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Correlations between Periodic Orbits and their Rôle in Spectral Statistics

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Abstract

We consider off-diagonal contributions to double sums over periodic orbits that arise in semiclassical approximations for spectral statistics of classically chaotic quantum systems. We identify pairs of periodic orbits whose actions are strongly correlated. For a class of systems with uniformly hyperbolic dynamics, we demonstrate that these pairs of orbits give rise to a τ^2 contribution to the spectral form factor $K(\tau)$ which agrees with random matrix theory. Most interestingly, this contribution has its origin in a next-to-leading-order behaviour of a classical distribution function for long times.

1. Introduction

Quantum systems with disorder or with a chaotic classical counterpart share the remarkable property that energy levels, eigenfunctions, transition amplitudes, or transport quantities exhibit universal features. They are independent of the details of the individual system and depend only on its symmetries. In order to see this universality, it is necessary to consider statistical properties, such as fluctuations in the distributions of energy levels. It was originally conjectured [1] and is by now numerically well established that the spectral correlations of classically chaotic quantum systems, in the semiclassical limit $\hbar \to 0$, agree with correlations between eigenvalues of random matrices [2].

While such a connection with random matrix theory has been proven for disordered systems using field theoretical methods [3] (in the so-called ergodic regime), it remains an outstanding problem in the theory of clean (disorder-free) quantum systems with a chaotic classical limit. It has been proposed to extend the field theoretical approaches for disordered conductors in order to treat clean chaotic systems as well [4,5]. However, computing energy level statistics for a given single chaotic system requires to replace the ensemble average over impurity configurations, inherent to disordered devices, by an average over an appropriate range of energy. This causes difficulties which are still discussed controversially (for a recent collection of related review articles see Ref. [6]).

Semiclassical theory being based on the Gutzwiller trace formula [7] represents the other approach towards an understanding of spectral statistics. It provides the most direct link between spectral quantities of the quantum Hamiltonian and properties of the chaotic dynamics of the corresponding classical system. In view of the fact that semiclassical theory can approximate quantum energy levels with a precision at least of the order of the mean level spacing, semiclassics should be appropriate to cope with spectral correlations, at least on energy scales larger than the mean level distance.

A central quantity to characterize spectral statistics is the spectral two-point correlation function, $R(\eta)$, involving a product of two densities of states with energy separation η. A semiclassical approach to $R(\eta)$ is based on approximating the densities of states by the trace formula, which expresses them by sums over contributions from classical periodic trajectories. Hence a computation of $R(\eta)$ involves the evaluation of a double sum over classical trajectories. Along this line, semiclassical theory has been applied [8–10] to better understand the observed universality in quantum energy spectra. It was shown [9] that by including only pairs of orbits with themselves or their time-reversed partner, the so-called diagonal approximation, and by employing mean properties of classical trajectories [8], the energy level correlator agrees with random matrix theory in the limit of long-range correlations. These results were extended in Ref. [10] to describe the leading oscillatory behaviour of $R(\eta)$ by linking it to the diagonal approximation. To access the spectral regime beyond these asymptotic results, the subject of this article, requires the direct calculation of off-diagonal contributions from pairs of different classical paths, and necessitates further insight into classical correlations between trajectories. Although the existence of such correlations has been observed in several systems [11–13], a deeper understanding of the origin of these correlations in generic systems and a systematic semiclassical evaluation of the correlation function $R(\eta)$ or its Fourier transform, the spectral form factor $K(\tau)$, is still missing.

With this article we approach this open question and point out classical correlations between periodic orbits and their rôle for spectral statistics in the semiclassical limit. We present pairs of different, but closely related periodic orbits in two-dimensional systems, and we provide evidence that they are relevant for the first correction to the diagonal approximation for the spectral form factor. These orbit pairs involve trajectories which exhibit self-intersections with small intersection angles. They resemble ballistic analogues of corresponding objects in disordered systems [14,15], i.e. diffusons and cooperons that are connected at Hikami boxes [16]. We note, however, that we deal here with entirely classical paths, while the notion of the Hikami box involves (quantum) scattering, i.e. non-classical processes [15].

