Predicate Logic – Part I

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Big Picture

- Where are we at the moment?
  - Study of Logical Statements and Forms
    - Propositional Logic
    - Predicate Logic
- Before the break: Intro. To Predicate Logic
  - Predicate as Open Sentence with Variables
  - Ways to turn Predicates to Statements
    - Substituting specific values for Predicate Variables
    - Quantification
      - Universal
      - Existential
  - Checking the Validity of Quantified Statements

Session Outline

- Predicate Logic (Continued)
  - Negating Quantified Statements
  - Universal Conditional Statements
  - Multiply-Quantified Statements
  - Valid and Invalid Forms of Quantified Statements

Negation of Universal Statements

Let *S*: "All mathematicians wear glasses".

What's the ~*S* ? (The negation of *S*)

- No mathematicians wear glasses?
- Mathematicians do not wear glasses?
- It's not the case that "All mathematicians wear glasses"
- ~*S*: "There is at least one mathematician who does not wear glasses."

Negation of Universal Statements

- Be careful with possible ambiguity in English.
  - All mathematicians do not wear glasses.
    - What! You say that all mathematicians wear glasses? Nonsense! All mathematicians do NOT wear glasses!
      - It's not the case that all mathematicians wear glasses.
  - All mathematicians do not wear glasses.
    - No mathematicians wear glasses.
- Don't just insert the word "not" to negate a quantified statement!
- Use "It's not the case that"

Negation of Universal Statements

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$$

- "It's not the case that for all x in D, $P(x)$" is logically equivalent to "There exists (at least one) x in D such that $\sim P(x)$"
- The negation of a universal statement is logically equivalent to an existential statement.

Example:
Negation of Universal Statements

- All mathematicians wear glasses.
  - ∀ mathematicians x, x wear glasses
- It’s not the case that all mathematicians wear glasses.
  - ~(∀ mathematicians x, x wear glasses)
  - ≡∃ mathematician x such that x does not wear glasses
  - ≡ There exists at least one mathematician who does not wear glasses.

Negation of Existential Statements

$$\sim(\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D, \sim P(x)$$

- "It's not the case that there exists x in D such that $P(x)$" is logically equivalent to "For all x in D, $\sim P(x)$"
- The negation of an existential statement is logically equivalent to a universal statement.

Example:
Negation of Existential Statements

- Some snowflakes are the same
  - ∃ snowflakes x, y such that x and y are the same.
- It's not the case that some snowflakes are the same
  - ~(∃ snowflakes x, y such that x and y are the same) ≡
  - ∀ snowflakes x and y, x is not the same as y
- Not a single snowflake is the same as any other
- No snowflakes are the same
- All snowflakes are different.

Exercise

Write formal negations for the following statements:

- ∀ primes p, p is odd.
  - ∃ a prime p such that p is not odd
- ∃ a triangle T such that the sum of the angles of T equals 200°.
  - ∀ triangles T, the sum of the angles of T does not equal 200°

Exercise

Write formal and informal negations of the following statements.

- No politicians are honest
- All computer programs are finite.
- Some computer hackers are over 40.
- The number 1,357 is divisible by some integer between 1 and 37.

Negation of Universal Conditional Statements

Week #2:

- $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Predicate Logic:

- $\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x)$

Just Now:

- $\sim(\forall x \in D, P(x)) \equiv \exists x \in D$ such that $\sim P(x)$

$\therefore \sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists x \in D$ such that $(P(x) \wedge \sim Q(x))$

- "It's not the case that for All x in D if P(x) then Q(x)" is logically equivalent to "There is at least one x in D such that P(x) and ~Q(x)"

Exercise

What's the negation of the following statement?

- ∀ people p, if p is blond, then p has blue eyes.

Exercise

Write an informal negation of the following statement:

- If a computer program has more than 100,000 lines, then it contains a bug.

