Predicate Logic II

CSX2008 Mathematics Foundation for Computer Science

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Multiply-Quantified Statements

Consider the following two statements:

- ∀ people *x*, ∃ a person *y* such that *x* loves *y*.
  - Given any person, you can find someone whom this person loves
  - Everyone loves someone
- ∃ a person *y* such that ∀ people *x*, *x* loves *y*.
  - You can find someone who is loved by everyone.
  - There is someone that everyone loves.

Key Points:

- The Order of Quantifications usually dictates the interpretation, hence the truth of the statements.
- In a statement containing both ∀ and ∃, changing the order of the quantifiers *usually* changes the truth of the statement.

When the Order does not Matter

- Consider the following two statements:
  - ∀ real numbers ***x*** and ∀ real numbers ***y***, $x + y = y + x$
  - ∀ real numbers ***y*** and ∀ real numbers ***x***, $x + y = y + x$
- Any difference?
  - Both statements mean the same
  - ∀ real numbers x and y, $x + y = y + x$

Key Points:

- If one quantifier immediately follows another quantifier of the same type, then the order of the quantifiers does not affect the interpretation.

Truth Values of Multiply-Quantified Statement (1)

$\forall$ people *x*, $\exists$ a person *y* such that *x* loves *y*.

$\exists$ a person *y* such that $\forall$ people *x*, *x* loves *y*.

- How to determine the truth values?
  - Given any person *x*, you can find at least one person *y* whom this person loves (first statement)
  - You can find a person *y* who is loved by everyone (second statement)
  - Whoever person *x* is chosen, *x* loves *y*

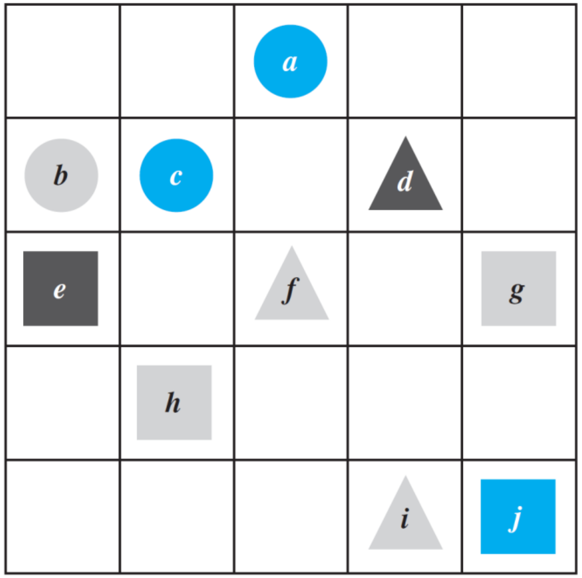
Truth Values of Multiply-Quantified Statement (2)

Establishing the truth of following statement forms

- $\forall x \in D, \exists y \in E$ such that $P(x, y)$
  - Imagine that you allow your opponent to pick whatever element x in D and then you must be able to find an element y in E that makes $P(x, y)$ for that particular x.

- $\exists y \in D$ such that $\forall x \in E, P(x, y)$
  - Are you able to find one particular y in D that will make $P(x, y)$ no matter what x in E your opponent might choose to challenge you with?

Exercise



Determine if the following statements are true or false in the Tarski world shown:

a) For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.

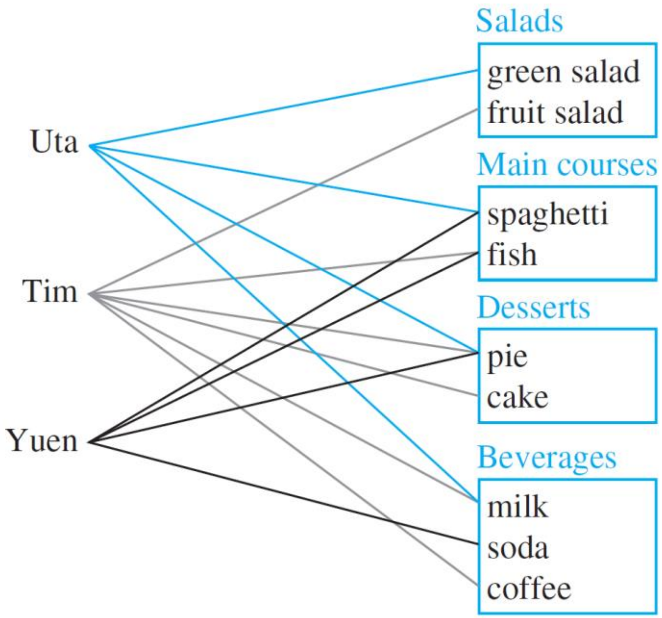
b) There is a triangle *x* such that for all circles *y*, *x* is a to the right of *y*.

c) For every square *x* there is a triangle *y* such that *x* and *y* have different colors.

d) There exists a triangle *y* such that for every square *x*, *x* and *y* have different colors.

