Language of Mathematics

CSX2008 Mathematics Foundation for Computer Science

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Session Outline

- Brief Introduction to Fundamental Concepts and Notations
- Language of Mathematics
  - Variables
  - Mathematical Statements
    - Universal, Conditional and Existential
  - Sets, Subsets, Ordered Pairs, Cartesian Products
  - Relation as Subset of Cartesian Product
  - Function as a form of Relation
  - Function as a machine with input / output specification

Language of Mathematics

- Imagine yourself working on a mathematical problem.
- Initially, you have little idea how to solve it.
- So you start by exploring examples, drawing pictures, rephrasing the problem statements, etc.
- Why?
  - So that you can understand more of its details and figure out the reasons behind possible solutions.
- The more you understand the problem and become closer to a solution, the more you need a language that expresses your ideas clearly, precisely and unambiguously.
- Why?
  - So that you can produce a solution that can be shown to work correctly without any doubt.
- And the solution needs to be expressed precisely.

Example:
Language of Mathematics

- A group of 12 school children accompanied by 3 teachers made a study visit to the Science Museum. The total admission cost for them was $162. Another group of 8 children and 3 adults paid $122 in total for the admission to the museum.

  How much was the admission for each child and adult?

- Understand the problem and express the problem in the language of mathematics:
  - $12x + 3y = 162$
  - $8x + 3y = 122$

- Language of Mathematics expresses your ideas clearly, precisely and unambiguously

Example: Writing Sentences with Variables

- Are there numbers with the property that the sum of their squares equals the square of their sum?
- Are there numbers *a* and *b* with the property that $a^2 + b^2 = (a + b)^2$?
- Are there numbers *a* and *b* such that $a^2 + b^2 = (a + b)^2$?
- Do there exist any numbers *a* and *b* such that $a^2 + b^2 = (a + b)^2$?
- $\exists a, b \in \boldsymbol{R}$ such that $a^2 + b^2 = (a + b)^2$?

Example: Writing Sentences with Variables

- Given any real number, its square is nonnegative.
- Given any real number r, r^2 is nonnegative.
- For any real number r, $r^2 \geq 0$.
- For all real numbers r, $r^2 \geq 0$.
- $\forall r \in \boldsymbol{R}, r^2 \geq 0$.

Some Important Kinds of Mathematical Statements

- **Universal Statements** (for all, every)
  - States that a certain property is true for all elements in a set
  - All positive numbers are great than zero
  - $\forall x \in R^{+}, x > 0$
- **Existential Statements** (there exists)
  - States that there is at least one element of a set that satisfies a given property
  - There is a prime number that is even
  - $\exists p \in$ Prime Numbers such that p is even.
- **Conditional Statements** (if … then …)
  - States that if one thing is true then some other thing also has to be true
  - If 378 is divisible by 18, then 378 is divisible by 6
  - 378 is divisible by 18 $\rightarrow$ 378 is divisible by 6

Combined Statements

- **Universal Conditional**
  - for all … if … then …
  - For all real numbers r, if $r > 5$, then $r^2 > 25$.
  - $\forall r \in \boldsymbol{R}, (r > 5) \rightarrow (r^2 \geq 25)$.
- **Universal Existential**
  - For all … , there exists …
  - For all nonzero real numbers r, there is a reciprocal for r.
  - $\forall r \in \boldsymbol{R}^{\text{nonzero}}, \exists s \in \boldsymbol{R}$ such that $r \times s = 1$.
- **Existential Universal**
  - There exists … such that for all …
  - There exists an integer a such that for every real number b, $a \times b = 0$.
  - $\exists a \in \boldsymbol{Z}$ such that $\forall b \in \boldsymbol{R}$, $a \times b = 0$.

