SECOND ORDER OPTIMALITY CONDITIONS FOR NONLINEAR PARABOLIC BOUNDARY CONTROL PROBLEMS

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1. Introduction

In this paper we shall derive sufficient second order optimality conditions for a nonlinear parabolic boundary control problem with constraints on the control and the state. By means of a semigroup technique we extend the results of /4/ and /8/ to the case of a domain of arbitrary dimension and additional state-constraints.

Let $D \subset R^n$ be a bounded domain with sufficiently smooth, say C_∞ -, boundary Γ, such that D is locally at one side of Γ. Moreover, we are given real numbers $T > 0$, $\mu \geq 0$, $u_1 < u_2$, t_i, c_i, $i = 1,...,k$, $t_i \in (0,T]$, real functions $q \in W_p^\sigma(D)$, $f_i \in W_p^\sigma(D)$, $i=1,...,k$ (p and σ will be specified later), and a real function $b = b(t,r,x,u): [0,T] \times R \times R \times [u_1,u_2] \longrightarrow R$. We assume that b is twice continuously differentiable with respect to (x,u) and fulfils a Caratheodory type condition: The continuity of b and of its partial derivatives with respect to (x,u) is uniform with respect to (t,r), and these functions are measurable with respect to (t,r) for all fixed (x,u).

We are going to investigate the following optimal control problem:

Minimize

$$\int_D (w(T,r) - q(r))^2 dr + \mu \int_0^T u(t,r)^2 dS_r dt$$

subject to

$$w_t(t,r) = \Delta w(t,r) - w(t,r) \qquad \text{in D}$$
$$w(0,r) = 0 \qquad \text{in D} \qquad\qquad (1.1)$$
$$\partial w / \partial n(t,r) = b(t,r,w(t,r),u(t,r)) \text{ on } \Gamma,$$

$t \in (0,T]$ (by $\partial / \partial n$ the outward normal derivative is denoted), and to

$$u_1 \leq u(t,r) \leq u_2 \quad \text{a.e. on } [0,T] \times \Gamma,$$
$$\int_D f_i(r)w(t_i,r)dr \leq c_i , \quad i = 1,...,k.$$

The control u is supposed to be measurable, thus $u \in U_\infty = L_\infty((0,T) \times \Gamma)$, and the state w is defined as mild solution of (1.1) (see section 2) with $w \in C([0,T], W_p^\sigma(D))$.

We assume that b is linear with respect to u,

$$b(t,r,x,u) = b_1(t,r,x) + b_2(t,r,x)u. \qquad (1.2)$$

For many boundary conditions with background in mathematical physics this is no serious restriction. For instance, boundary conditions describing different phenomena of heat exchange of the type

$$\partial w / \partial n = a(t,r,w)(u - w),$$
$$\partial w / \partial n = u(\vartheta - w), \text{ or}$$
$$\partial w / \partial n = a(u^4 - w^4)$$

can be covered by (1.2) (take $u := u^4$ as a new control in the last case).

2. Transformation to a mathematical programming problem

According to the assumptions on b the mapping

$$(w(r),v(r)) \longmapsto b(t,r,w(r),v(r)) =: \mathscr{B}(t,w,v)(r)$$

is twice continuously Fréchet-differentiable from $C(\bar{D}) \times L_p(\Gamma)$ to $L_p(\Gamma)$ for almost all t, $1 \leq p \leq \infty$. Moreover, for abstract functions $x \in X := C([0,T], C(\Gamma))$, $u \in U_{\nu,p} := L_\nu((0,T), L_p(\Gamma))$, by

$$(x(t),u(t)) \longmapsto \mathscr{B}(t,x(t),u(t)) =: B(x,u)(t)$$

a twice continuously Fréchet-differentiable mapping from $X \times U_{\nu,p}$ to $U_{\nu,p}$ is defined, $1 \leq \nu < \infty$, $1 \leq p < \infty$, an abstract Nemytskii operator.

