# On Necessary Optimality Conditions for Optimal Control Problems

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## 1 Introduction and Definitions

Optimal control problems, where optimality is understood as optimality of a given cost functional, can be regarded as constrained optimization problems, where the equality constraint is given by the state equation of the control problem.

Consequently it is an old idea to derive necessary optimality conditions for control problems as e.g. maximum principles, and adjoint equations from necessary conditions of constrained (nonlinear) optimization, i.e. Kuhn-Tucker Theorems.

That this approach can be useful has been demonstrated successfully for optimal control problems with ODE-state equation by Dubovitsky-Miljutin and others since about 30 years.

Control Problems with PDE state equation too can be treated by methods from nonlinear optimization, in particular Kuhn-Tucker theorems as the works of *Mackenroth*, *Barbu* and *Troeltzsch* ([27], [19], [2] ,among others) indicate. Remark 3.5 gives a short survey.

The generality of this method of approach has the advantage that it is applicable to many nonlinear control problems, with or without state constraints. But this generality is also responsible for the drawback of the method: One always has to verify the central hypotheses of the Kuhn-Tucker Theorem, i.e. the constraint qualifications. This generally, for control problems, amounts to proving existence of a solution of the *linearized* state equation. The purpose of this paper is to put this in an more exact way.

In section 2, a general Kuhn Tucker Theorem is derived and a discussion and comparison of the constraint qualifications used to establish this theorem is given. In section 3 it will be shown, how it can be used to derive necessary optimality conditions for nonlinear optimal control problems. The example considered there is a control problem where the cost functional may be nonsmooth and the equation of the state of the system is a nonlinear parabolic PDE.

**Definition 1.1** Let $f : X \to \mathbb{R}$ be locally Lipschitz continuous at $x \in X$ with a constant $\lambda$ , i.e. let assume there is $|f(u)-f(v)| \leq \lambda\|u-v\|$ for all $u, v$ in a neighborhood of $x$. Then the Clarke-derivative is given by

$$f^{\circ}(x,h) = \lim_{\epsilon \to 0_+} \sup \{ \theta^{-1}(f(v+\theta h) - f(v)) \mid 0 < \theta \leq \epsilon, \|v - x\| \leq \epsilon \}.$$

This limit exists for every $h \in X$ and is always sublinear in $h$ (see e.g. [6]). The *directional derivative* of $f$ (if it exists) is defined as

$$df(x,h) \quad := \quad \lim_{\theta \to 0_+} \theta^{-1}(f(x+\theta h) - f(x)).$$

This derivative is in general nonconvex, so we define the gereralized derivative of $f$ as

$$Df(x,h) \quad = \quad \begin{cases} df(x,h), \text{ if } df(x,\cdot) \text{ is sublinear} \\ \quad f^{\circ}(x,h) \text{ else.} \end{cases} \tag{1}$$

The subdifferential of $f$ then is

$$\partial f(x) \quad := \quad \{ x^* \in X^* \mid x^*(h) \leq Df(x,h) \; \forall\, h \in X \}.$$

If $Z$ is a topological vector space and $g : X \to Z$ a given mapping then the *directional derivative* of $g$ (if it exists) is defined as

$$Dg(x,h) \quad := \quad \lim_{\theta \to 0_+} \theta^{-1}(g(x+\theta h) - g(x)).$$

If ist is *linear* in $h$ we will call $Dg(x, \cdot)$ the Gâteaux-derivative of $g$. In this paper we will only use mappings $g$ where $Dg$ is at least sublinear, so in this case (if additionally $Z$ is an order complete vector lattice with a semi ordering $\leq$ see e.g. [4] or [25]) we can define the subdifferential of $g$ as

$$\partial g(x) \quad := \quad \{ L \in \mathcal{L}(X,Y) \mid L(h) \leq Dg(x,h) \;\; \forall\, h \in X \}. \tag{2}$$

Here $\mathcal{L}(X,Y)$ denotes the set of linear continuous operators from $X$ to $Y$. It is also possible to define a kind of Clarke's derivative of $g$ which is always sublinear, by using a different notion of Lipschitz continuity (see e.g. [15] or [25]).

To approximate sets, we will employ the concept of Bouligand's tangent cone, usually referred to as contingent cone.

**Definition 1.2** Let $X$ be a normed space, $S \subseteq X$ be a nonempty subset and $x \in X$ be given. Then the *contingent cone* of $S$ at $x$ is defined as

$$T(S,x) := \left\{ h \in X \;\middle|\; \begin{array}{ll} \exists\, \{h_n\}_{n \in \mathbb{N}} \subseteq X, & h = \lim\limits_{n \to \infty} h_n, \\ \exists\, \{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ & t_n \to 0, \end{array} \; x + t_n h_n \in S \;\; \forall\, n \in \mathbb{N} \right\}.$$

If $S$ is starshaped at $x$ or even convex, then we will use the cone

$$K(S,x) \quad = \quad \bigcup \{ \lambda(s-x) \mid \lambda \in \mathbb{R}_+, \; s \in S \},$$

(it is well known, that the closure of this cone coincides with $T(S,x)$ if $S$ is starshaped at $x$.)

# 2 Necessary Optimality Conditions

**Theorem 2.1** *Let $X$ be a normed vector space, and $\emptyset \neq S \subseteq X$. Assume that $f : X \to \mathbb{R}$ is locally Lipschitz at $x \in X$ with constant $\lambda$ and suppose that $f(x) = \min\{f(s) \mid s \in S\}$. Then the following assertions hold:*

*(a)* $Df(x,h) \geq 0 \quad \forall\, h \in T(S,x)$,
*and, if the directional derivative of $f$ exists then*
$df(x,h) \geq 0 \quad \forall\, h \in T(S,x)$.

