A PROBLEM OF EXACT CONTROLLABILITY OF DISTRIBUTED SYSTEM : BOUNDARY CONTROL OBTAINED AS LIMIT OF INTERNAL CONTROL .

Caroline Fabre *

INTRODUCTION - We consider a problem of exact controllability of the following model :

$$(P)\quad \begin{cases} u'' + \Delta^2 u = h \\ u(0) = u^0 \ , u'(0) = u^1 \ ; u = \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma \end{cases} ,$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a C^2-boundary Γ and $\Sigma = \Gamma \times (0, T)$.

When $N = 2$, this system models the motion of a vibrating plate, in a very simplified way. Enrike Zuazua has solved, using J.L.Lions' H.U.M, the exact controllability problem of this system when the control is distributed and acts on an ε - neighborhood of a suitable part of the boundary (see [4]). We present here, a study of the passage to the limit when $\varepsilon \to 0$. We prove that in one dimension of space, (that is, for the beams' problem), we obtain at the limit the boundary control given by H.U.M which acts on the normal derivative. In space dimension > 1, the question is still open but to point out the difficulty we will state the problem in the general case.

We could consider other boundary conditions and for example $u = \Delta u = 0$ on the boundary.

We also have similar problems for other equations and, for example, one can refer to [1] and [2] concerning the wave equation . For the Schroedinger equation, we already have some results but they are not complete.

PRESENTATION OF THE PROBLEM -Let Ω be a bounded open set in $\mathbb{R}^N$ with a C^2 - boundary Γ, and let $\nu(y)$ be the unit exterior normal at a point y of Γ.

Let Γ^0 be a subset of Γ. We will say that Γ^0 satisfy (1.1) if there exists x^0 in $\mathbb{R}^N$ with $\Gamma^0 = \{y \in \Gamma \,/\, (y - x^0).\nu(y) > 0\}$.

For $T > 0$, we write $\Sigma = \Gamma \times (0, T)$, $\Sigma^0 = \Gamma^0 \times (0, T)$, $\omega_\varepsilon = \Omega \cap O_\varepsilon$ where $O_\varepsilon = \bigcup \left(B(x, \varepsilon) \,/\, x \in \Gamma^0\right)$ and $Q_\varepsilon = \omega_\varepsilon \times]0, T[$. If Γ^0 satisfy (1.1), for $y^0 \in H_0^2(\Omega)$ and $y^1 \in L^2(\Omega)$, by J.L.Lions ' H.U.M, applied to this problem by E.Zuazua (see [4]), there exists a control $v_\varepsilon \in L^2(Q)$ such that the solution of

$$\begin{cases} \psi_\varepsilon'' + \Delta^2 \psi_\varepsilon = v_\varepsilon \chi_{Q_\varepsilon} \\ \psi_\varepsilon(0) = y^0 \ , \psi_\varepsilon'(0) = y^1 \ ; \psi_\varepsilon = \dfrac{\partial \psi_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \end{cases} ,$$

where χ_{Q_ε} denotes the charasteristic function of Q_ε, satisfies $\psi_\varepsilon(T) = \psi_\varepsilon'(T) = 0$. We recall some results given by the construction of this control v_ε .

From a solution φ of the homogeneous equation :

$$(H)\ \begin{cases}\varphi'' + \Delta^2\varphi = 0\\ \varphi^0 \in L^2(\Omega)\,,\ \varphi^1 \in H^{-2}(\Omega)\,;\ \varphi = \dfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma\end{cases},$$

we define ψ as the solution of the backward equation,

$$(C)\ \begin{cases}\psi'' + \Delta^2\psi = \varphi\,\chi_{Q_\varepsilon}\\ \psi(T) = 0\,,\ \psi'(T) = 0\,;\ \psi = \dfrac{\partial\psi}{\partial\nu} = 0 \text{ on } \Sigma\end{cases}.$$

Then we consider the operator Λ_ε defined by $\Lambda_\varepsilon(\varphi^0, \varphi^1) = (\psi'(0), -\psi(0))$ from $L^2(\Omega) \times H^{-2}(\Omega)$ to $L^2(\Omega) \times H_0^2(\Omega)$. One can see that

$$\big\langle \Lambda_\varepsilon(\varphi^0, \varphi^1)\,;\,(\psi'(0), -\psi(0))\big\rangle = \langle\varphi^1, \psi(0)\rangle_{H^{-2}, H_0^2} - (\varphi^0, \psi'(0))_{L^2} = \int_{Q_\varepsilon} \varphi^2(x, t)\,dx\,dt\,.$$