We shall use the first and second order derivatives $B'(x_o,u_o)h$ and $B''(x_o,u_o)[h_1,h_2]$ at $(x_o,u_o) \in X \times U_\infty$ in the direction $h = (v,z) \in X \times U_\infty$. It holds

$$B'(x_o,u_o)h = B_x v + B_u z$$
$$(B_x v)(t,r) = b_x(t,r,x_o(t,r),u_o(t,r))v(t,r) =: b_x^o(t,r)v(t,r) \qquad (2.1)$$
$$(B_u z)(t,r) = b_u(t,r,x_o(t,r),u_o(t,r))z(t,r) =: b_x^o(t,r)z(t,r).$$

Thus the image of B_x, B_u is obtained by multiplication with a certain bounded and measurable function. Therefore, B_x and B_u can be extended continuously to continuous linear operators acting in $L_\nu((0,T), L_p(\Gamma))$. Analogously,

$$(B''(x_o,u_o)[h,h])(t,r) = (B_{xx}[v,v] + 2B_{xu}[v,z] + B_{uu}[z,z])(t,r)$$
$$= b_{xx}^o(t,r)v(t,r)^2 + 2b_{xu}^o(t,r)v(t,r)z(t,r) \qquad (2.2)$$

has bounded and measurable coefficients b_{xx}^o and b_{xu}^o (note that $b_{uu}^o = 0$ by (1.2)).

At next we shall introduce the notion of a mild solution to (1.1). Therefore, we take p and $\mathcal{G} \in R$ such that $n/p < \mathcal{G} < 1 + 1/p$ and define an operator A acting in $L_p(D)$ by

$$D(A) = \left\{ w \in W_p^2(D) \mid \partial w/\partial n = 0 \quad \text{on } \Gamma \right\},$$
$$Aw = -\Delta w + w, \quad w \in D(A).$$

It is known that A is densely defined and closed in $L_p(D)$, and that $-A$ is the infinitesimal generator of a strongly continuous and analytic semigroup $S(t)$ of linear continuous operators in $L_p(D)$.
Moreover, we need the "Neumann" operator N, which assigns to $g \in L_p(\Gamma)$ the solution w of the elliptic boundary value problem

$$\Delta w - w = 0 \qquad \text{in } D,$$
$$\partial w/\partial n = g \qquad \text{on } \Gamma.$$

N is continuous from $L_p(\Gamma)$ to $W_p^s(D)$ for $s < 1 + 1/p$.
A mild solution w of (1.1) is any function of $C([0,T], W_p^{\mathcal{G}}(D))$, which solves the Bochner integral equation

$$w(t) = \int_0^t AS(t-s)N\mathscr{B}(s, \mathcal{T}w(s), u(s))ds, \quad t \in [0,T], \tag{2.3}$$

where $\mathcal{T}$ denotes the trace operator. Note that $\mathcal{G} > n/p$ implies $w \in C([0,T], C(\bar{D}))$.
It can be shown by standard methods that (2.3) is uniquely solvable on $[0,T]$ for all $u_1 \leqq u \leqq u_2$, if $T > 0$ is sufficiently small (see /9/). For many types of boundary conditions, in particular for conditions describing heat exchange processes, it can be proved by maximum principle arguments that $|w(t,r)|$ is uniformly bounded. Then the solution of (2.3) exists globally.
(2.3) is not the actual state equation we are going to deal with. We introduce a new state function $x(t) = \mathcal{T}w(t)$ and obtain from (2.3)

$$x(t) = \int_0^t \mathcal{T} AS(t-s)N \mathscr{B}(s, x(s), u(s))ds, \tag{2.4}$$

which is our actual equation of state. We consider x as element of $C([0,T], C(\Gamma))$. Finally, we define linear operators K, C, C_i by

$$(Kf)(t) = \int_0^t \mathcal{T} AS(t-s)Nf(s)ds,$$
$$Cf = \int_0^T AS(T-s)Nf(s)ds, \qquad C_i f = \int_0^{t_i} AS(t_i-s)Nf(s)ds.$$

It is known from AMANN /2/ that

$$\|AS(t)N\|_{L_p(\Gamma) \to W_p^{\sigma}(D)} \leq c \ t^{-(1 + (\sigma - \sigma')/2)} \tag{2.5}$$

for $0 < \sigma < \sigma' < 1 + 1/p$. Therefore, K, C, and C_i are continuous in the following spaces: $K: U_{\nu,p} \to C([0,T], C(\Gamma))$, C, $C_i: U_{\nu,p} \to W_p^{\sigma}(D)$, if $\nu > 2(\sigma' - \sigma)^{-1}$.

