Functions

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Department of Computer Science
Vincent Mary School of Science and Technology
Assumption University

Session Outline

- Function as a form of a Relation
  - Definition, Notations, and Terminologies
- Relating Functions with Sets
- Boolean Functions
- Equality of Functions
- Properties of
  - One-to-One Functions
  - Onto Functions
  - One-to-One Correspondence
- Inverse Functions
- Identity Functions
- Composition of Functions

Definition: Function

- A function f from a set X to a set Y is a relation that satisfies two properties:
  - Every element in X is related to some element in Y
    - $\forall a \in X, \exists b \in Y$ such that $(a, b) \in \boldsymbol{f}$
  - No element in X is related to more than one element in Y.
    - $\forall a \in X$ and $\forall b,c \in Y$, $(a, b) \in \boldsymbol{f} \wedge (a, c) \in \boldsymbol{f} \rightarrow b = c$
- $f: X \rightarrow Y$
- X is the domain of the function
- Y is the co-domain of the function

Thus, given any element x in X, there is a unique element y in Y that is related to x by f.

Definition: Function

- f maps x to y
- f sends x to y
- f: $x \longrightarrow y$
- $(x \xrightarrow{f} y)$
- Total Participation on the domain side
- One-to-One or Many-to-One from domain to co-domain

Notations and Terminologies

- $f(x)$ represents the unique element to which f sends x
  - Read as "f of x"
  - The value of f at x.
  - The output of f for the input x.
  - The image of x under f.

- Careful with Notations:
  - f refers to the function itself
  - $f(x)$ refers the value of the function at x.

- Range of $f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$
  - the set of all values of f taken together
  - the image of the domain X under f

Notations and Terminologies

Given an element *y* in the co-domain *Y*, there may exist an element in the domain *X* with *y* as its image; i.e., *f*(*x*) = *y*.

- *x* is called an inverse image of *y* (a preimage of *y*).
- The set of all inverse images of *y* is called the inverse image of *y*
- The inverse image of *y* = {*x* ∈ *X* | *f*(*x*) = *y*, for some *x* in *X*}

• Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

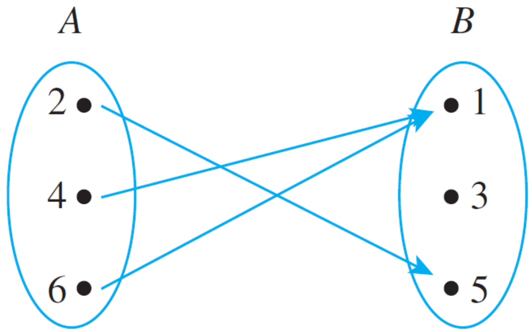
Exercise 1

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$.

Which of the relations R, S, and T defined below are functions from A to B?

- $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
- For all $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.
- T is defined by the arrow diagram below:

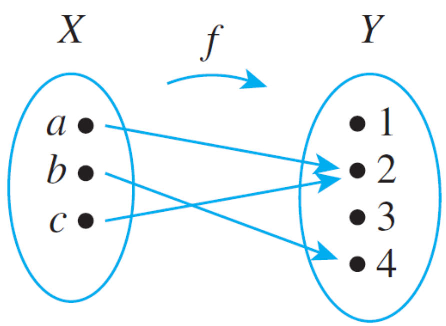
Draw an arrow diagram for S



Exercise 2

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$.

Let f be a function from X to Y defined by the arrow diagram shown:

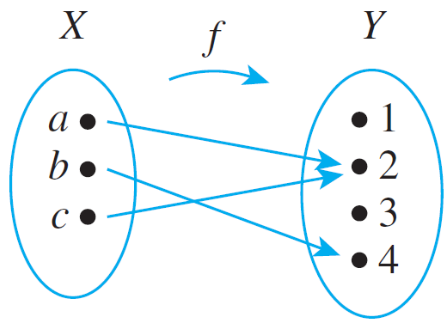


- Write the domain of f
- Write the co-domain f
- Find $f(a)$, $f(b)$, and $f(c)$
- What is the range of f?

Exercise 3

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$.

Let f be a function from X to Y defined by the arrow diagram shown:



- Is c an inverse image of 2?
- What is the inverse image of 2?
- Is b an inverse image of 3?
- Find the inverse images of 2, 4, and 1.
- Represent f as a set of ordered pairs.

