Counting and Probability

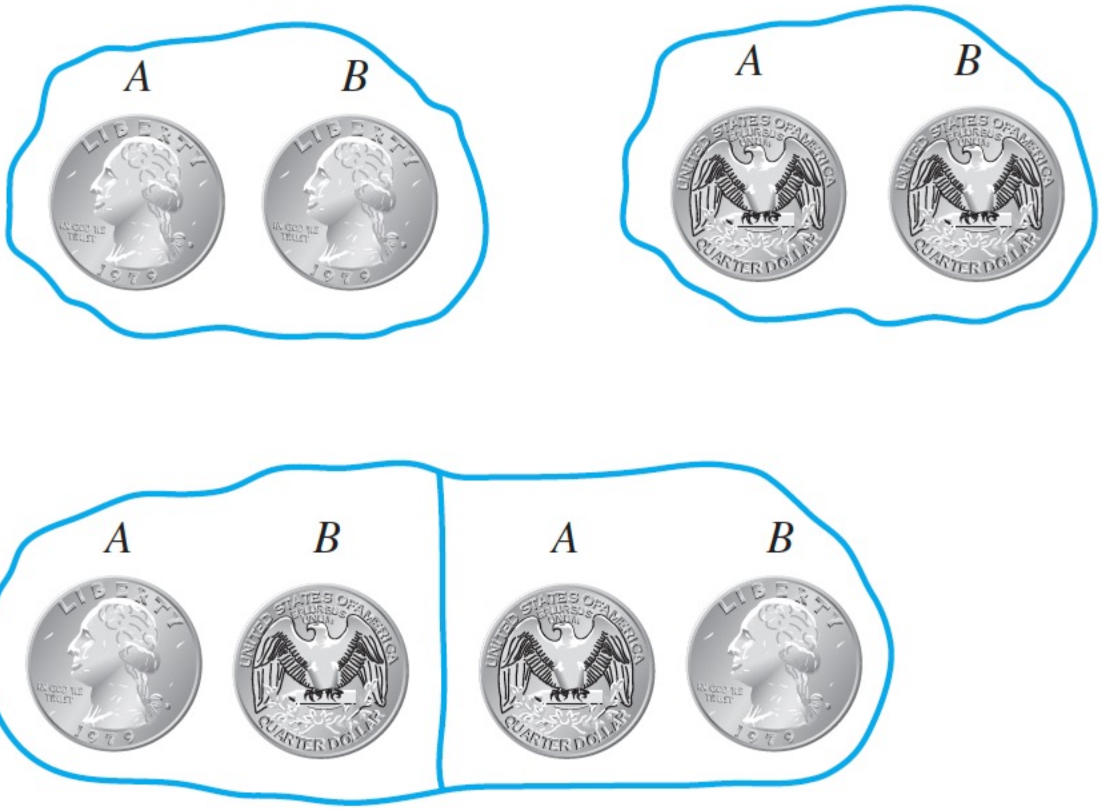
CSX2008 Mathematics Foundation for Computer Science

Department of Computer Science
Vincent Mary School of Science and Technology
Assumption University

Session Outline

- Probability
- Permutation
- Combination

Tossing Two Coins



Definition

- A random process, when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.
- A sample space is the set of all possible outcomes of a random process or experiment.
- An event is a subset of a sample space.
- In case an experiment has finitely many outcomes and all outcomes are equally likely to occur, the probability of an event (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes.

Equally Likely Probability

- If *S* is a finite sample space in which all outcomes are equally likely and *E* is an event in *S*, then the probability of *E*, denoted P(E), is

$$P(E) = \frac{N(E)}{N(S)}$$

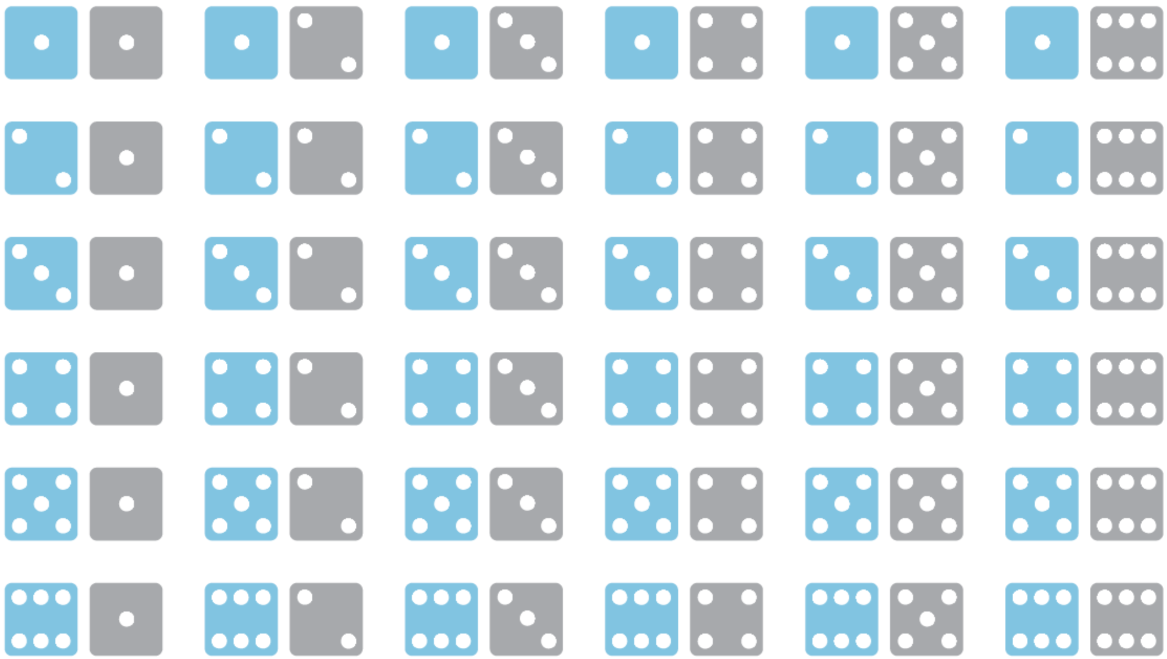
- where *N(A)* denotes the number of elements in *A* (for any finite set A).

