RELAXED CONTROLS FOR STOCHASTIC BOUNDARY VALUE PROBLEMS IN INFINITE DIMENSION

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Dedicated to Professor L. Cesari

ABSTRACT

The paper deals with the question of boundary controls for a class of linear abstract stochastic initial boundary value problems. The objective is to control the mean state trajectory and the corresponding covariance operator in the presence of both boundary and distributed noises. Both existence of optimal relaxed controls and necessary conditions of optimality are presented. The question of practical realization of relaxed controls is briefly discussed as closing remarks.

INTRODUCTION

Let Σ be an open bounded connected subset of R^n with smooth boundary, $X \equiv X(\Sigma)$ a Banach space of functions or generalized functions on Σ and $E \equiv E(\partial\Sigma)$ a Banach space of functions or generalized functions on $\partial\Sigma$. Consider the initial boundary value problem governed by a system of partial differential equations with L denoting the spatial differential operator and τ denoting the boundary differential operator

$$\text{IBVP}\begin{cases} \dfrac{\partial}{\partial t}\varphi = L\varphi + h, \quad t \geq 0, \\ \tau\varphi = g \\ \varphi(0) = \varphi_0. \end{cases} \tag{1}$$

We assume that $D(L) \subset D(\tau)$. The data h and g are X and E valued respectively and $\varphi_0 \in X$. Define

$$\begin{aligned} A &\equiv (L\,|_{\mathrm{Ker}\ \tau}) : D(A) \to X \\ R &\equiv (\tau\,|_{\mathrm{Ker}\ L})^{-1} : E \to X. \end{aligned} \tag{2}$$

Then the IBVP can be reformulated as an abstract Cauchy problem [1-5]:

$$\begin{cases} \dfrac{d\varphi}{dt} = A\varphi + \pi R g + h \\ \varphi(0) = \varphi_0 \end{cases} \tag{3}$$

where $\pi \equiv (\lambda I - A), \lambda \in \rho(A)(\neq \emptyset)$. More conveniently: setting $\varphi = \pi x$, we have,

$$\begin{cases} \dfrac{dx}{dt} = Ax + Rg + \Lambda h, \quad \Lambda = \pi^{-1} = R(\lambda, A) \\ x(0) = x_0 \qquad\qquad\qquad\qquad x_0 = \Lambda\varphi_0. \end{cases} \tag{4}$$

We consider

$$\begin{cases} \dfrac{dx}{dt} = Ax + Rg + \Lambda h \\ x(0) = x_0 \end{cases} \tag{5}$$

as the basic equation.

We are interested in the stochastic model controlled through the boundary. Suppose

$$\begin{aligned} h &= h_0 + \sigma_0 N_0 \ \ \text{(on spatial domain) } \Sigma \\ g &= f(u) + \sigma(u) N \ \ \text{(on boundary) } \partial\Sigma \end{aligned} \tag{6}$$

where N_0 and N are considered as the spatial and boundary noises modelled as the distributional derivatives of certain Wiener processes w_0 and w respectively.

The stochastic model is then given by:

$$\begin{cases} dx = Axdt + h_0 dt + Rf dt + \sigma_0 dw_0 + R\sigma dw \\ x(0) = x_0, \quad \varphi = \pi x. \end{cases} \tag{7}$$

We are interested in relaxed controls and hence the relaxed system,

$$\begin{cases} dx = Axdt + h_0 dt + R\nu_t(f) dt + \sigma_0 dw_0 + R\nu_t(\sigma) dw \\ x(0) = x_0, \quad \varphi = \pi x, \end{cases} \tag{8}$$

where,

$$\nu_t(\xi) \equiv \int_U \xi(u)\nu_t(du). \tag{9}$$

BASIC NOTATIONS AND ASSUMPTIONS

$$\begin{aligned} \text{State Space}: \quad & X \equiv \text{Hilbert Space} \\ \text{Boundary Space}: \quad & E \equiv \text{Hilbert Space} \\ \text{Control Space}: \quad & U \equiv \text{Compact Polish Space} \\ \text{State Space for Spatial Noise}: \quad & W_0 \equiv \text{Hilbert Space (Separable)} \\ \text{State Space for Boundary Noise}: \quad & W \equiv \text{Hilbert Space (Separable).} \end{aligned}$$

