A Stable Galerkin Reduced Order Model (ROM) for Compressible Flow

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Farhat Research Group (FRG) Seminar
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Outline

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CFD modeling of unsteady 3D flows is expensive!

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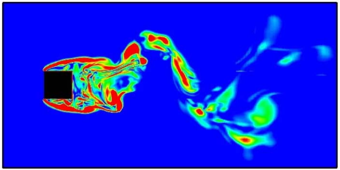

Applications in Fluid Dynamics:

- Predictive modeling across a parameter space (e.g., aeroelastic flutter analysis).
- System modeling for active flow control.
- Long-time unsteady flow analysis, e.g., fatigue of a wind turbine blade under variable wind conditions.

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Motivation for Numerical Analysis of ROMs

Use of ROMs in predictive applications raises questions about their stability & convergence.

- Projection ROM approach is an alternative discretization of the governing PDEs.
- Desired numerical properties of a ROM discretization:
  - **Consistency** (with continuous PDEs):
  - **Stability**:
  - **Convergence**:

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This talk focuses on how to construct a Galerkin ROM that is **stable** *a priori*

Outline

Step 1: Constructing the Modes

High-Fidelity CFD Simulations:

- Snapshot 1
- Snapshot 2
- ⋮
- Snapshot N

Step 1 →

Fluid Modal Decomposition (POD):

$$\boxed{\mathbf{u} = \sum_k a_k(t)\boldsymbol{\phi}_k(\mathbf{x})}$$

Step 2 →

Galerkin Projection of Fluid PDEs:

$$\boxed{\left(\phi_j, \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u})\right) = 0}$$

"Small" ROM ODE System:

$$\boxed{\dot{a}_k = f(a_1, \dots, a_M)}$$

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Proper Orthogonal Decomposition (POD), a.k.a. "Method of Snapshots"

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$$\max_{\phi \in H(\Omega)} \frac{\langle(\mathbf{u}, \phi)^2\rangle}{||\phi||^2} \tag{1}$$

where

$(\cdot, \cdot) \equiv$ inner product
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$$\mathbf{R}\phi \equiv \langle \mathbf{u}^k(\mathbf{u}^k, \phi) \rangle$$

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$$\mathbf{R} \equiv \langle \mathbf{u}^k(\mathbf{u}^k,\boldsymbol{\phi})\rangle$$

Solution to (1) is the set of M eigenfunctions $\{\boldsymbol{\phi}_i\}_{i=1}^M$ corresponding to the M largest eigenvalues $\lambda_1 \geq \cdots \geq \lambda_M$ of $\mathbf{R}$

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Properties of the POD Basis

- POD basis $\{\boldsymbol{\phi}_i\}_{i=1}^M$ is **orthonormal**: $(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j) = \delta_{ij}$.
- Average **energy** of projection of the snapshot ensemble onto the i^{th} mode is given by:

$$\langle (\mathbf{u}^k, \boldsymbol{\phi}_i)^2 \rangle = \lambda_i$$

$$\Rightarrow \text{energy of set } \{\boldsymbol{\phi}_i\}_{i=1}^M = \sum_{j=1}^M \lambda_j$$

- Truncated POD basis $\{\boldsymbol{\phi}_i\}_{i=1}^M$ describes more energy (on average) of the ensemble than any other linear basis of the same dimension.
- Given $M << N$ modes, ROM solution can be represented as a linear combination of these modes:

$$\underbrace{\mathbf{u}_M(\mathbf{x}, t)}_{\text{ROM solution}} = \sum_{i=1}^M \quad a_i(t) \quad \boldsymbol{\phi}_i(\mathbf{x}) \tag{2}$$

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Step 2: Galerkin Projection

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Galerkin Projection of (*Continuous*!) Equations

Governing System of PDEs:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{L}\mathbf{u} + \mathcal{N}_2(\mathbf{u}, \mathbf{u}) + \mathcal{N}_3(\mathbf{u}, \mathbf{u}, \mathbf{u}) \tag{3}$$

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- **Step 2.1:** Project (3) onto the modes ϕ_j in inner product $(\cdot, \cdot)$

$$\left(\phi_j, \frac{\partial \mathbf{u}}{\partial t}\right) = (\phi_j, \mathcal{L}\mathbf{u}) + (\phi_j, \mathcal{N}_2(\mathbf{u}, \mathbf{u})) + (\phi_j, \mathcal{N}_3(\mathbf{u}, \mathbf{u}, \mathbf{u})) \quad (4)$$

