The Hilbert Uniqueness Method: A Retrospective *

John E. Lagnese
Department of Mathematics
Georgetown University
Washington, DC 20057 USA

1 Introduction

The purpose of this paper is to give a brief overview of certain aspects of recent developments in the area of exact controllability of distributed parameter systems. Our starting point is a 1986 paper of J.-L. Lions in which is described a systematic, general method for attacking exact controllability problems for linear distributed parameter systems [14]. This method, called the *Hilbert Uniqueness Method* (HUM) by its author, provides a powerful, constructive means for solving a wide variety of exact controllability problems for partial differential equations. The reader is referred to [15], where HUM is systematically applied to a large and diverse collection of distributed parameter control problems.

It was subsequently pointed out in [11],[12],[13],[16],[17] and by others that HUM (and its first cousin, the *Reverse Hilbert Uniqueness Method* -RHUM) may be understood, at the abstract level, as a version of a well-known duality theory of exact controllability of linear evolutionary systems. (see e.g., [4, Theorem 2.1]). This observation cannot, however, account for the substantial progress made in exact controllability of distributed parameter systems since the introduction of HUM. Indeed, this success is precisely due to the *ad hoc*, distributed parameter systems approach to exact controllability adopted by Lions, based on new types of *a priori* estimates for solutions of various classes of partial differential equations that were originally developed outside of the immediate context of exact controllability theory.

Roughly speaking, the theoretical basis of HUM is the observation that if one has uniqueness of solutions of a linear evolutionary system in a Hilbert space it is possible to introduce a Hilbert space norm $\|\cdot\|_F$ based on the uniqueness property in such a way that the dual system is exactly controllable to the dual space F'. The exact controllability problem is thereby transfered to the problem of identifying or otherwise characterizing the couple F, F'. The latter is essentially a problem in partial differential equations when the original evolutionary system is a distributed parameter system: can *a priori* estimates of $\|\cdot\|_F$ be obtained in terms of norms in spaces which are both intrinsic to the given problem and which are readily identifiable? Fortunately, techniques developed in the early 1980's

*Research supported by the Air Force Office of Scientific Research through grant AFOSR 88-0337.

for deriving *a priori* estimates in the context of *uniform stabilization* of partial differential equations (e.g. [3],[8]) were available to provide a framework to attack the latter problem, at least for a number of distributed parameter control problems of interest. Indeed, one might speculate that it was an "observability" estimate for solutions of the wave equation with boundary observation (see [6]), derived by essentially the same multiplier methods as were originally employed in [3], that provided the catalyst for the introduction of HUM.

In fact, in practice it is common to first derive an *a priori* estimate leading to a uniqueness result and then to use that estimate as the starting point for the application of HUM. Each such estimate leads to *some* exact controllability theorem. However, the apparent emphasis of many authors on the derivation of *a priori* estimates has tended to obscure the simple duality principle underlying the method as well as the fact that the estimates themselves are not really part of the basic principle but rather are the means by which one identifies the space F or, more commonly, some other space G that is dense in F. (Of course, at the practical level identification of F is *the crucial point* since, otherwise, the exact controllability problem cannot be considered solved in any real sense.) Moreover, while the various estimates are obtained by similar methods (such as the use of multipliers), they appear to have a different structure from one problem to the next, and it is often difficult to discern any common threads running through them. Further, the control and state spaces that one is led to consider on the basis of the estimates often have extremely weak topologies and are certainly nonstandard in the context of classical distributed parameter systems. For example, certain estimates lead to control spaces that are not even spaces of distributions, and some components of the corresponding solutions may not be continuous functions of time into any space. One then must ask in what sense the exact controllability problem has been solved. To the uninitiated, each problem may appear to require a separate treatment.

In this paper the basic principle underlying the Hilbert Uniqueness Method will be described in an abstract framework general enough to be applicable to many distributed parameter control problems of interest. Our goal is to present HUM in a general manner that both retains the distributed parameter systems flavor of the method and parallels the way the method is actually employed in applications to specific control problems. In terms of our general description of HUM, we do not claim any particular novelty; what is done here is equivalent to what can already be found in the work of Lasiecka and Triggiani (see, e.g., [11],[12],[13],[17] and Remark 2.3 below), and at certain points we have exactly adopted their framework (as in the proof of Proposition 2.5 below), although at others we have taken a somewhat different point of view. In fact, the main motivation for this paper is Russell's review [16] of [15], and what we have attempted to do is extend the basic duality structure outlined briefly in [16] to a setting sufficiently general to cover a variety of interesting distributed parameter control problems, particularly boundary control problems.

The principle of HUM will be described in the next section in the context of the reachability problem for the first order linear control system $\dot{y} = \mathcal{A}y + \mathcal{B}u$. In Section 3 we consider the situation in which the first order system arises from a second order control system $\ddot{w} = Aw + Bu$, a common occurrence in practice. Naturally, stronger results obtain in this special case than hold in the general case and, in addition, it is possible to identify a "generic" space of reachable states in terms of spaces intrinsic to the second order system; that is to say, we can identify a particular lower bound for the controllability operator, in terms of a such spaces, that holds for many second order distributed parameter control systems. This estimate in some sense ties together the diverse collection of *a priori* estimates obtained in the process of applying HUM to specific distributed parameter control systems. Examples related to boundary control of elastic plates and of Maxwell's system are presented in Section 4 to illustrate how specific control problems can be framed within the general theory.

In preparing this paper, I have benefited greatly from discussions about HUM that I have had from time to time with G. Leugering, I. Lasiecka, D. L. Russell and, particularly, R. Triggiani. It is a pleasure to acknowledge their contributions. I also wish to thank A. Bensoussan for making available preprints of his related works [1],[2].

2 First Order Control Systems

Let $\mathcal{H}$ be a Hilbert space with dual space $\mathcal{H}'$. The scalar product between two elements h_1 and h_2 in $\mathcal{H}$ is denoted by $(h_1, h_2)_{\mathcal{H}}$, and the duality pairing between elements $h' \in \mathcal{H}'$ and $h \in \mathcal{H}$ is denoted by $\langle h', h\rangle_{\mathcal{H}}$. We denote by $\Lambda_{\mathcal{H}}$ the Riesz isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$. $\mathcal{H}'$ is itself a Hilbert space under the scalar product

$$(h_1', h_2')_{\mathcal{H}'} = (\Lambda_{\mathcal{H}}^{-1} h_1', \Lambda_{\mathcal{H}}^{-1} h_2')_{\mathcal{H}} = \langle h_1', \Lambda_{\mathcal{H}}^{-1} h_2'\rangle_{\mathcal{H}}.$$

Let $T > 0$ be fixed and $H^1(0,T;\mathcal{H}')$ be the Hilbert space consisting of functions $f : (0,T) \to \mathcal{H}'$ such that f and its strong derivative $\dot{f}$ ($\dot{f} = df/dt$) belong to $L^2(0,T;\mathcal{H}')$, topologized by

$$\left(\int_0^T [\|f(t)\|_{\mathcal{H}'}^2 + \|\dot{f}(t)\|_{\mathcal{H}'}^2]dt\right)^{1/2}.$$

We may identify $L^2(0,T;\mathcal{H})$ with the dual of $L^2(0,T;\mathcal{H}')$ and with this identification we have the dense and continuous embedding

$$L^2(0,T;\mathcal{H}) \subset (H^1(0,T;\mathcal{H}'))'.$$

We will usually write $L^2(\mathcal{H})$, $H^1(\mathcal{H}')$, etc., in place of $L^2(0,T;\mathcal{H})$, $H^1(0,T;\mathcal{H}')$, etc., when the value of T is clear from context.

Let $\mathcal{U}$ be another Hilbert space, $\mathcal{A}$ be a linear operator in $\mathcal{H}$ with domain $D_{\mathcal{A}}$, and $\mathcal{B} \in \mathcal{L}(\mathcal{U}', (H^1(\mathcal{H}'))')$. ($\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y.) We assume that $\mathcal{A}$ is the generator of a C_0-semigroup of bounded linear operators on $\mathcal{H}$. Consider the following control system:

$$\dot{y} = \mathcal{A}y + \mathcal{B}u, \quad y(0) = 0, \quad u \in \mathcal{U}'. \tag{2.1}$$

Our purpose is to identify or otherwise characterize the *reachable set*

$$\mathcal{R}_T = \{y(T) \,|\, u \in \mathcal{U}',\ y \text{ satisfies } (2.1)\}. \tag{2.2}$$

Remark 2.1. The choice of $(H^1(\mathcal{H}'))'$ as the space of control outputs is dictated primarily by applications to boundary control problems for partial differential equations that will be discussed in Section 4. This space is sufficiently general for many applications. However, one may treat more general classes of control outputs such as $(H^k(\mathcal{H}'))'$ or $(H^k(D_{\mathcal{A}'}))'$, $k \geq 0$, with only minor modifications of the theory presented below. Here $\mathcal{A}'$ denotes the dual operator of $\mathcal{A}$; $D_{\mathcal{A}'}$ is the domain of $\mathcal{A}'$ endowed with the graph norm of $\mathcal{A}'$.

Since the range of $\mathcal{B}$ is in a very weak space, the sense in which equation (2.1) is to be understood needs to be clarified. If $\mathcal{B}u$ is in the stronger space $L^2(\mathcal{H})$, the solution of (2.1) is unambiguously defined by the variation of constants formula

$$y(t) = \int_0^t S(t-s)(\mathcal{B}u)(s)\,ds, \quad 0 \leq t \leq T, \tag{2.3}$$

where $S(t), t \geq 0$, is the semigroup on $\mathcal{H}$ generated by $\mathcal{A}$. If $\phi^0 \in \mathcal{H}'$, from (2.3) we have

$$\begin{aligned} \langle \phi^0, y(T)\rangle_{\mathcal{H}} &= \int_0^T \langle \phi^0, S(T-s)(\mathcal{B}u)(s)\rangle_{\mathcal{H}}\,ds \\ &= \int_0^T \langle S'(T-s)\phi^0, (\mathcal{B}u)(s)\rangle_{\mathcal{H}}\,ds \\ &= \int_0^T \langle \phi(s), (\mathcal{B}u)(s)\rangle_{\mathcal{H}}\,ds = \langle \phi, \mathcal{B}u\rangle_{L^2(\mathcal{H})}, \end{aligned} \tag{2.4}$$

where $S'(t)$ is the dual semigroup of $S(t)$ and $\phi(s) = S'(T-s)\phi^0$. The dual semigroup acts in $\mathcal{H}'$ and is generated by the dual operator $\mathcal{A}'$ of $\mathcal{A}$. Therefore ϕ is a mild solution of

$$\dot{\phi}(t) = -\mathcal{A}'\phi(t), \ (t < T), \ \ \phi(T) = \phi^0. \tag{2.5}$$

The variational equation

$$\langle \phi^0, y(T)\rangle_{\mathcal{H}} = \langle \phi, \mathcal{B}u\rangle_{L^2(\mathcal{H})}, \quad \forall \phi^0 \in \mathcal{H}', \tag{2.6}$$

characterizes those states $y(T)$ that may be reached through the action of controls $u \in \mathcal{U}'$ such that $\mathcal{B}u \in L^2(\mathcal{H})$. A similar characterization will be given for the full set $\mathcal{R}_T$ and, simultaneously, the meaning of the solution of (2.1) when $\mathcal{B}u \in (H^1(\mathcal{H}'))'$ will be elucidated. This is done by the transposition method.