We first compute for a given pair of paths the difference in their classical actions. Employing statistics for the self-crossings of the trajectories we then semiclassically evaluate their contribution to the form factor. We find that this contribution vanishes when the leading long-time behaviour of the crossing statistics is applied. We then show that the next-order correction to the long-time asymptotics is important for spectral statistics. By numerical examinations on a Riemannian surface with constant negative curvature we find that this correction has indeed the form that is required for an agreement of spectral statistics with random matrix theory.

2. Off-diagonal contributions to the form factor

Gutzwiller's trace formula provides a link between the energy spectrum of a quantum system and the periodic orbits of its classical limit. It is a representation for the density of states in the form [7]

$$d(E) = \sum_n \delta(E - E_n) \approx \bar{d}(E) + \frac{1}{\pi\hbar} \mathrm{Re} \sum_\gamma A_\gamma \mathrm{e}^{\mathrm{i}S_\gamma(E)/\hbar} , \qquad (1)$$

where $\bar{d}(E)$ is the mean density of states and γ labels the periodic orbits of the classical system that is assumed to be chaotic. Each orbit contributes in terms of its classical action S_γ and its amplitude A_γ which depends on period, stability, and the number of self-conjugate points of the orbit.

One of the main reasons for the interest in the trace formula in recent years has been that it allows one to investigate theoretically the conjectured universality in the statistical distribution of energy levels. Consider, for example, the spectral form factor that is defined as the Fourier transform of the spectral two-point correlator,

$$K(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{\bar{d}(E)} \left\langle d_{\mathrm{osc}}(E + \eta/2) d_{\mathrm{osc}}(E - \eta/2) \right\rangle_E \mathrm{e}^{2\pi\mathrm{i}\eta\tau\bar{d}(E)} , \qquad (2)$$

where $d_{\mathrm{osc}}(E) = d(E) - \bar{d}(E)$. It is evaluated by averaging over an energy interval that is small in comparison to E but contains a large number of levels. For chaotic systems with time reversal symmetry, which we consider in the following, the form factor is expected to be identical with that of the Gaussian Orthogonal Ensemble (GOE) [2],

$$K^{\mathrm{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1 + 2\tau) & \text{if } \tau < 1, \\ 2 - \tau \log \dfrac{2\tau + 1}{2\tau - 1} & \text{if } \tau > 1, \end{cases} \qquad (3)$$

if τ is in the so-called universal regime $\tau > \tau_{\mathrm{erg}}$ (where $\tau_{\mathrm{erg}} = \mathcal{O}(\hbar^{d-1})$ in d dimensions). For small values of τ it has the expansion

$$K^{\mathrm{GOE}}(\tau) = 2\tau - 2\tau^2 + \dots . \qquad (4)$$

The semiclassical theory of spectral statistics has been developed in order to find an explanation for the observed agreement with random matrix statistics. Its aim is to attribute this universal property of the quantum system to generic properties of trajectories of the corresponding classical system. For the spectral form factor the semiclassical approximation is obtained by inserting the trace formula (1) into Eq. (2) and evaluating the Fourier transform in leading order of $\hbar$. This leads to a double sum over periodic orbits,

$$K(\tau) \approx \frac{1}{2\pi\hbar\bar{d}(E)} \sum_{\gamma,\gamma'} \left\langle A_\gamma A_{\gamma'}^* \, \mathrm{e}^{\mathrm{i}(S_\gamma - S_{\gamma'})/\hbar} \, \delta\!\left(T - \frac{T_\gamma + T_{\gamma'}}{2} \right) \right\rangle_E , \qquad (5)$$

where $T_\gamma = \partial S_\gamma / \partial E$ is the period of an orbit, and $\tau = T/(2\pi\hbar\bar{d}(E))$. Due to the exponential proliferation of the number of periodic orbits with their period, the double sum contains a huge number of pair terms. Most of the pairs consist of periodic orbits with actions that are uncorrelated, and their contributions cancel each other when summed over. It is expected that the non-vanishing contributions come from a relatively small number of pairs of orbits which are correlated.

The strongest correlation occurs between orbits which have identical actions. In the diagonal approximation only those pairs of orbits are considered which are identical or related by time inversion. Their contribution to the form factor can be evaluated by applying a classical sum rule for periodic orbits [8]. In this way one obtains the leading term of the GOE form factor for small values of τ: $K(\tau) \approx 2\tau$ [9].

