An introduction to Anderson localization and MBL

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Anderson localization

What began in 1958 ...

PHYSICAL REVIEW   VOLUME 109, NUMBER 5   MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

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... still remains relevant today

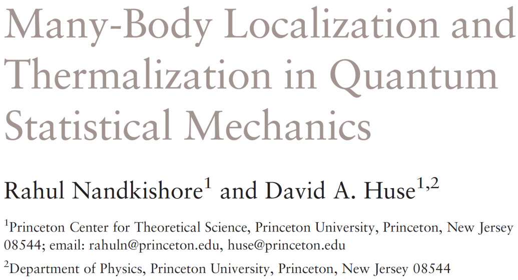
PHYSICAL REVIEW B **96**, 214201 (2017)

Anderson localization transitions with and without random potentials

Trithep Devakul and David A. Huse
Department of Physics, Princeton University, New Jersey 08544, USA
(Received 20 October 2017; published 6 December 2017)

MBL - the current "hot topic"

Published in 2015:



672 citations as of April 2018 acc. to Google Scholar.

Outline

Part 1: the Anderson localization

1. The basic concepts
2. The Anderson model
3. Numerical simulations

Part 2: MBL

1. The basics of ETH
2. MBL criteria
3. Numerical results - the level statistics of the tJ model

What is Anderson localization all about?

- Conduction in **NON-INTERACTING** systems with **DISORDER**
- Describes the role of **IMPURITIES**
- Completely different than the **Drude** model:

$$\sigma \propto l, \qquad l : \text{the mean-free path}$$

Anderson localization - the predictions

- for some disorder: $\sigma = 0$
- seminal paper by **P. W. Anderson (1958)** [1]
- Nobel prize in **1977**

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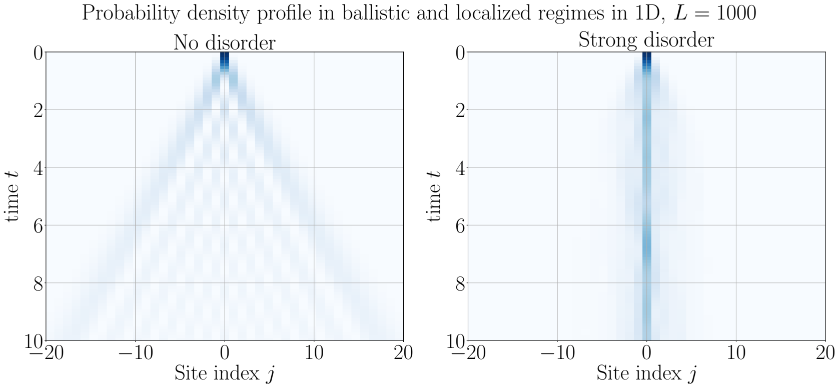
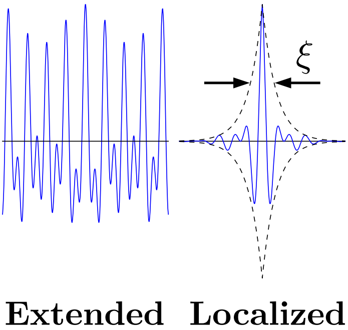


The basics

- **DISORDER** $\rightarrow$ **(eigen)**states can localize

Localization:



- A localized state:

$$|\psi(\mathbf{r})| \sim \exp\left(|\mathbf{r} - \mathbf{r}_0|/\xi\right)$$

- explains **vanishing** transport

The important keynotes

- An **interference** phenomenon
- Strong **dimensionality** dependence
- Energy dependence $\rightarrow$ the **mobility edge**

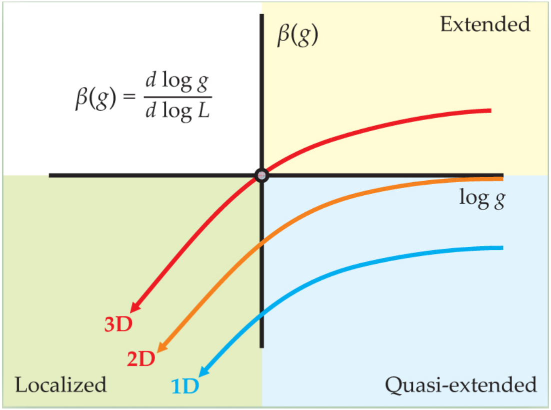
- scaling of the **conductance** g of a **hypercube** L^d [2]

- **Ohmic** conductor:

$$g = \sigma L^{d-2}$$

- **Localized** regime:

$$g \propto \exp(-L)$$



Transition between ext. and loc. states is only possible in 3D. Taken from [4].