$M(U) \equiv$ the space of probability measures on $B(U) \equiv$ Borel U.
$\mathcal{M} \equiv L_0([0,\infty), M(U)) \equiv$ the space of (weakly) Borel measurable functions from $[0,\infty)$ to $M(U)$ furnished with the Young topology τ_y, given by, $\nu^n \xrightarrow{\tau_y} \nu^0$ as $n \to \infty$ if for every $\xi \in C_b(U,Y)$ $(C_b(U,\mathcal{L}_s(Z,Y)))$

$$\int_J \int_U \xi(u)\nu_t^n(du) \xrightarrow{s(\tau_s)} \int_J \int_U \xi(u)\nu_t^0(du) \ \text{ in } Y \tag{10}$$

for each $J \subset I$, where Y is any Banach space ($\mathcal{L}_s(Z,Y) \equiv \mathcal{L}(Z,Y)$ furnished with strong operator topology). We assume

$$\left\{ \begin{array}{l} A \in G(X, M, \omega), M \geq 1, \omega \in R \\ Q_0 \equiv \text{Cov}.w_0 \in \mathcal{L}_n^+(W_0^*, W_0) = \mathcal{L}_n^+(W_0) \\ Q \equiv \text{Cov}.w \in \mathcal{L}_n^+(W^*, W) = \mathcal{L}_n^+(W) \\ f : U \to E, \quad \sigma : U \to \mathcal{L}(W, E). \end{array} \right\} \tag{11}$$

MOTIVATION

One of the physical problems that motivated us to this abstract stochastic boundary value problem is the boundary control of the Cantilever beam equation subject to random perturbations of the free end. This is described as follows:

$$\left\{ \begin{array}{l} \dfrac{\partial^2 y}{\partial t^2} + \Delta^2 y = u_1 + n_1 \equiv h, \quad x \in (0,l), t \geq 0 \\ y|_{x=0} = 0, \quad Dy|_{x=0} = 0 \\ D^2 y|_{x=l} = u_2 + h_2 \equiv g_1, \ D^3 y|_{x=l} = u_3 + n_3 \equiv g_2 \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in (0,l), \end{array} \right. \tag{12}$$

where u_1 is the distributed control, n_1 the distributed noise; u_2, u_3 are the boundary controls and n_2, n_3 are the boundary noises which may be induced by turbulent flow of a fluid past the cantilever end.

Introducing $\varphi_1 = y$, $\varphi_2 = y_t$,

$$L \equiv \begin{pmatrix} 0 & 1 \\ -\Delta^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{13}$$

$$\tau\varphi \equiv \begin{pmatrix} D^2\varphi|_{x=l} \\ D^3\varphi|_{x=l} \end{pmatrix} \tag{14}$$

and the state space

$$X \equiv H_l^2 \times L_2(0, l) \tag{15}$$

where,

$$H_l^2 \equiv \{\psi \in H^2 : \psi|_{x=0} = 0, (\psi_x = D\psi)|_{x=0} = 0\}$$

and the boundary space

$$E = R^2, \tag{16}$$

we can rewrite the above equation in the semiabstract form as follows:

$$\begin{cases} \dfrac{d\varphi}{dt} = L\varphi + h \\ \tau\varphi = g \\ \varphi(0, \cdot) \equiv \varphi_0. \end{cases} \tag{17}$$

Clearly $D(L) \subset D(\tau)$. Define A by $D(A) = \{\varphi \in X : L\varphi \in X \text{ and } \tau\varphi = 0\}$ and set $A\varphi = L\varphi$ for $\varphi \in D(A)$, that is, $A = L|_{\mathrm{Ker}(\tau)\cap X}$. Then define R as $R \equiv (\tau|_{\mathrm{Ker} L})^{-1}$ obtained by solving the equation

$$\begin{cases} L\varphi = 0 \\ \tau\varphi = g \quad \text{in } X. \end{cases} \tag{18}$$

One can easily verify that this equation has a unique solution.