Different Ways of Saying

- There is a positive integer that is less than or equal to every positive integer.
  - Existential Universal
- Some positive integer is less than or equal to every positive integer.
- There is a positive integer *m* that is less than or equal to every positive integer.
- There is a positive integer *m* such that every positive integer is greater than or equal to *m*.
- There is a positive integer *m* with the property that for all positive integers *n*, $m \le n$.
- $\exists m \in \mathbf{Z}^+$ such that $\forall n \in \mathbf{Z}^+$, $m \le n$

Different Ways of Saying

- Every real number has an additive inverse.
  - Universal Existential
- All real numbers have additive inverses
- For all real numbers *r*, there is an additive inverse for *r*.
- For all real numbers *r*, there is a real number *s* such that *s* is an additive inverse for *r*.
- $\forall r \in \mathbf{R}, \exists s \in \mathbf{R}$ such that $r + s = 0$

Different Ways of Saying

- For all animals *a*, if *a* is a dog, then *a* is a mammal.
  - Universal Conditional Statement
  - Domain of discourse for the variable *a* is "set of all animals"
- If *a* is a dog, then *a* is a mammal
- If an animal is a dog, then the animal is a mammal
  - Conditional Statement
  - Domain of discourse is implied to be "set of all animals"
- For all dogs *a*, *a* is a mammal
- All dogs are mammals
  - Universal Statement
  - Domain of discourse is "set of all dogs"

Understanding Mathematical Statements

- a statement shown below in a Calculus book:
  - the limit of a_n as n approaches infinity is L, where a_1, a_2, a_3, … is a sequence of real numbers

- **for all** positive real numbers ε, **there is** an integer N such that **for all** integers n, **if** $n > N$ **then** $-\varepsilon < a_n - L < \varepsilon$.
  - Universal existential universal conditional

Sets

Introduction to Sets

- Set as a mathematical term was introduced by Georg Cantor in 1879
- Informally, think of a set as a collection of elements
  - If *C* is the set of all the students in this class, then 'Taranjit' is an element of *C*.
- Notation
  - $x \in S$ means x is an element of the set S
  - $x \notin S$ means x is not an element of the set S
- **Axiom of extension**: a set is completely determined by what its elements are, *regardless of* the *frequency nor* the *order* of listing.

Set-Roster Notation

- Write all elements of the set between braces.
  - $A = \{ 3, 7, 9 \}$
  - $B = \{ 1, 2, 3, \ldots, 100 \}$
  - $C = \{ 1, 2, 3, \ldots \}$

Exercises

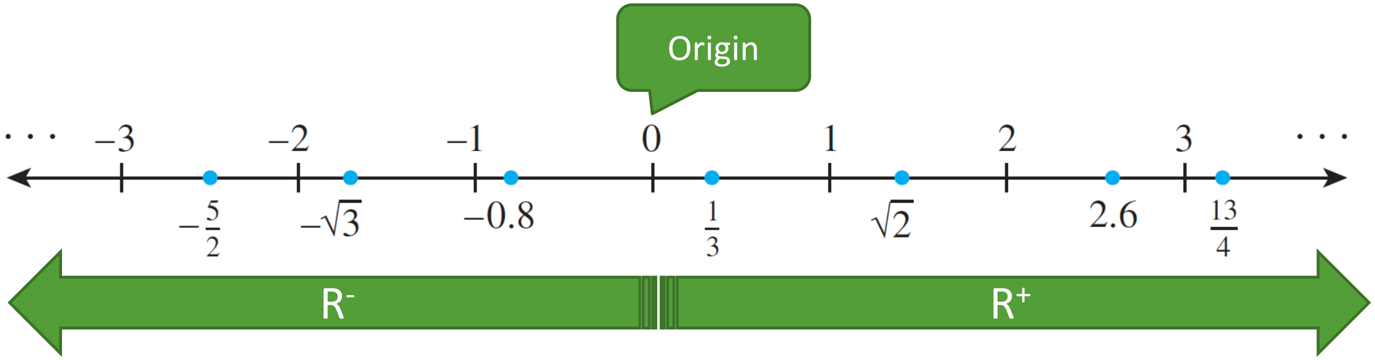
- Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$.
- What are the elements of A, B, and C?
  How are A, B, and C related?
- Is $\{0\} = 0$?
- How many elements are in the set $\{1, \{1\}\}$?
- For each nonnegative integer n,

  let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Common Sets of Numbers

- **R** Set of all real numbers
- **Z** Set of all integers
- **Q** Set of all rational numbers (quotients)
- $\mathbf{R}^{-}$ Set of negative real numbers
- $\mathbf{Z}^{+}$ Set of positive integers
- $\mathbf{Z}^{\text{nonneg}}$ Set of nonnegative integers, {0, 1, 2, …}
- **N** Set of natural numbers
  - could mean $\mathbf{Z}^{+}$ or $\mathbf{Z}^{\text{nonneg}}$ depending on writers
  - We avoid this terminology in this class