*(b)* *For every convex cone $K \subseteq T(S,x)$ we have*

$$0 \quad \in \quad \partial f(x) \quad - \quad K^*. \tag{3}$$

**Proof.**

(a) Suppose that there is some $h \in T(S,x)$ with the property $Df(x,h) < 0$, hence there is $\delta > 0$ with $Df(x,h) + 4\delta < 0$. Set

$$S(\epsilon) := \sup\{\,\theta^{-1}(f(v+\theta h) - f(v)) \mid 0 < \theta \leq \epsilon, \|v - x\| \leq \epsilon\,\}$$

By the definition of $Df$ there is $\epsilon > 0$ with $|S(\epsilon) - Df(x,h)| < \delta$. Because of $h \in T(S,x)$ there are sequences $\{h_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{\theta_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ for which

$$0 = \lim_{n\to\infty} \theta_n, \quad h = \lim_{n\to\infty} h_n \text{ and } x + \theta_n h_n \in S \quad \forall\, n \in \mathbb{N}.$$

Hence there is a number $n_0 \in \mathbb{N}$ with $\|h_n - h\| \leq \delta/\lambda$ and $\theta_n < \epsilon$ for $n > n_0$. Furthermore because $f$ is Lipschitz continuous ,

$$\|\theta_n^{-1}(f(x+\theta_n h) - f(x)) - \theta_n^{-1}(f(x+\theta_n h_n) - f(x))\| \leq \lambda\|h - h_n\|,$$

and hence

$$\begin{aligned} \theta_n^{-1}(f(x+\theta_n h_n) - f(x)) \quad &\leq \quad \theta_n^{-1}(f(x+\theta_n h) - f(x)) + \delta && (4)\\ \leq S(\epsilon) + \delta \quad &\leq \quad Df(x,h) + 2\delta < 0 && (5)\end{aligned}$$

for all $n > n_0$ in contradiction to $f(x)$ being minimal. The same argument applies to the directional derivative $df(x,h)$ if it exists, but we can omitt $S(e)$ from the proof.

(b) To prove (3) set $p(h) := Df(x,h)$. Observe that $p : X \to \mathbb{R}$ is sublinear and continuous. Hence the set

$$epi(p) := \{\,(h,r) \in X \times \mathbb{R} \mid p(h) \leq r\,\}$$

is convex and has nonempty interior in $X \times \mathbb{R}$ . Because statement (a) implies $p(k) \geq 0 \;\; \forall\, k \in K$ and hence

$$int\, epi(p) \;\cap\; \{\,(k,\alpha) \in X \times \mathbb{R} \mid k \in K, \alpha \leq 0\,\} \quad = \quad \emptyset. \tag{6}$$

By application of a separation theorem to this intersection we obtain the existence of a linear continuous functional $(x_1^*,\beta) \in (X \times \mathbb{R})^* = X^* \times \mathbb{R}$, $(x_1^*,\beta) \neq 0$ satisfying

$$x_1^*(h) + \beta r \quad \leq \quad 0 \quad \leq \quad x_1^*(k) + \beta\alpha \quad \forall\, h \in X, k \in K, r \geq p(h), \alpha < 0. \quad (7)$$

These inequalities imply $\beta \leq 0$. If $\beta = 0$ then (7) yields $x_1^* = 0$ in contradiction to $(x_1^*,\beta) \neq 0$. Hence we have $\beta < 0$. Set $x^* := -\frac{1}{\beta}x_1^*$ and conclude from (7)

$$x^* \quad \in \quad -K^* \quad \text{und} \quad x^*(h) \quad \leq \quad p(h) \quad \forall\, h \in X.$$

From the latter inequality follows $x^* \in \partial f(x)$ hence (3).

We now consider the situation that the set $S$ is given by explicit constraints. Let $Z_1$,$Z_2$ be partially ordered, $g_1: X \to Z_1$, $g_2$: $X \to Z_2$ be given mappings and suppose that $S_0 \subseteq X$ $D_1 \subseteq Z_1$ $D_2 \subseteq Z_2$ are nonempty sets. For $i = 1,2$ we define

$$\begin{aligned}
S_i &:= \{\, x \in X \mid g_i(x) \in D_i \,\}, & (8)\\
S &:= S_0 \cap S_1 \cap S_2, & (9)\\
P_i &:= \overline{co\,T}(D_i, g_i(x)), & (10)\\
P_i^* &:= \{\, z^* \in Z_i^* \mid z^*(p) \geq 0,\ \forall\, p \in P_i \,\}, & (11)\\
J_i &:= \{\, h \in X \mid L(h) \in P_i, \forall\, L \in \partial g_i(x) \,\}, & (12)\\
H_i &:= \{\, x^* \in X^* \mid \exists\, p^* \in P_i^*, L \in \partial g_i(x) \ mit\ x^* = p^* \circ L \,\}, & (13)\\
J_0 &:= \overline{co\,T}(S_0, x). & (14)
\end{aligned}$$

The sets $P_i^*$ are multiplier sets and the sets $H_i$ are sets of compositions of multipliers and subdifferentials. If in particular inequality and equality constraints are given, take $D_1 := -C$ and $D_2 = 0$, where $C \subseteq Z_1$ is a convex cone representing the inequality constraints.

**Lemma 2.2** *Let $K \subseteq X$ be a convex cone satisfying*

$$K \subseteq T(S,x) \quad \textit{and} \quad K^* = J_0^* + J_1^* + J_2^* \quad (15)$$

*and assume that the sets $H_1, H_2$ are $w^*$-closed. Then*

$$K^* \quad \subseteq \quad co\,H_1 + co\,H_2 + J_0^*. \quad (16)$$

**Proof.** In order to prove (16) we show $J_i^* = \overline{co\,H_i}^{w^*}$. Suppose that $x^* \in J_i^*$ with $x^* \notin \overline{co\,H_i}^{w^*} =: B$. By applying a separation theorem to the $w^*$-compact set $\{x^*\}$ and the $w^*$-closed set $B$ we conclude the existence of $h \in X$

$$x^*(h) \quad < \quad 0 \quad \leq \quad v^*(h) \quad \forall\, v^* \in co\,H_i.$$

This implies $L(h) \in P_i^{**} = P_i$ $\forall\, L \in \partial g_i(x)$ since $P$ is a closed and convex cone and hence $h \in J_i$ for i=1,2. But this implies $x^*(h) \geq 0$ in contradiction to $x^*(h) < 0$ from the inequality above. Let now be $x^* \in co\,H_i$ and $x^* \notin J_i^*$. Linear separation then yields the existence of some $h \in X$ with the property

$$x^*(h) \quad < \quad 0 \quad \leq \quad v^*(h) \quad \forall\, v^* \in J_i^*.$$

Because of $J_i^{**} = J_i$ this implies $h \in J_i$ which means $L(h) \geq 0 \ \ \forall\, L \in \partial g_i(x)$ and hence $z^* \circ L(h) \geq 0 \ \forall\, z^* \circ L \in H_i$. But this inequality is a contradiction to the separation inequality, since $x^* \in co\,H_i$ The last statement of the theorem follows from $coH_i$ being $w^*$–closed. ◇

The assumptions of this central Lemma are called *constraint qualifications* and they will, together with the following simple facts imply necessary optimality conditions for inequality and equality constrained problems.