Example 1: Deck of Cards

- An ordinary deck of cards contains 52 cards divided into four suits.
  - Diamonds ♦
  - Hearts ♥
  - Clubs ♣
  - Spades ♠
- Each suit contains 13 cards:
  - 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and A
  - J, Q, K are called face cards.
- What is the sample space of outcomes?
- What is the event that the chosen card is a black face card?
- What is the probability that the chosen card is a black face card?

Example 2: Rolling a Pair of Dice

- A die is one of a pair of dice.
  - A cube with six sides, each containing from one to six dots, called pips.
- Suppose we roll a blue die and a gray die together, how many outcomes will there be?



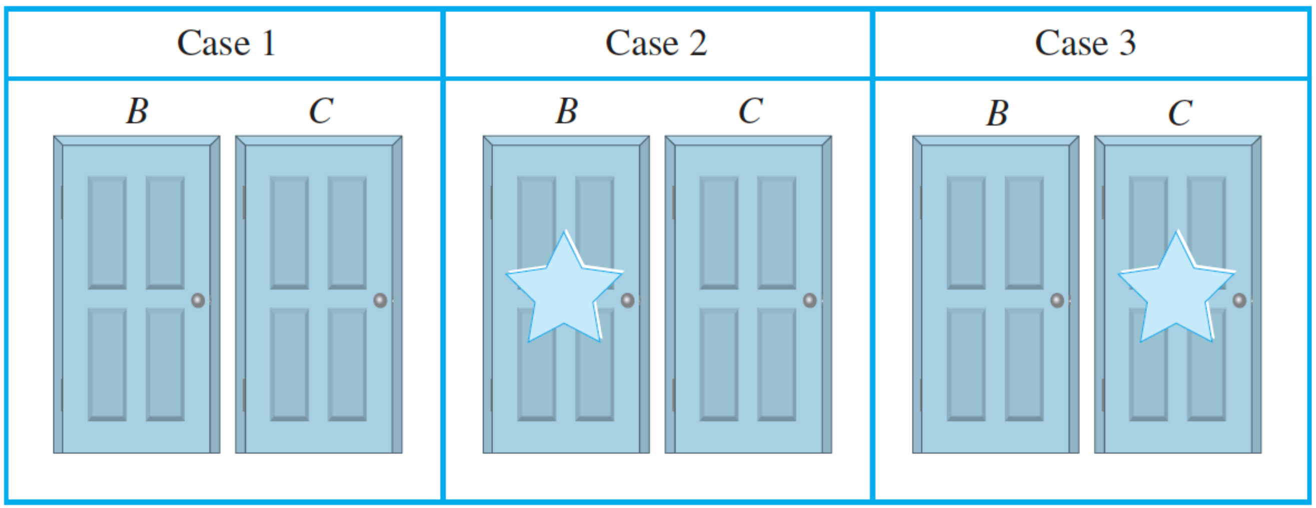
Example 2: Rolling a Pair of Dice

- Let the compact notation for  be 24.
- Use the compact notation to write the sample space S of possible outcomes.
- Use set notation to write the event E that the numbers showing face up have a sum of 6 and find the probability of this event.

Example 3: The Monty Hall Problem

- Imagine that you are in a game show. There are three doors on the set for a game show (A, B, and C).
- If you pick the right door, you win the prize.
- You pick door A.
- Monty Hall, the host of the show, then opens one of the other doors, and there's no prize behind it.
- He asks you whether you want to switch to the other closed door or stay with your original choice (door A).
- What should you do if you want to maximize your chance of winning the prize: stay with door A or switch—or would the likelihood of winning be the same either way?

Example 3:
The Monty Hall Problem



Counting the Elements of a List

- Theorem: The Number of Elements in a List
- If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Exercise

- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

100	101	102	103	104	105	106	107	108	109	110	$\cdots$	994	995	996	997	998	999
$\updownarrow$					$\updownarrow$					$\updownarrow$			$\updownarrow$				
$5\cdot 20$					$5\cdot 21$					$5\cdot 22$			$5\cdot 199$				

- What is the probability that a randomly chosen three-digit integer is divisible by 5?

Exercise

- Consider a pair of dice.
- Write each of the following events as a set and compute its probability.
  - The event that the sum of the numbers showing face up is 8.
  - The event that the sum of the numbers showing face up is at most 6.