Exercise

Uta: green salad, spaghetti, pie, milk
Tim: fruit salad, fish, pie, cake, milk, coffee
Yuen: spaghetti, fish, pie, soda



Write each of following statements informally and find its truth value.

a. ∃ an item *I* such that ∀ students *S*, *S* chose *I*.
b. ∃ a student *S* such that ∀ items *I*, *S* chose *I*.
c. ∃ a student *S* such that ∀ stations *Z*, ∃ an item *I* in *Z* such that *S* chose *I*.
d. ∀ students *S* and ∀ stations *Z*, ∃ an item *I* in *Z* such that *S* chose *I*.

Ambiguity in Everyday English

- Imagine you are visiting a factory that manufactures computer microchips. The factory guide tells you,
  - There is a person supervising every detail of the production process.

- What does he really mean?
  - There is one single person who supervises all the details of the production process.
    - ∃ a person *p* such that ∀ details *d, s* supervises *d*.
  - For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.
    - ∃ detail *d* and ∃ a person *p* such that *p* supervises *d*.

Ambiguity in Everyday English

- English expressions are open to ambiguity in their logical interpretation.
- Obviously, we need to select a particular interpretation to be able to determine the truth or falsity of the statement.
- Therefore, we may have to use context to try to ascertain the meaning as best we can.
  - context sensitive

English to Logical Statements

Rewrite the following informal statements into formal statements using quantifiers and variables:

- Every nonzero real number has a reciprocal
  - ∀ nonzero real numbers n, ∃ a real number r such that $nr = 1$.
  - $\forall n \in R^{\neq 0}, \exists r \in R$ such that $nr = 1$.
- There is a real number with no reciprocal
  - ∃ a real number r such that ∀ real numbers s, $rs \neq 1$.
- There is a smallest positive integer.
  - ∃ a positive integer m such that ∀ positive integers n, $m \leq n$
- There is no smallest positive real number.
  - ∀ positive real numbers x, ∃ a positive real number y such that $y < x$.

Negations of Multiply-Quantified Statements

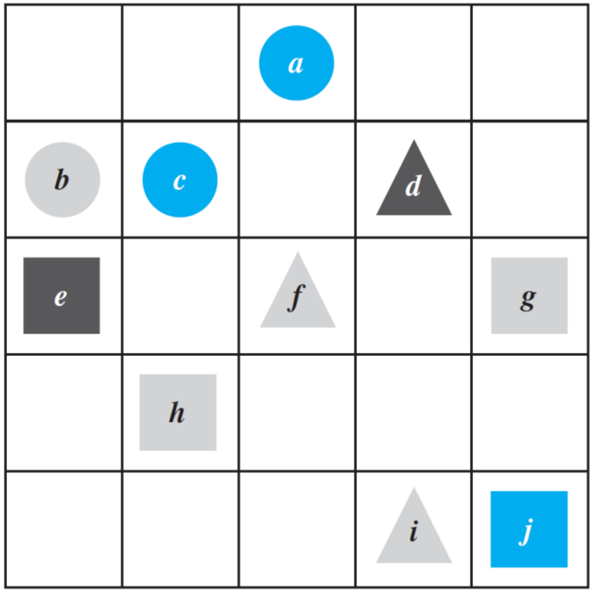
Earlier, we learned that

- $\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D$ such that $\sim P(x)$.
- $\sim(\exists x \text{ in } D$ such that $P(x)) \equiv \forall x \text{ in } D, \sim P(x)$.