Not Well-Defined Function

- Define a relation ***C*** from **R** to **R** as follows:

$$\text{For all } (x,y) \in \mathrm{R} \times \mathrm{R},\ (x,y) \in C \text{ means that } x^2 + y^2 = 1$$

- Is C a function?
- A "function" is said to be not well defined if it fails to satisfy the at least one of the requirements for being a function.
  - So "a not well-defined function" is not a function, in fact.

Exercise 4

Consider a function $f : \mathbf{Q} \rightarrow \mathbf{Z}$ defined by the formula f(m/n)=m for all integers m and n with $n \neq 0$.

Is f well defined? Why or why not?

- f is not well defined; in other words, f is not a function
- In the set **Q**, a same valued number has more than one representations. But f maps the same valued number to different integers.
- e.g., $\frac{1}{2}$ and $\frac{3}{6}$ are the two representations of the same valued number in **Q**.
- But f maps them to different values, namely 1 and 3.
- In fact, f is a Many-to-Many relation, not a function.

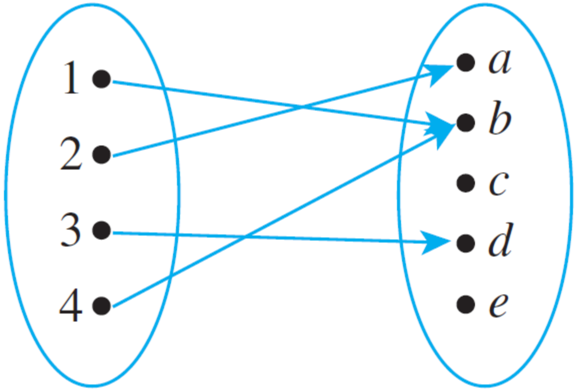
Functions and Sets

- Note that a function is defined as a form of a relation from the domain to the co-domain where every element of the domain is mapped to a unique element of the co-domain.
  - Domain: the set of inputs
  - Co-Domain: the set of possible outputs
- Not every element of the co-domain may be an image
  - Realize: the **range** $\subseteq$ **co-domain**
- We may be interested in a function acting on a certain subset of the domain or a subset of the co-domain.

Exercise 5

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$,

and define $F : X \to Y$ by the arrow diagram shown:



Let

$A = \{1, 4\}$

$C = \{a, b\}$

$D = \{c, e\}$.

Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Examples: Functions

Squaring is an example of a function

$$f : \mathbf{R} \rightarrow \mathbf{R} \text{ defined by the formula: } f(x) = x^2.$$

- What's the range of *f*?
  - $\mathbf{R}^{\text{nonneg}}$

- As we learned, a sequence is a function whose domain is a set of integers ≥ initial index.
  - 1, -1/2, 1/3, -1/4, 1/5, …, $(-1)^n/n+1$, …
    - $f(n) = (-1)^n \,/\, (n+1)$ when the domain is $\mathbf{Z}^{\text{nonneg}}$.
    - $f(n) = (-1)^{n+1} \,/\, n$ when the domain is $\mathbf{Z}^{+}$.

Example: Functions

Exponential Function with base b, with $b \neq 1$

- $\exp_b$: $\mathbf{R} \longrightarrow \mathbf{R}^+$ defined by the formula $\exp_b(x) = b^x$

Logarithm Function with base b, with $b \neq 1$

- $\log_b$: $\mathbf{R}^+ \longrightarrow \mathbf{R}$ defined by the formula $\log_b(x) = \log_b x$

Exercise 6

Let $A = \{a, b, c\}$.

Define a function F: $\mathscr{P}(A) \rightarrow \mathbf{Z}^{\text{nonneg}}$ as follows:

For each $X \in \mathscr{P}(A)$, $F(X)$ = the number of elements in X.

- Draw an arrow diagram for the function F.
- Find the value of $F(\{a,b\})$.
- What's the co-domain? What's the range?
- Find an inverse image of 2.
- Find the inverse image of 2.

Exercise 7

Let $K = \{0, 1\}$ and Let $\boldsymbol{S}$ be the **set of all strings over *K***.

