Optimization and Online Learning

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Overview

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Convexity

A set $\mathcal{X} \subseteq \mathbb{R}^d$ is convex if for any $x, y \in \mathcal{X}$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)y \in \mathcal{X}$.
For functions:,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \text{ for } x, y \in \mathbb{R}^d, \ \lambda \in [0,1].$$

Subgradient: $g \in \partial f(x)$ satisfies $f(y) \geq f(x) + \langle g, y - x \rangle \ \forall y$.

Convex Functions

$f : \mathbb{R}^d \to \mathbb{R}$ is convex iff $\mathrm{epi}(f)$ is convex. It is *closed* (lsc) iff $\mathrm{epi}(f)$ is closed.

- Supremum of convex closed $\to$ convex closed.
- Sum of convex functions is convex.
- Differentiable f is convex $\iff f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$.
- For twice-differentiable f, if $\nabla^2 f \succeq 0$ then f is convex.

Examples: $\|x\|$, $(x, y) \mapsto \|x - y\|$, $\|x\|^2$, $\exp(x)$, $-\log x$, $x \log x$, $x^\top A x$ for PSD A.

First-Order Optimality & Subgradients

- If f convex differentiable on convex $\mathcal{X}$ with a minimizer at x^*, then

$$x^* \in \arg\min_{x \in \mathcal{X}} f(x) \iff \langle \nabla f(x^*), x^* - x \rangle \leq 0 \quad \text{for all } x \in \mathcal{X}.$$

  Thus, for unconstrained problems: $\nabla f(x^*) = 0$.

- If subdifferentiable:

$$\langle g, x^* - x \rangle \leq 0 \quad \text{for all } x \in \mathcal{X} \text{ and all } g \in \partial f(x^*)$$

  and $0 \in \partial f(x^*)$, even with constraints.

Convexity in Machine Learning

Many problems:

- Linear regression $\min_x \{\|Ax - b\|_2^2\}$, compressed sensing $\min_x \{\|Ax - b\|_2^2 \mid \|x\|_1 \leq \rho\}$.
- Logistic Regression.
- SVMs.
- MLE for many problems.
- Convex Relaxations: $\min_X \|X\|_*$, subject to $X_{ij} = M_{ij}$, for $ij \in \text{SomeSet}$.
- Discrete Optimal Transport.
- Distributional Robust Optimization.
- ...

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Also useful for:

- Understanding convex problems is the first step to understand the non-convex world.
- Optimal algorithms for non-convex problems consist sometimes of the convex algorithms.
- Convex algorithms are helpful in practice for deep learning.

Other Regularity Properties

Lipschitz Continuity. $f : \mathcal{X} \to \mathbb{R}$ is G-Lipschitz w.r.t. $\|\cdot\|$ if

$$|f(x) - f(y)| \leq G\|x - y\|, \quad \forall x, y \in \mathcal{X}.$$

Implied for subdifferentiable f, if all subgradients g satisfy $\|g\|_* \leq G$.

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Smoothness. A differentiable function f is L-smooth w.r.t. $\|\cdot\|$ in $\mathcal{X}$ if

$$f(y) \;\le\; f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2, \quad \forall x, y \in \mathcal{X}.$$

Implied by ∇f being L-Lipschitz: $\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|$. Equivalent when f is convex.
It is implied by $v^T \nabla^2 f(z) v \le L\|v\|^2$ for all z. So for $\|\cdot\|_2$, it is implied if $\nabla^2 f(x) \preccurlyeq L$.

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Strong Convexity. f is μ-strongly convex w.r.t. $\|\cdot\|$ if

$$f(y) \;\ge\; f(x) + \langle \nabla f(x), y - x\rangle + \frac{\mu}{2}\|y - x\|^2, \quad \forall x, y \in \mathcal{X}.$$

Implies a unique minimizer and $\frac{\mu}{2}\|x - x^*\|^2 \le f(x) - f(x^*)$. Implied by $v^T \nabla^2 f(z) v \ge L\|v\|^2, \;\; \forall z$.

Towards Online Convex Optimization: Statistical Learning Theory

- Collect i.i.d. $Z_1, \dots, Z_n \sim \mathcal{P}$ in a set $\mathcal{Z}$.
- Choose action/prediction $a = a(Z_1, \dots, Z_n) \in \mathcal{A}$.
- Loss $\ell(a, Z) \geq 0$, define the *excess risk*

$$r_n = \mathbb{E}\left[\ell\big(a(Z_{1:n}), Z\big)\right] - \inf_{a \in \mathcal{A}} \mathbb{E}\left[\ell(a, Z)\right].$$

- Study upper/lower bounds on the risk r_n, and algorithms to attain them.