Real Number line



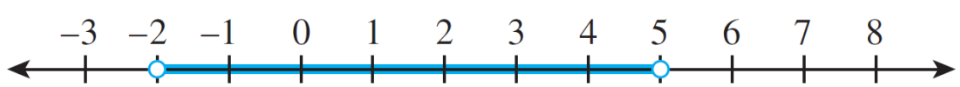
- Real number line is continuous (no holes)
  - How many real numbers are there between 0.4 and 0.5?
  - Can you define the smallest positive real number?
- Set of integers corresponds to a collection of points located at *fixed intervals* along the real number line
  - Integers are separated from each other: discrete
  - Can you define the smallest positive integer? What is it?
  - Every integer is a real number

Set Builder Notation

- Let *S* denote a set and let *P(x)* be a property that elements of S may or may not satisfy. We may define a new set to be the set of all elements *x* in *S* such that *P(x)* is true.
- We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

$\{x \in \mathbf{R} \mid -2 < x < 5\}$



$\{x \in \mathbf{Z} \mid -2 < x < 5\}$ {–1, 0, 1, 2, 3, 4}

$\{x \in \mathbf{Z}^{+} \mid -2 < x < 5\}$ {1, 2, 3, 4}

Subset Notations

- $A \subseteq B$
- $A \nsubseteq B$
- $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
- $A \subset B$

Subset Notations

- A ⊆ B
  - set A is a subset of set B *iff* every element of A is also an element of B
  - $\forall x \in$ A, $x \in$ B
  - A is contained in B
  - B contains A
- A ⊈ B
  - set A is not a subset of set B
  - $\exists x \in$ A such that $x \notin$ B
  - There is at least one element of A that is not in B.
- A = B ⇔ A ⊆ B and B ⊆ A
- A⊂B
  - set A is a proper subset of B *iff* every element of A is in B but there is at least one element of B that is not in A.

Exercise

Let $A = \mathbf{Z}^+$,

Z+ is a set of all positive integers

$B = \{ n \in \mathbf{Z} \mid 0 \le n \le 100 \}$ and

Z is a set of all integers

$C = \{ 100, 200, 300, 400, 500 \}$

Which of the following statements are true?

- $B \subseteq A$
- $C \subset A$
- C and B have at least one element in common
- $C \subseteq B$
- $C \subseteq C$
- $C \subset C$

Exercise

Which of the following are true statements?

- $2 \in \{1, 2, 3\}$
- $\{2\} \in \{1, 2, 3\}$
- $2 \subseteq \{1, 2, 3\}$
- $\{2\} \subseteq \{1, 2, 3\}$
- $\{2\} \subseteq \{\{1\}, \{2\}\}$
- $\{2\} \in \{\{1\}, \{2\}\}$

Ordered Pair

- (a, b)
  - An ordered pair consisting of a and b together with the specification that a is the first element of the pair and b is the second element.
- Equality of two ordered pairs (a, b) and (c, d)
  - $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$

Exercises: Ordered Pair

- Is (1, 2) = (2, 1) ?
- Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?
- What is the first element of (1, 1) ?

Cartesian Products

$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$

- Cartesian Product of A and B; read "A cross B"
- Set of all ordered pairs (a, b), where a is in A and b is in B.

