SHAPE DERIVATIVE OF DISCRETIZED PROBLEMS

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SUMMARY

This paper is concerned by numerical sensitivity analysis in problems governed by boundary-value problems described by partial differential equations (P.D.E.).

The P.D.E. solution y is approximated by the solution of finite element method (F.E.M) y_h.

In shape optimization, we are concerned by a cost functional $J(\Omega_h) = h(\Omega_h, M_h, y_h)$, where Ω_h is the discretized geometrical domain and M_h the nodes of the mesh.

This work is devoted to the numerical calculation of the derivative with respect to the coordinates of the nodes of the cost J associated to M_h we have a Q_ℓ Lagrange F.E.M and y is solution of a variational problem. For simplicity, we shall restrict the presentation to a second order problem.

The domain Ω_h is only described by the boundary nodes, we also discuss the use of the derivatives with respect to the internal nodes. The case P_1 Lagrange finite element is treated by Zolesio [6]. In this work, we treat the general case of Q_ℓ Lagrange finite element.

INTRODUCTION

The main objective of this paper, is to show how to obtain in shape optimization the derivatives with respect to the nodes as a particular case of the general theory of the gradient calculation developped in [1], [4] and [5], by a simple choice of the velocity field V well adapted to the situation.

The domain is only defined by the boundary nodes, and numerically the internal nodes move in an injustified way, we shall precise in the last part, the use of the derivatives with respect to the internal nodes.

The calculation of the derivative with respect to the domain Ω for boundary-value problems is realized by introducing the deformations T_t constructed from a velocity field V (particular velocity), this field is described by the functional space

$$E = C^0\left([0,\epsilon[,\ C_k\ (\mathbb{R}^n, \mathbb{R}^n)\right)\ k \geq 1$$

with this method, we envisage the general deformations $\Omega_t = T_t(\Omega)$ of the domain Ω, for V given in E.
The properties of T_t and Ω_t are widly studied. If U_t is solution in the configuration Ω_t, Γ_t of a boundary value problem, so $J(\Omega_t) = h(\Omega_t, u_t)$ is the functional to minimize with respect to the domain Ω.
The derivative of J at Ω in the V direction is given by $dJ(\Omega,V) = \frac{d}{dt} J(\Omega_t)|_{t=0}$
We establish in [4],[5] the properties of this derivative. In particular if Γ is regular, then $dJ(\Omega,V)$ is only dependent on the germ of $V(0)$ on the neighbourhood of Γ, if in addition the application $V \to dJ(\Omega,V)$ is linear, then from the structure theorem (Hadamard formula theorem, see [4], [5]), it exists a distribution g_n on the surface Γ (g_n is unique) such that the gradient G, vectorial distribution on $\mathbb{R}^n$, is given by $G = {}^t\gamma_\Gamma(g_n.n)$ (γ_Γ is the trace application or restriction on Γ) and the derivative with respect to the domain verifies :

$$(0.1) \qquad dJ(\Omega,V) = \langle G, V(0) \rangle_{D^k(\mathbb{R}^n,\mathbb{R}^n)' \times D^k(\mathbb{R}^n,\mathbb{R}^n)}$$

The calculation of the gradient G is generally done via the calculation of the eulerian derivative of the state equation.

$$\dot{u} = \frac{d}{dt}(u_t \circ T_t)\Big|_{t=0} \quad \text{(derivative in } H^m(\Omega))$$

For instance, for the simplest functional

$$J(\Omega) = \frac{1}{2}\int_\Omega (u - y_d)^2 \, dx$$

$\left(y_d \text{ is a given function on } \mathbb{R}^n\right)$, we have :

$$dJ(\Omega,V) = \int_\Omega (u-y_d)\left(\dot{u} - \nabla y_d.V(0)\right) dx + \frac{1}{2}\int_\Omega (u - y_d)^2 \operatorname{div} V(0) \, dx$$

However, introducing the derivative u' of the state u with respect to the domain on the V direction defined by

$$(0.2) \qquad u' = \dot{u} - \nabla u.V(0)$$

we obtain (Γ is supposed regular)

$$(0.3) \qquad dJ(\Omega,V) = \int_{\Omega}(u - y_d)\, u'\, dx + \frac{1}{2}\int_{\Gamma} (u - y_d)^2\, V(0).n\, d\sigma$$

this second expression (0,3) of dJ is more interesting since, closer to the structure noticed by the theorem, seeing that the second term on the right hand side already depends explicitly on the restriction V(0).n on Γ.

In fact, for elliptic and parabolic boundary value problems, we characterize u using the implicit function theorem, and we obtain the linearity and continuity of $V \longrightarrow \dot{u}$ and then of $V \longrightarrow u'$. We then prove the following results :

The derivative u' with respect to Ω on the V direction depends only on the restriction of V(0).n on Γ, (this result does not hold for $\dot{u}$). It means that u' is solution of boundary value problem with homogeneous conditions on Ω, and introducing in a classical way an adjoint state p in (0,3), we immediatly obtain the expression of the gradient G defined by (0,1).