Examples

- Linear regression: $\ell((w, b), (x, y)) = (w^\top x + b - y)^2$ w/ random (x, y).
- Binary classification for data in $R^n \times \{-1, 1\}$. Zero-one loss, logistic, hinge...

Towards Online Convex Optimization

- Data may arrive in a stream.
- We may need to take actions sequentially.
- Measure cumulative excess risk:

$$\sum_{t=1}^{T} \mathbb{E}\left[\ell(a_t, Z)\right] - T \inf_{a \in \mathcal{A}} \mathbb{E}\left[\ell(a, Z)\right], \text{ where } a_t = a(Z_{1:t-1}).$$

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- However, what if the distribution shifts with time? What if data is adversarial?
- We still can say obtain non-trivial bounds in that case!

Adversarial Online Learning Game: Regret minimization

1. Adversary picks loss $\ell_t : \mathcal{X} \to \mathbb{R}$
2. Learner picks $x_t \in \mathcal{X}$.
3. Learner suffers $\ell_t(x_t)$ and observes full, i.e. $\ell_t(\cdot)$, or partial feedback.

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Static Regret:

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Generalizes statistical bound by choosing $\ell_t(x) = \mathbb{E}\left[\ell(x, Z)\right]$.

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Dynamic Regret:

$$\text{Regret}_T(u_1, \ldots, u_T) = \sum_{t=1}^{T} \ell_t(x_t) - \sum_{t=1}^{T} \ell_t(u_t).$$

Typical bounds for dynamic regret depend on how much comparators move, like $\sum_{t=1}^{T-1} \|u_i - u_{i+1}\|$.

Peeking into the future

One-step-ahead regret. If you can see one step into the future, you may not regret anything.

Let $x_t^* = \arg\min_{x \in \mathcal{X}} \sum_{i=1}^{t} \ell_i(x)$, assuming it exists. Then by induction $\sum_{t=1}^{T} \ell_t(x_t^*) \leq \sum_{t=1}^{T} \ell_t(x_T^*)$.

Indeed, $T = 0$ is trivial, and if it holds for $T \geq 0$:

$$\sum_{t=1}^{T} \ell_t(x_t^*) + \ell_{T+1}(x_{T+1}^*) \overset{(IH)}{\leq} \sum_{t=1}^{T} \ell_t(x_T^*) + \ell_{T+1}(x_{T+1}^*) \leq \sum_{t=1}^{T} \ell_t(x_{T+1}^*) + \ell_{T+1}(x_{T+1}^*),$$

and so $\text{Regret}_T \leq 0$.

Example: Squared-Loss Prediction

Adversary plays

$$\ell_t(x) = (x - y_t)^2, \quad \text{for } y_t \in [0, 1].$$

Play

$$x_t^* = \frac{1}{t} \sum_{i=1}^{t-1} y_i.$$

Then

$$\text{Regret}_T \leq 2 \sum_{t=1}^{T} |x_t^* - x_{t-1}^*| \leq 2 \sum_{t=1}^{T} \frac{1}{t} = O(\log T).$$

- Sublinear regret on T, no stochastic assumption needed.
- But works well in the stochastic case too.

Betting on a coin

- Coin result can is $y_t \in \{-1, 1\}$. You play a percentage x_t of your wealth (with sign).
- **Win**: you get $2x_t$. **Lose**: you get nothing.
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- In expectation, you win $y_t(x_t W_t)$, so your new expected wealth W_{t+1} is

$$W_{t+1} = W_t + y_t x_t W_t = W_t(1 + y_t x_t) = W_0 \prod_{i=1}^{t} (1 + y_i x_i).$$

To simplify, consider you bet on a result $[-1, 1]$. The above holds. Taking logs:

$$\log(W_{t+1}) = \log(W_0) + \sum_{i=1}^{t} \log(1 + y_i x_i).$$

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- **Scenario**: adversary (or nature), played heads 85% of the time.
- You are as good $\{\text{heads}, \text{heads}, \dots\}$ up to the regret in log wealth.
- But no prior knowledge required.

The Experts Problem

- k experts, losses $g_t \in [-G, G]^k$.
- Actions in the simplex: $x_t \in \Delta^{k-1} = \{x \in \mathbb{R}^k_{\geq 0} \mid \sum_{i=1}^k x_i = 1\}$.
- Loss $\ell_t(x_t) = \langle g_t, x_t \rangle$.
- The regret is:

$$\text{Regret}_T = \sum_{t=1}^{T} \langle g_t, x_t \rangle - \min_{x \in \Delta^{k-1}} \sum_{t=1}^{T} \langle g_t, x \rangle = \sum_{t=1}^{T} \langle g_t, x_t \rangle - \min_{i \in [k]} \sum_{t=1}^{T} g_{t,i}.$$

