# STATE-CONSTRAINED CONTROL PROBLEMS OF QUASILINEAR ELLIPTIC EQUATIONS

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## 1   INTRODUCTION

In this work we continue the study of optimal control problems governed by quasilinear elliptic equations in divergence form. Up to now, we have only considered control constraints. Here we also consider state constraints (essentially of integral type). Our aim is to prove existence of solution and derive the optimality conditions. Moreover we utilize the optimality system to study the optimal control and optimal state regularity.

We consider the following differential operator:

$$Ay = -div(a(x, \nabla y)) + a_0(x, y)$$

where $a(x, \eta) = (a_1(x, \eta), \dots, a_n(x, \eta))$ and the associated Dirichlet problem:

$$\begin{cases} Ay = u & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases} \tag{1}$$

where $\Omega$ is a bounded and open subset of $R^n$ and $\Gamma = \partial\Omega$ is Lipschitz continuous.

In the sequel we will assume the conditions:

$$\begin{cases} a_j(\cdot, \eta) \text{ is a measurable function in } \Omega \\ a_j(x, \cdot) \text{ belongs to } C^1(R^n) \quad j = 1, \dots, n \end{cases} \tag{2}$$

$$\begin{cases} a_0(\cdot, s) \text{ is a measurable function in } \Omega \\ a_0(x, \cdot) \text{ belongs to } C^1(R) \end{cases} \tag{3}$$

$$\sum_{i,j=1}^{n} \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq \Lambda_1(k + |\eta|)^{\alpha-2}|\xi|^2 \tag{4}$$

$$\sum_{i,j=1}^{n} \left| \frac{\partial a_j}{\partial \eta_i}(x, \eta) \right| \leq \Lambda_2(k + |\eta|)^{\alpha-2} \tag{5}$$

$$0 \leq \frac{\partial a_0}{\partial s}(x,s) \leq f(|s|) \tag{6}$$

$$a_0(x,0) = a_j(x,0) = 0 \quad j = 1,\dots,n \tag{7}$$

for some $k \in [0,1]$, some $\alpha \in (1,+\infty)$, some strictly positive constants $\Lambda_1, \Lambda_2$, some positive and non-decreasing function $f$, all $x \in \Omega$, all $s \in R$ and all $\eta, \xi \in R^n$.

We make the following additional assumption on $\alpha$:

$$\alpha > n/2 \tag{8}$$

Under hypotheses 2–8, for each $u \in L^2(\Omega)$ there exists a unique solution $y_u \in W_0^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ of Dirichlet problem 1. Moreover there exists an upper bound of $\|y_u\|_{L^\infty(\Omega)}$ depending only on $A$, $\Omega$ and $\|u\|_{L^2(\Omega)}$, see J.M. Rakotoson [9]. Let us note that $L^2(\Omega)$ is included in $W^{-1,\beta}(\Omega)$ thanks to 8, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

In this work we extend some results related to the control problems of linear and semilinear equations with state constraints (see for instance J.F. Bonnans and E. Casas [1]). These problems are included in the case $\alpha = 2$.

This paper is organized as follows: in Section 2 we present the control problems; in Section 3, we study the conditions under which the functional $v \longrightarrow y_v$ is differentiable; in Sections 4 and 5, we derive the optimality systems in the differentiable and non differentiable cases respectively. In this last case we introduce a family of approximating problems and we pass to the limit in the correspondent optimality conditions. In Section 6, we prove the qualification of the control problems in certain particular situations. Finally, in Section 7, we utilize the optimality conditions to derive some regularity results of the optimal control and state.

# 2   THE CONTROL PROBLEMS

Let $K$ be a non-empty, convex and closed subset of $L^2(\Omega)$ and $J : L^2(\Omega) \longrightarrow R$ the cost functional defined by

$$J(v) = \frac{1}{2} \int_\Omega |y_v - y_d|^2 dx + \frac{\nu}{2} \int_\Omega |v|^2 dx$$

with $y_d$ a fixed element of $L^2(\Omega)$ and $\nu$ a non-negative constant.

We are concerned with the following optimal control problems:

**PROBLEM 1 (Differentiable Constraints of Integral Type)**

$$(P_1) \begin{cases} Minimize \ J(v) \\ \\ Subject \ to \ v \in K \ and \ \displaystyle\int_\Omega g_j(x,y_v(x))dx \leq \delta_j \ \ 1 \leq j \leq m. \end{cases}$$

**PROBLEM 2 (Non Differentiable Constraints of Integral Type)**

$$(P_2) \begin{cases} Minimize \ J(v) \\ \\ Subject \ to \ v \in K \ and \ \displaystyle\int_\Omega |y_v(x)|dx \leq \delta. \end{cases}$$

**PROBLEM 3 (Pointwise Constraints)**

$$(P_3) \begin{cases} \textit{Minimize } J(v) \\ \\ \textit{Subject to } v \in K \textit{ and } |y_v(x)| \leq \delta \quad \forall x \in \Omega. \end{cases}$$

Existence of solutions can be established in a standard form using the continuity of functional $v \in L^2(\Omega) \longrightarrow y_v \in W_0^{1,\alpha}(\Omega)$, the dominated convergence theorem (for $(P_1)$ and $(P_2)$) and the Sobolev inclusions (for $(P_3)$).

**THEOREM 1** *Let us suppose the following conditions:*

1. *There exist an admissible point for* $(P_i)$, $1 \leq i \leq 3$.

2. *For* $i = 1$, $g_j : \Omega \times R \longrightarrow R$ *is a Caratheodory function and there exist* $h \in L^1(\Omega)$ *and* $\phi : R^+ \longrightarrow R^+$ *non decreasing such that*

$$|g_j(x,y)| \leq h(x)\phi(|y|) \quad a.e. \ x \in \Omega \quad \forall y \in R \ 1 \leq j \leq m.$$

3. *Either* $K$ *is bounded in* $L^2(\Omega)$ *or* $\nu > 0$.

*Then there exists (at least) one solution of* $(P_i)$, $1 \leq i \leq 3$.