To motivate things, let y be a strong solution of (2.1), ϕ be the solution of

$$\dot{\phi}(t) = -\mathcal{A}'\phi(t) + g, \ (t < T), \ \ \phi(T) = \phi^0, \tag{2.7}$$

where $\phi^0 \in D_{\mathcal{A}'}$ and $g \in L^\infty(\mathcal{H}')$, $\dot{g} \in L^1(\mathcal{H}')$. Then ϕ is a strong solution of (2.7) and we have

$$\begin{aligned} \langle \phi, \mathcal{B}u\rangle_{L^2(\mathcal{H})} &= \int_0^T \langle \phi, \dot{y} - \mathcal{A}y\rangle_{\mathcal{H}} dt \\ &= \langle \phi^0, y(T)\rangle_{\mathcal{H}} - \langle g, y\rangle_{L^2(\mathcal{H})}. \end{aligned} \tag{2.8}$$

Equation (2.8) is essentially the definition of the solution of (2.1), provided we interpret the various brackets $\langle \cdot, \cdot \rangle$ as duality pairings in spaces different from those indicated in (2.8). For example, if $\mathcal{B}u \in (H^1(\mathcal{H}'))'$, the left bracket is to be interpreted in the $(H^1(\mathcal{H}'))' - H^1(\mathcal{H}')$ duality pairing

$$\langle \mathcal{B}u, \phi \rangle_{H^1(\mathcal{H}')},$$

provided $\phi \in H^1(\mathcal{H}')$. The duality pairings to be chosen on the right side of (2.8) depend on what must be assumed about ϕ^0 and g to assure that $\phi \in H^1(\mathcal{H}')$.

Lemma 2.1 *Assume that $\phi^0 \in D_{\mathcal{A}'}$ and $g \in L^1(D_{\mathcal{A}'}) \cap L^2(\mathcal{H}')$. Then the solution of (2.7) satisfies $\phi \in H^1(\mathcal{H}')$. Moreover,*

$$\|\phi\|_{H^1(\mathcal{H}')} \le C\left(\|\phi^0\|_{D(\mathcal{A}')} + \|g\|_{L^1(D_{\mathcal{A}'})} + \|g\|_{L^2(\mathcal{H}')}\right). \tag{2.9}$$

Proof. The assumptions $\phi^0 \in D_{\mathcal{A}'}$, $g \in L^1(D_{\mathcal{A}'})$, imply that the solution of (2.7) is strongly differentiable and satisfies the differential equation almost everywhere, $\dot{\phi} \in L^1(\mathcal{H}')$, and

$$\|\phi\|_{L^\infty(D_{\mathcal{A}'})} \le C\left(\|\phi^0\|_{D(\mathcal{A}')} + \|g\|_{L^1(D_{\mathcal{A}'})}\right).$$

If also $g \in L^2(\mathcal{H}')$ then

$$\dot{\phi} = -\mathcal{A}'\phi + g \in L^2(\mathcal{H}')$$

and

$$\|\dot{\phi}\|_{L^2(\mathcal{H}')} \le C\left(\|\phi^0\|_{D(\mathcal{A}')} + \|g\|_{L^1(D_{\mathcal{A}'})} + \|g\|_{L^2(\mathcal{H}')}\right). \ \square$$

We introduce the Banach space

$$\mathcal{X} = L^1(D_{\mathcal{A}'}) \cap L^2(\mathcal{H}')$$

with

$$\|g\|_{\mathcal{X}} = \|g\|_{L^1(D_{\mathcal{A}'})} + \|g\|_{L^2(\mathcal{H}')}.$$

One has the dense and continuous embeddings

$$L^2(D_{\mathcal{A}'}) \subset \mathcal{X}, \ \ \mathcal{X}' \subset L^2((D_{\mathcal{A}'})').$$

We now rewrite (2.8) as

$$\langle y(T), \phi^0 \rangle_{D_{\mathcal{A}'}} - \langle y, g \rangle_{\mathcal{X}} = \langle \mathcal{B}u, \phi \rangle_{H^1(\mathcal{H}')}, \ \ \forall \phi^0 \in D_{\mathcal{A}'}, \ \forall g \in \mathcal{X},$$

where ϕ satisfies (2.7).

Proposition 2.2 *If $\mathcal{B}u \in (H^1(\mathcal{H}'))'$, there is a unique pair*

$$(y^0, y) \in (D_{\mathcal{A}'})' \times \mathcal{X}'$$

such that

$$\langle y^0, \phi^0 \rangle_{D_{\mathcal{A}'}} - \langle y, g \rangle_{\mathcal{X}} = \langle \mathcal{B}u, \phi \rangle_{H^1(\mathcal{H}')}, \ \ \forall \phi^0 \in D_{\mathcal{A}'}, \ \forall g \in \mathcal{X}. \tag{2.10}$$

Proof. It is simply a matter of observing that the mapping $(\phi^0, g) \mapsto \langle \mathcal{B}u, \phi\rangle_{H^1(\mathcal{H}')}$ is linear and, according to Lemma 2.1, continuous from $D_{\mathcal{A}'} \times \mathcal{X}$ into $\Re$. □

By definition, the element y provided by Proposition 2.2 is the solution of (2.1) and y^0 is its value at T. This convention is justified by the following result, which also describes the sense in which y satisfies (2.1)

Proposition 2.3 *Let $(y^0, y) \in (D_{\mathcal{A}'})' \times L^2((D_{\mathcal{A}'})')$ satisfy*

$$\langle y^0, \phi^0\rangle_{D_{\mathcal{A}'}} - \int_0^T \langle y(t), g(t)\rangle_{D_{\mathcal{A}'}}\, dt = \langle \mathcal{B}u, \phi\rangle_{H^1(\mathcal{H}')}, \qquad \forall \phi^0 \in D_{\mathcal{A}'},\ \forall g \in L^2(D_{\mathcal{A}'}), \tag{2.11}$$

where ϕ is the solution of (2.7) corresponding to ϕ^0 and g. Then y satisfies, in the sense of distributions on $(0,T)$,

$$\frac{d}{dt}\langle y(t), \phi^0\rangle_{D_{\mathcal{A}'}} = \langle y(t), \mathcal{A}'\phi^0\rangle_{D_{\mathcal{A}'}} + ((\Lambda^{-1}\mathcal{B}u)(t), \phi^0)_{\mathcal{H}'} - \frac{d}{dt}(\frac{d}{dt}(\Lambda^{-1}\mathcal{B}u)(t), \phi^0)_{\mathcal{H}'},\quad \forall \phi^0 \in D_{(\mathcal{A}')^2}, \tag{2.12}$$

where Λ is the Riesz isomorphism of $H^1(\mathcal{H}')$ onto its dual. If, moreover, the map $t \to y(t) : [0,T] \to (D_{\mathcal{A}'})'$ is continuous at $t = T$ (resp., at $t = 0$), then

$$y(T) = y^0, \quad (\text{resp., } y(0) = 0.)$$

Proof. Let $\phi^0 \in D_{(\mathcal{A}')^2}$ and set

$$\phi(t) = \alpha(t)\phi^0, \quad g(t) = \dot\alpha(t)\phi^0 + \alpha(t)\mathcal{A}'\phi^0, \tag{2.13}$$

where

$$\alpha \in C^1([0,T]), \quad \alpha(0) = \alpha^0, \quad \alpha(T) = \alpha^1, \tag{2.14}$$

with α^0 and α^1 fixed, but arbitrary, constants. Then ϕ is the solution of

$$\dot\phi = -\mathcal{A}'\phi + g, \quad \phi(T) = \alpha^1\phi^0.$$

Substitution of (2.13) into (2.11) yields

$$\begin{aligned}\alpha^1\langle y^0, \phi^0\rangle_{D_{\mathcal{A}'}} \quad &- \quad \int_0^T \left[\dot\alpha(t)\langle y(t), \phi^0\rangle_{D_{\mathcal{A}'}} + \alpha(t)\langle y(t), \mathcal{A}'\phi^0\rangle_{D_{\mathcal{A}'}}\right] dt \\ &= \quad \langle \mathcal{B}u, \alpha(\cdot)\phi^0\rangle_{H^1(\mathcal{H}')}.\end{aligned} \tag{2.15}$$

We have

$$\begin{aligned}\langle \mathcal{B}u, \alpha(\cdot)\phi^0\rangle_{H^1(\mathcal{H}')} &= (\Lambda^{-1}\mathcal{B}u, \alpha(\cdot)\phi^0)_{H^1(\mathcal{H}')} \\ &= \int_0^T \left[\alpha(t)((\Lambda^{-1}\mathcal{B}u)(t), \phi^0)_{\mathcal{H}'} + \dot\alpha(t)(\frac{d}{dt}(\Lambda^{-1}\mathcal{B}u)(t), \phi^0)_{\mathcal{H}'}\right] \\ &= \int_0^T [\xi(t)\alpha(t) + \dot\xi(t)\dot\alpha(t)]dt,\end{aligned} \tag{2.16}$$

where

$$\xi(\cdot) = ((\Lambda^{-1}\mathcal{B}u)(\cdot), \phi^0)_{\mathcal{H}'} \in H^1(0,T). \tag{2.17}$$

