Dynamical Shape Control of Nonlinear Thin Rods

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1 Introduction

Dynamical shape control problems for linear partial differential equations have recently drawn much attention. The linear heat equation was studied in Cannarsa–Da Prato–Zolesio [1], where the feedback was constructed via the Hamilton–Jacobi–Bellman equation. Truchi–Zolesio [4] considered the linear wave equation. Closely related to the problem of dynamical shape control is the paper by Cannarsa–Da Prato–Zolesio [2], in which the damped linear wave equation was studied on a moving domain. In a recent paper, Sokolowski–Sprekels [3] considered the following problem:

Suppose a thin rod performs transversal oscillations which shall be damped out, and suppose the left part of the rod is fixed in such a way that it can be moved back and forth within the fixation, so that the length of the free part of the rod can be controlled dynamically. The objective is to move the rod in such a way that its tip is brought to rest at a prescribed point at a given time instant T. We sketch the situation in Figure 1. In contrast to the other works mentioned above, in [3] nonlinear constitutive laws were admitted. In particular, materials were considered that not only react to changes of the shear strain $\epsilon = u_x$ by a (possibly non-monotone) shear stress, but also to changes of the curvature of their crystal lattices by a couple stress. Consequently, the elastic potential Φ is assumed in the form

$$\Phi = \Phi(\epsilon, \epsilon_x) = F(\epsilon) + \frac{\gamma}{2}\epsilon_x^2, \tag{1.1}$$

where $\gamma > 0$, and where F is smooth and possibly non-convex.

Coming back to our problem, we observe that pulling the rod inside the fixation stabilizes the rod, and the oscillations cease completely if the rod has been pulled into the fixation. Since this is not feasible in practical applications, such as the stabilization of flexible structures (for instance in space), it is desirable to allow controls v (which denotes the deviation of the length of the rod from its initial length) where $v_t(t) > 0$, which means that the rod is pushed outside the fixation, i.e., the length of its free part is increased. Since increasing the length means to destabilize the structure, another mechanism is needed

Configuration at $t = 0$.
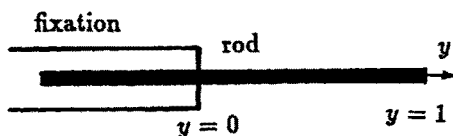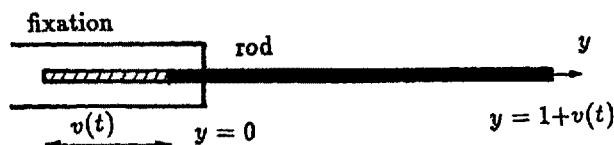

Configuration at $t > 0$.

Figure 1: Dynamical shape control of a thin rod.

to damp the oscillations in this situation. In this paper we assume that a damping mechanism becomes active at the tip of the rod if $v_t(t) > 0$; to be precise, we assume that the total stress σ at the tip is counteracted by means of the following boundary condition : $\sigma = G(v_t)u_t$, where u denotes the transversal displacement and G is a real–valued function satisfying

$$G(v) - v \geq \delta, \quad \forall v \in \mathbf{R}, \text{ with some } \delta > 0. \tag{1.2}$$

Typically,

$$G(v) = v_+ + \delta, \quad \forall v \in \mathbf{R}, \text{ with some } \delta > 0. \tag{1.3}$$

In the sequel, we give a report on the results obtained in [3] concerning the optimal control problem for the nonlinear thin rod.

2 Well–Posedness of the State Equations

Let $T > 0$ be fixed. We consider the initial–boundary value problem :

$$u_{tt} - (F'(u_y))_y + u_{yyyy} = g(y,t), \quad \text{in } Q_T(v), \tag{2.1}$$

$$u(0,t) = u_y(0,t) = 0 = u_{yy}(1+v(t),t), \quad 0 < t < T, \tag{2.2}$$

$$u_{yyy}(1+v(t),t) - F'(u_y(1+v(t),t)) = G(v_t(t))u_t(1+v(t),t), \quad 0 < t < T, \tag{2.3}$$

$$u(y,0) = u_0(y), \quad u_t(y,0) = u_1(y), \quad 0 < y < 1. \tag{2.4}$$

In terms of our problem, (2.1) is the balance law of linear momentum after the introduction of dimensionless variables and upon normalizing all physical constants to unity; g is a distributed (known) load, and u_0, u_1 stand for the initial displacement and velocity, respectively. Moreover, we have set

$$Q_T(v) = \{(y,t) \in \mathbf{R}^2 | 0 < t < T, 0 < y < 1 + v(t)\} \quad . \tag{2.5}$$

A typical situation is depicted in the following drawing:
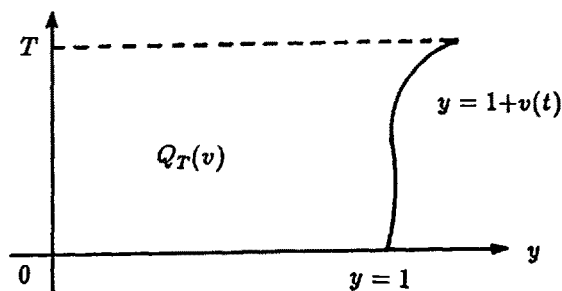
We always tacitly assume that the control v is small enough so that the rod is not pulled out of the fixation completely. For the data of the problem we generally assume :

(A1) $g \in H^1(0,T;L^2_{loc}(\mathbb{R}))$.

