SOME REGULARITY PROPERTIES FOR THE WAVE EQUATION RELATED TO AN EXACT CONTROLLABILITY PROBLEM

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1.INTRODUCTION.

If we consider the wave equation with Dirichlet boundary conditions, we know from a result of Zuazua using J.L.Lions' Hilbert Uniqueness Method (cf.[5]) that one can solve the problem of exact controllability when the control is distributed and acts on a neighborhood ω of a part Γ_0 of the boundary, if Γ_0 satisfies some suitable geometrical conditions and if the time of integration T is sufficiently large ($T > T_0$ where T_0 depends only on the diameter of Ω). We want to study here what happens when the set ω shrinks to Γ_0. Do we obtain, at the limit, an exact controllability problem where the control acts on the boundary and which problem? We will give a precise answer to these questions in section 2 without getting into the details of the proofs. In fact, these proofs require some new regularity results for solutions of the wave equation, which may be of more general interest, and which will be presented independently in section 3, with the details of the proof for one of them. These results have been announced in [3]

2.THE EXACT CONTROLLABILITY PROBLEM

We first consider a problem of exact controllability associated with the wave equation where the control is distributed and acts on a neighborhood of a part Γ_0 of the boundary satisfying suitable geometrical conditions.

More precisely, let Ω be a bounded regular open set in R^N with boundary Γ, $\nu(x)$ the unit exterior normal vector at a point $x \in \Gamma$ and let Γ_0 be a subset of Γ. We will sometimes assume that Γ_0 satisfies the following condition

$$\exists x_0 \in R^N, \Gamma_0 = \{x \in \Gamma, (x - x_0).\nu(x) > 0\}. \tag{2.1}$$

For $\epsilon > 0$, we consider the subset ω_ϵ of Ω defined by

$$\omega_\epsilon = \Big(\bigcup_{x\in\Gamma_0} B(x,\epsilon)\Big) \cap \Omega$$

and the wave equation (we denote by ' the time derivative and by χ_{ω_ϵ} the indicator function of ω_ϵ)

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$$\begin{cases} y'' - \Delta y = v.\chi_{\omega_\epsilon} \text{ in } Q_T = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \Gamma \times (0,T), \\ y(0) = y_0; y'(0) = y_1, \end{cases} \tag{2.2}$$

where $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $v \in L^2(Q_T)$ is the control which is distributed and acts only on ω_ϵ. Equation (2.2) has a unique solution $y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.

For fixed ϵ , the problem of exact controllability consists in finding a control $v \in L^2(Q_T)$ such that

$$y(T) = 0 \text{ and } y'(T) = 0,$$

and if possible such a control which, in addition minimizes $||v||_{L^2(Q_T)}$ among admissible controls.

Zuazua has shown, using J.L.Lions' H.U.M. (cf. [5]) that if $T > T_0$, where T_0 depends only on the diameter of Ω and not on ϵ, and if Γ_0 satisfies (2.1), there exists such a control. In fact, he finds a control v_ϵ given by the following optimality system.

Optimality system.

There exist $\tilde{\varphi}_0^\epsilon \in L^2(\Omega)$ and $\tilde{\varphi}_1^\epsilon \in H^{-1}(\Omega)$ such that, if $\tilde{\varphi}^\epsilon$ and ψ^ϵ are solutions of

$$\begin{cases} \tilde{\varphi}^{\epsilon''} - \Delta \tilde{\varphi}^\epsilon = 0 \text{ in } Q_T, \\ \tilde{\varphi}^\epsilon = 0 \text{ on } \Sigma, \\ \tilde{\varphi}^\epsilon(0) = \tilde{\varphi}_0^\epsilon; \tilde{\varphi}^{\epsilon\prime}(0) = \tilde{\varphi}_1^\epsilon, \end{cases} \tag{2.3}$$

$$\begin{cases} \psi^{\epsilon''} - \Delta \psi^\epsilon = \tilde{\varphi}^\epsilon.\chi_{\omega_\epsilon} \text{ in } Q_T, \\ \psi^\epsilon = 0 \text{ on } \Sigma, \\ \psi^\epsilon(T) = 0; \psi^{\epsilon\prime}(T) = 0, \end{cases} \tag{2.4}$$