Now we have all prerequisites to formulate the optimal control problem in the form of the following

Mathematical programming problem (P):

Minimize

$$F(x,u) = \|CB(x,u) - q\|_2^2 + \mu \|u\|_2^2$$

subject to $x \in X$,

$$x = KB(x,u)$$
$$g_i(x,u) = \langle \phi_i , C_iB(x,u) \rangle_p - c_i \leq 0, \ i = 1,\dots,k,$$
$$u \in U_{ad}.$$

In this setting, $\|\cdot\|_2$ denotes L_2-norms of the underlying spaces, $\langle \cdot , \cdot \rangle_p$ is the pairing between $L_{p'}(D)$ and $L_p(D)$, and $\phi_i \in L_p(D)^*$ are defined by

$$\langle \phi_i , v \rangle_p = \int_D f_i(r)v(r)dr .$$

Moreover,

$$U_{ad} = \left\{ u \in U_{\nu,p} \mid u_1 \leq u(t,r) \leq u_2 \right\} .$$

Here we assume $\nu > 2(\sigma' - \sigma)^{-1}$. It should be noticed that $L_\infty((0,T) \times \Gamma)$ can be continuously embedded into $L_\nu((0,T), L_p(\Gamma))$ $(1 \leq \nu, p < \infty)$, but not into $L_\infty((0,T), L_\infty(\Gamma))$ (cf. FATTORINI /3/). Each function $u \in U_{ad}$ can be represented by $u \in L_\infty((0,T) \times \Gamma)$. In the sequel we shall denote by $g(x,u)$ the column vector $(g_1(x,u),\dots,g_k(x,u))^t$.

3. Existence of optimal controls and first order necessary optimality conditions

The proof of existence is the first reason for the assumption of linearity of $b(t,r,x,u)$ with respect to u. We shall require additionally the following natural assumptions:

(A1) For all $u \in U_{ad}$ there is exactly one $x \in X$ with $x = KB(x,u)$.

(A2) The feasible set M,
$$M = \left\{ (x,u) \mid x = KB(x,u), \ g(x,u) \leq 0, \ u \in U_{ad} \right\}$$
is non-void.

(A1) is simply a restriction on T, as it was already pointed out above.

Theorem 1: Under the assumptions (A1), (A2) the control problem (P) admits at least one optimal solution (x_o, u_o).

Proof: The method is already standard. If (x_n, u_n) M is a minimizing sequence, then $u_n \longrightarrow u$ weakly in $U_{\nu, p}$ can be assumed. Compactness of K ensures strong convergence $x_n \longrightarrow x$ in X. The linearity of b with respect to u yields $x = KB(x, u)$, and from the weak lower semicontinuity of F the optimality of (x, u) is derived.   #

In an optimal pair (x_o, u_o) the component u_o is said to be an optimal control and x_o its corresponding state.
In order to define the notion of regularity, which is essential for any optimality condition, we introduce the so-called linearizing cone. An element $h = (v, z) \in X \times U_\infty$ belongs to L(M) iff

$$v = KB_x v + KB_u z \tag{3.1}$$

$$z \in \bigcup_{\lambda > 0} (U_{ad} - \{u_o\})$$

$$g'(x_o, u_o) h + r g(x_o, u_o) \leq 0, \quad r \geq 0.$$

Note that B_x, B_u are the partial derivatives of B at (x_o, u_o).
The set L(M) is said to be the linearizing cone of M at (x_o, u_o). The optimal control u_o is said to be regular, if it satisfies the assumption
(A3) There is a $\bar{h} = (\bar{v}, \bar{z}) \in X \times U_\infty$, such that $\bar{h}$ solves (3.1),
$\bar{z} \in U_{ad} - \{u_o\}$, and

$$g_i(x_o, u_o) + g_i'(x_o, u_o) \bar{h} < 0, \quad i = 1, \ldots, k. \tag{3.2}$$