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- Many problems reduce to experts with linear losses. Examples:
  - **Boosting**. k points, weak-learner error $\leq \frac{1}{2} - \gamma$.
  - Each point is an expert, play distribution x_t.
  - Loss $\ell_t(x_t) = 1 - \text{err}(h_t, x_t) \geq \frac{1}{2} + \gamma$.
  - $\text{Regret}_T / T < \gamma \implies$ strong learner.
  - **Linear Feasibility**. m constraints $A_i x \geq b_i - \varepsilon$.
  - Experts on constraints, distribution $x_t \in \Delta^{m-1}$.
  - Oracle tests $\langle x_t^\top A, y \rangle \geq x_t^\top b$.
  - Loss $\langle Ay_t - b, x_t \rangle$.
  - $\text{Regret}_T / T \leq \varepsilon \implies \bar{y}$ is ε-feasible.

Online Learning in Optimization: Online-to-Batch Conversion

- We can optimize by running OL algorithms on linear losses given by subgradients:

$$f\left(\frac{1}{T}\sum_{t=1}^{T} x_t\right) - f(x^*) \le \frac{1}{T}\sum_{t=1}^{T}(f(x_t) - f(x^*)) \le \frac{1}{T}\sum_{t=1}^{T}\langle g_t, x_t - x^*\rangle = \frac{\text{Regret}_T(x^*)}{T} \le \frac{\text{Regret}_T}{T}.$$

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- Non-uniform average version. Denote $A_t \stackrel{\text{def}}{=} \sum_{i=1}^{t} a_i$, for $a_i > 0$. Loss $\ell_t(\cdot) \equiv \langle a_t g_t, \cdot\rangle$. Then

$$f\left(\frac{\sum_{t=1}^{T} a_t x_t}{A_T}\right) - f(x^*) \le \frac{\sum_{t=1}^{T} a_t(f(x_t) - f(x^*))}{A_T} \le \frac{\sum_{t=1}^{T}\langle a_t g_t, x_t - x^*\rangle}{A_T} \le \frac{\text{Regret}_T}{A_T}.$$

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- In expectation. Sampling one point $\hat{x}$ uniformly at random in $\{x_1, \dots, x_T\}$ (or w/ the average; or in last iterate if you apply the trick in (Nesterov and Shikhman, 2015; Cutkosky, 2019)):

$$\mathbb{E}\left[f(\hat{x}) - f(x^*)\right] \le \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\langle g_t, x_t - x^*\rangle\right] = \frac{\mathbb{E}\left[\text{Regret}_T(x^*)\right]}{T} \le \frac{\mathbb{E}\left[\text{Regret}_T\right]}{T}.$$

Anytime Online-to-Batch Conversion

Run an OL algorithm $\mathcal{A}$ on linearized losses, but play averaged points:

$$\bar{x}_t = \frac{1}{A_t} \sum_{i=1}^{t} a_i \, x_i, \quad A_t = \sum_{i=1}^{t} a_i.$$

Send to $\mathcal{A}$ the loss

$$\ell_t(x) = \big\langle a_t \, \nabla f(\bar{x}_t), \, x \big\rangle.$$

Algorithm $\mathcal{A}$ plays x_{t+1}.
Then one shows

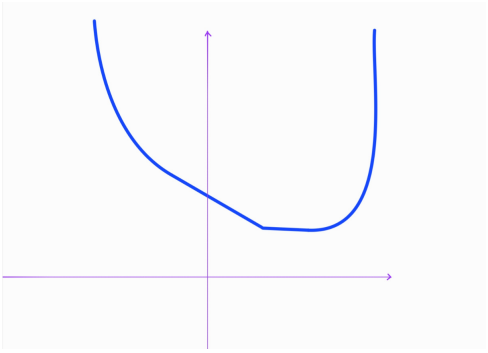
$$f(\bar{x}_T) - f(u) \;\leq\; \frac{R_T(u)}{A_T} - \sum_{t=1}^{T} \frac{A_{t-1}}{A_T} \, D_f(\bar{x}_{t-1}, \bar{x}_t).$$

Fenchel Dual

$$f^*(u) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^d} \{\langle u, x \rangle - f(x)\}.$$

$f(x) + f^*(u) \geq \langle u, x \rangle$ with equality $\iff$ $u \in \partial f(x)$.

If f is convex, closed and proper then $f^{**} = f$.