then

$$\psi^\epsilon(0) = y_0; \psi^{\epsilon\prime}(0) = y_1. \tag{2.5}$$

One can immediately see that by setting $v_\epsilon = \tilde{\varphi}^\epsilon$, the corresponding solution of (2.2) noted y_ϵ is equal to ψ^ϵ and satisfies, because of (2.4) , $y_\epsilon(T) = 0$ and $y_\epsilon'(T) = 0$. This gives a solution to the exact controllability problem. In the present work, we are interested in studying what happens when ϵ tends to 0.

Estimates.

First of all, one has of course to find estimates on the functions $\tilde{\varphi}_0^\epsilon$ and $\tilde{\varphi}_1^\epsilon$. They are given by the following lemmas.

Lemma 2.1.(C.Fabre [1])

There exists a constant C independent of ϵ such that

$$[||\tilde{\varphi}_0^\epsilon||_{L^2}^2 + ||\tilde{\varphi}_1^\epsilon||_{H^{-1}}^2]^{\frac{1}{2}} = \frac{C}{\epsilon^3}. \tag{2.6}$$

Proof of Lemma 2.1 is very difficult and interesting. It requires the results on the wave equation which will be given in section 3.

Lemma 2.2.
There exists a constant C independent of ϵ such that

$$\int_0^T \int_{\omega_\epsilon} |\tilde{\varphi}^\epsilon|^2 dxdt \leq \frac{C}{\epsilon^3}. \tag{2.7}$$

This result is simple to obtain by multiplying (2.4) by $\tilde{\varphi}^\epsilon$, integrating by parts and using Lemma 2.1.

These estimates suggest a change of notations. Let us write

$$\varphi_0^\epsilon = \epsilon^3 \tilde{\varphi}_0^\epsilon \quad ; \quad \varphi_1^\epsilon = \epsilon^3 \tilde{\varphi}_1^\epsilon \quad ; \quad \varphi^\epsilon = \epsilon^3 \tilde{\varphi}^\epsilon.$$

The optimality system becomes

$$\begin{cases} \varphi^{\epsilon\prime\prime} - \Delta\varphi^\epsilon = 0 \text{ in } Q_T, \\ \varphi^\epsilon = 0 \text{ on } \Sigma, \\ \varphi^\epsilon(0) = \varphi_0^\epsilon ; \varphi^{\epsilon\prime}(0) = \varphi_1^\epsilon, \end{cases} \tag{2.8}$$

$$\begin{cases} \psi^{\epsilon\prime\prime} - \Delta\psi^\epsilon = \frac{1}{\epsilon^3}\varphi^\epsilon . \chi_{\omega_\epsilon} \text{ in } Q_T, \\ \psi^\epsilon = 0 \text{ on } \Sigma, \\ \psi^\epsilon(T) = 0 ; \psi^{\epsilon\prime}(T) = 0, \end{cases} \tag{2.9}$$

with

$$\psi^\epsilon(0) = y_0 ; \psi^{\epsilon\prime}(0) = y_1,$$

and we have the estimates

$$||\varphi_0^\epsilon||_{L^2}^2 + ||\varphi_1^\epsilon||_{H^{-1}}^2 \leq C; \tag{2.10}$$

$$\frac{1}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} |\varphi^\epsilon|^2 dxdt \leq C. \tag{2.11}$$

We now want to study the limits of (2.8), (2.9) with the estimates (2.10), (2.11). To study the limit of (2.8) is easy. One can extract subsequences such that if ϵ tends to 0,

$$\varphi_0^\epsilon \rightarrow \varphi_0 \text{ in } L^2(\Omega) \text{ weak }, \varphi_1^\epsilon \rightarrow \varphi_1 \text{ in } H^{-1}(\Omega) \text{ weak },$$

and consequently

$$\varphi^\epsilon \rightarrow \varphi \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak}^* ,$$

with

$$\begin{cases} \varphi'' - \Delta\varphi = 0 \text{ in } Q_T, \\ \varphi = 0 \text{ on } \Sigma, \\ \varphi(0) = \varphi_0 ; \varphi'(0) = \varphi_1. \end{cases} \tag{2.12}$$

The main difficulty consists in studying the limit of (2.9) which is a wave equation with a singular right hand side. We can prove the following result.