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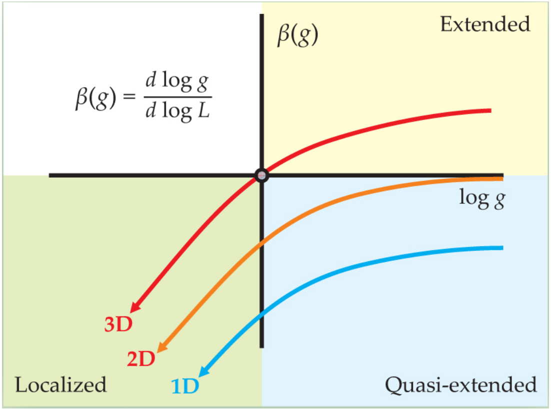

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The scaling theory

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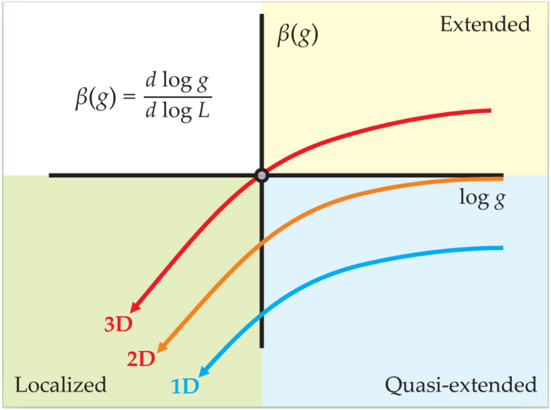


Transition between ext. and loc. states is only possible in 3D. Taken from [4].

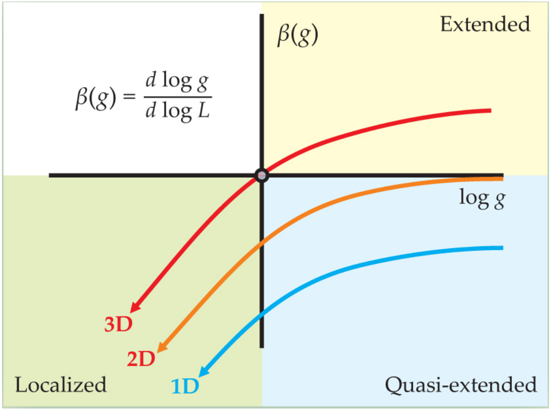
The scaling theory

1D, 2D

localization for any finite disorder

3D

localization for some critical disorder



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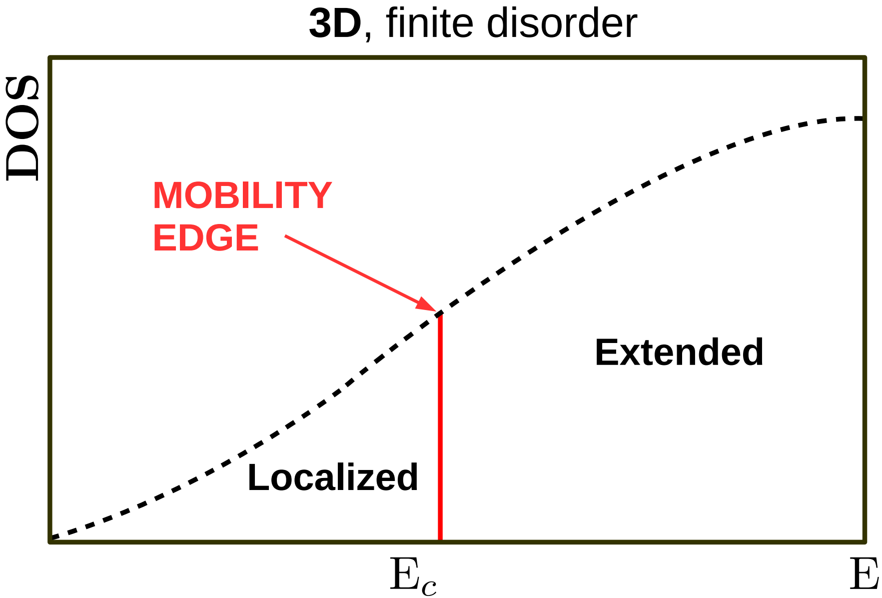