1. SHAPE OPTIMIZATION

1.1 General case

It would be well to distinguish two types of deformation : The first one adapted to small displacements, and the second to large displacements and to deformations governed by parameters. Let Ω included in $\mathbb{R}^n$ (to simplify, we will take n = 2). V is a field defined on the neighbourhood of U of $\bar{\Omega}$ (resp. on $[0,\epsilon[\times U$). The transformation T_t is defined by :

$$(1.1) \qquad T_t : x \longrightarrow x + t\, V(x) \qquad \left(\text{resp. } x(t) = x + \int_0^t V(s,\, x(q))\, ds\right)$$

we will find the properties of T_t in F. Murat and J. Simon [6] (resp. in Zolesio [4],[5]).

We will be more interested to the second deformations which, for small t, have the same properties as the first one, see T_t and T^{-1} are C^k, and preserve them when $t \longrightarrow +\infty$ while the field V is in the space R_{∞} (see [4]). In addition, for the shape optimization defined by parameters, it's easy to find the field V(t,x) which construct the same variation of the geometry as the variation of the parameters. So the derivation of the cost and states with respect to the parameters is simply a matter of a particular choice of the field V. For such situation see for instance Delfour, Payre, Zolesio

[2]. It's the analogous situation that we consider for the displacements of nodes in a mesh, however the coordinates of nodes are the parameters.

2.2 Preliminarities and notations

The first aspect, is that the triangulation $\mathcal{C}_h$ is established over the set $\bar{\Omega}$, i.e, the set $\bar{\Omega}$ is subdivised into a finite number of subsets K (see Ciarlet [9]) in such a way that the following properties are satisfied.

$$\Omega = \bigcup_{1 \leqslant k \leqslant M} K_k \quad .$$

M is the number of elements, and each element K_i is isoparametrically equivalent to a reference element $\hat{K}$ trough a mapping F_i, we shall use the usual correspondance

$$K_i \; : \; \hat{K} \longrightarrow K_i$$
$$\xi \longrightarrow \sum_{j=1}^{\ell} \hat{q}_j(\xi) \; M_j(K_i)$$

$F_i \in \left(Q_\ell(\hat{K})\right)^2$, where Q_ℓ is polynomial space of degree $\leqslant \ell$ and $M_j(K_i)$ the freedom degrees on the element K_i. So we are considering the general case of Q_ℓ Lagrange finite element. Let N_1 the number of nodes in the triangulation, and N_2 the number of freedom degrees which are not nodes of Ω. In what follows, we will consider only the gradient with respect to the nodes. Once the nodes are on position, the other freedom degrees are automatically known.

Let move a node M_j in the k-direction (k = 1,2)

$$\begin{aligned} M_j^t &= M_j + t\, \ell_k \\ &= \left(X_1^j,\, X_2^j\right) + t\, \ell_k \end{aligned}$$

ℓ_k is the euclidean basis in $\mathbb{R}^2$ ((0,1), (1,0)). This assumption will define a new triangulation $\mathcal{C}_h^t$, the elements are called K_i^t.

As for the element K_i, we define the mapping

$$K_i^t \; : \; \hat{K} \longrightarrow K_i^t$$
$$\xi \longrightarrow \sum \hat{q}_j(\xi) \; (M_j(K_i) + t \; \ell_k)$$

Let X_h a finite-dimensional subspace on the Hilert space (for example $H^1(\Omega)$) associated to the triangulation $\mathcal{C}_h$, and X_h^t to the triangulation $\mathcal{C}_h^t$. Since we are considering Lagrange finite element, the space X_h will be spanned by the functions v_i, $i=1,\ldots,N$ ($N = N_1+N_2$) such that.

$$\text{(2.1)} \quad \begin{cases} v_i \in \mathcal{C}^0\left(\bar{\Omega}\right) & \\ v_i/K_j \circ F_j \in Q_\ell(\hat{K}) & i \leqslant j \leqslant M \\ v_i(M_j) = \delta_{i,j} & 1 \leqslant j \leqslant N \end{cases}$$

In the same way, X_h^t is spanned by v_i^t

$$\text{(2.1)'} \quad \begin{cases} v_i^t \in \mathcal{C}^0\left\{\bar{\Omega}^t\right\} & \\ v_i^t\big/k_j^t \circ F_j^t \in Q_p(\hat{K}) & 1 \leqslant j \leqslant M \\ v_i^t\left(M_j^t\right) = \delta_{i,j} & 1 \leqslant j \leqslant N \end{cases}$$

2.3 <u>The deformation field</u>

As in the continuous case ([4], [5]), the goal is to determine the transformation T_t which maps Ω into Ω^t, via an adapted choice of velocity $V(t,.)$.

The transformation T_t defined by (0,1) is continuous, so we will determine it on each element K_i $1 \leqslant i \leqslant M$.

Let consider the situation where only the node i is noved in the k-direction.

