SHAPING THE REFERENCE INPUT RESPONSE OF LINEAR DISTRIBUTED PARAMETER SYSTEMS VIA OUTPUT FEEDBACK

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1. Introduction

In recent years a new representation of dynamic systems has been proposed by several authors. It has been restricted to lumped parameter systems so far and is characterized by generalized Fourier series expansions of the input, the state and the output, using a suitable orthogonal basis on a finite time interval. To this end, Paraskevopoulos et al. [1] and Vlassenbroeck et al. [2] cut off the time axis at some point $t = T$ and consider the system on the time interval $[0,T]$. In several papers the new representation is utilized for system analysis and identification [3], [4]. Franke [5] uses a different approach which avoids cutting off the time axis by introducing a nonlinearly distorted time coordinate

$$\tau = 1 - 2e^{-\alpha t}, \quad \alpha > 0, \tag{1}$$

thus mapping the interval $0 \leq t < \infty$ on the interval $-1 \leq \tau \leq 1$. The state equations

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}, \quad \underline{x}(0) = \underline{x}_o, \tag{2}$$

$$\underline{y} = \underline{C}\,\underline{x}, \tag{3}$$

$\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^p$, $\underline{y} \in \mathbb{R}^q$, $t \in [0,\infty)$, rewritten versus the τ-coordinate take the form

$$\alpha(1-\tau)\hat{\underline{x}}' = \underline{A}\,\hat{\underline{x}} + \underline{B}\,\hat{\underline{u}}, \quad \hat{\underline{x}}(-1) = \underline{x}_o, \tag{4}$$

$$\hat{\underline{y}} = \underline{C}\,\hat{\underline{x}}, \tag{5}$$

where $\tau \in [-1, 1]$. Then by inserting the series expansions

$$\hat{\underline{u}}(\tau) = \sum_k \underline{u}_k^* p_k(\tau), \quad \hat{\underline{x}}(\tau) = \sum_k \underline{x}_k^* p_k(\tau), \quad \hat{\underline{y}}(\tau) = \sum_k \underline{y}_k^* p_k(\tau), \tag{6}$$

into (4), where $p_k(\tau)$ are Legendre polynomials, and by applying Galerkin's method, one obtains the following algebraic relations between Fourier coefficients (for the case $\underline{x}_o = \underline{0}$):

$$\underline{x}_k^* = [(k+1)\alpha\underline{I} - \underline{A}]^{-1} \cdot \left\{(2k+1)\alpha(-1)^{k+1} \sum_{j=0}^{k-1} (-1)^j \underline{x}_j^* + \underline{B}\,\underline{u}_k^*\right\}, \tag{7}$$

$$\underline{y}_k^* = \underline{C}\,\underline{x}_k^*, \qquad k = 0, \ldots, N-1. \tag{8}$$

It can be shown that this Galerkin approximation minimizes the mean square state equation error for whatever choice of $\alpha > 0$ and N, provided the over-all system is stable.

Based on (7) and (8), a novel access to linear feedback control has been proposed in [6], [7] which is oriented at direct shaping of the reference input response by balancing the generalized Fourier coefficients in the closed loop. In the present paper the method will be extended to a class of linear distributed parameter systems (DPS).

2. A new representation of linear DPS

Let the infinite-dimensional system be given by its state equations

$$\partial\underline{x}(t,z)/\partial t = \underline{A}_z\underline{x}(t,z) + \underline{B}(z)\underline{u}(t), \tag{9}$$

$$\underline{y}(t) = \int_\Omega \underline{C}(z)\underline{x}(t,z)d\Omega, \tag{10}$$

where $\underline{u} \in \mathbb{R}^p$, $\underline{x} \in L_2(\Omega)$ and $\underline{y} \in \mathbb{R}^p$ are the control, the state and the output, respectively; $0 \le t < \infty$, $z \in \Omega$, where Ω is a simply connected finite spatial region. $\underline{A}_z$ is a linear matrix differential operator with respect to z, and $\underline{B}(z)$, $\underline{C}(z)$ are given space dependent matrices. Let the initial state be $\underline{x}(0,z) = \underline{x}_o(z)$ and the boundary conditions be formally homogeneous.