To go beyond the diagonal approximation requires the evaluation of pairs of different orbits which are not related by any symmetry. In this article we wish to provide an explanation where these off-diagonal contributions come from. In particular, we will discuss in detail the next term in the expansion (4) of the form factor for small τ, namely the term $-2\tau^2$. We will provide evidence that it can be obtained in two-dimensional systems from pairs of self-intersecting orbits with small opening angles and orbits in their close vicinity.

We start with some general considerations. For a chaotic system it is reasonable to expect that correlations exist only between periodic orbits which are close in coordinate space. One therefore needs a mechanism by which two or more periodic orbits can be obtained which are different but which are located almost everywhere in close vicinity to each other in coordinate space. Hints on their topology can be obtained from diagrams in perturbation theory for disordered systems [15,17], or from classical correlations between periodic and diffractive orbits [18]. The basic idea is that the periodic orbits consist of different segments. In each segment, an orbit follows very closely its neighbouring orbit or the time-reverse of this orbit, but the orbits differ in how these segments are connected. In order that the segments can be connected in different ways they must form loops. Thereby, one obtains a semiclassical loop expansion in close analogy to the loop expansion in diagrammatic perturbation theory.

Let us consider the simplest example. It consists of a pair of two periodic orbits with two loops in coordinate space as depicted in Fig. 1. The two orbits follow one loop in the same direction and the other loop in the opposite direction. For that reason these pairs can exist only in systems with time-reversal invariance.

In the following we argue that such pairs of classical periodic orbits indeed exist. Let us assume that the opening angle ε (we also call it crossing angle) is very small. As we will see later, it is sufficient to consider only this case. For small ε one can describe the outer orbit by linearizing the motion in the vicinity of the inner self-intersecting orbit. This leads to the following conditions for the outer periodic orbit:

$$\begin{aligned} \begin{pmatrix} \delta_2 \\ p(\gamma_2 + \varepsilon/2) \end{pmatrix} &= R \begin{pmatrix} \delta_1 \\ p(\gamma_1 - \varepsilon/2) \end{pmatrix} , \\ \begin{pmatrix} -\delta_2 \\ p(\gamma_2 - \varepsilon/2 \end{pmatrix} &= L \begin{pmatrix} -\delta_1 \\ p(\gamma_1 + \varepsilon/2) \end{pmatrix} . \end{aligned} \qquad (6)$$

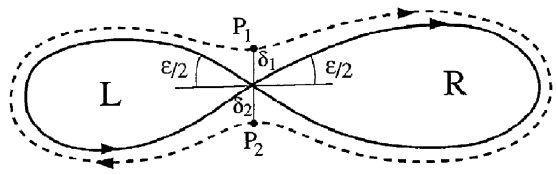

Fig. 1. An example of a self-intersecting classical periodic orbit with small opening angle ε, and its neighbouring periodic orbit. The local coordinate system is oriented along the middle of the opening angle ε.

L and R denote the two stability matrices of the left and right loop of the inner orbit, respectively, and p is the absolute value of the momentum at the crossing. The distances δ_1 and δ_2 are shown in the figure, and γ_1 and γ_2 denote the angles between the horizontal line and the tangents to the outer orbit in the points P_1 and P_2, respectively. The expression (6) consists of four inhomogeneous linear equations for the four unknown quantities δ_1, δ_2, γ_1, and γ_2 and can be solved. Therefore, the outer periodic orbit exists in the linearized approximation. A closer examination shows that it behaves in the following way. In point P_1 the outer orbit is exponentially close to the stable direction of the right inner loop. In the following, it approaches the inner loop exponentially fast for some time (for about half the loop) until it starts to deviate from it again. In point P_2 it then almost reaches the stable direction of the time-reverse of the left inner orbit, and again approaches it exponentially fast until it starts to deviate from the left loop again at about half the loop.

The action difference between both orbits can be obtained in the linearized approximation by expanding the action up to second order around the inner orbit. One finds

$$\Delta S(\varepsilon) \approx \frac{p\varepsilon}{2}(\delta_1 + \delta_2) \, . \tag{7}$$

The common solution of the equations in (6) leads to a linear relation between the angle ε and the distance $\delta_1 + \delta_2$,

$$\delta_1 + \delta_2 = \frac{R_{12}(\mathrm{Tr}L + 2) + L_{12}(\mathrm{Tr}R + 2)}{\mathrm{Tr}(L^i R) - 2} \, p\varepsilon \, , \tag{8}$$

so that the action difference in Eq. (7) depends quadratically on ε. In Eq. (8), L^i denotes the stability matrix for the time-inverse of the inner left loop. In terms of the matrix elements of L it is given by

$$L^i = \begin{pmatrix} L_{22} & L_{12} \\ L_{21} & L_{11} \end{pmatrix} . \tag{9}$$

The action difference $\Delta S(\varepsilon)$ predominantly originates from the region around the self-intersection.