Define a function D: $\boldsymbol{S} \to \boldsymbol{S}$ as follows:

> For each $s \in \boldsymbol{S}$, $D(s)$ = the string obtained from s by replacing each occurrence of 0 with 1 and each appearance of 1 with 0.

- What's the domain and co-domain of the function D?
- What is $D(00110)$?
- What is the image of 11100001 under D?
- What is the inverse image of 01101?

Functions defined on a Cartesian Product

Define a function M: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ as follows:

For all ordered pairs (a,b) of real numbers, $M(a,b) = ab$.

- What are $M(3,7)$, $M(-5, 4)$, $M(1/2,1/4)$, $M(\sqrt{2},\sqrt{2})$?
- What is the function M in simple English?

Functions defined on a Cartesian Product

Define a function R: $\mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \times \mathbf{R}$ as follows:

For all ordered pairs (a,b) of real numbers, $R(a,b) = (-a,b)$.

- What are $R(2,5)$, $R(-2,5)$, $R(3,-4)$?
- What does the function R do if the ordered pairs are the (x,y) coordinates of a Cartesian plane?

Functions Defined on a Cartesian Product

Consider the following definition of modulo function you learned when discussing the quotient-remainder theorem.

Given an integer *n* and a positive integer *d*, *n mod d* = the nonnegative integer remainder obtained when *n* is divided by d.

- What is the domain of the modulo function?
- What is the co-domain of the modulo function?

Definition: Boolean Function

- **Definition**

An **(n-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $\boxed{f: \{0, 1\}^n \to \{0, 1\}}$.

Example: Boolean Functions

Consider the following Boolean function f: $\{0,1\}^2 \rightarrow \{0,1\}$

$$f = \{((0,0),0),\ ((0,1),0),\ ((1,0),0),\ ((1,1),1)\}$$

- Draw an arrow diagram for the function f.
- Draw an input/output table for the function f.
- What is the logic defined by the function f?

Example: Boolean Functions

Consider the three-place Boolean function f: $\{0,1\}^3 \longrightarrow \{0,1\}$ defined by the following formula:

$$f(x_1,x_2, x_3) = (x_1 + x_2 + x_3)\ mod\ 2.$$

- Describe f using an input/output table.

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Equality of Functions

Proof:

Suppose $F(x) = G(x)$ for all $x \in X$. Then for all element x of X,

$$(x,y) \in F \Leftrightarrow y = F(x) \Leftrightarrow y = G(x) \Leftrightarrow (x,y) \in G.$$

F and G consists of exactly same elements and **hence $F = G$**.

Suppose $F = G$. Then for all element x of X,

$$y = F(x) \Leftrightarrow (x,y) \in F \Leftrightarrow (x,y) \in G \Leftrightarrow y = G(x).$$

Both $F(x) = y$ and $G(x) = y$; hence, we have that **$F(x) = G(x)$ for all $x \in X$**.

Exercise 9

Let $A = \{ 0, 1, 2 \}$ and define functions f and g from A to A as follows:

$$f(x) = (x^2 + x + 1) \; mod \; 3$$

$$g(x) = (x + 2)^2 \; mod \; 3$$

Does $f = g$?

Exercise 10

Let F: $\mathbf{R} \longrightarrow \mathbf{R}$ and G: $\mathbf{R} \longrightarrow \mathbf{R}$ be functions.

Define a new function $F + G$: $\mathbf{R} \longrightarrow \mathbf{R}$ as follows:

For all $x \in \mathbf{R}$, $(F + G)(x) = F(x) + G(x)$

And define a new function $G + F$: $\mathbf{R} \longrightarrow \mathbf{R}$ as follows:

For all $x \in \mathbf{R}$, $(G + F)(x) = G(x) + F(x)$.

Prove that $F + G = G + F$.

$$
\begin{aligned}
(F+G)(x) &= F(x) + G(x) && \text{by definition of } F+G \\
&= G(x) + F(x) && \text{by the commutative law for addition of real numbers} \\
&= (G+F)(x) && \text{by definition of } G+F
\end{aligned}
$$

Exercise 11

Let X and Y be sets and let F be a function from X to Y.

Let A and B be any subsets of X.