The Lagrange function L is defined by

$$L(x, u, y_1, y_2) = F(x, u) + \int_0^T \langle y_1(t), x(t) - (KB(x, u))(t) \rangle_p \, dt + y_2^t g(x, u),$$

where $y_1 \in L_{\nu'}((0, T), L_{p'}(\Gamma))$ and $y_2 \in R_+^k$ ($1/\nu + 1/\nu' = 1$, $1/p' + 1/p = 1$).
By definition, the equation $x = KB(x, u)$ is regarded in $X = C([0, T], C(\Gamma))$, thus the general theory of necessary optimality conditions would lead to a Lagrange multiplier $y_1 \in X^*$. However, this space can be avoided by an embedding technique, which mainly exploits the "smoothing property" $K: U_{\nu, p} \longrightarrow X$ (see /10/, thm. 1.3.2). In this way we arrive at the

Theorem 2: Let u_o be a regular optimal control with corresponding state x_o. Then there are $y_1 \in L_{\nu'}((0, T), L_{p'}(\Gamma))$, $y_2 \in R_+^k$ such that

$$L_x(x_o, u_o, y_1, y_2) = 0 \tag{3.3}$$

$$L_u(x_o, u_o, y_1, y_2)(u - u_o) = 0 \quad \forall u \in U_{ad} \tag{3.4}$$

$$y_2^t \, g(x_o, u_o) = 0 \tag{3.5}$$

<u>Proof</u>: The result follows from /10/, thm. 1.3.2 and the smoothing property of K.   #

Equation (3.3) is the <u>adjoint equation</u>

$$y_1(t) = B_x(t) \left\{ -C^*(w_o(T)-q) + \sum_{i=1}^{k} y_2^i C_i^* \Phi_i + K^* y_1 \right\}(t), \tag{3.6}$$

where C, C_i, and K are regarded as operators with image in $L_p(D)$ and $U_{y',p'}$, respectively, thus C^*, C_i^* : $L_{p'}(D) \to U_{y',p'}$, K^* : $U_{y',p'} \to U_{y',p'}$. $B_x(t)$ is the extension of B_x to $U_{y',p'}$. It is acting just by the multiplication of $\{...\}(t,r)$ with $b_x(t,r,x_o(t,r),u_o(t,r))$. Moreover, $w_o(T) := CB(x_o,u_o)$.

Actually, we have even more regularity of y_1:

<u>Theorem 3</u>: The assumption $q, f_i \in W_p^\sigma(D)$, $i = 1,...,k$, implies that $y_1 = y_1(t,r)$ is bounded and measurable on $(0,T) \times \Gamma$.

<u>Proof</u>: We show at first that $C^*(w_o(T)-q)(t,r)$ is continuous (to be more precise: it has a continuous representative). Take $v = v(t,r)$ from $U_{y,p}$. Then with $\bar{w} := w_o(T) - q \in W_p^\sigma(D)$

$$\langle \bar{w} , Cv \rangle_p = \langle \bar{w} , \int_0^T AS(T-s)Nv(s)ds \rangle_p$$

$$= \int_0^T \langle (AS(T-s)N)^* \bar{w} , v(s) \rangle_p \, ds$$

(here we regard $AS(T-s)N$ as operator from $L_p(\Gamma)$ to $L_p(D)$, hence $(AS(T-s)N)^*$: $L_{p'}(D) \to L_{p'}(\Gamma)$)

$$= \int_0^T \langle \tau S_{p'}(T-s)\bar{w} , v(s) \rangle_p \, ds$$

after an integration by parts, where $S_{p'}(t)$ denotes the semigroup generated by $A_{p'}$, which is the counterpart of A defined in $L_{p'}(D)$. The restriction of $S_{p'}$ to $W_p^\sigma(D)$ is a strongly continuous semigroup, too. Hence

$$\Psi(t) = S_{p'}(T-t)\bar{w}$$

belongs to $C([0,T],W_p^\sigma(D)) \subsetneq C([0,T],C(\bar{D}))$. Thus $(C^*\bar{w})(t) = \tau \Psi(t)$ is contained in $C([0,T],C(\Gamma))$.