# 3   SENSITIVITY ANALYSIS

In order to derive the optimality system, the main question to investigate is the differentiability of functional $v \longrightarrow y_v$. For this study we will assume that $k \neq 0$. In fact, for $\alpha > 2$ and $k = 0$ this application is not necessarily differentiable. An example can be found in [3].

Now, given $y \in W_0^{1,\alpha}(\Omega)$, let us define the space $H_0^y(\Omega)$ as the completion of $D(\Omega)$ with respect to the norm:

$$\|z\| = \left( \int_\Omega (k + |\nabla y|)^{\alpha-2} |\nabla z|^2 dx \right)^{1/2}$$

It may be easily verified that $H_0^y(\Omega)$ is a Hilbert space with the inner product

$$(z_1, z_2) = \int_\Omega (k + |\nabla y|)^{\alpha-2} \nabla z_1 \nabla z_2 dx$$

Moreover we have

$$W_0^{1,\alpha}(\Omega) \subset H_0^y(\Omega) \subset H_0^1(\Omega) \quad \text{if} \ \alpha \geq 2$$
$$H_0^1(\Omega) \subset H_0^y(\Omega) \subset W_0^{1,\alpha}(\Omega) \quad \text{if} \ \alpha \leq 2$$

with continuous injections.

More general spaces of this type (weighted Sobolev spaces) have been studied by C.V. Coffman et al. [5], M.K.V. Murthy and G. Stampacchia [8] and N.S. Trudinger [13].

In the next theorem we prove that the relationship between the control and the state is differentiable in some cases.

**THEOREM 2** *Let us suppose $k \neq 0$ and one of the following assumptions:*

*1. $\alpha \geq 2$.*

*2. $\alpha < 2$ and $n = 1$.*

*Let $F : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ (resp. $W_0^{1,\alpha}(\Omega)$ if $\alpha < 2$) be the functional defined by $F(u) = y_u$. Then $F$ is Gâteaux differentiable. Moreover, if $DF(u)v = z$, then $z$ belongs to $H_0^{y_u}(\Omega)$ and it is the unique solution in this space of problem*

$$
\begin{cases}
-div\left(\dfrac{\partial a}{\partial \eta}(x, \nabla y_u)\nabla z\right) + \dfrac{\partial a_0}{\partial s}(x, y_u)z = v & in \;\; \Omega \\[4mm]
\\
z = 0 & on \;\; \Gamma
\end{cases}
\tag{9}
$$

The proof of this theorem can be found in the paper of the authors [3]. Here we only mention some ideas. Given $u, v \in L^2(\Omega)$ and $0 < t < 1$, we consider the problems

$$
\begin{cases}
Ay_t = u + tv & in \;\; \Omega \\
y_t = 0 & on \;\; \Gamma
\end{cases}
$$

In case $\alpha \geq 2$, we prove that the sequence $\left\{\dfrac{y_t - y}{t}\right\}_{t>0}$ converges towards an element $z$ weakly in $H_0^{y_u}(\Omega)$ and strongly in $H_0^1(\Omega)$. This part requires a rather long development. For proving that $z \in H_0^{y_u}(\Omega)$ it is essential that

$$
\frac{y_t - y}{t} \in W_0^{1,\alpha}(\Omega) \subset H_0^{y_u}(\Omega).
$$

In case $\alpha < 2$, we can argue in the same form and we have also that $\dfrac{y_t - y}{t} \in W_0^{1,\alpha}(\Omega)$, but now $H_0^{y_u}(\Omega) \subset W_0^{1,\alpha}(\Omega)$ and then we can only prove that there exist subsequences converging to elements which are solutions of 9 in the distribution sense and belong to the space

$$
V_0^{y_u}(\Omega) = \left\{z \in W_0^{1,\alpha}(\Omega) : \int_{\Omega}(k + |\nabla y|)^{\alpha-2}|\nabla z|^2 dx < \infty\right\}
$$

In fact, for $\alpha < 2$, the differentiability of functional $F$ is equivalent to the equality of all these limit points. This is true if $D(\Omega)$ is dense in $V_0^{y_u}(\Omega)$ or, which is equivalent, if $z = 0$ is the unique solution in $V_0^{y_u}(\Omega)$ of PDE

$$
-div\left(\frac{\partial a}{\partial \eta}(x, \nabla y_u)\nabla z\right) + \frac{\partial a_0}{\partial s}(x, y_u)z = 0
$$

If $n = 1$, it may be easily verified that $\dfrac{\partial a}{\partial \eta}(x, \nabla y_u)\nabla z \in W^{1,\infty}(\Omega)$ and then $z = 0$. If $n > 1$, we do not know any positive or negative result about this question. In this context, it is interesting to mention a Serrin's paper [11] where it is proved the existence of non null solutions in $W_0^{1,\alpha}(\Omega)$, with $\alpha < 2$, for the homogeneous Dirichlet problem associated to a linear elliptic operator with bounded measurable coefficients.

# 4 OPTIMALITY CONDITIONS I

In this section we will study the control problems corresponding to the situation described in theorem 2: the functional $v \longrightarrow y_v$ is differentiable. Thus we suppose $k \neq 0$ and also $\alpha \geq 2$ or $\alpha < 2$ and $n = 1$.