Theorem 2.1.

If φ^ϵ is solution of the wave equation (2.8) and $\varphi^\epsilon \to \varphi$ in $L^\infty(0,T;L^2(\Omega))$ weak and if furthermore $1/\epsilon^3 \int_0^T \int_{\omega_\epsilon} |\varphi^\epsilon|^2 dxdt \le C$, then ψ^ϵ converges to ψ in $L^\infty(0,T;L^2(\Omega))$ weak*, $\psi^\epsilon(0)$ converges to $\psi(0)$ in $L^2(\Omega)$ weak, $\psi^{\epsilon\prime}(0)$ converges to $\psi'(0)$ in $H^{-1}(\Omega)$ weak, where ψ satisfies*

$$\begin{cases} \psi'' - \Delta\psi = 0 \ \textit{in} \ Q_T, \\ \psi_{/\Sigma-\Sigma_0} = 0; \psi_{/\Sigma_0} = -\dfrac{1}{3}\dfrac{\partial\varphi}{\partial\nu} \ \textit{where} \ \Sigma_0 = \Gamma_0 \times (0,T), \\ \psi(T) = 0; \psi'(T) = 0. \end{cases} \tag{2.13}$$

and we have

$$\psi(0) = y_0; \psi'(0) = y_1. \tag{2.14}$$

Proof of Theorem 2.1 is long. The outline is given in [6] and the details are exposed in [1] and [2]. It requires regularity results on the wave equation which are interesting and sharp and which will be given in section 3. Notice that the sense of $\partial\varphi/\partial\nu$ is not clear yet but it will be proved that $\partial\varphi/\partial\nu \in L^2(\Sigma_0)$. The result of Theorem 2.1 together with (2.12) says in particular that when ϵ tends to 0, the solution given by H.U.M. of the exact controllability problem where the control acts on ω_ϵ converges to the solution, given by H.U.M. again, of a problem of exact controllability with boundary control acting on the Dirichlet data on Σ_0. Notice that the functions ψ^ϵ vanish on the whole boundary Σ and they converge to a function ψ which, a priori, does not vanish on Σ_0. Then the convergence has to occur in weak spaces.

Remark 2.1.

One can study directly other equations with singular right hand sides, corresponding for example to the heat operator, the Schrödinger operator, the vibrating plates operator,... This leads to interesting questions which may have no relation with exact controllability. Some partial answers are given in [2].

3.REGULARITY RESULTS FOR THE WAVE EQUATION.

We are going to consider a situation which is analogous to the one encountered in the previous section, even if one could have taken into account more general situations.

We begin with a result giving the regularity "at the limit", for a sequence of very weak solutions of the wave equation satisfying estimates like (2.10), (2.11).

Let us consider a sequence of solutions φ^ϵ of the wave equation

$$\begin{cases} \varphi^{\epsilon\prime\prime} - \Delta\varphi^\epsilon = 0 \text{ in } Q_T, \\ \varphi^\epsilon = 0 \text{ on } \Sigma, \\ \varphi^\epsilon(0) = \varphi_0^\epsilon; \varphi^{\epsilon\prime}(0) = \varphi_1^\epsilon, \end{cases} \tag{3.1}$$

where

$$\varphi_0^\epsilon \to \varphi_0 \text{ in } L^2(\Omega) \text{ weak }, \varphi_1^\epsilon \to \varphi_1 \text{ in } H^{-1}(\Omega) \text{ weak },$$

$$\varphi^\epsilon \to \varphi \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak* }.$$

Theorem 3.1.

Under the above hypotheses (without assuming (2.1)), if in addition

$$\frac{1}{\epsilon^3}\int_0^T\int_{\omega_\epsilon}|\varphi^\epsilon|^2 dxdt \leq C,$$

then $\partial\varphi/\partial\nu \in L^2(\Sigma_0)$.