Exercises

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

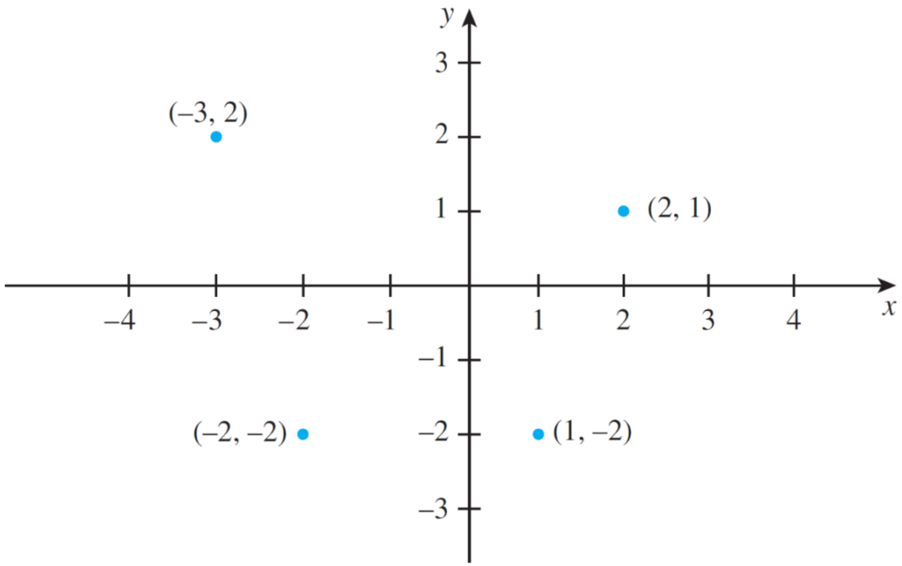
- Find $A \times B$
- Find $B \times A$
- Find $B \times B$
- How many elements are in $A \times B$, $B \times A$, and $B \times B$?
- Let R denote the set of all real numbers.
  Describe $R \times R$.

Future Topics

- Set vs. Multi-Set
- Sequence vs. Set

Cartesian Plane

- $R \times R$ is the set of all ordered pairs (x, y) where both x and y are real numbers
- If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $R \times R$ corresponds to a unique point in the plane
  - with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point.



Relations

Relations

- Two people are related by marriage
- Similarly, objects of mathematics may be related in various ways
  - Set A may be said to be related to a set B if A is a subset of B
  - A number x may be said to be related to a number y if $x < y$

Relations

- Let A and B be sets.

  A relation R from A to B is a subset of $A \times B$.

- Given an ordered pair (x,y) in $A \times B$.

  x is related to y by R, denoted xRy, *iff* (x,y) is in R

- $xRy \;\Leftrightarrow (x,y) \in R$
- $xRy \;\Leftrightarrow (x,y) \notin R$
- The set A is called the domain of R.
- The set B is called its co-domain.

Exercise

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows:

Given any $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{x-y}{2} \text{ is an integer.}$$

- State explicitly which ordered pairs are in $A \times B$ and which are in R.
- Is 1R3 ?
- Is 2R3 ?
- Is 2R2 ?
- What are the domain and co-domain of R?

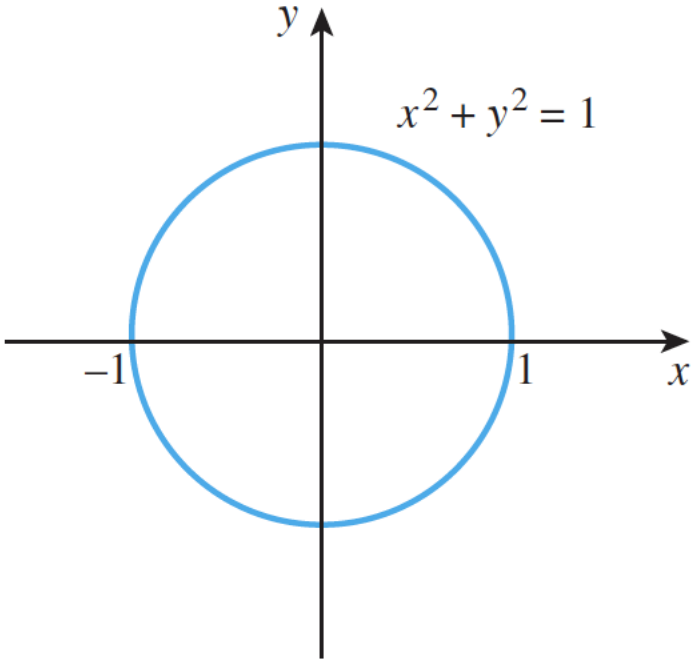
Exercise

Define a relation C from **R** to **R** as follows:

For any $(x, y) \in \mathbf{R} \times \mathbf{R}$, $(x, y) \in C \Leftrightarrow x^2 + y^2 = 1$

- Is $(1,0) \in C$?
- Is $(0,0) \in C$?
- Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$?
- Is $-2\ C\ 0$? Is $0\ C\ (-1)$? Is $1\ C\ 1$?
- What are the domain and co-domain of C?
- Draw a graph for C by plotting the points of C in the Cartesian plane