The operator R is a matrix of multiplication operators given by $R \equiv \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$ where

$$R_{11} = x^2/2, \; R_{12} = (x^3/6 - Lx^2/2), R_{21} = R_{22} = 0. \tag{19}$$

The equation (12) or equivalently (17) can then be written as an abstract evolution equation

$$\begin{cases} d\varphi = A\varphi dt + Budt + Bdw + \pi Ru_b dt + \pi R dw_b \\ \varphi(0) = \varphi_0 \end{cases} \tag{20}$$

where

$$u_b = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad w_b = \begin{pmatrix} w_2 \\ w_3 \end{pmatrix}, \tag{21}$$

with w_2, w_3 being standard wiener processes corresponding to the white noises n_2 and n_3 respectively.

Note that, in this example, the boundary conditions are partly absorbed in the state space and partly by the operator A.

RELAXED STOCHASTIC INITIAL BOUNDARY VALUE PROBLEM

Consider the boundary control system,

$$\begin{cases} dx = Axdt + R\nu_t(f)dt + R\nu_t(\sigma)dw \\ x(0) = x_0 \end{cases} \tag{22}$$

Objective 1

Find $\nu \in \mathcal{M}$ such that

$$J(\nu) \equiv \frac{1}{2} \int_0^\tau \{tr(PM_c) + \langle N_c(m - m^*, m - m^*_\cdot \rangle\} dt$$

is minimum where $m(\cdot)$ is the mean of the process $\{x\}$ and $P(\cdot)$ is the covariance operator corresponding to x. M_c and N_c are $\mathcal{L}_n^+(X)$ and $\mathcal{L}^+(X)$ valued functions respectively.

Lemma 1

Let $A \in G(X, M, \omega)$, $f \in C_b(U, E)$, $\sigma \in C_b(U, \mathcal{L}(W, E))$ and $R \in \mathcal{L}(E, X)$. Suppose also $\mathcal{F}(x_0) \perp \mathcal{F}_t^w$ for all $t > 0$ and $x_0 \in L_2(\Omega, \mathcal{F}, P; X)$. Then for each $\nu \in \mathcal{M}$, the mean vector $m(\cdot)$ and the covariance operator $P(\cdot)$ satisfies the following differential equations

$$\begin{cases} \dfrac{dm}{dt} = Am + R\nu_t(f) \\ m(0) = m_0, \end{cases} \tag{23}$$

$$\begin{cases} \dfrac{d}{dt}(P(t)\xi, \eta) = (A^*\xi, P\eta) + (P\xi, A^*\eta) + (R\nu_t(\sigma)Q\nu_t(\sigma^*)R^*\xi, \eta) \\ P(0) = P_0, \quad \xi, \eta \in D(A^*), \quad t \in I = [0, \tau]. \end{cases} \tag{24}$$

Unfortunately the τ_y-topology (Young topology) is too weak for the stochastic problem. We shall, instead, consider the topology of pointwise convergence τ_p, in the sense that, $\nu^n \xrightarrow{\tau_p} \nu$ as $n \to \infty$ if

$$\nu_t^n(\xi) \equiv \int_U \xi(u)\nu_t^n(du) \xrightarrow{s} \int_U \xi(u)\nu_t^0(du) \equiv \nu_t^0(\xi)$$

in Y as $n \to \infty$ for each $\xi \in C_b(U, Y)$ for almost all $t \geq 0$.

Lemma 2

Suppose the assumptions of Lemma 1 hold. Then, for each $\nu \in \mathcal{M}$, the equation (23) has a unique mild solution $m \in C(I, X)$ and the equation (24) has a weak solution $P \in C(I, \mathcal{L}_w^+(X))$. Further, $\nu \to m^\nu$ is continuous from $\mathcal{M}$ to $C(I, X)$ in τ_y-topology and $\nu \to P^\nu$ is continuous from $\mathcal{M}$ to $C(I, \mathcal{L}_w^+(X))$ in the τ_p-topology.