We are going to derive the optimality conditions for problems $(P_i)$. This is done by using the following result due to E. Casas:

**THEOREM 3** *Let $X, Y$ be Banach spaces, $\emptyset \neq K \subset X$, $C \subset Y$ convex sets with $\overset{\circ}{C} \neq \emptyset$. Let us consider the problem*

$$(P) \begin{cases} Minimize \;\; J(x) \\ Subject \; to \; x \in K \; and \; G(x) \in C \end{cases}$$

*where $G : X \longrightarrow Y$ and $J : X \longrightarrow (-\infty, +\infty]$ are two functions. Suppose that there exists a solution $\overline{x}$ of $(P)$ and that $G$ and $J$ are Gâteaux differentiable in $\overline{x}$. Then there exist a real number $\overline{\lambda} \geq 0$ and an element $\overline{\mu} \in Y'$ such that*

$$\overline{\lambda} + \|\overline{\mu}\| > 0 \tag{10}$$

$$< \overline{\mu}, y - G(\overline{x}) >_{Y', Y} \leq 0 \quad \forall y \in C \tag{11}$$

$$< \overline{\lambda} J'(\overline{x}) + [DG(\overline{x})]^* \overline{\mu}, x - \overline{x} >_{X', X} \geq 0 \quad \forall x \in K \tag{12}$$

*where $[DG(\overline{x})]^*$ denotes the adjoint functional of $DG(\overline{x})$. Moreover, if there exists $x_0 \in K$ such that*

$$G(\overline{x}) + DG(\overline{x}) \cdot (x_0 - \overline{x}) \in \overset{\circ}{C} \quad \text{(Slater Condition)}$$

*we can take $\overline{\lambda} = 1$.*

The proof of this theorem is a simple application of the well known separation theorem of convex sets to

$$A = \{(y, \lambda) \in Y \times R : \exists x \in K \text{ such that } y = G(\overline{x}) + DG(\overline{x}) \cdot (x - \overline{x})$$

$$\text{and } \lambda = J'(\overline{x}) \cdot (x - \overline{x})\}$$

$$B = \overset{\circ}{C} \times (-\infty, 0)$$

which are non-empty disjoint convex sets.

With the aid of previous theorem we can easily obtain the first order optimality system in the differentiable cases.

**THEOREM 4** *Let us suppose that for each $j \in \{1, \ldots, m\}$, $g_j$ is a Caratheodory function of class $C^1$ with respect to the second variable and there exist functions $h \in L^2(\Omega)$ and $\phi : R^+ \longrightarrow R^+$ non decreasing such that*

$$\left| \frac{\partial g_j}{\partial y}(x, y) \right| \leq h(x) \phi(|y|) \quad a.e. \; x \in \Omega \quad \forall y \in R \quad 1 \leq j \leq m. \tag{13}$$

*Let $\overline{u} \in K$ be a solution of $(P_1)$, then there exist elements $\overline{\lambda} \geq 0$, $\overline{y} \in W_0^{1,\alpha}(\Omega)$, $\overline{p} \in H_0^{\overline{y}}(\Omega)$ and $\overline{\mu}_j \geq 0$, $1 \leq j \leq m$, such that*

$$\begin{cases} -\mathrm{div}(a(x,\nabla\overline{y})) + a_0(x,\overline{y}) = \overline{u} & in \ \ \Omega \\ \qquad\qquad\qquad\qquad\quad \overline{y} = 0 & on \ \ \Gamma \end{cases} \tag{14}$$

$$\overline{\lambda} + \sum_{j=1}^{m} \overline{\mu}_j > 0 \tag{15}$$

$$\begin{cases} -\mathrm{div}\left(\left[\dfrac{\partial a}{\partial \eta}(x,\nabla\overline{y})\right]^T \nabla\overline{p}\right) + \dfrac{\partial a_0}{\partial s}(x,\overline{y})\overline{p} = \overline{\lambda}(\overline{y}-y_d) + \sum_{j=1}^{m}\overline{\mu}_j\dfrac{\partial g_j}{\partial y}(x,\overline{y}) & in \ \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad \overline{p} = 0 \quad on \ \ \Gamma \end{cases} \tag{16}$$

$$\overline{\mu}_j\left(\int_\Omega g_j(x,\overline{y}(x))dx - \delta_j\right) = 0 \quad 1 \leq j \leq m \tag{17}$$

$$\int_\Omega (\overline{p} + \overline{\lambda}\nu\overline{u})\cdot(v-\overline{u})dx \geq 0 \quad \forall v \in K \tag{18}$$

*Proof.* We consider $(P_1)$ as a particular case of problem $(P)$ of theorem 3 with $X = L^2(\Omega)$, $Y = R^m$, $C = (-\infty,\delta_1]\times\cdots\times(-\infty,\delta_m]$, $G(v) = (G_1(v),\ldots,G_m(v))$ and

$$G_j(v) = \int_\Omega g_j(x,y_v(x))dx$$

From theorem 2 it follows that the functionals $G_j$ are differentiable and

$$DG_j(\overline{u})\cdot v = \int_\Omega \frac{\partial g_j}{\partial y}(x,y_u)DF(\overline{u})vdx$$

Applying theorem 3, we deduce the existence of $\overline{\lambda} \geq 0$ and $\overline{\mu}_j \in R$, $1 \leq j \leq m$, such that

$$\overline{\lambda} + \sum_{j=1}^{m}|\overline{\mu}_j| > 0 \tag{19}$$

$$\sum_{j=1}^{m}\overline{\mu}_j\left(y_j - \int_\Omega g_j(x,\overline{y})dx\right) \leq 0 \quad \forall(y_1,\ldots,y_m) \in C \tag{20}$$

$$\overline{\lambda}J'(\overline{u})\cdot(v-\overline{u}) + \sum_{j=1}^{m}\overline{\mu}_j DG_j(\overline{u})\cdot(v-\overline{u}) \geq 0 \quad \forall v \in K \tag{21}$$

Denoting $DF(\overline{u})\cdot(v-\overline{u})$ by $z$, we have

$$J'(\overline{u})\cdot(v-\overline{u}) = \int_\Omega(\overline{y}-y_d)zdx + \nu\int_\Omega \overline{u}(v-\overline{u})dx.$$

Now let $\bar{p}$ be the unique solution of 16 in $H_0^{\bar{y}}(\Omega)$, then

$$\int_\Omega \left( \bar{\lambda}(\bar{y} - y_d) + \sum_{j=1}^m \bar{\mu}_j \frac{\partial g_j}{\partial y}(x,\bar{y}) \right) z\, dx = \int_\Omega \nabla z^T \left( \frac{\partial a}{\partial \eta}(x,\nabla \bar{y}) \right)^T \nabla \bar{p}\, dx + \int_\Omega z \frac{\partial a_0}{\partial s}(x,\bar{y})\bar{p}\, dx =$$

$$\int_\Omega \nabla \bar{p}^T \left( \frac{\partial a}{\partial \eta}(x,\nabla \bar{y}) \right) \nabla z\, dx + \int_\Omega \bar{p} \frac{\partial a_0}{\partial s}(x,\bar{y})\, dx = \int_\Omega \bar{p}(v-\bar{u})\, dx$$

Last equality follows from theorem 2. So from 21 we obtain 18.