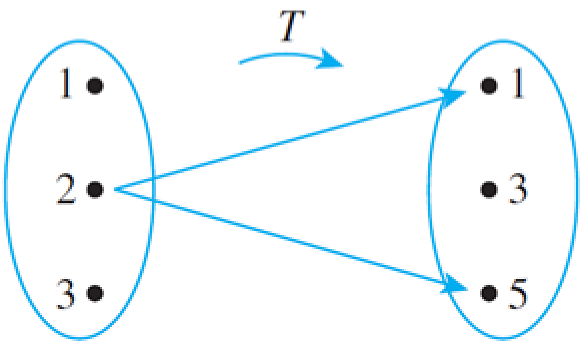
Exercise: Graph for Relation *C*



Arrow Diagram of a Relation

Let A = {1, 2, 3} and B = {1, 3, 5} and define relation T from A to B as follows:

$$T = \{\ (2,1), (2,5)\ \}$$



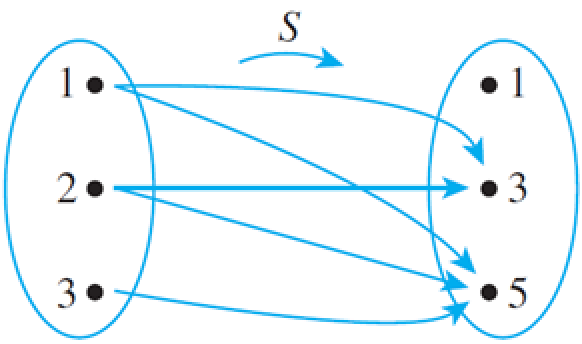
Exercise: Arrow Diagram

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

Let's define another relation S from A to B as follows:

For all $(x, y) \in A \times B$, $(x, y) \in S \Leftrightarrow x < y$

Draw an arrow diagram for the relation S



Functions

Functions (as a kind of Relations)

A function F from a set A to a set B is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.

2. For all elements x in A and y and z in B,

   if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

Every element of A is the first element of an ordered pair of F.

No two distinct ordered pairs in F have the same first element

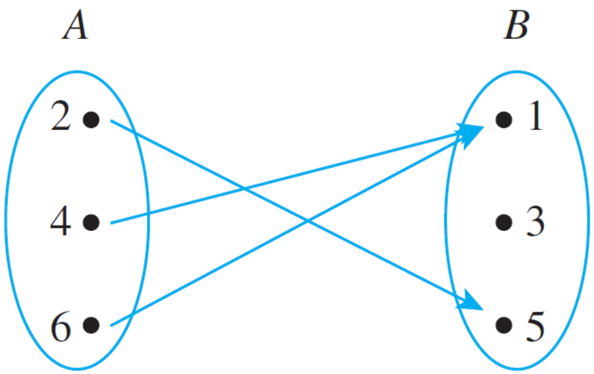
- Each and every element of A is related to exactly one element of B by the relation F
- Given any element x in A, the unique element in B that is related to x by F is denoted $F(x)$, read as "F of x".

Exercise

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$.

Which of the relations R, S, and T defined below are functions from A to B?

- $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
- For all $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.
- T is defined by the arrow diagram below:

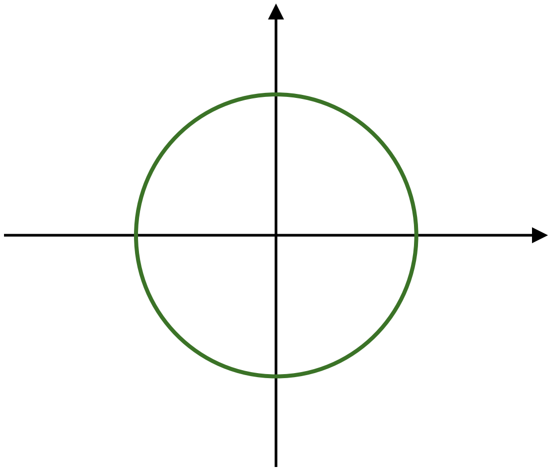


Exercise

- Define a relation from **R** to **R** as follows:

  For all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $(x, y) \in C$ means that $x^2 + y^2 = 1$

- Is C a function?