Therefore, if Γ_0 *satisfies (2.1) and* $T \geq T_0$*, this implies that* $\varphi_0 \in H_0^1(\Omega)$, $\varphi_1 \in L^2(\Omega)$ *and* φ *is a solution of finite energy of the wave equation.*

Proof.

We know that φ is a very weak solution of the wave equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 \text{ in } Q_T, \\ \varphi = 0 \text{ on } \Sigma, \\ \varphi(0) = \varphi_0; \varphi'(0) = \varphi_1. \end{cases}$$

and $\varphi \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$.

Let us define as in [5] the solutions Φ_0^ϵ and Φ_0 of

$$\Delta\Phi_0^\epsilon = \varphi_1^\epsilon, \Phi_0^\epsilon \in H_0^1(\Omega); \Delta\Phi_0 = \varphi_1, \Phi_0 \in H_0^1(\Omega),$$

and

$$\Phi^\epsilon(x,t) = \int_0^t \varphi^\epsilon(x,s)ds + \Phi_0^\epsilon(x); \Phi(x,t) = \int_0^t \varphi(x,s)ds + \Phi_0(x).$$

Then

$$\begin{cases} \Phi^{\epsilon\prime\prime} - \Delta\Phi^\epsilon = 0 \text{ in } Q_T, \\ \Phi^\epsilon = 0 \text{ on } \Sigma, \\ \Phi^\epsilon(0) = \Phi_0^\epsilon; \Phi^{\epsilon\prime}(0) = \varphi_0^\epsilon, \end{cases}$$

and

$$\begin{cases} \Phi'' - \Delta\Phi = 0 \text{ in } Q_T, \\ \Phi = 0 \text{ on } \Sigma, \\ \Phi(0) = \Phi_0; \Phi'(0) = \varphi_0, \end{cases}$$

Therefore, Φ^ϵ and Φ are solutions of finite energy and using a regularity result of J.L.Lions [5], we know that

$$\frac{\partial\Phi^\epsilon}{\partial\nu} \in L^2(\Sigma) \ , \ \frac{\partial\Phi}{\partial\nu} \in L^2(\Sigma) \text{ and } \frac{\partial\Phi^\epsilon}{\partial\nu} \to \frac{\partial\Phi}{\partial\nu} \text{ in } L^2(\Sigma) \text{ weak }.$$

This shows that $\partial\varphi^\epsilon/\partial\nu$ converges to $\partial\varphi/\partial\nu$ in $H^{-1}(0,T;L^2(\Gamma))$ weak. If $u \in \mathcal{D}(\Sigma_0)$ and if $<,>$ denotes the duality $\mathcal{D}'(\Sigma_0), \mathcal{D}(\Sigma_0)$, we have

$$\lim_{\epsilon \to 0} < \frac{\partial \varphi^\epsilon}{\partial \nu}, u > = < \frac{\partial \varphi}{\partial \nu}, u >,$$

and in order to prove the theorem, we want to show that

$$\exists C > 0, \forall u \in \mathcal{D}(\Sigma_0), | < \frac{\partial \varphi}{\partial \nu}, u > | \leq C ||u||_{L^2(\Sigma_0)}.$$

Using a covering of a neighborhood of Γ_0 and a partition of unity, we can localize the problem and assume that we work in a neighborhood U of a point of Γ_0. Then we can define a (local) change of coordinates

$$x = y - z\nu(y), y \in \Gamma_0, z \in R^+.$$

(y is the tangential coordinate and z the normal coordinate). The mapping

$$J^{-1} : x \to (y, z)$$

is a C^2 diffeomorphism, and if we write for a function v(x,t)

$$\hat{v}(z, y, t) = v(x, t),$$

then

$$\frac{\partial \hat{v}}{\partial z}(0, y, t) = -\frac{\partial v}{\partial \nu}(y, t).$$