<u>Definition 1.1</u>

For $1 \leqslant j \leqslant N_1$ where N_1 is the number of nodes, we define

$$V_j(t,x) = v_j^t(x).\ell_k$$

and

$$T_t(x) = x + \int_0^t V_j(s) \; x(s) \; ds$$

Lemma 1.1. The restriction of the transformation T_t to the element K_i is

$$T_t / K_i = F_i^t \circ F_i^{-1}$$

Proof. We will show that $\frac{d}{dt}(S_t) \circ (S^t)^{-1} = V_j(t,x)$ where $S^t = F_i^t \circ F_i^{-1}$.

From the definition of F_i and F_i^te have

$$F_i^t(\xi) = F_i(\xi) + t\,\hat{q}_j(\xi)\,\ell_k$$

$$\frac{d}{dt}\left(F_i^t \circ F_i^{-1}\right) = \left(\frac{d}{dt} F_i^t\right) \circ (F_i)^{-1} = \hat{q}_j \circ (F_i)^{-1}.\ell_k$$

$$\frac{d}{dt}(S^t) \circ (S^t)^{-1} = \hat{q}_j \circ (F_i)^{-1} \circ \left(F_i^{-t} \circ F_i^{-1}\right)^{-1}.\ell_k$$

$$= \hat{q}_j \circ \left(F_i^t\right)^{-1}.\ell_k = v_i^t / K_i^t \,.\, \ell_k \tag{2.3}$$

We deduce that S^t transforms K_i into K_i^t and its velocity $\left(\text{ie } \frac{d}{dt}(S^t) \circ (S^t)^{-1}\right)$

is given from (2.3) so T_t/S^t.

Lemma 1.2

Let v_i, v_i^t, $1 \leqslant i \leqslant N_i$ defined by (2.1) and (2.1)'. The basis function related to the nodes, then

$$v_i^t \circ T_t = v_i \tag{2.4}$$

Proof. We will verify (2.4) on each element K_j from the definition of v_i and v_i^t we have

$$(v_i/K_j) \circ F_j \in \hat{Q}(\hat{K}) \iff v_i/K_j \quad \hat{q} \circ F_j^{-1}$$

$$\left(v_i^t/K_j^t\right) \circ F_j^t \in \hat{Q}\ (\hat{K}) \iff v_i^t/\ K_j^t \quad \hat{q} \circ \left(F_j^t\right)^{-1}$$

however

$$v_i^t/K_j^t \circ T_t/K_j = v_i^t/K_j^t \circ \left(F_j^t \circ F_j^{-1}\right)$$

$$= \hat{q} \circ \left(F_j^t\right)^{-1} \circ \left(F_j^t \circ F_j^{-1}\right) = \hat{q} \circ F_j^{-1} = v_i/K_j$$

Corollary

For each node i $1 \leqslant i \leqslant N_1$ we have

$$\frac{d}{dt}\left(v_i^t \circ T_t\right) = 0 \iff \dot{v}_i^t = 0$$

then the basis function related to the nodes are convected.

Remark.

The lemma 1.2 still holds for $N_1+1 \leqslant i \leqslant N_1+N_2$, the freedom degrees which are not nodes of Ω, since

so $\dot{v}_i = 0$, for $1 \leqslant i \leqslant N1+N2$, and, the basis functions are convected.

The velocity field V discrebing the transformation T_t which applies Ω into Ω_t is spanned by the basis functions v_i related to the nodes $1 \leqslant i \leqslant N1$, so

$$v_t \in V \iff v^t(x,y) = \left(\sum_{i=1}^{N1} r_i \ v_i^t, \ \sum_{i=1}^{N1} s_i \ v_i^t\right) \tag{2.5}$$

3. DERIVATION WITH RESPECT TO THE NODES

In the general case, we are looking for partial derivative with respect to the coordinates of the nodes, for a functional $J(u,\Omega)$, where u is a discretized (generally noted u_h) solution of a variational problem defined on the domain Ω, called state problem.

We will see in this part how to derive bilinear forms, which are generally involved in such variational problems. To simplify the notation, we only consider the two-dimensional case n = 2.

3.1 Derivation of the state u

Let u_t an element of X_h^t, we set

$$u^t = u_t \circ T_t \tag{3.1}$$

which is an element of X_h defined by

$$u^t = \sum_{i=1}^{N} u_i(t)\ N_i \qquad\qquad N = N1 + N2$$

we note the derivative (when it exists)

$$\dot{u} = \frac{d}{dt} u^t \bigg|_{t=o} = \sum_{i=1}^{N} u_i^t (0 \text{à}\ v_i$$

$\dot{u}$ is the eulerion derivative, with respect to the field velocity V defined on (2.5).