To summarize, there is the following recipe for finding pairs of orbits as shown in Fig. 1. One has to look for periodic orbits which have self-intersections with a small crossing angle ε. The self-intersection divides an orbit into two loops. The prediction is that there exists a neighbouring periodic orbit which follows one loop in the same and the other loop in the opposite direction, and that it has a small action difference given by Eqs. (7) and (8). We tested this prediction for classical chaotic motion in the hyperbola billiard, for which we have a long list of periodic orbits available [19]. We looked for orbits with small crossing angles and checked whether the neighbouring orbits exist and have the predicted action difference. We found that this is indeed the case. For example, for an orbit pair involving a long orbit with a crossing angle of 2.6° we found the two lengths (corresponding to scaled actions) $l_1 = 24.08676$ and $l_2 = 24.08469$. The difference is $\Delta l = 0.00207$, compared to the theoretical value of $\Delta l_{\mathrm{th}} = 0.00208$ obtained from Eqs. (7) and (8).

In order to proceed we have to evaluate the number of self-intersections of periodic orbits and the distribution of the crossing angles. We start by calculating these quantities for general, non-periodic trajectories. The corresponding results for periodic orbits can be inferred by using the principle of uniformity [20,21]. According to it, averages of a quantity along generic non-periodic orbits lead to the same result as averages over all periodic orbits, if the latter are performed with relative weights which take into account the different stabilities of the periodic orbits. The derivation for the crossing angles is performed in the appendix by using the ergodic property of chaotic systems. As a result we find for a trajectory with time T that the average number of self-intersections with an opening angle in an interval $\mathrm{d}\varepsilon$ around ε ($0 \leq \varepsilon \leq \pi$) is given by

$$P(\varepsilon, T)\,\mathrm{d}\varepsilon \sim \frac{T^2\langle v^2\rangle}{\pi A}\frac{\sin\varepsilon}{2}\,\mathrm{d}\varepsilon \qquad \text{for} \qquad T \to \infty \, . \tag{10}$$

Here, $P(\varepsilon, T)$ is the density of crossings of opening angle ε for trajectories of time T, A is the accessible area at energy E, and $\langle v^2\rangle$ is the average of the velocity square over A (see the appendix for an accurate definition.) By an integration over ε it follows from Eq. (10) that the total number of self-intersections of a trajectory of time T increases as

$$N(T) \sim \frac{\langle v^2\rangle T^2}{\pi A} \qquad \text{for} \qquad T \to \infty \, . \tag{11}$$

We want to use this classical information to evaluate the contribution of the pairs of double-loop orbits to the spectral form factor. For general chaotic systems this requires further assumptions, in particular that the crossing-angle distribution is independent of elements of the stability matrices. Then one can show that the contribution to the form factor vanishes if the leading behaviour for large T, Eq. (10), is used. We will not perform this calculation for general systems. Instead, we will focus from now on onto a particular class of systems with uniformly hyperbolic dynamics for which the calculations are simpler and no further assumption is needed. This is the motion on Riemann surfaces of constant negative curvature [22]. There the periodic orbits do not have conjugate points, and they all possess the same Lyapunov exponent. These systems have the additional advantage that averages along periodic orbits need not be weighted in order to be identical to averages along generic trajectories. In these systems the stability matrix of an orbit of time T has the simple form

$$M = \begin{pmatrix} \cosh\lambda T & (m\lambda)^{-1}\sinh\lambda T \\ m\lambda\sinh\lambda T & \cosh\lambda T \end{pmatrix} , \tag{12}$$

where m is the mass of the particle and λ the Lyapunov exponent of the system. With Eq. (12) the action difference, Eq. (7) with (8), simplifies to

$$\Delta S(\varepsilon) \approx \frac{p^2 \varepsilon^2}{2m\lambda} \ . \tag{13}$$