**Lemma 2.3** *For $i = 1, 2$ :*

*(a)* $H_i \quad = \quad \bigcup \{\, L^*(P_i^*) \mid L^* \text{ adjoint of } L \in \partial g_i(x) \,\}$

*(b) If $\partial g_i(x) = \{L\}$ i.e. if the subdifferential consists of a single element, then the set $H_i$ is always convex.*

**Proof.** Immediate.

**Theorem 2.4** *Let $C \subseteq Z_1$ be a convex, closed cone and assume that $f : X \to I\!R$ is Lipschitz-continuous, that $g_1 : X \to Z_1$ and $g_2 : X \to Z_2$ have a sublinear directional derivative and a subdifferential. Let the assumptions of Lemma 2.2 be satisfied.*

*(a) Then there is $f^* \in \partial f(x)$ and for $j = 1, \ldots, n$ and $k = 1, \ldots, n$ there are linear operators $L_j^1 \in \partial g_1(x)$, $L_k^2 \in \partial g_2(x)$, and real numbers $\alpha_j \geq 0$ , $\beta_k \geq 0$ with $\sum_j \alpha_j = \sum_k \beta_k = 1$,*
*and linear functionals (i.e multipliers) $z_{1,j}^* \in P_1^*$, $z_{2,k}^* \in P_2^*$, with*

$$f^*(h) - \sum_j \alpha_j z_{1,j}^* \circ L_j^1(h) - \sum_k \beta_k z_{2,k}^* \circ L_k^2(h) \quad \geq \quad 0 \quad \forall\, h \in J_0 \tag{17}$$

*(b) (Smooth constraints) If we consider especially $D_1 = -C, D_2 = 0$ i.e.*
*$f(x) = \min\{x \in S_0 \mid g_1(x) \in -C, g_2(x) = 0\}$ and asssume that $g_1, g_2$ are Gâteaux-differentiable , then there is $f^* \in \partial f(x)$ and there exist multipliers $z_1^* \in -C^*, z_2^* \in Z_2^*$ such that $z_1^*(g_1(x)) = 0$ and*

$$f^*(h) - z_1^* \circ Dg_1(x,h) - z_2^* \circ Dg_2(x,h) \quad \geq \quad 0 \quad \forall\, h \in J_0 \tag{18}$$

*hold.*

**Proof.** Theorem 2.1 together with Lemma (2.2) implies

$$0_{X^*} \quad \in \quad \partial f(x) \quad - \quad K^* \quad \subseteq \quad \partial f(x) \quad -(co\,H_1 + co\,H_2 + J_0^*).$$

Considering the definition of the sets $H_1, H_2$ we obtain (a) (note that $x^* \in J_0^*$ is equivalent to $x^*(h) \geq 0 \ \forall\, h \in J_0$.) To prove statement (b), we note that by Lemma 2.3

(b) we obtain that the sets $H_1, H_2$ are convex since $g_1, g_2$ are Gâteaux-differentiable ($\partial g_i(x) = Dg_i(x,\cdot)$) Moreover we have to consider here $D_1 = C, D_2 = 0$ and hence it follows with standard arguments $P_1^* = (K(-C, -g_1(x))^* = -C^*$, $z_1^*(g_1(x)) = 0$ and $P_2^* = Z_2^*$. So, by the definition of $H_1, H_2$ we obtain the result (18)

Remark: The convex combinations of multipliers and derivatives seems to be typical for non-Gâteaux-differentiable problems and have their counterpart in the "Hamiltonian multipliers" Clarke's calculus of variations [6]. The assumptions of Lemma 2.2 used in the previous Theorem are abstract constraint qualifications (CQ) of a type introduced by *Guignard* in [10] . It will turn out in the sequel, that they are implied by the classical Slater conditions but that they are more general than those. In fact, as it has been demonstrated by *Bazaraa/Shetty* [3] even in finite dimensional spaces, Guignard's CQ is the weakest among several known CQ's, moreover Guignard's CQ does not assume the existence of interior points of $C$ and $S_0$ required by Slater's CQ (see also *Penot* [22]).

We will use the following Lemma.

**Lemma 2.5** *(a) For every convex cone $I$ with $\operatorname{int} I \neq \emptyset$ the dual cone $I^*$ is $w^*$-locally compact (or, equivalently, has a $w^*$-compact base).*

*(b) If $I^*$ and $J^*$ are convex closed cones and if $I^* \cap -J^* = \{0\}$ then the set $I^* + J^*$ is $w^*$-closed.*

**Proof.** The statement (a) is due to *Ky Fan* see e.g. [29]. The statement (b) is due to *Dieudonnée* (see e.g [13] or [12] Lemma 15 d.) $\diamond$

**Theorem 2.6** *Assume that $\operatorname{int} S_0 \neq \emptyset$ and $\operatorname{int} C \neq \emptyset$. Define $I_0 := \operatorname{int} J_0$ and $I_1 := \{\, h \in X \mid Dg_1(x,h) \in \operatorname{int} P_1 \,\}$ and $K := I_0 \cap I_1 \cap J_2$. If the cone $K$ satisfies the Slater condition $K \neq \emptyset$ and if additionally the tangential inclusion*

$$J_2 \quad \subseteq \quad T(S_2, x) \tag{19}$$

*holds, then $K$ satisfies Guignard's CQ, i.e. $K$ is a convex cone with the properties*

$$K \subseteq T(S,x), \quad K^* = (J_0 \cap J_1 \cap J_2)^*, \quad K^* = J_0^* + J_1^* + J_2^* \tag{20}$$

**Proof.** It is standard (convexity arguments) to prove the following statements:

$$\overline{I_0} = J_0, \quad \overline{I_1} = J_1, \tag{21}$$

$$K^* = (J_0 \cap J_1 \cap J_2)^*, \quad \overline{K} = J_0 \cap J_1 \cap J_2, \tag{22}$$

$$I_0 \cap I_1 \cap J_2 \quad \subseteq \quad T(S,x). \tag{23}$$

We will now deduce the equality $K^* = J_0^* + J_1^* + J_2^*$. Since

$$J_0^* + J_1^* + J_2^* \quad \subseteq \quad K = (J_0 \cap J_1 \cap J_2)^* \quad \subseteq \quad \overline{J_0^* + J_1^* + J_2^*}^{w^*}, \tag{24}$$

we only have to show that the set $J_0^* + J_1^* + J_2^*$ is $w^*$-closed, which will be done by Lemma 2.5. We now observe that $K \neq \emptyset$ implies $I_k^* \cap -J_2^*$ ($k = 0, 1$) and that $I_0^*, I_1^*$ are $w^*$–locally compact. Hence repeated application of (a),(b) of Lemma 2.5 gives $w^*$–closedness of the set $(J_0 \cap J_1 \cap J_2)^*$. $\diamond$