Completely analogous we find

$$(C_i^* \Phi_i)(t) = \begin{cases} S_{p'}(t_i-t)f_i, & 0 \le t \le t_i \\ \\ 0, & t_i < t \le T, \end{cases}$$

and $f_i \in W_p^{\sigma}(D)$ ensures that $(C_i^* \Psi_i)(t)$ is piecewise continuous on $[0,T]$ with values in $C(\Gamma)$, thus $(C_i^* \phi_i)(t,r)$ belongs to $L_{\infty}((0,T) \times \Gamma)$.

K^* admits the form

$$(K^* y_1)(t) = \int_t^T \tau A_p . S_p . (s-t) N_p . y_1(s) ds = \int_t^T k(t,s) y_1(s) ds,$$

and it can be shown that the restriction of $k(t,s)$ to $L_p(\Gamma)$ satisfies

$$\|k(t,s)\|_{L_p(\Gamma) \to C(\Gamma)} \leq c(t-s)^{-\alpha},$$

where $\alpha \in (0,1)$. (3.6) can be written

$$y_1(t) = \varphi(t) + B_x(t) \int_t^T k(t,s) y_1(s) ds, \qquad (3.7)$$

where $\varphi(t,r)$ is from $L_{\infty}((0,T) \times \Gamma) \hookrightarrow L_\nu((0,T), L_p(\Gamma))$ $\forall \nu < \infty$.

If y_1 belongs to $U_{\nu,p}$, then the right hand side of (3.7) is contained in $L_{\infty}((0,T) \times \Gamma)$, if ν is sufficiently large. Now a Neumann series argument yields $y_1 \in L_{\infty}((0,T) \times \Gamma)$.   #

4. Sufficient second order optimality conditions

The theory of second order optimality conditions for problems in function spaces was developed rapidly after the basic investigations by IOFFE /5/. In particular, MAURER /7/ worked out second order conditions for abstract optimal control problems with application to optimal control problems governed by systems of nonlinear ordinary differential equations. Such conditions were applied in many papers to different questions of optimal control of ordinary differential equations. Some corresponding references are quoted in /8/.

In the paper /4/ by GOLDBERG and the author necessary and sufficient second order optimality conditions have been derived for a control problem governed by the one-dimensional heat equation with nonlinear boundary condition subject to constraints on the control. Here we extend these results to the n-dimensional case and allow additional constraints on the state.

In what follows we shall use different norms of $h = (v,z) \in X \times U_{\infty}$:

We denote by $\|\cdot\|_{\nu,p}$ the natural norm of $L_\nu((0,T), L_p(\Gamma))$, $\|\cdot\|_2 :=$ $\|\cdot\|_{2,2}$, and by $\|\cdot\|_{\infty}$ the norm of X. We put

$$\|h\|_2 = \max(\|v\|_2, \|z\|_2), \quad \|h\|_{\infty,\nu,p} = \max(\|v\|_{\infty}, \|z\|_{\nu,p}).$$

We shall impose the following second order conditions on (x_o, u_o):

(SOC) There is a $\delta > 0$ such that

$$L''(x_o, u_o, y_1, y_2)[h,h] \geq \delta \|h\|_2^2 \tag{4.1}$$

for all $h \in L(M)$.

<u>Remark</u>: In (4.1) we can substitute $\|z\|_2$ for $\|h\|_2$, as v and z are connected by the linearized equation (3.1).

<u>Theorem 4</u>: Suppose that u_o is a regular control, such that $(x_o, u_o) \in M$, the first order necessary conditions (3.3-5) and the second order condition (SOC) are satisfied. Assume further that $q \in W_p^{\sigma}(D)$ and $f_i \in W_p^{\sigma}(D)$, $i = 1, \ldots, k$. Then there are constants $\alpha > 0$ and $\varepsilon > 0$, such that

$$F(x,u) - F(x_o, u_o) \geq \alpha (\|x - x_o\|_2^2 + \|u - u_o\|_2^2) \tag{4.2}$$

for all $(x,u) \in M$ with $\|(x-x_o, u-u_o)\|_{\infty, \nu, p} < \varepsilon$.

