Sequence and Recurrence Relation

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Big Picture

- Main Topic
  - Principle of Mathematical Induction as a method of Proof.
- Related Topics
  - Problem Solving using Recursive D&C
  - Defining Mathematical Objects Recursively
    - Initial Conditions and Recurrence Relations
  - Concept of Sequence and Series as examples of objects that can be recursively defined.
  - Mathematic Induction vs. Strong Mathematical Induction vs. Well-Ordering Principle

Session Outline

Sequences

- Definition and Properties
- Summation Notation and Product Notation
- Ways of defining Sequences
  - Listing vs. Explicit Formula vs. Recursive Definition

Recursive Logic and Problem Solving using Recursive D&C

- Forming Recursive Solution
  - Base Case, and Recurrence Relation

Sequence

Informally, a sequence is an ordered list of elements.

- Remind yourself of Sequence vs. Set from week #1
  - A, B, C, D, E, F (a sequence of length 6)
  - 2, 4, 5, 7, 5, 6, 8, 3, 2, 6 (sequence of length 10)
  - 2, 4, 6, 8, … (an infinite sequence)

More formally, a sequence is defined as a function.

- Remind yourself of “relation” and “function” from week #1

Sequence

- A sequence is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.
- We can also associate a position to each element of a sequence using a running number of integers

10	11	12	13	14	15
C	B	C	A	B	F

1	2	3	4	…
2	4	6	8	…

Sequence defined as a function.

Domain for the first example: Integers between 10 and 15
Domain for the second example: All integers greater than or equal to 1

- $F(1) = 2 = a_1$, $F(2) = 4 = a_2$, $F(3) = 6 = a_3$, $F(350) = 700 = a_{350}$, and so on
- $F(n) = 2 \cdot n = a_n$

Terminologies of Sequence

Consider the sequence shown below:

$$a_m \, , \, a_{m+1} \, , \, a_{m+2} \, , \, \ldots, \, a_n$$

- Each individual element a_k (a sub k) is called a term
- The k in a_k is called a subscript (index)
- a_m is the initial term and m is the initial subscript
- a_n is the final term and n is the final subscript
- $m \leq n$
- An infinite sequence ends with an ellipsis and does not have a specific final term, e.g. 2, 4, 6, 8, 10, …
- Explicit Formula (General Formula) for a sequence is a rule that shows how the values of a_k depend on k
  - $a_k = 2 \cdot k$ for all integers $k \geq 1$
  - Specification of the subscript range is essential in explicit formula

Exercise

Consider a sequence defined by each of the following explicit formulas.

- Write the first five terms of the sequence for each formula.

$$a_k = \frac{k}{k+1} \quad \text{for all integers } k \geq 1,$$

$$b_i = \frac{i-1}{i} \quad \text{for all integers } i \geq 2.$$

The same sequence may be defined by different explicit formulas.

An Alternating Sequence

Consider a sequence defined by the following explicit formula.

$$c_j = (-1)^j \text{ for all integers } j \geq 0.$$

Write the first six terms of the sequence

- $c_0 = (-1)^0 = 1$
- $c_1 = (-1)^1 = -1$
- $c_2 = (-1)^2 = 1$
- $c_3 = (-1)^3 = -1$
- $c_4 = (-1)^4 = 1$
- $c_5 = (-1)^5 = -1$

An infinite sequence may have a finite number of values.

Finding an Explicit Formula

Consider a sequence shown below

$$1, -\frac{1}{4}, \ \frac{1}{9}, \ -\frac{1}{16}, \ \frac{1}{25}, \ -\frac{1}{36}, \ldots$$

Find an explicit formula for the sequence

- We have to employ process of "Educated Guessing"
- Some common techniques
  - $1 = \frac{1}{1}$
  - $1, 4, 9, 16, 25, 36, \ldots = 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, \ldots$
  - $(-1)^j = 1$ when j is even and -1 when j is odd
  - $(-1)^{j+1} = 1$ when j is odd and -1 when j is even
- $a_k = \frac{(-1)^{k+1}}{k^2}$ for all integers $k \geq 1$
- $a_k = \frac{(-1)^k}{(k+1)^2}$ for all integers $k \geq 0$

Ways of Defining Sequences

Consider the following sequence: 3, 5, 7, …

- What do you think is the next term after 7?
  - 9 if it's a sequence of odd integers
  - 11 if it's a sequence of odd prime numbers

- This informal way of defining infinite sequences can be ambiguous.