3D, finite disorder
DOS
MOBILITY
EDGE
Extended
Localized
E_c
E

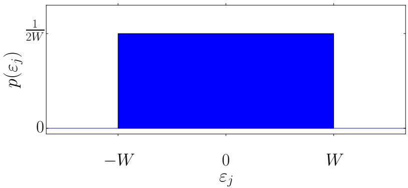
Our model - the **Anderson model**

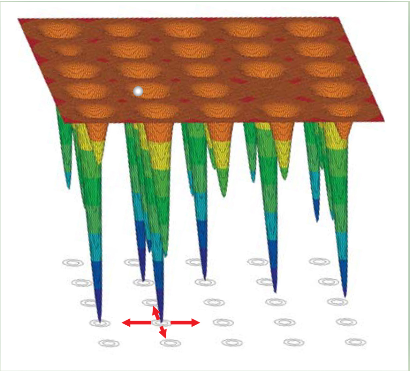
The Anderson Hamiltonian [1]

$$H = \sum_j \varepsilon_j c_j^\dagger c_j - V \sum_{\text{n.n.}} c_i^\dagger c_j + \text{h.c.}$$

Probability distribution of ε_j

$$p(\varepsilon_j) = \frac{1}{2W}\Theta(W - |\varepsilon_j|)$$

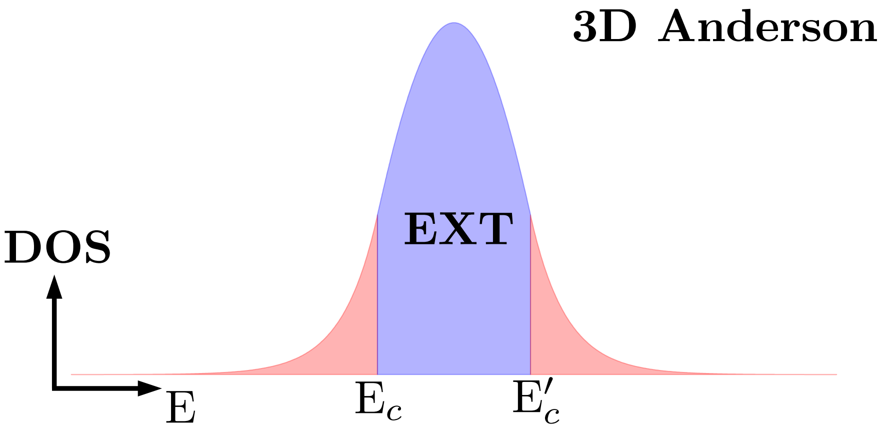




The model in 2D. Taken from [4].

The Anderson model

- Two mobility edges in 3D



My work - the numerical simulations

- How to extract features of the Anderson localization numerically?
  - implementation in 1D, 2D and 3D
  - calculations were run at the F-1 cluster at IJS
  - Full diagonalization and time evolution calculations
- Two localization criteria:
  - the **inverse participation ratio** (IPR)
  - the **absence of diffusion**