3.2 Transport and derivative of bilinear forms

As in the situation of shape optimization, the field V is defined on (2.5) and u_t, v_t in X_h^t, so on each element K_j^t we have

$$\begin{cases} \displaystyle\int_{K_j^t} \nabla u_t\ \nabla v_t\ dx = \int_{K_j} \left\langle A_j(t)\ \nabla u^t,\ \nabla(v_t \circ T_t)\right\rangle dx \\ \displaystyle\int_{K_j^t} F\ v_t\ dx = \int_{K_j} F \circ T_t . J_j^t\ v^t\ dx \end{cases}$$

where $A_j(t)$ is a (2x2) matrice defined by

$$\begin{cases} A_j(t) = J_j^t\ (DT_t)^{-1}\ {}^t(DT_t)^{-1} \qquad A_j(0) = Id \\ J_j^t \quad = \det\ (DT_t) \end{cases}$$

Defining A(t) and J_t on the domain Ω by

$$(3.2) \qquad \begin{aligned} A(t)/K_j &= A_j(t) \\ J_t/K_j &= J_j^t \end{aligned}$$

we obtain :

$$(3.3)\qquad \begin{aligned} \int_{\Omega_t} \nabla u_t\ \nabla v_t\ dx &= \int_{\Omega} \langle A(t)\ \nabla u^t,\ \nabla v^t \rangle\ dx \\ \int_{\Omega_t} F\ v_t\ dx &= \int_{\Omega} F \circ T_t \circ J_t\ v^t\ dx \end{aligned}$$

The matrix function $x \longrightarrow A(t,x)$ and the function $x \longrightarrow J(t,x)$ are continuous on Ω^t, and differentiable with respect to t at t = 0 on each element K_j so

$$A' = \frac{d}{dt} A(t,.)\Big|_{t=0} = \operatorname{div} V(0) - (DV(0) + {}^tDV(0))$$

So the derivative of bilinear form (3.3) is, at t=o :

$$(3.4)\qquad \begin{aligned} &\frac{d}{dt} \sum_j \int_{K_j^t} \langle A(t)\ \nabla u^t,\ \nabla v^t \rangle dx = \sum_j \int_{K_j} \left(\langle A'\nabla u,\ \nabla v \rangle + \left\langle \nabla \dot{u},\ \nabla v \right\rangle \right) dx \\ &\frac{d}{dt} \sum_j \int_{K_j^t} F \circ T_t\ .\ J_t\ v^t\ dx = \sum_k \int_{K_j} \operatorname{div}\ (FV(0))\ v\ dx \end{aligned}$$

In two-dimensional problem, where $v(t) = \left(v_i^t, 0\right)$ $\left(\text{resp. } \left(0,\ v_i^t\right)\right)$ $1 \leqslant i \leqslant N1$ the matrix A' has the following form.

$$(3.5)\qquad A' = \begin{pmatrix} \partial_1\ v_i & \partial_2\ v_i \\ \partial_2\ v_i & -\partial_1\ v_i \end{pmatrix} \qquad \left(\text{resp. } A' = - \begin{pmatrix} -\partial_2\ v_i & \partial_1\ v_i \\ \partial_1\ v_i & \partial_2\ v_i \end{pmatrix}\right)$$

3.3. <u>The Dirichlet Problem</u>

Consider $u_t \in V_h^t$ (subspace of $H_0^1\left(\Omega_h^t\right)$) solution of the system

$$(3.6)\qquad \int_{\Omega_h^t} \nabla u_t\ \nabla v_t\ dx = \int_{\Omega_t} F\ V_t\ dx \qquad \forall\ V_t \in V_h^t$$

F is given in $L^2(\mathbb{R}^2)$.

From (3.3) we have :

$$\begin{cases} u^t = u_t \circ T_t \in X_h \quad \text{subspace of } H^1_o(\Omega_h) \\ \displaystyle\int_{\Omega_h} \langle A(t)\ \nabla u^t,\ \nabla W \rangle\, dx = \int_{\Omega_h} F \circ T_t . J_t\ W\, dx \quad \forall W \in X_h \end{cases}$$

In the canonical basis of the vectorial space X_h, $u^t = \sum u_i(t)\ v_i$ where $u(t) = (u_1(t),\dots,u_N(t))$ is the solution of the linear system $a(t).u(t)=f$, $a(t)$ is the matrix

$$a(t)_{ij} = \int_{\Omega_h} \langle A(t)\ \nabla \ell_j,\ \nabla \ell_i \rangle\, dx$$

$$f_i = \int_{\Omega_h} F \circ T_t . J_t\ \ell_i\ dx$$

Let $\emptyset(t,u) = a(t).u-f$, defined from $\mathbb{R}\times\mathbb{R}^N$ into $\mathbb{R}^N$, we check that the implicit function theorem holds, the only verification is the calculation of the derivative : $\dfrac{\partial}{\partial t}\emptyset(t,u)$ given by

$$\frac{\partial}{\partial t}\emptyset(t,u) = a'(0).u-f$$

which is obvious using the expression of A' in (3.5) for a velocity field V.

<u>Proposition 3.1</u>

The application $t \longrightarrow u^t$ belongs to $C'([0,\epsilon[,\ V_h)$. Its derivative at t=o is characterized by the linear system.

$$\text{(3.7)} \quad \begin{cases} \dot{u} \in V_h \quad \forall\ v \in V_h \\ \displaystyle\int_{\Omega_h} \nabla \dot{u}\ \nabla v\ dx = -\int_{\Omega_h} \langle A'\ \nabla u,\ \nabla v \rangle + \int_{\Omega_h} -\mathrm{div}\ (FV(0))\ dx \end{cases}$$

To obtain the eulerian partial derivative $\dot{u}_{x_i}$, we select the velocity mentioned in (2.5). So A' has the form indicated in (3.5).