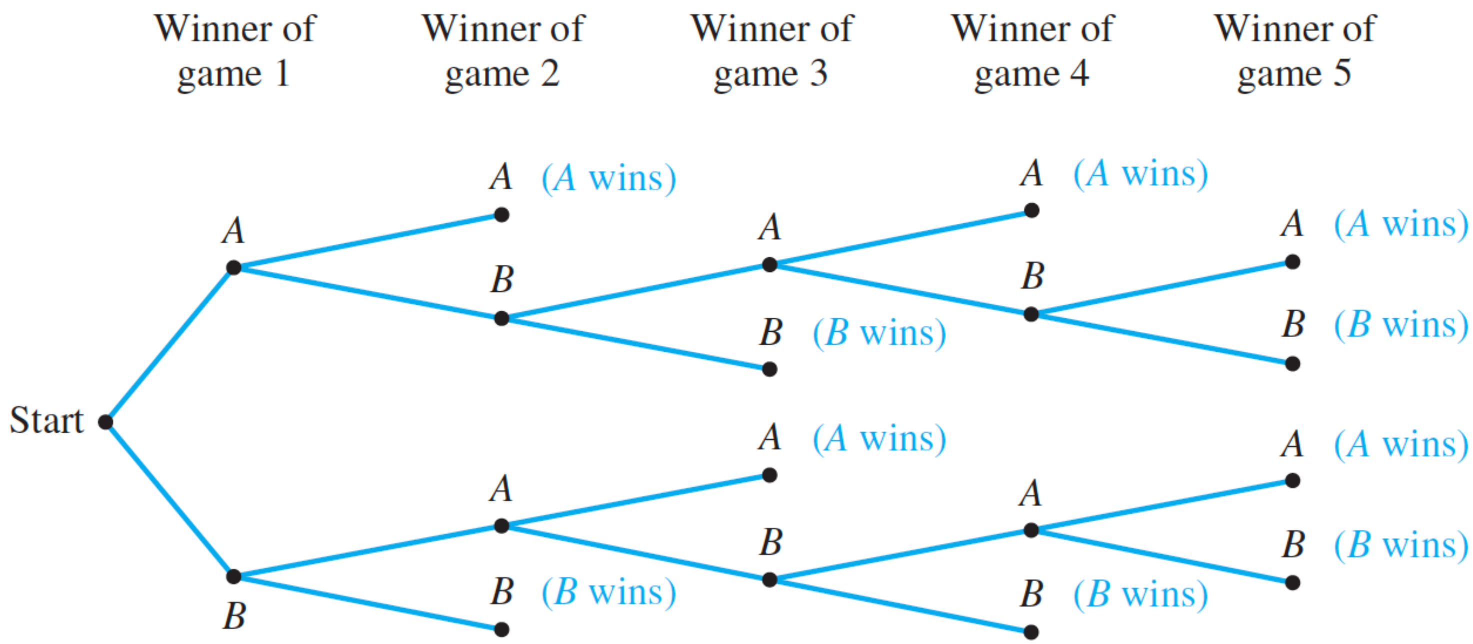
Possibility Trees

- A tree structure is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order.

Example 1: Tournament Play

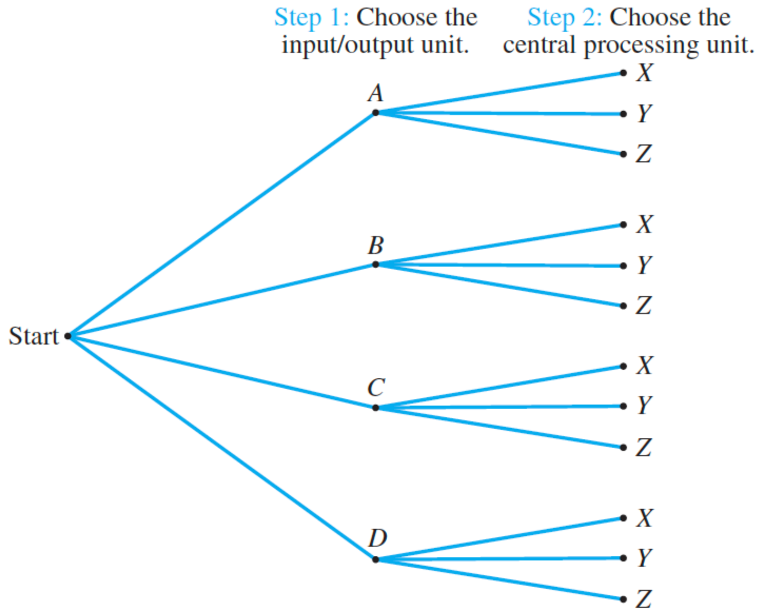
- Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games. One way in which this tournament can be played is for A to win the first game, B to win the second, and A to win the third and fourth games. Denote this by writing A–B–A–A.
- How many ways can the tournament be played?
- Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

Example 1: Tournament Play



Multiplication Rule

- Suppose a computer installation has four input/output units (A, B,C, and D) and three central processing units (X, Y, and Z).
- Any input/output unit can be paired with any central processing unit.
- How many ways are there to pair an input/output unit with a central processing unit?