If $u \in \mathcal{D}(\Sigma_0 \cap (U \times (0, T))$, let us define a regular function w with compact support in time, such that

$$w = 0 \text{ on } \Sigma_0 \ ; \ \frac{\partial w}{\partial \nu} = u \text{ on } \Sigma_0.$$

This is always possible by taking, for example

$$\hat{w}(z, y, t) = -z.u(y, t).$$

We then have

$$(\frac{1}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} |w|^2 dx dt)^{\frac{1}{2}} \leq M ||u||_{L^2(\Sigma_0)},$$

and of course

$$w' = 0 \text{ on } \Sigma_0 \ ; \ \frac{\partial w'}{\partial \nu} = u' \text{ on } \Sigma_0.$$

We can now write

$$
\begin{aligned}
< \frac{\partial \varphi^\epsilon}{\partial \nu}, u > &=< (\frac{\partial \Phi^\epsilon}{\partial \nu})', u >= - < \frac{\partial \Phi^\epsilon}{\partial \nu}, u' >= - \int_0^T \int_\Gamma \frac{\partial \Phi^\epsilon}{\partial \nu} u' dydt \\
&= -\frac{3}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} \Phi^\epsilon w' dxdt + (\frac{3}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} \Phi^\epsilon w' dxdt - \int_0^T \int_\Gamma \frac{\partial \Phi^\epsilon}{\partial \nu} u' dydt) \\
&= \frac{3}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} \varphi^\epsilon w dxdt + (\frac{3}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} \Phi^\epsilon w' dxdt - \int_0^T \int_\Gamma \frac{\partial \Phi^\epsilon}{\partial \nu} u' dydt) \\
&= A_\epsilon + B_\epsilon,
\end{aligned}
$$

where

$$
A_\epsilon = \frac{3}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} \varphi^\epsilon w dxdt
$$

$$
B_\epsilon = (\frac{3}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} \Phi^\epsilon w' dxdt - \int_0^T \int_\Gamma \frac{\partial \Phi^\epsilon}{\partial \nu} u' dydt).
$$

Using Cauchy-Schwarz inequality, we get

$$
|A_\epsilon| \le 3(\frac{1}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} |\varphi^\epsilon|^2 dxdt)^{\frac{1}{2}} (\frac{1}{\epsilon^3} \int_0^T \int_{\omega_\epsilon} |w|^2 dxdt)^{\frac{1}{2}} \le 3CM||u||_{L^2(\Sigma_0)}.
$$

Then

$$
\limsup_{\epsilon \to 0} |A_\epsilon| \le 3CM||u||_{L^2(\Sigma_0)}.
$$

We will now show that for a fixed u , $B_\epsilon \to 0$ if $\epsilon \to 0$. This proof is very technical.

$$
\begin{aligned}
B_\epsilon =& (\frac{3}{\epsilon^3} \int_0^T \int_0^\epsilon \int_{\Gamma_0} \hat{\Phi}^\epsilon(z,y,t) z u'(y,t) |JacJ(o,y)| dzdydt \\
&- \int_0^T \int_{\Gamma_0} \frac{\partial \hat{\Phi}^\epsilon}{\partial z}(0,y,t) u'(y,t) dydt) \\
&+ \frac{3}{\epsilon^3} \int_0^\epsilon z (\int_0^z < \frac{\partial \hat{\Phi}^\epsilon}{\partial z}(\xi,y,t), u'(y,t) \int_0^z \frac{\partial}{\partial z} |JacJ(s,y)| ds > d\xi) dz.
\end{aligned}
$$

One can show that $|JacJ(0,y)| = 1$ and if

$$
\begin{aligned}
C_\epsilon =& (\frac{3}{\epsilon^3} \int_0^T \int_0^\epsilon \int_{\Gamma_0} \hat{\Phi}^\epsilon(z,y,t) z u'(y,t) |JacJ(o,y)| dzdydt \\
&- \int_0^T \int_{\Gamma_0} \frac{\partial \hat{\Phi}^\epsilon}{\partial z}(0,y,t) u'(y,t) dydt),
\end{aligned}
$$

then

$$C_\epsilon = \frac{3}{\epsilon^3}\int_0^\epsilon z^2(\frac{1}{z}\int_0^z < \frac{\partial\hat{\Phi}^\epsilon}{\partial z}(\xi,y,t) - \frac{\partial\hat{\Phi}^\epsilon}{\partial z}(0,y,t), u'(y,t) > d\xi)dz.$$