Prove that $F(A \cup B) = F(A) \cup F(B)$

- Note that the question is not asking to show the equality of two functions.
- It is asking to show that the image of $(A \cup B)$ is equal to the union of image of A and the image of B when A and B are subsets of the domain.
- Note that the image of a subset S of the domain X is
  - $F(S) = \{ y \in Y \mid y = F(x) \text{ for some } x \text{ in } S \}$
- So the proof requires showing the equality of two sets.
  - Need to prove that the two sets are subsets of each other
  - $F(A \cup B) \subseteq F(A) \cup F(B)$ and $F(A) \cup F(B) \subseteq F(A \cup B)$

Exercise 11 (cont'd)

Proof:

Suppose X and Y are p.b.a.c sets and F is a p.b.a.c. function from X to Y. Suppose also that A and B are p.b.a.c. subsets of X.

Proof of $F(A \cup B) \subseteq F(A) \cup F(B)$

Provide your proof (on your Exercise book)

Proof of $F(A) \cup F(B) \subseteq F(A \cup B)$

Provide your proof (on your Exercise book)

Since both subset relations have been proved, it follows that
$F(A \cup B) = F(A) \cup F(B)$

One-To-One (Injective) Function

- (General) Function: either One-to-One or Many-to-One relation
- Injective Function: One-to-One relation only.
  - Each element of the Range is the image of exactly one element of the domain.

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F: X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

Not Injective Function

What does it mean for a function to be not injective**?**

A function $F\colon X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

Not Injective: many-to-one but not one-to-one.

Proving that a Function is One-to-One

How to Prove?

- When the domain is a finite set with small number of elements
  - For **all possible pairs** of x_1, x_2, show that if $x_1 \neq x_2$, then $F(x_1) \neq F(x_2)$
    - How long does it take to do that?
- When the domain is an infinite set (or a finite set with large number of elements.
  - **Suppose** x_1 and x_2 are p.b.a.c. elements of X **such that** $\boldsymbol{F(x_1) = F(x_2)}$.
  - Show that $x_1 = x_2$

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

$$\text{or, equivalently,} \qquad \text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Disproving that a Function is One-to-One

How to disprove?

- Find a counterexample
- Find elements x_1 and x_2 in X so that $F(x_1) = F(x_2)$ but $x_1 \neq x_2$

A function $F: X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

Exercise 12

Define f: $\mathbf{R} \longrightarrow \mathbf{R}$ be the following:

$$f(x) = 4x - 1 \text{ for all } x \in \mathbf{R}$$

Prove or Disprove that f is one-to-one.

Proof:

Suppose x_1 and x_2 are p.b.a.c. real numbers such that $f(x_1) = f(x_2)$.

Then $4x_1 - 1 = 4x_2 - 1$ by definition of f.

By adding 1 and dividing by 4 both sides, we get $x_1 = x_2$.

Q.E.D.

Exercise 13

Define g: $\mathbf{Z} \longrightarrow \mathbf{Z}$ be the following:

$$g(n) = n^2 \text{ for all } n \in \mathbf{Z}$$

Prove or disprove that g is one-to-one.

Answer:

Let $n_1 = 2$ and $n_2 = -2$.

Then by definition of g, $g(n_1) = g(2) = 2^2 = 4$ and $g(n_2) = g(-2) = (-2)^2 = 4$.

Hence $g(n_1) = g(n_2)$, but $n_1 \neq n_2$; therefore, g is not one-to-one

Onto (Surjective) Functions

(general) Functions: there may be an element of the co-domain that is not the image of any element in the domain; Range $\subseteq$ Co-Domain

Surjective Function: every element of the co-domain is the image of some element of its domain.

- Range = Co-Domain

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F\colon X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Not Surjective Function

- What does it mean for a function to be **not surjective**?

$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

- A function that is not onto has at least one element in the co-domain that is not the image of any element in the domain
- Range ≠ Co-domain; Range ⊂ Co-domain (proper subset)

Proving that a Function is Onto

- How to Prove?
  - When the co-domain is a finite set with small number of elements
    - For all element *y* of *Y*, show that there is an inverse image *x* in *X* for *y*
  - When the co-domain is an infinite set (or a finite set with large number of elements.
    - Suppose *y* is a p.b.a.c. element of *Y* (the co-domain)
    - Show that there is an inverse image *x* in *X* (the domain) for *y*;
      - There is an element *x* of *X* such that *F*(*x*) = *y*

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Disproving that a Function is Onto

- How to disprove?
  - Find a counterexample
  - Find an element y of Y such that $y \neq F(x)$ for any x in X.
  - Proof by Contradiction can be useful

$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

Exercise 14

Define f: $\mathbf{R} \rightarrow \mathbf{R}$ be the following:

$$f(x) = 4x - 1 \text{ for all } x \in \mathbf{R}$$

Prove or disprove that f is onto.

