A NONLINEAR ABEL INTEGRAL EQUATION

Dang Dinh Ang
Department of Mathematics, HoChiMinh City University
Ho Chi Minh City, Vietnam

Rudolf Gorenflo
Fachbereich Mathematik, Freie Universität Berlin
Arnimallee 2-6, D-1000 Berlin 33, Germany

Abstract

For the general nonlinear Abel integral equation

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}K(x,t,u(t))dt = f(x), \quad 0 \le x \le 1, \; 0 < \alpha < 1,$$

some theorems on existence and uniqueness of solutions in $L_p, 1 \le p \le \infty$, and in $C[0,1]$ are established. Furthermore, methods of regularization are described and stability estimates are given.

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1. Introduction

Consider the integral equation

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}K(x,t,u(t))dt = f(x), \qquad 0 \le x \le 1, \tag{1}$$

where $0 < \alpha < 1$ and $K : T \times I\!\!R \longrightarrow I\!\!R$ and $f : [0,1] \longrightarrow I\!\!R$ with

$$T = \{(x,t) | (x,t) \in I\!\!R^2, \; 0 \le t \le x \le 1\}$$

are given functions. The following will be tacitly assumed throughout:

(A1) $K \in C(T \times I\!R)$.

(A2) For $(t,w) \in [0,1] \times I\!R$, the function $x \longmapsto K(x,t,w)$ is differentiable on $[t,1]$ and $K_x \in C(T \times I\!R)$.

(A3) There exists a constant $c > 0$ such that

$$(K(x,x,w_1) - K(x,x,w_2))(w_1 - w_2) \geq c(w_1 - w_2)^2$$

for all x in $[0,1]$ and w_1, w_2 in $I\!R$.

(A4) K_x is Lipschitzian with respect to the third variable, i.e. there exists a constant $M > 0$ such that

$$|K_x(x,t,w_1) - K_x(x,t,w_2)| \leq M|w_1 - w_2|.$$

As an example of a function K satisfying (A1), (A2), (A3) and (A4), but not lying in $C^1(T \times I\!R)$, take

$$K(x,t,w) = w + \frac{1}{4}\,\frac{1}{1 + (x-t)^2 + |w|}.$$

We shall use operators J^β of fractional integration and D^β of fractional differentation. For $0 < \beta < 1$ we define

$$J^\beta u(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} u(t)dt, \; D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_0^x (x-t)^{-\beta} f(t)dt.$$

Note that $D^\beta = DJ^{1-\beta}$ with $D = \frac{d}{dx}$.

The purpose of this paper is to establish some existence and uniqueness results in $L_p(0,1)$, $1 \leq p \leq \infty$, and in $C[0,1]$ and to produce regularized solutions to equation (1). Stability estimates will be given.

We observe that equation (1) was studied by Branca [Br] for the case $\alpha = 1/2$, K in $C^1(T \times I\!R)$, with K_w bounded away from zero and K_x Lipschitzian with respect to the third variable. Branca's results were extended to the general case $0 < \alpha < 1$ by Brunner and Van der Houwen [BH] and, independently, by Gorenflo and Vessella [GV]. The basic tool used was Dini's implicit function theorem. Dini's theorem will not work here because of lack of differentiability. However, the kernel K generates a monotone operator, and we can make use of a general theorem on monotone operators. Our strategy is as follows. We start with an existence theorem in the L_2-case , then move to the L_∞-case and the continuous case, and end up with the L_1- and L_2-cases with stability estimates.

We first convert equation (1) into an equivalent equation, easier to handle. As in [GV], we apply the operator $J^{1-\alpha}$ to both sides of (1) to get

$$\int_0^x H(x,t,u(t))dt = J^{1-\alpha}f(x) \tag{2}$$

where

$$H(x,t,w) = \frac{\sin(\alpha\pi)}{\pi}\int_t^x (x-y)^{-\alpha}(y-t)^{\alpha-1}K(y,t,w)dy. \tag{3}$$

Note that H satisfies

(i) $H \in C(T \times \mathbb{R})$.