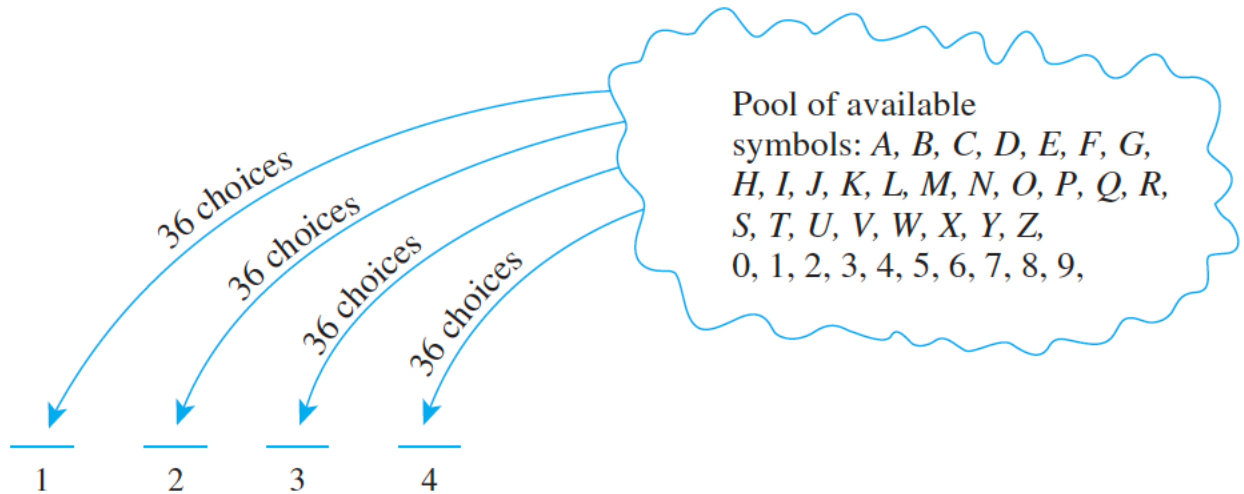


Multiplication Rule

- Theorem: The Multiplication Rule
- If an operation consists of k steps and
  - the first step can be performed in n_1 ways,
  - the second step can be performed in n_2 ways,
  - The k^{th} step can be performed in n_k ways,
- Then the entire operation can be performed in $n_1 n_2 \ldots n_k$ ways.

Example

- Number of Personal Identification Numbers (PINs)
- A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed.
- How many different PINs are possible?



Example

- Suppose A_1, A_2, A_3, and A_4 are sets with n_1, n_2, n_3, and n_4 elements, respectively.

- Show that the set A_1 x A_2 x A_3 x A_4 has $n_1*n_2*n_3*n_4$ elements.
  - Step 1: Choose the first element of the 4-tuple. n_1 ways
  - Step 2: Choose the second element of the 4-tuple. n_2 ways
  - Step 3: Choose the third element of the 4-tuple. n_3 ways
  - Step 4: Choose the fourth element of the 4-tuple. n_4 ways

  - By multiplication rule, there are $n_1*n_2*n_3*n_4$

Example

- We want to form a PIN using four symbols, either letters of the alphabet or digits, and supposing that repetition is not allowed.
- How many different PINs are there?
- If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbol?

Example

- Consider the set of all circuits with two input signals P and Q.
- For each such circuit an input/output table can be constructed, but two such input/output tables may have the same values.
- How many distinct input/output tables can be constructed for circuits with input/output signals P and Q?

P	Q	Output
1	1	1
1	0	0
0	1	1
0	0	0

P	Q	Output
1	1	0
1	0	0
0	1	1
0	0	0

Example

- Consider the following nested loop.

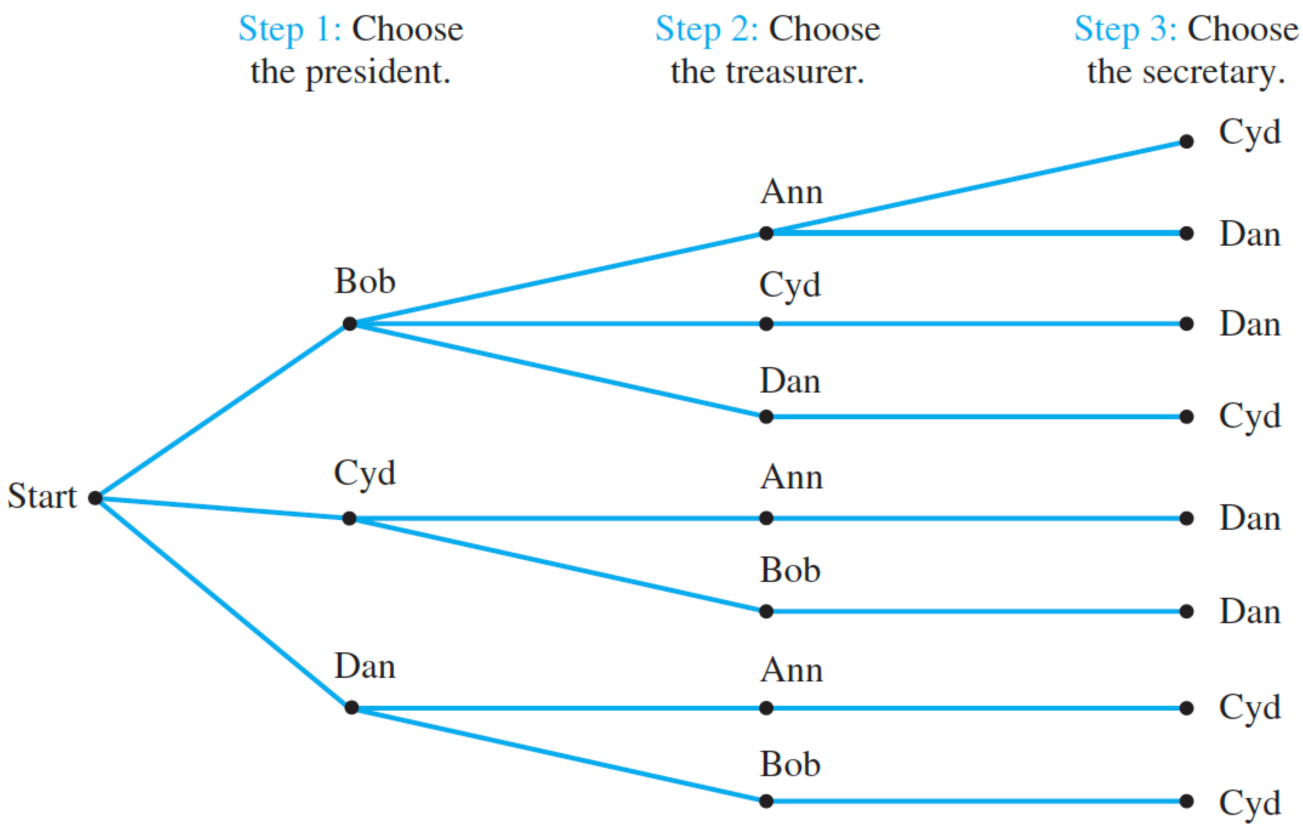
```
for i := 1 to 4
   for j := 1 to 3
      [Statements in body of inner loop.
       None contain branching statements
       that lead out of the inner loop.]
    next j
next i
```