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- **Step 2.2:** Substitute the modal decomposition $\mathbf{u}_M = \sum_{k=1}^{M} a_k(t)\phi_k(\mathbf{x}) \rightarrow \mathbf{u}$ into (4)

$$\dot{a}_k(t) = \sum_{l=1}^{M} a_l(\phi_k, \mathcal{L}(\phi_l)) + \sum_{l,m=1}^{M} a_l a_m (\phi_k, \mathcal{N}_2(\phi_l,\phi_m)) + \sum_{l,m,n=1}^{M} a_l a_m a_n (\phi_k, \mathcal{N}_3(\phi_l,\phi_m,\phi_n)) \tag{5}$$

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Continuous vs. Discrete Projection Approach

DISCRETE APPROACH

Governing Equations
$\mathbf{u}_t = \mathcal{L}u$

$\downarrow$

CFD Model
$\dot{\mathbf{u}}_N = \mathbf{A}_N \mathbf{u}_N$

$\downarrow$

Discrete Modal Basis Φ

$\downarrow$

Projection of CFD Model (Matrix Operation)

$\downarrow$

ROM
$\dot{\mathbf{a}} = \Phi^T \mathbf{A}_N \Phi \mathbf{a}$

CONTINUOUS APPROACH

Governing Equations
$\mathbf{u}_t = \mathcal{L}u$

$\downarrow$

CFD Model
$\dot{\mathbf{u}}_N = \mathbf{A}_N \mathbf{u}_N$

$\downarrow$

Continuous Modal Basis* $\phi_j(\mathbf{x})$

$\downarrow$

Projection of Governing Equations (Numerical Integration)

$\downarrow$

ROM
$\dot{a}_j = (\phi_j, \mathcal{L}\phi_k) a_k$

*Continuous functions space is defined using finite elements.

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Analyzed with the **Energy Method**:
uses an equation for the evolution of numerical solution “energy” to determine stability

$$||u_N(\mathbf{x},t)||_E \equiv \left\{ \begin{array}{c} \text{energy of } u_N \text{ in norm } ||\cdot||_E \\ \text{induced by inner product } (\cdot,\cdot)_E \end{array} \right\}$$

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$$||u_N(\mathbf{x},t)||_E \overset{?}{\leq} e^{\beta t}||u_N(\mathbf{x},0)||_E, \qquad \beta \in \mathbb{R}$$

3D Linearized Compressible Euler Equations

- Useful for aero-elasticity, aero-acoustics, flow instability analysis.

Linearization of Full Compressible Euler Equations

$$\mathbf{q}^T(\mathbf{x},t) \equiv \begin{pmatrix} u_1 & u_2 & u_3 & \zeta & p \end{pmatrix} \equiv \underbrace{\bar{\mathbf{q}}^T(\mathbf{x})}_{\text{mean}} + \underbrace{\mathbf{q}'^T(\mathbf{x},t)}_{\text{fluctuation}} \in \mathbb{R}^5$$

$$\Rightarrow \boxed{\frac{\partial \mathbf{q}'}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{q}'}{\partial x_i} + \mathbf{C}\mathbf{q}' = \mathbf{0}} \tag{6}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} \bar{u}_1 & 0 & 0 & 0 & \bar{\zeta} \\ 0 & \bar{u}_1 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_1 & 0 & 0 \\ -\bar{\zeta} & 0 & 0 & \bar{u}_1 & 0 \\ \gamma\bar{p} & 0 & 0 & 0 & \bar{u}_1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \bar{u}_2 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_2 & 0 & 0 & \bar{\zeta} \\ 0 & 0 & \bar{u}_2 & 0 & 0 \\ 0 & -\bar{\zeta} & 0 & \bar{u}_2 & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{u}_2 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} \bar{u}_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{u}_3 & 0 & \bar{\zeta} \\ 0 & 0 & -\bar{\zeta} & \bar{u}_3 & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{u}_3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \frac{\partial \bar{u}_1}{\partial x_1} & \frac{\partial \bar{u}_1}{\partial x_2} & \frac{\partial \bar{u}_1}{\partial x_3} & \frac{\partial \bar{p}}{\partial x_1} & 0 \\ \frac{\partial \bar{u}_2}{\partial x_1} & \frac{\partial \bar{u}_2}{\partial x_2} & \frac{\partial \bar{u}_2}{\partial x_3} & \frac{\partial \bar{p}}{\partial x_2} & 0 \\ \frac{\partial \bar{u}_3}{\partial x_1} & \frac{\partial \bar{u}_3}{\partial x_2} & \frac{\partial \bar{u}_3}{\partial x_3} & \frac{\partial \bar{p}}{\partial x_3} & 0 \\ \frac{\partial \bar{\zeta}}{\partial x_1} & \frac{\partial \bar{\zeta}}{\partial x_2} & \frac{\partial \bar{\zeta}}{\partial x_3} & -\nabla\cdot\bar{\mathbf{u}} & 0 \\ \frac{\partial \bar{p}}{\partial x_1} & \frac{\partial \bar{p}}{\partial x_2} & \frac{\partial \bar{p}}{\partial x_3} & 0 & \gamma\nabla\cdot\bar{\mathbf{u}} \end{pmatrix}$$