With (2.16), (2.17), identity (2.15) takes the form

$$\begin{aligned} \alpha^1\langle y^0, \phi^0\rangle_{D_{\mathcal{A}'}} &- \int_0^T \left[\dot\alpha(t)\langle y(t), \phi^0\rangle_{D_{\mathcal{A}'}} + \alpha(t)\langle y(t), \mathcal{A}'\phi^0\rangle_{D_{\mathcal{A}'}}\right] dt \\ &= \int_0^T [\xi(t)\alpha(t) + \dot\xi(t)\dot\alpha(t)]dt. \end{aligned} \tag{2.18}$$

This identity holds for every $\phi^0 \in D_{(\mathcal{A}')^2}$ and every α which satisfies (2.14). Since α^0 and α^1 are arbitrary, it follows from (2.18) that y satisfies, in the sense of distributions on $(0,T)$, the variational equation (2.12) and, in a weak sense made precise by (2.18), the end conditions

$$\langle y^0 - y(T), \phi^0\rangle_{D_{\mathcal{A}'}} = 0, \;\; \langle y(0), \phi^0\rangle_{D_{\mathcal{A}'}} = 0, \;\;\; \forall \phi^0 \in D_{\mathcal{A}'}. \tag{2.19}$$

If y is continuous from $[0,T]$ into $(D_{\mathcal{A}'})'$ then, in particular, (2.19) holds in the strict sense, so that

$$y(T) = y^0, \;\; y(0) = 0. \;\; \square$$

If we set $g = 0$ in (2.10), it follows that *elements* $y(T) \in \mathcal{R}_T$ *are characterized by the variational equation*

$$\langle y(T), \phi^0\rangle_{D_{\mathcal{A}'}} = \langle \mathcal{B}u, \phi\rangle_{H^1(\mathcal{H}')} = \langle u, \mathcal{B}'\phi\rangle_{\mathcal{U}} = (u, \Lambda_{\mathcal{U}}\mathcal{B}'\phi)_{\mathcal{U}'}, \;\;\; \forall \phi^0 \in D_{\mathcal{A}'}, \tag{2.20}$$

where ϕ satisfies (2.5). Let us introduce the *linear space of observations*

$$\mathcal{O} = \{u \,|\, u = \Lambda_{\mathcal{U}}\mathcal{B}'\phi, \;\; \phi \text{ satisfies (2.5) with } \phi^0 \in D_{\mathcal{A}'}\} \subset \mathcal{U}'.$$

It follows from (2.20) that controls $u \in \overline{\mathcal{O}}^{\perp}$, the orthogonal complement in $\mathcal{U}'$ of the closure of $\mathcal{O}$, simply steer the zero state to itself, therefore

$$\mathcal{R}_T = \{y(T) |\, u \in \overline{\mathcal{O}}, \;\; y \text{ satisfies (2.1)}\}.$$

Proposition 2.4 *Controls* $u \in \overline{\mathcal{O}}$ *driving* 0 *to a state* $y^0 \in \mathcal{R}_T$ *are unique. If* $u \in \overline{\mathcal{O}}$ *drives* 0 *to* y^0*, then* u *is the control of minimum norm among all controls in* $\mathcal{U}'$ *that drive* 0 *to* y^0.

Proof. If $u \in \overline{\mathcal{O}}$ and $v \in \overline{\mathcal{O}}$ drive 0 to the same state y^0, then

$$(u - v, \Lambda_{\mathcal{U}}\mathcal{B}'\phi)_{\mathcal{U}'} = 0, \;\; \forall \phi^0 \in D_{\mathcal{A}'},$$

hence $u - v \in \overline{\mathcal{O}}^{\perp}$, so that $u - v = 0$.

Let u be a control in $\mathcal{U}'$ that drives 0 to a state y^0. We may write

$$u = u_0 + u_1, \;\;\; u_0 \in \overline{\mathcal{O}}, \;\; u_1 \in \overline{\mathcal{O}}^{\perp}.$$

But u_1 drives 0 to itself, hence u_0 drives 0 to y^0 and $\|u\|^2_{\mathcal{U}'} \geq \|u_0\|^2_{\mathcal{U}'}$. $\square$

In order to proceed further, we need to impose the following hypothesis:

Observability Assumption:

$$\phi^0 \in D_{\mathcal{A}'}, \ \ \|\mathcal{B}'\phi\|_{\mathcal{U}} = 0 \Longleftrightarrow \phi^0 = 0. \tag{2.21}$$

If (2.21) holds, we may introduce a Hilbert norm

$$\|\phi^0\|_F = \|\mathcal{B}'\phi\|_{\mathcal{U}},$$

and a Hilbert space

$$F = \text{completion of } D_{\mathcal{A}'} \text{ in } \|\cdot\|_F.$$

The space F will, in general, depend on T. Since

$$\|\mathcal{B}'\phi\|_{\mathcal{U}}^2 \le C\|\phi\|_{H^1(\mathcal{H}')}^2 \le C_T\|\phi^0\|_{D_{\mathcal{A}'}}^2,$$

we have the dense and continuous embeddings

$$D_{\mathcal{A}'} \subset F, \ \ F' \subset (D_{\mathcal{A}'})'.$$

For $\psi^0 \in D_{\mathcal{A}'}$ we may define $\Lambda_F\psi^0 \in (D_{\mathcal{A}'})'$ by $y(T) = \Lambda_F\psi^0$, where $y(T)$ is the solution of (2.20) corresponding to $u = \Lambda_{\mathcal{U}}\mathcal{B}'\psi$, ψ denoting the solution of (2.5) with $\psi(T) = \psi^0$. (Λ_F corresponds to the controllability Grammian in the finite dimensional case.) From (2.20) we have

$$\langle \Lambda_F\psi^0, \phi^0\rangle_{D_{\mathcal{A}'}} = (\psi^0, \phi^0)_F, \ \ \forall \psi^0, \phi^0 \in D_{\mathcal{A}'}. \tag{2.22}$$

Therefore Λ_F extends to an operator in $\mathcal{L}(F, F')$, this extension being precisely the Riesz isomorphism of F onto F'. As a consequence we have

$$\mathcal{R}_T = \text{ range of } \Lambda_F \ = F'.$$

Remark 2.2. The uniqueness property (2.21), together with construction of the corresponding Hilbert space F whose dual characterizes the reachable states of (2.1), is the reason for the terminology *Hilbert Uniqueness Method.*

To find the control u_0 in $\overline{\mathcal{O}}$ that steers 0 to a given element $y^0 \in F'$, define $\psi^0 = \Lambda_F^{-1}y^0 \in F$, let ψ be the "solution" of

$$\dot{\psi} = -\mathcal{A}'\psi, \ \ (t < T), \ \ \psi(T) = \psi^0, \tag{2.23}$$

and define $u_0 = \Lambda_{\mathcal{U}}\mathcal{B}'\psi$. Then the above construction formally gives $y(T) = y^0$. However, when $\psi^0 \in F$, it is not clear how the solution of (2.23) is defined and, consequently, what is the meaning of u_0. However, u_0 can be made precise as follows. Take a sequence $\{\psi_n^0\} \in D_{\mathcal{A}'}$ such that $\psi_n^0 \to \psi^0$ in F. Then $u_n =: \Lambda_{\mathcal{U}}\mathcal{B}'\psi_n$ converges in $\mathcal{U}'$ (from the definition of the F-norm) and therefore

$$u_0 =: \lim_n \Lambda_{\mathcal{U}}\mathcal{B}'\psi_n$$

is a well-defined element in $\overline{\mathcal{O}}$. Furthermore, if $y_n^0 =: \Lambda_F\psi_n^0$ we have $y_n^0 \to y^0$ in F'. Let y_n and y be the solutions of

$$\dot{y}_n = \mathcal{A}y_n + \mathcal{B}u_n, \ \ y_n(0) = 0,$$

$$\dot{y} = \mathcal{A}y + \mathcal{B}u_0, \quad y(0) = 0.$$

By the definition of these solutions,

$$\langle y_n^0 - y(T), \phi^0\rangle_{D_{\mathcal{A}'}} - \langle y_n - y, g\rangle_{\mathcal{X}} = \langle \mathcal{B}u_n - \mathcal{B}u_0, \phi\rangle_{H^1(\mathcal{H}')}, \quad \forall\phi^0 \in D_{\mathcal{A}'}, \ \forall g \in \mathcal{X}.$$

It follows that

$$y_n \to y \ \text{ weak* in } \mathcal{X}', \quad y_n^0 \to y(T) \ \text{ weak* in } (D_{\mathcal{A}'})'.$$

Since $y_n^0 \to y^0$ strongly in F' and, *a fortiori*, in $(D_{\mathcal{A}'})'$, we have $y(T) = y^0$. That is, u_0 is the unique control in $\overline{\mathcal{O}}$ that drives 0 to y^0.

Remark 2.3. An alternative formulation of the foregoing description of the reachable set is given in [11],[12],[17] and may be described in terms of the control-to-state map $u \mapsto y(T) : \mathcal{U}' \mapsto (D_{\mathcal{A}'})'$. If fact, denoting this map by L_T, from (2.20) we deduce that $L_T \in \mathcal{L}(\mathcal{U}', (D_{\mathcal{A}'})')$. The kernal of L_T is $\overline{\mathcal{O}}^{\perp}$, and its dual operator satisfies

$$L_T'\phi^0 = B'\phi, \quad \forall\phi^0 \in D_{\mathcal{A}'},$$

where ϕ^0, ϕ satisfy (2.5). The observability assumption is equivalent to the hypothesis that L_T' is one-to-one which is in turn equivalent to $\overline{\mathrm{Rg}(L_T)} = (D_{\mathcal{A}'})'$, i.e., the system (2.1) is approximately controllable. Thus observability (sometimes called *distinguishability*; c.f. [4]) and approximate controllability are "dual" properties. The operator Λ_F is given in terms of L_T by

$$\Lambda_F = L_T\Lambda_{\mathcal{U}}L_T' \quad \text{on } D_{\mathcal{A}'}.$$

The minimum norm control that steers 0 to an element $y^0 \in \Lambda_F(D_{\mathcal{A}'})$ is

$$u_0 = \Lambda_{\mathcal{U}}B'\psi = \Lambda_{\mathcal{U}}L_T'\psi^0 = \Lambda_{\mathcal{U}}L_T'(L_T\Lambda_{\mathcal{U}}L_T')^{-1}y^0. \ \square$$