In the following we do not use Laplace transform methods. However, there will arise a relation to Green's function methods. Therefore, let $\underline{G}(s,z,\zeta)$ be the Green's matrix corresponding to $(s\underline{I} - \underline{A}_z)$. Hence, the complex valued input-output equations of the above system are

$$\underline{X}(s,z) = \int_\Omega \underline{G}(s,z,\zeta)\underline{B}(\zeta)d\Omega \cdot \underline{U}(s), \tag{11}$$

$$\underline{Y}(s) = \int_\Omega \underline{C}(z)\underline{X}(s,z)d\Omega. \tag{12}$$

Now (9) and (10) will be subject to time transformation (1) which yields

$$\alpha(1-\tau)\partial\hat{\underline{x}}(\tau,z)/\partial\tau = \underline{A}_z\hat{\underline{x}}(\tau,z) + \underline{B}(z)\hat{\underline{u}}(\tau), \tag{13}$$

$$\hat{\underline{y}}(\tau) = \int_\Omega \underline{C}(z)\hat{\underline{x}}(\tau,z)d\Omega, \qquad -1 \leq \tau \leq 1, \tag{14}$$

with formally homogeneous boundary conditions and initial state $\hat{\underline{x}}(-1, z) = \underline{x}_o(z)$.

Fourier series expansions are quite common in the analysis of DPS, however they are usually applied with respect to spatial coordinates. Here we use expansions with respect to τ,

$$\hat{\underline{u}}(\tau) = \sum_k \underline{u}_k^* p_k(\tau) = \underline{U}^*\underline{p}(\tau), \tag{15}$$

$$\hat{\underline{x}}(\tau,z) = \sum_k \underline{x}_k^*(z) p_k(\tau) = \underline{X}^*(z)\underline{p}(\tau), \tag{16}$$

$$\hat{\underline{y}}(\tau) = \sum_k \underline{y}_k^* p_k(\tau) = \underline{Y}^*\underline{p}(\tau), \tag{17}$$

where again $p_k(\tau)$ are Legendre polynomials. By inserting these series into (13) and by minimizing the mean square equation error via Galerkin's method to be applied with respect to τ, one obtains a boundary value problem for each Fourier coefficient $\underline{x}_k^*(z)$. The solution of this problem is quite analogous to (7), namely (for the case $\underline{x}_o(z) = \underline{O}$)

$$\underline{x}_k^*(z) = [(k+1)\alpha\underline{I} - \underline{A}_z]^{-1}\Big\{(2k+1)\alpha(-1)^{k+1} \cdot \sum_{j=0}^{k-1} (-1)^j \underline{x}_j^*(z) + \underline{B}(z)\underline{u}_k^*\Big\}, \tag{18}$$

$$\underline{y}_k^* = \int_\Omega \underline{C}(z)\underline{x}_k^*(z)d\Omega, \qquad k = 0, \dots, N-1. \tag{19}$$

In (18), $(^{-1})$ denotes the inverse operator which in view of (11) can be rewritten using Green's matrix:

$$\underline{x}_k^*(z) = \int_\Omega \underline{G}[(k+1)\alpha,z,\zeta]\cdot\Big\{(2k+1)\alpha(-1)^{k+1} \cdot \sum_{j=0}^{k-1} (-1)^j \underline{x}_j^*(\zeta) + \underline{B}(\zeta)\underline{u}_k^*\Big\}d\Omega, \quad k=0, \dots, N-1. \tag{20}$$

However, in contrast to (11), all equations are real valued now. It should be remarked that of course $(k+1)\alpha$, $k = 0, \dots, N-1$, are required to be in the resolvent set of operator $\underline{A}_z$. This can be met by suitable choice of α.

It can be ovserved that eqs. (20) as well as eqs. (7) have a triangular structure which allows evaluation in a recursive manner. If for example, the control is given by its Fourier coefficients $\underline{u}_k^*$, then the $\underline{x}_k^*(z)$ can be obtained from (20) and the $\underline{y}_k^*$ from (19); and vice versa, if the output is prescribed by its Fourier coefficients $\underline{y}_k^*$, then the $\underline{u}_k^*$ and $\underline{x}_k^*(z)$ can also be computed from (19), (20) in a recursive manner.

An important question to be treated next is how many Fourier coefficients to be considered. To this end we define the relative degree of DPS.

