

THE EULER-BERNOULLI PLATE IS EXACTLY CONTROLLABLE VIA BENDING MOMENTS ONLY

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I. Introduction and Statement of the Problem

Let Ω be a bounded open set in \mathbb{R}^2 with a sufficiently smooth boundary, Γ . We consider the following model of the Euler-Bernoulli equation with control action in the bending moments:

$$w_{tt} + \Delta^2 w = 0 \quad \text{in } Q_T = (0, T) \times \Omega \quad (1.1.a)$$

$$\begin{aligned} w(0, \cdot) &= w_0 \in H_0^1(\Omega) \\ w_t(0, \cdot) &= w_1 \in H^{-1}(\Omega) \end{aligned} \quad \text{in } \Omega \quad (1.1.b)$$

$$w|_{\Sigma} = 0 \quad \text{in } \Sigma_T = (0, T) \times \Gamma \quad (1.1.c)$$

$$\Delta w + (1 - \mu) B w = u \in L_2(\Sigma_T) \quad \text{in } \Sigma_T \quad (1.1.d)$$

In (1.1.d), the boundary operator B takes the form

$$B w = -\frac{\partial^2 w}{\partial t^2} - k \frac{\partial w}{\partial \eta} \quad (1.2)$$

where $\eta = (n_1, n_2)$ is the outward normal, $\tau = (-n_2, n_1)$, and k is the curvature of Γ . The constant μ , $0 < \mu < 1/2$, represents Poisson's ratio.

We consider the problem of exact controllability of (1.1), i.e. given $(w_{T,1}, w_{T,2}) \in H_0^1(\Omega) \times H^{-1}(\Omega)$, we want to determine $T > 0$ and $u \in L_2(\Sigma_T)$ such that the solution, (w, w_t) , of (1.1) satisfies

$$w(T) = w_{T,1}; \quad w_t(T) = w_{T,2}. \quad (1.3)$$

The problem of exact controllability for Euler-Bernoulli models with control on the boundary has attracted a lot of attention in recent years ([L-1], [L-2], [L-L], [L-T-1], [L-T-2],

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[L-T-3], [L-T-4]). However, in all these cases, the control action, u , is acting through different boundary conditions than (1.1.c)-(1.1.d). In [L-1], the Euler-Bernoulli model with control acting only through Neumann boundary conditions has been treated. The cases when only one control is active in Dirichlet boundary conditions or else control is active in both Dirichlet and Neumann boundary conditions are considered in [L-T-1] and [L-T-4]. Three papers which deal with the same boundary condition as in (1.1.c)-(1.1.d), but with $B \equiv 0$, are [L-1], [L-T-2], and [L-T-5]. But in [L-1] and [L-T-2], in order to obtain the exact controllability results on spaces of optimal regularity, it was necessary to assume that two controls are acting, one in each boundary condition. This limitation was due to a technical difficulty arising in certain controllability estimates, where, in order to dispense with a "higher order term", the simplest way was to add the second control. Since then, the question of whether the Euler-Bernoulli problem can be controlled with only one active control (e.g. bending) has become an open and physically appealing problem. A solution to exact controllability (and uniform stabilization) with control only in $\Delta w|_{\Sigma}$ is found in [L-T-5], but in spaces of finite energy, not of optimal regularity. At the same time, it has been recognized that in order to settle this question in spaces of optimal regularity, new controllability estimates are needed. The first progress in this direction is in [L-2] where the exact controllability for the model (1.1) with *one control* only was established. Although [L-2] settles the question of controllability with one control, $u \in L_2(\Sigma_T)$, the space of reachable states obtained is strictly smaller than $H_0^1(\Omega) \times H^{-1}(\Omega)$. However, regularity results for (1.1) give

$$u \in L_2(\Sigma_T) \Rightarrow (w, w_t) \in C[0, T; H_0^1(\Omega) \times H^{-1}(\Omega)] \quad (1.4)$$

(see [L-T-6]). Hence, $H_0^1(\Omega) \times H^{-1}(\Omega)$ is the natural, and in fact the optimal, space of exact controllability.

The first result where exact controllability in Δw is established on the optimal space $H_0^1(\Omega) \times H^{-1}(\Omega)$ with one control only is a recent reprint, [L-3]. But [L-3] only considers the case when $B \equiv 0$. In fact, the assumption $B \equiv 0$ is critical for the analysis in [L-3] which is based on reduction of the plate problem to the Schrödinger problem. If $B \neq 0$, the problem loses a natural symmetry and this reduction is no longer valid.