- How many times will the inner loop be iterated when the algorithm is implemented and run?

Example

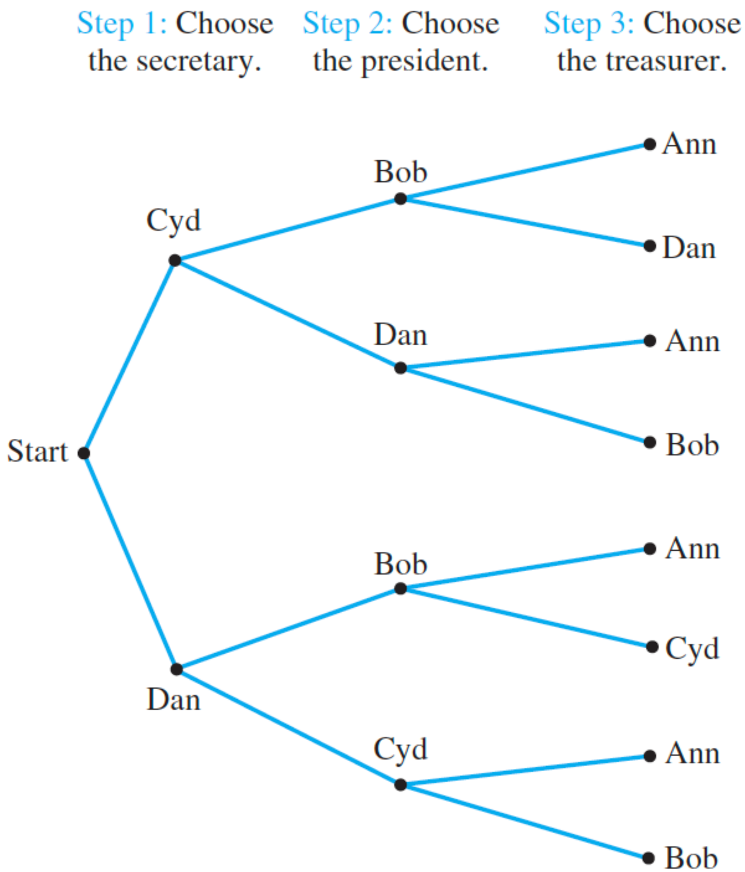
- Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people:
  - Ann, Bob, Cyd, and Dan
- Suppose that Ann cannot be president and either Cyd or Dan must be secretary.
- How many ways can the officers be chosen?

Example



Step 1: Choose the president.
Step 2: Choose the treasurer.
Step 3: Choose the secretary.
Start
Bob
Cyd
Dan
Ann
Cyd
Dan
Ann
Bob
Ann
Bob
Cyd
Dan
Dan
Cyd
Dan
Dan
Cyd
Cyd

Example



Step 1: Choose the secretary.
Step 2: Choose the president.
Step 3: Choose the treasurer.
Start
Cyd
Dan
Bob
Dan
Bob
Cyd
Ann
Dan
Ann
Bob
Ann
Cyd
Ann
Bob

Permutation

- A permutation of a set of objects is an ordering of the objects in a row.
- For example, the set of elements a, b, and c has six permutations.

abc acb cba bac bca cab

- In general, given a set of n objects, how many permutations does the set have?
- Form a permutation as an n-step operation:
  - Step 1: Choose an element to write first.
  - Step 2: Choose an element to write second.
  - ...
  - Step *n*: Choose an element to write n^{th}.

Permutation

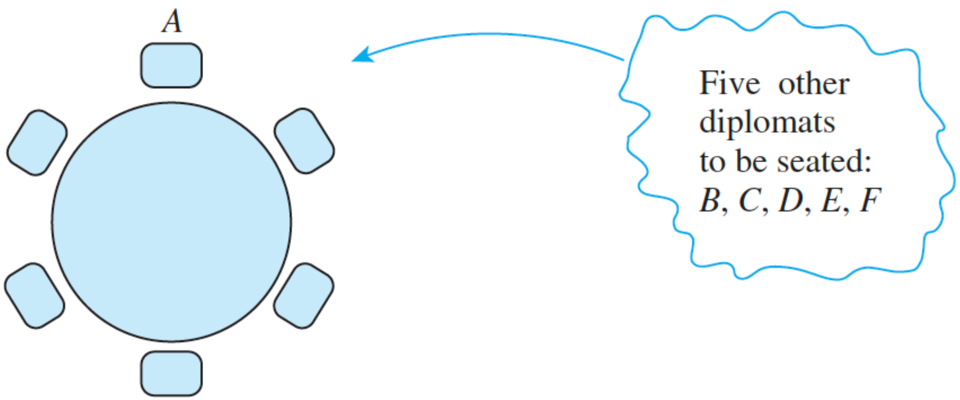
- Theorem
- For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Exercise

- How many ways can the letters in the word COMPUTER be arranged in a row?
- How many ways can the letters in the word COMPUTER be arranged if the letters CO must remain next to each other (in order) as a unit?
- If letters of the word COMPUTER are randomly arranged in a row, what is the probability that the letters CO remain next to each other (in order) as a unit?

