Propositional Logic

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Session Outline

- Arguments and Argument Forms
- Valid and Invalid Arguments
- True vs. Valid vs. Sound
- Common Forms of Valid Arguments
  - Modus Ponens and Modus Tollens
  - Generalization and Specialization
  - Conjunction and Elimination
  - Transitivity
  - Division into Cases
  - Contradiction Rule
- Common forms of Fallacies (Invalid Arguments)

Logical Argument and Argument Forms

- An argument is a sequence of statements aimed at demonstrating the truth of assertion. (not a quarrel)
  - Conclusion: the assertion at the end of sequence
  - Premises: preceding statements
    - *aka* assumptions or hypothesis
- Logical Arguments (Logical Forms)
  - Logic can be used to formally determine if an argument is valid or invalid.
  - Logic cannot be used to determine if each atomic statement is true or not.

Form of Arguments vs. Contents of Arguments

- Logical analysis does not help determine the truth of an argument's contents.
    - individual premises and the conclusion
- But can help analyze the form of argument to determine whether the truth of the conclusion follows *necessarily* from the truth of the premises.

Example

All cats have big bellies.

Tom is a cat.

Therefore, Tom has a big belly.

Argument Form vs. Argument Content

- Argument #1
  - If the program syntax is faulty or if program execution results in division by zero, then the computer will generate an error message. Therefore, if the computer does not generate an error message, then the program syntax is correct and program execution does not result in division by zero.
- Argument #2
  - If x is a real number such that x < -2 or x > 2, then x2 > 4. Therefore, if x2 $\not>$ 4, then x $\not<$ -2 and x $\not>$ 2.
- Both have the same Argument Form
  - If p or q, then r
  - ∴ if not r, then not p and not q.
- Is Argument #1 valid? Is Argument #2 valid?

Validity of Arguments

If Tom is a cat, then Tom is a feline.

Tom is a cat.

Therefore, Tom is a feline

if p then q

p

$\therefore q$

$\boldsymbol{p} \rightarrow \boldsymbol{q}$

$\boldsymbol{p}$

$\therefore q$

Is the argument valid?

How to determine if an argument form is valid or invalid?

We have to look at the argument form

Validity of Arguments

- Valid Argument Forms guarantee that truth of the conclusion *necessarily* follows from the truth of all the premises.
  - Construct a truth table showing the truth values of all the premises and the conclusion.
  - If the conclusion in every critical row is true, then it is valid.
    - Critical rows: rows of the truth tables where all the premises are true

if ***p*** then ***q***

p

∴ ***q***

$p \rightarrow q$

p

$\therefore q$

		premises		conclusion	
p	q	$p \rightarrow q$	p	q	
T	T	T	T	T	← critical row
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	

This argument and any argument having the same form is valid.

Exercise

- Is the following argument form valid or invalid?

$$\begin{array}{c} p \lor q \\ \sim p \\ \therefore q \end{array}$$

- Valid Arguments guarantees that truth of the conclusion necessarily follows from the truth of all the premises.

		premises		*conclusion*	
p	q	$p \lor q$	$\sim p$	q	
T	T	T	F	T	
T	F	T	F	F	
F	T	T	T	T	*critical row*
F	F	F	T	F	

The truth table shows that in the only situation (represented by row 3) in which both premises are true, the conclusion is also true. Therefore, the the second version of elimination is valid.

Exercise

- Is the following argument form valid or invalid?

$$p \rightarrow q$$

$$\sim p$$

$$\therefore \sim q$$

- Valid Arguments guarantees that truth of the conclusion necessarily follows from the truth of all the premises.

Exercise

- Is the following argument form valid or invalid?

$$p \to q \vee \sim r$$

$$q \to p \wedge r$$

$$\therefore p \to r$$

Exercise

- Is the following argument form valid or invalid?

$$
\begin{array}{c}
p \vee q \\
p \rightarrow r \\
q \rightarrow r \\
\therefore r
\end{array}
$$

Valid vs. True vs. Sound

If Tom is a cat, then Tom is a feline.

Tom is a cat.

Therefore, Tom is a feline

if p then q

p

$\therefore q$

- A valid argument with true premises; consequently a true conclusion
  - Note that a valid argument with true premises always leads to a true conclusion, i.e. In a valid argument, a true conclusion necessarily follows from true premises.
- An example of sound argument.
  - An argument is called sound *iff* it is valid and all the premises are true.
  - An argument that is not sound is called unsound.