- Extra Exercise for your workbook
  - Write the original statement above in a formal way
  - Write a formal negation of the original statement

Quantified Statements as Propositions

- Be reminded that by quantifying predicates, we turn predicates into statements (propositions)
- If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, . . . , x_n\}$, then the statement

$$\exists x \in D \text{ such that } Q(x)$$

is logically equivalent to

$$Q(x_1) \lor Q(x_2) \lor \ldots \lor Q(x_n)$$

Negations of Quantified Statements (De Morgan's Laws)

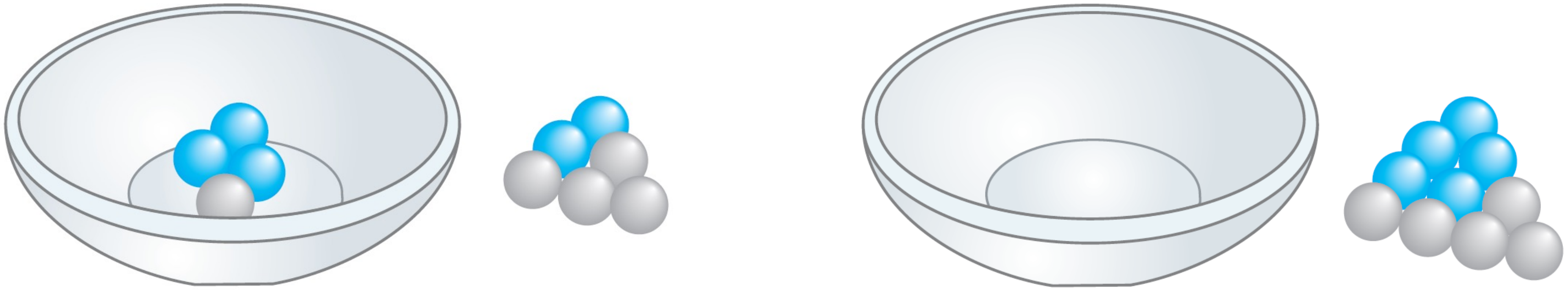
$\sim(Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)) \equiv \sim Q(x_1) \vee \sim Q(x_2) \vee \ldots \vee \sim Q(x_n)$

- $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D$ such that $\sim Q(x)$

$\sim(Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)) \equiv \sim Q(x_1) \wedge \sim Q(x_2) \wedge \ldots \wedge \sim Q(x_n)$

- $\sim(\exists x \in D$ such that $Q(x)) \equiv \forall x \in D, \sim Q(x)$

Vacuous Truth of Universal



All the balls in the bowl are blue

- True or false?

What if the bowl is empty? True or False? Why?

- A statement is false *iff* its negation is true.
- Negation of “All the balls in the bowl are blue” is
  - “There exists a ball in the bowl that is not blue”.
  - “There is at least one non-blue ball in the bowl.”
- “There exists a ball in the bowl that is not blue” can be true only if there actually is at least one non-blue ball in the bowl.
  - But there is none in the bowl. The bowl is empty.
  - So the negation is false;
- Therefore, the statement “All the balls in the bowl are blue” is true
- This logic is the essence of Vacuous Truth or True by Default.

Vacuous Truth

Vacuous Truth of Conditional Statements

- If Chiangmai is the capital city of Korea then “Se Won Kim” is handsome
  - Is the statement True or False?
  - The statement (as a whole) is true by default because the antecedent (hypothesis) is false.
    - Whether “Se Won Kim” is handsome or not is irrelevant to the truth value of the whole statement

Vacuous Truth of Universally Quantified Statements

- $\forall x \in D, P(x)$ is vacuously true when D is empty
- $\forall x \in D, P(x) \rightarrow Q(x)$ is vacuously true when D is empty or $P(x)$ is false for every x in D (i.e., $\forall x \in D, \sim P(x)$)
- Asserts that all members of the empty set have a certain property.