Theorem 3

Let $\mathcal{M}_p$ be a subset of $\mathcal{M}$, compact in the τ_p-topology and suppose the assumptions of Lemma 1 hold and

$$M_c \in L_\infty(I, \mathcal{L}_n^+(X)), N_c \in L_\infty(I, \mathcal{L}^+(X)) \text{ and } m^* \in C(I, X).$$

Then there exists a $\nu^0 \in \mathcal{M}_p$ for which

$$J(\nu^0) \leq J(\nu) \quad \text{for all } \nu \in \mathcal{M}_p.$$

Proof

Follows from the facts that $\nu \to J(\nu)$ is continuous from $\mathcal{M}_p$ to R, $J(\nu) \geq 0$ and that $\mathcal{M}_p$ is compact in τ_p-topology.

Objective 2 (Time Optimal Control)

Consider the uncontrolled system

$$d\xi = A\xi dt + \sigma_0 dw_0, \quad \xi(0) = x_0 \tag{25}$$

and the controlled system,

$$\begin{cases} dx = Axdt + \sigma_0 dw_0 + R\nu_t(f)dt + R\nu_t(\sigma)dw \\ x(0) = x_0. \end{cases} \tag{26}$$

Let P_ξ and P_x^ν denote the covariance operators corresponding to the processes $\{\xi\}$ and $\{x\}$ respectively.

Let $P_\xi^\infty \equiv w \cdot \lim_{t\to\infty} P_\xi(t)$. The problem is to find a $\nu \in \mathcal{M}_p$ such that

$$P_x^\nu(t^0) = P_\xi^\infty \quad \text{in minimum time } t^0.$$

Lemma 4 [6]

Consider the uncontrolled system (25) and suppose $A \in G(X, M, -\delta)$ for some $\delta > 0$ and $\sigma_0 \in \mathcal{L}(W_0, X)$. Then there exists a $P_\xi^\infty \in \mathcal{L}_n^+(X)$ such that

$$P_\xi(t) \xrightarrow{\tau_{w_0}} P_\xi^\infty \quad \text{as } t \to \infty,$$

and further,

$$\mu_{\xi,t} \xrightarrow{w^*} \mu_\xi^\infty \text{ as } t \to \infty,$$

where $\mu_{\xi,t}(\Gamma) \equiv \text{Prob.}\{\xi(t) \in \Gamma\}, \Gamma \in B(X)$ and μ_ξ^∞ is a countably additive Gaussian measure with covariance operator P_ξ^∞.

Note: We use the convention $\inf(\emptyset) = \infty$.

Theorem 5

Suppose the assumptions of Lemma 1 and Lemma 4 hold and further, there exists a $u^0 \in U$ such that $f(u^0) = 0, \sigma(u^0) = 0$. suppose there exists a $\nu \in \mathcal{M}_p$ such that

$$t(\nu) \equiv \inf\{t \geq 0 : P_x^\nu(t) = P_\xi^\infty\} < \infty.$$

Then there exists a $\nu^0 \in \mathcal{M}_p$ such that

$$t_0 \equiv t(\nu^0) \leq t(\nu) \ \ \forall \nu \in \mathcal{M}_p.$$

Proof

The proof is standard and follows from the expression,

$$\begin{aligned}
&(P^\nu(t)\xi, \eta) = (P_0 T^*(t)\xi, T^*(t)\eta) \\
&+ \int_0^t \langle \sigma_0 Q_0 \sigma_0^* T^*(t-\theta)\xi, T^*(t-\theta)\eta\rangle d\theta \\
&+ \int_0^t \langle \nu_\theta(\sigma) Q \nu_\theta(\sigma^*) R^* T^*(t-\theta)\xi, R^* T^*(t-\theta)\eta\rangle d\theta, \quad \xi, \eta \in D(A^*)
\end{aligned}$$

and the fact that T^* is also a C_0-semigroup in X.