Finding negations of multiply-quantified statements

- Apply the above rules step by step from left to right
- $\sim(\forall x \text{ in } D, \exists y \text{ in } E$ such that $P(x, y))$
  - $\exists x \text{ in } D$ such that $\sim(\exists y \text{ in } E$ such that $P(x, y))$
  - $\exists x \text{ in } D$ such that $\forall y \text{ in } E, \sim P(x, y)$
- $\sim(\exists x \text{ in } D$ such that $\forall y \text{ in } E, P(x, y))$
  - $\forall x \text{ in } D, \sim(\forall y \text{ in } E, P(x, y))$
  - $\forall x \text{ in } D, \exists y \text{ in } E$ such that $\sim P(x, y)$

Exercise



Write a negation for each of the following statements, and determine which is true; the given statement or its negation.

a. For all squares *x*, there is a circle *y* such that *x* and *y* have the same color.

b. There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.

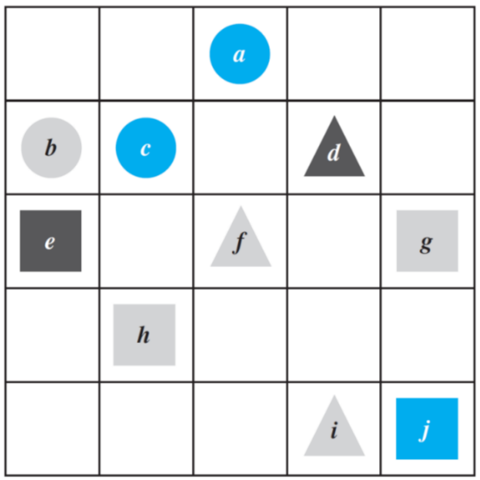
Formal Notation for First-Order Logic

- Language of First Order Logic is basically Predicate Logic we have been learning so far, but uses purely symbolic notations for quantifiers, variables, predicates and logical connectives. In particular, it gets rid of the word "such that" in existential statements.
  - $\exists x$ in D such that $P(x)$ is written as $\exists x\ (x \in D \wedge P(x))$
  - $\forall x$ in D, $P(x)$ is written as $\forall x\ (x \in D \rightarrow P(x))$

- The formal symbolic notation of First Order Logic is used in
  - Theory of Computation
  - Artificial Intelligence
  - Formal Languages and Automata
  - Program Verification,
  - and etc.

Good for automating the process of logical evaluation as programed procedure.

Exercise



Triangle(x), Circle(x), Square(x)

Blue(x), Gray(x), Black(x)

RightOf(x, y), Above(x, y)

SameColorAs(x, y)

$x = y$

x equals to *y*

Write the following statements and their negations using the language of First Order Logic

a. For all circles *x*, *x* is above *f*.

b. There is a square *x* such that *x* is black.

c. For all circles *x*, there is a square *y* such that *x* and *y* have the same color.

d. There is a square *x* such that for all triangles *y*, *x* is to right of *y*.

Exercise (cont'd)

- For all circles *x*, *x* is above *f*.

- There is a square *x* such that *x* is black.

Exercise (cont'd)

- For all circles *x*, there is a square *y* such that *x* and *y* have the same color.

Exercise (cont'd)

- There is a square *x* such that for all triangles *y*, *x* is to right of *y*.

Valid Forms of Arguments

Modus Ponens	$p \to q$ p • q	**Elimination**	**a.** $p \vee q$ $\sim q$ • p **b.** $p \vee q$ $\sim p$ • q
Modus Tollens	$p \to q$ $\sim q$ • $\sim p$	**Transitivity**	$p \to q$ $q \to r$ • $p \to r$
Generalization	**a.** p • $p \vee q$ **b.** q • $p \vee q$	**Proof by** **Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ • r
Specialization	**a.** $p \wedge q$ • p **b.** $p \wedge q$ • q		
Conjunction	p q • $p \wedge q$	**Contradiction Rule**	$\sim p \to c$ • p

Important Valid Argument Forms in Predicate Logic

- Universal Instantiation
  - *aka* Universal Elimination, Universal Specification
- Universal Modus Ponens
  - By combining Universal Instantiation and Modus Ponens (from Propositional Logic)
- Universal Modus Tollens
  - By combining Universal Instantiation and Modus Tollens (from Propositional Logic)

Universal Instantiation

$\forall x \in D, P(x)$

$c \in D$

$\therefore\ P(c)$

If a property or condition is true for all elements in a domain, then it is true for any particular element in the domain.