On the other hand, using that $G(\bar{y}) \in C$ and taking in 20

$$y_j < \int_\Omega g_j(x,\bar{y}(x))\, dx, \quad y_k = \int_\Omega g_k(x,\bar{y}(x))\, dx \quad \text{if } k \neq j$$

it follows that $\bar{\mu}_j \geq 0 \ \forall j \in \{1,\ldots,m\}$ and then 19 coincides with 15. Finally, if we take

$$y_j = \delta_j, \quad y_k = \int_\Omega g_k(x,\bar{y}(x))\, dx \quad \text{if } k \neq j$$

we derive 17. $\square$

**THEOREM 5** *Let $\bar{u}$ be a solution of $(P_2)$. Then there exist a real number $\bar{\lambda} \geq 0$ and elements $\bar{y} \in W_0^{1,\alpha}(\Omega)$, $\bar{p} \in H_0^{\bar{y}}(\Omega)$ and $\bar{\mu} \in L^\infty(\Omega)$ verifying 14, 18 and*

$$\bar{\lambda} + \|\bar{\mu}\| > 0 \qquad (22)$$

$$\begin{cases} -div\left( \left[ \frac{\partial a}{\partial \eta}(x,\nabla \bar{y}) \right]^T \nabla \bar{p} \right) + \frac{\partial a_0}{\partial s}(x,\bar{y})\bar{p} = \bar{\lambda}(\bar{y} - y_d) + \bar{\mu} \quad \text{in } \Omega \\[2mm] \bar{p} = 0 \quad \text{on } \Gamma \end{cases} \qquad (23)$$

$$\int_\Omega \bar{\mu}(y - \bar{y})\, dx \leq 0 \quad \text{if } \|y\|_{L^1(\Omega)} \leq \delta \qquad (24)$$

*Proof.* In this case it is enough to take $X = L^2(\Omega)$, $Y = L^1(\Omega)$, $C = $ the ball of $L^1(\Omega)$ with center at 0 and radio $\delta$ and $G(v) = y_v$. $\square$

**REMARK 1** *It is possible to consider more general constraints of type*

$$\int_\Omega g(x,y_u(x))\, dx \leq \delta$$

*where $g : \Omega \times R \longrightarrow R$ is a measurable function with respect to the first variable and a convex function with respect to the second one, $g(\cdot,0) \in L^1(\Omega)$ and there exists $h \in L^\infty(\Omega)$ such that*

$$|g(x,y) - g(x,z)| \leq h(x)|y-z| \quad a.e. \ x \in \Omega \ \ \forall y,z \in R$$

*Nevertheless, it is necessary that the closed convex*

$$C = \{y \in L^1(\Omega) : \int_\Omega g(x,y(x))\, dx \leq \delta\}$$

*has a non empty interior.*

Before stating the optimality conditions for problem $(P_3)$, let us introduce some notations. Let $C_0(\Omega)$ be th space of real and continuous functions on $\overline{\Omega}$ that are null on $\Gamma$, endowed with the supremum-norm $\| \cdot \|_{L^\infty(\Omega)}$ and let $M(\Omega)$ be the space of real and regular Borel measures on $\Omega$ endowed with the norm

$$\|\mu\|_{M(\Omega)} = |\mu|(\Omega)$$

where $|\mu|$ is the total variation measure of $\mu$, see W. Rudin [10]. As consequence of Riesz representation theorem, it is known that $M(\Omega)$ is the dual space of $C_0(\Omega)$ with the duality product defined by

$$< \mu, y >= \int_\Omega y(x) d\mu(x) \quad \forall y \in C_0(\Omega)$$

**THEOREM 6** *Let us suppose $n = 1$ and let $\overline{u}$ be a solution of $(P_3)$. Then there exist $\overline{\lambda} \geq 0$, $\overline{y} \in W_0^{1,\alpha}(\Omega)$, $\overline{p} \in H_0^{\overline{y}}(\Omega)$ and $\overline{\mu} \in M(\Omega)$ verifying 14, 22, 23, 18 and*

$$\int_\Omega (y(x) - \overline{y}(x)) d\overline{\mu}(x) \leq 0 \quad \forall y \in \overline{B}_\delta(0) \tag{25}$$

*where $\overline{B}_\delta(0)$ is the ball of $C_0(\Omega)$ with center at 0 and radio $\delta$.*

*Proof.* It follows from theorem 3 with $X = L^2(\Omega)$, $Y = C_0(\Omega)$, $C = \overline{B}_\delta(0)$ and $G(v) = y_v$.

**REMARKS 1** *1. From 25 it is possible to obtain certain properties of $\overline{\mu}$ as in E. Casas [2].*

*2. There exist two difficulties for considering the case $n > 1$: the first one is that, in general, the states are not continuous functions; the second one is the non existence of solution in $H_0^{\overline{y}}(\Omega)$ of problem 9 when the second term belongs to $M(\Omega)$. Let us note that*

$$H_0^{\overline{y}}(\Omega) \subset C_0(\Omega) \quad \text{if and only if} \quad n = 1.$$

# 5 OPTIMALITY CONDITIONS II

In this section we will assume that one of the following conditions is satisfied:

1. $n > 1$, $\alpha < 2$ and $k \neq 0$.

2. $\alpha > 2$ and $k = 0$.