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Symmetrized Compressible Euler Equations & Symmetry Inner Product

Energy stability can be proven following "symmetrization" of the linearized compressible Euler equations.

- Linearized hyperbolic compressible Euler system is "symmetrizable".
- Pre-multiply equations by symmetric positive definite matrix:

$$\mathbf{H} = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ 0 & 0 & 0 & \bar{\rho}\alpha^2 & \frac{1+\alpha^2}{\gamma\bar{p}} \end{pmatrix} \Rightarrow \boxed{\mathbf{H}\frac{\partial \mathbf{q}'}{\partial t} + \mathbf{H}\mathbf{A}_i \frac{\partial \mathbf{q}'}{\partial x_i} + \mathbf{H}\mathbf{C}\mathbf{q}' = \mathbf{0}}$$

- $\mathbf{H}$ is called the "symmetrizer" of the system: $\mathbf{H}\mathbf{A}_i$ are all symmetric.
- Define the "symmetry" inner product and "symmetry" norm:

$$(\mathbf{q}'^{(1)}, \mathbf{q}'^{(2)})_{(\mathbf{H},\Omega)} \equiv \int_\Omega [\mathbf{q}'^{(1)}]^T \mathbf{H}\mathbf{q}'^{(2)} d\Omega, \qquad ||\mathbf{q}'||_{(\mathbf{H},\Omega)} \equiv (\mathbf{q}', \mathbf{q}')_{(\mathbf{H},\Omega)} \tag{7}$$

Stability in the Symmetry Inner Product

$$\begin{aligned}
\frac{d}{dt}||\mathbf{q}'||^2_{(\mathbf{H},\Omega)} &= -\int_\Omega [\mathbf{q}']^T \mathbf{H}\left[\mathbf{A}_i \frac{\partial \mathbf{q}'}{\partial x_i} + \mathbf{C}\mathbf{q}'\right] d\Omega \\
&= -\int_{\partial\Omega} [\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS + \int_\Omega [\mathbf{q}']^T \left(\frac{\partial}{\partial x_i}(\mathbf{H}\mathbf{A}_i) - \mathbf{H}\mathbf{C} - \mathbf{C}^T\mathbf{H}\right)\mathbf{q}' d\Omega \\
&= -\int_{\partial\Omega} [\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS + \int_\Omega [\mathbf{q}']^T \mathbf{H}^{-T/2}\mathbf{B}\mathbf{H}^{T/2}\mathbf{q}' d\Omega \\
&\leq -\int_{\partial\Omega} [\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS + \beta\left(\mathbf{q}',\mathbf{q}'\right)_{(\mathbf{H},\Omega)} \\
&\leq \beta||\mathbf{q}'||^2_{(\mathbf{H},\Omega)} \quad \text{if } \int_{\partial\Omega}[\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS \geq 0 \text{ (well-posed BCs)}
\end{aligned}$$

where β is an upper bound on the eigenvalues of

$$\mathbf{B} \equiv \mathbf{H}^{-T/2}\frac{\partial(\mathbf{H}\mathbf{A}_i)}{\partial x_i}\mathbf{H}^{-1/2} - \mathbf{H}^{1/2}\mathbf{C}\mathbf{H}^{-1/2} - (\mathbf{H}^{1/2}\mathbf{C}\mathbf{H}^{-1/2})^T$$

Stability in the Symmetry Inner Product

$$\boxed{\frac{d}{dt}||\mathbf{q}'||^2_{(\mathbf{H},\Omega)}} = -\int_\Omega [\mathbf{q}']^T\mathbf{H}\left[\mathbf{A}_i\frac{\partial \mathbf{q}'}{\partial x_i} + \mathbf{C}\mathbf{q}'\right] d\Omega$$