- Every dog has four legs
- Fido is a dog
- Therefore, Fido has four legs

$\forall\ x \in D$, x has four legs, where D is a set of dogs

Fido $\in D$

$\therefore$ Fido has four legs

Notice the implied general domain of dogs

Example: Universal Instantiation

- What is 5^2?
- But more importantly, why?
- Applying Universal Instantiation:

  $\forall x \in \boldsymbol{R}, x^2 = x \cdot x$

  $5 \in \boldsymbol{R}$

  $\therefore 5^2 = 5 \cdot 5 = 25$

Universal Modus Ponens

- $\forall x \in D, P(x) \longrightarrow Q(x)$
- $\quad c \in D \wedge P(c)$
- $\therefore \; Q(c)$

$\forall x$, if $P(x)$ then $Q(x)$

$P(c)$ for a particular c

$\therefore Q(c)$

Implied Domain Membership

- If x makes $P(x)$ true then x makes $Q(x)$ true
- c makes $P(x)$ true.
- $\therefore c$ makes $Q(x)$ true.

Implied Universal Quantification and Domain Membership

Example: Universal Modus Ponens

- If an integer is even, then its square is even
- k is an integer that is even.
- Therefore, k^2 is even

- $\forall x \in \mathbf{Z}$, if x is even then x^2 is even
- k is an even integer
- $\therefore k^2$ is even

Universal Instantiation vs. Universal Modus Ponens

Every dog has four legs

Fido is a dog

Therefore, Fido has four legs

$\forall$ dogs x, x has four legs

Fido is a dog

$\therefore$ Fido has four legs

- Universal Instantiation, but can be stated as Universal Modus Ponens by enlarging the Domain with a Superclass.

$\forall$ animals x, if x is a dog, then x has four legs

Fido is a dog

$\therefore$ Fido has four legs

Exercise

Rephrase the following argument as Universal Modus Ponens

- All men are mortal
- Socrates is a man
- ∴ Socrates is mortal

Looks natural as Universal Instantiation, but can be re-phrased as Universal Modus Ponens.

Universal Modus Tollens

$\forall x \in D, P(x) \rightarrow Q(x)$

$c \in D \wedge \sim Q(c)$

$\therefore \sim P(c)$

$\forall x$, if $P(x)$ then $Q(x)$

$\sim Q(c)$ for a particular c

$\therefore \sim P(c)$

Implied Domain Membership

If x makes $P(x)$ true then x makes $Q(x)$ true

c does NOT make $Q(x)$ true.

$\therefore$ c does NOT makes $P(x)$ true.

Implied Universal Quantification and Domain Membership

Examples: Universal Modus Tollens

- Every dog has four legs

  Nemo does not have four legs.

  Therefore, Nemo is not a dog.

- If a number is an integer, then it is a rational.

  k is a particular number that is not a rational number.

  Therefore, *k* is not an integer.

Exercise: Recognizing the Forms

Identify the form of following arguments and rewrite them as statements with quantifiers, predicate symbols and variables.

- Pigs can't fly.
- Wilbur is a pig.
- Therefore, Wilbur can't fly.

- All human beings are mortal.
- Zeus is not mortal.
- Therefore, Zeus is not human.

Exercise: Recognizing the Forms

Identify the form of following arguments and rewrite them as statements with quantifiers, predicate symbols and variables.

- All good drivers are very alert.
  People who are drunk are not very alert.
  Therefore, people who are drunk are not good drivers.
- Let $G(x)$ be "x is a good driver",
  let $A(x)$ be "x is very alert", and
  let D stand for "People who are drunk"

  $\forall x$, if $G(x)$ then $A(x)$

  $\sim A(D)$

  $\therefore \sim G(D)$

Exercise: Recognizing the Forms

Identify the form of following argument and rewrite them as statements with quantifiers, predicate symbols and variables.

- No good car is cheap
- An Aston Martin is a good car
- Therefore, An Aston Martin is not cheap

All good cars are not cheap

Exercise: Recognizing the Forms

Identify the form of following argument and rewrite it as statements with quantifiers, predicate symbols and variables.

- No polynomials functions have horizontal asymptotes.
- This function has a horizontal asymptote.
- Therefore, this function is not a polynomial function.

- No polynomials functions have horizontal asymptotes.
- Every polynomial function does not have a horizontal asymptote
- All polynomial functions do not have horizontal asymptotes.