(ii) H_x is continuous on $T \times \mathbb{R}$ and is Lipschitz continuous with respect to the third variable with the same Lipschitz constant as for K_x.

(iii) $H(x,x,w) = K(x,x,w)$.

Differentiating (2) with respect to x, we get

$$K(x,x,u(x)) + \int_0^x H_x(x,t,u(t))dt = D^\alpha f(x). \tag{4}$$

Observe that if $J^{1-\alpha}f$ is absolutely continuous and if $u \in L_1 = L_1(0,1)$, then the foregoing differentiation process is valid. Furthermore, if $f(0) = 0$ and f is absolutely continuous, then $J^{1-\alpha}f$ is absolutely continuous. However, $J^{1-\alpha}f$ need not be absolutely continuous if f is only in L_1.

We have converted equation (2) into (4), which is an equation of second kind. This is possible if $J^{1-\alpha}f$ is absolutely continuous. In the absence of absolute continuity, one has to work with an equation of first kind. In the latter case, some kind of regularization is in order. This problem will be taken up in the final portion of the paper.

2. Existence and Uniqueness Results

We first give an existence theorem in the L_2-case.

Theorem 1: *Suppose $D^\alpha f$ is in $L_2 = L_2(0,1)$. Then equation (4) has a unique solution in L_2. If, furthermore, $J^{1-\alpha}f(0) = 0$, then this solution is the unique solution of (1).*

Proof: For $w = w(t)$ given in L_2, consider the following equation in $u = u(x)$:

$$K(x,x,u(x)) = -\int_0^x H_x(x,t,w(t))dt + D^\alpha f(x). \tag{5}$$

Since w is in L_2, the first term of the right hand side is continuous, and in particular in L_2. Thus the right hand side is in L_2. Let $A : L_2 \longrightarrow L_2$ be defined by

$$Au(x) = K(x, x, u(x)).$$

Then we can show that A is monotone. In fact, we have

$$< Au - Av, u - v > \;\; \geq \;\; c \parallel u - v \parallel_2^2, \tag{6}$$

Here $< \cdot, \cdot >$ is the L_2 inner product, and $\parallel \cdot \parallel_2$ is the L_2 norm. By (6), A is monotone and furthermore

$$< Au, u > / \parallel u \parallel_2 \longrightarrow \infty \text{ for } \parallel u \parallel_2 \longrightarrow \infty. \tag{7}$$

Since H_x is Lipschitzian with respect to the third variable, A takes bounded sets into bounded sets. Finally, A is weakly continuous on lines. Hence by Theorem 2.1 of [L], p.171, there exists an element u in L_2 such that

$$Au(x) = -\int_0^x H_x(x, t, w(t))dt + D^\alpha f(x). \tag{8}$$

The solution is clearly unique. Moreover, we have

$$c|u_1(x) - u_2(x)| \leq |Au_1(x) - Au_2(x)| = |v_1(x) - v_2(x)|. \tag{9}$$

Thus

$$|A^{-1}v_1(x) - A^{-1}v_2(x)| \leq |v_1(x) - v_2(x)|/c.$$

Consider the operator $A^{-1}B$, where B is defined as

$$Bw(x) = -\int_0^x H_x(x, t, w(t))dt + D^\alpha f(x). \tag{10}$$

As shown earlier

$$u(x) = A^{-1}Bw(x). \tag{11}$$

We shall prove that $A^{-1}B$ has a unique fixed point $u = A^{-1}Bu$, and that u can be computed by successive approximation. Put

$$\begin{aligned} u_0(x) &= 0, \\ &\vdots \\ u_n(x) &= A^{-1}Bu_{n-1}(x) \,, \end{aligned}$$

i.e.

$$u_n(x) = A^{-1}(-\int_0^x H_x(x,t,u_{n-1}(t))dt + D^\alpha f(x)). \tag{12}$$

Then, for $n \geq 1$, we have

$$|u_{n+1}(x) - u_n(x)| \leq (\frac{M}{c})^n \frac{1}{n!} \int_0^x |u_1(t)|dt \quad .$$