<u>Proof</u>: To prove theorems of this type, the so-called two-norm discrepancy plays a decisive role. In $X \times U_\infty$ we shall work with $\|\cdot\| :=$ $\|\cdot\|_{\infty, \nu, p}$ and the L_2-norm $\|\cdot\|_2$. Moreover, we shall denote by $r_j(h,E)$ the j-th order remainder term of a certain differentiable mapping E (at $(x_o, \dot{u}_o)$).
According to MAURER /7/ we have to verify the following conditions:
If $\|h\| \to 0$ $(h = (v,z) \in X \times U_\infty)$, then

(a) $|r_1(g,h)| / \|h\|_2 \to 0$,

(b) $|r_2(L,h)| / \|h\|_2^2 \to 0$,

and there is a $c > 0$ such that

(c) $|L''(x_o, u_o, y_1, y_2)[h_1, h_2]| \leq c \|h_1\|_2 \|h_2\|_2 \quad \forall h \in X \times U_\infty$.

Then the theorems 3.1 and 3.5 of /7/ yield the statement of the theorem. From the Taylor formula for $b = b(t,r,x,u)$ we obtain

$$\|r_1(B,h)\|_2 \leq \alpha(h) \|h\|_2 \tag{4.3}$$

$$\|r_2(B,h)\|_1 = \beta(h) \|h\|_2^2 \tag{4.4}$$

where $\alpha(h), \beta(h) \to 0$ for $\|h\| \to 0$, and $\|\cdot\|_1$ is the norm of $L_1((0,T), L_1(\Gamma))$.
For (4.4) the linearity of b with respect to u is essential. This can be seen as follows: We assume for short that $b(t,r,x,u) = b_1(t,r,x)u$, denote by $b_{1,xx}^{\vartheta}(t,r)$ the function $b_{1,xx}(t,r,x_o(t,r) + \vartheta v(t,r))$, and introduce similar expressions for the other derivatives. Then for $\vartheta \in (0,1)$

$$\|(B''(x_o + \vartheta v, u_o + \vartheta z) - B''(x_o, u_o))[h,h]\|_1 =$$

$$= \| (b_{1,xx}^{\vartheta}(u_o + \vartheta z) - b_{1,xx}^o u_o) v^2 + 2(b_{1,x}^{\vartheta} - b_{1,x}^o)vz \|_1$$

$$\leq \| (b_{1,xx}^{\vartheta} - b_{1,xx}^o)u_o \|_\infty \|v\|_2^2 + \| b_{1,xx}^{\vartheta} v \|_\infty \|v\|_2 \|z\|_2 +$$

$$+ 2\| b_{1,x}^{\vartheta} - b_{1,x}^o \|_\infty \|v\|_2 \|z\|_2.$$

For $\|v\|_\infty \to 0$ the $\|\cdot\|_\infty$-factors tend to zero, thus (4.4) follows easily.

To verify (a), we consider

$$r_1(g_i, h) = \Big\langle \phi_i , \int_0^{t_i} AS(t_i - s)Nr_1(B,h)ds \Big\rangle_p$$

$$= \int_0^{t_i} \Big\langle (AS(t_i - s)N)^* \phi_i , r_1(B,h) \Big\rangle_p ds.$$

Along the lines of the proof of theorem 3 we deduce that the left hand side in the duality brackets is bounded and measurable. Hence by (4.3)

$$|r_1(g_i, h)| \leq c \alpha(h) \|h\|_2,$$

implying (a).

L" is given by

$$L''(x_o, u_o, y_1, y_2)[h_1, h_2] = F''(x_o, u_o)[h_1, h_2] - \Big\langle y_1 , KB''(x_o, u_o)[h_1, h_2] \Big\rangle_p$$

where $h_i = (v_i, z_i)$ and

$$F''(x_o, u_o)[h_1, h_2] = 2\Big\langle CK(B_x v_1 + B_u z_1) , CK(B_x v_2 + B_u z_2)\Big\rangle_2$$

$$+ 2\Big\langle w_o(T) - q , CKB''(x_o, u_o)[h_1, h_2]\Big\rangle_2.$$

To show (b) we confine ourselves to the part $f(x,u) := \Big\langle y_1 , KB(x,u)\Big\rangle_p$ of L. We have

$$r_2(f, h) = \int_0^T \Big\langle y_1(t) , \int_0^t \tau AS(t-s)Nr_2(B,h)(s)ds \Big\rangle_p dt$$

$$= \int_0^T \Big\langle \int_t^T (\tau AS(s-t)N)^* y_1(s)ds , r_2(B,h)(t)\Big\rangle_p dt.$$