Therefore we do not know if the functional $v \longrightarrow y_v$ is differentiable (case 1.) or we know that it is not differentiable (case 2.). To deal with these situations we introduce a family of approximating problems $(P_\epsilon)$ that belongs to the differentiable case, we obtain the optimality system for each $(P_\epsilon)$ and we pass to the limit in these systems when $\epsilon \to 0$.

For each $\epsilon > 0$ let us consider the perturbed differential operator

$$A_\epsilon y = -\epsilon \Delta y + Ay$$

$A_\epsilon$ satisfies the hypotheses 2–7 (with exponent 2 if $\alpha < 2$). Hence, given $u \in L^2(\Omega)$ there exists a unique solution $y_\epsilon(u) \in W_0^{1,\alpha}(\Omega)$ (resp. $H_0^1(\Omega)$ if $\alpha < 2$) of problem

$$\begin{cases} A_\epsilon y = u & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

Let $\overline{u}$ be a solution of $(P_i)$, $1 \leq i \leq 3$. Associated to this solution we define the cost functional

$$J_\epsilon(v) = \frac{1}{2} \int_\Omega |y_\epsilon(v) - y_d|^2 dx + \frac{\nu}{2} \int_\Omega |v|^2 dx + \frac{1}{2} \int_\Omega |v - \overline{u}|^2 dx$$

and the corresponding control problems

$$(P_1^\epsilon) \begin{cases} \text{Minimize} \quad J_\epsilon(v) \\ \text{Subject to} \quad v \in K \text{ and } \int_\Omega g_j(x, y_\epsilon(v)) dx \leq \delta_j(\epsilon) \quad 1 \leq j \leq m \end{cases}$$

with $\delta_j(\epsilon) = \max \left\{ \delta_j, \int_\Omega g_j(x, y_\epsilon(\overline{u})) dx \right\}$, $1 \leq j \leq m$.

$$(P_2^\epsilon) \begin{cases} \text{Minimize} \quad J_\epsilon(v) \\ \text{Subject to} \quad v \in K \text{ and } \int_\Omega |y_\epsilon(v)| dx \leq \delta(\epsilon) \end{cases}$$

with $\delta(\epsilon) = \max \left\{ \delta, \int_\Omega |y_\epsilon(\overline{u})| dx \right\}$.

$$(P_3^\epsilon) \begin{cases} \text{Minimize} \quad J_\epsilon(v) \\ \text{Subject to} \quad v \in K \text{ and } |y_\epsilon(v)(x)| \leq \delta(\epsilon) \ \forall \, x \in \Omega \end{cases}$$

with $\delta(\epsilon) = \max \left\{ \delta, \|y_\epsilon(\overline{u})\|_{L^\infty(\Omega)} \right\}$.

Let us observe that we have enlarged the set of state constraints in such a way that we can guarantee the existence of admissible points for each $(P_i^\epsilon)$: in fact $\overline{u}$ is an admissible point. Now we can deduce the existence of a solution $u_\epsilon$ of problem $(P_i^\epsilon)$ as in theorem 1. On the other hand, taking into account the Gâteaux differentiability of $J_\epsilon$, we can obtain the optimality conditions for $(P_i^\epsilon)$, $1 \leq i \leq 3$, as in theorems 4–6. For instance, if $i = 2$ this system is given by the following theorem.

**THEOREM 7** *For each $\epsilon > 0$ there exists (at least) one solution $u_\epsilon$ of $(P_2^\epsilon)$. Moreover there exist elements $\lambda_\epsilon \geq 0$, $y_\epsilon \in W_0^{1,\alpha}(\Omega)$ (resp. $H_0^1(\Omega)$ if $\alpha < 2$), $p_\epsilon \in H_0^{y_\epsilon}(\Omega)$ (resp. $H_0^1(\Omega)$) and $\mu_\epsilon \in L^\infty(\Omega)$ verifying*

$$\lambda_\epsilon + \|\mu_\epsilon\| > 0 \tag{26}$$

$$\begin{cases} -div[\epsilon\nabla y_\epsilon + a(x, \nabla y_\epsilon)] + a_0(x, y_\epsilon) = u_\epsilon & in \ \Omega \\ y_\epsilon = 0 & on \ \Gamma \end{cases} \tag{27}$$

$$\begin{cases} -div\left( \left[ \epsilon I + \frac{\partial a}{\partial \eta}(x, \nabla y_\epsilon) \right]^T \nabla p_\epsilon \right) + \frac{\partial a_0}{\partial s}(x, y_\epsilon) p_\epsilon = \lambda_\epsilon(y_\epsilon - y_d) + \mu_\epsilon & in \ \Omega \\ p_\epsilon = 0 & on \ \Gamma \end{cases} \tag{28}$$

$$\int_\Omega \mu_\epsilon(y - y_\epsilon)dx \le 0 \quad if \quad \|y\|_{L^1(\Omega)} \le \delta(\epsilon) \tag{29}$$

$$\int_\Omega (p_\epsilon + \lambda_\epsilon[\nu u_\epsilon + u_\epsilon - \overline{u}])(v - u_\epsilon)dx \ge 0 \quad \forall v \in K \tag{30}$$

*where I denotes the identity matrix $n \times n$.*

Similar theorems can be formulated for problems $(P_1^\epsilon)$ and $(P_3^\epsilon)$. In order to pass to the limit in these optimality systems we use the following results:

**THEOREM 8 (E. Casas and L.A. Fernández [3])** *For each $\epsilon > 0$ let $(y_\epsilon, v_\epsilon)$ belong to $W_0^{1,\alpha}(\Omega) \times L^2(\Omega)$ (resp. $H_0^1(\Omega) \times L^2(\Omega)$ if $\alpha < 2$) and satisfy*

$$\begin{cases} -div[\epsilon \nabla y_\epsilon(v_\epsilon) + a(x, \nabla y_\epsilon(v_\epsilon))] + a_0(x, y_\epsilon(v_\epsilon)) = v_\epsilon & in \ \Omega \\ y_\epsilon(v_\epsilon) = 0 & on \ \Gamma \end{cases}$$