The main contribution of this paper is that we treat a complete boundary operator, (1.1.d), as it arises in physical models and which includes moments of inertia realistically present in the system. For this model, we shall prove that, with one control acting as a bending moment, exact controllability holds on the maximal space $H_0^1(\Omega) \times H^{-1}(\Omega)$. In our case, because the boundary operator includes the term B , major new technical difficulties occur. The techniques of [L-3], which are based on microlocal estimates for the Schrödinger equation, are not applicable. Instead, we shall prove the necessary controllability estimates for the plate equation directly. The main technical contribution is the proof of new regularity

estimates for the traces of the solutions to the perturbed plate equation (see Lemma 3.2).

Below we formulate our main result:

Theorem 1 : For any $T > 0$ and any $(w_0, w_1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$,
 $(w_{T,1}, w_{T,2}) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ there exists $u \in L_2(\Sigma_T)$ such that the solution of (1.1)
satisfies

$$(w(T), w_t(T)) = (w_{T,1}, w_{T,2}).$$

Remark 1.1 : If we consider the Kirchoff model with finite speed of propagation instead of the Euler-Bernoulli model, then the question of controllability with only one control is a simpler one. Indeed, the solutions to Kirchoff models display more regularity in the time variable. As a result, the controllability estimates are easier to obtain (see [L-T-3]). In view of this, our results assert that, among other things, the Kirchoff model in the limit, i.e., when the speed of propagation becomes infinite, displays the same controllability properties as the model with finite speed of propagation.

Remark 1.2 : The result of Theorem 1, together with regularity property (1.4), allow us to use abstract Riccati theory (see [F-L-T]). This, in turn, provides us with a solution to the stabilization problem where the feedback operator is based on the solution to the Algebraic Riccati Equation.

Remark 1.3 : One could consider a more general case when only a portion of the boundary is available to the control problem. The techniques of this paper can be easily adapted to provide a solution to this problem, assuming the boundary, Γ , satisfies some rather natural geometric conditions.

The paper is organized in the following way. In the second section, we provide some background material and we state the controllability inequality. The third and fourth sections are devoted to the proof of this inequality.

II. Background Material and Controllability Inequality

It is enough to prove Theorem 1 for some $T > T_0 > 0$. Indeed, once we have the result for $T > T_0$, then an independent argument as in [L-1] allows us to deduce the same result for an arbitrary $T > 0$.

We find it convenient to represent the solution to (1.1) in semigroup form. To accomplish this, we introduce the following operators:

Define $A: L_2(\Omega) \rightarrow L_2(\Omega)$ by:

$$Ay = \Delta^2 y \quad \mathcal{D}(A) = \{y \in H^4(\Omega): y|_{\Gamma} = 0, \Delta y + (1-\mu)By|_{\Gamma} = 0\}. \quad (2.1)$$

Define $G: L_2(\Gamma) \rightarrow L_2(\Omega)$ by:

$$Gg = v \text{ iff } \begin{cases} \Delta^2 v = 0 \\ v|_{\Gamma} = 0 \\ \Delta v + (1-\mu)Bv = g \end{cases}. \quad (2.2)$$

The sine and cosine operators corresponding to A will be respectively denoted by:

$$S(t): L_2(\Omega) \rightarrow L_2(\Omega) \quad \text{and} \quad C(t): L_2(\Omega) \rightarrow L_2(\Omega). \quad (2.3)$$

Define $\mathcal{L}_T^i: L_2(\Sigma_T) \rightarrow L_2(\Omega)$, $i = 1, 2$, by:

$$\begin{aligned} \mathcal{L}_T^1 u &\equiv \int_0^T AS(T-\tau) Gu(\tau) d\tau \\ \mathcal{L}_T^2 u &\equiv \int_0^T AC(T-\tau) Gu(\tau) d\tau \end{aligned} \quad (2.4)$$

By the same arguments as those in [L-T-6], we can show that the operator

$$\mathcal{L}_T = \begin{bmatrix} \mathcal{L}_T^1 u \\ \mathcal{L}_T^2 u \end{bmatrix} \in \mathcal{L}(L_2(\Sigma_T) \rightarrow C[0, T; H_0^1(\Omega) \times H^{-1}(\Omega)]). \quad (2.5)$$

The solution to (1.1) can now be written as:

$$\begin{cases} w(t) = C(t)w_0 + S(t)w_1 + (\mathcal{L}_T^1 u)(t) \\ w_t(t) = -AS(t)w_0 + C(t)w_1 + (\mathcal{L}_T^2 u)(t) \end{cases} \quad (2.6)$$

Thus, equation (1.1) is exactly controllable if and only if the operator \mathcal{L}_T is from $L_2(\Sigma_T)$ onto $H_0^1(\Omega) \times H^{-1}(\Omega)$. The latter is equivalent to the statement: there exists a constant $C_T > 0$ such that

$$|\mathcal{L}_T^* v|_{L_2(\Sigma_T)} \geq C_T |v|_{H_0^1(\Omega) \times H^{-1}(\Omega)} \quad \forall v \in H_0^1(\Omega) \times H^{-1}(\Omega). \quad (2.7)$$

Our next step is to compute \mathcal{L}_T^* .