**Remark 2.7** (a) The Slater CQ is sometimes stated in the equivalent form

$$\exists\ h \in int\, S_0 \ : \ Dg_1(x,h) \in int\, K(-C, g_1(x)), \qquad Dg_2(x,h) = 0. \tag{25}$$

(b) If no equality constraints are present and we just consider affine-linear inequality constraints i.e. the feasible set is $S := \{ x \in X \mid g(x) = Ax - b \in -C \}$, $A$ linear, then for

$$K \quad := \quad J \quad = \quad \{ h \in X \mid Dg(x,h) = Ah \in K(S,x) \}$$

one does not need Slater CQ because the inclusion $K \subseteq T(S,x)$ is always true, hence this hypothesis of Theorem 2.4 holds. ( Note that $h \in J$ implies $\lambda Ah \in -C - g(x)\ \ \forall \lambda \geq 0$ hence $g(x + \lambda h) = Ax + \lambda Ah - b \in -C$ if $g(x) \in -C$ which is true since $x$ is feasible. Hence $h \in T(S,x)$.)

(c) The hypothesis $J_2 \subseteq T(S_2, x)$ (19 ) for the equality constraints, called here tangential inclusion is probably the hardest one to prove, it is the essence of the famous Theorem of *Ljusternik*. In the next theorem we cite two modern references which extend the original one. But apart from these general conditions for (19 ) to hold, this condition can sometimes be verified *directly* if $g_2$ is an integral equation or a differential equation (see [25]).

**Theorem 2.8** *Let $X, Z$ be Banach spaces, $Z_0 \subseteq Z$ a given set and $g : X \to Z$ a mapping, $x \in S := \{ x \in X \mid g(x) \in Z_0 \}$ and assume that $g$ is continuous in $x$. Let $U(x)$ denote some neighborhood of $x$.*

*(a) (Penot [23],Theorem 3.1 ) If the strict directional derivative $Dg(u,\cdot)$ of $g$ exists for all $u \in U(x)$, and is regularly surjective and continuous (i.e. it's inverse is Lipschitz continuous) then*

$$J := \{ h \in X \mid Dg(x,h) \in T(Z_0, g(x)) \} \quad \subseteq \quad T(S,x). \tag{26}$$

*$T$ as always denotes Bouligands tangent cone. (If $g$ is Gâteaux-differentiable in $U(x)$ and $Dg(x,\cdot)$ is continous and surjective, then it is regularly surjective).*

*(b)* Kirsch/Warth/Werner,*[14]* Theorem *1.13 We consider especially $Z_0 = 0$. If $g$ is Gâteaux-differentiable on some $U(x)$ and $Dg(x,\cdot)$ is surjective then*

$$J := \{ h \in X \mid Dg(x,h) = 0 \} \quad \subseteq \quad T(S,x). \tag{27}$$

Part (a) of this theorem obviously can be used for inequality and for equality constraints. Another Theorem of this kind is given by *Frankowska* in [8].
We now discuss various sufficient conditions which imply that $H_1, H_2$ is $w^*$-closed.

**Theorem 2.9** *Let $X, Z_1, Z_2$ be Banach spaces.*

*(a) If $H_2 \;=\; \bigcup \{ L^*(P_2^*) \mid L^*$ adjoint of $L \in \partial g_2(x) \} = X^*$, then $H_2$ is convex and $w^*$-closed. If one $L \in \partial g_2(x)$ is a closed operator (i.e. L(X) is closed) and injective, then $H_2 = X^*$, and this set is convex and $w^*$-closed.*

*(b) Let now $\partial g_2(x) = \{L\}$. Then the set $H_2$ is always convex and the following holds.*

*(i) If $Z_2$ is finite dimensional then the set $H_2$ is always $w^*$-closed.*

*(ii) If $L$ is closed then $H_2$ is $w^*$-closed and norm-closed.*

*(c) If $\partial g_1(x) = \{L\}$ and $\operatorname{int} C \neq \emptyset$ then the set $H_1 = L^*(-C^*)$ is a convex cone with $w^*$-compact base.*

**Proof.**

(a) Because $L$ is injective it follows that $\overline{L^*(Z_2^*)}^{w^*} = X^*$, (see e.g. [21] Theorem 4.12 and Corollaries) and closedness of $L$ is equivalent to closedness of $L^*$ in the norm-topology and in the $w^*$-topology (see e.g. [21] Theorem 4.14.) hence $L^*(Z_2^*) = X^* = H_2$ and this last set is obviously convex and $w^*$-closed.

(b) (i) The set $H_2 = L^*(Z_2^*)$ is a finite dimensional subspace and hence $w^*$-closed.

(ii) Same argument as in (a).

(c) See [29]. ◇

# 3 Optimal Control Problem

In this section we discuss with the aid of an (abstract) example of an optimal control problem having a nonlinar parabolic equation of state how the necessary optimality conditions of Theorem 2.4 can be used to derive necessary conditions for this control problem.

Let $\Omega \subseteq I\!R^n$ be bounded with boundary $\partial\Omega$, let $I = [0, b] \subseteq I\!R_+$ be an interval and $W$ and $U$ be spaces of functions on $\Omega \times I$. We consider the following general problem of optimal (distributed) control on $\Omega \times I$
(**Problem (P)**): Minimize the functional $f : W \times U \to I\!R$, $f = f_1 + f_2$ given by

$$f_1(y,u) \;:=\; f_1(y(b)), \qquad f_2(y,u) \;:=\; \int_I F(y(t), u(t), t)\, dt$$

over all functions $y(x,t), u(x,t)$, $(x,t) \in \Omega \times I$, $(y,u) \in W \times U$ satisfying the following nonlinear parabolic equation with initial conditions and homogeneous Dichlet boundary conditions.

$$\begin{aligned} y_t(x,t) - \mathbf{div}[G_2(y(x,t), \nabla y(x,t))] - B(u(x,t)) &= 0 \qquad (x,t) \in \Omega \times I \quad (28)\\ y(x,0) - y_0(x) &= 0 \qquad x \in \Omega, \quad (29)\\ y(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times I \quad (30)\\ u \in U_0 \subseteq U = L^s(\Omega \times I),\ 1 \le s \le \infty & \quad \text{given.} \quad (31) \end{aligned}$$

