Mathematical Induction

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Session Outline

Principle of Mathematical Induction

- What, When and How

Forming a Recursive Solution and Proving its correctness

Big Picture

- Main Topic
  - Principle of Mathematical Induction as a method of Proof.
- Related Topics
  - Problem Solving using Recursive D&C
  - Defining Mathematical Objects Recursively
    - Initial Conditions and Recurrence Relations
  - Concept of Sequence and Series as examples of objects that can be recursively defined.
  - Mathematic Induction vs. Strong Mathematical Induction vs. Well-Ordering Principle

Principle of Mathematical Induction

- Consider a set S containing some integers.
- Suppose that the following two statements about the set S are true:
  - $a \in S$ (a must be an integer since $a \in S$)
  - For all integers $k \geq a$, if $k \in S$ then $k + 1 \in S$
- What can we conclude about the set S?
  - S must contain every integer greater than equal to a
- Why?
- Principle of Mathematical Induction

Principle of Mathematical Induction

Principle of Mathematical Induction

Let $P(n)$ be a $\boxed{\text{property that is defined for integers } n}$, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.

2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.

Principle of Mathematical Induction (cont’d)

- This form of arguments is called Mathematical Induction and its validity is taken as an axiom.
  - Any argument of this form is a valid argument by mathematical induction
- Very useful and common method of checking conjectures about the outcomes of processes that occur repeated and according to definite patterns.
  - Used to prove that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer

Example

- Consider the following series:

$$\sum_{k=1}^{n} k$$

n	The sum	The sum
1	**1**	**1** = (1 x 2) / 2
2	1+2 = **3**	**3** = (2 x 3) / 2
3	1+2+3 = **6**	**6** = (3 x 4) / 2
4	1+2+3+4 = **10**	**10 =** (4 x 5) / 2
5	1+2+3+4+5 = **15**	**15 =** (5 x 6) / 2
6	**15** + 6 **= 21**	**21 =** (6 x 7) / 2
7	**21** + 7 **= 28**	**28 =** (7 x 8) / 2
8	**28** + 8 **= 36**	**36 =** (8 x 9) / 2
n	????????	**(*n* x (*n*+1)) / 2** ????

- The outcomes of the process of finding the sum of the first *n* positive integers seems to follow certain pattern
- Our careful observation seems to suggest that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Unproven proposition that appears to be correct

- That's our conjecture
- Is it really true for all integers $n \geq 1$?
- How can we formally prove it?

Proof by Mathematical Induction

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.

Proof by Mathematical Induction

- To prove by Method of Mathematical Induction
- Basis Step: Show that $P(a)$ is true
- Inductive Step: Show that for all integers $k \geq a$, $P(k) \rightarrow P(k + 1)$
  - Suppose $P(k)$ is true for a p.b.a.c integer $k \geq a$
    - Inductive Hypothesis
  - Then show that $P(k + 1)$ is true
    - Inductive Conclusion
  - In other words, show that truth of $P(k + 1)$ necessarily follows from the truth of $P(k)$

Example

Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ by Mathematical Induction

Proof (by mathematical induction):

Let the property $P(n)$ be the equation $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Basis Step:

Observe that $\sum_{k=1}^{1} k = 1$ (left hand side of the equation)

and $\frac{1(1+1)}{2} = \frac{2}{2} = 1$ (right hand side of the equation).

Hence $P(1)$ is true.

particular but arbitrarily chosen

Inductive Step:

Suppose that m is a p.b.a.c integer with $m \geq 1$ such that $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$

We need to show that

$$\sum_{k=1}^{m+1} k = \frac{(m+1)((m+1)+1)}{2} = \frac{(m+1)(m+2)}{2}$$

Example (cont'd)

Inductive Step (cont'd):

We need to show that $\sum_{k=1}^{m+1} k = \frac{(m+1)((m+1)+1)}{2} = \frac{(m+1)(m+2)}{2}$

Observe that

$$\begin{aligned}\sum_{k=1}^{m+1} k &= \sum_{k=1}^{m} k + (m+1) \\ &= \frac{m(m+1)}{2} + (m+1) \\ &= \frac{m(m+1)}{2} + \frac{2(m+1)}{2} \\ &= \frac{m^2+m+2m+2}{2} \\ &= \frac{m^2+3m+2}{2} \\ &= \frac{(m+1)(m+2)}{2}\end{aligned}$$

By the inductive hypothesis

- So $\sum_{k=1}^{m+1} k = \frac{(m+1)(m+2)}{2}$ as was to be shown.