By using Eqs. (10) and (13), we can evaluate the contribution of the pairs of orbits to the spectral form factor. We do this by summing over all intersections of angle ε that occur in periodic orbits, and then integrate over ε. We account for an additional degeneracy factor of two in Eq. (5) owing to time reversal invariance. Furthermore, we take twice the real part, since there is a corresponding complex conjugate term in the double sum over periodic orbits. Altogether we obtain the following expression:

$$\begin{aligned} K_{\rm off}^{(2)}(\tau) &\approx \frac{4}{2\pi\hbar\bar{d}(E)} \mathrm{Re} \int_0^\infty \mathrm{d}\varepsilon \sum_\gamma |A_\gamma|^2 P(\varepsilon, T) \\ &\quad \times \exp(\mathrm{i}\Delta S(\varepsilon)/\hbar)\delta(T - T_\gamma) \\ &\approx \frac{4}{2\pi\hbar\bar{d}(E)} \mathrm{Re} \int_0^\infty \mathrm{d}\varepsilon \int_0^\infty \mathrm{d}T' \rho(T') \frac{T'^2}{\exp(\lambda T')} P(\varepsilon, T) \\ &\quad \times \exp(\mathrm{i}\Delta S(\varepsilon)/\hbar)\delta(T - T') \\ &\approx \frac{4}{2\pi\hbar\bar{d}(E)} \mathrm{Re} \int_0^\infty \mathrm{d}\varepsilon \frac{v^2 T^3}{\pi A} \frac{\varepsilon}{2} \exp\left(\frac{\mathrm{i}p^2\varepsilon^2}{2\hbar m\lambda}\right) \\ &= 0 \end{aligned} \tag{14}$$

It has been evaluated by replacing the sum over periodic orbits by an integral with density

$$\rho(T) \sim \frac{\exp(\lambda T)}{T} \quad \text{for} \quad T \to \infty \ , \tag{15}$$

and by using $|A_\gamma|^2 \approx T_\gamma^2 \exp(-\lambda T_\gamma)$. In the limit $\hbar \to 0$, the main contribution to the integral comes from angles ε close to zero. For obtaining the leading semiclassical contribution we could therefore take only the first term in the Taylor expansion of $\sin\varepsilon$. Furthermore, we extended the integral to infinity. (For convergence questions it should be considered with a momentum p that has a small positive imaginary part that is sent to zero after the integral is performed.) The result vanishes since the result of the integration is purely imaginary.

After a closer inspection of expression (14) it is, however, not surprising that the result gives zero. In order to perform the semiclassical limit, one has to translate the time T into τ by the relation $T = 2\pi\hbar\bar{d}(E)\tau$, and then take the limit $\hbar \to 0$. Since in two-dimensional systems $\bar{d}(E) \sim mA/(2\pi\hbar^2)$, one finds that the expression in (14) is of order $\mathcal{O}(\hbar^{-1})$. Without taking the real part the expression would diverge in the limit $\hbar \to 0$. Moreover, it would be of order τ^3 whereas we believe that it should be the lowest-order off-diagonal contribution and thus be of order τ^2.

These considerations indicate that the correct contribution to the form factor might arise from a $1/T$ correction to the asymptotic law (10) for the classical density $P(\varepsilon, T)$. In the following we first show that a multiplicative correction term of the form

$$1 - \frac{4\Delta T}{T} \qquad \text{with} \qquad \Delta T = -\frac{1}{\lambda}\log(c\,\varepsilon) \ , \tag{16}$$

where c is an arbitrary constant, leads to the random matrix result. We then confirm numerically that such a correction indeed exists.

Multiplying the integrand in Eq. (14) by $(-4\Delta T/T)$ leads to

$$\begin{aligned} K_{\rm off}^{(2)}(\tau) &\approx \frac{4}{2\pi\hbar\bar{d}(E)} \mathrm{Re} \int_0^\infty \mathrm{d}\varepsilon \frac{v^2 T^3}{\pi A} \frac{\varepsilon}{2} \exp\left(\frac{\mathrm{i}p^2\varepsilon^2}{2\hbar m\lambda}\right) \frac{4\log(c\varepsilon)}{\lambda T} \\ &= \frac{4}{2\pi\hbar\bar{d}(E)} \mathrm{Re} \int_0^\infty \mathrm{d}\eta \frac{2i\hbar T^2}{\pi m A} \exp(-\eta) \log\sqrt{\mathrm{i}} \\ &= -2\tau^2, \end{aligned} \tag{17}$$

where, after the change of the integration variable, all further arguments of the logarithm could be neglected since they lead to a vanishing contribution. The final result agrees with the random matrix expression in Eq. (4).