Although the reachable set at time T is identified as the space F' (when the observability assumption is satisfied), the reachability problem is still not solved in any practical sense unless F', or a dense subspace of it, can be identified in terms of spaces intrinsic to the problem. In order to do this, it is necessary and sufficient to establish *a priori* estimates of the form

$$\|\mathcal{B}'\phi\|_{\mathcal{U}}^2 \geq c^2\|\phi^0\|_{\mathcal{H}_1}^2, \quad \forall\phi^0 \in D_{\mathcal{A}'}. \tag{2.24}$$

More precisely, one has

Proposition 2.5 *Let $\mathcal{H}_1$ be a Hilbert space such that $D_{\mathcal{A}'}$ is continuously and densely embedded in $\mathcal{H}_1$. If* (2.24) *holds, then $\mathcal{H}_1'$ is in the reachable set of* (2.1). *Conversely, if $\mathcal{H}_1'$ is in the reachable set of* (2.1) *and if*

$$D = \{u \in \mathcal{U}' \,|\, y(T) = L_T u \in \mathcal{H}_1'\} \ \text{ is dense in } \mathcal{U}',$$

then (2.24) *must hold.*

Proof. If (2.24) holds then

$$F \subset \mathcal{H}_1, \quad \mathcal{H}_1' \subset F',$$

hence $\mathcal{H}_1'$ is in the reachable set of (2.1). For the converse, define $\mathcal{L}_T = L_T|_D$. It is a densely defined, closed linear operator from $D \subset \mathcal{U}'$ into $\mathcal{H}_1'$. Its dual $\mathcal{L}_T'$: $\mathrm{Dom}(\mathcal{L}_T') \subset \mathcal{H}_1 \mapsto \mathcal{U}$ satisfies $D_{\mathcal{A}'} \subset \mathrm{Dom}(\mathcal{L}_T')$ and $\mathcal{L}_T'|_{D_{\mathcal{A}}'} = L_T'$. It is standard theory that $\mathrm{Rg}(\mathcal{L}_T) = \mathcal{H}_1'$ if, and only if, $\mathcal{L}_T'$ is continuously invertible. The latter property is equivalent to (2.24). □

Property (2.24) is sometimes referred to as *continuous $\mathcal{H}_1$ observability*. Thus exact controllability to $\mathcal{H}_1$ and continuous $\mathcal{H}_1$ observability are equivalent properties, a duality relation that has frequently been pointed out in the control literature (e.g., [4],[11],[12],[17] in infinite dimensional contexts). Therefore, deriving *a priori* estimates of the form (2.24) is the crucial issue in exact controllability problems. An application of the last proposition to Maxwell's system of will be given in Section 4.

In cases where $\mathcal{A}$ is a skew-adjoint operator arising from a linear, second order system (a situation considered in the next section), there is an intrinsic space $\mathcal{H}_1$ for which the estimate (2.24) often can be established. In fact, it is sometimes possible to obtain two-sided estimates of $\|\cdot\|_F$ in terms of $\|\cdot\|_{\mathcal{H}_1}$, so that $F = \mathcal{H}_1$ in such cases.

3 Second Order Control Systems

Let V and H be Hilbert spaces with V dense and continuously embedded in H. We identify H with its dual space so that we have as usual $V \subset H \subset V'$. Let A be the Riesz isomorphism of V onto V', and set

$$D_A = \{v \in V \mid Av \in H\}, \quad \Delta_A = \{v \in V \mid Av \in V\},$$

$$\|v\|_{D_A} = \|Av\|_H, \quad \|v\|_{\Delta_A} = \|Av\|_V.$$

D_A and Δ_A are Hilbert spaces and A is an isomorphism of D_A onto H and of Δ_A onto V. We have the algebraic and topological inclusions

$$\Delta_A \subset D_A \subset V \subset H \subset V' \subset D_A' \subset \Delta_A'.$$

Furthermore, A extends to an isomorphism of H onto D_A' and of V' onto Δ_A'. The spaces V', D_A' and Δ_A' are Hilbert spaces under the scalar product

$$(v, w)_X = (A^{-1}v, A^{-1}w)_Y,$$

where (X, Y) stands for any one of the pairs (V', V), (D_A', H) or (Δ_A', V').

Let $\mathcal{U}$ be a Hilbert space whose dual $\mathcal{U}'$ will be the space of controls, and let $T > 0$ be fixed. In keeping with the notation of Section 2, we shall write $L^2(H)$, $H^1(V)$, etc., in place of $L^2(0,T;H)$, $H^1(0,T;V)$, etc. We identify $L^2(H)$ with its dual space, so that

$$H^1(V) \subset L^2(H) \subset (H^1(V))'.$$

Let $B \in \mathcal{L}(\mathcal{U}', (H^1(V))')$. We consider the following control problem:

$$\ddot{w} + Aw = Bu, \quad w(0) = \dot{w}(0) = 0, \quad u \in \mathcal{U}'. \tag{3.1}$$

We wish to identify the reachable set

$$\mathcal{R}_T = \{(w(T), \dot{w}(T)) | \, u \in \mathcal{U}', \ w \text{ satisfies } (3.1)\}. \tag{3.2}$$

The problem (3.1) may be transformed to a first order system of the type considered in Section 2. Set

$$y_1 = w, \ \ y_2 = \dot{w}, \ \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}.$$

Define

$$\mathcal{H} = H \times V', \ \ (\text{so that } \mathcal{H}' = H \times V).$$

Then $\mathcal{B} \in \mathcal{L}(\mathcal{U}', (H^1(\mathcal{H}'))')$ and (3.1) becomes

$$\dot{y} = \mathcal{A}y + \mathcal{B}u, \ \ y(0) = 0. \tag{3.3}$$

Lemma 3.1 *The operator $\mathcal{A}$, as an unbounded operator in $H \times V'$ with domain $V \times H$, is skew-adjoint and, therefore, generates a group of unitary operators on $H \times V'$.*

Proof. It is obvious that $\mathcal{A}$ is an isomorphism of $V \times H$ onto $H \times V'$ and, therefore, $\mathcal{A}$ is closed. So we have only to check that $-\mathcal{A} \subset \mathcal{A}^*$, the adjoint of $\mathcal{A}$. If $y = (y_1, y_2)$ and $z = (z_1, z_2)$ are in $V \times H$ we have

$$\begin{aligned} (\mathcal{A}y, z)_{\mathcal{H}} &= (y_2, z_1)_H - (Ay_1, z_2)_{V'} \\ &= (y_2, z_1)_H - \langle z_2, y_1 \rangle_V \\ &= (y_2, z_1)_H - (z_2, y_1)_H = -(y, \mathcal{A}z)_{\mathcal{H}}. \ \square \end{aligned}$$

The theory of Section 2 may therefore be applied to (3.3). The problem (3.3) has a unique solution in the sense of Proposition 2.2 and, if the observability assumption (2.20) is satisfied, the reachable set may be identified with the space F'. However, because of the special structure of the operator $\mathcal{A}$, one can obtain stronger results than those provided by Section 2.

First, we need to identify the dual operator $\mathcal{A}'$ of $\mathcal{A}$.

Lemma 3.2 *The dual of $\mathcal{A}$ is the operator in $H \times V$ defined by*

$$\mathcal{A}' = \begin{pmatrix} 0 & -A \\ I & 0 \end{pmatrix}, \ \ D_{\mathcal{A}'} = V \times D_A.$$

Proof. the operator $\mathcal{A}'$ is related to the adjoint operator $\mathcal{A}^* = -\mathcal{A}$ by

$$\mathcal{A}' = \Lambda_{\mathcal{H}} \mathcal{A}^* \Lambda_{\mathcal{H}}^{-1},$$

where $\Lambda_{\mathcal{H}}$ is the Riesz isomorphism of $\mathcal{H} = H \times V'$ onto $\mathcal{H}' = H \times V$. Thus

$$\Lambda_{\mathcal{H}} = \begin{pmatrix} I & 0 \\ 0 & A^{-1} \end{pmatrix}, \ \ \Lambda_{\mathcal{H}}^{-1} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},$$

so that $\mathcal{A}'$ is the indicated matrix. The domain of $\mathcal{A}'$ consists of elements $\phi = (\phi_1, \phi_2)$ such that $\Lambda_{\mathcal{H}}^{-1}\phi = (\phi_1, A\phi_2) \in D(\mathcal{A}) = V \times H$; thus $D(\mathcal{A}') = V \times D_A$. $\square$

Proposition 3.3 *The solution $y = (y_1, y_2)$ of (3.3) (guaranteed by Proposition 2.2) satisfies*

$$y_1 \in C(V'), \ \ y_2 = \dot{y}_1 \in Y' \subset L^2(D'_A),$$

where $Y = L^1(D_A) \cap L^2(V)$,

$$\|f\|_Y = \|f\|_{L^1(D_A)} + \|f\|_{L^2(V)}.$$

In addition, $w = y_1$ satisfies, in the sense of distributions on $(0,T)$,

$$\frac{d}{dt}\langle w(t), \phi_1^0\rangle_V - \langle \dot{w}(t), \phi_1^0\rangle_{D_A} = 0, \ \ \forall \phi_1^0 \in D_A, \tag{3.4}$$

$$\begin{aligned}\frac{d}{dt}\langle \dot{w}(t), \phi_2^0\rangle_{D_A} + \langle w(t), A\phi_2^0\rangle_V &= \langle(\Lambda^{-1}Bu)(t), \phi_2^0\rangle_V \\ &\quad -\frac{d}{dt}\langle\frac{d}{dt}(\Lambda^{-1}Bu)(t), \phi_2^0\rangle_V, \ \ \forall \phi_2^0 \in \Delta_A,\end{aligned} \tag{3.5}$$

where Λ denotes the Riesz isomorphism of $H^1(V)$ onto $(H^1(V))'$.