Notes: Showing Equality of two sides

- We need to show that

$$\sum_{k=1}^{m+1} k = \frac{(m+1)((m+1)+1)}{2} = \frac{(m+1)(m+2)}{2}$$

- When showing the equality of any equation:
  - We can show a step-by-step of transformation of one side to match the other side (previous slide)
  - We can show that both the left side and the right side of the equation transform to the same value (textbook).
  - However, don't make use of what needs to be shown as part of your transformation!

- e.g. Need to show that $x = y$
  - $x + 1 = y + 1$
  - $x + 1 - 1 = y + 1 - 1$
  - Therefore, $x = y$

Putting Things Together

- Problem solving using Recursive Logic (D&C Recursively)
  - Break the problem into smaller subproblems
  - Express the solution to the original problem in terms of the solutions to the subproblems
    - Assume solutions to the subproblems exist and solve the original problem.
  - Keep breaking down the subproblems using the same strategy until subproblems are small and trivial enough to be solved by a simple method.
- The proposed solution is just a conjecture for now.

Proof by Mathematic Induction

- Formal proof of recursive solution
- Steps of proof by Mathematical Induction
  - Define the property to be proved.
  - Identify the Basis Step and show its correctness.
  - Formulate the Inductive Step and show its correctness.
    - Identify the Inductive Hypothesis.
    - Identify the Inductive Conclusion.
    - Show the conclusion necessarily follows from the hypothesis.

Exercise

- In the Banana Republic, there are only two types of coins: B3 and B5. Can we pay for any price of products as long as the prices are at least B8 using only the two types of coins?

Price	How to pay for it
8	3 + 5
9	3 + 3 + 3
10	5 + 5
11	3 + 3 + 5
12	3 + 3 + 3 + 3
13	3 + 5 + 5
14	3 + 3 + 3 + 5
15	5 + 5 + 5
16	3 + 3 + 5 + 5
17	3 + 3 + 3 + 3 + 5
n	?????

If we continue with the table on the left, it seems we can!

But how can we know for sure that any integer ≥ 8 can be expressed as a sum of 3s and 5s?

Exercise (cont'd)

- How can we solve the problem using Recursive D&C?
  - Assume we can pay for a p.b.a.c. value k using only B3 and B5 coins for any integer $k \geq 8$.
  - Try to find a solution to paying for $k + 1$ using the solution to paying for k assumed above.
  - Keep breaking in this fashion.
  - When do we stop breaking and why?
    - When k is 8 because the problem defines $k \geq 8$
    - And we have a simple solution: $8 = 3 + 5$.

Exercise (cont'd)

- Let's assume we can pay for a p.b.a.c. value k using only B3 and B5 coins for any integer $k \geq 8$.
- How can pay for $k + 1$ using only B3 and B5?
- There are only two types of coins: B3 and B5.
- Therefore, any particular way to pay for a price of K will either:
  - No B5 coin used at all
    - Replace three B3 coins with two B5 coins
      - B9 → B10
      - Can pay for K+1
  - At least one B5 coin is used.
    - Replace one B5 coin with two B3 coins
      - B5 → B6
      - Can pay for K+1

Must contain at least three B3 coins since the K is at least B8.

Exercise (cont'd)

- Prove that for all integer values of $n \geq 8$, n can be expressed as a sum of 3s and 5s.

- Proof (by mathematical Induction):

  Let the property $P(n)$ be the following sentence:

  n can be expressed as a sum of 3s and/or 5s.

  Basis Step:

  $P(8)$ is true since $8 = 3 + 5$.

  Inductive Step:

  Suppose $P(k)$ is true for any integer $k \geq 8$.

  That is suppose that k can be expressed as a sum of 3s and/or 5s for a p.b.a.c integer $k \geq 8$.

Exercise (Cont'd)

We must show that (k + 1) can be expressed as a sum of 3s and/or 5s.

Case 1: At least one 5 is used in the expression of k as a sum of 3s and/or 5s.