Important Reminder

- Remind yourself of the differences among “Valid” vs. “True” vs. “Sound” from last week.
- Universal Instantiation, Universal Modus Ponens, Universal Modus Tollens are valid argument forms.
- Argument Form is valid means:
  - No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting statements are all true, then the conclusion is also true.
  - Truth of the conclusion follows necessarily (inescapably) from the truth of the its premises.
- An argument is valid if and only if its form is valid
- Validity is a property of Argument Forms
  - If an argument is valid (or invalid), then so is every other arguments that has the same form.

Invalid Argument Form: Converse Error

Converse Error (in Predicate Logic)

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $c \in D \wedge Q(c)$
- $\therefore\ P(c)$

$\forall x$, if $P(x)$ then $Q(x)$

$Q(c)$ for a particular c

$\therefore P(c)$

- If x makes $P(x)$ true then x makes $Q(x)$ true
- c makes $Q(x)$ true.
- $\therefore c$ makes $P(x)$ true.

Invalid Argument Forms: Inverse Error

Inverse Error (in Predicate Logic)

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $c \in D \wedge \sim P(c)$
- $\therefore \sim Q(c)$

$\forall x$, if $P(x)$ then $Q(x)$

$\sim P(c)$ for a particular c

$\therefore \sim Q(c)$

- If x makes $P(x)$ true then x makes $Q(x)$ true
- c does NOT make $P(x)$ true.
- $\therefore$ c does NOT makes $Q(x)$ true.

Exercise

Is the following argument valid or Invalid? Why?

- Every dog has four legs.
- Tom has four legs.
- Therefore, Tom is a dog.

$\forall x$, if $P(x)$ then $Q(x)$

$Q(c)$ for a particular c

$\therefore P(c)$

An example of Converse Error in argument

Exercise

Is the following argument valid or Invalid? Why?

- Every dog has four legs.
- Tom is not a dog.
- Therefore, Tom does not have four legs.