Proof. The solution of (3.3) guaranteed by Proposition 2.2 satisfies

$$(y_1^0, y_2^0) \in (D_{A'})' = V' \times D'_A, \ \ (y_1, y_2) \in \mathcal{Y}',$$

where

$$\mathcal{Y} = (L^1(V) \cap L^2(H)) \times (L^1(D_A) \cap L^2(V)).$$

Let $\phi = (\phi_1, \phi_2)$ be the solution of the system (corresponding to (2.7))

$$\begin{cases} \dot{\phi}_1 = A\phi_2 + g_1, & \dot{\phi}_2 = -\phi_1 + g_2, \\ \phi_1(T) = \phi_1^0, & \phi_2(T) = \phi_2^0. \end{cases} \tag{3.6}$$

The solution of (3.3) is defined by (2.10), i.e.,

$$\begin{aligned}\langle(y_1^0, y_2^0), (\phi_1^0, \phi_2^0)\rangle_{D_{A'}} \ &- \ \langle(y_1, y_2), (g_1, g_2)\rangle \\ &= \ \langle \mathcal{B}u, \phi\rangle_{H^1(\mathcal{H}')} = \langle Bu, \phi_2\rangle_{H^1(V)}.\end{aligned} \tag{3.7}$$

We have purposely omitted for the time being reference to the specific spaces in the second duality pairing of the left side of (3.7). The regularity possessed by the solution of (3.7) is determined by the spaces that $\phi^0 = (\phi_1^0, \phi_2^0)$ and $g = (g_1, g_2)$ must belong to in order to assure that the linear functional $(\phi^0, g) \mapsto \langle Bu, \phi_2\rangle_{H^1(V)}$ is continuous. But it is standard that if $(\phi_1^0, \phi_2^0) \in D_{A'} = V \times D_A$ and if $g_1 \in L^1(V)$, $g_2 \in L^1(D_A) \cap L^2(V)$, then

$$\|\phi_1\|_{L^\infty(V)} + \|\phi_2\|_{L^\infty(D_A)} + \|\dot{\phi}_2\|_{L^2(V)} \leq C\left(\|(\phi_1^0, \phi_2^0)\|_{V \times D_A} + \|g_1\|_{L^1(V)} + \|g_2\|_Y\right).$$

Thus $(\phi^0, g) \mapsto \langle Bu, \phi_2\rangle_{H^1(V)}$ is linear and continuous from $(V \times D_A) \times (L^1(V) \times Y)$ into $\Re$. It follows that the unique solution of (3.7) satisfies, in particular, $y_1 \in L^\infty(V')$. The duality pairing in the second term of (3.7) is between $L^\infty(V') \times Y'$ and $L^1(V) \times Y$. One may pass from $L^\infty(V')$ to $C(V')$ by a standard approximation technique.

One has $y_2 = \dot{y}_1$ since

$$\int_0^T [(y_1(t), \dot{g}_2(t))_V + \langle y_2(t), g_2(t)\rangle_{D_A}]\, dt = 0, \quad \forall g_2 \in C_0^\infty(0,T;D_A). \tag{3.8}$$

In fact, set $\phi_1^0 = \phi_2^0 = 0$, $g_1 = \dot{g}_2$ in (3.6). Then

$$\ddot{\phi}_2 = -\dot{\phi}_1 + \dot{g}_2 = -A\phi_2, \ (t < T), \quad \phi_2(T) = \dot{\phi}_2(T) = 0.$$

Therefore $\phi_2 \equiv 0$ and we obtain (3.8) from (3.7).

Equations (3.4) and (3.5) may be derived by choosing in (3.7)

$$g_1(t) = -\alpha_2(t)A\phi_2^0 + \dot{\alpha}_1(t)\phi_1^0, \quad g_2(t) = \alpha_1(t)\phi_1^0 + \dot{\alpha}_2(t)\phi_2^0,$$

where α_i is an arbitrary $C^1([0,T])$ function and $(\phi_1^0, \phi_2^0) \in D_A \times \Delta_A$. Then $\phi_1(t) = \alpha_1(t)\phi_1^0$ and $\phi_2(t) = \alpha_2(t)\phi_2^0$ satisfy the adjoint system (3.6) and have end values $\phi_1(T) = \alpha_1(T)\phi_1^0$, $\phi_2(T) = \alpha_2(T)\phi_2^0$, respectively. If these quantities are then substituted into (3.7), equations (3.4) and (3.5) result. In addition, since w is continuous into V', the traces of w at at $t = 0$ and at $t = T$ equal 0 and y_1^0, respectively. If $\dot{w}$ is continuous into D_A' (this is not guaranteed by Proposition 3.3), then the traces of $\dot{w}$ agree with 0 and y_2^0 at $t = 0$ and $t = T$, respectively. □

The observability assumption (2.21), in the present context, takes the form

$$(\phi_1^0, \phi_2^0) \in V \times D_A, \quad \|B'\phi_2\|_{\mathcal{U}} = 0 \Longleftrightarrow \phi_1^0 = \phi_2^0 = 0, \tag{3.9}$$

where $\phi = (\phi_1, \phi_2)$ is the solution of

$$\begin{cases} \dot{\phi}_1 = A\phi_2, \quad \dot{\phi}_2 = -\phi_1, \quad (t < T), \\ \phi_1(T) = \phi_1^0, \quad \phi_2(T) = \phi_2^0. \end{cases} \tag{3.10}$$

Under assumption (3.9), we may introduce a norm

$$\|(\phi_1^0, \phi_2^0)\|_F = \|B'\phi_2\|_{\mathcal{U}}$$

and a space

$$F = \text{completion of } V \times D_A \text{ in } \|\cdot\|_F.$$

We have

$$V \times D_A \subset F, \quad F' \subset V' \times D_A',$$

and F' coincides with the set of reachable states of (3.1) at time T. Let Λ_F be the Riesz isomorphism of F onto F'. If $(w^0, w^1) \in F'$, define

$$(\phi_1^0, \phi_2^0) = \Lambda_F^{-1}(w^0, w^1), \quad u_0 = \Lambda_{\mathcal{U}} B'\phi_2,$$

where (ϕ_1, ϕ_2) solves (3.10). Then u_0 is the control of minimum $\mathcal{U}'$ norm among all controls driving $(0,0)$ to (w^0, w^1), where the dynamics are described by (3.1). Since, in general, it will not be the case that $\phi_2 \in H^1(V)$ when $(\phi_1^0, \phi_2^0) \in F$, the precise meaning of u_0 is

$$u_0 = \lim_n \Lambda_{\mathcal{U}} B'\phi_{2,n} \ \text{ strongly in } \mathcal{U}',$$

where $(\phi_{1,n}, \phi_{2,n})$ is the solution of (3.10) corresponding to $(\phi_{1,n}^0, \phi_{2,n}^0)$, where $(\phi_{1,n}^0, \phi_{2,n}^0) \subset V \times D_A$ converges to (ϕ_1^0, ϕ_2^0) in F.

Corollary 3.4 *If*

$$\|B'\phi_2\|_{\mathcal{U}}^2 \geq c^2\|(\phi_1^0, \phi_2^0)\|_{V'\times H}^2, \quad \forall(\phi_1^0, \phi_2^0) \in V \times D_A, \tag{3.11}$$

where (ϕ_1, ϕ_2) *solves* (3.10), *then the reachable set of* (3.1) *at time* T *contains* $V \times H$ *. Conversely, if* $V \times H$ *is in the reachable set at time* T *and if*

$$\{u \in \mathcal{U}' \,|\, (w(T), \dot{w}(T)) \in V \times H\} \quad \text{is dense in } \mathcal{U}',$$

then (3.11) *must hold.*

The *a priori* estimate (3.11) is one that is known to hold for many boundary control problems for hyperbolic partial differential equations. An illustration of this is provided in the first example of the next section.

4 Examples

In this section two examples will be presented to illustrate how the above theory applies to specific distributed parameter control systems.

4.1 Kirchhoff plate equation with boundary controls acting in shear force and in bending and twisting moments

Let Ω be a bounded region in $\Re^2$ with smooth boundary Γ. Let $X_0 = (x_0, y_0)$ be a fixed but otherwise arbitrary point of $\Re^2$, and set

$$\Gamma_+ = \{X \in \Gamma \,|\, (X - X_0)\cdot\nu > 0\}, \quad \Gamma_- = \Gamma - \Gamma_+,$$

where ν denotes the unit normal to Γ pointing towards the exterior of Ω. Note that $\Gamma_\pm$ depend on the choice of X_0. We consider the plate equation

$$\ddot{w} - \gamma^2\Delta\ddot{w} + \gamma^2\Delta^2 w = 0 \quad \text{in } Q = \Omega \times (0,T) \tag{4.1}$$

with boundary conditions

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Sigma_- = \Gamma_- \times (0,T), \tag{4.2}$$

$$\begin{cases} \gamma^2[\Delta w + (1-\mu)P_1 w] = u_0, \\ \gamma^2\left[\dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)P_2 w - \dfrac{\partial \ddot{w}}{\partial \nu}\right] = -u_2 + \dfrac{\partial u_1}{\partial \tau} \quad \text{on } \Sigma_+ = \Gamma_+ \times (0,T), \end{cases} \tag{4.3}$$

and initial conditions

$$w(\cdot,0) = \frac{\partial w}{\partial t}(\cdot,0) = 0 \quad \text{in } \Omega. \tag{4.4}$$

In the above, Δ is the ordinary Laplacian in $\Re^2$, γ^2 is a constant of order $O(h^2)$, h denoting the uniform thickness of the plate, and $\mu \in (0,1)$ is another constant (Poisson's ratio).

We specifically assume that $\Gamma_\pm \neq \emptyset$. τ is the positively oriented unit tangent vector to Γ_+, and P_1 and P_2 are boundary operators which satisfy the Green's formula

$$(\Delta^2 u, v)_{L^2(\Omega)} = a(u,v) + \int_\Gamma \left[v\left(\frac{\partial \Delta u}{\partial \nu} + (1-\mu)P_2 u\right) - (\Delta u + (1-\mu)P_1 u)\frac{\partial v}{\partial \nu}\right] d\Gamma$$

where

$$a(u,v) = \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 + 2\mu \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 \right] dx dy.$$

The specific forms of these operators may be found in [5] or in [10]. The quantities u_0, u_1 and u_2 are the controls. They correspond, respectively, to a bending moment about the tangent vector to Γ, a twisting moment about the normal to Γ and to an edge shear force acting perpendicularly to the faces of the plate.

