 Y_\ell^m( \theta , \varphi ) = \sqrt{{(2\ell+1)\over 4\pi}{(\ell-m)!\over (\ell+m)!}}  \, P_\ell^m ( \cos{\theta} ) \, e^{i m \varphi } 

Y_\ell^{m*} (\theta, \varphi) = (-1)^m Y_\ell^{-m} (\theta, \varphi),

\sum_{m=-\ell}^\ell Y_{\ell m}^*(\theta,\varphi) \, Y_{\ell m}(\theta,\varphi) = \frac{2\ell + 1}{4\pi}


\begin{align}
& {} \quad \int Y_{l_1m_1}(\theta,\varphi)Y_{l_2m_2}(\theta,\varphi)Y_{l_3m_3}(\theta,\varphi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&  =
\sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}
\begin{pmatrix}
  l_1 & l_2 & l_3 \\[8pt]
  0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  l_1 & l_2 & l_3\\
  m_1 & m_2 & m_3
\end{pmatrix}
\end{align}



(2j+1)\sum_{m_1 m_2}
\begin{pmatrix}
  j_1 & j_2 & j\\
  m_1 & m_2 & m
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_2 & j'\\
  m_1 & m_2 & m'
\end{pmatrix}
=\delta_{j j'}\delta_{m m'}.



\sum_{j m} (2j+1)
\begin{pmatrix}
  j_1 & j_2 & j\\
  m_1 & m_2 & m
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_2 & j\\
  m_1' & m_2' & m
\end{pmatrix}
=\delta_{m_1 m_1'}\delta_{m_2 m_2'}.


\sum_m (-1)^{j-m}
\begin{pmatrix}
  j & j & J\\
  m & -m & 0
\end{pmatrix} = \sqrt{2j+1}~ \delta_{J0}


c^k(\ell,m,\ell',m')=\int d^2\Omega \ Y_\ell^m(\Omega)^* Y_{\ell'}^{m'}(\Omega) Y_k^{m-m'}(\Omega)


\begin{align}
c^k(\ell,m,\ell',m') &= c^k(\ell,-m,\ell',-m')\\
&=(-1)^{m-m'}c^k(\ell',m',\ell,m)\\
&=(-1)^{m-m'}\sqrt{\frac{2\ell+1}{2k+1}}c^\ell(\ell',m',k,m'-m)\\
& = (-1)^{m'}\sqrt{\frac{2\ell'+1}{2k+1}}c^{\ell'}(k,m-m',\ell,m).\\
\sum_{m=-\ell}^{\ell} c^k(\ell,m,\ell,m)  &=  (2\ell+1)\delta_{k,0}.\\
\sum_{m=-\ell}^\ell \sum_{m'=-\ell'}^{\ell'} c^k(\ell,m,\ell',m')^2  &=  \sqrt{(2\ell+1)(2\ell'+1)}\cdot c^k(\ell,0,\ell',0).\\
\sum_{m=-\ell}^\ell c^k(\ell,m,\ell',m')^2 & =  \sqrt{\frac{2\ell+1}{2\ell'+1}}\cdot c^k(\ell,0,\ell',0).\\
\sum_{m=-\ell}^\ell c^k(\ell,m,\ell',m')c^k(\ell,m,\tilde\ell,m')  &=  \delta_{\ell',\tilde\ell}\cdot\sqrt{\frac{2\ell+1}{2\ell'+1}}\cdot c^k(\ell,0,\ell',0).\\
\sum_m c^k(\ell,m+r,\ell',m) c^k(\ell,m+r,\tilde\ell,m)  &=  \delta_{\ell,\tilde\ell} \cdot \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2k+1}\cdot c^k(\ell,0,\ell',0).\\
\sum_m c^k(\ell,m+r,\ell',m)c^q(\ell,m+r,\ell',m)  &= \delta_{k,q}\cdot\frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2k+1}\cdot c^k(\ell,0,\ell',0).
\end{align}