(A2) $u_0 \in H^3(0,1)$, $u_1 \in H^2(0,1)$, $u_0(0) = u_0'(0) = 0$;
the compatibility conditions of sufficiently high order are satisfied.

(A3) $G \in C(\mathbb{R})$, and with some fixed $\epsilon > 0$, $G(x) - x \geq \epsilon$, $\forall x \in \mathbb{R}$.

(A4) $F \in C^4(\mathbb{R})$, and there exist a nonnegative function F_1 and positive constants β_i, $i = 1,\ldots,6$, such that

(i) $\beta_1 F_1(x) - \beta_2 \leq F(x) \leq \beta_3 F_1(x) + \beta_4$,$\forall x \in \mathbb{R}$,

(ii) $xF'(x) \leq \beta_5 F(x) + \beta_6$, $\forall x \in \mathbb{R}$.

(A5) $v \in H^3(0,T)$, $v(0) = v_t(0) = 0$, and, with some $M \in (0,1)$, $v(t) \geq -M$, $\forall t \in [0,T]$.

Note that (A4) holds if F is an even polynomial with positive coefficient at the highest order; the condition $v(t) \geq -M$ means that the rod is never pulled into its fixation completely.

Next we transform the problem onto the fixed domain Ω_T, where $\Omega = (0,1)$ and, for $t > 0$, $\Omega_t = \Omega \times (0,t)$. To this end, introduce the new coordinate $x = \frac{y}{1+v(t)}$. The new unknown function $z(x,t) = u(y,t) = u(x(1+v(t)),t)$ satisfies the system

$$z_{tt} - \frac{2xv_t}{1+v}z_{xt} + z_x\left(x\left(\frac{v_t}{1+v}\right)^2 - \left(\frac{xv_t}{1+v}\right)_t\right) + \left(\frac{xv_t}{1+v}\right)z_{xx}$$

$$-\frac{1}{1+v}\left(F'\left(\frac{z_x}{1+v}\right)\right)_x + \frac{1}{(1+v)^4}z_{xxxx} = f(x,t), \quad \text{in} \quad \Omega_T , \qquad (2.6)$$

$$z(0,t) = z_x(0,t) = 0 = z_{xx}(1,t), \quad 0 < t < T, \tag{2.7}$$

$$\frac{1}{(1+v(t))^3}z_{xxx}(1,t) - F'\left(\frac{z_x(1,t)}{1+v(t)}\right)$$
$$= G(v_t(t))\left(z_t(1,t) - \frac{v_t(t)}{1+v(t)}z_x(1,t)\right), \quad 0 < t < T, \tag{2.8}$$

$$z(x,0) = u_0(x), \quad z_t(x,0) = u_1(x), \quad 0 < x < 1. \tag{2.9}$$

Here we have set : $f(x,t) \equiv g(x(1+v(t)),t)$.

We now derive a weak formulation of (2.6)–(2.9). To this end, we test (2.6) by any function $\varphi \in V$, where

$$V = \{\varphi \in H^2(0,1)|\varphi(0) = \varphi'(0) = 0\} \quad . \tag{2.10}$$

Then

$$[z_{tt}(t) + a_2(t)z_x(t) + a_3(t)z_{xx}(t), \varphi] - [a_1(t)z_t(t), \varphi_x] + [a_4(t)F'\left(\frac{z_x(t)}{1+v(t)}\right), \varphi_x]$$
$$+ [a_5(t)z_{xx}(t), \varphi_{xx}] + a_1(1,t)z_t(1,t)\varphi(1) + a_6(t)z_t(1,t)\varphi(1) + a_7(t)z_x(1,t)\varphi(1)$$
$$- [f(t), \varphi] = 0 \quad , \quad \forall \varphi \in V \quad . \tag{2.11}$$