<u>Proposition 3.2</u>

u is solution of the Dirichlet problem (3.6), $M_i(x_i,y_i)$ $1 \leqslant i \leqslant N$, the nodes of the triangulation of Ω_h, then the Eulerian partial derivative

$\dot{u}_{x_1}$ is solution of

$$\dot{u}_{x_1} \in V_h \qquad \forall v \in V_h$$

$$\int_{\Omega_h} \nabla\dot{u}_{x_1} \nabla V \, dx = - \int_{\Omega_h} (\partial_1 \ell i \, (\partial_1 u \partial_1 v - \partial_2 u \partial_2 v) + \partial_2 \ell i \, (\partial_1 u \, \partial_2 v + \partial_2 u \, \partial_1 v))dx$$

$$+ \int_{\Omega_h} \partial_1 (F\ell i) V \, dx$$

the Eulerian partial derivative $\dot{u}_{y_1}$ is solution of

$$\dot{u}_{y_1} \in V_h \qquad \forall v \in V_h$$

$$\int_{\Omega_h} \nabla\dot{u}_{y_1} \nabla v \, dx = - \int_{\Omega_h} (\partial_1 \ell i(\partial_1 u \partial_2 v + \partial_2 u \partial_1 v) + \partial_2 \ell i(\partial_2 u \partial_2 v - \partial_1 u \partial_1 v))dx$$

$$+ \int_{\Omega_h} \partial_2 (F\ell i) \, v \, dx$$

Remark

This technique to obtain $\dot{u}$ can be extended to Newman problems. In the numerical example, we will consider the mixt problem.

4. DERIVATIVE OF FUNCTIONAL COST WITH RESPECT TO THE NODES

Let consider a functional defined on the domain Ω for instance

$$(4.1) \qquad J(\Omega_h) = \frac{1}{2} \int_{\Omega_h} (u-z)^2 \, dx \qquad u \in X_h$$

where u is solution of the Dirichlet problem (3.6). Let V the velocity fiel defined in (2.5) and T_t defined in the lemma 1.2

$$J\left(\Omega_h^t\right) = \frac{1}{2} \int_{\Omega_h} \left(u^t - z \circ T_t\right) J_t \, dx$$

The derivative with respect to t at t=o is

$$(4.2) \qquad dJ\,(\Omega_h, V) = \int_{\Omega_h} \left[\left(\dot{u} - \nabla z . V(0) \right) (u-z) + \frac{1}{2} (u-z)^2 \text{ div } V(0) \right] dx$$

we introduce an adjoint state defined by the linear system, $p \in X_h$

$$(4.3) \qquad \int_{\Omega_h} \nabla p \; \nabla v \; dx = \int_{\Omega_h} (u-z) v \; dx \qquad \forall \; v \in X_h$$

using (4.3) into (4.2), we have

$$(4.4) \qquad dJ(\Omega_h, V) = \int_{\Omega_h} \nabla \dot{u} \; \nabla p \; dx + \int_{\Omega_h} \left[-\nabla z . V(o)(u-z) + \frac{1}{2}(u-z)^2 \text{ div } V(o) \right] dx$$

this expression is valid for a general velocity field V, to calculate the partial derivative with respect to the node coordinates x_i and y_i, we select the velocity V mentioned in (2.5) the expression is then simplified, and we have

Theorem 4.1

Let u and p the solutions of the linear systems (3.6) and (4.3), and $J(\Omega_h)$ the functional cost defined by (4.1), x_i and y_i are the coordinates of the node M_i in the triangulation of Ω_h. Then the partial derivatives of H with respect to x_i and y_i are :

$$\frac{\partial}{\partial x_i} J(\Omega_h) = + \int_{\Omega_h} [\partial_1 \ell i \; (\partial_1 u \partial_1 p - \partial_2 u \partial_2 p) + \partial_2 \ell i \; (\partial_1 u \partial_1 p + \partial_2 u \partial_2 p)] dx$$

$$(4.5) \qquad + \int_{\Omega_h} \partial_1 \; (F \ell i) p \; dx - \int_{\Omega_h} \ell i \; \partial_1 z (u-z) \; dx + \frac{1}{2} \int_{\Omega_h} (u-z)^2 \partial_1 \ell i \; dx$$

$$\frac{\partial}{\partial y_i} J(\Omega_h) = - \int_{\Omega_h} [\partial_2 \ell i \; (\partial_1 u \partial_1 p - \partial_2 u \partial_2 p) - \partial_1 \ell i (\partial_1 u \partial_2 p + \partial_2 u \partial_1 p)] dx$$

$$+ \int_{\Omega_h} \partial_2 (F \ell i) p \; dx - \int_{I_h} \ell i \; \partial_2 z \; (u-z) dx + \frac{1}{2} \int_{\Omega_h} (u-z)^2 \partial_2 \; \ell i \; dx$$

these derivatives have been obtained by O. Pironneau [3]. however, the technique used here is systematic and respects the general theory of deformation as in [1], [4], [5], [6]. The partial derivatives appear as a particular case of the general situation.