In this case, replace the 5 with 3 + 3 in the expression; the result will be the expression of (k + 1) as a sum of 3s and/or 5s.

Case 2: k is expressed as a sum of 3s only.

In this case, 3 must appear at least three times in the expression since $k \geq 8$.

So remove the expression 3 + 3 + 3 and replace them with 5 + 5; the result will be the expression of (k + 1) as a sum of 3s and/or 5s.

Thus in either case (k + 1) can be expressed as a sum of 3s and/or 5s. Q.E.D.

Notes: Terminologies

- Deductive Reasoning
  - Use of laws of logical reasoning to infer a specific conclusion from general statements
  - Basically what we have been studying so far.
    - Propositional and Predicate Logic, Rules of Inferences, Direct and Indirect Proof Methods and Mathematical Induction
    - Note that Mathematical Induction is a form of deductive reasoning
  - Logically valid way to formally prove or disprove conjectures
- Inductive Reasoning
  - Formulation of general statements from specific observations
  - Not a logically valid way to prove or disprove
  - Used to form conjectures

Notes: Terminologies

- We use both Inductive Reasoning and Deductive Reasoning in Science
  - Use Inductive reasoning to make conjectures
  - Use Deductive reasoning to test conjectures and prove or disprove them
  - Refer back to examples on sum of first *N* positive integers and B3 and B5 coins problem to see how we use Inductive Reasoning to make conjectures and make use of Mathematical Induction (Deductive Reasoning) to test the conjectures

Exercise

Consider the following sequence defined recursively:

- $a_k = a_{k-1} + 2$ for all integers $k \geq 1$
- $a_0 = 1$

- What is a_{500}?
- Requires finding all a_k from $k = 1$ up to $k = 500$. Why?
- What if we can figure out an explicit formula for the sequence?
- How can we find an explicit formula for a sequence defined recursively?
  - One possible technique: Method of Iteration
    - Try finding each term progressively and see if you can observe a pattern emerging.
    - Form of Inductive Reasoning

Exercise (cont'd)

Consider the following sequence defined recursively:

$a_k = a_{k-1} + 2$ for all integers $k \geq 1$

$a_0 = 1$

An attempt to find an explicit formula:

$a_0 = 1 \quad = 1 + 0 \cdot 2$

$a_1 = a_0 + 2 = 1 + 2 \quad = 1 + 1 \cdot 2$

$a_2 = a_1 + 2 = (1 + 2) + 2 = 1 + 2 + 2 \quad = 1 + 2 \cdot 2$

$a_3 = a_2 + 2 = (1 + 2 + 2) + 2 = 1 + 2 + 2 + 2 \quad = 1 + 3 \cdot 2$

$a_4 = a_3 + 2 = (1 + 2 + 2 + 2) + 2 = 1 + 2 + 2 + 2 + 2 \quad = 1 + 4 \cdot 2$

$a_5 = a_4 + 2 = (1 + 2 + 2 + 2 + 2) + 2 = 1 + 2 + 2 + 2 + 2 + 2 \quad = 1 + 5 \cdot 2$

So it seems there is a pattern emerging.

Can you guess what a_n is?

Observation

- It seems like $a_n = 1 + n \cdot 2 = 1 + 2n$ for all integers $n \geq 0$.
- Note that we are using an inductive reasoning to make a guess here
- But that's just our guess that seems to be correct (a conjecture).
- How can we formally prove it's actually true?
  - Prove it by Method of Mathematical Induction
  - Deductive Reasoning

Exercise

Prove that the sequence defined recursively in the previous slide has a following explicit formula:

$$a_n = 1 + n \cdot 2 = 1 + 2n \text{ for all integers } n \geq 0.$$

- $a_k = a_{k-1} + 2$ for all integers $k \geq 1$
- $a_0 = 1$

Exercise (cont'd)

Proof (by Mathematical Induction):

Let the property $P(n)$ be the following equation: $a_n = 1 + 2n$

Basis Step: Initial condition

Observe that $a_0 = 1$ and $1 + 2 \cdot 0 = 1$. Thus $a_0 = 1 + 2 \cdot 0$ and hence $P(0)$ is true.