Proof:

Suppose y is a p.b.a.c. real number and let $x = (y + 1) / 4$.

Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers so $x \in \mathbf{R}$

Observe that $f(x) = f((y + 1) / 4)$ by substitution

$= 4\cdot((y + 1) / 4) - 1$ by definition of f

$= 4\cdot(y + 1) / 4 - 1 = (y + 1) - 1 = y$.

[note that we have shown that there is an inverse image x for a p.b.a.c y]

Exercise 15

Define g: $\mathbf{Z} \to \mathbf{Z}$ be the following rules:

$$g(n) = 4n - 1 \text{ for all } n \in \mathbf{Z}$$

Prove or Disprove that g is onto.

Answer:

Observe that 0 is an element of the co-domain, that is $0 \in \mathbf{Z}$.

Suppose g were onto, then there exists an integer n such that $g(n) = 0$.

By the definition of g, $4n - 1 = 0$, which implies that $4n = 1$.

Thus, $n = 1/4$, which is not an integer and contradicting the supposition.

Therefore, g is not onto.

One-to-One Correspondences (Bijective Functions)

- Given any x in X, there is a unique image y of x under F;
  - Since F is a function.
- Given any y in Y, there is a unique inverse image x of y in X
  - Since F is onto and one-to-one
- A one-to-one correspondence (bijective function) sets up unique pairings between elements of X and Y.
  - Each (and every) element of X is paired with exactly one element of Y
  - Each (and every) element of Y is paired with exactly one element of X

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

One-to-One Correspondences (Bijective Functions)

Important Implication:

- Let $F: X \rightarrow Y$ be a one-to-one correspondence.
- If there are k elements in the set X, how many elements are there in Y?

k elements

Proving that a Function is a One-to-One Correspondence

- To prove a function is bijective, we need to prove that it is both injective and surjective.

Example

Let A = { 0, 1 } and let T be the set of all strings over A.

Define a function g: $T \to T$ by the rule:

For $\forall s \in T$, $g(s)$ = the string obtained by writing the characters of s in reverse order.

Prove that g is a one-to-one correspondence.

Proof: g is one-to-one

Suppose s_1 and s_2 are p.b.a.c strings in T such that $g(s_1) = g(s_2)$.

If two strings are equal when written in reverse order, then they must be equal to start with.

Hence, $g(s_1) = g(s_2)$ implies that $s_1 = s_2$

Example (cont'd)

Proof: g is onto

Suppose $\boldsymbol{t}$ is a p.b.a.c string in T (the co-domain) and let $s = g(\boldsymbol{t})$.

[Note that to prove g is onto, we need to show that we can find a string s in T (the domain) such that $g(s) = \boldsymbol{t}$].

Observe that when the order of the characters of a string is reversed once and then reversed again, the original string is obtained.

Thus $g(s) = g(g(\boldsymbol{t})) = \boldsymbol{t}$ [as was to be shown]

Exercise 16

Define a function F: $\mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \times \mathbf{R}$ as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R},\ F(x, y) = (x + y, x - y)$$

Prove that F is a bijection.

Prove that
F is injective
(one-to-one)

Prove that
F is surjective
(onto)

Definition: Inverse Function

Theorem 7.2.2

Suppose $\boxed{F\colon X \to Y}$ is a $\boxed{\text{one-to-one correspondence}}$; that is, suppose F is one-to-one and onto. Then there is a function $\boxed{\boldsymbol{F^{-1}\colon Y \to X}}$ that is defined as follows:

Given any element y in Y,

$$\boxed{\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.}$$

In other words,

$$\boxed{F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).}$$

- **Definition**

The function F^{-1} of Theorem 7.2.2 is called the **inverse function** for F.