Localization criteria - the IPR

The definition

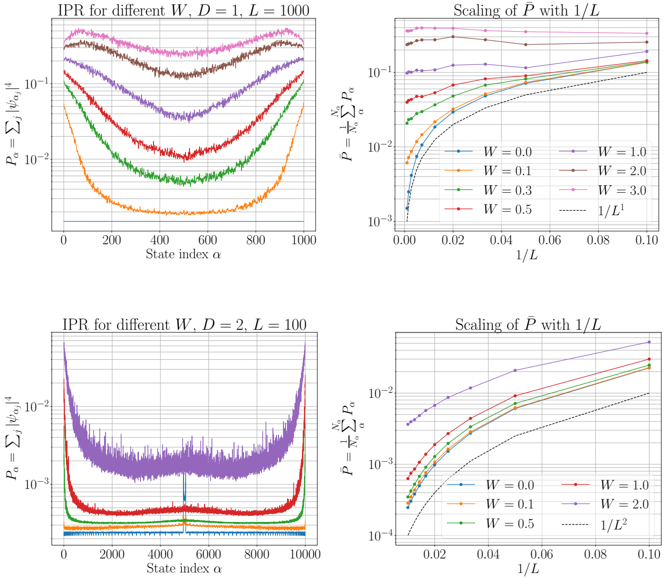
$$P^{-1} = \sum_{\mathbf{r}} |\psi(\mathbf{r})|^4, \qquad \|\psi\| = 1.$$

- sum over the lattice sites

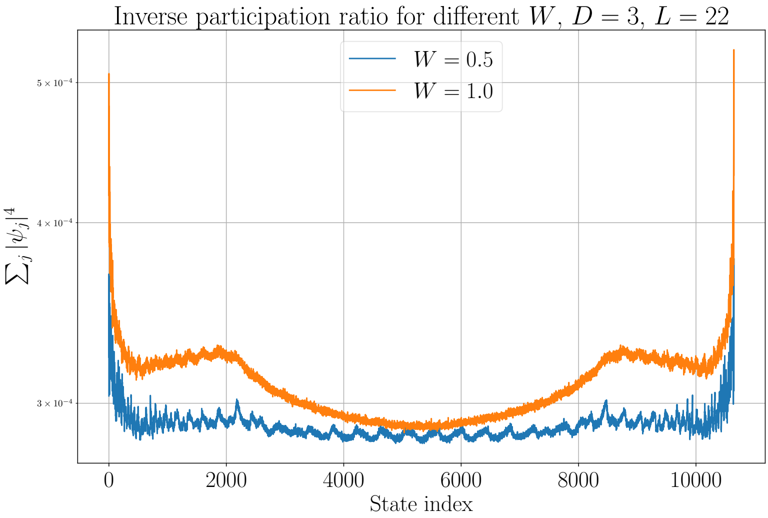
If L large enough

$$P^{-1} \propto \begin{cases} 1/L^d, & \text{extended} \\ \text{const}, & \text{localized} \end{cases}$$

IPR - the results



IPR 3D

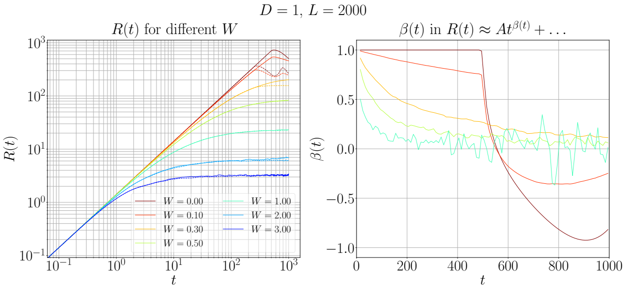

Inverse participation ratio for different W, $D = 3$, $L = 22$
$W = 0.5$
$W = 1.0$
5×10^{-4}
4×10^{-4}
3×10^{-4}
$\sum_j |\psi_j|^4$
0
2000
4000
6000
8000
10000
State index

Time evolution

$$|\psi, t + \mathrm{d}t\rangle = \exp\left(-\mathrm{i}\hat{H}\,\mathrm{d}t\right)|\psi, t\rangle,$$