3. Infinite-dimensional systems with finite relative degree

For the finite-dimensional system (2), (3) the relative degree d_i with respect to output y_i is defined as

$$d_i = \min\left\{k \left| \underline{c}_i^T \underline{A}^{k-1} \underline{B} \neq \underline{0}^T, \quad k = 1, \ldots, n \right.\right\}. \tag{21}$$

Now for the infinite-dimensional system (9), (10) assume that for some finite integer k

$$\int_\Omega \underline{c}_i^T(z) \underline{A}_z^{k-1} \underline{B}(z) d\Omega \neq \underline{0}^T.$$

Then the relative degree d_i with respect to output y_i will be defined as [8]

$$d_i = \min\left\{k \left| \int_\Omega \underline{c}_i^T(z) \underline{A}_z^{k-1} \underline{B}(z) d\Omega \neq \underline{0}^T, \; k = 1, 2, \ldots \right.\right\}. \tag{22}$$

The simple meaning of d_i in (21) and (22) is that there is at least one component of $\underline{u}$ acting directly on the d_i-th derivative of $y_i(t)$.

For example, we have $d_i = 1$, if

$$\int_\Omega \underline{c}_i^T(z) \underline{b}_j(z) d\Omega \neq 0 \quad \text{for some } j.$$

Obviously, finite d_i requires colocated or overlapping spatial supports for actuators and sensors.

Example: Euler-Bernoulli beam equation

$$\frac{\partial^2 x}{\partial t^2} + \frac{\partial^4 x}{\partial z^4} = u_1(t)\delta(z-z_1) + u_2(t)\delta(z-z_2), \quad 0 < z < 1, \tag{23}$$

with forces u_1 and u_2 acting pointwise at z_1 and z_2, respectively.

Boundary conditions:

$$x(t,0) = x(t,1) = 0, \tag{24}$$

$$\left.\frac{\partial^2 x}{\partial z^2}\right|_{z=0} = \left.\frac{\partial^2 x}{\partial z^2}\right|_{z=1} = 0. \tag{25}$$

Let the velocity $\partial x/\partial t$ be measured in colocated points:

$$y_1(t) = \left.\frac{\partial x}{\partial t}\right|_{z_1}, \quad y_2(t) = \left.\frac{\partial x}{\partial t}\right|_{z_2}. \tag{26}$$

Then by introducing the state variables

$$x_1(t,z) = \frac{\partial^2 x}{\partial z^2}, \quad x_2(t,z) = \frac{\partial x}{\partial t}, \tag{27}$$

the system matrices are

$$\underline{A}_z = \begin{bmatrix} 0 & \partial^2/\partial z^2 \\ -\partial^2/\partial z^2 & 0 \end{bmatrix}, \qquad \underline{B}(z) = \begin{bmatrix} 0 & 0 \\ \delta(z-z_1) & \delta(z-z_2) \end{bmatrix},$$

$$\underline{C}(z) = \begin{bmatrix} 0 & \delta(z-z_1) \\ 0 & \delta(z-z_2) \end{bmatrix},$$

and therefore the relative degrees here are

$$\underline{d_1 = d_2 = 1}. \tag{28}$$

4. Controller design by balancing generalized Fourier coefficients

In the following we restrict ourselves to infinite-dimensional systems with finite relative degree. If the relative degree is finite, it turns out to be a small integer in most situations. This is favourable for the controller design by balancing a small number of Fourier coefficients in the closed loop.

Let the system (9), (10) be augmented by a linear feedback control law, in the simplest case

$$\underline{u}(t) = \underline{K}_0 \underline{w}(t) - \underline{K}\,\underline{y}_M(t), \tag{29}$$

where

$$\underline{y}_M(t) = \int_\Omega \underline{C}_M(z)\underline{x}(t,z)d\Omega \in \mathbb{R}^q \tag{30}$$

is a vector of measured variables, $\underline{w}(t)$ is the reference input, and $\underline{K}$, $\underline{K}_o$ are constant matrices to be determined.