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- Exact solutions to the linearized Euler equations satisfy:

$$\boxed{||\mathbf{q}'(\mathbf{x},t)||_{(\mathbf{H},\Omega)} \leq e^{\beta t}||\mathbf{q}'(\mathbf{x},0)||_{(\mathbf{H},\Omega)}}$$

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- It turns out that the Galerkin approximation $\mathbf{q}'_M = \sum_{i=1}^{M} a_k(t)\boldsymbol{\phi}_k(\mathbf{x})$ satisfies the same energy expression as for the continuous equations:

$$\boxed{||\mathbf{q}'_M(\mathbf{x},t)||_{(\mathbf{H},\Omega)} \leq e^{\beta t}||\mathbf{q}'_M(\mathbf{x},\mathbf{0})||_{(\mathbf{H},\Omega)}}$$

i.e., it is stable.

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&= -\int_{\partial\Omega} [\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS + \int_\Omega [\mathbf{q}']^T \mathbf{H}^{-T/2}\mathbf{B}\mathbf{H}^{T/2}\mathbf{q}' d\Omega \\
&\leq -\int_{\partial\Omega} [\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS + \beta\left(\mathbf{q}', \mathbf{q}'\right)_{(\mathbf{H},\Omega)} \\
&\leq \beta||\mathbf{q}'||^2_{(\mathbf{H},\Omega)} \quad \text{if } \int_{\partial\Omega} [\mathbf{q}']^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}' dS \geq 0 \text{ (well-posed BCs)}
\end{aligned}
$$

where β is an upper bound on the eigenvalues of

$$
\mathbf{B} \equiv \mathbf{H}^{-T/2}\frac{\partial(\mathbf{H}\mathbf{A}_i)}{\partial x_i}\mathbf{H}^{-1/2} - \mathbf{H}^{1/2}\mathbf{C}\mathbf{H}^{-1/2} - (\mathbf{H}^{1/2}\mathbf{C}\mathbf{H}^{-1/2})^T
$$

- Exact solutions to the linearized Euler equations satisfy:

$$
||\mathbf{q}'(\mathbf{x},t)||_{(\mathbf{H},\Omega)} \leq e^{\beta t}||\mathbf{q}'(\mathbf{x},0)||_{(\mathbf{H},\Omega)}
$$

- It turns out that the Galerkin approximation $\mathbf{q}'_M = \sum_{i=1}^M a_k(t)\boldsymbol{\phi}_k(\mathbf{x})$ satisfies the same energy expression as for the continuous equations:

$$
||\mathbf{q}'_M(\mathbf{x},t)||_{(\mathbf{H},\Omega)} \leq e^{\beta t}||\mathbf{q}'_M(\mathbf{x},\mathbf{0})||_{(\mathbf{H},\Omega)}
$$

i.e., it is stable.

- For uniform base flow, the Galerkin scheme satisfies the strong stability estimate:

$$
||\mathbf{q}'_M(\mathbf{x},t)||_{(\mathbf{H},\Omega)} \leq ||\mathbf{q}'_M(\mathbf{x},0)||_{(\mathbf{H},\Omega)}
$$

Stability in the Symmetry Inner Product (cont'd)

- Stability analysis dictates that we use the symmetry inner product

$$\begin{aligned}(\mathbf{q}'^{(1)}, \mathbf{q}'^{(2)})_{(\mathbf{H},\Omega)} \quad & \equiv \int_\Omega [\mathbf{q}'^{(1)}]^T \mathbf{H} \mathbf{q}'^{(2)} d\Omega \\ & = \int_\Omega \Big[\bar{\rho}\mathbf{u}'^{(1)} \cdot \mathbf{u}'^{(2)} + \alpha^2 \gamma \bar{\rho}^2 \zeta'^{(1)} \zeta'^{(2)} \\ & + \tfrac{1+\alpha^2}{\gamma \bar{p}} + \alpha^2 \bar{\rho}(\zeta'^{(2)} p'^{(1)} + \zeta'^{(1)} p'^{(2)}) \Big] \; d\Omega\end{aligned}$$

to compute the POD modes and perform the Galerkin projection.

Practical Implication of Stability Analysis

Symmetry inner product ensures that any "bad" modes will not introduce spurious non-physical numerical instabilities into the Galerkin approximation.

- Galerkin projection step is stable for *any* basis in the symmetry inner product!