We can integrate in time again and define

$$\theta^\epsilon(x,t) = \int_0^t \Phi^\epsilon(x,s)ds + \theta_0^\epsilon,$$

where

$$\Delta\theta_0^\epsilon = \varphi_0^\epsilon, \theta_0^\epsilon \in H_0^1(\Omega).$$

Then θ^ϵ is bounded in $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, which shows that $\partial\hat{\theta}^\epsilon/\partial z$ is bounded in $H^1(0,\epsilon_0;L^2(\Sigma_0))$, and $\xi \to \partial\hat{\theta}^\epsilon/\partial z\ (\xi,y,t)$ is continuous at $\xi = 0$, uniformly in ϵ , with values in $L^2(\Sigma_0)$. As $\partial\hat{\Phi}^\epsilon/\partial z = (\partial\hat{\theta}^\epsilon/\partial z\ )'$, we see that $\xi \to \partial\hat{\Phi}^\epsilon/\partial z\ (\xi,y,t)$ is continuous at $\xi = 0$, uniformly in ϵ , with values in $H^{-1}(\Sigma_0)$, and this implies that $C_\epsilon \to 0$ if $\epsilon \to 0$.

The same argument also shows that $\xi \to \partial\hat{\Phi}^\epsilon/\partial z\ (\xi,y,t)$ is bounded on $[0,\epsilon_0]$, uniformly in ϵ , with values in $H^{-1}(0,T;L^2(\Gamma_0))$ and on the other hand, one can show, because of the regularity of Ω that $\partial/\partial z|JacJ(s,y)|$ is bounded on $[0,\epsilon_0]$ with values in $L^\infty(\Gamma_0)$. Therefore

$$| < \frac{\partial\hat{\Phi}^\epsilon}{\partial z}(\xi,y,t), u'(y,t)\int_0^z \frac{\partial}{\partial z}|JacJ(s,y)|ds > | \le Mz$$

and if

$$D_\epsilon = \frac{3}{\epsilon^3}\int_0^\epsilon z(\int_0^z < \frac{\partial\hat{\Phi}^\epsilon}{\partial z}(\xi,y,t), u'(y,t)\int_0^z \frac{\partial}{\partial z}|JacJ(s,y)|ds > d\xi)dz$$

we have

$$|D_\epsilon| \le \frac{3}{\epsilon^3}\int_0^\epsilon \frac{M}{2}z^3 dz \le \frac{3M\epsilon}{8} \to 0 \text{ if } \epsilon \to 0.$$

As $B_\epsilon = C_\epsilon - D_\epsilon$, this shows that for fixed u in $\mathcal{D}(\Sigma_0)$, $B_\epsilon \to 0$ if $\epsilon \to 0$, and the proof of Theorem 3.1 is complete.

Remark 3.1.

The regularity result given in Theorem 3.1 is only valid for the limit. Each function φ^ϵ cannot, in general, be as regular as the limit φ. The precise behavior in ϵ of the L^2-norm of φ^ϵ in $\omega_\epsilon \times (0,T)$ gives an additional information about the regularity of the limit φ.

We next state a result giving the precise behavior, near the boundary, of the solutions of finite energy of the wave equation, and which appears as a reciproqual of Theorem 3.1, even if it is not really the case.

For $(f,u_0,u_1) \in L^1(0,T;L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ let u be the solution of

$$\begin{cases} u'' - \Delta u = f \text{ in } Q_T, \\ u = 0 \text{ on } \Sigma, \\ u(0) = u_0; u'(0) = u_1; \end{cases} \tag{3.2}$$

Then $u \in C(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega))$ (u has finite energy) and the mapping $(f, u_0, u_1) \to u$ is linear and continuous.

Theorem 3.2.