Exercise

Determine if each of the following statements are true or false. Explain why.

- If you are President of the US, then I am Father Christmas!
- All mobile phones in the room are turned off.
  - 3 phones are turned off and 2 turned on
- Assume there are no phones at all in the room
  - All mobile phones in the room are turned off
  - All mobile phones in the room are turned on
  - All mobile phones in the room are turned on and off
  - All mobile phones in the room are turned on or off.

Inverse, Converse and Contrapositive of Conditional Statements

First, review of what we learned on Conditional Statement.

- Let S be $p \longrightarrow q$
- Inverse of S
  - $\sim p \longrightarrow \sim q$ $\quad (p \longrightarrow q \not\equiv \sim p \longrightarrow \sim q)$
- Converse of S
  - $q \longrightarrow p$ $\quad (p \longrightarrow q \not\equiv q \longrightarrow p)$
- Contrapositive of S
  - $\sim q \longrightarrow \sim p$ $\quad (p \longrightarrow q \equiv \sim q \longrightarrow \sim p)$

Inverse, Converse and Contrapositive of Universal Conditional Statements

Extension to Universal Conditional Statement

- Let S be $\forall x \in D, P(x) \to Q(x)$
- Inverse of S
  - $\forall x \in D, \sim P(x) \to \sim Q(x)$
  - Note that $\forall x \in D, P(x) \to Q(x) \not\equiv \forall x \in D, \sim P(x) \to \sim Q(x)$
- Converse of S
  - $\forall x \in D, Q(x) \to P(x)$
  - Note that $\forall x \in D, P(x) \to Q(x) \not\equiv \forall x \in D, Q(x) \to P(x)$
- Contrapositive of S
  - $\forall x \in D, \sim Q(x) \to \sim P(x)$
  - Note that $\forall x \in D, P(x) \to Q(x) \equiv \forall x \in D, \sim Q(x) \to \sim P(x)$

Exercise

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

- If a real number is greater than 2, then its square is greater than 4.

- First write it formally: $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.
- Contrapositive:
  - $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$
  - If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.
  - Verify that the statement and its contrapositive are logically equivalent!

Exercise (Cont'd)

- Formal Statement: $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.
- Converse:
  - $\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.
  - If the square of a real number is greater than 4, then the number is greater than 2.
  - Verify that the converse is not logically equivalent to the statement by giving a counterexample
- Inverse:
  - $\forall x \in \mathbf{R}$, if $x \leq 2$ then $x^2 \leq 4$
  - If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.
  - Verify that the inverse is not logically equivalent to the statement by giving a counterexample

Alternative Expressions for Universal Conditional Statements

- $\forall x, P(x)$ is a sufficient condition for $Q(x)$
  - $\forall x, P(x) \rightarrow Q(x)$ ($\forall x$, if $P(x)$ then $Q(x)$)
- $\forall x, P(x)$ is a necessary condition for $Q(x)$
  - $\forall x, \sim P(x) \rightarrow \sim Q(x)$ ($\forall x$, if $\sim P(x)$ then $\sim Q(x)$)
  - $\forall x, Q(x) \rightarrow P(x)$ ($\forall x$, if $Q(x)$ then $P(x)$)
- $\forall x, P(x)$ only if $Q(x)$
  - $Q(x)$ is a necessary condition for $P(x)$
  - $\forall x, \sim Q(x) \rightarrow \sim P(x)$ ($\forall x$, if $\sim Q(x)$ then $\sim P(x)$)
  - $\forall x, P(x) \rightarrow Q(x)$ ($\forall x$, if $P(x)$ then $Q(x)$)

Exercise

Rewrite the following statements as quantified conditional statements.

- Squareness is a sufficient condition for rectangularity
- Being at least 35 years old is a necessary condition for being President of the United States

Exercise

Rewrite the following statement as quantified conditional statements.

- A product of two numbers is 0 only if one of the number is 0.