Steps to Obtain a Stable Compressible Fluid ROM

- Galerkin-project the equations in the symmetry inner product (7):

$$\left(\boldsymbol{\phi}_j, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} + \left(\boldsymbol{\phi}_j, \mathbf{A}_i \frac{\partial \mathbf{q}'_M}{\partial x_i}\right)_{(\mathbf{H},\Omega)} + (\boldsymbol{\phi}_j, \mathbf{C}\mathbf{q}'_M)_{(\mathbf{H},\Omega)} = 0 \quad (8)$$

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- Galerkin-project the equations in the symmetry inner product (7):

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- Integrate second term in (8) by parts

$$\left(\boldsymbol{\phi}_j, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} = \int_\Omega \left[\frac{\partial}{\partial x_i}[\boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i] - \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{C}\right] \mathbf{q}'_M \, d\Omega - \int_{\partial\Omega} \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_n \quad \mathbf{q}'_M \quad dS$$

Steps to Obtain a Stable Compressible Fluid ROM

- Galerkin-project the equations in the symmetry inner product (7):

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- Integrate second term in (8) by parts and apply boundary conditions:

$$\left(\boldsymbol{\phi}_j, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} = \int_{\Omega} \left[\frac{\partial}{\partial x_i}[\boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i] - \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{C}\right] \mathbf{q}'_M \, d\Omega - \int_{\partial\Omega} \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i n_i \, \mathbf{q}'_M \, dS$$

Insert boundary conditions into boundary integrals (weak implementation)

* Energy stability is maintained if the boundary conditions are such that $\int_{\partial\Omega} \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}'_M \, dS \geq 0$.

Steps to Obtain a Stable Compressible Fluid ROM

- Galerkin-project the equations in the symmetry inner product (7):

$$\left(\boldsymbol{\phi}_j, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} + \left(\boldsymbol{\phi}_j, \mathbf{A}_i \frac{\partial \mathbf{q}'_M}{\partial x_i}\right)_{(\mathbf{H},\Omega)} + (\boldsymbol{\phi}_j, \mathbf{C}\mathbf{q}'_M)_{(\mathbf{H},\Omega)} = 0 \quad (8)$$

- Integrate second term in (8) by parts and apply boundary conditions:

$$\left(\boldsymbol{\phi}_j, \frac{\partial \mathbf{q}'_M}{\partial t}\right)_{(\mathbf{H},\Omega)} = \int_\Omega \left[\frac{\partial}{\partial x_i}[\boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i] - \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{C}\right] \mathbf{q}'_M \, d\Omega - \int_{\partial\Omega} \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i n_i \, \mathbf{q}'_M \, dS$$

Insert boundary conditions into boundary integrals (weak implementation)

* Energy stability is maintained if the boundary conditions are such that $\int_{\partial\Omega} \boldsymbol{\phi}_j^T \mathbf{H}\mathbf{A}_i n_i \mathbf{q}'_M \, dS \geq 0$.

- Substitute modal decomposition $\mathbf{q}'_M = \sum_k a_k(t)\boldsymbol{\phi}_k(\mathbf{x})$ to obtain an $M \times M$ linear dynamical system of the form $\dot{\mathbf{a}} = \mathbf{K}\mathbf{a}$

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Outline

Numerical Implementation of Fluid ROM

- So far, all analysis is for continuous and smooth basis functions, and exact evaluation of inner product integrals.

Stability-Preserving Discrete Implementation:

- Define solution snapshots and POD basis functions using a piecewise smooth finite element representation:

$$\mathbf{q}_e'^h(\mathbf{x}) = \sum_{i=1}^{N_n} N_i(\mathbf{x})\mathbf{q}_i'$$

- Apply Gauss quadrature rules $\left(\int_\Omega f(\mathbf{x})d\Omega = \sum_{j=1}^{n^{quad}} \omega_j f(\mathbf{x}_j)\right)$ of sufficient accuracy to exactly integrate the inner products:

$$(\mathbf{u},\mathbf{v})_{(\mathbf{H},\Omega^e)} = \int_{\Omega^e} [\mathbf{u}]^T \mathbf{H}\mathbf{v}d\Omega^e = [\mathbf{u}_e^h]^T \mathbf{W}^e \mathbf{v}_e^h$$

where $\mathbf{w}_{kl}^e\mathbf{I}$ with $\mathbf{w}_{kl}^e = \sum_{j=1}^{n^{quad}} \mathbf{H}_e^h N_k^e(\mathbf{x}_j)N_l^e(\mathbf{x}_j)\omega_j$ is the $(k,l)^{th}$ block of $\mathbf{W}^e$.