The spaces $L^2(\Omega)$, $H^k(\Omega)$, $L^2(\Gamma)$ denote the standard real L^2 and Sobolev spaces over Ω or Γ as notation implies. We set

$$H = \{v \mid v \in H^1(\Omega),\ v = 0 \ \text{ on } \Gamma_-\},$$

$$V = \{v \mid v \in H^2(\Omega),\ v = \partial v/\partial \nu = 0 \ \text{ on } \Gamma_-\}.$$

The norms in these spaces are taken to be

$$\|v\|_H = \left(\int_\Omega (v^2 + \gamma^2 |\nabla v|^2) dx dy\right)^{1/2}, \quad \|v\|_V = [\gamma^2 a(v,v)]^{1/2}.$$

That $\|\cdot\|_V$ is a norm equivalent to the standard $H^2(\Omega)$ norm is a version of Korn's Lemma. We further choose

$$\mathcal{U} = (L^2(\Sigma_+))^3$$

and we identify $\mathcal{U}$ with its dual space $\mathcal{U}'$.

If one forms the $L^2(Q)$ scalar product of (4.1) with a test function $\phi \in L^2(0,T;V) =: L^2(V)$ and uses the above Green's formula one obtains

$$(\ddot{w}, \phi)_{L^2(H)} + (w,\phi)_{L^2(V)} = \int_0^T \int_{\Gamma_+} \left(u_2 \phi + u_1 \frac{\partial \phi}{\partial \tau} + u_0 \frac{\partial \phi}{\partial \nu}\right) d\Gamma dt.$$

This variational equation is the same as

$$\ddot{w} + Aw = Bu \quad \text{in } L^2(V'), \ \ w(0) = \dot{w}(0) = 0. \tag{4.5}$$

where $u = (u_0, u_1, u_2) \in \mathcal{U}$ and where $B \in \mathcal{L}(\mathcal{U}, L^2(V'))$ is defined by

$$(Bu, \phi)_{L^2(V)} = \int_0^T \int_{\Gamma_+} \left(u_2 \phi + u_1 \frac{\partial \phi}{\partial \tau} + u_0 \frac{\partial \phi}{\partial \nu}\right) d\Gamma dt, \ \ \forall \phi \in L^2(V).$$

Problem (4.5) has a unique solution with $(w, \dot{w}) \in C([0,T]; H \times V')$.

Let us write down the inequality (3.11). The dual operator B' is defined by

$$B'\phi = \left(\frac{\partial\phi}{\partial\nu}, \frac{\partial\phi}{\partial\tau}, \phi\right)\Big|_{\Sigma_+}, \quad \forall\phi \in L^2(V),$$

so that (3.11) is

$$\|B'\phi_2\|_{\mathcal{U}}^2 = \int_0^T \int_{\Gamma_+} (\phi_2^2 + |\nabla\phi_2|^2) d\Gamma dt \geq c^2 \|(\phi_1^0, \phi_2^0)\|_{V' \times H}^2, \tag{4.6}$$
$$\forall(\phi_1^0, \phi_2^0) \in V \times D_A.$$

Set $\eta = \phi_2 \in C([0,T]; D_A)$. Then $\dot{\eta} = -\phi_1 \in C([0,T]; V)$ and

$$\ddot{\eta} + A\eta = 0 \quad \text{in } C([0,T]; H), \quad \eta(T) = \phi_2^0 \in D_A, \quad \dot{\eta}(T) = -\phi_1^0 \in V, \tag{4.7}$$

and (4.6) is equivalent to

$$\int_0^T \int_{\Gamma_+} (\eta^2 + |\nabla\eta|^2) d\Gamma dt \geq c^2 \|(\eta^0, \eta^1)\|_{H \times V'}^2, \quad \forall(\eta^0, \eta^1) \in D_A \times V. \tag{4.8}$$

However, (4.7) signifies that η is a solution of

$$\ddot{\eta} - \gamma^2 \Delta\ddot{\eta} + \gamma^2 \Delta^2 \eta = 0 \quad \text{in } Q,$$

$$\eta = \frac{\partial\eta}{\partial\nu} = 0 \quad \text{on } \Sigma_-,$$

$$\begin{cases} \gamma^2[\Delta\eta + (1-\mu)P_1\eta] = 0, \\ \gamma^2\left[\dfrac{\partial\Delta\eta}{\partial\nu} + (1-\mu)P_2\eta - \dfrac{\partial\ddot{\eta}}{\partial\nu}\right] = 0 \quad \text{on } \Sigma_+, \end{cases}$$

with final data in the space $D_A \times V$. It follows from the estimate in [10, Lemma V.5.1] by the trick of "weakening the norm" that (4.8) is satisfied for all sufficiently large T whenever η is a solution of the last system with data having the indicated regularity.

It follows from the general theory that the reachable reachable set F' of the system (4.1)–(4.4) contains $V \times H$, where F is completion of $V \times D_A$ in the norm

$$\|(\phi_1^0, \phi_2^0)\|_F = \left(\int_0^T \int_{\Gamma_+} (\phi_2^2 + |\nabla\phi_2|^2) d\Gamma dt\right)^{1/2}, \quad T > T_0.$$

Given $(w^0, w^1) \in V \times H$, the minimum norm control in $\mathcal{U}$ that drives $(0,0)$ to (w^0, w^1) at time T is defined by

$$u_0 = \frac{\partial\eta}{\partial\nu}\Big|_{\Sigma_+}, \quad u_1 = \frac{\partial\eta}{\partial\tau}\Big|_{\Sigma_+}, \quad u_2 = \eta|_{\Sigma_+},$$

η given by (4.7) with $(\phi_1^0, \phi_2^0) = \Lambda_F^{-1}(w^0, w^1)$.

4.2 Maxwell's equations with control acting through a tangentially flowing current in the boundary

Let $\Omega \subset \Re^3$ be a bounded region with smooth boundary Γ. We consider Maxwell's system

$$\begin{cases} \dot{E} - \text{curl } H = 0, \ \ \dot{H} + \text{curl } E = 0, \\ \text{div } E = \text{div } H = 0 \quad \text{in } Q, \end{cases} \tag{4.9}$$

$$\nu \times H = u \quad \text{on } \Sigma, \tag{4.10}$$

$$H(0) = E(0) = 0, \tag{4.11}$$

where $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. The control u represents a density of current flowing tangentially in Γ at each instant t.

We set

$$\mathcal{L}^2(\Omega) = (L^2(\Omega))^3, \ \ \mathcal{H}^k(\Omega) = (H^k(\Omega))^3, \ \ \mathcal{L}^2(\Gamma) = (L^2(\Gamma))^3, \ \ \mathcal{H}^k(\Gamma) = (H^k(\Gamma))^3.$$

The norm and scalar product in $\mathcal{L}^2(\Omega)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We also introduce the spaces (the notation is adopted from [7])

$$J = \text{ closure in } \mathcal{L}^2(\Omega) \text{ of } \{\chi | \chi \in C^\infty(\overline{\Omega}), \text{ div } \chi = 0\},$$

$$\hat{J} = \text{ closure in } \mathcal{L}^2(\Omega) \text{ of } \{\chi | \chi \in C_0^\infty(\Omega), \text{ div } \chi = 0\}.$$

For $k \geq 1$ we set

$$J_\nu^k = \{\chi | \chi \in J \cap \mathcal{H}^k(\Omega), \ \nu \cdot \chi = 0 \ \text{ on } \Gamma\},$$
$$J_\tau^k = \{\chi | \chi \in J \cap \mathcal{H}^k(\Omega), \ \nu \times \chi = 0 \ \text{ on } \Gamma\},$$

with the topology in each case that inherited from $\mathcal{H}^1(\Omega)$. In addition, define

$$J_\nu^* = \{\chi | \chi \in J_\nu^2, \nu \times \text{curl } \chi = 0 \text{ on } \Gamma\},$$

$$J_\tau^* = \{\chi | \chi \in J_\tau^2, \nu \cdot \text{curl } \chi = 0 \text{ on } \Gamma\},$$

each with the $\mathcal{H}^2(\Omega)$ topology. We have the dense and continuous embeddings

$$J_\tau^* \subset J_\tau^1 \subset J, \quad J_\nu^* \subset J_\nu^1 \subset \hat{J}.$$

If J (resp., $\hat{J}$) is identified with its dual space, we therefore also have

$$J \subset (J_\tau^1)' \subset (J_\tau^*)', \quad \hat{J} \subset (J_\nu^1)' \subset (J_\nu^*)'.$$

The mapping $\phi \mapsto \text{curl } \phi$ is an isomorphism from X onto Y, where (X, Y) stands for any one of the pairs

$$(J_\tau^k, J_\nu^{k-1}), \ \ (J_\nu^k, J_\tau^{k-1}), \ \ (J_\tau^*, J_\nu^1), \ \ (J_\nu^*, J_\tau^1), \ \ (k \geq 1),$$

and where $J_\nu^0 = \hat{J}$, $J_\tau^0 = J$ (see [7]). Therefore we may renorm J_ν^1, J_τ^1, J_ν^* and J_τ^* by setting

$$\|\phi\|_{J_\nu^1} = \|\text{curl } \phi\|, \ \ \|\phi\|_{J_\tau^1} = \|\text{curl } \phi\|, \ \ \|\phi\|_{J_\nu^*} = \|\text{curl curl } \phi\|, \ \ \|\phi\|_{J_\tau^*} = \|\text{curl curl } \phi\|.$$

These norms are equivalent to the corresponding Sobolev norms. Since

$$(\text{curl }\phi, \psi) = (\phi, \text{curl }\psi), \quad \forall \phi \in J^1_\tau, \psi \in J^1_\nu,$$

the map curl extends to a isomorphism of J onto $(J^1_\nu)'$ and of $\hat{J}$ onto $(J^1_\tau)'$. We have

$$\begin{cases} \langle \text{curl }\phi, \psi \rangle_{J^1_\nu} = (\phi, \text{curl }\psi), & \forall \phi \in J, \psi \in J^1_\nu, \\ \langle \text{curl }\phi, \psi \rangle_{J^1_\tau} = (\phi, \text{curl }\psi), & \forall \phi \in \hat{J}, \psi \in J^1_\tau. \end{cases} \tag{4.12}$$

In addition, curl extends to an element in $\mathcal{L}((J^1_\nu)', (J^*_\tau)')$ and in $\mathcal{L}((J^1_\tau)', (J^*_\nu)')$, both isomorphisms, through the formulas

$$\begin{cases} \langle \text{curl }\phi, \psi \rangle_{J^*_\tau} = \langle \phi, \text{curl }\psi \rangle_{J^1_\nu}, & \forall \phi \in (J^1_\nu)', \psi \in J^*_\tau, \\ \langle \text{curl }\phi, \psi \rangle_{J^*_\nu} = \langle \phi, \text{curl }\psi \rangle_{J^1_\tau}, & \forall \phi \in (J^1_\tau)', \psi \in J^*_\nu. \end{cases} \tag{4.13}$$

To obtain the abstract formulation of (4.9), (4.10), let $(\phi, \psi) \in L^2(0, T; J^1_\tau \times J^1_\nu) := L^2(J^1_\tau \times J^1_\nu)$, and form

$$\begin{aligned} 0 &= \int_0^T [(\dot{H} + \text{curl } E, \phi) + (\dot{E} - \text{curl } H, \psi)]dt \qquad (4.14) \\ &= \int_0^T [((\dot{H}, \dot{E}), (\phi, \psi))_{J\times J} - ((H, E), (\text{curl }\psi, -\text{curl }\phi))_{J\times J}]dt \\ &\quad - \int_0^T \int_\Gamma \psi \cdot u \, d\Gamma dt. \end{aligned}$$

We now consider various choices of control and state spaces.