$\forall x$, if $P(x)$ then $Q(x)$

$\sim P(c)$ for a particular c

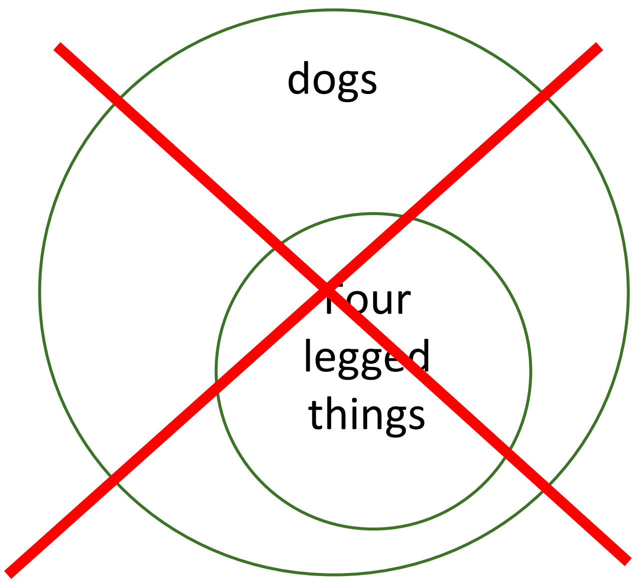
$\therefore \sim Q(c)$

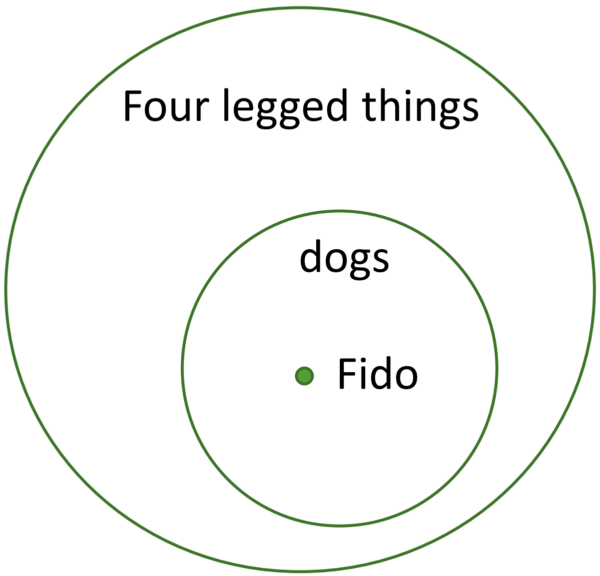
An example of Inverse Error in argument

Using Venn Diagram to Test for Validity

Informal, but it can help you analyze

- Every dog has four legs
- Fido is a dog
- Therefore, Fido has four legs





Using Venn Diagram to Test for Validity

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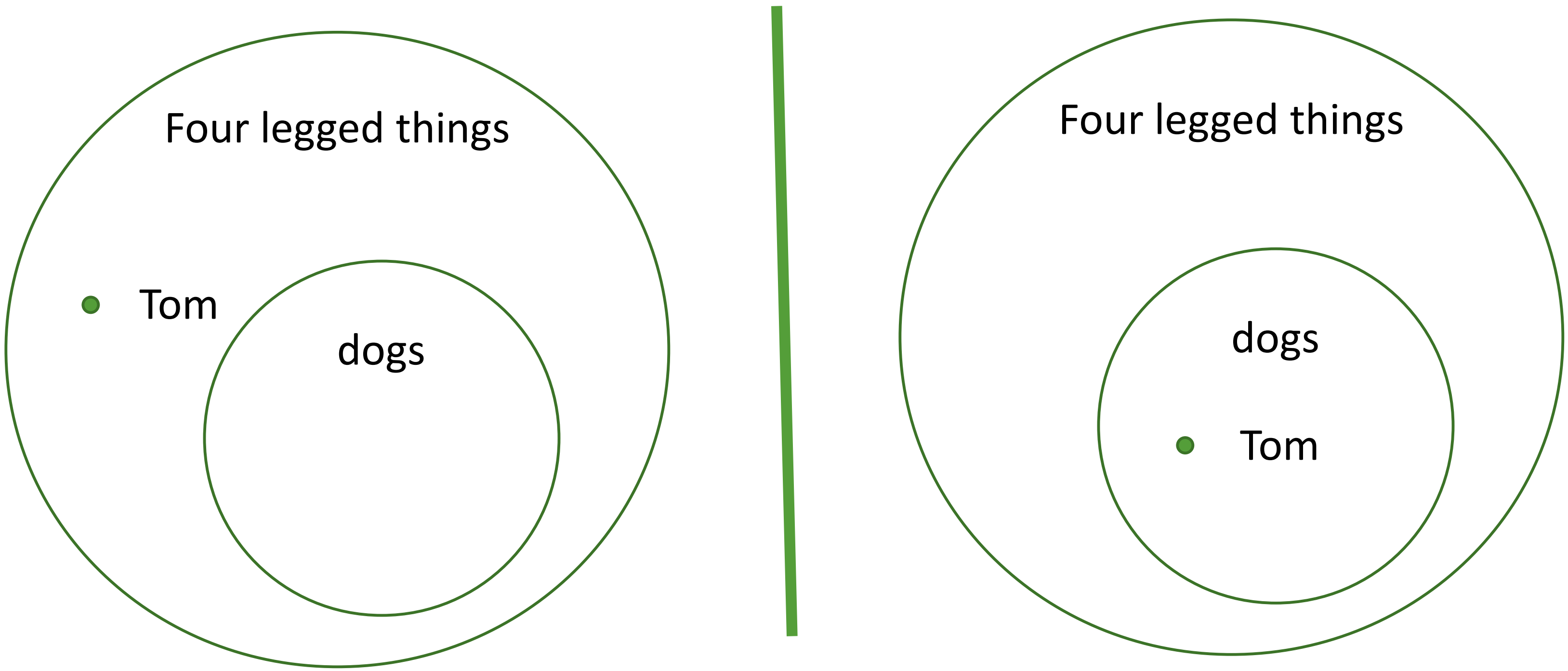
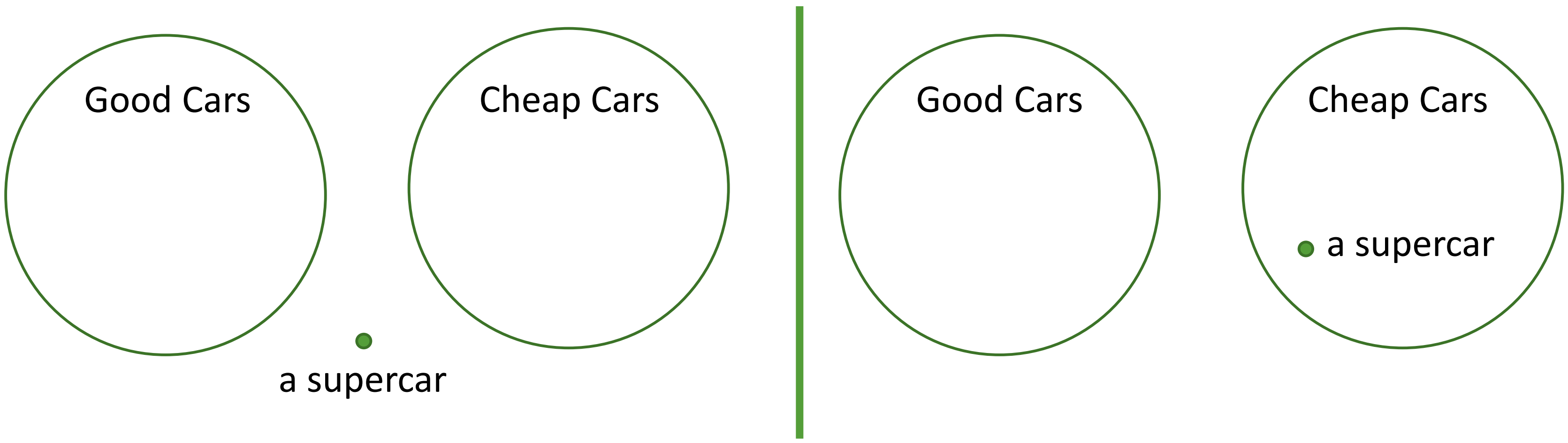