Ways of Defining Sequences

Precise and formal ways of defining sequences:

- Explicit Formula
- Recursive Definition

Explicit Formula

- Consider the following sequence:

$$a_k = 2 \cdot k + 1 \quad \text{for all integers } k \geq 1$$

- Sequence defined by an explicit formula is precise and each term can be found in constant time.

- However, finding an explicit formula for a sequence can be sometimes difficult or even impossible.

Recursive Definition

- Consider the following sequence:

$$a_1 = 3 \text{ and } a_k = a_{k-1} + 2 \text{ for all integers } k \geq 2$$

- The above defines the sequence recursively, which consists of:
  - one or more initial conditions
  - a recurrence relation
- Sequence defined recursively is precise but finding the value of a specific term requires knowing the values of all the previous terms.
- Sometimes, finding a recursive definition of a sequence can be easier than finding an explicit formula.

Recursive Definition

- A way of defining an object in terms of itself.
- A very common technique in mathematics and CS.
- Example: Recursive Definition of Factorial
  - $N! = N \cdot (N-1)!$ for all integer $N \geq 1$
    - Recurrence Relation (Inductive Clause)
  - $0! = 1$
    - Initial Conditions (Base Cases)
- Sequences can be defined recursively by specifying a recurrence relation and initial conditions.

Recursive Definition

• Definition

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$. The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on k.

When defining a recurrence relation a_k in terms of a_{k-i} ,

- ensure $k - i \geq 0$ (for the smallest possible value of k)
- ensure initial conditions cover up to a_{i-1}

Exercise

Consider the following sequence defined recursively.

$$c_k = c_{k-1} + kc_{k-2} + 1 \text{ for all integers } k \geq 2$$

$$c_0 = 0 \text{ and } c_1 = 2$$

- Find c_2, c_3, and c_4.

Exercise

Consider the following sequence defined recursively.

$c_{k+1} = c_k + (k+1)c_{k-1} + 1$ for all integers $k \geq 1$

$c_0 = 0$ and $c_1 = 2$

- What can we say about this sequence and previous one?

Exercise

Consider the following recurrence relation.

$$a_k = 3a_{k-1} \text{ for all integers } k \geq 2$$

- Find a_2 , a_3 , a_4 when the initial condition is that $a_1 = 2$.
  - 2, 6, 18, 54, 162, …
- Find a_2 , a_3 , a_4 when the initial condition is that $a_1 = 1$.
  - 1, 3, 9, 27, 81, …

Exercise

- Consider the following sequence defined with an explicit formula

$$t_n = 2 + n \text{ for all integers } n \geq 0$$

- Find t_0, t_1, t_2, t_3, t_{100}, t_i for any $i \geq 0$

Exercise

Show that this sequence satisfies the following recurrence relation:

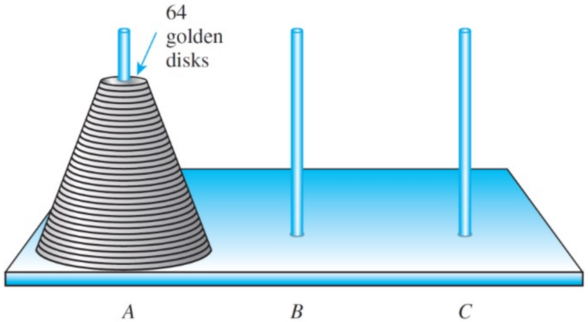
$$t_k = 2t_{k-1} - t_{k-2} \text{ for all integers } k \geq 2$$

- Substitute k, k-1 and k-2 in place of n in the definition:
  - $t_k = 2 + k$
  - $t_{k-1} = 2 + (k - 1)$
  - $t_{k-2} = 2 + (k - 2)$
- We can do above substitution for all integers $k \geq 2$ (Why?)
- Then $2t_{k-1} - t_{k-2}$
  $= 2(2 + (k - 1)) - (2 + (k - 2))$
  $= 2(1 + k) - k$
  $= 2 + 2k - k$
  $= 2 + k$
  $= t_k$ for all integer $k \geq 2$. Q.E.D.