Here, $[\cdot, \cdot]$ denotes the inner product in $L^2(0,1)$, and we have used the abbreviations

$$a_1(x,t) = \frac{-2xv_t(t)}{1+v(t)} \quad , \quad a_2(x,t) = x\left(\frac{v_t(t)}{1+v(t)}\right)^2 + \left(\frac{xv_t(t)}{1+v(t)}\right)_t,$$
$$a_3(x,t) = \frac{xv_t(t)}{1+v(t)} \quad , \quad a_4(t) = \frac{1}{1+v(t)} \quad , \quad a_5(t) = \frac{1}{(1+v(t))^4} \quad ,$$
$$a_6(t) = \frac{G(v_t(t))}{1+v(t)} \quad , \quad a_7(t) = -\frac{v_t(t)G(v_t(t))}{(1+v(t))^2} \quad . \tag{2.12}$$

Next we introduce our notion of a weak solution to the system (2.1)–(2.4). To this end, we integrate (2.11) with respect to t. We define:

Definition :

A function u is called a *weak solution* of (2.1)–(2.4), if $u(y,t) = u(x(1+v(t)),t) = z(x,t)$, where z has the following properties :

(i) $z \in L^\infty(0,T;H^2(0,1))$, $z_t \in C(0,T;L^2(0,1))$, $z_x \in L^\infty(\Omega_T)$, $z_t(1,\cdot) \in L^2(0,T)$.

(ii) For all $\varphi \in V$ and for all $\tau \in (0,T]$ there holds

$$[z_t(\tau), \varphi] - [u_1, \varphi] + \int_0^\tau [a_5(t)z_{xx}(t), \varphi_{xx}]dt - \int_0^\tau [a_1(t)z_t(t), \varphi_x]dt$$

$$+ \int_0^\tau [a_2(t)z_x(t) + a_3(t)z_{xx}(t), \varphi]dt + \int_0^\tau \left[a_4(t)F'\left(\frac{z_x(t)}{1+v(t)}\right), \varphi_x \right] dt$$

$$+ \int_0^\tau [(a_6(t) + a_1(1,t))z_t(1,t)\varphi(1) + a_7(t)z_x(1,t)\varphi(1)]dt$$

$$- \int_0^\tau [f(t), \varphi]dt = 0. \qquad (2.13)$$

(iii) $z(x,0) = u_0(x)$, for all $x \in [0,1]$.

We have the following existence result (cf. [3], Theorems 2.1 and 2.3):

Theorem 2.1 *Under the assumptions (A1)–(A5), the system (2.1)–(2.4) has a weak solution. If, in addition, $G \in C^1(\mathbb{R})$, then the weak solution satisfies*

$$z_{tt} \in L^\infty(0,T; L^2(0,1)), z_{xxt} \in L^\infty(0,T; L^2(0,1)),$$
$$z_{xt} \in L^\infty(\Omega_T), z_{tt}(1,\cdot) \in L^2(0,T). \qquad (2.14)$$

The following result states that the weak solution depends Lipschitz continously on the domain parameter $v \in H^3(0,T)$. In particular, the weak solution is unique. We have (cf. [3], Theorem 2.4):

Theorem 2.2 *Let $G \in C^1(\mathbb{R})$, and let (A1)–(A4) be satisfied. Suppose $z^{(i)}$ is any weak solution associated with the domain parameter $v^{(i)} \in H^3(0,T)$, where $v^{(i)}$ satisfies (A5), $i = 1,2$. Then for $z = z^{(1)} - z^{(2)}, v = v^{(1)} - v^{(2)}$, there holds the inequality*

$$\sup_{t \in (0,T)} (\|z_t(t)\|^2 + \|z_{xx}(t)\|^2) + \|z_x\|_{L^\infty(\Omega_T)} + \int_0^t z_t^2(1,s)\,ds$$
$$\leq C \|v\|_{H^2(0,T)}^2 \quad , \qquad (2.15)$$

where C depends only on the data and $\|v^{(i)}\|_{H^3(0,T)}$, $i = 1,2$.

3 The Optimal Control Problem

Let $K \subset H^3(0,T)$ denote some convex set. We consider a dynamical shape control problem for the nonlinear thin rod :

Minimize the cost functional

$$I(v) = \frac{1}{2} \int_0^T \int_0^{1+v(t)} (u_t^2 + u_{xx}^2)\,dx\,dt + \frac{\alpha}{2} \text{II}(v,v) \quad , \qquad (3.1)$$

where $\alpha > 0$, $\text{II}(v,v)^{1/2}$ is a norm or a seminorm on $H^3(0,T)$, and u the weak solution of (2.1)–(2.4) corresponding to v (we assume that the assumptions of Theorem 2.2 are satisfied).

