Graphs, Graph Algorithms, and Optimization
Problem set 9

W 24/25

- **GitHub** https://github.com/JanPastorek/1-AIN-413-22-Graphs

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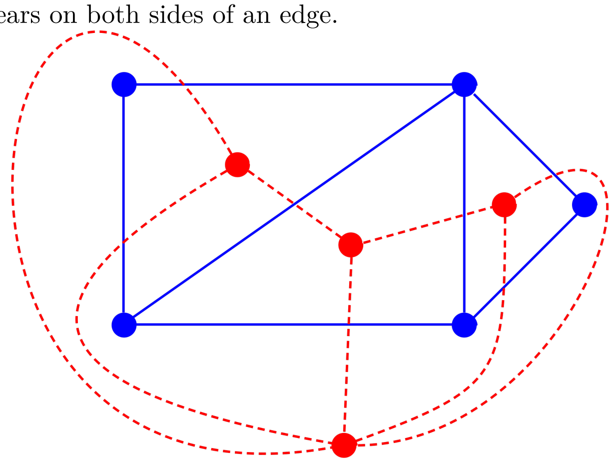
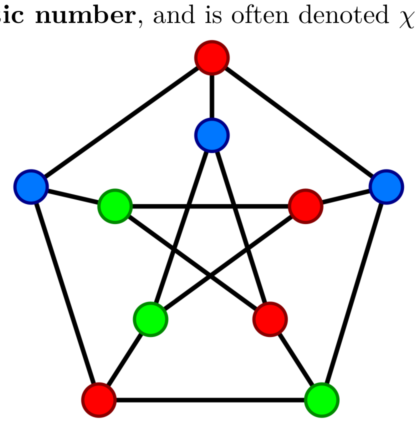
Today's exercises are based on:

- Stanoyevitch, A. (2011). *Discrete structures with contemporary applications*. CRC Press, Taylor & Francis Group.
- West, D. B. (2001). *Introduction to graph theory* (2nd ed). Prentice Hall.

Recall:

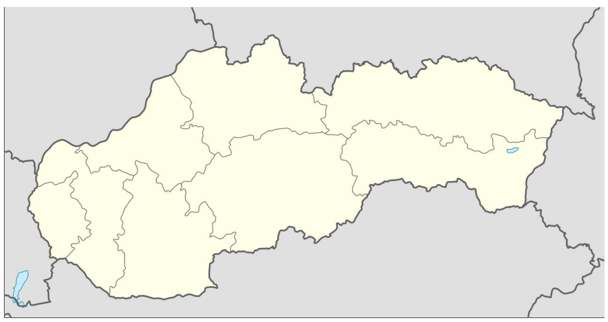
- graph G is a **planar graph**, if it can be drawn in a plane without intersection of edges.
- **dual graph** of a planar graph G is a graph that has a vertex for each face of G. The dual graph has an edge for each pair of faces in G that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.
  - 
- The smallest number of colors needed to color a graph G is called its **chromatic number**, and is often denoted $\chi(G)$.
  - 

Problem 0. [Any questions?]

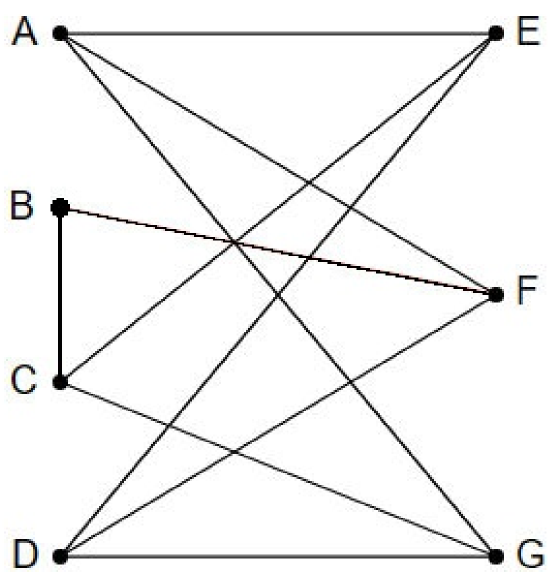
Is there anything unclear from the lectures?

Problem 1. [Planar graphs]

1. Construct a planar graph out of this map of Slovakia (ignore the boundaries of other states)



2. Construct dual graph to the planar graph from 1.
3. Is this graph planar?



3. Can you think of any graph, whose dual graph is isomorphic to the original graph? Hint: "regularity".

Problem 2. [Reasoning about planar graphs]

6.1.23. Theorem. If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Proof: It suffices to consider connected graphs; otherwise we could add edges. Euler's Formula will relate $n(G)$ and $e(G)$ if we can dispose of f.

Proposition 6.1.13 provides an inequality between e and f. Every face boundary in a simple graph contains at least three edges (if $n(G) \geq 3$). Letting $\{f_i\}$ be the list of face lengths, this yields $2e = \sum f_i \geq 3f$. Substituting into $n - e + f = 2$ yields $e \leq 3n - 6$.

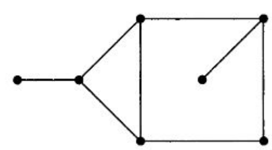
1. Use the theorem to prove that $K_5, K_{3,3}$ are nonplanar.
2. Solve **6.1.1, 6.1.2**

6.1.1. (−) Prove or disprove:
a) Every subgraph of a planar graph is planar.
b) Every subgraph of a nonplanar graph is nonplanar.

6.1.2. (−) Show that the graphs formed by deleting one edge from K_5 and $K_{3,3}$ are planar.

6.1.3. (−) Determine all r, s such that $K_{r,s}$ is planar.

6.1.4. (−) Determine the number of isomorphism classes of planar graphs that can be obtained as planar duals of the graph below



6.1.5. (−) Prove that a plane graph has a cut-vertex if and only if its dual has a cut-vertex.

6.1.6. (−) Prove that a plane graph is 2-connected if and only if for every face, the bounding walk is a cycle.

6.1.7. (−) A **maximal outerplanar graph** is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph. Let G be a maximal outerplanar graph with at least three vertices. Prove that G is 2-connected.

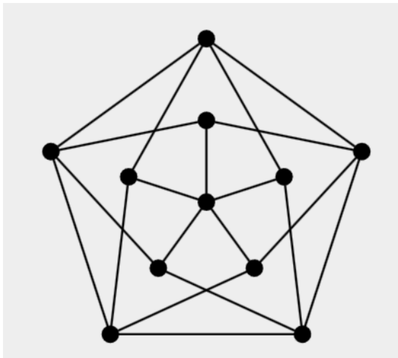
6.1.8. (−) Prove that every simple planar graph has a vertex of degree at most 5.

6.1.9. (−) Use Theorem 6.1.23 to prove that every simple planar graph with fewer than 12 vertices has a vertex of degree at most 4.

6.1.10. (−) Prove or disprove: There is no simple bipartite planar graph with minimum degree at least 4.

Problem 3. [Chromatic numbers]

1. Determine the chromatic number of the following graph.



1. If I make disjoint union of graphs, what is their chromatic number?