In the following we examine numerically whether a correction to the law (10) of the form (16) exists. For that purpose we consider a Riemannian surface in form of an octagon. We choose random trajectories, follow them for a fixed length L and determine the mean density of the intersection angles. Since we are interested in the correction to the asymptotic form of this density we need to have good statistics. The numerics is carried out with 50 million randomly chosen trajectories of length $L = 100$, and half a million trajectories of length $L = 1000$ (the area of the system is $A = 4\pi$). We use dimensionless units in which $v = \lambda = 1$ and $L = T$.

First we test the asymptotic law (10) by calculating

$$p(\varepsilon, T) := P(\varepsilon, T) \frac{\pi A}{T^2 \langle v^2 \rangle} \ . \tag{18}$$

For long times this function should agree with a normalized distribution of crossing angles of the form $\sin(\varepsilon)/2$, and we compare it to this curve in Figs. 2(a) and (c). In the first of these two figures the results for trajectories of length $L = 100$ are presented. Here one can still see a small difference between the two curves. When going to the results for longer trajectories of length $L = 1000$ in Fig. 2(c), however, this difference cannot be discerned any more.

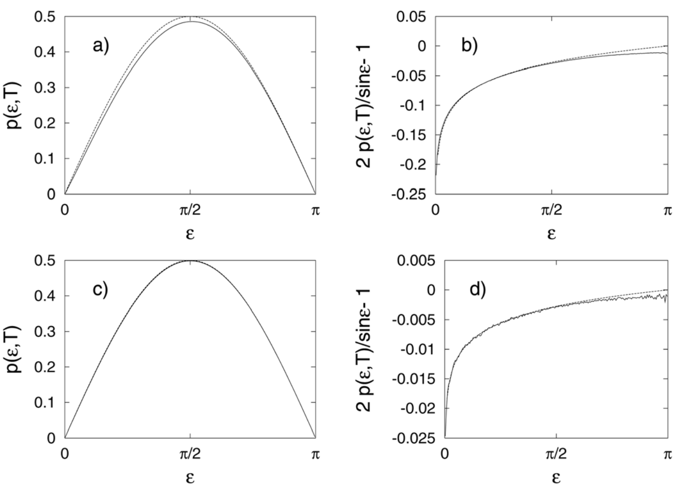

Fig. 2. (a) and (c): The distribution of crossing angles $p(\varepsilon, T)$ (full line) in comparison to $\sin(\varepsilon)/2$ (dashed line), evaluated along trajectories of length $L = 100$ and $L = 1000$, respectively. (b) and (d): The deviation of $p(\varepsilon, T)$ from $\sin(\varepsilon)/2$ (full line) in comparison with the log-distribution (dashed line) that is described in the text, evaluated along trajectories of length $L = 100$ and $L = 1000$, respectively.

In Figs. 2(b) and (d) we investigate the deviation from the asymptotic law (10). We do this by plotting the distribution $2p(\varepsilon, T)/\sin\varepsilon - 1$. As argued above we expect that this deviation is responsible for the τ^2 term of the spectral form factor. In order to obtain an agreement with random matrix theory, the plotted function must have the form $(4\log\varepsilon + \text{const})/(\lambda T)$ for small ε as follows from Eq. (16). This expression contains only one free additive parameter, which is of no relevance for the form factor. We fitted this parameter and plotted this curve also (dashed lines in Figs. 2(b) and (d)).

The agreement between the different curves in Fig. 2(b) and also in Fig. 2(d) is remarkable. The random matrix prediction requires only an agreement for small values of ε, but there is an excellent agreement almost over the whole range $0 \leq \varepsilon \leq \pi$. In our opinion this result is the first clear indication on the origin of the off-diagonal contributions to the form factor in the perturbative regime near $\tau = 0$.