Objective 3

Find $\nu^0 \in \mathcal{M}$ such that $J(\nu^0) \leq J(\nu)$ for $\nu \in \mathcal{M}$ where

$$J(\nu) = \frac{1}{2} \int_0^\tau \{tr(P(t)M_c) + \langle N_c(m(t) - m^*(t)), m(t) - m^*(t)\rangle\} dt$$

subject to the dynamic constraints:

$$\begin{cases}
\dfrac{dm}{dt} = Am(t) + R\nu_t(f), \quad m(0) = m_0 \\
\dfrac{d}{dt}(P\xi, \eta) = (A^*\xi, P\eta) + (P\xi, A^*\eta) + \langle Q\nu_t(\sigma^*)R^*\xi, \nu_t(\sigma^*)R^*\eta\rangle, \quad \text{for } 0 \leq t \leq \tau < \infty \\
P(0) = P_0, \quad \xi, \eta \in D(A^*).
\end{cases}$$

Theorem 6

Suppose the assumptions of Lemma 1 hold and further,

$$m^* \in C(I,X),\ M_c \in L_\infty(I,\mathcal{L}_n^+(X)),\ \text{and}\ N_c \in L_\infty(I,\mathcal{L}^+(X)).$$

Then, in order that the triple $\{\nu^0, m^0, P^0\} \in \mathcal{M} \times C(I,X) \times C(I,\mathcal{L}_n^+(X))$ be optimal, it is necessary that the following equations and inequalities hold:

$$\begin{cases} \dfrac{dm^0}{dt} = Am^0 + R\nu_t^0(f),\quad m^0(0) = m_0 \\ \dfrac{d}{dt}(P^0\xi,\eta) = (A^*\xi, P^0\eta) + (P^0\xi, A^*\eta) + \langle Q\nu_t^0(\sigma^*)R^*\xi, \nu_t^0(\sigma^*)R^*\eta\rangle \\ P^0(0) = P_0,\quad \xi,\eta \in D(A^*),\quad \text{for } 0 \le t \le \tau < \infty \end{cases} \tag{27}$$

$$\begin{cases} -\dfrac{dp}{dt} = A^*p + N_c(t)(m^0(t) - m^*(t)), p(\tau) = 0 \\ \dfrac{d}{dt}(S(t)\xi,\eta) = (S(t)\xi, A\eta) + (A\xi, S(t)\eta) + (M_c(t)\xi,\eta) \\ S(\tau) = 0,\quad \xi,\eta \in D(A) \end{cases} \tag{28}$$

$$J^{(1)}(\nu^0,\nu-\nu^0) \equiv \int_0^T \{tr[(\nu_t(\sigma)-\nu_t^0(\sigma))Q\nu_t^0(\sigma^*)R^*SR] + \langle \nu_t(f)-\nu_t^0(f), R^*p\rangle\}dt \ge 0 \tag{29}$$

for all $\nu \in \mathcal{M}$.

Proof of theorem 6(outline)

Let $\nu^0 \in \mathcal{M}$ be optimal and $\nu \in \mathcal{M}$ arbitrary, and define

$$\nu^\varepsilon = \nu^0 + \varepsilon(\nu - \nu^0),\quad 0 \le \varepsilon \le 1.$$

Let m^ε and m^0 correspond to ν^ε and ν^0 respectively. Then one shows that

$$\frac{m^\varepsilon - m^0}{\varepsilon} \xrightarrow{\varepsilon\downarrow 0} \tilde{m} \ \text{ in } C(I,X), I = [0,T]$$

where $\tilde{m}$ is the mild solution of

$$(\tilde{m}) \qquad \frac{d\tilde{m}}{dt} = A\tilde{m} + R(\nu_t(f) - \nu_t^0(f)),\quad \tilde{m}(0) = 0.$$