Suppose that Λ_ε is invertible and consider $\left(\widetilde{\varphi}_\varepsilon^0, \widetilde{\varphi}_\varepsilon^1\right) = \Lambda_\varepsilon^{-1}\left(y^1, -y^0\right)$. Denote by $\widetilde{\varphi}_\varepsilon$ the solution of (H) with initial data $\left(\widetilde{\varphi}_\varepsilon^0, \widetilde{\varphi}_\varepsilon^1\right)$ and by ψ_ε the solution of (C) with $\widetilde{\varphi}_\varepsilon$ in the right hand side. Then by definition of Λ_ε, $\psi_\varepsilon(0) = y^0$ and $\psi_\varepsilon'(0) = y^1$, and $\psi_\varepsilon(T) = \psi_\varepsilon'(T) = 0$ so this solves the control problem when the control acts in an ε - neighborhood of Γ^0. To prove that the operator Λ_ε is invertible, we use the Lax-Milgram theorem. For this, we establish the equivalence between the $L^2(\Omega) \times H^{-2}(\Omega)$ norm of the initial data of solutions of (H) and the L^2 - norm in Q_ε of these solutions. This has been done by E.Zuazua in [4] and more exactly he proved the two following results :

Theorem 1 - There exists a constant C depending only of the geometry of Ω and T *such that for every solution of* (H) *with initial data* $\left(\theta^0, \theta^1\right) \in H_0^2 \times L^2$, *we have*

$$\| \theta^0 \|_{H_0^2}^2 + \| \theta^1 \|_{L^2}^2 \le C\left[\frac{1}{\varepsilon^5}\int_{Q_\varepsilon}\left(\theta'^2(x, t) + \theta^2(x, t)\right)dx\,dt\right].$$

By a compactness argument, he deduces from this theorem the following one :

Theorem 2 - There exists a constant C_ε *depending on* Ω ,T *and* ε *such that for every solution of* (H) *with initial data* $\left(\varphi^0, \varphi^1\right) \in L^2 \times H^{-2}$, *we have*

$$\| \varphi^0 \|_{L^2}^2 + \| \varphi^1 \|_{H^{-2}}^2 \leq C_\varepsilon \int_{Q_\varepsilon} \varphi^2(x, t) \; dx \, dt.$$

REMARK - We have no longer any estimate on C_ε.

This proves that Λ_ε is invertible and it allows us to control in an ε - neighborhood of Γ^0.

The problem is now to find what happens when $\varepsilon \to 0$ that means to study the convergence of the problem

$$(CE_\varepsilon) \begin{cases} \psi_\varepsilon'' + \Delta^2 \psi_\varepsilon = \widetilde{\varphi}_\varepsilon \chi_{Q_\varepsilon} \\ \psi_\varepsilon(T) = 0 \, , \, \psi_\varepsilon'(T) = 0 \, ; \, \psi_\varepsilon = \dfrac{\partial \psi_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \end{cases},$$

where,

$$(H_\varepsilon) \begin{cases} \widetilde{\varphi}_\varepsilon'' + \Delta^2 \widetilde{\varphi}_\varepsilon = 0 \\ \widetilde{\varphi}_\varepsilon^0 \in L^2(\Omega) \, , \, \widetilde{\varphi}_\varepsilon^1 \in H^{-2}(\Omega) \, ; \, \widetilde{\varphi}_\varepsilon = \dfrac{\partial \widetilde{\varphi}_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \end{cases},$$

and,

$$\left\langle \widetilde{\varphi}_\varepsilon^1, y^0 \right\rangle_{H^{-2}, H_0^2} - (\widetilde{\varphi}_\varepsilon^0, y^1)_{L^2} = \int_{Q_\varepsilon} \widetilde{\varphi}_\varepsilon^2 (x, t) \, dx \, dt.$$

To study this question, we need estimates on the right hand side of (CE_ε) and for this we have to get estimates on the problem (H_ε). We will see later on that the "good" functions to consider are not $\widetilde{\varphi}_\varepsilon$ (they are not bounded in $L^2(Q_\varepsilon)$) but $\varphi_\varepsilon = \varepsilon^5 \widetilde{\varphi}_\varepsilon$. Indeed, for $N = 1$, we will show that $\left(\varphi_\varepsilon^0 , \varphi_\varepsilon^1\right)$ is bounded in $L^2(\Omega) \times H^{-2}(\Omega)$ and that

$$(1.2) \qquad \frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \varphi_\varepsilon^2 (x, t) \; dx \, dt \leq M,$$

where M is independent of ε.

