Relations

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Session Outline

- Relation: Definition and Notations
- The Inverse of a Relation
- Binary relations and N-ary relations
- A relation on a Set
- Relation properties
  - Reflexive
  - Symmetric
  - Transitive

Relations

• Definition

Let A and B be sets. A **relation R from A to B** is a $\boxed{\text{subset of } A \times B}$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the $\boxed{\text{domain}}$ of R and the set B is called its $\boxed{\text{co-domain.}}$

Given an ordered pair (x,y) in $A \times B$, **x is related to y by R**, denoted $\boldsymbol{x\ R\ y}$, iff (x,y) is in R

- $\boldsymbol{x\ R\ y} \Leftrightarrow (x, y) \in R$
- $\boldsymbol{x \not{R} y} \Leftrightarrow (x, y) \notin R$

Example

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x,y) \in A \times B$, $(x,y) \in R \Leftrightarrow \frac{x-y}{2}$ is an integer.

- List the elements of $A \times B$
- What are the domain and the co-domain of R?
- Is $1\ R\ 3$? Is $2\ R\ 3$? Is $2\ R\ 2$?
- List the elements of R.
- Draw the arrow diagram for R.

Exercise 1

Define a relation L from **R** to **R** as follows:

For all real numbers x, y, $x \; L \; y \Leftrightarrow x < y$.

- Is 57 L 53? Is (-17) L (-14)? Is 143 L 143? Is (-17) L 3?
- Draw the graph of L as a subset of the Cartesian plane **R** x **R**.

Exercise 2

Let $X = \{a, b, c\}$. Find the power set of X, i.e., $P(X)$

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Define a relation S from $P(X)$ to $P(X)$ as follows:

For all sets A and B in $P(X)$, $(A, B) \in S \Leftrightarrow A$ has at least as many elements as B.

- Is $\{a, b\}\ S\ \{b, c\}$?
- Is $\{a\}\ S\ \emptyset$?
- Is $\{b, c\}\ S\ \{a, b, c\}$?
- Is $\{c\}\ S\ \{a\}$?

Exercise 3

Define a relation E from Z to Z as follows:

$$\text{For all } (m, n) \in Z \times Z,\ m\ E\ n \Leftrightarrow m - n \text{ is even}$$

- Is 4 E 0? Is 2 E 6? Is 3 E (-3)? Is (5,2) $\in E$?
- List five integers that are related by E to 1.

Exercise 4

Define a relation E from Z to Z as follows:

$$\text{For all } (m, n) \in Z \times Z,\ m\ E\ n \Leftrightarrow m - n \text{ is even}$$

Prove that if n is any odd integer, then n E 1.

Proof:

Suppose n is a p.b.a.c. odd integer.

Then $n = 2k + 1$ for some integer k.

By definition of E, $n\ E\ 1$ if and only if $n - 1$ is even.

Since $n = 2k + 1$, $\boldsymbol{n - 1} = (2k + 1) - 1 = \boldsymbol{2k}$.

Observe that $2k$ is even since k is an integer.

Thus $n - 1$ is even, and therefore, $n\ E\ 1$ by definition of E

The inverse of a Relation

If R is a relation from A to B, then the inverse relation R^{-1} from B to A is defined by interchanging the elements of all the ordered pairs of R.

• **Definition**

Let R be a relation from A to B. Define the $\boxed{\text{inverse relation } R^{-1} \text{ from } B \text{ to } A}$ as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

$$\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.$$

Example

Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be a relation from A to B:

$$\text{For all } (x, y) \in A \times B,\ x\ R\ y \Leftrightarrow x \mid y.$$

- Find all the elements of R and R^{-1}.
  - $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$
  - $R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$

y is divisible by x

- Draw arrow diagrams for R and R^{-1}.

- Describe R^{-1} in words
  - $y\ R^{-1}\ x \Leftrightarrow y$ is a multiple of x

Exercise

Define a relation L from **R** to **R** as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R},\ x\ L\ y \Leftrightarrow y = 2|x|$$

- Is $2\ L\ 4$?
- Is $-2\ L\ 4$?
- Is $4\ L^{-1}\ 2$?
- Is $4\ L^{-1}\ -2$?
- List 5 elements in the relation L.
- List 5 elements in the relation L^{-1}.