Recursive Logic (Recursion)

- Recursion is one of the central ideas in Computer Science
  - A form of Divide and Conquer
- To solve a problem recursively means:
  - Find a way to break the problem into smaller subproblems each having the same form as the original problem
  - Assume the existence of solutions to the smaller problems
  - Define the solution to the original problem in terms of solutions to the smaller subproblems
  - Keep doing this until the each of the broken subproblems is trivial enough to be solved by a simple method.
  - Then solutions to each subproblem can be used together to form a solution to the original problem.

Tower of Hanoi: Famous Example



- The goal of the game:
  - To move all the disks from one pole to another pole
- The rules of the game:
  - Only one disk can be moved at a time
  - No disk of a larger size can be placed on top of a smaller one
- The problem to be solved:
  - What’s the minimum number of moves required to transfer all the disks from one pole to another?

Tower of Hanoi (cont’d)

- What’s the minimum number of moves required to transfer N disks from one pole to another?
- Applying D&C strategy and thinking recursively
  - Moving N disks from Source Pole to Target Pole
    - Moving $N – 1$ disks from “Source Pole” to “Temp Pole”
    - Moving the largest disk (disk # N) from “Source Pole” to “Target Pole”
    - Moving $N – 1$ disks from “Temp Pole” to “Target Pole”
  - But how do we move $N – 1$ disks from “source Pole” to Temp Pole”?
    - Well, this smaller problem is same as the original problem, **but smaller**
    - How to solve it? Use the same solution used to move N disks!
  - But how do we move $N – 2$, $N – 3$, $N – 4$, $N – 5$, etc disks?
  - Keep breaking and sooner or later we are left with a very small problem: move 1 disk!

Tower of Hanoi (cont'd)

- Let M_N be the minimum number of moves to move N disks
- Assume we know the answer to $M_{N\text{-}1}$
  - minimum moves for $N - 1$ disks
- Then $M_N = M_{N-1} + 1 + M_{N-1} = 2M_{N-1} + 1$ for all integers $N \geq 2$
- We also know $M_1 = 1$.
- What is M_1, M_2, M_3, M_4, M_5, M_6 , M_{64}?
- $M_{64} = 1.844674 \times 10^{19}$ moves required!
- Assume 1 second to move 1 disk, it will take about 584.5 billion years to move 64 disks!

Fibonacci Number: Famous Example

- We have a single pair (male and female) of rabbits at the beginning of a year.
- Rabbit pairs are not fertile during their first month of life but give birth to one new (male/female) pair at the end of every month thereafter.
- Assume no rabbits dies.
- How many rabbit pairs will we have at the end of the year?

Fibonacci Number: Famous Example

- How can we solve this problem using recursive D&C?
- The number of rabbit pairs at the end of month k.
  - Number of rabbit pairs at the end of previous month $(k - 1)$ plus
  - Number of newly born rabbit pairs at the end of this month (k)
    - Number of rabbit pairs at the end of two previous months $(k - 2)$
    - Number of fertile pairs this month = number of alive pairs at $k - 2$

Fibonacci Number: Famous Example

- Let F_k be the number of rabbit pairs at the end of month k.
- Then $F_k = F_{k-1} + F_{k-2}$
  - Do you see why? Look at the previous slide!
- We also know $F_0 = 1$ and $F_1 = 1$
  - Initially 1 pair at the beginning of the year.
  - Rabbit pairs do not give birth during the first month of life. So we have the same single pair we started with at the end of the first month.
- Define the Fibonacci number recursively:
  - $F_k = F_{k-1} + F_{k-2}$ for all integers $k \geq 2$
    - Recurrence Relation
  - $F_0 = 1$ and $F_1 = 1$
    - Initial Conditions
- Find the values of F_2, F_3, F_4, F_5, F_6, F_7.

What have we looked at so far?

- Concept of Sequence
- Three different ways of defining sequence
  - Listing of terms in the order of the sequence
  - Explicit Formula
  - Recursive Definition
- Concept of D&C and Recursive Logic
  - Tower of Hanoi
  - Fibonacci Number

What’s the point?

- Solutions to certain problems can be stated as sequences defined recursively and their correctness can be formally proved by a method called Mathematical Induction.

- The fact that sequences can be defined recursively is equivalent to the fact that Mathematical Induction works as a method Proof.