Exercise

- At a meeting of diplomats, the six participants are to be seated around a circular table.
- Since the table has no ends to confer particular status, it doesn't matter who sits in which chair. But it does matter how the diplomats are seated relative to each other.
  - In other words, two seatings are considered the same if one is a rotation of the other.
- How many different ways can the diplomats be seated?



Permutations of Selected Elements

- An r-permutation of a set of n elements is an ordered selection of r elements taken from the set of n elements.

- The number of r-permutations of a set of n elements is denoted $P(n, r)$.

Theorem:

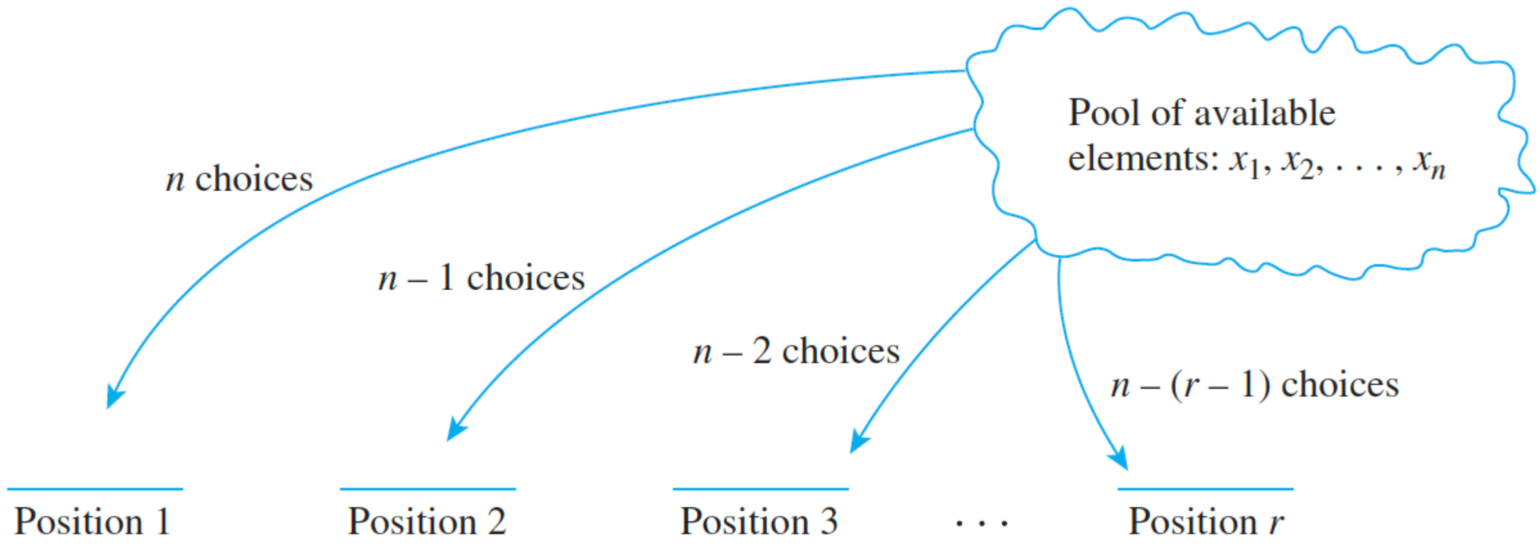
- If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

- or, equivalently,

$$P(n,r) = \frac{n!}{(n-r)!}$$

Concept of the Proof

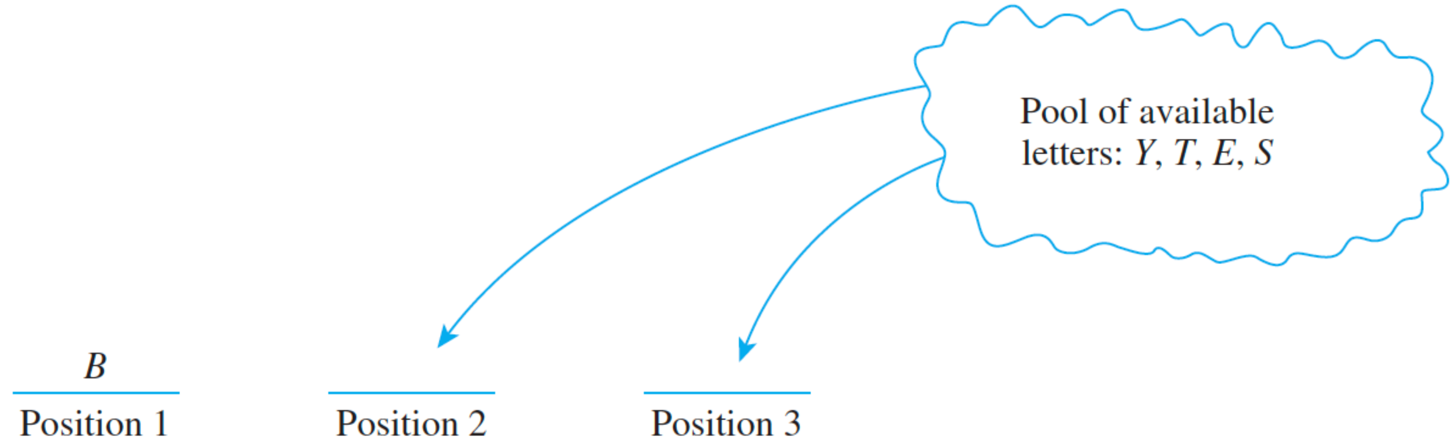


Exercise

a. Evaluate P(5, 2).
b. How many 4-permutations are there of a set of seven objects?
c. How many 5-permutations are there of a set of five objects?