To get this last estimate, we will need results concerning the solution of (1) and precisely :

(i) we'll have to describe exactly the behaviour near the boundary of solutions with finite energy,

(ii) we will see that in problems like (H_ε), if the initial data are bounded and if the condition (1.2) is fulfilled, then the solution of the limit problem has a finite energy,

(iii) we 'll also have to study the convergence of linear forms of the type :

$$(u^0 , u^1, h) \in H_0^2(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega)) \to \frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \varphi_\varepsilon (x, t) \, u(x, t) \, dx \, dt ,$$

where u is defined by (P).

Having those 3 points, we can show that problem (CE_ε) converges and exhibit the limit.

As (iii) uses again (i) and (ii), we begin by these two last points : at the moment we know how to solve (i) in one dimension of space but not for $N \geq 2$ and the difficulty is here in the sense that we could solve the problem if we had this result.

(i) BEHAVIOUR NEAR THE BOUNDARY OF THE SOLUTIONS OF (P) WITH FINITE ENERGY .

In this section $\Omega = [0,\ 1]$ and $\Gamma^0 = \Gamma = \{0\} \cup \{1\}$.
We are interested here in solutions of finite energy of (P) that is for $u^0 \in H_0^2(\Omega)$, $u^1 \in L^2(\Omega)$ and $h \in L^1(0, T; L^2(\Omega))$, we consider u solution of

$$(P)\begin{cases} u'' + \Delta^2 u = h \\ u(0) = u^0\ ,\ u'(0) = u^1\ ;\ u = \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma \end{cases} .$$

We recall that J.L.Lions proved that, in this context, $\Delta u \in L^2(\Sigma)$ (see [3]) and that the linear mapping $(u^0, u^1, h) \in H_0^2(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega)) \rightarrow \Delta u \in L^2(\Sigma)$ is continuous

We consider the expression $\dfrac{1}{\varepsilon^5}\displaystyle\int_{Q_\varepsilon} u^2(x, t)\, dx\, dt$: it may look strange but as we will see, it appears naturally in our problem and on an other side, for very regular solutions "we have" (this is false of course but it gives an idea of the meaning of this expression) :

for $x \in]0, \varepsilon[$, $\qquad u^2(x, t) \approx x^4\left(\dfrac{\partial^2 u}{\partial x^2}(0, t)\right)^2$

so that, $\qquad \dfrac{1}{\varepsilon^5}\displaystyle\int_{Q_\varepsilon} u^2(x, t)\, dx\, dt \approx \dfrac{1}{5}\int_0^T \left(\Delta u(0, t)\right)^2 d\sigma\, dt$.

This explains (with the regularity result of J.L.Lions) the statement of the following theorem :

Theorem 3 - There exists a constant C *depending only on* T *such that for every solution of* (P) *with finite energy, we have :*

$$\frac{1}{\varepsilon^5}\int_{Q_\varepsilon} u^2(x, t)\, dx\, dt \leq C\left(\| u^0 \|^2_{H_0^2} + \| u^1 \|^2_{L^2} + \| h \|^2_{L^1(0, T; L^2(\Omega))}\right) .$$

Of course, we hope to have the same result in dimension 2 but there still remain some technical difficulties.
To have an idea of the proof, one can refer to [1] where we show a similar result for the wave equation.
This theorem is essential for our problem and we will use it many times.

