Graphs, Graph Algorithms, and Optimization
Problem set 6

W 24/25

- **GitHub** https://github.com/JanPastorek/1-AIN-413-22-Graphs
- **New Home project is already on Moodle!**

Problem -1. [Previous week - TSP]

Stanoyevitch 9.2.35 (a) Fix a positive integer n. Construct a TSP for which the nearest neighbor heuristic produces a tour of weight greater than 10^n times the weight of an optimal tour.

(b) Repeat Part (a) for the nearest insertion method.

(c) Repeat Part (a) for the furthest insertion method.

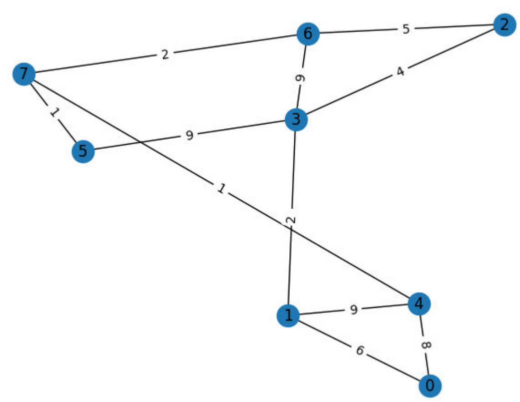
(d) Repeat Part (a) for the cheapest insertion method.

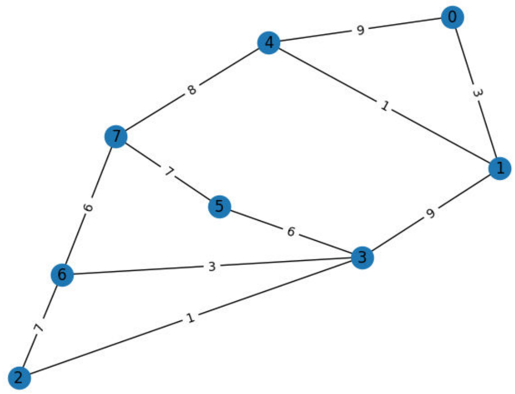
- program these methods

Stanoyevitch 9.2.10 (Random TSP Generator) Write a program that will randomly generate Euclidean distance TSPs and that has the following syntax: [W, Points] = TSPRand(n,d). Here, the first input n is an integer greater than 1, and the second optional input d is a positive number (default value 10). The program will generate n points randomly distributed in the square $0 \leq x, y \leq d$. These points will be the vertices of the complete graph for the TSP. The first output variable is the $n \times n$ edge-weight matrix W whose entries give the Euclidean distance between corresponding pairs of points. The second output variable, Points is an $n \times 2$ matrix whose rows give the x- and y-coordinates of the vertices.

Problem 0. [Any questions?]

Is there anything unclear from the lectures or about the home project?





Problem 1. [Simulating algorithms]

- (a) Simulate each shortest-path algorithm on the graphs above .
- (b) How to read the shortest paths from the output of the algorithms? Provide an idea of the algorithm.

Dijkstra algorithm

```
def dijkstra(v1):
    # in array D are current distances of all vertices from v1
    D[all vertices] = inf      # initialize to infinity & and continuously decrease
    D[v1] = 0
    queue = priority queue for all vertices using keys of D
    while queue:
        v1 = queue.remove_min()
        for v2 in neighbours(v1):
            if v2 in queue:           # v2 was not yet processed
                if D[v1] + weight(v1, v2) < D[v2]: # here is why weights can NOT be negativ
                    D[v2] = D[v1] + weight(v1, v2)
                    change the key D[v2] in queue for v2
    return D     # array of distances of all vertices from v1
```

Johnson algorithm

Idea of the algortihm:

Step 1: adding a base vertex

Step 2: Reweighting the edges using Bellman algorithm to positive - if a certain path is the shortest path between those vertices before reweighting, it must also be the shortest path between those vertices after reweighting

Step 3: finding all pairs shortest path on the reweighted Graph (, and reweighting back)**

```
JOHNSON(G, w)
compute G′, where G′.V = G.V ∪ {s},
      G′.E = G.E ∪ {(s, v) : v ∈ G.V}, and
      w(s, v) = 0 for all v ∈ G.V
if BELLMAN-FORD(G′, w, s) == FALSE
      print "the input graph contains a negative-weight cycle"
else for each vertex v ∈ G′.V
            set h(v) to the value of δ(s, v)
                  computed by the Bellman-Ford algorithm
      for each edge (u, v) ∈ G′.E
            ŵ(u, v) = w(u, v) + h(u) − h(v)
      let D = (d_uv) be a new n × n matrix
      for each vertex u ∈ G.V
            run DIJKSTRA(G, ŵ, u) to compute δ̂(u, v) for all v ∈ G.V
            for each vertex v ∈ G.V
                  d_uv = δ̂(u, v) + h(v) − h(u)
      return D
```

- (b) (Cormen) 25.3-2 What is the purpose of adding the new vertex s to V , yielding V'?