Numerical Implementation of Fluid ROM (cont'd)

- AERO-F was used to generate the CFD simulations, using unstructured tetrahedral meshes.
- Piecewise-linear finite elements were used to represent snapshot data and POD modes
- $\mathbf{H}$ was taken to be piecewise constant over each element.
- A computer code was written that reads in the snapshot data written by AERO-F, assembles the necessary finite element representation of the snapshots, computes the numerical quadrature for evaluation of inner products, and projects the equations onto the modes.
- ROMs integrated in time using RK-4 scheme with same time step that was used in the CFD computation.

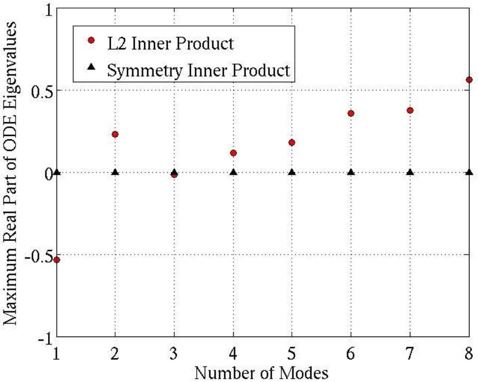
Numerical Stability & Convergence Tests

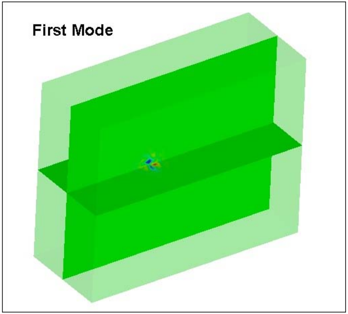
- To test *a posteriori* the **stability** of a ROM dynamical system $\dot{\mathbf{a}} = \mathbf{K}\mathbf{a}$, check the Lyapunov condition:

$$\boxed{\max_i \mathcal{R}\{\lambda_i(\mathbf{K})\} \leq 0?}$$

- To test *a posteriori* the **convergence** of a ROM solution $\mathbf{q}'_M \rightarrow \mathbf{q}'_{CFD}$ as $M \rightarrow \infty$, check:
  - $(\mathbf{q}'_M, \boldsymbol{\phi}_j)_{(\mathbf{H},\Omega)} = \left(\sum_{i=1}^{M} a_i \boldsymbol{\phi}_i, \boldsymbol{\phi}_j\right)_{(\mathbf{H},\Omega)} = \boxed{a_j \rightarrow (\mathbf{q}'_{CFD}, \boldsymbol{\phi}_j)_{(\mathbf{H},\Omega)}?}$
  - $\boxed{\langle ||\mathbf{q}'_M - \mathbf{q}'_{exact}||_{(\mathbf{H},\Omega)}\rangle \rightarrow \langle ||\mathbf{q}'_{CFD} - \mathbf{q}'_{exact}||_{(\mathbf{H},\Omega)}\rangle?}$

Test Case 1: Purely Random Basis





- Uniform base flow: physically stable to any linear disturbance.
- Each mode is a random disturbance field that decays to 0 at the domain boundaries.
- Model problem for modes dominated by numerical error: extreme case of “bad” modes.

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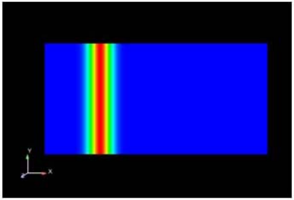
Test Case 2: 1D Acoustic Pressure Pulse

- 1D acoustic pressure pulse prescribed as the initial condition in $\Omega = (0,20) \times (-5,5) \times (0,1)$:

$$p'|_{t=0} = -\bar{\rho}\bar{c}e^{-(x-5)^2}, \quad u_1'|_{t=0} = u_3'|_{t=0} = 0$$