(ii) REGULARITY OF THE LIMIT OF NON REGULAR PROBLEMS

In this section, $N \geq 1$ and Γ^0 is any subset of Γ with measure > 0. Suppose that we have

$$(H_\varepsilon) \begin{cases} \varphi_\varepsilon'' + \Delta^2 \varphi_\varepsilon = 0 \\ \varphi_\varepsilon^0 \in L^2(\Omega)\,,\ \varphi_\varepsilon^1 \in H^{-2}(\Omega)\,;\ \varphi_\varepsilon = \dfrac{\partial \varphi_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \end{cases},$$

with $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$ bounded in $L^2(\Omega) \times H^{-2}(\Omega)$.
Then one can say that the functions φ_ε converge (after extraction of a subsequence) for the weak - * topology of $L^\infty(0, T\,; L^2(\Omega))$ to φ solution of

$$(H) \begin{cases} \varphi'' + \Delta^2 \varphi = 0 \\ \varphi^0 \in L^2(\Omega)\,,\ \varphi^1 \in H^{-2}(\Omega)\,;\ \varphi = \dfrac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Sigma \end{cases}$$

where φ^0 and φ^1 are the weak limit of φ_ε^0 and φ_ε^1 . As we have already mentionned, our functions φ_ε will also satisfy (1.2) and this condition gives the following regularity for the limit :

Theorem 4 - If $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$ *are bounded in* $L^2(\Omega) \times H^{-2}(\Omega)$ *and if furthermore*

$$\sup_\varepsilon \frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \varphi_\varepsilon^2(x, t)\ dx\, dt \leq M,$$

*then the weak - * limit* φ *of the functions* φ_ε *satisfies* $\Delta\varphi \in L^2(\Sigma^0)$. *If* Γ^0 *satisfies* (1.1) *then* $\varphi^0 \in H_0^2(\Omega)$, $\varphi^1 \in L^2(\Omega)$ *and* φ *has a finite energy.*

REMARK - The last point is easily given by the following results of J.L.Lions and E.Zuazua (see [3] , [4]) :

Theorem 5 - If Γ^0 *satisfies* (1.1), *the* $H_0^2(\Omega) \times L^2(\Omega)$ *-norm of the initial data of* (H) *is equivalent to the* $L^2(\Sigma^0)$ *-norm of* $\Delta\varphi$ *on the boundary .*

(iii) STUDY OF THE LIMIT OF SOME LINEAR FORMS

In this section, $\Omega = [0, 1]$ and Γ^0 satisfies (1.1), which means, for example, that $\Gamma^0 = \{0\}$.
We introduce the following linear forms :

$$L_\varepsilon : H_0^2(\Omega) \times L^2(\Omega) \times L^1(0, T ; L^2(\Omega)) \to \mathbb{R}$$

$$(u^0, u^1, h) \to \frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \varphi_\varepsilon(x, t)\, u(x, t)\, dx\, dt$$

where u is the solution of (P) and φ_ε satisfies $\varphi_\varepsilon'' + \Delta^2 \varphi_\varepsilon = 0$, $\varphi_\varepsilon(0) = \varphi_\varepsilon^0 \in L^2(\Omega)$, $\varphi'_\varepsilon(0) = \varphi_\varepsilon^1 \in H^{-2}(\Omega)$, $\varphi_\varepsilon = \frac{\partial \varphi_\varepsilon}{\partial \nu} = 0$ on Σ with initial data bounded in $L^2(\Omega) \times H^{-2}(\Omega)$.

We want to study their convergence and to exhibit their limit. If we assume that φ_ε satisfies (1.2), using theorem 3 and Holder 's inequality, one can easily see that $(L_\varepsilon)_\varepsilon$ is bounded in $H^{-2}(\Omega) \times L^2(\Omega) \times L^\infty(0, T ; L^2(\Omega))$. So we already know that under these assumptions and after extraction of a subsequence, $(L_\varepsilon)_\varepsilon$ converges for the weak - * topology of $H^{-2}(\Omega) \times L^2(\Omega) \times L^\infty(0, T ; L^2(\Omega))$ to $L \in H^{-2}(\Omega) \times L^2(\Omega) \times L^\infty(0, T ; L^2(\Omega))$.
We will see that these hypotheses are those which emerge naturally from the control problem.
Having the convergence, it remains to exhibit the limit which is given by the following

Theorem 6 - Under the above assumptions, the linear forms L_ε *converge for the weak - * topology of* $H^{-2}(\Omega) \times L^2(\Omega) \times L^\infty(0, T ; L^2(\Omega))$ *to* L *with*

$$L(u^0, u^1, h) = \frac{1}{5} \int_{\Sigma^0} \Delta\varphi\, \Delta u\, d\sigma\, dt$$

where φ *is (after extraction of a subsequence) the limit, in a weak sense, of* φ_ε.

One can see that if we had theorem 3 in any dimension of space, we would also have this theorem and by the way all the results from the beginning of this paper.