4.2.1 Exact controllability to $J \times \hat{J}$ with $L^2(\mathcal{L}^2(\Gamma))$ controls, under a geometric condition on Γ.

We choose as the control space

$$\mathcal{U}' = L^2(\mathcal{L}^2(\Gamma)) = \mathcal{U},$$

and the state space

$$\mathcal{H} = (J^1_\tau)' \times (J^1_\nu)'.$$

We identify $J \times \hat{J}$ with its dual space, so that $\mathcal{H}' = J^1_\tau \times J^1_\nu$. Define an operator $\mathcal{A}$ in $\mathcal{H}$ by $D_\mathcal{A} = J \times \hat{J}$,

$$\mathcal{A} = \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix}. \tag{4.15}$$

By using (4.12) and the properties of curl enunciated above, one sees that $\mathcal{A}$ is a skew-adjoint operator in $\mathcal{H}$. The dual of $\mathcal{A}$ is therefore given by

$$\mathcal{A}' = -\Lambda_\mathcal{H} \mathcal{A} \Lambda_\mathcal{H}^{-1} = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}, \tag{4.16}$$

$$D_{\mathcal{A}'} = J^*_\tau \times J^*_\nu,$$

where $\Lambda_{\mathcal{H}}$ is the canonical isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$ and satisfies

$$\Lambda_{\mathcal{H}}^{-1} = \begin{pmatrix} \text{curl curl} & 0 \\ 0 & \text{curl curl} \end{pmatrix}.$$

We define the control operator $\mathcal{B}$ by

$$\langle \mathcal{B}u, (\phi,\psi)\rangle = \int_0^T \int_\Gamma \psi \cdot u \, d\Gamma dt, \quad \forall (\phi,\psi) \in L^2(J_\tau^1 \times J_\nu^1). \tag{4.17}$$

We have

$$|\langle \mathcal{B}u, (\phi,\psi)\rangle| \le C\|u\|_{\mathcal{U}'}\|\psi\|_{L^2(J_\nu^1)} \le C\|u\|_{\mathcal{U}'}\|(\phi,\psi)\|_{L^2(\mathcal{H}')},$$

so that $\mathcal{B} \in \mathcal{L}(\mathcal{U}', L^2(\mathcal{H}))$. The dual operator $\mathcal{B}' \in \mathcal{L}(L^2(\mathcal{H}'), \mathcal{U})$ is given by

$$\mathcal{B}'(\phi,\psi) = \psi|_\Sigma, \tag{4.18}$$

In view of (4.17), (4.14) may be written

$$((\dot{H},\dot{E}),(\phi,\psi))_{L^2(J\times\hat{J})} = \langle (H,E), \mathcal{A}'(\phi,\psi)\rangle_{L^2(\mathcal{H}')} + \langle \mathcal{B}u, (\phi,\psi)\rangle_{L^2(\mathcal{H}')},$$

that is to say,

$$\dot{y} = \mathcal{A}y + \mathcal{B}u \quad \text{in } L^2(\mathcal{H}), \tag{4.19}$$

where $y = (H,E)$. With the initial condition $y(0) = 0$, (4.19) has a unique solution in $C([0,T];\mathcal{H})$.

The observability condition (2.21) is:

$$(\phi^0,\psi^0) \in J_\tau^* \times J_\nu^*, \quad \int_0^T \int_\Gamma |\psi|^2 d\Gamma\, dt = 0 \Leftrightarrow \phi^0 = \psi^0 = 0,$$

where

$$\begin{cases} \dot{\phi} + \text{curl}\ \psi = 0, \quad \dot{\psi} - \text{curl}\ \phi = 0, \\ \text{div}\ \phi = \text{div}\ \psi = 0 \quad \text{in } Q, \end{cases} \tag{4.20}$$

$$\phi(T) = \phi^0, \quad \psi(T) = \psi^0 \quad \text{in } \Omega. \tag{4.21}$$

The following result is proved in [9, Lemma 3.3].

Theorem 4.1 *Assume that Γ is star-shaped with respect to some point in $\Re^3$. Then there exists $T_0 > 0$ such that for all $T > T_0$*

$$\int_0^T \int_\Gamma |\psi|^2 d\Gamma\, dt \ge c^2(T - T_0)\|(\phi^0,\psi^0)\|^2_{J\times\hat{J}}, \quad \forall (\phi^0,\psi^0) \in J_\tau^* \times J_\nu^*.$$

With the space F defined as the completion of $J_\tau^* \times J_\nu^*$ in the norm

$$\|(\phi^0,\psi^0)\|_F = \left(\int_0^T \int_\Gamma |\psi|^2 d\Gamma\, dt\right)^{1/2},$$

we therefore have

$$F \subset J \times \hat{J} \subset F',$$

and $J \times \hat{J}$ is in the reachable set of (4.9)–(4.11). Given $(H^0,E^0) \in J \times \hat{J}$, the minimum norm control in $L^2(\mathcal{L}^2(\Gamma))$ that drives $(0,0)$ to (H^0,E^0) is given by

$$u = \psi|_\Gamma$$

with (ϕ,ψ) given by (4.20), (4.21) and with $(\phi^0,\psi^0) = \Lambda_F^{-1}(H^0,E^0)$.

4.2.2 Exact controllability to $(J_\tau^1)' \times (J_\nu^1)'$ with $(H^1(\mathcal{L}^2(\Gamma)))'$ controls, under a geometric condition on Γ.

Here we choose

$$\mathcal{U} = H^1(\mathcal{L}^2(\Gamma))$$

so that, identifying $L^2(\mathcal{L}^2(\Gamma))$ with its dual space, the control space is

$$\mathcal{U}' = (H^1(\mathcal{L}^2(\Gamma)))'. \tag{4.22}$$

The choices of $\mathcal{H}$, $\mathcal{A}$ and $D_{\mathcal{A}}$ are the same as in the last subsection. The only difference is the operator $\mathcal{B}$, now defined by the duality pairing

$$\langle \mathcal{B}u, (\phi, \psi)\rangle = \langle u, \psi\rangle_{H^1(\mathcal{L}^2(\Gamma))}.$$

We have

$$\langle \mathcal{B}u, (\phi,\psi)\rangle \le \|u\|_{(H^1(\mathcal{L}^2(\Gamma)))'}\|\psi\|_{H^1(\mathcal{L}^2(\Gamma))} \le C\|u\|_{(H^1(\mathcal{L}^2(\Gamma)))'}\|(\phi,\psi)\|_{H^1(\mathcal{H}')}.$$

Therefore $\mathcal{B} \in \mathcal{L}(\mathcal{U}', (H^1(\mathcal{H}'))')$, and the abstract formulation of (4.9), (4.10) is

$$\dot{y} = \mathcal{A}y + \mathcal{B}u \ \text{ in } (H^1(\mathcal{H}'))'.$$

With $y(0) = 0$, the last equation has a unique solution whose properties are delineated in Propositions 2.2 and 2.3 above.

The dual operator $\mathcal{B}' \in \mathcal{L}((H^1(\mathcal{H}'), \mathcal{U})$ is again given by (4.18). To identify the reachable set of the system we have to consider

$$\|(\phi^0,\psi^0)\|_F^2 = \|\mathcal{B}'(\phi,\psi)\|_{\mathcal{U}}^2 = \int_0^T\int_\Gamma (|\psi|^2 + |\dot{\psi}|^2)d\Gamma\, dt,$$

where (ϕ, ψ) satisfies (4.20), (4.21). According to [9, Lemma 3.1], we have

Theorem 4.2 *Assume that Γ is star-shaped with respect to some point in $\Re^3$. Then there exists $T_0 > 0$ such that for all $T > T_0$,*

$$\int_0^T\int_\Gamma (|\psi|^2 + |\dot{\psi}|^2)d\Gamma\, dt \ge c^2(T-T_0)\|(\phi^0,\psi^0)\|^2_{J_\tau^1 \times J_\nu^1}, \quad \forall (\phi^0,\psi^0) \in J_\tau^* \times J_\nu^*.$$

It follows that the reachable set of (4.9)-((4.11), with controls satisfying (4.22), contains $(J_\tau^1 \times J_\tau^1)'$. Given (H^0, E^0) in this space, the minimum norm control is given by

$$u_{\min} = \Lambda_{\mathcal{U}}\psi|_\Sigma,$$

where (ϕ, ψ) is given by (4.20), (4.21) with $(\phi^0, \psi^0) = \Lambda_F^{-1}(H^0, E^0)$, and where $\Lambda_{\mathcal{U}}$ the canonical isomorphism of $\mathcal{U}$ onto $\mathcal{U}'$. For $\xi \in \mathcal{U}$ one may write

$$\Lambda_{\mathcal{U}}\xi = \xi - \frac{d^2\xi}{dt^2},$$

where $\frac{d^2\xi}{dt^2} \in \mathcal{L}(\mathcal{U}, \mathcal{U}')$ is *defined* by

$$\langle \frac{d^2\xi}{dt^2}, \eta\rangle = -\int_0^T\int_\Gamma \dot{\xi}\cdot\dot{\eta}\, d\Gamma\, dt, \quad \forall \xi, \eta \in \mathcal{U}.$$