The design procedure aims at direct shaping of the reference input response. To this end the following steps are proposed:

(a) Rewrite the controller equations, similar to the plant equations, in terms of Fourier coefficients:

$$\underline{u}_k^* = \underline{K}_o\underline{w}_k^* - \underline{K}\,\underline{y}_{Mk}^*, \tag{31}$$

$$k = 0, \ldots, N-1.$$

$$\underline{y}_{Mk}^* = \int_\Omega \underline{C}_M(z)\underline{x}_k^*(z)d\Omega. \tag{32}$$

(b) Select reference input $\underline{w}(t)$, e.g. unit step function, hence $\underline{W}^*$.
(c) Select $\alpha > 0$.
(d) Prescribe $\underline{Y}^*$ by prescribing $\underline{y}(t)$. (Number of Fourier coefficients oriented at the relative degrees d_i, see [7] and exemplary discussion in the next section).
(e) Calculate $\underline{U}^*(\alpha)$ and $\underline{X}^*(z,\alpha)$ from plant equations (19), (20).
(f) Calculate $\underline{Y}_M^*(\alpha)$ from (32).
(g) Calculate controller parameters $\underline{K}_o$, $\underline{K}$ from (31).

It should be emphasized that the above procedure requires nothing else than solving linear equations, and this overcomes a main drawback of Riccati design or eigenvalue assignment.

It should also be pointed out that the design procedure does not necessarily imply stability, although it has a stabilizing tendency whenever $\underline{y}(t)$ is prescribed well-damped in step (d), see the example in the next section. In any case, stability should be examined in a final step, e.g. via Ljapunov's direct method.

5. Numerical example

As an example we consider the active damping of the Euler-Bernoulli beam, eqs. (23) - (28), by feedback.

Using the abbreviations

$$\dot{\underline{y}}(t) = \begin{bmatrix} \partial x/\partial t|_{z_1} \\ \partial x/\partial t|_{z_2} \end{bmatrix}, \quad \underline{y}(t) = \begin{bmatrix} x(t,z_1) \\ x(t,z_2) \end{bmatrix}, \tag{33}$$

the following alternative feedback laws will be discussed:

$$\text{I)}\quad \underline{u}(t) = \underline{K}_o\underline{w}(t) - \underline{K}\,\dot{\underline{y}}(t), \tag{34}$$

$$\text{II)}\quad \underline{u}(t) = \underline{K}_o\underline{w}(t) - \underline{K}_P\underline{y}(t) - \underline{K}_D\dot{\underline{y}}(t). \tag{35}$$

Eq. (35) can be regarded as a multivariable PD-Controller with feedback of both deflection and velocity.

The design objectives are

- Stabilization with guaranteed spillover prevention,
- Matching of a simple closed-loop transfer model including noninteraction.

5.1 Stability

It can be shown by Ljapunov's direct method:
Controller (34) stabilizes the beam equation asymptotically if

(i) $\varphi_i(z_k) \neq 0, \quad i = 1, 2, 3, \ldots$
$\qquad\qquad\qquad k = 1, 2,$

where $\varphi_i(z)$ are the eigenfunctions of the beam equation, and

(ii) $\underline{K}$ is any symmetric and positive definite matrix.

For control law (35) to stabilize the beam equation asymptotically, condition (ii) has to be replaced by

(iii) $\underline{K}_P$ and $\underline{K}_D$ are any symmetric and positive definite matrices.

5.2 Model matching

According to (28) the relative degrees with respect to velocities $\dot{y}_1$ and $\dot{y}_2$ are $d_1 = d_2 = 1$, hence the relative degrees with respect to deflections y_1 and y_2 are $\tilde{d}_1 = \tilde{d}_2 = 2$. This motivates prescription of a simple and well damped response to unit step input

$$\underline{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}: \qquad \hat{\underline{y}}(\tau) \overset{!}{=} \begin{bmatrix} 0.25(1+\tau)^2 \\ 0 \end{bmatrix}, \; -1 \le \tau \le 1. \tag{36}$$

The polynomial $0{,}25(1+\tau)^2$ is the simplest one which on the one hand meets the system's transient abilities, characterized by $\tilde{d}_1 = 2$, and on the other hand meets the steady state requirement

$$\hat{\underline{y}}(+1) = \underline{w}, \quad \text{hence } \underline{y}(t\to\infty) = \underline{w}. \tag{37}$$

By inserting the time transformation (1) into (36) one obtains the original function

$$\underline{y}(t) = \begin{bmatrix} (1 - e^{-\alpha t})^2 \\ 0 \end{bmatrix} \quad 0 \le t < \infty. \tag{38}$$

From (38) it can be seen that the input-output behaviour to be matched is a second order lumped parameter model. Moreover, the time scaling parameter α to be selected has a very simple meaning: $(-\alpha)$ is the dominant pole of the finite-dimensional model to be matched.