Proposition 2.1 : With $v = (v_0, v_1)$,

$$\mathcal{L}_T^* v = G^* A [S(T-t)A^{1/2}v_0 + C(T-t)A^{-1/2}v_1] \quad (2.8)$$

or, in a partial differential equation form,

$$\mathcal{L}_T^* v = \frac{\partial}{\partial \eta} \Psi \quad (2.9)$$

where $\Psi(t)$ is the solution to

$$\begin{cases} \Psi_{tt} = \Delta^2 \Psi \\ \Psi|_{\Gamma} = 0 \\ \Delta \Psi + (1-\mu) B \Psi = 0 \\ \Psi(T) = A^{1/2} v_1, \quad \Psi_t(T) = A^{-1/2} v_0 \end{cases} \quad (2.10)$$

Proof : From the interpolation result of [G-1], $\mathcal{D}(A^{1/4}) = H_0^1(\Omega)$. Therefore,

$$|y|_{H_0^1(\Omega)} = |A^{1/4} y|_{L_2(\Omega)} \quad \text{and} \quad |y|_{H^{-1}(\Omega)} = |A^{-1/4} y|_{L_2(\Omega)}.$$

By using the above fact and Fubini's theorem, it is easily seen that

$$\langle L_T^* u, v \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)} = \langle u, G^* A [S(T-t) A^{1/2} v_0 + C(T-t) A^{-1/2} v_1] \rangle_{L_2([0, T]) \times \Gamma}. \blacksquare$$

We find it convenient to introduce another change of variable. Let

$$z \equiv A_D^{-1} \Psi \quad (2.11)$$

where Ψ satisfies (2.10) and

$$A_D \Psi \equiv \Delta \Psi \quad \forall \Psi \in H_0^1(\Omega) \cap H^2(\Omega).$$

Clearly,

$$z|_{\Gamma} = \Delta z|_{\Gamma} = 0. \quad (2.12)$$

We can easily show that $z(t)$ satisfies the equation

$$\begin{aligned} z_{tt} + \Delta^2 z &= (1-\mu) D \left(k \frac{\partial}{\partial \eta} \Delta z \right) \\ \text{where } Dg = v \quad \text{iff} \quad &\begin{cases} \Delta v = 0 \text{ in } \Omega \\ v|_{\Gamma} = g \end{cases} \end{aligned} \quad (2.13)$$

Moreover, since for all t

$$z(t) = A_D^{-1} \Psi(t) \quad \text{and} \quad z_t(t) = A_D^{-1} \Psi_t(t),$$

we can find, using interpolation theory ([G-1]), that

$$\begin{aligned} |A_D^{3/2} z(t)|_{L_2(\Omega)} &= |A_D^{3/2} A_D^{-1} \Psi(t)|_{L_2(\Omega)} = |A_D^{1/2} \Psi(t)|_{L_2(\Omega)} - |A^{1/4} \Psi(t)|_{L_2(\Omega)} \\ |A_D^{1/2} z_t(t)|_{L_2(\Omega)} &= |A_D^{-1/2} \Psi_t(t)|_{L_2(\Omega)} - |A^{-1/4} \Psi_t(t)|_{L_2(\Omega)}. \end{aligned} \quad (2.14)$$

Since

$$\frac{d}{dt} \left[|A^{1/4} \Psi(t)|_{L_2(\Omega)}^2 + |A^{-1/4} \Psi_t(t)|_{L_2(\Omega)}^2 \right] = 0, \quad (2.15)$$

by (2.14) and (2.15), we obtain that for any t_1, t_2 ,

$$|A_D^{3/2} z(t_1)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(t_1)|_{L_2(\Omega)}^2 - |A_D^{3/2} z(t_2)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(t_2)|_{L_2(\Omega)}^2. \quad (2.16)$$

Noting that

$$\frac{\partial}{\partial \eta} \Delta z = \frac{\partial}{\partial \eta} \Psi \quad (2.17)$$

and recalling Proposition 2.1, inequality (2.7) can be equivalently expressed as

Lemma 2.1 : (Controllability Inequality)

The result of Theorem 1 holds iff the following inequality is satisfied: there exists $T > 0$ and $C_T > 0$ such that for any $z(t)$ satisfying both equation (2.13) with boundary conditions (2.12) and the equivalence relation (2.16), we have

$$\left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_T)} \geq C_T \left[|A_D^{3/2} z(T)|_{L_2(\Omega)} + |A_D^{1/2} z_t(T)|_{L_2(\Omega)} \right]. \quad (2.18)$$