Valid vs. True vs. Sound

If Jerry is a cat, then Jerry is a feline.

Jerry is a cat.

Therefore, Jerry is a feline

if p then q

p

$\therefore q$

- The argument form is valid
- One of the premises is false (Jerry from Tom and Jerry is not a cat)
  - Which one is it?
- The conclusion is false
- Unsound argument

Valid vs. True vs. Sound

If Jerry is a cat, then Tom is a cat.

Jerry is a cat.

Therefore, Tom is a cat

if p then q

p

$\therefore q$

- The argument form is valid
- One of the premises is false
  - Which one?
  - Why is the first premise true?
- The conclusion is true
- Unsound argument

Valid vs. True vs. Sound

If Simba is a cat, then Simba is a feline.

Simba is a feline.

Therefore, Simba is a cat

if p then q

q

$\therefore p$

- The argument form is invalid (Exercise: Check via truth table)
- Its premises are true (Simba from Lion King is a lion, a feline)
  - Explain why the first premise is true
- And the conclusion is false
- Unsound argument

Valid vs. True vs. Sound

If Bangkok is a big city, then Bangkok has tall buildings.

Bangkok has tall buildings

Therefore, Bangkok is a big city.

if p then q

q

$\therefore p$

- The argument form is **invalid** (**converse error**)
- Its premises are **true**
- And the conclusion is **true**
- **Unsound argument**

Valid vs. True vs. Sound

- Validity is a property of Argument Forms
  - If an argument is valid (or invalid), then so is every other argument that has the same form.
- Valid arguments
  - true premises always lead to a true conclusion
    - Cannot have all true premises, but false conclusion
  - can have true or false conclusion upon false premises
- Invalid arguments
  - Can have any combination of true/false premises and true/false combination, but
  - The argument is invalid therefore we can't be sure the conclusion is (really) true.
- Sound Arguments
  - valid argument with true premises; consequently true conclusion (since it's a valid argument with all true premises)
- We can be only assured of the actual truth of a conclusion when we know that the argument is sound,
  - i.e. when we know that the argument is valid and all the premises are true.

Rules of Inference

- Generalization
- Specialization
- Conjunction
- Elimination
- Transitivity
- Division into Cases
- Syllogism (a Major and a Minor Premises and a Conclusion)
  - Modus Pones
  - Modus Tollens
- Contradiction Rules

Generalization (Addition)

$$\begin{array}{l} \quad p \\ \therefore p \vee q \end{array} \qquad\qquad \begin{array}{l} \quad q \\ \therefore p \vee q \end{array}$$

- First, check the validity of the form using truth tables.
- Now how do we make use of it?

Example: Generalization

- Call every Upperclassmen to a meeting
- Somchai is a Junior student.
- Should I call Somchai?
  - Yes, Why?
- Upperclassmen = Junior or Senior student
- Junior, therefore Upperclassmen (Junior or Senior)

Specialization (Simplification)

$$p \wedge q \qquad\qquad p \wedge q$$
$$\therefore p \qquad\qquad \therefore q$$

- First, check the validity of the form using truth tables.
- Now how do we make use of it?

Example: Specialization

- I need someone who can speak Korean
- Eugene can speak Korean and Thai
- Can Eugene help me?
  - Yes, why?
- Eugene can speak Korean and he can speak Thai
- Therefore, Eugene can speak Korean

Conjunction

$$p$$

$$q$$

$$\therefore p \wedge q$$

- Check the Validity of the Argument Form!

In formal logic, connectives can connect only complete statement.

Example: Conjunction

- Eugene can speak Korean
- Eugene can speak Thai
- Therefore, Eugene can speak Korean and Eugene can speak Thai
- Therefore, Eugene can speak Korean and Thai
  - Typical English expression

Elimination (Disjunctive Syllogism)

$$p \vee q$$
$$\sim q$$
$$\therefore p$$

$$p \vee q$$
$$\sim p$$
$$\therefore q$$

- Check validity through truth tables, first
- If there are only two possibilities (p v q) and one of them is not possible ($\sim q$), then the other must be possible (p)

Example: Elimination

- $x - 3 = 0$ **or** $x + 2 = 0$
- $x \geq 0$
- Therefore, x = ?
  - $x = 3$

Transitivity (Hypothetical Syllogism)

$p \rightarrow q$	if *p* then *q*
$q \rightarrow r$	if *q* then *r*
$\therefore p \rightarrow r$	Therefore, if *p* then *r*

- Check validity through truth tables, first
- Is the argument sound?
  - The form is valid.
  - Is the major premise (first premise) true?
  - Is the minor premise (second premise) true?
  - So what can we conclude?