The operators $\mathbf{div}[..]$ and $\nabla$ are with respect to $x$, and $G_2$ and $B$ are given expressions which may be nonlinear. (We will not distinguish here between the mapping $G_2 : I\!R \times I\!R^n \to I\!R^n$ and it's associated Nemytski operator.) The solutions of (28) -(30)

should be understood as weak solutions, i.e. we use reflexive function spaces $H$,$V$ of functions over $\Omega$ with the property

$$V \subseteq H \subseteq V^*, \tag{32}$$

( for example let $V = W_0^{1,r}(\Omega)$ for some $r > 1$ and $H$ some $L^p(\Omega)$ space such that (32) holds, which space one finally has to choose depends on the existence theory of (28 - 30) for an explicitly given $G_2$). Moreover we use

$$W \subseteq W^{1,p,q}(I;V,V^*) = \{\, y \mid \|y\|_V \in L^p(I), \|y_t\|_{V^*} \in L^q(I), p^{-1} + q^{-1} = 1 \,\}. \tag{33}$$

Then with the abbreviation $( z \mid v ) = \int_\Omega z(x)v(x)\,dx$, we see (by using Green's formula) that the equations

$$\Gamma(y,u,v)(t) \quad := \quad ( y_t \mid v ) + \int_\Omega G_2(y,\nabla y)\cdot\nabla v \; + B(u)v\,dx \quad = \quad 0 \quad \forall\, v \in V. \tag{34}$$

represent (28) and (30). So we have the equality constraint

$g_2(y,u) = (g_2^1(y,u), g_2^2(y,u)) = 0$ where $g_2^1$ given by (34) and $g_2^2$ is defined by (29). Hence we set

$$g_2(y,u) = \begin{pmatrix} g_2^1(y,u) \\ g_2^2(y,u) \end{pmatrix} = \begin{pmatrix} y_t - \operatorname{div}[G_2(y,\nabla_x y)] - B(u) \\ y(x,t) - y_0(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and apply Theorem 2.4 to the problem

$$\text{minimize } f(y,u) \text{ subject to } g_2(y,u) = 0,\; u \in U_0.$$

This is possible if we assume that

(A1) Problem (P) has at least one solution, i.e. there exists one optimal state-control pair $(y,u)$.

(A2) The constraint qualifications of Theorem 2.4 are satisfied. This assumption can be reduced to the assumption that the linearized operator $D_y g$ is surjective, which amounts to the solvability of a *linear parabolic* equation (see below).

Both assumptions have to be verified if one wants to apply the necessary condition to a concrete problem. In e.g [24] the existence of solutions of quasilinear problems is investigated. With "Linearization " we mean the following

(A0) (Differentiability) Assume that $G_2 : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R} \to \mathbb{R}$ is directionally differentiable in these spaces and with bounded difference quotients (this holds e.g. if $G_2, B$ are lipschitz continuous) with *linear* directional derivatives. From $f$ we assume Lipschitz-continuity (in $H$-norm.)

Then from (A0) by Lebesgues' Theorem on dominated convergence we can deduce the following : For directions $(h,k) \in W \times U$ : the Gâteaux-derivative $Dg_2((y,u),(h,k))$ exists and is given by

$$\begin{aligned}
\langle D_y g_2^1(y,u)(h) \mid v \rangle_V &= \int_\Omega h_t v + D_y G_2((y,\nabla y);(h,\nabla h)) \cdot \nabla v \, dx && (35)\\
&= \int_\Omega h_t v + \{D_1 G_2(w)h + D_2 G_2(w) \cdot \nabla h\} \nabla v \, dx && (36)\\
&=: (h_t \mid v)_H + \langle A(t)h \mid v \rangle_V \quad \forall v \in V && (37)
\end{aligned}$$

where we set $w := w(x,t) := (y(x,t), \nabla y(x,t))$, and

$$D_1 G_2(y,\nabla y) := \frac{\partial}{\partial y} G_2(y,\nabla y) \quad D_2 G_2(y,\nabla y) := \frac{\partial}{\partial(\nabla y)} G_2(y,\nabla y). \tag{38}$$

Equation (37) should be understood as the Definition of the linear Operator $A(t) : V \to V^*$. So the operator $D_y g_2^1(y,h) = h_t + A(t)h$, is linear in $h$. The other derivatives of $g_2$ are obvious, we obtain :

$$Dg_2(h,k) = \begin{pmatrix} D_y g_2^1(h) & +D_u g_2^1(k) \\ D_y g_2^2(h) & +D_u g_2^2(k) \end{pmatrix} = \begin{pmatrix} h_t + A(t)h & + D_u B(k) \\ h(0,x) & +0 \end{pmatrix}. \tag{39}$$

According to [6] the derivatives of $f$ read as follows: There are linear functionals ( derivatives)

$$\begin{aligned}
\phi_y \in \partial_y f_1(y(b)), \quad \phi_y(h) &= \langle \phi_y \mid h \rangle_H && (40)\\
\psi_y \in \partial_y F(y(t),u(t)), \quad \psi_y(h) &= \int_I \langle \psi_y \mid h \rangle_V dt && (41)\\
\psi_u \in \partial_u F(y(t),u(t)), \quad \psi_y(k) &= \int_I \langle \psi_u \mid k \rangle_U dt && (42)
\end{aligned}$$

for all $h \in W, k \in U$. If $f$ is additionally Gâteaux-differentiable then we can take even $\phi_y = D_y f_1(y,\cdot)$ and so on.

We can now replace (A2) by a more specific assumption :

(A2') Assume that $int U_0 \neq \emptyset, D_u B(U_0) \subseteq Z_2$ holds and that the operator $A(t) : V \to V^*$ (i.e. the linearization of $G_2$ ) has such properties that $\forall z = (z_1,z_2) \in Z_2 = L^q(I;V^*) \times H$, there is $h \in W$ with $h = 0$ on $\partial\Omega \times I$ and

$$\begin{aligned}
D_y g_2^1(h) &= h_t + A(t)h &&= z_1 && (43)\\
D_y g_2^2(h) &= h(0,x) &&= z_2, && (44)
\end{aligned}$$

i.e. a weak solution of the linear parabolic problem exists for all appropriate functions $(z_1,z_2)$, i.e. we assume that $D_y g(y,\cdot) : W \to Z_2$ is surjective.

From this assumptions we can now deduce that the constraint qualifications are satisfied.