Proof : Follows at once from (2.7), (2.9), and (2.11). Indeed, it is enough to note that from (2.10), (2.14), and (2.15) we have

$$|v_1|_{H^{-1}(\Omega)} \sim |A^{-1/4} A^{1/2} \Psi(T)|_{L_2(\Omega)} = |A^{1/4} \Psi(T)|_{L_2(\Omega)} \sim |A_D^{3/2} z(T)|_{L_2(\Omega)}.$$

$$|v_0|_{H_0^1(\Omega)} \sim |A^{1/4} A^{-1/2} \Psi_t(T)|_{L_2(\Omega)} \sim |A_D^{1/2} z_t(T)|_{L_2(\Omega)}. \blacksquare$$

III. Proof of Controllability Inequality

Lemma 3.1 : Let z be the solution to (2.12)-(2.13) with the property (2.16). Then

$$\left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_T)}^2 + \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Sigma_T)}^2 \geq C_T \left[|A_D^{3/2} z(T)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(T)|_{L_2(\Omega)}^2 \right]. \quad (3.1)$$

Proof : By multiplying both sides of equation (2.13) by $\vec{h} \cdot \nabla (\Delta z)$, where $\vec{h} = \vec{x} - \vec{x}_0$, $\vec{x}_0 \in \mathbb{R}^2$, and using the boundary conditions (2.12), we can find

$$\begin{aligned} \int_{Q_T} \left\{ |\nabla z_t|^2 + |\nabla(\Delta z)|^2 \right\} d\Omega dt &\leq |(z_t, \vec{h} \cdot \nabla(\Delta z))_\Omega|^T + \frac{n}{2} \left| \int_{Q_T} \left\{ |\nabla z_t|^2 - |\nabla(\Delta z)|^2 \right\} d\Omega dt \right| \\ &\quad + \frac{3}{2} M_h \int_{\Sigma_T} \left| \frac{\partial z_t}{\partial \eta} \right|^2 d\Gamma dt + \frac{3}{2} M_h \int_{\Sigma_T} \left| \frac{\partial(\Delta z)}{\partial \eta} \right|^2 d\Gamma dt \\ &\quad + (1-\mu) \left| \left\langle \frac{\partial(\Delta z)}{\partial \eta}, D^*(\vec{h} \cdot \nabla(\Delta z)) \right\rangle_{L_2(\Sigma_T)} \right|. \end{aligned} \quad (3.2)$$

To bound the last term on the right-hand side of equation (3.2), note that

$$D^* \in L(L_2(\Omega) \rightarrow L_2(\Gamma)).$$

Therefore,

$$(1-\mu) \left| \left\langle k \frac{\partial(\Delta z)}{\partial \eta}, D^*(\vec{h} \cdot \nabla(\Delta z)) \right\rangle_{L_2(\Sigma_T)} \right| \\ \leq (1-\mu) M_k \left\{ \frac{1}{4\epsilon} \left| \frac{\partial(\Delta z)}{\partial \eta} \right|_{L_2(\Sigma_T)}^2 + \epsilon M_h^2 M_D^2 \left| \nabla(\Delta z) \right|_{L_2(Q_T)}^2 \right\}, \quad (3.3)$$

where M_k , M_h , and M_D are constants depending respectively on k , \vec{h} , and the operator D . To bound the second term on the right-hand side of equation (3.2), we use the multiplier Δz to get

$$\left| \int_{Q_T} \left\{ |\nabla z_t|^2 - |\nabla(\Delta z)|^2 \right\} d\Omega dt \right| \leq (1-\mu) \frac{M_k}{4\epsilon} \left| \frac{\partial(\Delta z)}{\partial \eta} \right|_{L_2(\Sigma_T)}^2 + \epsilon C \left| \nabla(\Delta z) \right|_{L_2(Q_T)}^2. \quad (3.4)$$

Considering the first term on the right-hand side of (3.2), we can show, using (2.16), that

$$\left| (z_t, \vec{h} \cdot \nabla(\Delta z))_{\Omega} \right|_0^2 \leq \frac{M_h}{2\epsilon} |z_t|_{C[0,T;L_2(\Omega)]}^2 + \epsilon \left[|A_D^{3/2} z(T)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(T)|_{L_2(\Omega)}^2 \right]. \quad (3.5)$$

By substituting equations (3.3)-(3.5) into (3.2) and using the property (2.16), we arrive at

$$\begin{aligned} & \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_T)}^2 + \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Sigma_T)}^2 + |z_t|_{C[0,T;L_2(\Omega)]}^2 \\ & \leq C_T \left[|A_D^{3/2} z(T)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(T)|_{L_2(\Omega)}^2 \right]. \end{aligned} \quad (3.6)$$

By the well known compactness argument, we can show that

$$|z_t|_{C[0,T;L_2(\Omega)]}^2 \leq C \left[\left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Omega)}^2 + \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Omega)}^2 \right] \quad (3.7)$$

which, together with (3.6), gives us (3.1). ■

Remark 3.1 : Inequality (3.1) together with the techniques of [L-T-3] imply the regularity result of (2.5).

