) $K_{\frac{1}{2}}(m) = \{m\}$

b) $K_1(m) = \{m\}$

c) $K_2(m) = M$

d) $\overline{K}_{\frac{1}{2}}(m) = \{m\}$

e) $\overline{K}_1(m) = M$

f) $\overline{K}_2(m) = M$

f) $S_{\frac{1}{2}}(m) = \emptyset$

g) $S_1(m) = M \setminus \{m\}$

h) $S_2(m) = \emptyset$

Naloga: Ugotovi ali so naslednje množice odprte / zaprte:

a) $A = \{\frac{1}{n};\ n \in \mathbb{N}\} \subset \mathbb{R}$

b) $\mathbb{Z} \subset \mathbb{R}$

c) $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$

d) $\{(x,y);\ (x-1)^2 + (y-1)^2 < 1\} \subset \mathbb{R}^2$

e) $\{(x,y);\ (x-1)^2 + (y-1)^2 \leq 1\} \subset \mathbb{R}^2$

f) $\{(x,y);\ (x-1)^2 + (y-1)^2 \leq 1,\ y < 1\} \subseteq \mathbb{R}^2$

g) $K_{\frac{1}{2}}(\sin) \subset (\mathcal{C}[0,1], d_\infty)$

h) polinomi $\subset (\mathcal{C}[0,1], d_\infty)$

i) $\{$funkcije $f;\ f(0) > 0\} \subset (\mathcal{C}[0,1], d_\infty)$

a) 

$\mathbb{R}$ ← krogle so intervali $(a-r, a+r)$

A ni odprta množica, ker ne vsebuje nobenega intervala.

Zaprta? $a \in A \Rightarrow a \in \partial A$, ker $K_r(a) \cap A \ni a$ in $K_r(a) \not\subset A$

$\partial A = A \cup \{0\}$

$0 \in \partial A$: $K_r(0) \cap A^c \ni 0$

$K_r(0) \cap A \neq \emptyset$

$0 \notin A$ in $0 \in \partial A \Rightarrow A$ NI zaprta

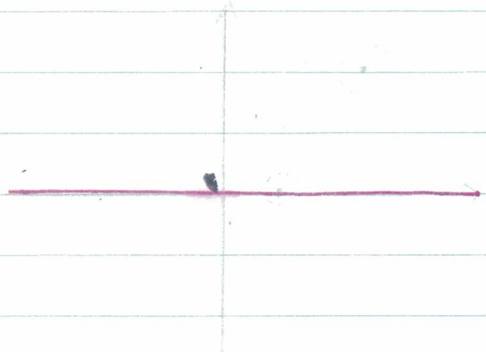
b) 

← krogle v $\mathbb{R}$ so intervali

$\mathbb{Z}$ ni odprta (ne vsebuje nobenega intervala)

$x \notin \mathbb{Z} \Rightarrow x \notin \partial \mathbb{Z}$

Velja: $\partial \mathbb{Z} \subset \mathbb{Z}$, $\partial \mathbb{Z} = \mathbb{Z}$ $\Rightarrow$ $\mathbb{Z}$ je zaprta c)



$\mathbb{R}\times\{0\} = \{(x,y);\ x\in\mathbb{R},\ y=0\}$

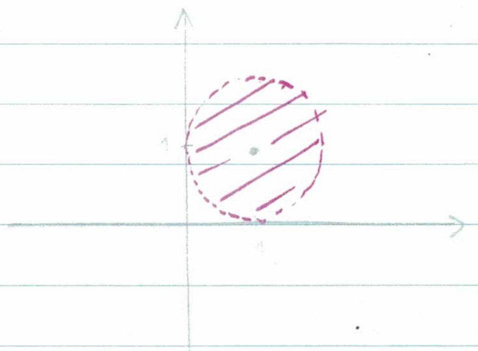
V $\mathbb{R}^2$ so krogle krogi.

↘ Ni odprta, ker ne vsebuje nobenega kroga.

$x \notin A \Rightarrow x \notin \partial A \Rightarrow$ A je zaprta $\quad \partial A \subset A$

$A \subset \partial A \longrightarrow \mathring{A} = \partial A$

d)



$A = K_1((1,1))$

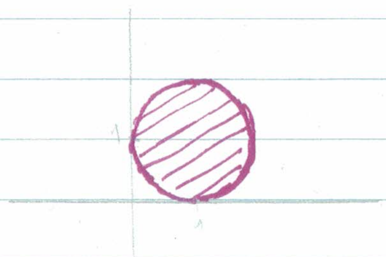
Trditev: Odprta krogla je odprta množica. ↗ $v(M,d)$

$\Rightarrow$ A je odprta

Ali je zaprta? $(1,0) \in \partial A \setminus A \Rightarrow$ A ni zaprta

najdeš točko, ki je na robu, ni pa v A-ju

e)



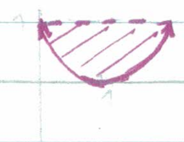
$A = \overline{K}_1((1,1))$

Trditev: Zaprta krogla v (M,d) je zaprta množica.

$\Rightarrow$ A je zaprta množica.

Ali je odprta? $(1,0) \in A,\ (1,0) \notin \mathring{A} \Rightarrow$ A ni odprta.

f)



odprtost: $(1,0) \in A,\ (1,0) \notin \mathring{A} \Rightarrow$ A ni odprta

zaprtost: $(1,1) \in \partial A,\ (1,1) \notin A \Rightarrow$ A ni zaprta

$\tilde{c}$) $\mathbb{Q} \cap (0,1) \subset \mathbb{R}$:

$\frac{1}{2} \in \underbrace{\{\mathbb{Q}\cap(0,1)\}}^{=C}$, toda $K_\varepsilon(\frac{1}{2}) \not\subseteq C \Rightarrow C$ NI odprta

$\frac{\sqrt{2}}{2} \notin C$, toda $\forall \varepsilon>0:\ K_\varepsilon(\frac{\sqrt{2}}{2}) \cap C \neq \emptyset \Rightarrow C$ NI zaprta