**Lemma 3.1** *Assume (A2') . Then the constraint qualifications of theorem 2.4 are fulfilled.*

**Proof.** In the following read $Dg := D_{y,u}g$. Set $K = intJ_0 \cap J_2$, where $intJ_0 = int(K(W,w) \times K(U_0,u)) = W \times intK(U_0,u)$, (since $W$ is a linear space we have $K(W,w) = W$) and $J_2 = \{ (h,k) \mid Dg_2((y,u);(h,k)) = 0 \}$. According to Theorems 2.6 , 2.8, 2.9 we have to show that $K \neq \emptyset$ holds and that the mapping $Dg((y,u),\cdot)$ is surjective, since then we can deduce $J_2 \subseteq T(S_2,(y,u))$ (Theorem 2.8(a)) where $S_2 := \{ (y,u) \in W \times U \mid g_2(y,u) = 0 \}$. By (A2') the operator $D_y g_2 : W \to Z_2$ is surjective hence it follows that $Dg_2 : W \times U \to Z_2$ is surjective and in particular a closed operator. Hence by Theorem 2.8 we have $J_2 \subseteq T(S_2,(y,u))$ and by Theorem 2.9 the set $H_2$ is $w^*$-closed. Now we show that $K \neq \emptyset$ holds. By (A2') we have for even for every $k \in int\, K(U_0,u)$ that $z_2 := D_u B(k) \in Z_2$ and hence that there exists a solution $h \in W$ of the linear equation

$$Dg_2(h,k) = \begin{pmatrix} h_t + A(t)h + D_u B(u,k) \\ h(0,x) + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that there exists

$$(h,k) \in intJ_0 = W \times \, int\, K(U_0,u) \text{ such that } Dg_2((y,u);(h,k)) = 0,$$

which is exactly $K \neq \emptyset$.

This Lemma now permits the application of Theorem 2.4 in order to to derive the necessary optimality conditions for Problem (P) under (A1), (A2').

**Theorem 3.2** (Necessary Conditions) *Assume that hypotheses (A0), (A1) and (A2') hold, such that $(y,u)$ is a solution of (P). Then there exist derivatives (see (37) to (42)) and Multipliers $z_2^* = (z_{2,1}^*, z_{2,2}^*) \in Z_2^* = (L^q(I;V^*) \times H)^* = L^p(I;V) \times H^*$ i.e. $z_{2,1}^* \in L^p(I;V)$, $z_{2,2}^* \in H^*$ such that the following adjoint equation and variational inequality hold.*

$$\begin{aligned} \psi_y(t) &= -\frac{\partial}{\partial t} z_{2,1}^*(t) + [A(t)]^* z_{2,1}^*(t) \quad t \in I && (45)\\ \phi_y &= z_{2,1}^*(b) && (46)\\ z_{2,2}^* &= z_{2,1}^*(0) && (47)\end{aligned}$$

*where*

$$\langle A(t)h \mid v \rangle_V = \int_\Omega [D_1 G_2(y,\nabla y)h(x) + D_2 G_2(y,\nabla y)\cdot \nabla h(x)]\nabla v(x)\,dx \qquad (48)$$

$$\int_I \Big\langle \psi_u - [D_u B]^* z_{2,1}^* \;\Big|\; k(t) \Big\rangle_H dt \;\geq\; 0 \quad \forall k \in K(U_0,u) = \cup_{\lambda \geq 0} \lambda(U_0 - u) \qquad (49)$$

*If especially $U_0 = \{ u \in L^\infty(\Omega \times I) \mid |u(x,t)| \leq 1 \text{ a.e} \}$ then there holds a pointwise maximum principle (a.e. in $\Omega \times I$.*

$$\Big(\psi_u - [D_u B]^* z_{2,1}^*(x,t)\Big)\, u(x,t) = \min_{|\alpha| \leq 1} \Big(\psi_u - [D_u B]^* z_{2,1}^*(x,t)\Big)\, \alpha \qquad (50)$$

*and a weak Bang-Bang principle*

$$u(x,t) = \begin{cases} 1 & (x,t) : \psi_u(x,t) - [D_u B]^* z_{2,1}^*(x,t) < 0 \\ -1 & (x,t) : \psi_u(x,t) - [D_u B]^* z_{2,1}^*(x,t) > 0. \end{cases} \qquad (51)$$

**Proof.** We set $X := W \times U$, $Z_2 = L^q(I; V*) \times H$. Then by the assumptions for the mapping $f : X \to I\!R$, $g_2 : X := W \times U \to Z_2$ and $D_{y,u} g_2 : X := W \times U \to Z$ the constraint qualifications are satisfied. The application of Theorem 2.4 then produces multipliers $z_2^* = (z_{2,1}^*, z_{2,2}^*) \in Z_2^*$ such that for all $h \in W$, $k \in K(U_0, u)$ we have

$$\psi_y(h) + \phi_y(h) - z_{2,1}^* \circ D_y g_2^1(h) - z_{2,2}^* \circ D_y g_2^2(h) \quad = \quad 0 \tag{52}$$

$$\psi_u(k) - z_{2,2}^* \circ D_u g_2^1(k) \quad \geq \quad 0 \tag{53}$$

Inserting the computed derivatives of (37 ) to (42) and using the notation $v^* \in V^*, v \in V, v^*(v) = \langle\, v^* \mid v \,\rangle_V$ and noting moreover, that
$\langle\, z_{2,1}^* \mid v^* \,\rangle_{V*} = \langle\, v^* \mid z_{2,1}^* \,\rangle_V$ since $V$ is reflexive ($V^{**} = V$) we then obtain

$$\langle\, \phi_y \mid h(b) \,\rangle_H + \int_I \langle\, \psi_y \mid h \,\rangle_V - \langle\, z_{2,1}^* \mid h_t + A(t)h \,\rangle_{V*}\, dt - \langle\, z_{2,2}^* \mid h(0) \,\rangle_H \quad = \quad 0, \tag{54}$$

$$\int_I \Big\langle\, \psi_u(t)k(t) \;\Big|\; k(t) \,\Big\rangle_H - \Big\langle\, z_{2,1}^* \;\Big|\; D_u Bk(t) \,\Big\rangle_H dt \quad \geq \quad 0 \quad \forall\, k \in K(U_0, u). \tag{55}$$