Looking at inequality (3.1), it is clear that by eliminating the term $\left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Sigma_T)}^2$ from (3.1), inequality (2.18) will follow. In fact, this is the main difficulty and novelty in this paper.

Let $\alpha > 0$ be a given constant and define $\Sigma_{T_\alpha} \equiv \Gamma \times [-\alpha, T+\alpha]$.

Lemma 3.2 : For any $\varepsilon > 0$,

$$\begin{aligned} \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Sigma_T)}^2 &\leq C_T, \varepsilon \left[\left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})}^2 + |z(T)|_{L_2(\Omega)}^2 + |z_t(T)|_{L_2(\Omega)}^2 \right] \\ &\quad + \varepsilon \left[|A_D^{3/2} z(T)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(T)|_{L_2(\Omega)}^2 \right]. \end{aligned} \quad (3.8)$$

Assuming that Lemma 3.2 is valid, we shall now prove the controllability inequality (2.18).

Proof of Lemma 2.1 : Combining the results of Lemma 3.1 and Lemma 3.2 we obtain, since ε can be chosen to be arbitrarily small,

$$\begin{aligned} &|A_D^{3/2} z(T)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(T)|_{L_2(\Omega)}^2 \\ &\leq C_T \left[\left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})}^2 + |z(T)|_{L_2(\Omega)}^2 + |z_t(T)|_{L_2(\Omega)}^2 \right]. \end{aligned} \quad (3.9)$$

Using our compactness argument, we obtain the estimate

$$|z(T)|_{L_2(\Omega)}^2 + |z_t(T)|_{L_2(\Omega)}^2 \leq C_T \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})}^2. \quad (3.10)$$

Combining (3.10) with the equivalence relation in (2.16), we find

$$|A_D^{3/2} z(T+\alpha)|_{L_2(\Omega)}^2 + |A_D^{1/2} z_t(T+\alpha)|_{L_2(\Omega)}^2 \leq \bar{C}_T \int_{-\alpha}^{T+\alpha} \left| \frac{\partial}{\partial \eta} \Delta z \right|_{\Gamma}^2 dt. \quad (3.11)$$

Finally, introducing the new variable $\tilde{z}(t) \equiv z(t-\alpha)$ in (3.11) yields

$$|A_D^{3/2} \tilde{z}(T+2\alpha)|_{L_2(\Omega)}^2 + |A_D^{1/2} \tilde{z}_t(T+2\alpha)|_{L_2(\Omega)}^2 \leq \bar{C}_T \int_0^{T+2\alpha} \left| \frac{\partial}{\partial \eta} \Delta \tilde{z} \right|_{\Gamma}^2 dt. \quad (3.12)$$

Since both \tilde{z} and z are solutions to the same problem, (2.12)-(2.13) and (2.16), we obtain the statement of Lemma 2.1 with T replaced by $T+2\alpha$. ■

Next we must prove Lemma 3.2.

Proof of Lemma 3.2 : Using the variation of parameter formula to write the solution of (2.12)-(2.13), we obtain

$$z(t) = e^{iA_D(T-t)} \tilde{z}_0 + e^{-iA_D(T-t)} \tilde{z}_1 - A_D^{-1} \int_t^T \frac{1}{2i} (e^{iA_D(t-\tau)} - e^{-iA_D(t-\tau)}) Df(\tau) d\tau \quad (3.13)$$

$$\text{where } \begin{cases} f \equiv (1-\mu) k \frac{\partial}{\partial \eta} \Delta z \\ \tilde{z}_0 \equiv \frac{1}{2} z_0 + \frac{1}{2} A_D^{-1} z_1, \\ \tilde{z}_1 \equiv \frac{1}{2} z_0 - \frac{1}{2} A_D^{-1} z_1 \end{cases}$$

Define the following:

$$\begin{cases} A_1 \equiv \frac{\partial}{\partial \eta} A_D e^{iA_D(T-t)} \tilde{z}_0 \\ A_2 \equiv \frac{\partial}{\partial \eta} A_D e^{-iA_D(T-t)} \tilde{z}_1 \\ B_1 \equiv \frac{1}{2} \frac{\partial}{\partial \eta} \int_t^T e^{iA_D(t-\tau)} Df(\tau) d\tau \\ B_2 \equiv \frac{1}{2} \frac{\partial}{\partial \eta} \int_t^T e^{-iA_D(t-\tau)} Df(\tau) d\tau. \end{cases} \quad (3.14)$$