Thus

$$\| u_{n+1} - u_n \|_2 \leq (M/c)^n \frac{1}{n!} \| u_1 \|_2 \, . \tag{13}$$

Hence, (u_n) converges in L_2 to a function u, which, by the continuity of $A^{-1}B$, is a fixed point of $A^{-1}B$, i.e.,

$$u = A^{-1}Bu,$$

or equivalently,

$$K(x,x,u(x)) + \int_0^x H_x(x,t,u(t))dt = D^\alpha f(x). \tag{14}$$

The L_2-solution is unique since $(A^{-1}B)^n$ is a contraction for n large. This completes the proof of Theorem 1.

We next consider the L_∞-case and the continuous case.

Theorem 2: *Suppose $D^\alpha f$ is in $L_\infty = L_\infty(0,1)$. Then equation (4) has a unique solution in L_∞. If u_1, u_2 are the solutions of (4) corresponding to $D^\alpha f_i$, $i = 1,2$, then the following holds:*

$$\| u_1 - u_2 \|_\infty \leq e^{-1} \exp(\frac{M}{c}) \| D^\alpha f_1 - D^\alpha f_2 \|_\infty \, . \tag{15}$$

Proof: Let u be the L_2-solution of (4). It is sufficient to show that u ist in L_∞. Now, the second term of the left hand side of (4) is bounded since it is continuous. Since the right hand side is (essentially) bounded by hypothesis, is follows that $K(x,x,u(x))$ is essentially bounded. Then

$$c \| u \|_\infty \leq \| K(x,x,u(x)) - K(x,x,0) \|_\infty + \| K(x,x,0) \|_\infty \, .$$

Hence u ist in L_∞. Now, let u_1, u_2 be the L_∞-solutions of (4) corresponding to $D^\alpha f_1, D^\alpha f_2$. Then,

$$|u_1(x) - u_2(x)| \leq \| D^\alpha f_1 - D^\alpha f_2 \|_\infty /c + (M/c) \int_0^x |u_1(t) - u_2(t)|dt \, .$$

By Gronwall's inequality,

$$\| u_1 - u_2 \|_\infty \leq \frac{e^{M/c}}{c} \| D^\alpha f_1 - D^\alpha f_2 \|_\infty .$$

QED.

Theorem 3: *Suppose $D^\alpha f$ is continuous on $[0,1]$. Then there exists a unique continuous solution of (4). If u_i is the continuous solution of (4) corresponding to $D^\alpha f_i$, $i = 1,2$, then the following holds:*

$$|u_1(x)-u_2(x)| \leq \frac{1}{c}|D^\alpha f_1(x)-D^\alpha f_2(x)|+\frac{M}{c^2}\int_0^x \exp[\frac{M}{c}(x-s)]\cdot|D^\alpha f_1(s)-D^\alpha f_2(s)|ds. \quad (16)$$

Proof: Let u be the L_2-solution of (4). Then, since the right hand side of (4) and the second term of the left side are continuous, it follows that $K(x,x,u(x))$ is continuous. Denoting it by $h(x)$, we have

$$|K(x',x',u(x)) - K(x',x',u(x')| = |K(x',x',u(x)) - K(x,x,u(x)) + h(x) - h(x')|.$$

Hence

$$|u(x) - u(x')| \leq c^{-1}|K(x',x',u(x)) - K(x,x,u(x))| + c^{-1}|h(x) - h(x')|.$$

Thus $u(x') \longrightarrow u(x)$ for $x' \longrightarrow x$. We have just proved that u is continuous.

For a stability estimate, let u_i be the continuous solution of (4) corresponding to $D^\alpha f_i$, $i = 1,2$. Then we have

$$|u_1(x) - u_2(x)| \leq \frac{1}{c}|D^\alpha f_1(x) - D^\alpha f_2(x)| + \frac{M}{c}\int_0^x |u_1(t) - u_2(t)|dt.$$

By Gronwall's generalized inequality [Hi]:

$$|u_1(x) - u_2(x)| \leq \frac{1}{c}|D^\alpha f_1(x) - D^\alpha f_2(x)| + \frac{M}{c^2}\int_0^x \exp(\frac{M}{c}(x - s))|D^\alpha f_1(s) - D^\alpha f_2(s)|ds.$$

This concludes the proof of Theorem 3.