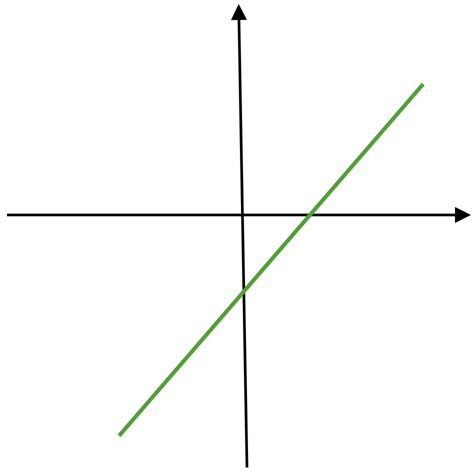


Exercise

- Define a relation from **R** to **R** as follows:

For all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $(x, y) \in L$ means that $y = x - 1$

- Is L a function?



Functions as Machines with I/O specification

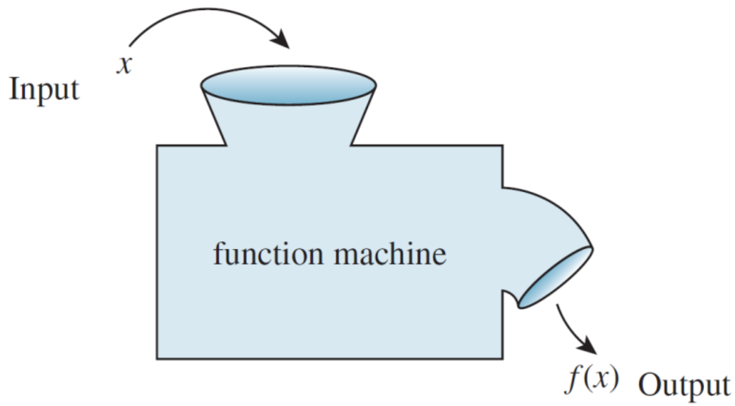
A function F from a set X to set Y

- Every element of X is related to exactly one element of Y by F.
- F sends (or maps) each x in X to a unique y in Y;

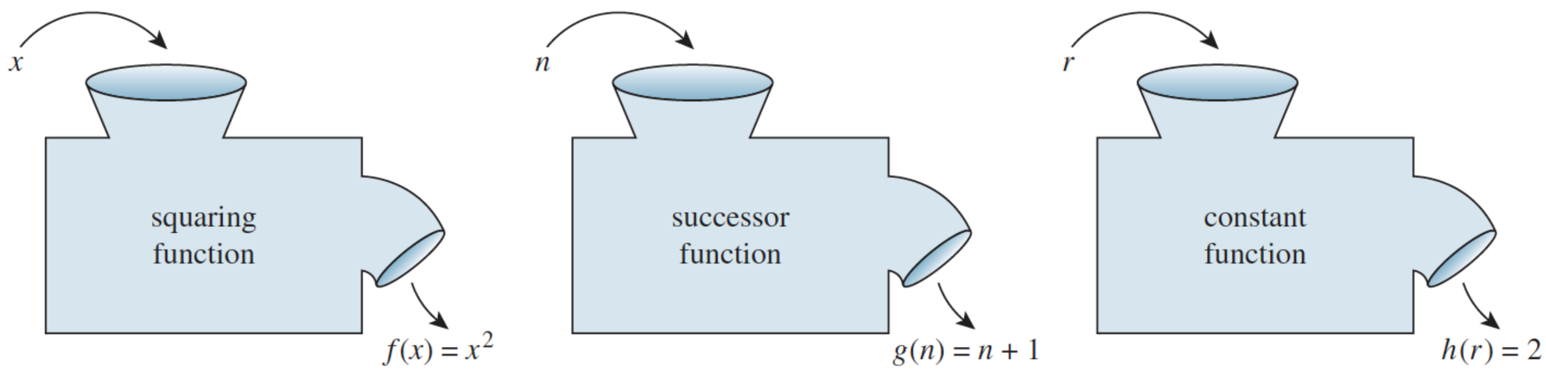
$$\boldsymbol{F: X \to Y}$$

- Let $F(x) = y$
  - y is the unique element in Y that is related to x by F
  - y is the value of F at x (y is the image of x under F)
  - y is the output of F for the input x