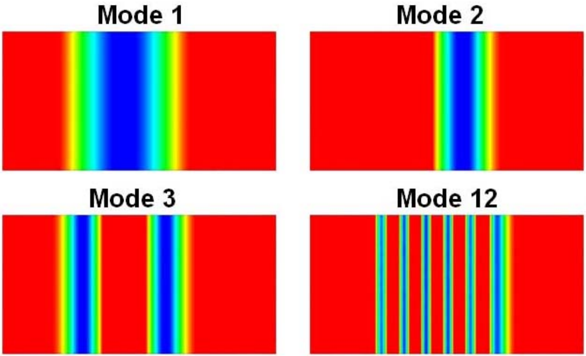
CFD animation: pressure



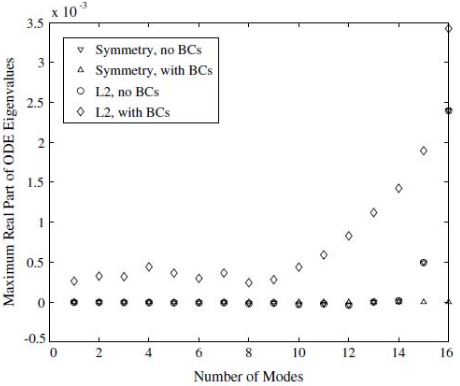
- Uniform base flow, $M_\infty \equiv \bar{u}/\bar{c} = 0.5$ in the x–direction (pulse propagates in x-direction with velocity $\bar{u} + \bar{c}$).
- Slip wall boundary conditions applied on constant y and z boundaries.

POD Modes for 1D Acoustic Pressure Pulse Example

- CFD simulation run until $T_{tot} = 5.25$ (non-dimensional time) using 512 time steps.
- Snapshots taken every 8 time steps ($N = 64$ snapshots).
- $M = 4$ POD modes captured 85.5% of energy; $M = 8$ POD modes captured 99.5% of total ensemble energy.



Stability for 1D Acoustic Pressure Pulse Example



- Four Galerkin schemes:
  1. Symmetry inner product with BCs.
  2. Symmetry inner product without BCs.
  3. L^2 inner product with BCs.
  4. L^2 inner product without BCs.

Only the symmetry inner product with BCs produces a stable ROM for all M $\left(\max_i \mathcal{R}\{\lambda_i(\mathbf{K})\} < 10^{-9}\right)$

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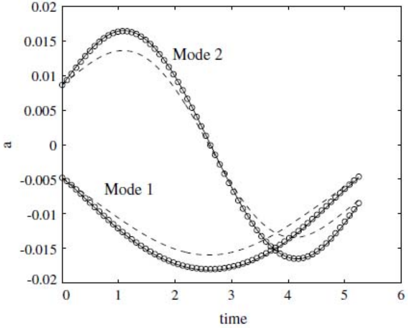
Convergence of the ROM for the 1D Acoustic Pressure Pulse Example



Convergence check:

$$\mathbf{q}'_M = \sum_{i=1}^{M} \boxed{a_i(t)}\ \boldsymbol{\phi}_i(\mathbf{x})$$

?

$$\Pi_M \mathbf{q}'_{CFD} = \sum_{i=1}^{M} \boxed{(\mathbf{q}'_{CFD}, \boldsymbol{\phi}_i)_{(\mathbf{H},\Omega)}}\ \boldsymbol{\phi}_i(\mathbf{x})$$

- Figure shows symmetry ROM (with BCs) coefficients a_i vs. $(\mathbf{q}'_{CFD}, \boldsymbol{\phi}_i)_{(\mathbf{H},\Omega)}$ [- - 4 mode ROM; – 8 mode ROM; ∘ CFD solution].
- Symmetry ROM (with BCs) appears to be convergent as the number of modes increases.

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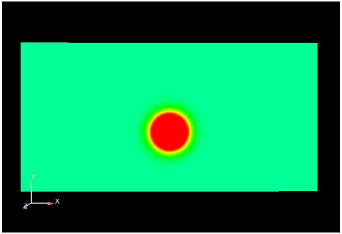
Test Case 3: 2D Pressure Pulse

- Reflection of cylindrical Gaussian pressure pulse in $\Omega = (0,20) \times (-5,5) \times (0,1)$:

$$p'|_{t=0} = e^{-(x-10)^2-(y+1)^2}, \quad u_1'|_{t=0} = u_2'|_{t=0} = u_3'|_{t=0} = 0$$



- Uniform base flow, $M_\infty = 0.25$ in x–direction.
- Slip wall boundary conditions applied on constant y and z boundaries.