We are now in the position to get the estimate on the controls $\tilde{\varphi}_\varepsilon$.

(iv) ESTIMATE ON THE CONTROLS

From now on, $\Omega =]0, 1[$ and $\Gamma^0 = \{0\}$.
We begin by some recalls : y^0 and y^1 are fixed in H_0^2 and L^2. $\tilde{\varphi}_\varepsilon$ is solution of

$$(H_\varepsilon) \begin{cases} \tilde{\varphi}_\varepsilon'' + \Delta^2 \tilde{\varphi}_\varepsilon = 0 \\ \tilde{\varphi}_\varepsilon^0 \in L^2(\Omega)\, ,\ \tilde{\varphi}_\varepsilon^1 \in H^{-2}(\Omega)\, ;\ \tilde{\varphi}_\varepsilon = \dfrac{\partial \tilde{\varphi}_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \end{cases},$$

with ,

(1.3) $\left\langle \Lambda_\varepsilon(\varphi^0, \varphi^1); (\psi'(0), -\psi(0)) \right\rangle = \langle \widetilde{\varphi}^1_\varepsilon, y^0 \rangle_{H^{-2}, H_0^2} - (\widetilde{\varphi}^0_\varepsilon, y^1)_{L^2} = \int_{Q_\varepsilon} \widetilde{\varphi}^2_\varepsilon(x, t)\, dx\, dt$.

To get an estimate on the controls, we have to describe how the constant C_ε of Theorem 2 depends on ε. This is given by

Theorem 7 - There exists a constant C depending only on T such that for every solution of (H) *with initial data* $(\varphi^0, \varphi^1) \in L^2 \times H^{-2}$, *we have*

$$(1.4) \qquad \| \varphi^0 \|^2_{L^2} + \| \varphi^1 \|^2_{H^{-2}} \leq C \left[\frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \varphi^2(x, t)\ dx\, dt \right],$$

or equivalently, there exists a constant C depending only on T such that for every solution of (H) *with initial data* $(\theta^0, \theta^1) \in H_0^2 \times L^2$, *we have*

$$(1.5) \qquad \| \theta^0 \|^2_{H_0^2} + \| \theta^1 \|^2_{L^2} \leq C \left[\frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \theta'^2(x, t)\, dx\, dt \right].$$

This theorem uses (i), (ii) and (iii) and it would be true in any dimension of space if we had (i). The proof of this result will be given later on.

Assume Theorem 7, we write $\varphi_\varepsilon = \varepsilon^5 \widetilde{\varphi}_\varepsilon$, φ_ε is solution of an homogeneous beam 's equation, and using (1.3), one can show that $(\varphi^0_\varepsilon, \varphi^1_\varepsilon)$ is bounded in $L^2 \times H^{-2}$ and that the condition (1.2) is satisfied.

We can now study the limit of (CE_ε) .

(v) PASSAGE TO THE LIMIT IN THE PROBLEM OF CONTROL

We rewrite the system (CE_ε) as follows : $\begin{cases} \psi_\varepsilon'' + \Delta^2 \psi_\varepsilon = \dfrac{1}{\varepsilon^5} \varphi_\varepsilon \chi_{Q_\varepsilon} \\ \psi_\varepsilon(T) = 0\ ,\ \psi_\varepsilon'(T) = 0\ ;\ \psi_\varepsilon = \dfrac{\partial \psi_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \end{cases}$.

By definition of φ_ε, we have $\psi_\varepsilon(0) = y^0$ and $\psi_\varepsilon'(0) = y^1$ which are fixed in H_0^2 and L^2.
The hypotheses on the functions φ_ε are

$$\begin{cases} \varphi_\varepsilon'' + \Delta^2 \varphi_\varepsilon = 0 \\ \varphi_\varepsilon = \dfrac{\partial \varphi_\varepsilon}{\partial \nu} = 0 \text{ on } \Sigma \\ (\varphi_\varepsilon^0, \varphi_\varepsilon^1) \text{ is bounded in } L^2 \times H^{-2} \\ \sup\limits_\varepsilon \left(\dfrac{1}{\varepsilon^5} \displaystyle\int_{Q_\varepsilon} \varphi_\varepsilon^2(x,t)\, dx\, dt \right) < \infty \end{cases}$$

