Logic of Quantified Statements

CSX2008 Mathematics Foundation for Computer Science

Kwankamol Nongpong, Ph.D.
Department of Computer Science
Vincent Mary School of Science and Technology
Assumption University

Session Outline

- Introduction to Predicate Logic
  - Universal Quantification
  - Existential Quantification
  - Implicit Quantifications in Statements

Review

- Recall that "He is a college student" is not a statement.
  - Why?
  - It may be either true or false depending on the value of the pronoun he.
- "$x + y$ is greater than 0" is not a statement either.
  - Why?
  - Because its truth value depends on the values of the variables x and y.

Predicate

- In grammar: predicate refers to the part of a sentence that gives information about the subject.
  - "James is a student at Bedford College".
- In logic: predicates can be obtained by removing some or all of the nouns from a statement.
  - Let *P* stand for "is a student at Bedford College"
  - Let *Q* stand for "is a student at."
  - "*x* is a student at Bedford College" can be symbolized as $P(x)$
  - "*x* is a student at *y*" can be symbolized as $Q(x, y)$
  - For simplicity, we define a predicate to be a predicate symbol together with suitable predicate variables

P and Q are predicate symbols

The Universal Quantifier: $\forall$

- A predicate is turned into a statement by means of quantification.
- Common English Expressions for the Universal Quantifier
  - for all, for every, for any, for arbitrary, for each, given any
- It's important that the domain of the quantified predicate variable is clearly specified because the truth value of the statement could depend on the domain.
  - $\forall$ students x
  - $\forall x \in S$

Examples

- x majors in Computer Science
  - A predicate; not a statement (proposition)
- Somchai majors in Computer Science
  - Assign a specific value "Somchai" to x
  - A predicate turned into a statement
- ∀ students *x*, *x* majors in Computer Science
  - All students major in Computer Science
- $\forall x \in S$, x majors in Computer Science
  - If we let *S* be the set of all students in Assumption University

Universal Statements

Universal Statements

Truth Value of Universally Quantified Statement

- Let *Q(x)* be a predicate and *D* the domain of *x*.
- A universal statement is a statement of the form

$$\forall x \in D, Q(x)$$

- It is defined to be true *iff Q(x)* is true for every *x* in *D*.
- It is defined to be false *iff Q(x)* is false for at least one *x* in *D*.
  - A value of x for which *Q(x)* is false is called a counterexample to the universal statement.

Exercise

- Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement $\forall x \in D$, $x^2 \geq x$. Show that this statement is true.

- Consider the statement $\forall x \in \mathbf{R}$, $x^2 \geq x$. Find a counterexample to show that this statement is false.

The Existential Quantifier: ∃

- Another way to quantify a predicate variable is to specify an existence of at least one such element.
- Common English expressions for Existential Quantifier
  - there exists, there is a, we can find a, there is at least one, for some, for at least one.
- For instance,
  - There is a student in CS2101 class.
  - ∃ a person p such that p is a student in CS2101
  - $\exists p \in P$ such that p is a student in CS2101
    - where P is a set of all people.
  - $S(p)$ if we let S stand for "is a student in CS2101"

Truth Value of Existentially Quantified Statement

- Let *Q(x)* be a predicate and *D* the domain of *x*.
- A existential statement is a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

- It is defined to be true *iff Q(x)* is true for at least one *x* in *D*.
- It is false *iff Q(x)* is false for all *x* in *D*.

Exercise

- Consider the statement $\exists m \in Z+$ such that $m^2 = m$.
- Show that this statement is true.

- Let $E = \{5, 6, 7, 8\}$.
- Consider the statement $\exists m \in E$ such that $m^2 = m$.
- Show that this statement is false.

Formal vs. Informal Language

$$\forall x \in R, x^2 \geq 0$$

- What does the above formal statement mean?
- Express it in more informal ways.
  - For any real number x, x^2 is nonnegative.
  - x^2 is nonnegative for any real number x
  - All real numbers have nonnegative squares.
  - Every real number has a nonnegative square
  - Any real number has a nonnegative square
  - The square of each real number is nonnegative.