By partial integration of the product $z_{2,1}^* h_t$ in (54) with respect to $t$ (see e.g. [9] p. 147) and taking the adjoint operator $A^*$ of $A(t)$ we obtain for all $h \in W$

$$\langle\, \phi_y \mid h(b) \,\rangle_H + \langle\, z_{2,1}^*(b) \mid h(b) \,\rangle_H - \langle\, z_{2,1}^*(0) \mid h(0) \,\rangle_H + \tag{56}$$

$$+ \quad \textstyle\int_I \langle\, \psi_y \mid h \,\rangle_V - \langle\, -\frac{\partial}{\partial t} z_{2,1}^* + A^*(t) \mid h \,\rangle_V \, dt - \langle\, z_{2,2}^* \mid h(0) \,\rangle_H \quad = \quad 0, \tag{57}$$

which is equivalent to

$$\langle\, \phi_y - z_{2,1}^*(b) \mid h(b) \,\rangle_H - \langle\, z_{2,1}^*(0) - z_{2,2}^* \mid h(0) \,\rangle_H + \tag{58}$$

$$+ \qquad \textstyle\int_I \langle\, \psi_y \mid h \,\rangle_V - \langle\, -\frac{\partial}{\partial t} z_{2,1}^* + A^*(t) \mid h \,\rangle_{V*} \, dt \quad = \quad 0. \tag{59}$$

From this we deduce by standard arguments (variation over appropriate $h$) the adjoint equation (45) with the endpoint condition. The multiplier $z_{2,2}^*$ obviously can be eliminated by expressing it by $z_{2,1}^*(0)$. By taking the adjoint of $D_u B$ we obtain directly the variational inequality (49). If we consider now the special control set $U_0$ we we obtain (50) and (51) from (49) by noting that $k \in K(U_0, u)$ means $k = \lambda(v - u)$, $\lambda \geq 0, v \in U_0$ and varying over all functions $|v(x,t)| \leq 1$.

Some typical examples for the functional $f_1$ and it's derivative: Let $y_d : \Omega \to I\!R$ be the desired final state of the system, which is given, then

$$f_1(y) = \|y(b) - y_d\|^2_{L^2(\Omega)} \quad \Rightarrow \quad \langle\, \phi_Y \mid h \,\rangle_H = \int_\Omega 2(y(x,b) - y_d(x))h(x,b)\, dx$$

$$f_1(y) = \|y(b) - y_d\|_{L^1(\Omega)} \quad \Rightarrow \quad Df(y,h) = \int_\Omega I_N(x)|h| + (1 - I_N(x))\mathrm{sgn}(y)h\, dx$$

(read always $h = h(x,b)$, and , for the second, nonsmooth functional,

$$\langle\, \phi_Y \mid h \,\rangle_H \quad = \quad \int_\Omega I_N(x)\alpha(x)h(x,b) + (1 - I_N(x))\ \mathrm{sgn}(y)h(x,b)\, dx,$$

where $I_N$ is the indicator function of the set $N = \{\, (x,t) \in \Omega \times I \ \mid y(x,t) = 0 \,\}$ and $\alpha$ is a function with $|\alpha(x)| \leq 1$ *a.e.*.

**Remark 3.3'** (Extensions) In the example above we have not considered constraints on the state which can be formulated as inequality constraints. If we consider additionally $g_1(y(x,t)) \leq 0$ i.e. $g_1(y(x,t)) \in -C$ with $C := \{ y \mid y(x,t) \geq 0 \ (x,t) \in \Omega \times I \}$ we encounter two problems: One is to prove that the constraint qualifications of Theorem 2.4 hold, the second one is to characterize the dual space of $Z_1$ where $g_1 : Y \to Z_1$. These two problems are of course connected. If we want to use the Slater Constraint qualification, we have to use a function cone $C$ which has nonempty interior. This is done by *Mackenroth, Troeltzsch* [19] and [27], by taking $Z_1 = C^0(I;E) =$ set of continuous functions from $I$ to $E$, where $E$ again has to be a function cone of functions over $\Omega$ who has a nonempty topological interior i.e. $E = C^0(\Omega)$, because in $L^p$-spaces the cone of (a.e.) positive functions does *not* have this property. Having made this choice one then has to prove (or assume) that there is $h, k$ with $Dg_2((y,u),(h,k)) = 0$ (i.e. $h, k$ is the solution of the linear PDE in (A2') and *additionally* satisfies the system of strict inequalities $g_1(y,u)(t) + Dg_1((y,u),(h,k))(t) \in -int\, C_E$. This implies that $K \neq \emptyset$ in Theorem 2.6 hence the inclusion $K \subseteq T(S,x)$ used in Theorem 2.4 holds. The dual $Z_1^*$ of $Z_1 = C^0(I;E)$ is ,according to [19] $NBV(I;E^*)$, i.e. consists of normalized functions with bounded variation that are borel measures with values in $E^*$, and then one apply the Kuhn Tucker Theorem and proceed as [19] to develop the adjoint equations. The problem there is always to evaluate the multiplier $z_1^*$. (If $g_1(y)$ depends only on $t$, e.g. $g_1(y) = -\|y(t)\|_{L^p(\Omega)} + \alpha$ then the situation is slightly easier,because one then has only $Z_1 = C(I)$, $Z_1^* =$ positive Borel-measures.) In particular, if one has only a weak existence theory for the PDE, the inequality constraint $g_1$ may not depend from $\nabla y$ since then its image $Z_1$ is some $L^p$ space where Slater CQ cannot be proved. The same problem arises if the state $y$ is given by a *variational inequality* as e.g.

$$\int_\Omega G_1(y,\nabla y)(w(x) - y(x)) + G_2(y,\nabla y)(\nabla w(x) - \nabla y(x))) \, dx \geq 0 \quad w \in W_0 \subseteq W$$

instead of the PDE (28) e.g. if one considers obstacle problems or optimal shape design problems, see e. g. [20], [31], [11]. Here we have an inequality constrained problem where Slater CQ (if $G$ depends *nonlinearly* on $y$) seams difficult to prove and where the Guignard CQ might be used better.

In the following example we give a constructive procedure how this CQ can be verified directly in an inequality constrained problem without using Theorem 2.8.