Exercise

- How many different ways can three of the letters of the word BYTES be chosen and written in a row?
- How many different ways can this be done if the first letter must be B?



Addition Rule

- Theorem: The Addition Rule
- Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k).$$

Example

- A computer access password consists of at least one but not more than three letters chosen from the 26 in the alphabet with repetitions allowed.
- How many different passwords are possible?
- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

Probability of the Complement of an Event

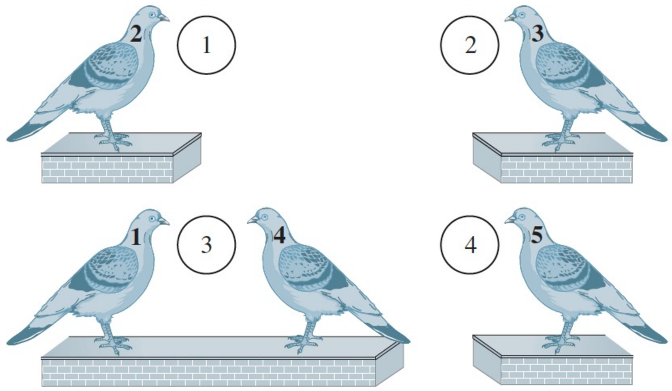
- Formula: Probability of the Complement of an Event
- If S is a finite sample space and A is an event in S, then

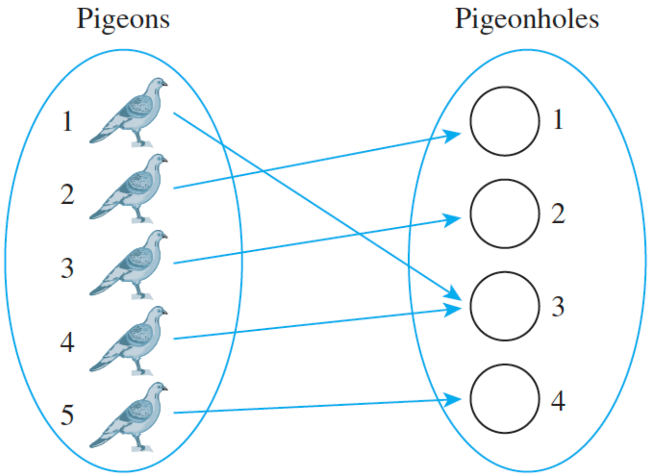
$$P(A^c) = 1 - P(A).$$

Exercise

- In the computer language Python, identifiers must start with one of 53 symbols:
  - either one of the 52 letters of the upper- and lower-case Roman alphabet or an underscore (_).
- The initial character may stand alone, or it may be followed by any number of additional characters chosen from a set of 63 symbols:
  - The 53 symbols allowed as an initial character plus the ten digits.
- Certain keywords (such as *and*, *if*, *print*, and so forth), are set aside and may not be used as identifiers.
  - In one implementation of Python there are 31 such reserved keywords, none of which has more than eight characters.
- How many Python identifiers are there that are less than or equal to eight characters in length?

Pigeonhole Principle





Pigeonhole Principle

- A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.

Example

- In a group of six people, must there be at least two who were born in the same month?
- In a group of thirteen people, must there be at least two who were born in the same month? Why?
- Among the residents of New York City, must there be at least two people with the same number of hairs on their heads? Why?

Pigeonhole Principle

- Theorem: The Pigeonhole Principle
- For any function f from a finite set X with n elements to a finite set Y with m elements, if n > m, then f is not one-to-one.

Combination

- Let n and r be nonnegative integers with r ≤ n.
- An r-combination of a set of *n* elements is a subset of *r* of the *n* elements. The symbol

$$\binom{n}{r},$$

- which is read "n choose r ," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Exercise

- Let S = {Ann, Bob, Cyd, Dan}. Each committee consisting of three of the four people in S is a 3-combination of S.
- List all such 3-combinations of S.
- What is $\binom{4}{3}$?

Ordered vs. Unordered Selection

- In an ordered selection, it is not only what elements are chosen but also the order in which they are chosen that matters.
  - Two ordered selections are said to be the same if the elements chosen are the same and also if the elements are chosen in the same order.
  - An ordered selection of r elements from a set of n elements is an r-permutation of the set.
- In an unordered selection, it is only the identity of the chosen elements that matters.
  - Two unordered selections are said to be the same if they consist of the same elements, regardless of the order in which the elements are chosen.
  - An unordered selection of r elements from a set of n elements is the same as a subset of size r or an r-combination of the set.

Exercise: Unordered Selection

- How many unordered selections of two elements can be made from the set {0, 1, 2, 3}?
  - 2-combination
  - {0, 1}, {0, 2}, {0, 3} subsets containing 0
  - {1, 2}, {1, 3} subsets containing 1 but not already listed
  - {2, 3} subsets containing 2 but not already listed.