3. Discussion and Conclusions

This clear-cut numerical result suggests to draw the following conclusions:

(*i*) It is possible to systematically evaluate off-diagonal contributions to the spectral form factor by the semiclassical method.
(*ii*) The τ^2 term of the spectral form factor is indeed related to the eight-shaped orbits in Fig. 1
(*iii*) The τ^2 term originates from the next-to-leading asymptotic form of the distribution of crossing angles for large T.

For general systems the calculations will be more complicated. One reason is that Maslov indices are present. A second reason is that the stabilities of the orbits are, in general, different. As a consequence, it is not the pure distribution of crossing angles which matters, but a distribution which depends also on Maslov indices and elements of the stability matrices along the loops. A complete analytical derivation of the τ^2 contribution, purely on the basis of classical chaotic dynamics, remains to be performed.

The fact that the contribution (14) vanishes in the semiclassical limit and that only the term (17) arising from corrections to the long-time distribution of crossing angles prevails, shows at least a formal analogy with the situation when evaluating two-loop corrections to the density correlator for a diffusive system using diagrammatic perturbation theory. There, the contribution from dressed square Hikami boxes vanishes to leading order and only the next-order expansion in energy leads to the final result [15].

The problem to compute off-diagonal contributions to the spectral form factor is closely related to corresponding questions which involve energy averages over products of advanced and retarded Green functions. One prominent example is mesoscopic quantum transport. For clean chaotic systems a semiclassical theory, which adequately and quantitatively describes weak localization, is still lacking [23–25]. The question to obtain the τ^2 term in the spectral form factor has much in common with this long-lasting problem to semiclassically compute weak-localization corrections in ballistic quantum transport. There, it was already suggested to consider certain pairs of initially close classical orbits [24,26]. Proceeding in the same way as for the pairs of correlated periodic two-loop orbits we have computed the action difference for orbit pairs relevant to quantum transport. It also scales quadratically with the self-intersection angle ε, but with a different prefactor. Again deviations from the asymptotic form of crossing distributions must be included to get non-zero results [27].

Another field of application of our findings are mesoscopic Andreev billiards, i.e. ballistic cavities coupled to superconducting leads. Semiclassical approaches to the proximity effect in these systems so far rely on the diagonal approximation [28–30] and an extension to off-diagonal paths along the lines presented here appears promising.

To conclude, we have shown that in chaotic systems a class of off-diagonal pairs of periodic orbits exists which evidently exhibit action correlations. We have demonstrated for systems with uniformly hyperbolic dynamics that in the perturbative regime (corresponding to the small τ expansion) these orbit pairs yield a τ^2 contribution to the spectral form factor which agrees with the random matrix result. Our results for the two-loop orbits demonstrate that the semiclassical theory is a powerful tool to deal with spectral statistics of individual disorder-free quantum systems. We believe that, for systems with time-reversal symmetry, the higher-order contributions to the form factor involve periodic orbits with three and more loops. For systems without time-reversal symmetry the considered two-loop orbits do not exist, and contributions from orbits with more loops should cancel mutually. A systematic computation of higher-order contributions from multi-loop periodic-orbit configurations remains as a challenging future program.

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4. Appendix: The density of angles of self-intersection

In this appendix we derive the leading asymptotic form of the density $P(\varepsilon, T)$ for long times T. The quantity $P(\varepsilon, T)\,\mathrm{d}\varepsilon$ is defined as the average number of self-intersections with an opening angle in an interval $\mathrm{d}\varepsilon$ around ε of trajectories of time T in a chaotic system. For the derivation we employ the ergodicity theorem which can be formulated in the form

$$\int_0^T \mathrm{d}t\, f(\boldsymbol{q}(t), \boldsymbol{p}(t)) \sim T\, \frac{\int \mathrm{d}^2q\, \mathrm{d}^2p\; \delta(E - H(\boldsymbol{q}, \boldsymbol{p}))\, f(\boldsymbol{q}, \boldsymbol{p})}{\int \mathrm{d}^2q\, \mathrm{d}^2p\; \delta(E - H(\boldsymbol{q}, \boldsymbol{p}))}\,, \quad T \to \infty\,. \tag{19}$$

It states that for almost all initial conditions the integration of a sufficiently smooth function f along a trajectory of time T is, asymptotically for large T, given by T times the phase space average of this quantity. We choose in this appendix a simple, heuristic derivation by employing the ergodicity theorem for a function f that has the form of a delta-function. A rigorous derivation would require the use of smoothed quantities.