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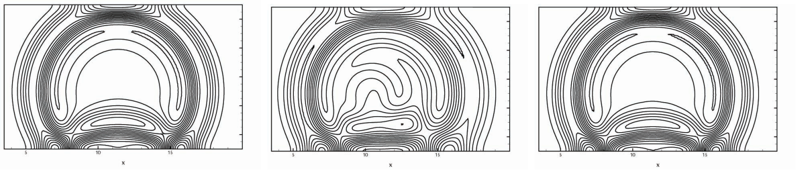
Results for the 2D Pressure Pulse Example

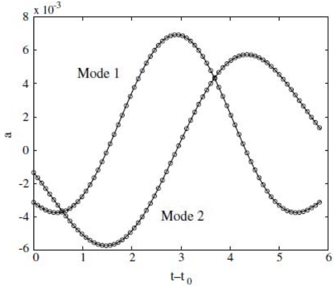
- CFD simulation run until $T_{tot} = 6.4$ (non-dimensional time) using 624 time steps.
- Snapshots taken every 4 time steps starting at time $t = t_0 = 0.57$.
- 6 mode basis captures 97.4% of total ensemble energy.
- Good qualitative agreement between CFD solution and 6 mode symmetry ROM (with BCs) on large scale.
- Excellent agreement between CFD solution and 14 mode symmetry ROM (with BCs).
- Symmetry ROM (with BCs) is stable – vs. L^2 ROM, which experienced instability when more than 6 or 7 modes were used.

Pressure contours at $t - t_0 = 5.0$.

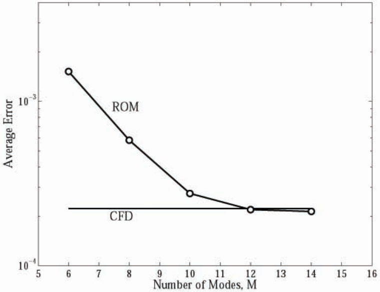


CFD 6 mode ROM 14 mode ROM

Convergence of the ROM for the 2D Pressure Pulse Example



a_i vs. $(\mathbf{q}'_{CFD}, \phi_i)_{(\mathbf{H},\Omega)}$ for $M = 12$ (– 12 mode ROM; ∘ CFD solution)



Time-average error of the symmetry ROM solution as a function of M, compared with the time-average error in the CFD solution

- Tests demonstrate numerically the convergence of the symmetry ROM with BCs.
- For $M \geq 12$, ROM gives solution trajectory that is slightly closer to exact solution than the CFD solution.

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Outline

Summary

- A Galerkin ROM in which the *continuous* equations are projected onto the modal basis in a *continuous* inner product is proposed.
- For this *continuous* Galerkin projection approach, the choice of inner product is crucial to stability.
- For linearized, compressible flow, Galerkin projection in the "symmetry" inner product leads to an approximation that is numerically stable for any choice of basis.
- A weak enforcement of the boundary conditions preserves stability, provided they are well-posed.
- A numerical implementation using finite elements that preserves stability is presented.
- Numerical stability of some POD/Galerkin ROMs constructed using this scheme is examined on several model problems.

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Further Work

- A structure ROM governed by the non-linear plate equations was also developed (Segalman *et al.*).
- ROM convergence was examined mathematically, and *a priori* error estimates for the ROM solution error were derived (Kalashnikova & Barone 2010 *in press*).
- Extension of symmetry inner product methods to non-linear equations using an interpolation procedure to handle efficiently the non-linear terms (e.g., “best points interpolation procedure” of Peraire, Nguyen, *et al.*).

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References

[1] I. Kalashnikova, M.F. Barone. “On the Stability and Convergence of a Galerkin Reduced Order Model (ROM) of Compressible Flow with Solid Wall and Far-Field Boundary Treatment”. *Int. J. Numer. Meth. Engng* (in print).

[2] M.F. Barone, I. Kalashnikova, M.R. Brake, D.J. Segalman. “Reduced Order Modeling of Fluid/Structure Interaction”. *Sandia National Laboratories Report, SAND No. 2009-7189.* Sandia National Laboratories, Albuquerque, NM (2009).

[3] M.F. Barone, I. Kalashnikova, D.J. Segalman, H. Thornquist. “Stable Galerkin Reduced Order Models for Linearized Compressible Flow”. *J. Comput. Phys.* **288** (2009) 1932–1946.

[4] M.F. Barone, D.J. Segalman, H. Thornquist, I. Kalashnikova. “Galerkin Reduced Order Models for Compressible Flow with Structural Interaction”. *AIAA Paper No. 2008−0612 , 46th AIAA Aerospace Science Meeting and Exhibit,* Reno, NV (January 2008).

(can be downloaded from my website:
`www.stanford.edu/~irinak/pubs.html`)

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References

[1] I. Kalashnikova, M.F. Barone. “On the Stability and Convergence of a Galerkin Reduced Order Model (ROM) of Compressible Flow with Solid Wall and Far-Field Boundary Treatment”. *Int. J. Numer. Meth. Engng* (in print).

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Thank You!

References

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Questions?

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