To pass to the limit in (CE_ε), we consider ψ_ε as defined by transposition from solutions of (P) with finite energy so we have for all $(u^0, u^1, h) \in H_0^2(\Omega) \times L^2(\Omega) \times L^1(0, T; L^2(\Omega))$,

$$\langle \psi_\varepsilon, h \rangle_{L^\infty(0,T;L^2(\Omega));L^1(0,T;L^2(\Omega))} = -\langle y^0, u^1 \rangle_{L^2} + \langle y^1, u^0 \rangle_{H^{-2}; H_0^2} + \frac{1}{\varepsilon^5} \int_{Q_\varepsilon} \varphi_\varepsilon(x,t)\, u(x,t)\, dx\, dt$$

In the right hand side of this equality, we recognize exactly the linear forms L_ε that we have already introduced. We have seen in theorem 6 that under the assumptions on φ_ε, L_ε converges in a weak sense and we have exhibited the limit. This proves that $(\psi_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0, T; L^2(\Omega))$ and if we note ψ its limit for the weak - * topology of $L^\infty(0, T; L^2(\Omega))$ (after extraction of a subsequence), we have

Theorem 8 - ψ is solution of

$$\begin{cases} \psi'' + \Delta^2 \psi = 0 \\ \psi(T) = 0,\ \psi'(T) = 0 \\ \psi = 0 \text{ on } \Sigma \\ \dfrac{\partial \psi}{\partial \nu} = \dfrac{1}{5} \Delta\varphi \text{ on } \Sigma^0 = \{0\} \times (0, T) \\ \dfrac{\partial \psi}{\partial \nu} = 0 \text{ on } \Sigma - \Sigma^0 = \{1\} \times (0, T) \\ \psi(0) = y^0 \text{ and } \psi'(0) = y^1 \end{cases},$$

where the limit φ of φ_ε satisfies (by Theorem 4) $\Delta\varphi(0, t) \in L^2(0, T)$ and has a finite energy.

We obtain at the limit the boundary control given by H.U.M which acts on the normal derivative (see [3] for T big enough and [4] for all T > 0).

REMARK - J.L Lions ' H.U.M gives a boundary control in $L^2(0, T)$ for initial data in $L^2(0, 1)$ and $H^{-2}(0, 1)$. To reach these spaces, we approach the initial data y^0 and y^1 in $L^2(0, 1)$ and $H^{-2}(0, 1)$ by sequences of regular initial data y_n^0 and y_n^1, then we control in ε-neighborhoods of Γ^0, and we pass to the limit first when ε tends to zero and then when n tends to infinity.

PROOF OF THEOREM 7

We have N = 1 and Γ^0 satisfy (1.1).
For $(\varphi^0, \varphi^1) \in L^2 \times H^{-2}$ and φ solution of (H), we will denote by

$$\theta(x, t) = \int_0^t \varphi(x, s)\, ds + \chi \quad \text{where} \quad \begin{cases} \Delta^2 \chi = -\varphi^1 \\ \chi \in H_0^2 \end{cases}.$$

θ is a solution of (H) with finite energy, $\theta' = \varphi$ and $\|\varphi^0\|_{L^2}^2 + \|\varphi^1\|_{H^{-2}}^2$ is equivalent to $\|\theta^0\|_{H_0^2}^2 + \|\theta^1\|_{L^2}^2$. This justifies the equivalence between (1.4) and (1.5).
We are going to prove (1.5) by a counter - argument ; suppose that (1.5) is false, then there exists a sequence $(\varepsilon_n)_n$ of real non negative numbers which tends to zero and sequences $(\theta_n^0)_n \subset H_0^2$, $(\theta_n^1)_n \subset L^2$ such that

$$(1.6)\quad E^2(\theta_n) = \|\theta_n^0\|_{H_0^2}^2 + \|\theta_n^1\|_{L^2}^2 = 1$$

and

$$(1.7)\quad \frac{1}{\varepsilon_n^5} \int_0^T \int_0^{\varepsilon_n} \theta_n'^2(x, t)\, dx\, dt \to 0.$$

By (1.6), $\theta_n^0 \to \theta^0 \in H_0^2$ and $\theta_n^1 \to \theta^1 \in L^2$ respectively for the weak topologies of H_0^2 and L^2 (and after extraction of subsequences). By continuity with respect to the initial data, θ_n converges in $L^\infty(0, T ; H_0^2(0, 1))$ weak - * to the solution θ of (H) with (θ^0, θ^1) as initial data and θ_n converges in $L^\infty(0, T ; L^2(0, 1))$ weak - * to $\dot{\theta}$.