To treat this control problem, we transform the state equations and the cost functional onto the fixed domain $\Omega_T = (0,1) \times (0,T)$. To simplify the exposition somewhat, we consider the problem :

Problem (P) :

Minimize the cost functional

$$J(v) = \frac{1}{2} \int_0^T \int_0^1 (z_t^2 + z_{xx}^2) \, dx \, dt$$

$$+ \frac{\beta}{2} \int_0^T |G(v_t(t))z_t(1,t)|^2 \, dt + \frac{\alpha}{2} \, \text{II}(v,v) \tag{3.2}$$

over the closed and convex set

$$K = \{ v \in H^3(0,T) | \, v(0) = v_t(0) = v_{tt}(0) = 0 \,, \, v(T) = L \,,$$
$$v_t(T) = v_{tt}(T) = 0 \,, \, v(t) \geq -M \,, \, 0 \leq t \leq T \} \quad . \tag{3.3}$$

We thus want to bring the tip of the rod at time T to a rest at the prescribed position $1 + L$ (where, of course, $L \geq -M$) while keeping the total energy of the rod, penalized by two terms representing the cost of the control action, at a minimal value. We have (cf. [3], Theorem 3.1) the result:

Theorem 3.1 *Suppose that $G \in C^1(\mathbb{R})$, and suppose (A1)–(A4) are satisfied. Furthermore, let for $C > 0$ the sets $\{ v \in K \, | \, \text{II}(v,v) \leq C \}$ be bounded in $H^3(0,T)$. Then the problem (P) has an optimal solution $v^* \in K$, and the necessary conditions of optimality are satisfied in the sense that there exist (z^*, p^*) which satisfy the system :*

1. State Equation :

$$\int_0^1 [z_{tt}^*(t) + a_1^*(t)z_{xt}^*(t) + a_2^*(t)z_x^*(t) + a_3^*(t)z_{xx}^*(t)]\varphi \, dx$$

$$+ \int_0^1 a_4^*(t)F'\left(\frac{z_x^*(t)}{1+v^*(t)}\right)\varphi_x \, dx + \int_0^1 a_5^*(t)z_{xx}^*(t)\varphi_{xx} \, dx - \int_0^1 f(t)\varphi \, dx$$

$$+ (a_6^*(t)z_t^*(1,t) + a_7^*(t)z_x^*(1,t))\varphi(1) = 0 \quad,$$
$$\text{for all} \quad \varphi \in V \quad \text{and a.e.} \quad t \in (0,T) \quad, \tag{3.4}$$

with the initial conditions

$$z^*(x,0) = u_0(x) \quad, \quad z_t^*(x,0) = u_1(x) \quad, \quad 0 \leq x \leq 1 \quad . \tag{3.5}$$

2. Adjoint-State Equation :

$$\int_0^1 [p_{tt}^*(t)\eta - (p^*(t)a_1^*(t))_t\eta_x + p^*(t)(a_2^*(t)\eta_x + a_3^*(t)\eta_{xx})$$

$$+a_4^*(t)F''\left(\frac{z_x^*(t)}{1+v^*(t)}\right)\frac{p_x^*(t)}{1+v^*(t)}\eta_x + a_5^*(t)p_{xx}^*(t)\eta_{xx}]\,dx$$

$$-(a_6^*(t)p^*(1,t))_t\eta(1) + a_7^*(t)p^*(1,t)\eta_x(1)$$

$$= \beta(\,(G(v_t^*(t)))^2 z_t^*(1,t))_t\,\eta(1) + \int_0^1 (z_{tt}^*(t)\eta - z_{xx}^*(t)\eta_{xx})\,dx\quad,$$

$$\text{for all}\quad \eta \in V\quad \text{and a.e.}\quad t \in (0,T)\quad, \tag{3.6}$$

with final conditions

$$p^*(x,T) = 0\quad,\quad p_t^*(x,T) = z_t^*(T)\quad. \tag{3.7}$$

3. Optimality Conditions :

$$\int_0^1 \{\quad p^*(t)\quad [a_1'(v^*;v-v^*(t))z_{xt}^*(t) + a_2'(v^*;v-v^*(t))z_x^*(t)$$

$$+a_3'(v^*;v-v^*(t))z_{xx}^*(t)] + a_4'(v^*;v-v^*(t))F'\left(\frac{z_x^*(t)}{1+v^*(t)}\right)p_x^*(t)$$

$$-a_4(v^*)F''\left(\frac{z_x^*(t)}{1+v^*(t)}\right)\frac{p_x^*(t)}{(1+v^*(t))^2}(v-v^*(t))$$

$$+a_5'(v^*;v-v^*(t))z_{xx}^*(t)p_{xx}^*(t)\quad\}\,dx$$

$$+a_6'(v^*;v-v^*(t))z_t^*(1,t)p^*(1,t) + a_7'(v^*;v-v^*(t))z_x^*(1,t)p^*(1,t)$$

$$+\alpha\,\Pi(v^*(t),v-v^*(t)) \geq 0\quad,$$

$$\text{for all}\quad v \in K\quad \text{and a.e.}\quad t \in (0,T)\quad. \tag{3.8}$$

(Here the expressions $a_j'(v^*;v-v^*(t))$,$1 \leq j \leq 7$, have their obvious meanings.)

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