Similarly one proves that

$$\frac{P^\varepsilon - P^0}{\varepsilon} \xrightarrow{\tau_{w_0}} \tilde{P} \ \text{ as } \varepsilon \downarrow 0 \ \text{ in } C(I,\mathcal{L}_w^+(X))$$

where $\tilde{P}$ satisfies

$$(\tilde{P}) \qquad \begin{cases} \dfrac{d}{dt}(\tilde{P}\xi,\eta) = (A^*\xi, \tilde{P}\eta) + (\tilde{P}\xi, A^*\eta) + \langle Q(\nu - \nu^0)(\sigma^*)R^*\xi, \nu^0(\sigma^*)R^*\eta\rangle_{W,W^*} \\ \tilde{P}(0) = 0. \end{cases}$$

$$(\tilde{J}) \qquad J'(\nu^0, \nu - \nu^0) = 1/2 \int_0^T \{tr(\tilde{P}M_c) + \langle N_c(m^0 - m^*), \tilde{m}\rangle_{X^*,X}\} dt.$$

Using the variational equations $(\tilde{m}), (\tilde{P})$ and $(\tilde{J})$ and introducing the adjoint equations one obtains the result.

Remark 7

If one prefers to work with $\varphi = \pi x$, we may introduce

$$Y = [D(A^*)] \equiv (D(A^*), \text{ graph norm})$$

$$Y^* = [D(A^*)]^* \equiv \text{ dual of } Y.$$

Then

$$\mu_t^\varphi = \mu_t^x \pi^{-1} \quad \text{and}$$

$$J(\nu) = 1/2 \int_0^T \{tr(P_\varphi(t)M_c) + \langle N_c(m_\varphi - m^*), m_\varphi - m^*\rangle_{Y^*,Y}\} dt$$

where

$$M_c \in C(I, \mathcal{L}_n^+(Y^*, Y)),$$

$$N_c \in C(I, \mathcal{L}^+(Y^*, Y)),$$

$$\int_{Y^*} (\varphi, \xi)\mu_t^\varphi(d\varphi) = \int_X (x, \pi^*\xi)\mu_t^x(dx), \quad \xi \in Y$$

$$= (m_t^x, \pi^*\xi)_{X,X^*},$$

$$\int_{Y^*} (\varphi, \xi)^2 \mu_t^\varphi(d\varphi) = \int_X (x, \pi^*\xi)^2 \mu_t^x(dx)$$

$$= (P_x(t)\pi^*\xi, \pi^*\xi)_{X,X^*}$$

$$= (\pi P_x(t)\pi^*\xi, \xi)_{Y^*,Y}.$$

Remark 8

The results presented here also hold under the following (weaker) assumptions:

(i) X is a reflexive Banach space, or X is not necessarily reflexive, but $\overline{D(A^*)} = X^*$

and

(ii) E is a Banach space, W_0 and W are separable Banach spaces.

PRACTICAL REALIZATION OF RELAXED CONTROLS

The controls $\{\nu_t\}$ can be approximated by sums of Dirac measures as

$$\nu_t(du) \cong \sum_{i=1}^N \alpha_i(t)\delta_{u_i}(du), \quad N < \infty$$

where $\{u_i\} \in U$.

In that case the inequality (29) takes the form

$$\int_0^T \{\sum_i(\alpha_i(t)-\alpha_i^0(t))\Big[\sum_{j=1}^n \alpha_j^0(t)tr[\sigma(u_i)Q\sigma^*(u_j)R^*SR] + \langle f(u_i), R^*p(t)\rangle_{E,E^*}\Big]\}dt \geq 0 \tag{30}$$

for α's satisfying

$$\sum \alpha_i(t) = 1, \quad 0 \leq \alpha_i \leq 1.$$

Let α^n denote the value of α at the nth stage and ν^n be the corresponding relaxed control. Using this ν^n in (27) and (28) one obtains m^n, P^n, p^n and S^n. Substituting these in (30) and denoting the expression within the bracket $\Big[\ \Big]$ by β_i^n one can obtain the update for α as

$$\alpha_i^{n+1} = \alpha_i^n - \varepsilon\beta_i^n, \ 1 \leq i \leq N, \quad \text{for } \varepsilon > 0,$$

giving

$$J(\alpha^{n+1}) = J(\alpha^n) - \varepsilon\|\beta^n\|^2.$$

This way one can obtain an approximate realization of optimal relaxed controls.

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