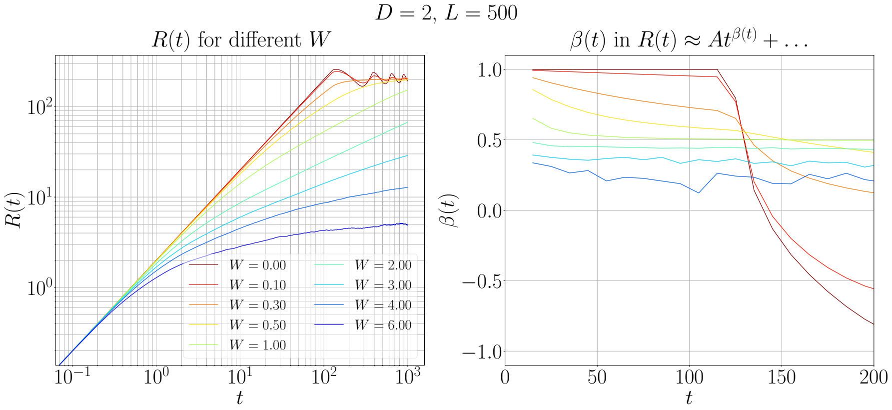
$$\hat{R}^2 = \sum_{\mathbf{r}_j} \mathbf{r}_j^2 \hat{n}_{\mathbf{r}_j},$$

$$\beta(t) = \frac{\mathrm{d}\log R}{\mathrm{d}\log t}$$



$$R(t) = \sqrt{\langle\psi, t|\hat{R}^2|\psi, t\rangle - \langle\psi, 0|\hat{R}^2|\psi, 0\rangle}$$

Absence of diffusion, 2D



Many-body localization

What happens when INTERACTIONS are included?

- Localization for low T, weak interactions
- Localization for any T, any interaction strength

Annals of Physics 321 (2006) 1126-1205

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Metal–insulator transition in a weakly interacting many-electron system with localized single-particle states

D.M. Basko [a,b,*], I.L. Aleiner [b], B.L. Altshuler [a,b,c]

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PHYSICAL REVIEW B **75**, 155111 (2007)

Localization of interacting fermions at high temperature

Vadim Oganesyan*
Department of Physics, Yale University, New Haven, Connecticut 06520, USA

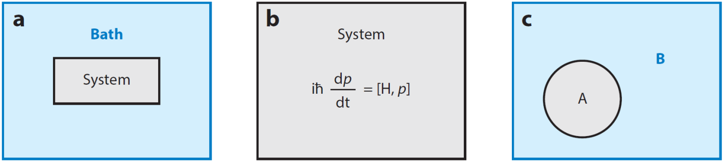
David A. Huse†
Department of Physics, Princeton University, Princeton, New Jersey 08544, USA
(Received 28 December 2006; published 23 April 2007)

MBL - keynotes

- Occurs in **closed** quantum many-body systems
- MBL systems fail to thermally equilibrate
- The **eigenstate thermalization hypothesis (ETH)** is not valid in such systems

Thermalization in closed quantum systems

- closed system $\rightarrow$ no coupling to an external **reservoir**



Nandkishore, Huse, 2015

- Thermalization in such systems **is possible** if subsystem **B** can act as a reservoir for the subsystem **A**

The eigenstate thermalization hypothesis

Thermalization in a system $\iff$ its **EIGENSTATES** $|m\rangle$ are **"THERMAL"**

Eigenstate expectation values equal the ensemble averages at a given temperature:

$$\langle m|\hat{O}|m\rangle = \langle \hat{O} \rangle_T$$

Not valid for **INTEGRABLE** and **MBL** systems

Hallmarks of MBL

- **ABSENCE** of **ERGODICITY**

Abanin, Altman, Bloch, Serbyn, 2018

- **ENTANGLEMENT ENTROPY:**
  - Eigenstates with area-law entanglement
  - Entanglement grows logarithmically in time

- **CHARACTERISTIC ENERGY LEVEL STATISTICS**
  - The subject of my current numerical investigations

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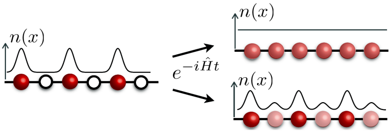

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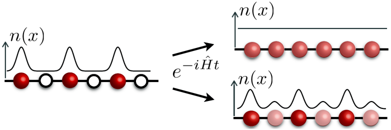

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PHYSICAL REVIEW B **77**, 064426 (2008)

Many-body localization in the Heisenberg *XXZ* magnet in a random field

Marko Žnidarič,[1] Tomaž Prosen,[1] and Peter Prelovšek[1,2]
[1]*Department of Physics, FMF, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia*
[2]*Jožef Stefan Institute, Jamova 39, SI-1000 Ljubljana, Slovenia*
(Received 31 August 2007; revised manuscript received 8 November 2007; published 25 February 2008)