Therefore

$$u_{\min} = \psi|_\Sigma - \left.\frac{d^2\psi}{dt^2}\right|_\Sigma.$$

4.2.3 Exact controllability to $(J_\tau^1)' \times (J_\nu^1)'$ without geometric restrictions.

When the star-shapedness requirement on Γ is removed, it is necessary to work with control and state spaces with weaker topologies than before. Accordingly, we choose

$$\mathcal{H} = (J_\tau^* \times J_\nu^*)'.$$

The operator $\mathcal{A}$ is still given by the matrix (4.15), but with

$$D_{\mathcal{A}} = (J_\tau^1 \times J_\nu^1)'.$$

By using (4.13) it is seen that $\mathcal{A}$ is a skew-adjoint operator in $\mathcal{H}$. Its dual $\mathcal{A}'$ is an operator in $\mathcal{H}' = J_\tau^* \times J_\nu^*$ given by the matrix in (4.16) with

$$D_{\mathcal{A}'} = \{(\phi^0, \psi^0) | (\phi^0, \psi^0) \in J_\tau^* \times J_\nu^*, \text{ curl } \phi^0 \in J_\nu^*, \text{ curl } \psi^0 \in J_\tau^*\}.$$

The control space will be the dual of

$$\mathcal{U} = H^1(\mathcal{L}^2(\Gamma)) \oplus L^2(U),$$

where U is a certain Hilbert space which satisfies $\mathcal{H}^1(\Gamma) \subset U \subset \mathcal{L}^2(\Gamma)$. Thus

$$\mathcal{U}' = (H^1(\mathcal{L}^2(\Gamma)))' \oplus L^2(U') \subset (H^1(\mathcal{L}^2(\Gamma)))' \oplus L^2(\mathcal{H}^{-1}(\Gamma)).$$

To define U, we introduce the closed subspace of $\mathcal{L}^2(\Gamma)$

$$\mathcal{L}_\tau^2 = \{\chi \in \mathcal{L}^2(\Gamma) | \nu \times \chi = 0 \text{ on } \Gamma\}.$$

For smooth functions χ defined in $\overline{\Omega}$ we have (see [9, Section 4.3])

$$\text{curl } \chi|_\Gamma = \nu \times \frac{\partial \chi}{\partial \nu} + \sigma \times \chi$$

where $\sigma = (\sigma_1, \sigma_2.\sigma_3)$ is a formally self-adjoint tangential operator of order one on Γ:

$$\int_\Gamma \tilde{\chi} \cdot (\sigma \times \chi) d\Gamma = \int_\Gamma (\sigma \times \tilde{\chi}) \cdot \chi \, d\Gamma, \quad \forall \chi, \tilde{\chi} \in \mathcal{H}^1(\Gamma).$$

If, therefore, $\chi \in \mathcal{L}_\tau^2$ then

$$\int_\Gamma \tilde{\chi} \cdot \text{curl } \chi|_\Gamma d\Gamma = \int_\Gamma \tilde{\chi} \cdot (\sigma \times \chi) d\Gamma = \int_\Gamma (\sigma \times \tilde{\chi}) \cdot \chi \, d\Gamma. \tag{4.23}$$

Consequently, if $\chi \in \mathcal{H}^1(\Gamma)$, we may *define* curl $\chi \in \mathcal{L}_\tau^2$ by

$$(\text{curl } \chi, \tilde{\chi})_{\mathcal{L}_\tau^2} = \langle \sigma \times \tilde{\chi}, \chi \rangle_{\mathcal{H}^1(\Gamma)}, \quad \forall \tilde{\chi} \in \mathcal{L}_\tau^2.$$

Then curl$\in \mathcal{L}(\mathcal{H}^1(\Gamma), \mathcal{L}_\tau^2)$. In particular, we have $\nu \times \text{curl } \chi = 0$ for all $\chi \in \mathcal{H}^1(\Gamma)$.

We introduce on $\mathcal{H}^1(\Gamma)$ the norm

$$\|\chi\|_U = \left(\|\text{curl } \chi\|^2_{\mathcal{L}^2(\Gamma)} + \|\chi\|^2_{\mathcal{L}^2(\Gamma)}\right)^{1/2} \tag{4.24}$$

and define U as the completion of $\mathcal{H}^1(\Gamma)$ in $\|\cdot\|_U$. For $u = u_0 + u_1 \in \mathcal{U}'$ we define the control operator by

$$\langle \mathcal{B}u, (\phi, \psi)\rangle = \langle u_0, \psi\rangle_{H^1(\mathcal{L}^2(\Gamma))} + \langle u_1, \psi\rangle_{\mathcal{L}^2(U)}, \quad \forall(\phi, \psi) \in L^2(J_\tau^* \times J_\nu^*).$$

We have

$$\begin{aligned} |\langle \mathcal{B}u, (\phi, \psi)\rangle| &\le \|u_0\|_{(H^1(\mathcal{L}^2(\Gamma)))'}\|\psi\|_{H^1(\mathcal{L}^2(\Gamma))} + \|u_1\|_{L^2(U')}\|\psi\|_{L^2(U)} \\ &\le C\|u\|_{\mathcal{U}'}\left[\|(\phi,\psi)\|_{H^1(J_\tau^1\times J_\nu^1)} + \|(\phi,\psi)\|_{L^2(J_\tau^*\times J_\nu^*)}\right]. \end{aligned}$$

Therefore we have, in particular, $\mathcal{B} \in \mathcal{L}(\mathcal{U}', (H^1(J_\tau^* \times J_\nu^*))') = \mathcal{L}(\mathcal{U}', (H^1(\mathcal{H}'))')$, so that the theory of Section 2 may be applied. To do so, we have to consider

$$\begin{aligned} \|(\phi^0, \psi^0)\|_F^2 &= \|\mathcal{B}'(\phi,\psi)\|_{\mathcal{U}} = \|\phi\|^2_{H^1(\mathcal{L}^2(\Gamma))} + \|\psi\|^2_{L^2(U)} \\ &= \int_0^T \int_\Gamma (2|\psi|^2 + |\dot\psi|^2 + |\text{curl } \psi|^2)d\Gamma\, dt, \end{aligned}$$

where (ϕ, ψ) satisfy (4.20), (4.21) with $(\phi^0, \psi^0) \in D_{\mathcal{A}'}$. According to [9, Lemma 3.2] we have

$$\int_0^T \int_\Gamma (|\psi|^2 + |\dot\psi|^2 + |\text{curl } \psi|^2)d\Gamma\, dt \ge c^2(T - T_0)\|(\phi^0, \psi^0)\|^2_{J_\tau^1\times J_\nu^1}, \quad \forall(\phi^0, \psi^0) \in D_{\mathcal{A}'},$$

provided $T > T_0$ with a suitable T_0. The reachable set of our problem therefore contains $(J_\tau^1)' \times (J_\nu^1)'$. If (H^0, E^0) is in this space then, with $(\phi^0, \psi^0) = \Lambda_F^{-1}(H^0, E^0)$, the control of minimum norm in $\mathcal{U}'$ steering $(0, 0)$ to (H^0, E^0) is given by

$$u_{\min} = u_0 + u_1$$

where, as in the last subsection,

$$u_0 = \psi|_\Sigma - \left.\frac{d^2\psi}{dt^2}\right|_\Sigma,$$

and where

$$u_1 = \Lambda_{\mathcal{U}_1}\mathcal{B}'(\phi, \psi) = \Lambda_{\mathcal{U}_1}\psi|_\Sigma,$$

$\Lambda_{\mathcal{U}_1}$ denoting the canonical isomorphism from $\mathcal{U}_1 := L^2(U)$ onto $\mathcal{U}_1' = L^2(U')$. From (4.23) and the definition (4.24) of the norm on U it is seen that

$$\Lambda_{\mathcal{U}_1}\chi = \chi + \sigma \times \text{curl } \chi, \quad \forall\chi \in \mathcal{U}_1.$$

Therefore

$$u_1 = \psi|_\Sigma + \sigma \times \text{curl } \psi|_\Sigma = \psi|_\Sigma + \sigma \times \dot\phi|_\Sigma.$$

References

[1] A. Bensoussan, *On the general theory of exact controllability for skew-symmetric operators*, preprint.

[2] A. Bensoussan, *Some remarks on the exact controllability of Maxwell's equations*, preprint.

[3] G. Chen, *Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain*, J. Math. Pures Appl., 58 (1979), 249–274.

[4] S. Dolecki and D. L. Russell, *A general theory of observation and control*, SIAM J. Control and Opt., 15, (1977), 185–220.

[5] G. Duvaut and J.-L. Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972.

[6] L. F. Ho, *Observabilité frontière de l'equation des ondes*, C.R. Acad. Sci. Paris Sér. I, 302 (1986), 443–446.

[7] O. A. Ladyzhenskaya and V. A. Solonikov, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics*, J. Soviet Nath., 8, (1977), 384–422.

[8] J. E. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Diff. Eqs. 50, (1983), 163–182.

[9] J. E. Lagnese, *Exact boundary controllability of Maxwell's equations in a general region*, SIAM J. Control and Opt., 27, (1989), 374–388.

[10] J. E. Lagnese and J.-L. Lions, *Modelling, Analysis and Control of Thin Plates*, Recherches en Mathématiques Appliquées, Vol. 6, Masson, Paris, 1988.

[11] I. Lasiecka, *Controllability of a viscoelastic Kirchhoff plate*, Internat. Ser. in Numerical Math., 91 (1989), 237–247.

[12] I. Lasiecka and R. Triggiani, *Exact controllability of the wave equation with Neumann boundary control*, Appl. Math. and Opt., 19 (1989), 243–290.

[13] I. Lasiecka and R. Triggiani, *Exact controllability of the Euler-Bernoulli equation with controls in the Dirichlet and Neumann boundary conditions: a nonconservative case*, SIAM J. Control and Opt., 27 (1989), 330–372.

[14] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed parameter systems*, SIAM Review, 30 (1988), 1–68.

[15] J.-L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués. Tome 1, Contrôlabilité Exacte; Tome 2, Perturbations*, Recherches en Mathématiques Appliquées, Vols. 8 and 9, Masson, Paris, 1988.

[16] D. L. Russell, Review of *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, Bull. Amer. Math. Soc., 22 (1990), 353–356.

[17] R. Triggiani, *Exact boundary controllability on $L^2(\Omega) \times H^{-1}(\Omega)$ of the wave equation with Dirichlet boundary control action of a portion of the boundary, and related problems*, Appl. Math. and Opt., 18 (1988), 241–277.