Relations between Permutations and Combinations

- Theorem:
- The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

- or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

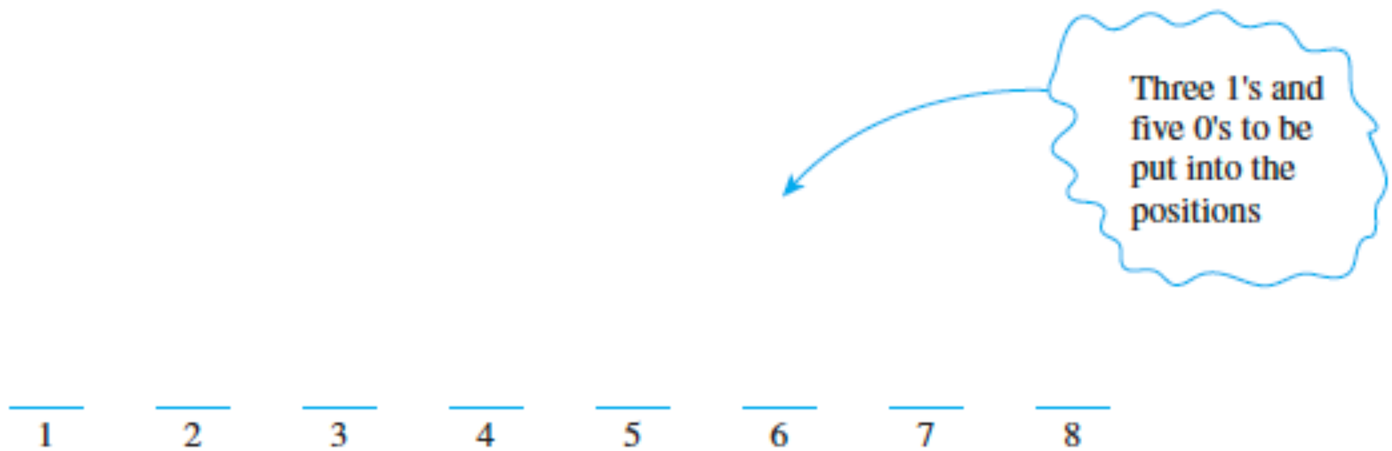
- where n and r are nonnegative integers with r ≤ n.

Exercise

- Suppose the group of twelve consists of five men and seven women.
- How many five-person teams can be chosen that consist of three men and two women?
- How many five-person teams contain at least one man?
- How many five-person teams contain at most one man?

Exercise

- How many eight-bit strings have exactly three 1’s?

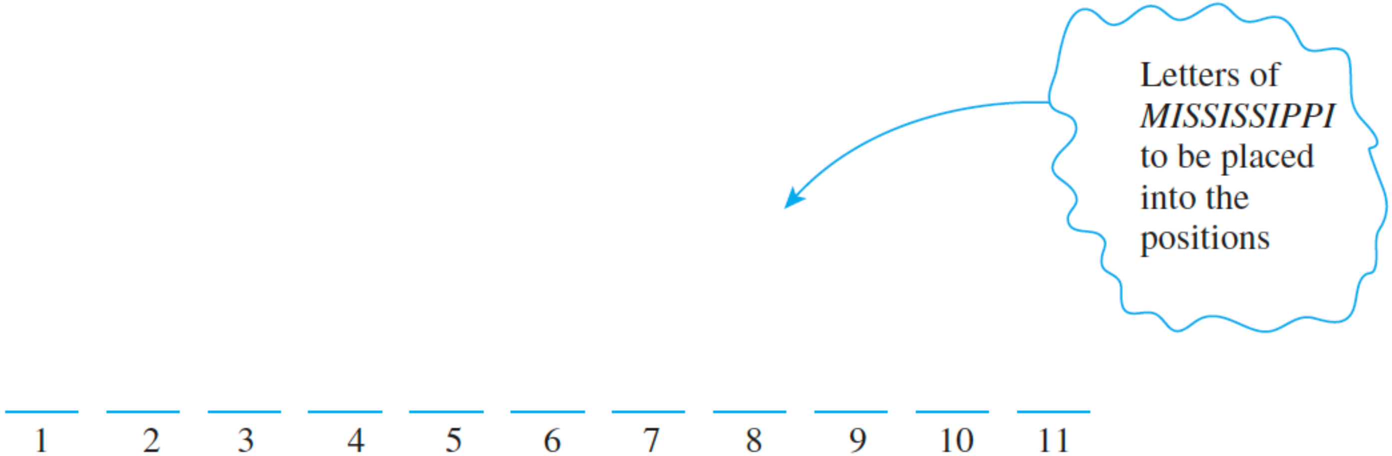


Exercise

- Consider various ways of ordering the letters in the word MISSISSIPPI:

  IIMSSPISSIP, ISSSPMIIPIS, PIMISSSSIIP, and so on.

- How may distinguishable orderings are there?



Exercise (cont'd)

- Arranging MISSISSIPPI
- Step 1: Choose a subset of four positions for the S's.
- Step 2: Choose a subset of four positions for the I 's.
- Step 3: Choose a subset of two positions for the P's.
- Step 4: Choose a subset of one position for the M.

Permutations with Sets of Indistinguishable Objects

- Theorem: Permutations with Sets of Indistinguishable Objects
- Suppose a collection consists of n objects of which

  n_1 are of type 1 and are indistinguishable from each other

  n_2 are of type 2 and are indistinguishable from each other

  ...

  n_k are of type k and are indistinguishable from each other,

- and suppose that $n_1 + n_2 + . . . + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n - n_1}{n_2}\binom{n - n_1 - n_2}{n_3}\cdots\binom{n - n_1 - n_2 - \ldots n_{k-1}}{n_k} = \frac{n!}{n_1!\,n_2!\,n_3!\ldots n_k!}$$