5. MOTION OF INTERNAL NODES

Given the position of the nodes on the boundary, in numerical computations, the position of the internal nodes id deduced from a regular meshing, this gives a good finite element approximation for the state and adjoint problems (3.5) and (4.3).

This procedure defines explicitly and some times implicitly an application Θ such that

$$\Theta : \mathbb{R}^{2\mu} \longrightarrow \mathbb{R}^{2q}$$

μ is the number of boundary noces $i = 1,\ldots,\mu$
q is the number of internal nodes $i = \mu+1,\ldots,\mu+q$.
The application Θ express the position of internal nodes, in terms of those on the boundary.

$$\Theta(x_1,y_1,\ldots,x_\mu,y_\mu) = (x_{\mu+1},\ y_{\mu+1},\ldots,x_{\mu+q},\ y_{\mu+q})$$

Assume, the application Θ derivable, and $D\Theta$ defined by

$$\frac{\partial x_j}{\partial x_i},\ \frac{\partial x_j}{\partial y_i},\ \frac{\partial y_j}{\partial x_i},\ \frac{\partial y_j}{\partial y_i}$$

$1 \leqslant i \leqslant \mu \quad \mu+1 \leqslant j \leqslant \mu+q$
the elements of the jacobian matrix $D\Theta$.
Let the case, where the matric $D\Theta$ is known explicitly, so the functional to minimize is

$$j_\theta = j_\Theta(x_1,y_1,\ldots,x_\mu,y_\mu) = J(\Omega(x_1,y_1,\ldots,x_\mu,y_\mu),\ \Omega(x_1,y_1,\ldots,x_\mu,y_\mu))$$

this functional depends on the application Θ, so its derivative is also depending on Θ, in fact it depends only on the matrix $D\Theta$ for $1 \leqslant i \leqslant \mu$.

$$\frac{\partial}{\partial x_i} j_\theta = \frac{\partial}{\partial x_i} J(\Omega) + \sum_{\ell=\mu+1}^{\mu+q} \frac{\partial}{\partial x_\ell} J(\Omega) \frac{\partial x_\ell}{\partial x_i} + \frac{\partial}{\partial y_\ell} J(\Omega) \frac{\partial y_\ell}{\partial x_i}$$

(5.1)

$$\frac{\partial}{\partial y_i} j\theta = \frac{\partial}{\partial y_i} J(\Omega) + \sum_{\ell=\mu+1}^{\mu+q} \frac{\partial}{\partial x_\ell} J(\Omega) \frac{\partial x_\ell}{\partial y_i} + \frac{\partial}{\partial y_\ell} J(\Omega) \frac{\partial y_\ell}{\partial y_i}$$

the derivatives $\frac{\partial}{\partial x_\ell} J$, $\frac{\partial}{\partial y_\ell} J$, $\frac{\partial}{\partial x_i} J$, $\frac{\partial}{\partial y_i} J$, are calculated from (4.5) with only one state adjoint P.

<u>Remark</u>.

The gradient of j_θ is only dependent on the matrix $D\Theta$. So it is not necessary to know explicitly the application θ to calculate the gradient of j_θ in (5.2).

Let give an example for the calculation of the gradient when the application θ and $D\Theta$ are known explicitly.

Consider the domain Ω defined by :

$$\Omega = \{(x,y) \;/\; a \leqslant x \leqslant b \;\; ; \;\; 0 \leqslant y \leqslant g(x)\}$$

where g is a continuous positive function, piecewise linear on [a,b].

For the triangulation, each vertical segment is divised into (m+1) equal segments (this is the choice of the application Θ). So, the nodes are the points $(x_{ij}, y_{ij}) = \left(x_i, j \cdot \frac{1}{m+1} y_i\right)$, $1 \leqslant i \leqslant n$; $0 \leqslant j \leqslant m+1$, and the parameters defining the geometry are the coordinates of the nodes belonging to the graph of g, $M_i = (x_i, y_i)$ $1 \leqslant i \leqslant n$. The nodes on the triangulation are then defined as a function of those points, and satisfy

$$\frac{\partial}{\partial x_k} y_{ij} = 0 \qquad\qquad \frac{\partial}{\partial y_k} y_{ij} = \frac{1}{n+1}\,\delta_{kj}$$

$$\frac{\partial}{\partial x_k} x_{ij} = \delta_{ki} \qquad\qquad \frac{\partial}{\partial y_k} x_{ij} = 0$$

which give the jacobian matrix $D\theta$, and the partial derivative (5.1) are for $1 \leqslant i \leqslant n$.

$$\frac{\partial}{\partial x_j} j_\theta(x_1, y_1, \ldots, x_n, y_n) = \frac{\partial}{\partial x_i} J(\Omega) + \sum_{j=o}^{m+1} \frac{\partial}{\partial x_{ij}} J(\Omega)$$

$$\frac{\partial}{\partial y_i} j_\theta(x_1, y_1, \ldots, x_n, y_n) = \frac{\partial}{\partial y_i} J(\Omega) + \sum_{j=o}^{m+1} \frac{\partial}{\partial y_{ij}} J(\Omega).$$

The derivatives $\frac{\partial}{\partial x_{ij}} J$, $\frac{\partial}{\partial y_{ij}} J, \ldots,$ are calculated from (4.5).