Example: Transitivity

- If 18,486 is divisible by 18, then 18,486 is divisible by 9.
- If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.
- Therefore, if 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

Case Analysis: Division into Cases

$$p \vee q$$

$$p \longrightarrow r$$

$$q \longrightarrow r$$

$$\therefore r$$

- Make sure you check the validity of this argument form using truth table as an exercise.
- How many critical rows?
- We know at least one of two possibilities is true ($p \vee q$)
- We also know first possibility leads to a certain conclusion and the second possibility also leads to the same conclusion ($p \longrightarrow r$ and $q \longrightarrow r$)
- Then, the conclusion must be also true ($\therefore r$)

Example

- Let x be nonzero real number. Show that $x^2 > 0$.
  - $x > 0$ or $x < 0$ (by trichotomy property and elimination)
  - if $x > 0$, then $x^2 > 0$
  - if $x < 0$, then $x^2 > 0$
  - Therefore, $x^2 > 0$

Modus Ponens

$$p \longrightarrow q$$

$$p$$

$$\therefore q$$

- Validation via truth table
- Method of Affirming

Example: Modus Ponens

- If Tom is a cat, then Tom is a feline.
- Tom is a cat.
- Therefore, Tom is a feline

Modus Tollens

$$p \longrightarrow q$$

$$\sim q$$

$$\therefore \sim p$$

- Remind yourself that a conditional statement and its contrapositive are logically equivalent.
  - $p \longrightarrow q \equiv \sim q \longrightarrow \sim p$
- Method of Denying

Example

- If Tom is a cat, then Tom is a feline.
- Tom is not a feline.
- Therefore, Tom is a not a cat!
  - Why?
  - Assume Tom is a cat, then by the major premise, Tom would be a feline.
  - But the minor premise states that Tom is not a feline, leading to contradiction
  - Hence, Tom cannot be a cat.

Exercise

- If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
- There are more pigeons than there are pigeonholes.
- Therefore, ___________________________.
  - What's the form of argument?

- If 870,232 is divisible by 6, then it is divisible by 3.
- 870,232 is not divisible by 3.
- Therefore, ___________________________.
  - What's the form of argument?

Contradiction Rule

$$\sim p \rightarrow \mathbf{c}$$

$$\therefore\ p$$

- First, recall "tautology" and "contradiction" from last week
- If the supposition that "*p* is false" leads to a contradiction, then *p* must be true.
  - Check the validity of the form via truth tables

			premises	conclusion
p	$\sim p$	$\mathbf{c}$	$\sim p \rightarrow \mathbf{c}$	p
T	F	F	T	T
F	T	F	F	F

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

Example: Proof by Contradiction

- Show that "There is no largest even integer."
- Proof by Contradiction:
  - Suppose that there were a largest even integer. Let's call it k.
  - Since k is even, $k = 2n$, for some integer n.
  - Consider the number $k + 2$; $k + 2 = (2n) + 2 = 2(n + 1)$.
  - So, $k + 2$ is even
    - because $n + 1$ is an integer, and a number that equals to twice any integer is an even number.
  - However, $k + 2$ is larger than k, and this contradicts our supposition that k was the largest even integer.
  - Therefore, there is no largest even integer.

$$\sim p \longrightarrow \mathbf{c}$$
$$\therefore\ p$$

Example: Use of Rules of Inference

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Example: Rules of Inference

RK = I was reading the newspaper in the kitchen.

GK = My glasses are on the kitchen table.

SB = I saw my glasses at breakfast.

RL = I was reading the newspaper in the living room.

GC = My glasses are on the coffee table.

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.
b. If my glasses are on the kitchen table, then I saw them at breakfast.
c. I did not see my glasses at breakfast.
d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Example: Rules of Inference

1. $RK \rightarrow GK$	by (a)
$GK \rightarrow SB$	by (b)
$\therefore RK \rightarrow SB$	by transitivity
2. $RK \rightarrow SB$	by the conclusion of (1)
$\sim SB$	by (c)
$\therefore \sim RK$	by modus tollens
3. $RL \vee RK$	by (d)
$\sim RK$	by the conclusion of (2)
$\therefore RL$	by elimination
4. $RL \rightarrow GC$	by (e)
RL	by the conclusion of (3)
$\therefore \boldsymbol{GC}$	by modus ponens

Thus the glasses are on the coffee table.