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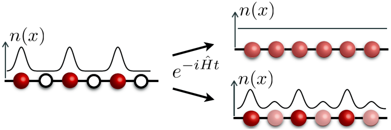

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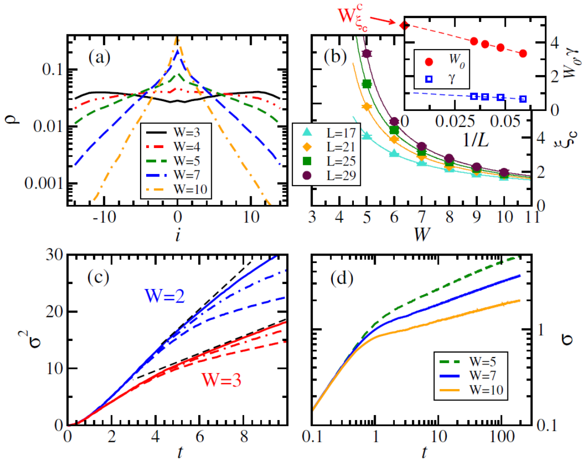
The tJ model

The model Hamiltonian

$$H = -t\sum_{i,\sigma}\left(\tilde{c}^{\dagger}_{i,\sigma}\tilde{c}_{i+1,\sigma} + c.c.\right) + J\sum_{i} S_i S_{i+1} + \sum_{i} w_i S^z_i + \sum_{i,\sigma} h_i n_{i,\sigma}$$

- The projected fermion operators: $\tilde{c}_{i,\sigma} = (1 - n_{i,-\sigma})c_{i,\sigma}$
- w_i, h_i: spin and hole disorder, box distributions with parameters W and H
- We consider the 1D PBC case with $S^z = 0$ for a single hole and for finite doping as well

Studies of the hole (sub)diffusion in the tJ model

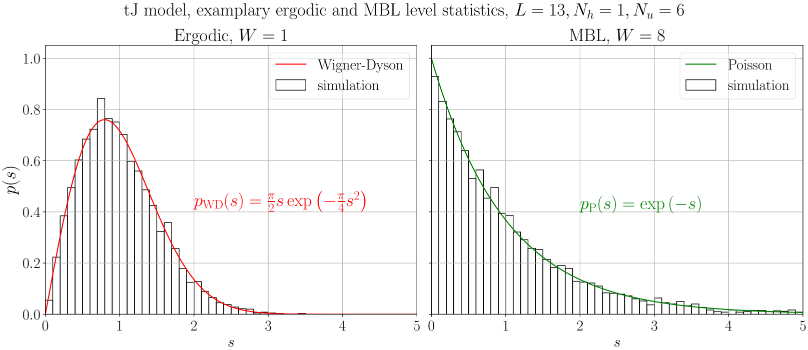


Lemut, Bonča, Mierzejewski, PRL **119** (2017)

Studies of the level statistics

- We study the **spectral statistics** of the adjacent energy levels of the tJ Hamiltonian
- We lean on the findings of the **random matrix theory (RMT):**
    - The **ergodic/quantum chaotic** case: spectral statistics of the Gaussian orthogonal ensemble (**GOE**)
    - **MBL** case: no level repulsion, the adjacent energy levels are distributed according to the **Poisson** distribution
- Further reading:
    - Oganesyan, Huse, PRB **75**, 15511 (2007)
    - Y.Y. Atas *et. al.*, PRL **110**, 084101 (2013)

Studies of the level statistics



L - system size, N_h - number of holes, N_u - number of up spins.