Functions as Machines with I/O specification



Examples: Functions



$f(x) = x^2$ (f: $x \longrightarrow x^2$)

$g(n) = n + 1$ (g: $n \longrightarrow n + 1$)

$h(r) = 2$ (h: $r \longrightarrow 2$)

Equality of Functions

- Note that a relation is a subset of Cartesian product and a function is a special kind of relation.
- A function is a relation in which each and every element of the domain maps to exactly one element in the co-domain.
- Specifically, if f and g are functions from a set A to a set B, then

$$f = \{ (x, y) \in A \times B \mid y = f(x) \}$$

and

$$g = \{ (x, y) \in A \times B \mid y = g(x) \}$$

- It follows that

f **equals** g, written $f = g$, *iff*, $f(x) = g(x)$ **for all** $\boldsymbol{x}$ in A.

Exercise

- Define $f : \mathbf{R} \longrightarrow \mathbf{R}$ and $g: \mathbf{R} \longrightarrow \mathbf{R}$ by the following formulas:

$$\begin{aligned} f(x) &= |x| &\quad \text{for all } x \in \mathbf{R} \\ g(x) &= \sqrt{x^2} &\quad \text{for all } x \in \mathbf{R} \end{aligned}$$

- Does $f = g$?

Proofs

What is a proof?

A proof is a valid argument that establishes the truth of a mathematical statement.

Methods of Proofs

- $\forall x(P(x) \rightarrow Q(x))$
- Direct Proofs
  - Assume a hypothesis is true and then logically deduces a conclusion.
  - Begin with the premises, continue with a sequence of deductions, and end with the conclusion.
  - Direct Proof Methods:
    - Proof by exhaustion
    - Proof by Induction
- Indirect Proofs
  - Do not start with the premises and end with the conclusion.
  - Indirect Proof Methods:
    - Proof by Contraposition
    - Proof by Contradiction

Proof Techniques

- Proof by Contraposition
- Proof by Induction – consists of two parts
  - Basis
  - Induction step
- Proof by Contradiction – assume the theorem is false and show that such an assumption leads to a false consequence
- Proof by Construction – demonstrate how to construct the object
- Proof by Exhaustion – exhaust all possibilities
- Proof by Cases – cover all possible cases

Example: Direct Proof

- “If n is an odd integer, then n^2 is odd.”
- Proof:
  Assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd.

  By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2.

  When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k$
  $2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (i.e., it is one more than twice an integer).

  Consequently, we have proved that if n is an odd integer, then n
  2 is an odd integer.

Example: Proof by Contraposition

- “If n is an integer and $3n + 2$ is odd, then n is odd.”

- Proof:
  Assume that the conclusion of the conditional statement “If $3n + 2$ is odd, then n is odd” is false; namely, assume that n is even.

  Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

  This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true.

  Our proof by contraposition succeeded; we have proved the theorem “If $3n + 2$ is odd, then n is odd”.

Example: Proof by Contradiction

- "If $3n + 2$ is odd, then n is odd."
- Proof:
  Let p be "$3n + 2$ is odd" and q be "n is odd."
  To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n + 2$ is odd and that n is not odd.

  Because n is not odd, we know that it is even. Because n is even, there is an integer k such that $n = 2k$.

  This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Because $3n + 2$ is $2t$, where $t = 3k + 1$, $3n + 2$ is even.

  Note that the statement "$3n + 2$ is even" is equivalent to the statement $\neg p$, because an integer is even if and only if it is not odd. Because both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd.

Example: Proof by Exhaustion

- Prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.
- Proof:
  We need verify the inequality $(n + 1)^3 \geq 3^n$ when n = 1, 2, 3, and 4.

  For n = 1, we have $(n + 1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$.

  For n = 2, we have $(n + 1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$.

  For n = 3, we have $(n + 1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$.

  For n = 4, we have $(n + 1)^3 = 5^3 = 125$ and $3^n = 3^4 = 81$.

  In each of these four cases, we see that $(n + 1)^3 \geq 3^n$. We have used the method of exhaustion to prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Existence Proofs

- $\exists x P(x)$
- Constructive Existence Proof
- Nonconstructive Existence Proof