We finally consider the L_p-case.

Theorem 4: *Suppose $D^\alpha f$ is in L_1. Then there exists a unique L_1-solution of (4). If u_i is the L_1 -solution of (4) corresponding to $D^\alpha f_i$, $i = 1,2$, then the following holds:*

$$\| u_1 - u_2 \|_1 \leq \frac{1}{c}\exp(\frac{M}{c}) \| D^\alpha f_1 - D^\alpha f_2 \|_1 . \quad (17)$$

Proof: Let (g_n) be a sequence of continuous functions converging in L_1 to $D^\alpha f$. By Theorem 3, if u_n is the continuous solution of (4) corresponding to g_n in the right hand side, the following holds:

$$|u_n(x) - u_m(x)| \leq \frac{1}{c}|g_n(x) - g_m(x)| + \frac{M}{c^2}\int_0^x \exp(\frac{M}{c}(x-t))|g_n(t) - g_m(t)|dt.$$

Integrating over x from 0 to 1 gives

$$\| u_n - u_m \|_1 \leq \frac{1}{c} \| g_n - g_m \|_1 + \frac{M}{c^2}\int_0^1 \int_0^x \exp(\frac{M}{c}(x-t))|g_n(t) - g_m(t)|dt\, dx \qquad (18)$$

$$= \frac{1}{c} \| g_n - g_m \|_1 + \frac{M}{c^2}\int_0^1 \int_t^1 \exp(\frac{M}{c}(x-t))dx|g_n(t) - g_m(t)|dt \leq \frac{1}{c}e^{M/c} \| g_n - g_m \|_1 .$$

Thus (u_n) is a Chauchy sequence in L_1, which converges to u, say. It is easily seen that u is the L_1-solution of (4). The stability estimate (17) is derived by considering sequences of continuous functions g_n^1, g_n^2 converging in L_1 to $D^\alpha f_1, D^\alpha f_2$ respectively, and passing to the limits in [18], with u_n, u_m replaced by u_n^1, u_n^2 respectively. QED.

Theorem 5: *Suppose $D^\alpha f$ is in L_2. Then the L_2 -solution of (4), which exists (and is unique) by Theorem 1, is stable with respect to variations in $D^\alpha f$. In fact, if u_i is the L_2-solution of (4) corresponding to $D^\alpha f_i$, $i = 1,2$, then the followings holds:*

$$\| u_1 - u_2 \|_2 \leq \frac{1}{c}e^{M/c} \| D^\alpha f_1 - D^\alpha f_2 \|_2 . \qquad (19)$$

Proof: We can (and shall) assume that $D^\alpha f_1$ and $D^\alpha f_2$ are continuous. The general case is obtained by passing to the limit. For $D^\alpha f_i \equiv g_i$ continuous, $i = 1,2$, the corresponding (continuous) solutions u_1, u_2 of (4) satisfy, by Theorem 3,

$$|u_1(x) - u_2(x)| \leq \frac{1}{c}|g_1(x) - g_2(x)| + \frac{M}{c^2}\int_0^x \exp(\frac{M}{c}(x-t))|g_1(t) - g_2(t)|dt. \qquad (20)$$

Consider the second term in the right hand side of (20) and denote it by Q, for brevity. Squaring and using Schwarz's inequality give

$$Q^2 \leq (\frac{M}{c^2})^2 e^{2Mx/c}\int_0^x \exp(-\frac{Mt}{c})dt \int_0^x \exp(-\frac{Mt}{c})|g_1(t) - g_2(t)|^2 dt$$

$$= \frac{M}{c^3} e^{2Mx/c}(1 - e^{-Mx/c}) \int_0^x \exp(-\frac{Mt}{c})|g_1(t) - g_2(t)|^2 dt.$$