Due to the second order polynomial, $\hat{\underline{y}}(\tau)$ contains only three Fourier coefficients:

$$\underline{y}_0^* = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}, \quad \underline{y}_1^* = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad \underline{y}_2^* = \begin{bmatrix} 1/6 \\ 0 \end{bmatrix}. \tag{39}$$

In the same way, if the reference input is selected as

$$\underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

the Fourier coefficients of the output are prescribed as

$$\underline{y}_0^* = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}, \quad \underline{y}_1^* = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \quad \underline{y}_2^* = \begin{bmatrix} 0 \\ 1/6 \end{bmatrix}. \tag{40}$$

This model includes noninteraction.

Balancing the Fourier coefficients (39), (40) in the closed loop yields a set of linear equations for the controller matrices (see step (g) in section 4). These equations are overdetermined, and therefore they are solved approximately in the least square sense.

5.3 Numerical results

The following points of measurement and control will be selected,

$$z_1 = 1 - \sqrt{2}/2 \approx 0.293, \qquad z_2 = \sqrt{2}/2 \approx 0.707, \tag{41}$$

which makes the problem symmetric with respect to z.

First, controller (34) will be computed by selecting $\alpha = 5$:

$$\underline{K}(5) = \begin{bmatrix} 57.1 & -46.6 \\ -46.6 & 57.1 \end{bmatrix} \quad \underline{K}_0(5) = \begin{bmatrix} 225.2 & -186.8 \\ -186.8 & 225.2 \end{bmatrix}.$$

Obviously $\underline{K}(5)$ is positive definite which guarantees asymptotic stability of the closed-loop system. Simulation results can be seen from Figures 1, 2 and 3. In Fig. 2 the response of the actual system is compared to the response of the model versus the τ-coordinate. The

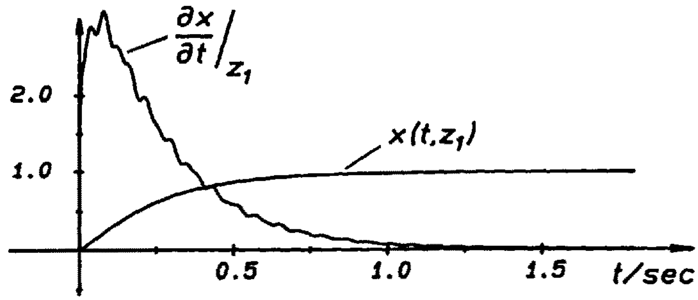

Fig. 1. Response to $\underline{w}^T = (1,0)$ at point z_1 with feedback law (34), $\alpha = 5$

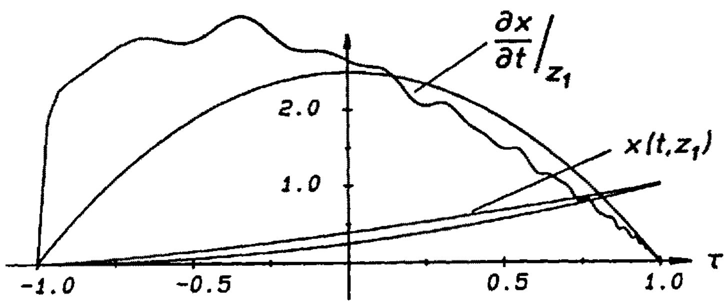

Fig. 2. Transformation of Fig. 1 on the τ-coordinate

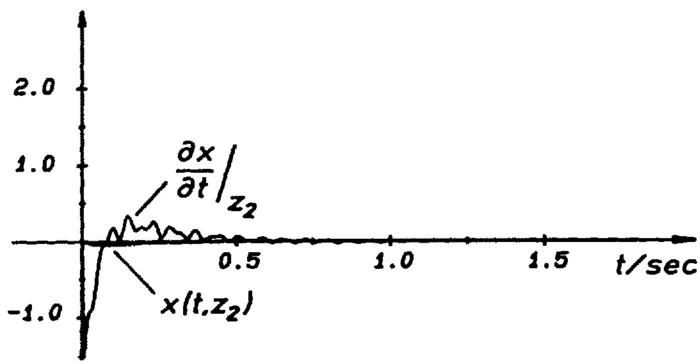

Fig. 3. Response to $\underline{w}^T = (1,0)$ at point z_2 with feedback law (34), $\alpha = 5$