Then

$$\begin{aligned} \frac{\partial}{\partial \eta} z_t(t) &= -i(A_1 - A_2) - (B_1 + B_2); \\ \frac{\partial}{\partial \eta} \Delta z(t) &= (A_1 + A_2) + i(B_1 - B_2). \end{aligned}$$

Hence

$$|\frac{\partial}{\partial \eta} z_t(t, x)|^2 - |\frac{\partial}{\partial \eta} \Delta z(t, x)|^2 = -4 \operatorname{Re} B(t, x) \quad (3.15)$$

$$\begin{aligned} \text{where } B(t, x) &\equiv (A_1 + iB_1)(\overline{A_2 - iB_2}) \\ &= A_1 \bar{A}_2 - iA_1 \bar{B}_2 + iA_2 \bar{B}_1 - B_1 \bar{B}_2. \end{aligned} \quad (3.16)$$

Let $\Phi(t) \in C_0^\infty(\mathbb{R}^2)$ be such that $\Phi(t) \equiv 1$ on $[0, T]$ and $\Phi(t) \equiv 0$ on $(-\infty, -\alpha) \cup (T+\alpha, \infty)$.

Then by (3.15),

$$\begin{aligned} \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Sigma_T)}^2 &\leq \int_{-\infty}^{\infty} \Phi(t) \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Gamma)}^2 dt = \int_{-\infty}^{\infty} \Phi(t) \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Gamma)}^2 dt \\ &- 4 \operatorname{Re} \int_{-\infty}^{\infty} \Phi(t) \int_{\Gamma} B(t, x) dx dt \leq \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_T)}^2 + 4 \left| \int_{-\infty}^{\infty} \Phi(t) \int_{\Gamma} B(t, x) dx dt \right|. \end{aligned} \quad (3.17)$$

By using the same arguments as in [L-3], we show that

$$\left| \int_{-\infty}^{\infty} \Phi(t) \int A_2 \bar{A}_1 dx dt \right| \leq C_T \left[|z_0|_{L_2(\Omega)}^2 + |z_1|_{L_2(\Omega)}^2 \right]. \quad (3.18)$$

In order to estimate the remaining three terms on the right hand side of (3.16), we need the following result.

Proposition 3.1 : Let v be a solution of

$$\begin{cases} v_t = iA_D v - Dg \\ v(T) = v_0 \in \mathcal{D}(A_D^{1/2}) \end{cases} \quad (3.19)$$

Then

$$\left| \frac{\partial}{\partial \eta} v \right|_{L_2(\Sigma_T)} \leq C_T \left[|A_D^{1/2} v_0|_{L_2(\Omega)}^2 + |g|_{L_2(\Sigma_T)} \right]. \quad (3.20)$$

Proposition 3.1 will be proven in the next section.

Remark 3.2 : Notice that inequality (3.20) does not follow from standard regularity theory for the Schrödinger equation. In fact, with $g \in L_2(\Sigma_T)$, one has $Dg \in L_2[0, T; H^{1/2-\epsilon}(\Omega)]$ and standard regularity theory gives $v \in C[0, T; H^{1/2-\epsilon}(\Omega)]$. This result would at most imply $\frac{\partial}{\partial \eta} v \in L_2[0, T; H^{-1/2-\epsilon}(\Omega)]$. Instead, (3.20) allows us to gain one additional derivative on the boundary for $\frac{\partial}{\partial \eta} v$.

Assuming the validity of Proposition 3.1, we now continue with the proof of Lemma 3.2.

Applying the result of Proposition 3.1 with $g \equiv 0$ we obtain

$$\begin{aligned} |A_1|_{L_2(\Sigma_{T_\alpha})}^2 + |A_2|_{L_2(\Sigma_{T_\alpha})}^2 &\leq C_T \left[|A_D^{3/2} \bar{z}_0|_{L_2(\Omega)}^2 + |A_D^{3/2} \bar{z}_1|_{L_2(\Omega)}^2 \right] \\ &\leq C_T \left[|A_D^{3/2} z_0|_{L_2(\Omega)}^2 + |A_D^{1/2} z_1|_{L_2(\Omega)}^2 \right]. \end{aligned} \quad (3.21)$$

Again applying the result of Proposition 3.1, but now with $v_0 \equiv 0$, we find

$$|B_1|_{L_2(\Sigma_{T_\alpha})}^2 + |B_2|_{L_2(\Sigma_{T_\alpha})}^2 \leq C_T |f|_{L_2(\Sigma_{T_\alpha})}^2 = C_T \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})}^2. \quad (3.22)$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(t) \int [|A_1 \bar{B}_2| + |A_2 \bar{B}_1| + |B_1 \bar{B}_2|] dx dt \\ \leq \int_{\Sigma_{T_\alpha}} [|A_1| |B_2| + |A_2| |B_1| + |B_1| |B_2|] \\ \leq C_T \left[|A_D^{3/2} z_0|_{L_2(\Omega)} + |A_D^{1/2} z_1|_{L_2(\Omega)} + \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})} \right] \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})}. \end{aligned} \quad (3.23)$$