Integrating the latter quantity from 0 to 1 gives, using Fubini's theorem and rearranging,

$$\frac{M}{c^3} \int_0^1 (e^{Mx/c} - 1) \int_0^x \exp(\frac{M}{c}(x-t))|g_1(t) - g_2(t)|^2 dt\, dx \tag{21}$$

$$\leq \frac{M}{c^3}(e^{M/c} - 1) \int_0^1 \int_t^1 \exp(\frac{M}{c}(x-t))dx|g_1(t) - g_2(t)|^2 dt \leq \frac{1}{c^2}(e^{M/c} - 1)^2 \| g_1 - g_2 \|_2^2 .$$

Hence the L_2-norm of the right hand side of (20) is majorized by $\frac{1}{c}e^{M/c} \| g_1 - g_2 \|_2$. Thus

$$\| u_1 - u_2 \|_2 \leq \frac{1}{c} e^{M/c} \| g_1 - g_2 \|_2 . \tag{22}$$

QED.

Remark 1: From what precedes, it is clear that if $D^\alpha f \in L_p$, $1 < p < \infty$, then (4) admits a unique L_p-solution. Furthermore, the L_p-solution is stable with respect to variations in $D^\alpha f$, and stability estimates of the type (19) can be derived by using Hölder's inequality instead of Schwarz's inequality. Combining with the estimates (15) and (17) one then has

$$\| u_1 - u_2 \|_p \leq \frac{1}{c} e^{M/c} \| D^\alpha f_1 - D^\alpha F_2 \|_p \quad \text{for} \ \ 1 \leq p \leq \infty.$$

We do not pursue this matter further.

Remark 2: It is observed that if $u \in L_1$, then the left hand side of (2) is an absolutely continuous function. As a first consequence, equation (2) is equivalent to equation (4). A second consequence is that (2) and (because of equivalence with (2)) (1) has a continuous (resp. L_p) solution only if $J^{1-\alpha} f(0) = 0$ and $J^{1-\alpha} f$ has a derivative that is continuous (resp. in L_p).

3. Regularization

Consider equation (2). We have seen in Section 2 that if $J^{1-\alpha} f$ is absolutely continuous, then (2), an equation of first kind, is equivalent to (4), an equation of second kind. The problem is then well posed in the sense that if $D^\alpha f$ belongs to $C[0,1]$ or $L_p(0,1)$, then a unique solution exists in the corresponding function space and depends continuously on $D^\alpha f$. We are now considering the case where $J^{1-\alpha} f$ is not supposed to be absolutely continuous but simply to be continuous or in L_p, and is known only approximately. In the case of the classical (linear) Abel equation, it is known that the problem is ill-posed in the usual (and most useful) function spaces. It can be shown that in the present non-linear case, the problem is also ill-posed in the usual function spaces. Hence some kind of regularization is required.

In the sequel, it will be assumed that g is in L_2 (resp. L_1) and that g_0 is an L_2-function (resp. L_1-function) such that

$$\| g - g_0 \|_2 \leq \epsilon \quad (\text{resp.} \quad \| g - g_0 \|_1 \leq \epsilon). \tag{23}$$

It is assumed that g_0 is absolutely continuous with a derivative g_0' in L_2 or L_1. Let u be the solution of (4) corresponding to g_0' in the right hand side (g_0 in place of $J^{1-\alpha}f$, g_0' in place of $D^\alpha f$). It is our purpose to "construct" a function that depends continuously on g and is δ-close to u where $\delta = \delta_{(\epsilon)} \longrightarrow 0$ for $\epsilon \longrightarrow 0$. Such a function will be called a *regularized solution* of (2).

It will be convenient to put

$$Jv(x) = \int_0^x v(t)dt \quad \text{for} \quad v \quad \text{in} \quad L_1(0,1).$$

Our regularization problem here consists in approximating the derivative of a function. We give two sample results.