error between model and actual system is due to the approximations made. From Fig. 3 it can be seen that the noninteraction is quite favourable.

Feedback law (35) is underlying the simulations in Figures 4, 5 and 6. The error between model and actual system is now smaller than in the previous case, because the controller has more parameters to meet the specified Fourier coefficients. Again, z_1 and z_2 have been selected according to (41), however $\alpha = 50$ now. The gain matrices ob-

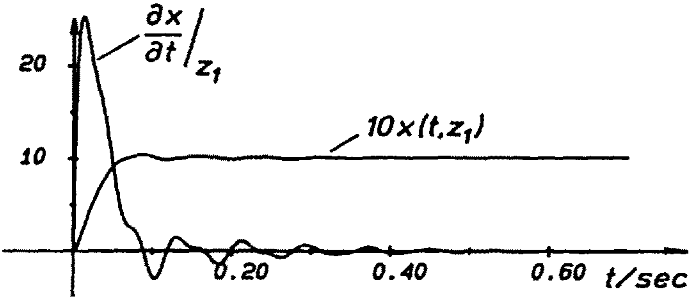

Fig. 4. Response to $\underline{w}^T = (1,0)$ at point z_1 with feedback law (35), $\alpha = 50$

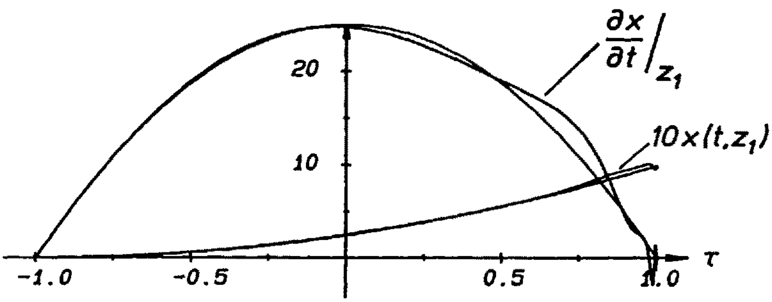

Fig. 5. Transformation of Fig. 4 on the τ-coordinate

tained by balancing Fourier coefficients are

$$\underline{K}_P(50) = \begin{bmatrix} 1098 & 443 \\ 443 & 1098 \end{bmatrix}, \quad \underline{K}_D(50) = \begin{bmatrix} 34.9 & 7.7 \\ 7.7 & 34.9 \end{bmatrix},$$

$$\underline{K}_0(50) = \begin{bmatrix} 1324 & 257 \\ 257 & 1324 \end{bmatrix}.$$

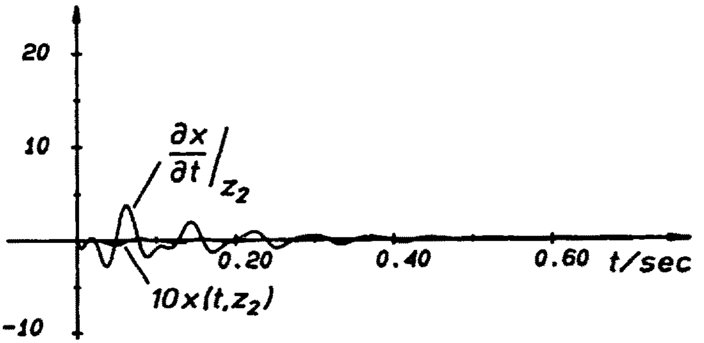

Fig. 6. Response to $\underline{w}^T = (1,0)$ at point z_2 with feedback law (35), $\alpha = 50$

Asymptotic stability is guaranteed, since $\underline{K}_P$ and $\underline{K}_D$ are positive definite.

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