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Exercise

- There is an island containing two types of people: knights who always tell the truth and knaves who always lie.
- You visit the island and are approached by two natives who speak to you as follows:
  - A says: B is a knight
  - B says: A and I are of opposite type.
- Make use of Rules of Inference discussed today to figure out who A and B are.

Fallacies and Common Forms of Invalid Arguments

- A fallacy is an error in reasoning that results in an invalid argument.
- Two common fallacies
  - Converse Error
    - Mistaking that a conditional statement and its converse is logically equivalent
  - Inverse Error
    - Mistaking that a conditional statement and its inverse is logically equivalent

Converse Error

$p \to q$

q

$\therefore p$

Be careful not to make these logical errors!

Inverse Error

$p \to q$

$\sim p$

$\therefore \sim q$

Example: Converse Error

- If Simba is a cat, then Simba is a feline.
- Simba is a feline.
- Therefore, Simba is a cat

- If Bangkok is a big city, then Bangkok has tall buildings.
- Bangkok has tall buildings
- Therefore, Bangkok is a big city.

$$p \to q$$
$$q$$
$$\therefore p$$

Example: Inverse Error

- If interest rates are going up, stock market prices will go down.
- Interest rates are not going up.
- Therefore, Stock market prices will not go down.

- If it barks, it's a dog.
- It does not bark
- Therefore, it's not a dog.

$$p \to q$$
$$\sim p$$
$$\therefore \sim q$$

Reminder: Mathematical Logic

- Propositional Logic
  - *aka* Propositional Calculus, Statement Calculus
  - What we have studied so far
    - Symbolic Analysis of Propositions (statements) formed with Logical connectives
- Predicate Logic
  - *aka* Predicate Calculus
  - What we will be studying now and next week
    - Symbolic Analysis of Propositions (statements) containing variables that are “quantified”
    - Two common Quantifiers
      - Universal: $\forall$ (For All)
      - Existential: $\exists$ (There exists)

Reminder: Sentence vs. Statement

- Propositions (Statements) are Closed Sentences
  - a sentence that can be determined either true or false
  - The sun is shining
  - $3 + 4 = 9$
- Open Sentence
  - a sentence whose truth cannot be determined because the truth or falsity depends on some other unknown facts.
  - He majors in Computer Science
    - Unknown facts: Who is “he”?
  - $x + y > 17$
    - Unknown facts: What’s the value of x? What’s the value of y?

Predicate

- Predicate is an open sentence in which unknown facts are expressed as variables.
  - He majors in Computer Science
  - ***x*** majors in Computer Science
    - A predicate with one predicate variable x
- A predicate becomes a proposition (statement) when specific values are substituted for the variables.
  - The domain of a predicate variable is the set of all values that may be substituted in place of the variable. (what's the domain of x in above predicate?)
- We may also use symbols to represent predicates
  - Let *P* stand for "majors in Computer Science"
  - $P(x)$ symbolically denotes x majors in Computer Science
  - Note that $P(x)$ can be true or false depending value of x

Exercise

Let P stand for "is a student at AU"

Let Q stand for "is a student at"

- Express the predicate "x is a student at AU" symbolically
- Express the predicate "x is a student at y" symbolically

Let $P(x)$ be the predicate "$x^2 > x$", where the domain of the predicate variable x is $\mathbf{R}$.

- Write $P(2)$, $P(1/2)$ and $P(-1/2)$ and indicate the truth values of the each statement.

Truth Set of a Predicate

- If $P(x)$ is a predicate and x has domain D,
- The truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.
- The truth set of $P(x)$ is denoted

$$\{ x \in D \mid P(x) \}$$

Exercise

- Let $Q(n)$ be the predicate "*n* is a factor of 8".
- Find the truth set of $Q(n)$ when
  - The domain of n is the set $\mathbf{Z}^+$
    - {1, 2, 4, 8}
  - The domain of n is the $\mathbf{Z}$
    - {1, 2, 4, 8, -1, -2, -4, -8}