Theorem 6: *Let g and g_0 be in $L_2(0,1)$ such that*

$$\| g - g_0 \|_2 \leq \epsilon. \tag{24}$$

Suppose

$$g_0(x) = \int_0^x v(t)dt \tag{25}$$

where $v \in H^1(0,1)$ with

$$\| v \|_2 + \| v' \|_2 \leq E. \tag{26}$$

For $\beta = \sqrt{\epsilon/E}$, let v_β be given by

$$v_\beta = (\beta I + J)^{-1} g \tag{27}$$

(with I as identity operator) and let u_β be the solution of the equation

$$K(x, x, u_\beta(x)) + \int_0^x H_x(x, t, u_\beta(t))dt = v_\beta(x). \tag{28}$$

Suppose u is the solution of the equation

$$\int_0^x H(x,t,u(t)dt = g_0(x). \tag{29}$$

Then

$$\| u - u_\beta \|_2 \leq (3/c)e^{M/c}\sqrt{E\epsilon}. \tag{30}$$

Remark 3: For application of Theorem 6 it is desirable to specify bounds on $g_0' = v$ in term of bounds on u. We propose to do this follows.

Put $h(x,w) = H(x,x,w)$ (which is $= K(x,x,w)$). Suppose $h : [0,1] \times I\!R \longrightarrow I\!R$ is C^1 and $K_{xx}(x,t,w)$ is continuous on $T \times I\!R$. Assume that u, the solution of (29), is in $H^1(0,1)$ with $\| u \|_2 + \| u' \|_2 \leq c$. Put

$$E = |h|_\infty + \| H_x \|_\infty + |h_x|_\infty + c|h_w|_\infty + |H_{xx}|_\infty + \| H_x \|_\infty$$

where

$$|\cdot| = \sup\{|\cdot| \ \ |0 \leq x \leq 1, |w| \leq c\}\ ,$$

$$\|\cdot\|_\infty = \sup\{|\cdot| \ \ |(x,t) \in T,\ |w| \leq c\}.$$

Then $\| v \|_2 + \| v' \|_2 \leq E$.

Proof: It can be shown (cf. [HA 1] and [Go] for methods of estimation) that $\| v - v_\beta \|_2 \leq 3\sqrt{E\epsilon}$. Combining this with (19) we have (30). QED.

Theorem 7: *Let g, g_o satisfy $\| g - g_0 \|_1 \leq \epsilon$. Suppose $g_0(x) = \int_0^x v(t)dt$ where v is of bounded variation with $var(v) \leq E$ where $var(v)$ is the total variation of v on $[0,1]$. For $0 < h < E/4$, put*

$$g_h(x) = \frac{1}{h}(g(x+h) - g(x)) \ \ \textit{if} \ \ 0 \leq x \leq 1-h, = \frac{1}{h}(g(x) - g(x-h)) \ \ \textit{if} \ \ 1-h < x \leq 1.$$

Let u be the solution of the equation

$$\int_0^x H(x,t,u(t)dt = g_0(x) \tag{31}$$

and let u^h be the solution of

$$K(x,x,u^h(x)) + \int_0^x H_x(x,t,u^h(t))dt = g_h(x). \tag{32}$$

Then

$$\| u^h - u \|_1 \leq (4/c)e^{M/c}\sqrt{E\epsilon}. \tag{33}$$

Proof: It is shown in [HA 2] that $\| v - g_h \|_1 \leq 4\sqrt{E\epsilon}$. Combining this with (17), we have (31).

Remark 4: *Suppose K is in C^1 $(T \times I\!R)$ and $K_x(x,t,w)$ is Lipschitzian with respect to x, with Lipschitz constant L. Assume that u, the solution of (31), is of bounded variation, var $(u) \leq c$. Put*

$$E = cM_0 + 2 \| H_x \|_\infty + \| H_t \|_\infty + L,$$

with M_0 as Lipschitz constant of $K(x,t,w)$ with respect to w and

$$\| \cdot \|_\infty = \sup\{ | \cdot | \quad |(x,t) \in T, \quad |w| \leq c\}.$$

Then var $(v) \leq E$.

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