6 Application to a naval hydrodynamical problem

6.1 The continuous case.

To test the gradient (5.4) with respect to the nodes, we will give two numerical examples related to a naval hydrodynamical problem, and concerning the search of a free surface, which is the water wave. In the first example the water wave is generated by a flow arround an obstacle resting at the bottom Fig 6.1. For the second, the obstacle is piercing the free surface Fig 6.4.

Let explain the physical phenomena, and deduce the equations and boundary conditions governing the flow. In what follows, we will consider a stationary irrotational and incompressible flow in a domain $\Omega \subset \mathbb{R}^2$. So we define a stream funtion Ψ, such that the velocity $V = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$.

Problem with obstacle at the bottom.

To simplify, we consider the flow uniform at upstream and downstream infinity (see Cahouet [12], Cahouet-Lenoir [11]), and write the equations concerning a flow arround an obstacle resting at the bottom, in adimensional stream function formulation.

$$
\begin{array}{ll}
\Delta\Psi = 0 & \Omega \\
\Psi = 0 & B \\
\partial\Psi/\partial n = 0 & S_1, S_2 \\
\partial\Psi/\partial n = a(y) & S
\end{array}
\qquad (6.1)
$$

where $a(y) = (1 - (2/F{*}F){*}(y-1))^{1/2}$

Let explain in a physical way the meaning of the equations (6.1) as done in [12].

Equation in Ω

Since the flow is irrotational, the vorticity vanishes, so the stream function is harmonic.

Condition at the bottom B.

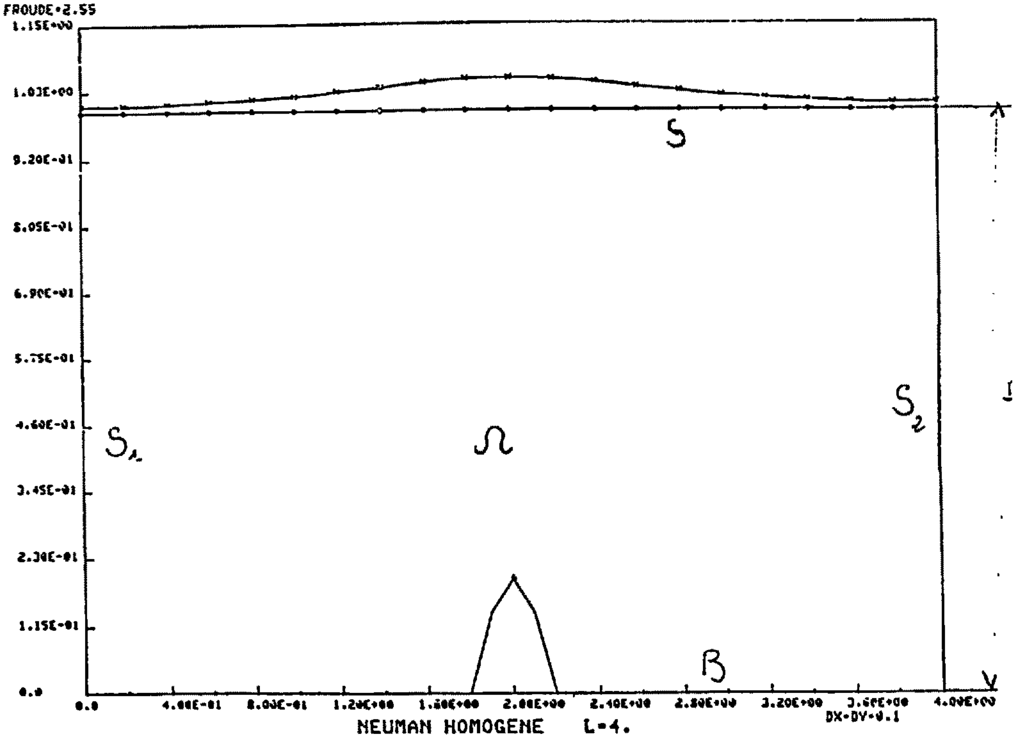
The bottom B is a streamline, this implies that Ψ is constant on B, we will fix arbitrary its value to zero.