Inductive Step:

Suppose $a_k = 1 + 2k$ for a p.b.a.c integer $k \geq 0$.

We need to show that $a_{k+1} = 1 + 2(k + 1) = 1 + 2k + 2 = 2k + 3$

Observe that

$$
\begin{aligned}
a_{k+1} &= a_k + 2 && \text{(by the recursive definition)} \\
&= (1 + 2k) + 2 && \text{(by inductive hypothesis)} \\
&= 2k + 3
\end{aligned}
$$

[Therefore, $a_{k+1} = 2k + 3$ as was to be shown.]

In-Class Exercise

- Recall the Tower of Hanoi problem we solved using Recursive Logic.
- The minimum number of moves required to move N disks is:
  - $M_N = M_{N-1} + 1 + M_{N-1} = 2M_{N-1} + 1$ for all integers $N \geq 2$
  - $M_1 = 1$.
- What's the minimum number of moves to move 32 disks?
- To find M_n recursively, we need to find M_2, M_3, …, all the way to M_n
- Can you find an explicit formula for M_n?

In-Class Exercise (cont'd)

- $M_1 = 1$
- $M_2 = 2M_1 + 1 = 2(1) + 1 = 2 + 1$
- $M_3 = 2M_2 + 1 = 2(2 + 1) + 1 = 2 \cdot 2 + 2 + 1 = 2^2 + 2 + 1$
- $M_4 = 2M_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$
- $M_5 = 2M_4 + 1 = 2(2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$
- $M_6 = 2M_5 + 1 = 2(2^4 + 2^3 + 2^2 + 2 + 1) + 1 = 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1$
  - Note that $2^5 + 2^4 + 2^3 + 2^2 + 2 + 1 = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$
- $M_7 = ???$
- $M_N = ???$

Does it apply to M_2 and M_1 as well?

Can you observe the pattern?

In-Class Exercise (cont'd)

- Can you find an explicit formula for M_n? (Inductive Reasoning)
- Can you observe the pattern? Does it apply to M_2 and M_1 as well?
  - It seems that $M_N = 2^{N-1} + 2^{N-2} + 2^{N-3} + \ldots + 2^2 + 2 + 1$

    $= \frac{2^N - 1}{2-1} = \frac{2^N - 1}{1}$

    $= 2^N - 1$
  - If you can't see why $M_N = 2^{N-1} + 2^{N-2} + 2^{N-3} + \ldots + 2^2 + 2 + 1 = \frac{2^N - 1}{2-1}$, ensure you study the topic of "Geometric Series".
- We conjectured that $M_N = 2^N - 1$ for all integers $N \geq 1$. Formally Prove it.
  - Do it now on your exercise book.

In-Class Exercise (cont'd)

Prove that the following recursive definition for the minimum number of moves required to move N disks for the Tower of Hanoi problem:

$$M_N = M_{N-1} + 1 + M_{N-1} = 2M_{N-1} + 1 \text{ for all integers } N \geq 2$$

$$M_1 = 1.$$

has the explicit formula: $M_N = 2^N - 1$ for all integers $N \geq 1$

In-Class Exercise (cont'd)

Proof (by Mathematical Induction):

Let the property $P(n)$ be the following equation: $M_N = 2^N - 1$

Basis Step:

Observe that $M_1 = 1$ (by the initial condition of the recursive definition)

Note that $M_1 = 2^1 - 1 = 2 - 1 = 1$; Therefore $P(1)$ is true.

Inductive Step:

Suppose $M_k = 2^k - 1$ for a p.b.a.c. integer $k \geq 1$.

[We need to show that $M_{k+1} = 2^{k+1} - 1$]

Observe that $M_{k+1} = 2M_k + 1$ as $k+1 \geq 2$ since $k \geq 1$. [by the recursive definition]

$= 2(2^k - 1) + 1$ [by the inductive hypothesis]

$= 2 \cdot 2^k - 2 \cdot 1 + 1 = 2^{k+1} - 2 + 1$ [by basic algebra]

$= 2^k - 1$ [as was to be shown]

Exercise

Observe that

$$\frac{1}{1\cdot 3} = \frac{1}{3}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} = \frac{2}{5}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{3}{7}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} = \frac{4}{9}$$

- Guess a general formula and prove it by mathematical induction.