Combining (3.17), (3.18), and (3.23) gives us

$$\begin{aligned} \left| \frac{\partial}{\partial \eta} z_t \right|_{L_2(\Sigma_T)}^2 &\leq C_T \left| \frac{\partial}{\partial \eta} \Delta z \right|_{L_2(\Sigma_{T_\alpha})}^2 + C_T \left[|z_0|_{L_2(\Omega)}^2 + |z_1|_{L_2(\Omega)}^2 \right] \\ &\quad + \epsilon \left[|A_D^{3/2} z_0|_{L_2(\Omega)}^2 + |A_D^{1/2} z_1|_{L_2(\Omega)}^2 \right] \end{aligned} \quad (3.24)$$

where ϵ can be taken to be arbitrarily small. Thus, the proof of Lemma 3.2 is complete. ■

IV. Proof of Proposition 3.1

Let

$$v(t) = e^{iA_D(T-t)} v_0 + \int_t^T e^{iA_D(T-\tau)} Dg(\tau) d\tau = v_1(t) + v_2(t).$$

From the above equation, we can see that $v(t)$ satisfies equation (3.15).

Step 1 : We multiply equation (3.15) by $\vec{h} \cdot \nabla v$, where $\vec{h}|_{\Gamma} = \vec{\eta}$, and integrate by parts. This gives

$$\left| \frac{\partial}{\partial \eta} v \right|_{L_2(\Sigma_T)} \leq C_T \left[|Dg|_{L_1(0,T; \Omega)} + |A_D^{1/2} v|_{L_\infty(0,T; \Omega)} \right]. \quad (4.1)$$

Proof : Using the multiplier $\vec{h} \cdot \nabla \bar{v}$, we obtain

$$\begin{aligned} \operatorname{Im} \int_{Q_T} v_t \vec{h} \cdot \nabla \bar{v} d\Omega dt - \frac{1}{2} \int_{\Sigma_T} \left| \frac{\partial v}{\partial \eta} \right|^2 d\Gamma dt + \int_{Q_T} H |\nabla v|^2 d\Omega dt \\ - \frac{1}{2} \int_{Q_T} |\nabla v|^2 \operatorname{div} \vec{h} d\Omega dt = - \operatorname{Im} \int_{Q_T} Dg \vec{h} \cdot \nabla \bar{v} d\Omega dt. \end{aligned} \quad (4.2)$$

Let

$$F = \vec{h} \bar{\Psi} \Psi_t.$$

Then

$$\operatorname{div} F = \bar{\Psi} \Psi_t, \operatorname{div} \vec{h} + (\vec{h} \cdot \nabla \Psi) \Psi_t + (\vec{h} \cdot \nabla \Psi_t) \bar{\Psi}.$$

Therefore, by using the divergence theorem, we can find

$$\begin{aligned} \operatorname{Im} \int_{Q_T} v_t \vec{h} \cdot \nabla \bar{v} d\Omega dt &= \frac{1}{2} \int_{\Sigma_T} \bar{v} v_t d\Gamma dt - \frac{1}{2} \int_{Q_T} \bar{v} v_t \operatorname{div} \vec{h} \\ &\quad - \frac{1}{2} \int_{\Omega} [\bar{v}(T) \vec{h} \cdot \nabla v(T) - \bar{v}(0) \vec{h} \cdot \nabla v(0)]. \end{aligned} \quad (4.3)$$

Combining equations (4.2) and (4.3), we get

$$\begin{aligned}
\frac{1}{2} \int_{\Sigma_T} \left| \frac{\partial v}{\partial \eta} \right|^2 d\Gamma dt &= -\frac{1}{2} \int_{Q_T} \bar{v} v_t \operatorname{div} \vec{h} d\Omega dt \\
&\quad + \frac{1}{2} \int_{Q_T} [\bar{v}(0) \vec{h} \cdot \nabla v(0) - \bar{v}(T) \vec{h} \cdot \nabla v(T)] d\Omega + \int_{Q_T} H |\nabla v|^2 d\Omega dt \quad (4.4) \\
&\quad - \frac{1}{2} \int_{Q_T} |\nabla v|^2 \operatorname{div} \vec{h} d\Omega dt + \operatorname{Im} \int_{Q_T} D g \vec{h} \cdot \nabla \bar{v} d\Omega dt.
\end{aligned}$$

Next, using the multiplier \bar{v} with equation (3.15), we obtain

$$\left| \int_{Q_T} v_t \bar{v} d\Omega dt \right| \leq (1 + C\varepsilon) \|\nabla v\|_{L_\infty[0,T;L_2(\Omega)]}^2 + \frac{1}{4\varepsilon} \|Dg\|_{L_1[0,T;L_2(\Omega)]}^2. \quad (4.5)$$