*Assume that $v_\epsilon \to u$ weakly in $L^2(\Omega)$ as $\epsilon \to 0$, then $y_\epsilon(v_\epsilon) \to y_u$ in $W_0^{1,\alpha}(\Omega)$ as $\epsilon \to 0$ and there exists a constant $C > 0$ such that*

$$\|y_\epsilon\|_{L^\infty(\Omega)} \le C \quad \forall \epsilon > 0.$$

**THEOREM 9** *Let $u_\epsilon$ be a solution of $(P_i^\epsilon)$, $1 \le i \le 3$. Set $\overline{y} = y_{\overline{u}}$ and $y_\epsilon = y_\epsilon(u_\epsilon)$. Then we have*

$$u_\epsilon \to \overline{u} \quad in \ \ L^2(\Omega)$$
$$y_\epsilon \to \overline{y} \quad in \ \ W_0^{1,\alpha}(\Omega)$$
$$J_\epsilon(u_\epsilon) \to J(\overline{u})$$

*as $\epsilon \to 0$.*

The proof utilizes the same arguments that theorem 4.4 of [3]. Let us point out that if $y_\epsilon \to y_u$ in $W_0^{1,\alpha}(\Omega)$ and $\{y_\epsilon\}_{\epsilon>0}$ satisfies the state constraints of problems $(P_i^\epsilon)$, then $y_u$ satisfies the state constraints of $(P_i)$ thanks to the convergence of $y_\epsilon(\overline{u})$ towards $\overline{y}$ in $W_0^{1,\alpha}(\Omega)$.

## 5.1 PASSAGE TO THE LIMIT: $\alpha < 2$ and $k \ne 0$

Since we are assuming $\alpha < 2$ and 8, then obviously $n$ must be less than or equal to 3. However the case $n = 1$ has been studied in section 3, thus $n = 2$ or 3 in this section.

Next results can be proved using the same argumentation of [3], taking into account the following facts. For the family of problems $(P_2^\epsilon)$ we can suppose without loss of generality that $\lambda_\epsilon + \|\mu_\epsilon\|_{L^\infty(\Omega)} = 1$: in other case, it is enough to divide the expressions 28-30 by $\sigma_\epsilon = \lambda_\epsilon + \|\mu_\epsilon\|_{L^\infty(\Omega)}$ and to rename $p_\epsilon$, $\lambda_\epsilon$ and $\mu_\epsilon$ instead of $\sigma_\epsilon^{-1}p_\epsilon$, $\sigma_\epsilon^{-1}\lambda_\epsilon$ and $\sigma_\epsilon^{-1}\mu_\epsilon$. So there exist elements $\overline{\lambda} \ge 0$, $\overline{\mu} \in L^\infty(\Omega)$ and subsequences (denoted in the same for     such that

$$\lambda_\epsilon \to \overline{\lambda} \quad and \quad \mu_\epsilon \to \overline{\mu} \quad weakly^* \ in \ L^\infty(\Omega).$$

Hence, we can pass to the limit in 27–30. Now it remains to prove 22. Suppose that $\overline{\lambda} = 0$, then $\|\mu_\epsilon\|_{L^\infty(\Omega)} \to 1$. We are going to see that $\overline{\mu} = 0$ is not a weak* limit point of $\{\mu_\epsilon\}_{\epsilon>0}$. Let us consider $y_0 \in L^1(\Omega)$ such that

$$r = \delta - \|y_0\|_{L^1(\Omega)} > 0.$$

In virtue of 29 we have

$$\int_\Omega \mu_\epsilon y \, dx \leq \int_\Omega \mu_\epsilon(y_\epsilon - y_0) dx \quad \text{if } \|y\|_{L^1(\Omega)} \leq r$$

Taking supremum in the first term, we get

$$r\|\mu_\epsilon\|_{L^\infty(\Omega)} \leq \int_\Omega \mu_\epsilon(y_\epsilon - y_0) dx$$

Letting $\epsilon$ tend to 0, we obtain

$$0 < r \leq \int_\Omega \overline{\mu}(\overline{y} - y_0) dx$$

therefore $\overline{\mu} \neq 0$.

For $(P_1^\epsilon)$ the arguments are similar. In this way we obtain the following theorems:

**THEOREM 10** *Let us suppose 13 and let $\overline{u}$ be a solution of $(P_1)$. Then there exist elements $\overline{\lambda} \geq 0$, $\overline{p} \in W_0^{1,\alpha}(\Omega)$ and $\overline{\mu}_j \geq 0$, $1 \leq j \leq m$ satisfying together with $\overline{u}$ and $\overline{y}$ the system 14–18. Moreover*

$$\int_\Omega \nabla\overline{p}^T \left(\frac{\partial a}{\partial \eta}(x, \nabla\overline{y})\right)^T \nabla\overline{p} \, dx + \int_\Omega \frac{\partial a_0}{\partial s}(x, \overline{y})\overline{p}^2 dx \leq \int_\Omega \left(\overline{\lambda}(\overline{y} - y_d) + \sum_{j=1}^m \overline{\mu}_j \frac{\partial g_j}{\partial y}(x, \overline{y})\right)\overline{p} \, dx$$

**THEOREM 11** *Let $\overline{u}$ be a solution of $(P_2)$. Then there exist $\overline{\lambda} \geq 0$, $\overline{p} \in W_0^{1,\alpha}(\Omega)$ and $\overline{\mu} \in L^\infty(\Omega)$ satisfying together with $\overline{u}$ and $\overline{y}$ the system 14, 22–24 and 18. Moreover*

$$\int_\Omega \nabla\overline{p}^T \left(\frac{\partial a}{\partial \eta}(x, \nabla\overline{y})\right)^T \nabla\overline{p} \, dx + \int_\Omega \frac{\partial a_0}{\partial s}(x, \overline{y})\overline{p}^2 dx \leq \int_\Omega [\overline{\lambda}(\overline{y} - y_d) + \overline{\mu}]\overline{p} \, dx$$