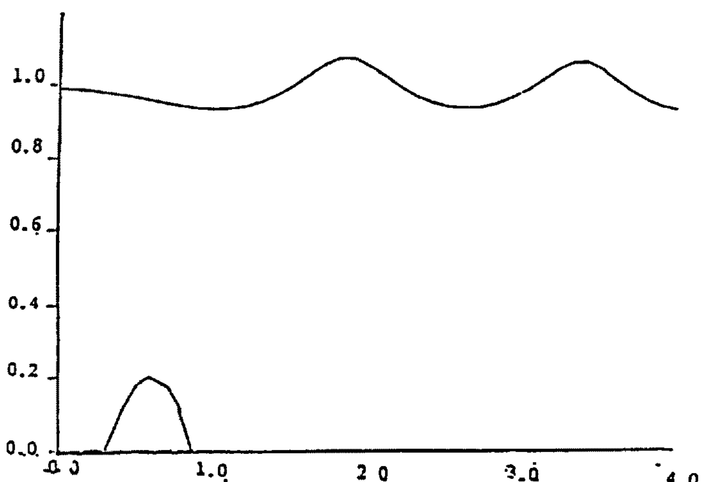
Conditions on lateral boundaries.

The flow is uniform at upstream, and dowstream infinity. If the domain is large enough compared to the obstacle, So the homogenous Newman conditions seem reasonable. This is only valid when the Froude number F in (6.1) is greather than 1. In this case the problem (6.1) is variational and symetrical. For F<1 (6.1) is not any more variational, and non symetrical. So we will get more complicated boundary conditions on the lateral sides. This is described in detail in (12).

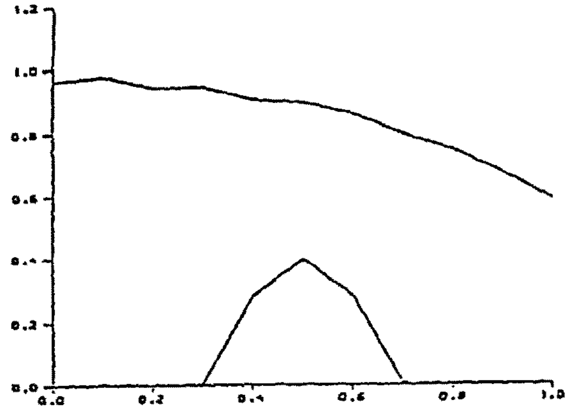
Conditions on the free surface.

Along the free surface S, we dispose on two boundary conditions . The first one is deduced from the Bernouilli equation, and is related to the continuity of the pressure through the free surface, this is given in (6.1), where F is the Froude number and y(x) the adimensional equation of S. The second condition explains the fact that S is a streamline, so Ψ is constant on S. The constant is fixed by the corresponds to Fig 6.3 with $F=0.4$ and the amplitude of the sinusoidal obstacle is 0.4. The gradient method converges after 40 iterations, and the cost decreases from $J_0 = 0.270$ to $J_{40} = 0.12.10^{-4}$. For the search of the water wave in naval hydrodynamical problem, or in a general free surface problem, we should first test the fixed point method, because it's faster and less expensive. The convergence is geometrical, for this method, at each iteration we only need to solve a linear system corresponding to the state problem. In case this method fails, the gradient method may converge, at least for some cases. The gradient method is more expensive, at each iteration, we solve two linear systems, the state and the adjoint state, and the convergence as it is known, is not geometrical.

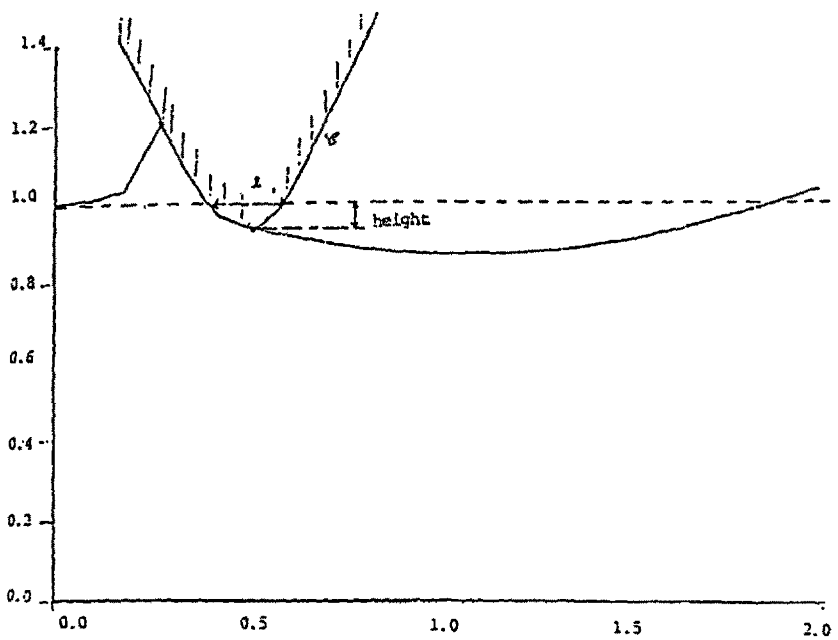




height of obstacle = 0.2 Froude = 0.5



height obst = 0.4 $F = 0.5$



Froude = 0.64 height of obstacle = 0.05 ℓ=0.2

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