By combining (4.5) with (4.4), we obtain our desired inequality, (4.1). ■

Step 2 : Take $g \equiv 0$ in (3.15). Since v_1 is the solution to the resulting problem, we will use the multiplier $\bar{v}_{1,t}$. This yields

$$|A_D^{1/2} v_1(t)|_{L_2(\Omega)}^2 = \text{constant} = |A_D^{1/2} v_0|_{L_2(\Omega)}^2. \quad (4.6)$$

Hence, for all $x \in H_0^1(\Omega)$,

$$\sup_t |A_D^{1/2} e^{iA_D t} x|_{L_2(\Omega)}^2 = |A_D^{1/2} x|_{L_2(\Omega)}^2 \quad (4.7)$$

and therefore,

$$|A_D^{1/2} v_1|_{L_\infty[0,T;L_2(\Omega)]} = |A_D^{1/2} v_0|_{L_2(\Omega)}. \quad (4.8)$$

Thus, applying step 1 with $g \equiv 0$, we find

$$\left| \frac{\partial}{\partial \eta} v_1 \right|_{L_2(\Sigma_T)}^2 \leq C_T |A_D^{1/2} v_0|_{L_2(\Omega)}^2. \quad (4.9)$$

Step 3 : We shall prove

$$|v_2|_{L_\infty[0,T;H_0^1(\Omega)]} \leq C_T \|g\|_{L_2(\Sigma_T)}. \quad (4.10)$$

We define the closed and densely defined operator $L: L_2(\Sigma_T) \rightarrow L_2(Q_T)$ by:

$$(Lf)(t) = A_D \int_t^T e^{iA_D(t-\tau)} Df(\tau) d\tau.$$

Then we can easily show that

$$(L^* g)(t) \equiv D^* A_D \int_0^t e^{-iA_D(t-\tau)} g(\tau) d\tau.$$

If we let

$$\Psi(t) \equiv \int_0^t e^{-iA_D(t-\tau)} g(\tau) d\tau,$$

then $\Psi(t)$ satisfies

$$\begin{cases} \Psi_t = -iA_D \Psi + g \\ \Psi(0) = 0. \end{cases} \quad (4.11)$$

Similarly to the proof in step 1, we can find

$$|\frac{\partial}{\partial \eta} \Psi|_{L_2(\Sigma_T)} \leq C_T \left[|g|_{L_1[0,T; L_2(\Omega)]} + |A_D^{1/2} \Psi|_{L_\infty[0,T; L_2(\Omega)]} \right]. \quad (4.12)$$

But from (4.6), we have

$$|A_D^{1/2} \Psi(t)|_{L_2(\Omega)} \leq \int_0^t |A_D^{1/2} g(\tau)|_{L_2(\Omega)} d\Omega \leq |A_D^{1/2} g|_{L_1[0,T; L_2(\Omega)]}. \quad (4.13)$$

Combining (4.12) and (4.13) yields

$$|\frac{\partial}{\partial \eta} \Psi|_{L_2(\Sigma_T)} \leq C_T \left[|A_D^{1/2} g|_{L_1[0,T; L_2(\Omega)]} \right] = C_T |g|_{L_1[0,T; H_0^1(\Omega)]} \quad (4.14)$$

which means that

$$L^* \in \mathcal{L}(L_1[0,T; H_0^1(\Omega)] \rightarrow L_2(\Sigma_T)). \quad (4.15)$$

Hence,

$$L \in \mathcal{L}(L_2(\Sigma_T) \rightarrow L_\infty[0,T; H^{-1}(\Omega)]). \quad (4.16)$$

Let

$$Kf \equiv A_D^{-1} Lf.$$

Because of the regularity of A_D^{-1} , (4.16) is equivalent to the statement

$$K \in \mathcal{L}(L_2(\Sigma_T) \rightarrow L_\infty[0,T; H_0^1(\Omega)]). \quad (4.17)$$

In particular, this means that

$$|v_2|_{L_\infty[0,T; H_0^1(\Omega)]} \leq C_T |g|_{L_2(\Sigma_T)} \quad (4.18)$$

as desired.

Step 4 : Combining (4.7) and (4.10), we find

$$|A_D^{1/2} v|_{L_\infty[0,T; L_2(\Omega)]} \leq C_T \left[|A_D^{1/2} v_0|_{L_2(\Omega)} + |g|_{L_2(\Sigma_T)} \right]. \quad (4.19)$$

By substituting (4.19) into (4.1) and recalling that $D \in \mathcal{L}(L_2(\Gamma) \rightarrow L_2(\Omega))$, the desired result of Proposition 3.1 is found. ■

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