Exercise

Define a relation L from $\mathbf{R}$ to $\mathbf{R}$ as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R},\ x\ L\ y \Leftrightarrow y = 2|x|$$

- Draw the graph of L in the Cartesian plane.
- Is L a function?
- Draw the graph of L^{-1} in the Cartesian plane.
- Is L^{-1} a function?

Relations

- Let A and B be sets.
- A relation R from A to B is a subset A x B.
- Given an ordered pair (x, y) in A x B, x is related to y by R, written $x\ R\ y$, *iff*, (x, y) is in R.
- The set A is called the domain of R.
- The set B is called the co-domain of R.

N-ary Relations

- Given sets $A_1, A_2, \ldots, A_n$, an n-ary relation R on $A_1 \text{ x } A_2 \text{ x } \ldots \text{ x } A_n$ is a subset of $A_1 \text{ x } A_2 \text{ x } \ldots \text{ x } A_n$
- *n*-ary relations are foundation of "Relational Database Model"
- *n*-ary relation R on $A_1 \times A_2 \times \ldots \times A_n$
- An element $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \ldots \times A_n$ is called an *n*-tuple
- Binary, ternary and qu are special cases of *n*-ary relations.

What we call a "relation" as **a subset of a Cartesian product of *two* sets** is actually a form of relation called "***binary* relation**".

Relation on a Set

- A relation on a set A is a relation from A to A.
- It's a form of binary relation where the domain and the co-domain is the same set.
- We say, a Relation R is defined on a set A when the set A is both the domain and the co-domain.

Example

Let $A = \{3, 4, 5, 6, 7, 8\}$ and

Define a relation R on A as follows:

$$\text{For all } x, y \in A, x \, R \, y \Leftrightarrow 2 \mid (x - y).$$

- Is $3 \, R \, 3$?
- Is $3 \, R \, 4$?
- Is $3 \, R \, 5$?
- Is $3 \, R \, 6$?
- Is $3 \, R \, 7$?
- Is $3 \, R \, 8$?

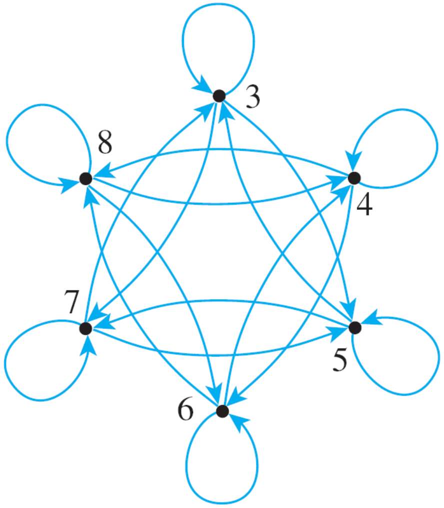
Example

Let $A = \{3, 4, 5, 6, 7, 8\}$ and

Define a relation R on A as follows:

$$\text{For all } x, y \in A,\ x\ R\ y \Leftrightarrow 2 \mid (x - y).$$

- Find all the elements of R.
- Draw **the directed graph** of the relation R.



Reflexivity, Symmetry, Transitivity

- **Definition**

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x\ R\ x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, ***if*** $x\ R\ y$ then $y\ R\ x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, ***if*** $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

1. R is reflexive	$\Leftrightarrow$	for all x in A, $(x, x) \in R$.
2. R is symmetric	$\Leftrightarrow$	for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive	$\Leftrightarrow$	for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

All the definitions are for a relation *R* on a set *A*

Equivalence Relation

- Let A be a set and R a relation on A.
- R is an equivalence relation, *iff*, R is reflexive, symmetric, and transitive.