## 5.2 PASSAGE TO THE LIMIT: $\alpha > 2$ and $k = 0$

To treat case $k = 0$, that corresponds to a degenerate equation, we need assume some additional hypotheses that guarantee $C^1$-regularity of states:

$$\begin{cases} a_j \in C^1(\Omega \times R^n) \quad j = 1, \ldots, n \\[2mm] \displaystyle\sum_{i,j=1}^N \left|\frac{\partial a_j}{\partial x_i}(x, \eta)\right| \leq \Lambda_2 |\eta|^{\alpha-1} \\[2mm] 0 < \Lambda_3 \leq \frac{\partial a_0}{\partial s}(x, s) \leq f(|s|) \end{cases} \tag{31}$$

for all $x \in \Omega$, all $s \in R$ and all $\eta \in R^n$, where $f$ is a positive and non-decreasing function, and also the boundedness of the controls. So we will take in this section

$$K = \{v \in L^2(\Omega) : -\infty < m \le v(x) \le M < +\infty \quad a.e. \ x \in \Omega\}$$

Using a Tolksdorf's result [12], we deduce that $y_u \in C^{1,\sigma}(\Omega)$ for each $u \in K$ and some $\sigma \in (0,1)$. Now arguing as in the previous subsection and in the paper [3] we can prove the following theorems.

**THEOREM 12** *Let $\overline{u}$ be a solution of $(P_1)$ and let us assume 13. Then there exist elements $\overline{\lambda} \ge 0$, $\overline{\mu}_j \ge 0$, $1 \le j \le m$, and $\overline{p} \in L^2(\Omega) \cap H^1_{loc}(\Omega_0)$, where*

$$\Omega_0 = \{x \in \Omega : |\nabla \overline{y}(x)| > 0\}$$

*satisfying together with $\overline{u}$ and $\overline{y}$ the system 14–15, 17–18 and*

$$- div \left( \left[ \frac{\partial a}{\partial \eta}(x, \nabla \overline{y}) \right]^T \nabla \overline{p} \right) + \frac{\partial a_0}{\partial s}(x, \overline{y})\overline{p} = \overline{\lambda}(\overline{y} - y_d) + \sum_{j=1}^{m} \overline{\mu}_j \frac{\partial g_j}{\partial y}(x, \overline{y}) \quad in \quad \Omega_0 \quad (32)$$

**THEOREM 13** *Let $\overline{u}$ be a solution of $(P_2)$. Then there exist elements $\overline{\lambda} \ge 0$, $\overline{\mu} \in L^\infty(\Omega)$ and $\overline{p} \in L^2(\Omega) \cap H^1_{loc}(\Omega_0)$, where $\Omega_0$ is given as in previous theorem, satisfying together with $\overline{u}$ and $\overline{y}$ the system 14, 22, 24, 18 and*

$$- div \left( \left[ \frac{\partial a}{\partial \eta}(x, \nabla \overline{y}) \right]^T \nabla \overline{p} \right) + \frac{\partial a_0}{\partial s}(x, \overline{y})\overline{p} = \overline{\lambda}(\overline{y} - y_d) + \overline{\mu} \quad in \quad \Omega_0 \quad (33)$$

**THEOREM 14** *Let $\overline{u}$ be a solution of $(P_3)$ and let us assume $n = 1$. Then there exist elements $\overline{\lambda} \ge 0$, $\overline{\mu} \in M(\Omega)$ and $\overline{p} \in L^2(\Omega) \cap H^1_{loc}(\Omega_0)$, where $\Omega_0$ is given as in theorem 12, satisfying together with $\overline{u}$ and $\overline{y}$ the system 14, 22, 25, 18 and*

$$- div \left( \left[ \frac{\partial a}{\partial \eta}(x, \nabla \overline{y}) \right]^T \nabla \overline{p} \right) + \frac{\partial a_0}{\partial s}(x, \overline{y})\overline{p} = \overline{\lambda}(\overline{y} - y_d) + \overline{\mu} \quad in \quad \Omega_0 \quad (34)$$

# 6 CONSTRAINT QUALIFICATIONS

The optimality system can be viewed as being degenerate when $\overline{\lambda} = 0$ because the characteristic elements of functional $J$ to be minimized ($y_d$ and $\nu$) do not appear. Several supplementary conditions can be proposed under which it is possible to assert that $\overline{\lambda} \ne 0$ (in the terminology of F. H. Clarke [4] the problem is "normal"): for instance the Slater condition, see theorem 3. Other conditions are the following

**THEOREM 15**  *1. If $K = L^2(\Omega)$ and $\left\{\dfrac{\partial g_j}{\partial y}(x, \overline{y})\right\}_{j=1}^m$ are linearly independent in $\Omega$ (resp. $\Omega_0$), then problem $(P_1)$ is normal.*

*2. If $K = L^2(\Omega)$ then problems $(P_2)$ and $(P_3)$ are normal.*

*3. If $K$ is bounded in $L^2(\Omega)$ or $\nu > 0$ and if there exists an admissible control for problem $(P_1^\delta)$, then problem $(P_1^\gamma)$ is normal for almost every $\gamma \in R^m$ such that $\gamma_j \geq \delta_j,\ 1 \leq j \leq m$.*

*4. If $K$ is bounded in $L^2(\Omega)$ or $\nu > 0$ and if there exists an admissible control for problem $(P_i^\delta)$, with $i = 2$ or $3$, then problem $(P_i^\gamma)$ is normal for almost every $\gamma \in R$ such that $\gamma \geq \delta$.*

*Proof.* 1) If $\overline{\lambda} = 0$ it follows from 18 that $\overline{p} = 0$. Now, thanks to 16 or 32, we conclude that
$$\sum_{j=1}^m \overline{\mu}_j \frac{\partial g_j}{\partial y}(x, \overline{y}) = 0 \quad \text{in } \Omega \ (\text{ or } \Omega_0)$$
then $\overline{\mu}_j = 0 \ \forall j$ which contradicts 15.

