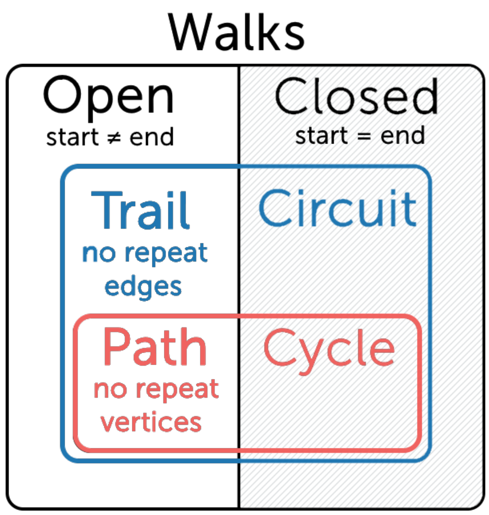
Graphs, Graph Algorithms, and Optimization
Problem set 5

W 24/25

GitHub https://github.com/JanPastorek/1-AIN-413-22-Graphs

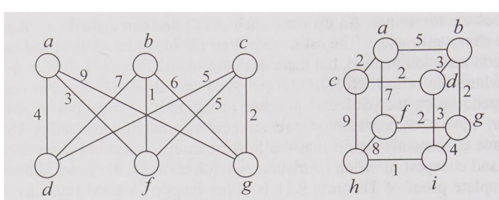


Problem 0. [Any questions?]

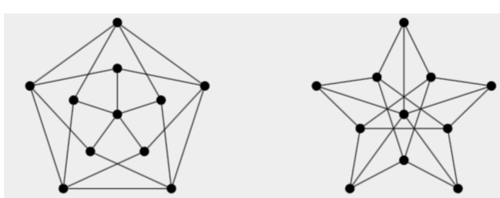
Is there anything unclear from the lectures or about the home project?

Problem 1. [Eulerian cycles]

(a) Determine whether these graphs are Eulerian and find an Euler tour if one exists. (if it is not Eulerian, determine whether there is at least Euler path, and which one it is)



(b) Determine the girth and circumference of these graphs



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1. For what values of n does the graph K_n contain an Euler trail? An Euler tour? A Hamilton path? A Hamilton cycle?

2. (a) For what values of m and n does the complete bipartite graph $K_{m,n}$ contain an Euler tour?

   (b) Determine the length of the longest path and the longest cycle in $K_{m,n}$, for all m, n.

3. (a) Find a graph such that every vertex has even degree but there is no Euler tour.

   (b) Find a disconnected graph that has an Euler tour.

4. Let G be a connected graph that has an Euler tour. Prove or disprove the following statements.

   (a) If G is bipartite then it has an even number of edges.

   (b) If G has an even number of vertices then it has an even number of edges.

   (c) For edges e and f sharing a vertex, G has an Euler tour in which e and f appear consecutively.

5. Show that if $k > 0$ then the edge set of any connected graph with $2k$ odd-degree vertices can be split into k trails.

6. Prove that in any connected graph G, there is a walk that uses each edge exactly twice.

7. Let G be a connected graph with an even number of edges such that all the degrees are even. Prove that we can color each of the edges of G red or blue in such a way that every vertex has the same number of red and blue edges touching it.

8. Prove that the following Fleury's algorithm finds an Euler tour or an Euler trail if it is possible. Analyse the time complexity of this algorithm.

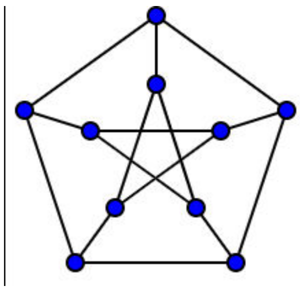
   (a) If there are 0 odd vertices, start anywhere. If there are 2 odd vertices, start at one of them.

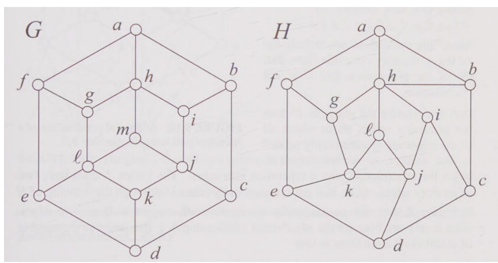
   (b) At each step choose the next edge in the path to be one whose deletion would not disconnect the graph, unless there is no such edge, in which case pick the remaining edge left at the current vertex.

   (c) Stop when you run out of edges.

Problem 2. [Hamiltonian cycles]

(a) Determine if the following graphs are Hamiltonian. If they are, find a Hamiltonian tour on these graphs. If no, is there a Hamiltonian path?





(b) Use Ore's theorem to give a short proof of the fact that any n-vertex graph G with more than $\binom{n-1}{2} + 1$ has a Hamilton cycle.