Using Venn Diagram to Test for Validity

- No good car is cheap
- An Aston Martin is a good car
- Therefore, An Aston Martin is not cheap

Using Venn Diagram to Test for Validity

- No good car is cheap
- A supercar is not a good car
- Therefore, a supercar is not cheap



Other Valid Forms involving Universal Quantification

- By combining Universal Instantiation with the Rules of Inference for Propositional Logic, we can create more valid argument forms involving universal quantification.
  - See Section 2.3 of the textbook
- For example, we can generate Universal Transitivity
  - Universal Instantiation + Transitivity

Universal Transitivity

Formal Version	*Informal Version*
$\forall x\, P(x) \rightarrow Q(x)$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\, Q(x) \rightarrow R(x)$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
$\bullet\ \forall x\, P(x) \rightarrow R(x)$.	$\bullet$ Any x that makes $P(x)$ true makes $R(x)$ true.

Valid Forms of Arguments

Modus Ponens	$p \to q$ p • q	**Elimination**	**a.** $p \vee q$ $\sim q$ • p **b.** $p \vee q$ $\sim p$ • q
Modus Tollens	$p \to q$ $\sim q$ • $\sim p$	**Transitivity**	$p \to q$ $q \to r$ • $p \to r$
Generalization	**a.** p • $p \vee q$ **b.** q • $p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ • r
Specialization	**a.** $p \wedge q$ • p **b.** $p \wedge q$ • q		
Conjunction	p q • $p \wedge q$	**Contradiction Rule**	$\sim p \to c$ • p

Deductive Reasoning

- What we have studied so far is Deductive Reasoning.
- The process of reasoning from one or more premises to reach a logically certain conclusion.
- Rules of inference

Abductive Reasoning

- Unlike Deductive reasoning, the truths of premises do not guarantee the conclusion, but indicate possibility.
- The form is a variation on Converse Error.
- Technicians troubleshooting the causes of a problem uses similar reasoning as do doctors making diagnosis based on observed symptoms of patients.
- Artificial Intelligence, Expert Systems, etc.
- Probabilistic Abductive Reasoning in Bayesian Network

Example: Abductive Logic

- A detective (Sherlock Holmes) observes the following:
  - All members of the Blue Dragon gang visits the temple once a week.
  - Somchai was seen visiting the same temple.
- Can the detective conclude that Somchai is a member of the Blue Dragon gang with a certainty?
  - No, that would be an example of Converse Error
  - But, it does raise the possibility of Somchai being a member.