All the definitions are for
a relation R on a set A

Equivalence Class

- Let A be a set and R an equivalence relation on A.
- For each element a in A, the equivalence class of a (the class of a), denoted $[a]$ is the set of all elements x in A such that x is related to a by R.

$$[a] = \{ x \in A \mid x \, R \, a \}$$

All the definitions are for a relation R on a set A

Example: Equivalence Relation

- Consider the problem of grouping a set of people in a conference room into groups of people having the same birthdays.
- Let A be the set of people in the conference room
- We can define a relation R on A as follows:
  - For all $x, y \in A$, $x \; R \; y \Leftrightarrow x$ and y have the same birthday.
- Then followings about the relation R are true:
  - Every person has the same birthday as himself/herself.
  - If Chai has the same birthday as Jane, then Jane surely has the same birthday as Chai.
  - If Chai has the same birthday as Jane and Jane has the same birthday as Kim, then Chai has the same birthday as Kim.

Example: Equivalent Classes

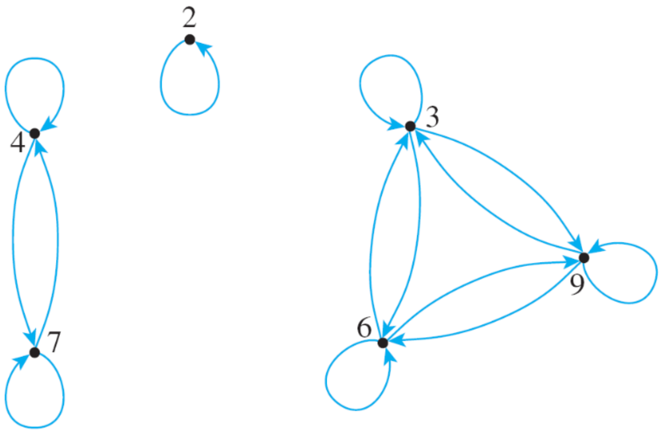
- Every person has the same birthday as himself/herself.
  - For all $x \in A$, $x\ R\ x$ (R is reflexive)
- If Chai has the same birthday as Jane, then Jane surely has the same birthday as Chai.
  - For all $x, y \in A$, if $x\ R\ y$ then $y\ R\ x$ (R is symmetric)
- If Chai has the same birthday as Jane and Jane has the same birthday as Kim, then Chai has the same birthday as Kim.
  - For all $x, y, z \in A$, if $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z$ (R is transitive)
- All the people in the conference room who has the same birthday as Chai is called the equivalence class of Chai
- $[Chai] = \{x \in A \mid x\ R\ Chai\}$ equivalence class of *Chai* under R

Example

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows:

For all $x, y \in A$, $3 \mid (x - y)$

- Find all elements of R
- Draw the directed graph of R.
- Is R an equivalence relation?
  - Is R reflexive?
  - Is R symmetric?
  - Is R transitive?



Example

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A,\ \ 3 \mid (x - y)$$

Find the distinct equivalence classes of the relation R.

$[2] = \{x \in A \mid x\, R\, 2\} = \{x \in A \mid 3 \text{ divides } (x - 2)\} = \{2\}$

$[3] = \{x \in A \mid x\, R\, 3\} = \{x \in A \mid 3 \text{ divides } (x - 3)\} = \{3, 6, 9\}$

$[4] = \{x \in A \mid x\, R\, 4\} = \{x \in A \mid 3 \text{ divides } (x - 4)\} = \{4, 7\}$

$[6] = \{x \in A \mid x\, R\, 6\} = \{x \in A \mid 3 \text{ divides } (x - 6)\} = \{3, 6, 9\}$

$[7] = \{x \in A \mid x\, R\, 7\} = \{x \in A \mid 3 \text{ divides } (x - 7)\} = \{4, 7\}$

$[9] = \{x \in A \mid x\, R\, 9\} = \{x \in A \mid 3 \text{ divides } (x - 9)\} = \{3, 6, 9\}$

- Therefore, distinct equivalence classes of R are $\{2\}$, $\{3,6,9\}$ and $\{4,7\}$
- Note that distinct equivalence classes of R is a partition of A.