3) It is a consequence of F.H. Clarke [4, theorem 3] applied to the function
$$\phi : R^m \longrightarrow (-\infty, +\infty]$$
$$\phi(\gamma) = \min\left\{ J(v) : v \in K \ \text{ and } \ \int_\Omega g_j(x, y_v(x)) dx \leq \gamma_j,\ 1 \leq j \leq m \right\}$$

Let us remark that $J$ and $G_j$ are locally Lipschitz functions from $L^2(\Omega)$ to $(-\infty, +\infty]$ because $v \longrightarrow y_v$ is a Lipschitz functional from $L^2(\Omega)$ to $H_0^1(\Omega)$ if $\alpha \geq 2$ and a locally Lipschitz functional from $L^2(\Omega)$ to $W_0^{1,\alpha}(\Omega)$ if $\alpha < 2$.

The proof of 2) and 4) can be carry out in the same way as 1) and 3) respectively.

# 7   REGULARITY OF THE OPTIMAL CONTROL AND STATE

In this section our aim is to derive some regularity results of optimal control and state from the previous optimality systems. Firstly we establish the boundedness of adjoint state $\overline{p}$.

**THEOREM 16 (E. Casas and L.A. Fernández [3])** *Let us suppose $k \neq 0$. If $\overline{\lambda} \neq 0$ assume also that $y_d \in L^\rho(\Omega) \cap L^2(\Omega)$, with $\rho > n/2$ and furthermore*
$$\rho > \frac{\alpha}{2\alpha - 2} \quad \text{if} \quad \alpha < 2 \text{ and } n = 2$$
$$\rho > \frac{3\alpha}{5\alpha - 6} \quad \text{if} \quad \alpha < 2 \text{ and } n = 3$$
*then $\overline{p} \in L^\infty(\Omega)$.*

**COROLLARY 1** *Let us suppose that $k$ and $\overline{\lambda}$ are non null, $y_d \in L^\rho(\Omega) \cap L^2(\Omega)$, with $\rho$ as in the previous theorem, $\nu$ is strictly positive and $K$ coincides with one of the following sets*

$$K_1 = \left\{ v \in L^2(\Omega) : \|v\|_{L^2(\Omega)} \leq 1 \right\} \quad or$$

$$K_2 = \left\{ v \in L^2(\Omega) : -\infty \leq m \leq v(x) \leq M \leq +\infty \quad a.e. \ x \in \Omega \right\}$$

*then $\overline{u}$ belongs to $H^1(\Omega) \cap L^\infty(\Omega)$ if $\alpha \geq 2$ (resp. $W^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ if $\alpha < 2$).*

*Proof.* The inequality 18 characterizes $\overline{u}$ as the proyection of $-\dfrac{\overline{p}}{\nu\overline{\lambda}}$ in the convex $K$. If $K = K_1$ it follows easily that

$$\overline{u}(x) = -\frac{\overline{p}(x)}{\|\overline{p}\|_{L^2(\Omega)}} \quad a.e. \ x \in \Omega \ \text{ if } \ \|\overline{p}\|_{L^2(\Omega)} > \nu\overline{\lambda}$$

$$\overline{u}(x) = -\frac{\overline{p}(x)}{\nu\overline{\lambda}} \quad a.e. \ x \in \Omega \ \text{ if } \ \|\overline{p}\|_{L^2(\Omega)} \leq \nu\overline{\lambda}$$

In the same way, if $K = K_2$ we deduce that

$$\overline{u}(x) = \max\left\{ m, \min\left\{ -\frac{\overline{p}(x)}{\nu\overline{\lambda}}, M \right\} \right\} \quad a.e. \ x \in \Omega.$$

The assertion follows from previous relations and theorem 16. □

When $K = K_2$ and $\overline{\lambda}$ or $\nu$ are null, it is easy to obtain from 18 that

$$\overline{u}(x) = \begin{cases} m & \text{if } \overline{p}(x) > 0 \\ M & \text{if } \overline{p}(x) < 0 \end{cases}$$

Thus $\overline{u}$ is essentially of Bang-Bang type. However, when $K = K_1$, we can still deduce some regularity properties of optimal control:

**COROLLARY 2** *Let us suppose $k \neq 0$, $\overline{\lambda}\nu = 0$, $\overline{p} \neq 0$ and $K = K_1$. Then $\overline{u}$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ if $\alpha \geq 2$ (resp. $W_0^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ if $\alpha \leq 2$).*

*Proof.* From 18 it can be deduced that $\overline{u}(x) = -\dfrac{\overline{p}(x)}{\|\overline{p}\|_{L^2(\Omega)}}$ $a.e.$ $x \in \Omega$, which permits to conclude the assertion. □

Provided $a_j, a_0, y_d$ and $\Gamma$ are sufficiently smooth, we can combine the previous results with some regularity results of O. Ladyzhenskaya and N. Ural'tseva [7] and P. Tolksdorf [12] to obtain that

$$\overline{y} \in C^{0,\sigma}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega) \quad \text{for some} \quad \sigma \in (0,1) \quad \text{and} \quad \overline{p}, \overline{u} \in C(\Omega)$$

For $\alpha = 2$, M. Giaquinta and E. Giusti [6] have proved that $\overline{y} \in C^{1,\sigma}(\overline{\Omega})$. In this case, we can deduce that $\overline{p} \in C^{0,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$ (see O. Ladyzhenskaya and N. Ural'tseva [7, theorem 14.1]) and we obtain higher regularity for $\overline{y}, \overline{p}$ and $\overline{u}$ (depending on $K$) with the aid of the usual bootstrap argumentation.

In case $k = 0$, we have supposed that $K$ is a bounded subset of $L^\infty(\Omega)$, thus we know that $\overline{y} \in C^{1,\sigma}(\Omega)$. As in the preceding case we can deduce that $\overline{p}$ and $\overline{u}$ are continuous in $\Omega_0$.

For $(P_3)$ we have only considered the one dimensional case. So the regularity f $\overline{u}, \overline{y}$ and $\overline{p}$ is a consequence of the Sobolev inclusions.

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