GOE: Wigner-Dyson statistics

MBL: Poisson statistics

Quantitative analysis

- Gaps between **adjacent** many body levels:

$$\delta_n = E_{n+1} - E_n \geq 0$$

- We define the **gap ratio**:

$$0 \leq r_n = \min\{\delta_n, \delta_{n-1}\} / \max\{\delta_n, \delta_{n-1}\} \leq 1$$

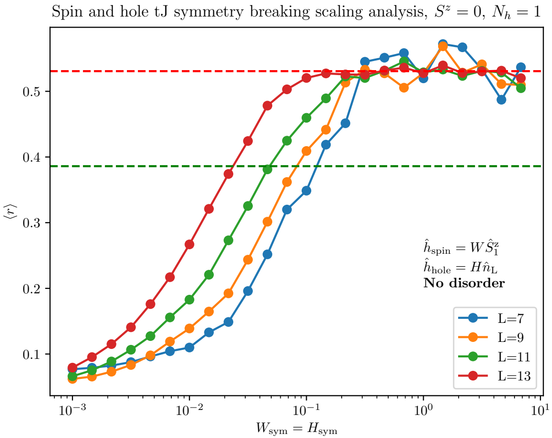
- **GOE** and **Poisson** average values $\langle r \rangle$ are well known:

$$\langle r \rangle_{\mathrm{GOE}} = 0.5307, \qquad \langle r \rangle_{\mathrm{P}} = 2 \ln 2 - 1 \approx 0.3863$$

Numerical results - single hole case

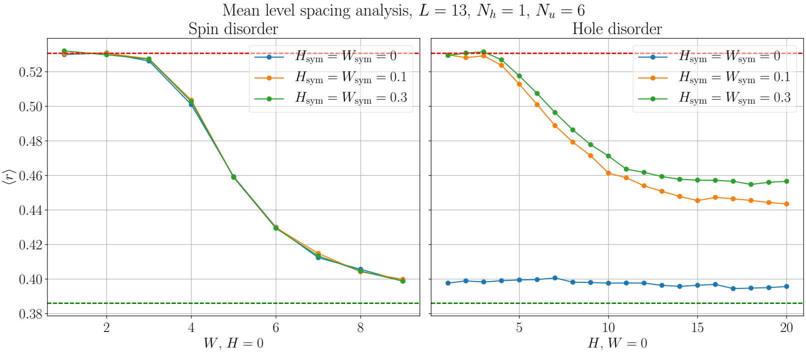
The effect of local symmetry-breaking terms

Finite size scaling analysis



Numerical results - single hole case

Varying spin (left) and hole (right) disorder:

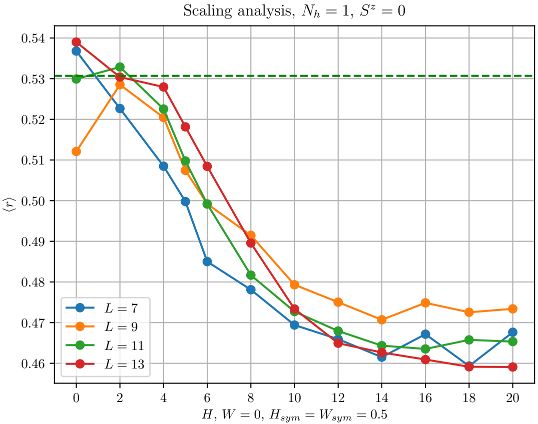


Averaging over different realizations of disorder(s) is performed to obtain final results.

Numerical results - single hole case

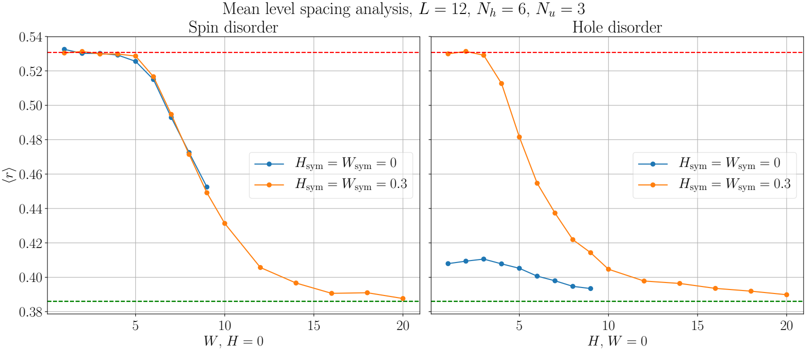
Finite size scaling analysis

Hole disorder, no spin disorder.



Numerical results - finite doping case

Varying spin (left) and hole (right) disorder:



Finite hole doping: emergence of MBL for any type of disorder?

References and sources of images

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