Formal vs. Informal Language

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m$$

- Rewrite the above formal statement in a more informal way.
  - There exists at least one positive integer m such that $m^2 = m$
  - $m^2 = m$ for some positive integer m
  - There exists at least one positive integer whose square is equal to itself.
  - There is a positive integer whose square is equal to itself.
  - We can find at least one positive integer equal to its own square
  - Some positive integer equals its own square.
  - Some positive integers equal their own squares.

Exercise

- Rewrite each of the following statements in formal form using quantifiers and variables.
  - Every triangle has three sides
    - $\forall$ triangles t, t has three sides
    - $\forall t \in T$, t has three sides (where T is the set of all triangles)
  - No dogs have wings
    - $\forall$ dogs d, d does not have wings
    - $\forall d \in D$, d does not have wings (where D is the set of all dogs)

Exercise

- Rewrite each of the following statements in formal form using quantifiers and variables.
  - Some programs are structured
    - ∃ a program p such that p is structured
    - $\exists p \in P$ such that p is structured (where P is the set of all programs)
  - The square of each real number is nonnegative.
    - $\forall x \in \mathbf{R}, x^2 \geq 0$
  - Some positive integers equal their own squares.
    - $\exists m \in \mathbf{Z}^+$ such that $m^2 = m$

Universal Conditional Statement

$$\forall x, P(x) \longrightarrow Q(x)$$

- $\forall x$, if $P(x)$ then $Q(x)$
- For all values of x in the domain D, if the predicate $P(x)$ is true, then the predicate $Q(x)$ is also true.
- $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$
  - For all real numbers, if it is greater than 2, then its square is greater than 4
  - If a real number is greater than 2 then its square is greater than 4.
  - Whenever a real number is greater than 2, its square is greater than 4.
  - The square of any real number greater than 2 is greater than 4.
  - The square of all real numbers greater than 2 are greater than 4.

Universal Conditional Statement

Rewrite the following statements in the form

$\forall$_______________, if ______________then____________

- If a real number is an integer, then it is a rational number
  - $\forall x \in \mathbf{R}$, if $x \in \mathbf{Z}$ then $x \in \mathbf{Q}$
- All bytes have eight bits
  - $\forall x$, if x is a byte, then x has eight bits.
- Note that it is common to omit explicit specification of the domain of predicate variables in universal conditional statements when the domain can be generalized objects
  - $\forall x$, if x is a byte, then x has eight bits.
  - $\forall x$, if x is a fire truck, then x is not green.

Domain Specification and Equivalent Forms

- Universal Conditional Statements can be transformed into Universal Statements and vice versa by changing the Domain of Variables.

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x) \qquad\qquad \forall x \in D,\ Q(x)$$

- When $D \subseteq U$, more specifically $D = \{ x \in U \mid P(x) \}$

- When D is the truth set of $P(x)$

Example

- $\forall x \in \mathbf{R}$, if $x \in \mathbf{Z}$ then $x \in \mathbf{Q}$
- $\forall x \in \mathbf{Z}, x \in \mathbf{Q}$
  - Both statements mean “All integers are rational”

- All squares are rectangles.
  - Write it as a universal conditional statement
  - Write it as a universal statement

Domain Specification and Equivalent Forms

$$\exists x \in U \text{ such that } P(x) \text{ and } Q(x)$$

can be written as

$$\exists x \in D \text{ such that } Q(x),$$

where $D = \{ x \in U \mid P(\text{x}) \}$, D is the truth set of $P(x)$

Example

"There is an integer that is both prime and even"

- Let $P(n)$ stand for "n is prime"
- Let $E(n)$ stand for "n is even"
- $\exists n \in \mathbf{Z}$ such that $P(n) \wedge E(n)$
- $\exists$ a prime number n such that $E(n)$
- $\exists$ an even number n such that $P(n)$