Inverse Function

- The proof of the above theorem follows immediately from the definition of one-to-one and onto.
  - given any element y in Y, there is an element x in X with $F(x) = y$ because F is onto
  - x is unique because F is one-to-one.

Inverse Function

Note the following

- $F: X \rightarrow Y$ has an inverse function *iff* F is a bijective function
- Inverse function of F (denoted F^{-1}) is a function from Y to X.
  - $F^{-1}: Y \rightarrow X$
- The inverse function $F^{-1}: Y \rightarrow X$ is also a bijection.

Proof that the inverse function $F^{-1}: Y \rightarrow X$ is a bijection.

Inverse Function is Bijection

Proof of F^{-1} is one-to-one:

Suppose y_1 and y_2 are p.b.a.c. elements of Y such that $F^{-1}(y_1) = F^{-1}(y_2)$ and let $x = F^{-1}(y_1) = F^{-1}(y_2)$. We need to show that $y_1 = y_2$.

Then $x \in X$ and by definition of F^{-1},

$F(x) = y_1$ since $x = F^{-1}(y_1)$ and

$F(x) = y_2$ since $x = F^{-1}(y_2)$

Therefore, $y_1 = y_2$ [as was to be shown]

- Note that we can say $y_1 = y_2$ since F is a function.

Proof of F^{-1} is onto:

Suppose x is a p.b.a.c. element in X. Let $y = F(x)$.

Then $y \in Y$ and by definition of F^{-1}, $F^{-1}(y) = x$. [as was to be shown]

Exercise 17

Let A = { 0, 1 } and let T be the set of all strings over A.

Define a function g: $T \rightarrow T$ by the rule:

g(s) = the string obtained by writing the characters of s in reverse order.

- We previously proved that g is a one-to-one correspondence.
- So there must exist the inverse function for g.

Find the g^{-1}.

Exercise 17 (cont'd)

Note that if the characters of *t* are written in reverse order and then written in reverse order again, the original string is obtained.

Thus given any string $\boldsymbol{t}$ in T,

$g^{-1}(\boldsymbol{t})$ = the value (output) of g^{-1} for t

= the unique string that, when written in reverse order, equals $\boldsymbol{t}$

= the string obtained by writing the characters of $\boldsymbol{t}$ in reverse order

= $g(\boldsymbol{t})$.

Hence $g^{-1}: T \longrightarrow T$ is the same as g, or in other words, $g^{-1} = g$

Exercise 18

Define a function f: $\mathbf{R} \rightarrow \mathbf{R}$ by the formula

$$f(x) = 4x - 1 \text{ for all real numbers } x.$$

- Does the function f have an inverse function?
- Prove that f is a bijection. (on your exercise book)
- Find the inverse function of f

Note that f^{-1}: $\mathbf{R} \rightarrow \mathbf{R}$ and by definition of f^{-1}

$f^{-1}(y)$ = the unique real number x such that $f(x) = y$

Observe that $f(x) = 4x - 1$ by definition of f.

So if $f(x) = y$ then $y = 4x - 1$

Implying that $x = (y + 1) / 4$

Hence $f^{-1}(y) = (y + 1) / 4$.

Exercise 19

Define a function F: $\mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \times \mathbf{R}$ as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R},\ F(x, y) = (x + y, x - y)$$

- We proved that F is a one-to-one correspondence.

Find the inverse function F^{-1}: $\mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \times \mathbf{R}$ for F.

Exercise 19 (cont'd)

By the definition of F^{-1}

[Ask yourself what the output of F^{-1} is for a p.b.a.c. input to the function F^{-1} in terms of the function F]

$F^{-1}(w,z)$ = the uniqe ordered pair (x,y) such that $F(x, y) = (w,z)$

Observe that $F(x, y) = (x + y, x - y)$ by definition of F.

So if $F(x, y) = (w,z)$, then $(w,z) = (x + y, x - y)$.

Implying that $w = x + y$ and $z = x - y$.

Solving the system of the equations,

we get $x = \frac{w+z}{2}$ and $y = \frac{w-z}{2}$.

So $F^{-1}(w,z) = (\frac{w+z}{2}, \frac{w-z}{2})$.

Identity Function

Consider the following function

$$f(x) = x.$$

- The function send each input accepted directly as the output without changing it anyway.
- Thus, the domain and co-domain are the same

Given a set X, define a function I_X from X to X by

$$I_X(x) = x \text{ for all } x \text{ in } X.$$

The function is called the identity function on X.