Let u be the solution (of finite energy) of the wave equation (3.2). Then (without assuming (2.1)) there exists a constant $C > 0$ independent of ϵ such that

$$\frac{1}{\epsilon^3}\int_0^T \int_{\omega_\epsilon} |u|^2 dxdt \le C[||f||^2_{L^1(0,T;L^2(\Omega))} + ||u_0||^2_{H_0^1(\Omega)} + ||u_1||^2_{L^2(\Omega)}] = CE, \tag{3.3}$$

where E is the "energy".

Idea of the proof.

We just give here a sketch of the proof. The complete details will be given in [4]. We use an extension of the Rellich multiplier method. As Ω is regular, we know that there exist functions $h^\epsilon \in W^{2,\infty}(\overline{\Omega}; R^N)$ such that $h^\epsilon_{/\Gamma_0} = \nu$. Multiplying (3.2) by $2h^\epsilon.\nabla u - (div h^\epsilon).u$ and integrating by parts gives

$$2\int_0^T \int_\Omega \frac{\partial h_k^\epsilon}{\partial x_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k} dxdt = -\int_0^T \int_\Omega u \frac{\partial u}{\partial x_j}\frac{\partial^2 h_k^\epsilon}{\partial x_j \partial x_k} dxdt + R_1, \tag{3.4}$$

where the terms in R_1 will be easily bounded in terms of the energy, thanks to the regularity result of J.L.Lions [5], saying that $\partial u/\partial \nu \in L^2(\Sigma)$ and that its norm is bounded by the energy. As in the proof of Theorem 3.1, we first localize the problem, then use the change of variables

$$x = y - z\nu(y), y \in \Gamma, z \in R^+, y = p(x).$$

Then, we take

$$h^\epsilon(x) = \begin{cases} \rho^\epsilon(z).\nu(y) \text{ if } x \in \omega_\epsilon, \\ 0 \text{ if } x \in \Omega - \omega_\epsilon, \end{cases}$$

where $\rho^\epsilon \in W^{2,\infty}(0,\epsilon)$ is a decreasing function such that

$$\begin{cases} \rho^\epsilon(\epsilon) = 0; \rho^{\epsilon\prime}(\epsilon) = 0, \\ ||\rho^\epsilon||_{L^\infty} = O(1); ||\rho^{\epsilon\prime}||_{L^\infty} = O(\frac{1}{\epsilon}). \end{cases}$$

From (3.4) we obtain

$$2\int_0^T \int_\Gamma \int_0^\epsilon |\frac{\partial \rho^\epsilon}{\partial z}|(\frac{\partial \hat{u}}{\partial z})^2 |Jac(J)| dzdydt = \int_0^T \int_\Gamma \int_0^\epsilon \frac{\partial^2 \rho^\epsilon}{\partial z^2}\hat{u}\frac{\partial \hat{u}}{\partial z}|Jac(J)| dzdydt + R_2,$$

where R_2 plays the same role as R_1.

Setting

$$G(\epsilon)=\frac{1}{\epsilon}\int_0^T\int_\Gamma\int_0^\epsilon(\frac{\partial\hat{u}}{\partial z})^2|Jac(J)|dzdydt=\frac{1}{\epsilon}\int_0^T\int_{\omega_\epsilon}|\nabla u(x,t).\nu(p(x))|^2dxdt \text{ if } \epsilon>0,$$

$$G(0)=\int_0^T\int_\Gamma|\frac{\partial u}{\partial\nu}|^2dydt,$$

G is continuous at 0 and we can prove, by choosing suitable functions ρ^ϵ that there exists $\delta>0$ such that

$$\sup_{\epsilon\in[0,\delta]}G(\epsilon)\leq C.E,$$

Where C is independent of ϵ and u. Then, by a simple argument of integration with respect to the normal variable z, this shows the result of Theorem 3.2.

Remark 3.2.

Similar results have recently been obtained by C.Fabre [2] for the Schrödinger equation and the equation of vibrating plates or beams associated with the operator $\partial^2/\partial t^2+\Delta^2$.

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