The density $P(\varepsilon, T)$ is defined as

$$P(\varepsilon, T) = \left\langle \frac{1}{2}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \; |J| \, \delta(\boldsymbol{q}(t) - \boldsymbol{q}(t')) \, \delta(\varepsilon - |\measuredangle[\boldsymbol{v}(t), \boldsymbol{v}(t')]|) \right\rangle \tag{20}$$

where the average is taken over different initial conditions and $\measuredangle[\boldsymbol{v}(t), \boldsymbol{v}(t')]$ denotes the angle between $\boldsymbol{v}(t)$ and $\boldsymbol{v}(t')$. The quantity J is the Jacobian for the transition from the argument of the first delta-function to t and t'

$$\begin{aligned} |J| &= |\dot{x}(t)\dot{y}(t') - \dot{x}(t')\dot{y}(t)| = |\boldsymbol{v}(t) \times \boldsymbol{v}(t')| \\ &= v(t)\, v(t') \sin |\measuredangle[\boldsymbol{v}(t), \boldsymbol{v}(t')]| \, . \end{aligned} \tag{21}$$

We apply the ergodicity theorem twice to replace the time integrals by phase space averages. In the following we are more general than in the remaining article by allowing Hamiltonians of the form $H = (1/2m)(\boldsymbol{p} - (e/c)\boldsymbol{A})^2 + V(\boldsymbol{q}) = (m/2)v^2 + V(\boldsymbol{q})$ with the possibility of a vector potential that breaks time reversal symmetry. Inserting Eq. (19) into Eq. (20) results in

$$\begin{aligned} P(\varepsilon, T) &\sim \frac{T^2}{2} \times \\ &\frac{\int \mathrm{d}^2q\mathrm{d}^2p\mathrm{d}^2q'\mathrm{d}^2p'\delta(E - H(\boldsymbol{q},\boldsymbol{p}))\delta(E - H(\boldsymbol{q}',\boldsymbol{p}'))|\boldsymbol{v} \times \boldsymbol{v}'|\delta(\boldsymbol{q} - \boldsymbol{q}')\delta(\varepsilon - |\langle[\boldsymbol{v},\boldsymbol{v}']|)}{\int \mathrm{d}^2q\mathrm{d}^2p\mathrm{d}^2q'\mathrm{d}^2p'\delta(E - H(\boldsymbol{q},\boldsymbol{p}))\delta(E - H(\boldsymbol{q}',\boldsymbol{p}'))} \\ &= \frac{T^2}{2}\frac{\int \mathrm{d}^2qv\mathrm{d}v d\phi \mathrm{d}^2q'v'\mathrm{d}v'\delta(E - H(\boldsymbol{q},\boldsymbol{p}))\delta(E - H(\boldsymbol{q}',\boldsymbol{p}'))2vv'\sin\varepsilon\delta(\boldsymbol{q} - \boldsymbol{q}')}{\int \mathrm{d}^2qv\mathrm{d}v d\phi \mathrm{d}^2q'v'\mathrm{d}v'\mathrm{d}\phi'\delta(E - H(\boldsymbol{q},\boldsymbol{p}))\delta(E - H(\boldsymbol{q}',\boldsymbol{p}'))} \\ &= \frac{T^2}{2\pi A^2}\sin\varepsilon \int \mathrm{d}^2q\frac{2}{m}(E - V(\boldsymbol{q})). \end{aligned} \tag{22}$$

Here the integration variables have been changed from Cartesian coordinates for the momenta to polar coordinates for the velocities. A is the accessible area at energy E and we define $\langle v^2 \rangle$ as average of the velocity square over A:

$$A = \int_{V(\boldsymbol{q}) \le E} \mathrm{d}^2q \, , \qquad \langle v^2 \rangle = \frac{1}{A}\int_{V(\boldsymbol{q}) \le E} \mathrm{d}^2q \, \frac{2}{m}(E - V(\boldsymbol{q})) \, . \tag{23}$$

With this definition we arrive at the final result

$$P(\varepsilon, T) \sim \frac{T^2 \langle v^2 \rangle}{\pi A}\frac{\sin\varepsilon}{2} \, , \qquad T \to \infty \, , \tag{24}$$

where ε varies between 0 and π. The T^2-dependence of the total number of crossings (without explicit prefactor) has been proven in Ref. [32].

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