Implicit Quantification

- Be aware that mathematical writing contains many example of implicitly quantified statements.
- If a number is an integer then it is a rational number.
  - Implicit universal quantification
  - $\forall x \in \mathbf{Z}, x \in \mathbf{Q}$, or
  - $\forall x \in \mathbf{R}$, if $x \in \mathbf{Z}$ then $x \in \mathbf{Q}$
- The number 24 can be written as a sum of two even integers.
  - Implicit existential quantification
  - $\exists$ even integers m and n such that $24 = m + n$

Exercise

- In an algebra text book, you find the following:

$$(x + 1)^2 = x^2 + 2x + 1$$

  - What does it mean?
  - Rewrite it as formal statement using a quantifier

- On the same book, you find the following

$$\text{Solve } 3x - 4 = 5$$

  - What does it mean?
  - Rewrite it as a formal statement using a quantifier

Symbolic Notation for Implicit Quantification

Let $P(\mathrm{x})$ and $Q(\mathrm{x})$ be predicates and suppose the common domain of x is D.

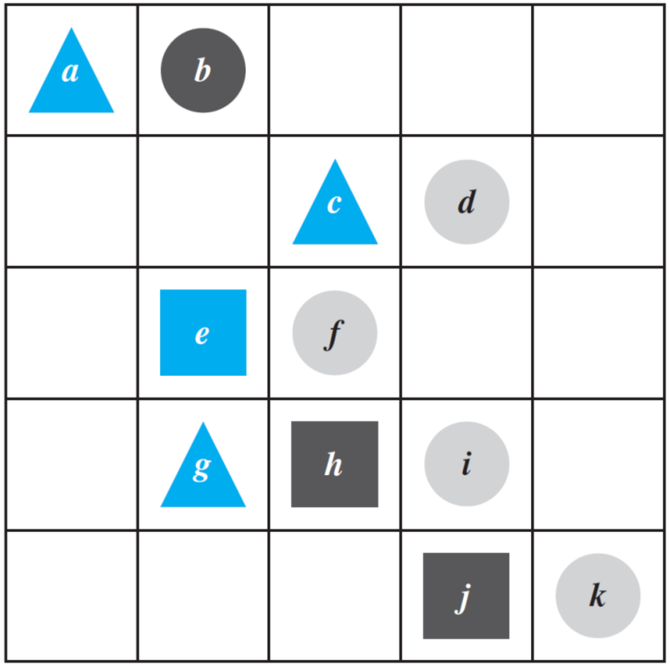
- $P(\mathrm{x}) \Rightarrow Q(\mathrm{x})$ means $\forall x \in D, P(\mathrm{x}) \longrightarrow Q(\mathrm{x})$
  - Every element in the truth set of $P(x)$ is in the truth set of $Q(x)$
- $P(\mathrm{x}) \Leftrightarrow Q(\mathrm{x})$ means $\forall x \in D, P(\mathrm{x}) \leftrightarrow Q(\mathrm{x})$
  - $P(x)$ and $Q(x)$ have identical truth sets
- e.g. In a context of a book where *x* has been used to indicate a real number, the following

$$x > 2 \Rightarrow x^2 > 4$$

means

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4$$

Exercise: Tarski's World



Predicate Symbols

- Triangle(x)
  - x is a triangle
  - Square(x), Circle(x), etc.
- Blue(y)
  - y is blue
  - Gray(y), Black(y), etc.
- RightOf(x, y)
  - x is to the right of y
    (but possibly on a different row)

Determine the truth value of each of the following statements. The implied domain for all variables is the set of objects in the Tarski world.

- $\forall t$, Triangle(t) $\longrightarrow$ Blue(t)
- $\forall x$, Blue(x) $\longrightarrow$ Triangle(x)
- $\exists y$ such that Square(y) $\wedge$ RightOf($\boldsymbol{d}$, y)
- $\exists z$ such that Square(z) $\wedge$ Gray(z)