**Example 3.4** Let $g : Y := L^2(\Omega \times I) \to L^1(\Omega \times I)$ be given by $g(y) := y^2(x,t) - c$ where $c > 0$ is a given, positive number. We want to consider the nonlinear mapping $g$ as inequality constraint, i.e. we consider the feasible set $S := \{ y \in Y \mid g(y) = |y^2(x,t)|^2 - c \leq 0 \ a.e. \}$. The Slater CQ cannot be verified since $g(y) \in -C$ where the cone $C = \{ y \in L^2(\Omega \times I) \mid y(x,t) \geq 0 \ a.e. \}$ has empty interior. Take a fixed $y \in S$ which is equivalent to $g(y) \in -C$. Note that $Dg(y,h) = 2yh$ and that, by linearity of $Dg(x,\cdot)$ ,we have
$J = \{ h \in Y \mid g(y) + Dg(y,h) \leq 0 a.e. \} = \{ h \in Y \mid Dg(y,h) \in K(-C, g(x)) \}$.
We want to prove directly that the inclusion

$$K \quad := \quad J \quad = \quad \{ h \in Y \mid g(y) + Dg(y,h) \leq 0 a.e. \} \quad \subseteq \quad T(S,y),$$

which is central in Theorem 2.4, holds. Note that $Dg(y,h) = 2yh$, Theorem 2.8 is not applicable since $Dg(y,\cdot)$ is in general not surjective ( it's surjective if the set $\{ (x,t) \mid y(x,t) = 0 \}$ has measure zero). Take $h \in J$. In order to prove $h \in T(S,y)$ we have to find sequences $h_n \in Y$, $\lambda_n \in \mathbb{R}_+$ with $h_n \to h$ and $y + \lambda_n h_n \in S$ i.e. $g(y+\lambda_n h_n) \leq 0$ *a.e.* Set $M = \{ (x,t) \mid y^2(x,t) = c \}$, $N = \{ (x,t) \mid y^2(x,t) < c \}$ and for every $n \in \mathbb{N}$ set $M_n = M \cap \{ (x,t) \mid h(x,t) \leq n \}$, $N_n = \{ (x,t) \mid y^2(x,t) \leq c - n^{-1} \} \cap \{ (x,t) \mid h(x,t) \leq n \}$. It's clear that (modulo sets of measure zero) $M \cup N = \Omega \times I$, $M_n \subseteq M, N_n \subseteq N$. Let $I_N, I_M, I_{M_n}, I_{N_n}$ be the indicator functions of those sets, i.e. $I_M(x,t) = 1$ if $(x,t) \in M$, $I_M(x,t) = 0$ if $(x,t) \notin M$ and so on. Then with $n \to \infty$ we have $I_{M_n} \to I_M$, $I_{N_n} \to I_N$ pointwise. Now we choose the sequence $\lambda_n \geq 0$ such that $\lambda_n \leq \min\{n^{-1}\sqrt{c}\,, n^{-2}(4\sqrt{c}+1)^{-1}\}$. and set $h_n := I_{M_n}h + I_{N_n}h$. Then $h_n \to h$ pointwise and, by Lebesgues' Theorem on dominated convergence, $h_n \to h$ in $L^2(\Omega \times I)$. Now it remains to show $g(y+\lambda_n h_n) \leq 0$. Read always $h = h(x,t), y = y(x,t)$. If $(x,t) \in M_n \subseteq M$ we deduce from $h \in J$ that $\mathrm{sgn}(y) = -\,\mathrm{sgn}(h)$ hence by $\lambda_n \leq n^{-1}\sqrt{c}$ , we obtain $(y+\lambda_n h_n)^2 \leq c$ which is the assertion. If $(x,t) \in N_n \subseteq N$ we have $|y| \leq \sqrt{c - n^{-1}}$, and $h \leq n$ and we obtain from $\lambda_n \leq n^{-2}(4\sqrt{c}+1)^{-1}$ that

$$(y+\lambda_n h_n)^2 = y^2 + 2\lambda_n h_n y + \lambda_n^2 h_n^2 \quad \leq \quad c - n^{-1} + 2\lambda_n n\sqrt{c-n^{-1}} + \lambda_n^2 n^2 \quad \leq \quad c$$

holds. Hence we have for all $(x,t) \in M_n \cup N_n$ that $g(y+\lambda_n h_n) \leq 0$. On $\Omega \times I \setminus M_n \cup N_n$ we have by construction $h_n = 0$ hence $g(y+\lambda_n h_n) = g(y) \leq 0$. Thus the Constraint Qualification is proved. It should be clear that such a construction can be carried through for more complicated operators $g$, even a nonsmooth one.

**Remark 3.5** Comparable necessary conditions for nonlinear elliptic or parabolic control problems (nonlinearity in the functional $f$ or in the state equation (PDE) or in both) have been obtained by several authors with approaches related to necessary conditions of optimization theory. The following is a short and incomplete survey indicating the problems and the employed techniques.

(a) For a linear PDE and the quadratic functional $f(y,u) = f(u) = \|y(u) - z_d\|_H^2 + (u|u)_U$ quadratic already *Lions* in [16] obtains necessary conditions by considering the *linear* mapping $u \to y(u)$ and directly introducing the adjoint state into the necessary optimality condition $D_u f(y(u),u) = 0$ (or $\geq 0$ if there is a constraint on the control. This general approach can also be used successfully for nonlinear problems as *Casas/Fernandez* in [5] show for a control problem with nonlinear elliptic PDE e.g. $-|\nabla y|^{p-2}\nabla y = B(u)$. But here the nonlinearity of the map $u \to y(u)$ requires additional investigations concerning differentiability.

(b) For convex functional and linear PDE with state constraints (= inequality contraint) *Mackenroth* [19] uses a convex duality theorem of Rockafellar to obtain the adjoint equation.

(c) For the nonlinear PDE

$$y_t - \Delta y - y^3 = u$$

(which might have explosive solutions) and the convex cost functional
$f(y,u) = f(u) = \|y(u) - z_d\|_H^6 + (u|u)_{L^2(\Omega\times I)}$
*Lions* in [18] proves (among other properties) necessary conditions using a *penalty method.*

(d) Instead of considering the PDE $y_t - A(t)y = B(u)$ as equality constraint one can, at least if this equation is linear or semilinear and $A$ generates a $C_0$- semigroup $S(t,s)$ use the integral equation

$$y(t) = S(t,0)y_0 + \int_0^t S(t,s)u(s)\ ds.$$

This is done by in the book of *Barbu/Precupanu* [2] if $f$ is convex and $A$ linear by using convex duality theory and by *Tröltzsch* [26], [27], [28] if the PDE is semilinear and $f$ is Fréchet-differentiable by using the *nonlinear* integral equation,

$$y(t) = S(t,0)y_0 + \int_0^t S(t,s)B(y(s),u(s))\ ds$$

and applying a Kuhn-Tucker Theorem of [32].

(d) An other approach , which more based on variational calculus is used by *Fattorini* in [7].

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