Lemma 1 - We have $\theta^0 \in H^4 \cap H_0^2$ *and* $\theta^1 \in H_0^2$

proof - we apply theorem 4 to the functions θ'_n and we get $\frac{\partial \theta'}{\partial x}(0, t) \in L^2(0, T)$ so θ' has a finite energy .

Lemma 2 - $\theta^0 = 0$ *and* $\theta^1 = 0$ *hence* $\theta = 0$.

Proof - We consider the following linear continuous forms : (the continuity is given by Theorem 3)

$$\Lambda_n : H_0^2 \times L^2 \times L^1(0, T ; L^2) \to \mathbb{R}$$

$$(u^0, u^1, h) \in H_0^2 \times L^2 \times L^1(0, T ; L^2) \to \frac{1}{\varepsilon_n^5} \int_0^T \int_0^{\varepsilon_n} \theta'_n(x, t)\, u(x, t)\, dx\, dt$$

where u is solution of (P).

From theorem 6, Λ_n converges in a weak - * sense to

$$\Lambda(u^0, u^1, h) = \frac{1}{5} \int_0^T \frac{\partial \theta'}{\partial x}(0, t) \frac{\partial u}{\partial x}(0, t)\, dt.$$

But, from (1.7) and theorem 3,

$$\| \Lambda_n \| \leq c \left(\frac{1}{\varepsilon_n^5} \int_0^T \int_0^{\varepsilon_n} \theta'^2_n(x, t)\, dx\, dt \right)^{1/2} \to 0,$$

hence we have

$$\forall (u^0, u^1, h) \in H_0^2 \times L^2 \times L^1(0, T ; L^2)) \ , \qquad \int_0^T \frac{\partial \theta'}{\partial x}(0, t) \frac{\partial u}{\partial x}(0, t)\, dt = 0.$$

Taking $u = \theta'$ (which has a finite energy) we easily deduce that $\theta = 0$.

Lemma 3 - There exists a constant c depending only on T such that

$$\forall n \textit{ and } \ \forall t \in (0, T) \quad \frac{1}{\varepsilon_n^5} \int_0^{\varepsilon_n} \theta_n^2(x, t)\, dx \leq c\, E^2(\theta_n) = c\,.$$

Proof - We use again theorem 3 to get

$$\forall n \quad \frac{1}{\varepsilon_n^5} \int_0^T \int_0^{\varepsilon_n} \theta_n^2(x, t)\, dx \leq c\, E^2(\theta_n) = c\,.$$

Now,

$$\theta_n(x, t) = \theta_n(x, s) + \int_s^t \theta'_n(x,r)\, dr$$

thus

$$\theta_n^2(x, t) = \theta_n^2(x, s) + \left(\int_s^t \theta'_n(x,r)\, dr \right)^2 + 2\, \theta_n(x, s) \int_s^t \theta'_n(x,r)\, dr\,.$$

Taking into account the sign of the middle term, we have for every $0 < \gamma < 1$,

$$\forall s, \forall n \quad \frac{T}{\varepsilon_n^5} \int_0^{\varepsilon_n} \theta_n^2(x, s)\, dx \leq c + \gamma T \frac{1}{\varepsilon_n^5} \int_0^{\varepsilon_n} \theta_n^2(x, s)\, dx + \frac{1}{\gamma} \frac{T^2}{\varepsilon_n^5} \int_0^T \int_0^{\varepsilon_n} \theta_n'^2(x, t)\, dx\, dt \quad ,$$

where c depends only on T. By (1.7), we deduce Lemma 3.

We introduce ψ_n obtained from θ_n by integration in time just as described above.
$\psi_n^0 = \chi_n \in H^4 \cap H_0^2$ and $\psi_n^1 = \overset{0}{\theta_n} \in H_0^2$ and they both strongly converge to zero in H_0^2 and L^2.

Lemma 4 - There exists a real positive number p and a subsequence $(n_k)_k$ *such that*

$$\forall t \in (0, T) \qquad \frac{1}{\varepsilon_{n_k}^5} \int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x, t)\, dx \underset{k\,\infty}{\rightarrow} p$$

(the subsequence is the same for all t).