Problem 2. [Choice of algorithms]

- (a) Apart from finding shortest paths, what can you use these algorithms for?
- (b) Decide which algorithm would you use for these graphs (consider both all-pairs-shortest-path problem APSP, and single-source-shortest-path SSSP problem)? Make timed simulations on random graphs of that many vertices and edges with all the algorithms in Python.

1. **Small Sparse Graph**:
   - Description: A graph with 20 vertices and 30 edges.
   - Characteristics: Sparse connectivity, no negative weights, no cycles.
2. **Small Dense Graph**:
   - Description: A graph with 20 vertices and 190 edges (a nearly complete graph).
   - Characteristics: Dense connectivity, no negative weights, possible cycles.
3. **Medium Graph with Negative Weights**:
   - Description: A graph with 100 vertices and 300 edges, including some negative weights.
   - Characteristics: Medium size, sparse, negative weights, possible cycles.
4. **Large Sparse Graph**:
   - Description: A graph with 1000 vertices and 2000 edges.

- Characteristics: Large, sparse, no negative weights, possible cycles.

5. **Graph with Many Negative Cycles**:
   - Description: A graph with 50 vertices and 120 edges, several negative cycles.
   - Characteristics: Medium size, negative cycles.
6. **Directed Acyclic Graph (DAG)**:
   - Description: A DAG with 200 vertices and 400 edges.
   - Characteristics: No cycles, directed edges, no negative weights.
7. **Weighted Tree**:
   - Description: A tree graph with 100 vertices, 99 edges.
   - Characteristics: Tree structure (acyclic), potentially negative weights.
8. **Large Dense Graph**:
   - Description: A graph with 1000 vertices and 499,500 edges (a nearly complete graph).
   - Characteristics: Large, dense, no negative weights, many cycles.
9. **Random Graph with Negative Weights**:
   - Description: A graph with 500 vertices and 1000 edges, random distribution of negative weights.
   - Characteristics: Medium size, sparse, negative weights, possible cycles.
10. **DiGraph**:
    - Description: A DiGraph with 500 vertices and 1000 edges

Runtime:

Bellman-Ford $O(|V|.|E|)$

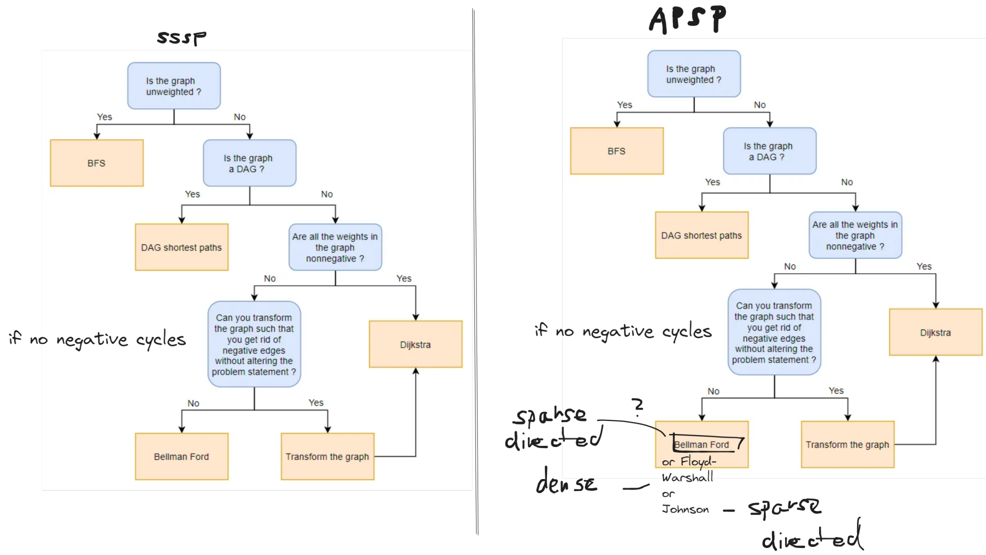
Dijkstra's (with list) $O(|V|^2)$

Dijkstra's (with heap) $O(|V| \log_2(|V|))$

Floyd-Warshall $O(|V|^3)$

Johnson's $O(|E| \cdot |V| + |V|^2 \cdot \log_2(|V|)) = O(\text{Bellman-Ford} + \text{V} \cdot \text{Dijkstra})$

Hints:



- **For graphs with negative weight edges, the single source shortest path problem needs Bellman-Ford to succeed**. Otherwise **Dijkstra**.
- For **dense graphs and the all-pairs problem, Floyd-Warshall should be used**. Negative weights are allowed.
- **For sparse graphs and the all-pairs problem, it might be obvious to use Johnson's algorithm**.
  - However, **if there are no negative edge weights, then it is actually better to use Dijkstra's algorithm with binary heaps in the implementation**. Running Dijsktra's from each vertex will yield a better result.
- **Bellman and Johnson works on directed only**. So either you have to change your graph or choose a diff. algorithm.