Composition of Functions

Consider the operations of taking an integer n, as input and then

- First, incrementing it by 1: $n + 1$
- Then squaring it: $(n + 1)^2$

Hence, the function F: Z ⟶ Z defined as $F(n) = (n + 1)^2$ can be considered as

Composition of two functions

- F_1: Z ⟶ Z defined as $F_1(n) = n + 1$
- F_2: Z ⟶ Z defined as $F_2(n) = n^2$
  - $F(n) = F_2(F_1(n))$

Composition of Functions

Composition of f and g

- $g \circ f = g(f(x))$

- f is the first function and g is the second function

Note that the composition can be formed only if the output of the first function is acceptable input to the second function

- The Range of the first function $\subseteq$ Domain of the second function

- For $g \circ f$ defined as above, the Range of $f \subseteq$ Domain of g

Exercise 20

Let f: Z $\rightarrow$ Z be defined as $f(n) = n + 1$ and

Let g: Z $\rightarrow$ Z be defined as $g(n) = n^2$

- Find the composition $g \circ f$
- Find the composition $f \circ g$
- Is $g \circ f = f \circ g$?
  - $(g \circ f)(1) = ?$
  - $(f \circ g)(1) = ?$

In general, the composition of functions is NOT commutative.

Composition with the Identity Function

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, \mathrm{w}\}$, and suppose $f: X \rightarrow Y$ is given by the arrow diagram.

Can we compose $f \circ I_X$?
Can we compose $f \circ I_y$?
Can we compose $I_Y \circ f$?
Can we compose $I_X \circ f$?
Find $f \circ I_X$ and $I_Y \circ f$.

$f \circ I_X = f = I_Y \circ f$

Composition with the Identity Function

Proof of $f \circ I_X = f$

Suppose f is a p.b.a.c. function from a set X to a set Y, and I_X is the identity function on X.

Then, for all x in X, $(f \circ I_X)(x) = f(I_X(x)) = f(x)$

Hence $f \circ I_X = f$ by definition of equality of functions.

Proof of $f = I_Y \circ f$

Suppose f is a p.b.a.c. function from a set X to a set Y, and I_Y is the identify function on Y.

Then, for all x in X, $(I_Y \circ f)(x) = I_y(f(x)) = f(x)$

Hence $f = I_Y \circ f$ by definition of equality of functions.

Composing a Function with Its Inverse

Let f be a p.b.a.c bijection from a set X to a set Y

- What is $f^{-1} \circ f$?
- What is $f \circ f^{-1}$?
- Note that f is both one-to-one and onto
  - Therefore, f has an inverse function f^{-1}

$(f^{-1} \circ f)(a) = ?$
$(f^{-1} \circ f)(b) = ?$
$(f^{-1} \circ f)(c) = ?$

$f^{-1} \circ f = I_X$

$(f \circ f^{-1})(x) = ?$
$(f \circ f^{-1})(y) = ?$
$(f \circ f^{-1})(z) = ?$

$f \circ f^{-1} = I_Y$

Composing a Function with its Inverse

Proof of $f^{-1} \circ f = I_X$

[Your Exercise]

Proof of $f \circ f^{-1} = I_Y$

[Your Exercise]

Composition of One-to-One Functions

Suppose f: $X \longrightarrow Y$ and g: $Y \longrightarrow Z$ are both one-to-one functions.

Show that $g \circ f$ is one-to-one.

Proof Idea:

- How to show $g \circ f$ is one-to-one?
  - $g \circ f$ is one-to-one $\Leftrightarrow \forall x_1, x_2 \in X$, if $(g \circ f)(x_1) = (g \circ f)(x_2)$ then $x_1 = x_2$

Proof: [Your Exercise]

Composition of Onto Functions

Suppose f: $X \longrightarrow Y$ and g: $Y \longrightarrow Z$ are both onto functions.

Show that $g \circ f$ is onto.

Proof Idea:

- How to show $g \circ f$ is onto?
  - $g \circ f : X \longrightarrow Z$ is onto $\Leftrightarrow \forall z \in Z, \exists x \in X$ such that $g \circ f(x) = z$

Proof: [Your Exercise]