Proof - By a similar method than in Lemma 3, one can easily show that there exists c depending only on T such that

$$\forall t \in (0, T), \forall n \qquad \frac{1}{\varepsilon_n^5} \int_0^{\varepsilon_n} \psi_n^2(x, t)\, dx \leq c\,.$$

For $t = 0$, we can find a subsequence $(n_k)_k$ such that $\frac{1}{\varepsilon_{n_k}^5} \int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x, 0)\, dx \underset{k\,\infty}{\rightarrow} p$ where $p \geq 0$.

Then, we consider the following linear forms :

$$L_{n_k}^t : H_0^2 \times L^2 \times L^1(0, T; L^2) \rightarrow \mathbb{R}$$

$$(u^0, u^1, h) \in H_0^2 \times L^2 \times L^1(0, T; L^2)) \rightarrow \frac{1}{\varepsilon_{n_k}^5} \int_0^t \int_0^{\varepsilon_{n_k}} \psi_{n_k}(x, s)\, u(x, s)\, dx\, ds\,,$$

where u is solution of (P). From Lemma 3 and Theorem 3, these forms are continuous and they are uniformly bounded in t . As the initial data of ψ_{n_k} converge strongly to zero in H_0^2 and L^2, we have $L_{n_k}^t(\psi_{n_k}^0, \psi_{n_k}^1, 0) \rightarrow 0$. But, by integration by part in time, we obtain

$$2\, L_{n_k}^t(\psi_{n_k}^0, \psi_{n_k}^1, 0) = \frac{1}{\varepsilon_{n_k}^5} \int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x, t)\, dx - \frac{1}{\varepsilon_{n_k}^5} \int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x, 0)\, dx\,,$$

which proves that $\frac{1}{\varepsilon_{n_k}^5} \int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x, t)\, dx$ also converges to p and this for all t .

Lemma 5 - The constant $p = 0$.

Proof - Denote by $f_k(t) = \frac{1}{\varepsilon_{n_k}^5} \int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x, t)\, dx$

We have $f_k \rightarrow p$ everywhere, $\forall t$, $|f_k(t)| \leq c$ where c is independent of t, k. Then, by Lebesgue 's theorem, $\int_0^T f_k(t)\, dt \rightarrow p\,T$.

But, by another way,

$$\int_0^T f_k(t)\,dt = \frac{1}{\varepsilon_{n_k}^5}\int_0^T\int_0^{\varepsilon_{n_k}} \psi_{n_k}^2(x,s)\;dx\,ds \le c\left(\|\psi_{n_k}^0\|_{H_0^1}^2 + \|\psi_{n_k}^1\|_{L^2}^2\right) \to 0$$

This proves that p = 0 and Lemma 5 .

End of the proof - We consider $\Lambda_{n_k}(\psi_{n_k}^0, \psi_{n_k}^1, 0)$.
On one hand, we have $\left|\Lambda_{n_k}(\psi_{n_k}^0, \psi_{n_k}^1, 0)\right| \to 0$ by (1.7) and Lemma 4, and on an other hand, we have by integration by part in time :

$$\Lambda_{n_k}(\psi_{n_k}^0, \psi_{n_k}^1, 0) = \frac{1}{\varepsilon_{n_k}^5}\int_0^{\varepsilon_{n_k}} \psi_{n_k}(x,T)\,\theta_{n_k}(x,T)\,dx - \frac{1}{\varepsilon_{n_k}^5}\int_0^{\varepsilon_{n_k}} \psi_{n_k}(x,0)\,\theta_{n_k}(x,0)\,dx -$$

$$- \frac{1}{\varepsilon_{n_k}^5}\int_0^T\int_0^{\varepsilon_{n_k}} \theta_{n_k}^2(x,t)\,dx\,dt \;.$$

Using Lemma 3, 4, 5 and Holder 's inequality, one can see that the boundary terms in time tend to zero when k goes to infinity . We then deduce that

$$\frac{1}{\varepsilon_{n_k}^5}\int_0^T\int_0^{\varepsilon_{n_k}} \theta_{n_k}^2(x,t)\,dx\,dt \to 0$$

and we conclude with E. Zuazua 's Theorem 1 which contradicts (1.6).

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* Département de mathématique et d'informatique de l'université d'Orléans
et C M A P, Ecole Polytechnique, 91128 Palaiseau Cedex.