$y_1(t)$ is bounded and measurable, hence $y_1 \in L_\nu((0,T), L_p(\Gamma))$, too. Thus the integral in the brackets is continuous, as ν, p are sufficiently large. Therefore,

$$|r_2(f,h)| \leq c \|r_2(B,h)\|_1 \leq c \beta(h) \|h\|_2^2$$

by (4.4). Analogously

$$|f''(x_o, u_o)[h_1, h_2]| = \int_0^T \Big\langle \int_t^T (\tau AS(t-s)N)^* y_1(s)ds, B''(x_o, u_o)[h_1, h_2](t)\Big\rangle_p dt$$

$$\leq c \| B''(x_o, u_o)[h_1, h_2] \|_1 \leq c \|h_1\|_2 \|h_2\|_2$$

is obtained. The discussion of the part F(x,u) of L is analogous, but tedious. In the estimations, the continuity of K as operator in

$L_2((0,T),L_2(\Gamma))$ must be used as a basic tool. The technique is
mainly along the lines of /8/.   #

<u>Remarks</u>:

(i) Thus, u_o is a locally optimal control with respect to the L_p-
norm, as the mapping $u \mapsto x = x(u)$ is continuous from $U_{\nu,p}$ to X.

(ii) Theorem 4 remains true without the assumption of linearity of b
with respect to u, if $\|(x - x_o, u - u_o)\|_{\infty,\infty} < \varepsilon$ is required.
This, however, is a very hard restriction (see section 5).

5. Second order conditions and approximation

It is known that sufficient second order optimality conditions can be
supposed to show the stability of solutions of mathematical programming
problems or optimal control problems with respect to perturbations of
the given data. In particular, this refers to the numerical approxima-
tion of optimal control problems governed by nonlinear ordinary
differential equations. Here we mention the paper by ALT /1/, where
optimal control is considered as a particular case of a more general
class of problems. Moreover, we refer to the investigations on stability
by MALANOWSKI /6/.
The integral equation method permits to extend this approach also to
control problems governed by parabolic partial differential equations
in one-dimensional domains with nonlinear boundary conditions, see
TRÖLTZSCH /8/. Similarly, the sufficient second order conditions of
section 4 apply to show strong L_2-convergence of optimal controls, if
the nonlinear parabolic control problem of section 1 is approximated by
a suitable numerical method. Then we have to solve (1.1) by a finite
difference or finite element technique, and the set of controls must be
discretized, too, for instance by piecewise linear or smoother functions.

A discussion of these aspects of approximation would go beyond the scope
of this paper. We shall only explain, where the linearity of b is needed:
It is the discretization of controls, which is responsible for this
assumption.

Let $U^n_{ad} \subset U_\infty$, $n \in N$, be a set of discretized controls,
$S_i \subset L_\nu((0,T),L_p(\Gamma))$, $i = 1,2$. We define with a certain norm $\|\cdot\|$

$$d(u,S_1) = \inf_{v \in S_1} \|u - v\|$$

$$d(S_2,S_1) = \sup_{u \in S_2} d(u,S_1).$$

In order to show convergence of approximating controls one has to assume

$$d(u_o, U_{ad}^n) \leq \alpha(n), \tag{5.1}$$

$$d(U_{ad}, U_{ad}^n) \leq \alpha(n), \tag{5.2}$$

where $\alpha(n) \to 0$, $n \to \infty$. The norm $\| \cdot \|$, which underlies the definition of the distance d, must comply with the differentiability of $B(x,u)$. The choice $\| \cdot \| := \| \cdot \|_\infty$ would not cause restrictions on the nonlinearity of $b(t,r,x,u)$. But then, as a rule, (5.1) can only be satisfied, if u_o is sufficiently smooth (cf. ALT /1/). For $\| \cdot \| = \| \cdot \|_{\nu,p}$ assumption (5.1) can be fulfilled, but then differentiability with respect to this norm and the assumptions (a) - (b) on the remainder term lead to the requirement of linearity with respect to u. We refer to /8/.

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