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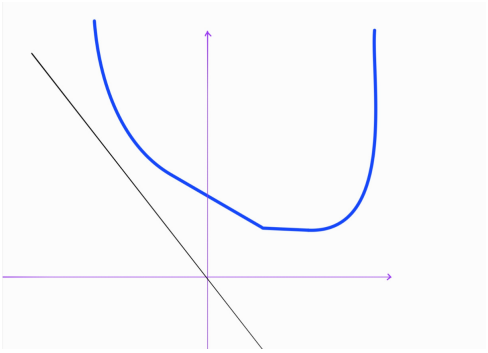

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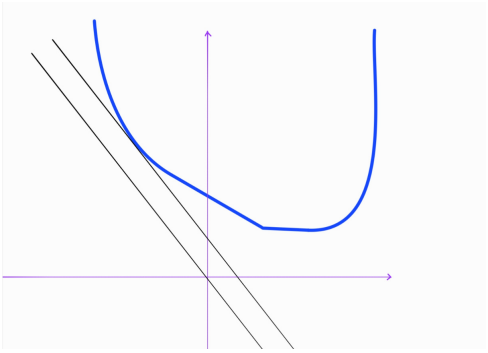

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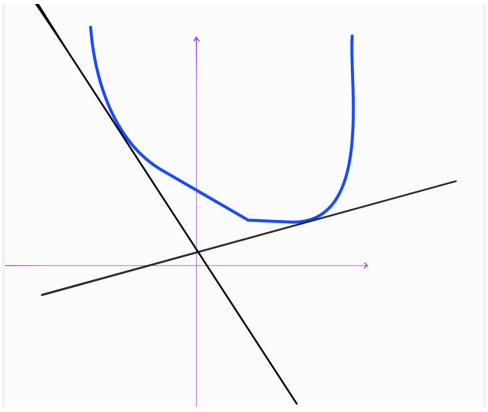

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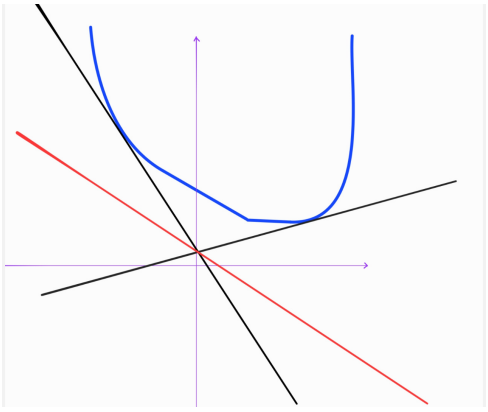

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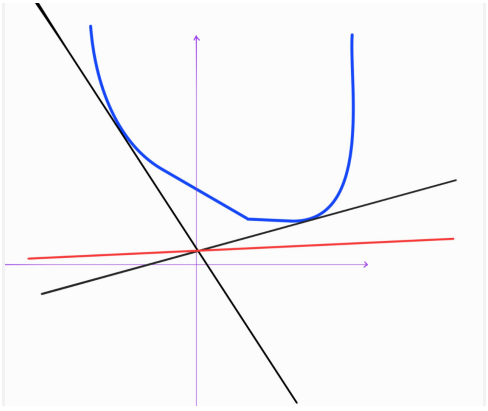

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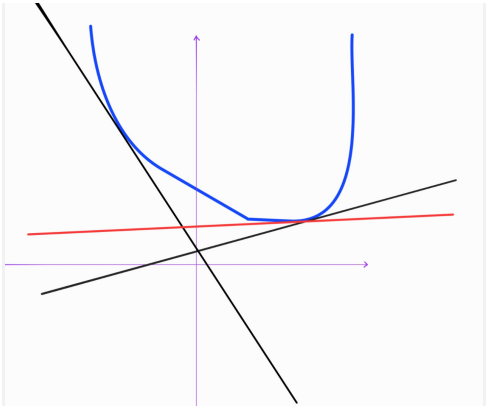

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Example: The Fenchel dual of $\frac{1}{2}\|x\|_p^2$ is $\frac{1}{2}\|x\|_{p^*}^2$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. Including $p = 1$, $p^* = \infty$.

Legendre Functions

Definition

A closed convex proper function f is *Legendre* if

- strictly convex, differentiable on $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$,
- $\|\nabla f(x)\| \to \infty$ as $x \to \partial \operatorname{dom} f$.

Example

- $f(x) = x \log x - x + \mathcal{I}_{[0,\infty)}(x)$,
- $f'(x) = \log(x)$ for $x \in (0, \infty)$, $\quad f''(x) = \frac{1}{x} > 0$,
- $(f^*)'(u) = \exp(u)$ for $u \in \mathbb{R}$, $\quad (f^*)''(u) = \exp(u) > 0$,
- $f^*(u) = \exp(u)$ for $u \in \mathbb{R}$.

Bregman Divergence

For differentiable convex $\psi : \mathcal{X} \to \mathbb{R}$,

$$D_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla\psi(y), x - y \rangle.$$

- **Non-negativity:** $D_\psi(x, y) \geq 0$, equals 0 iff $x = y$.
- **Three-point identity:** $D_\psi(x, y) = D_\psi(z, y) + D_\psi(x, z) + \langle \nabla\psi(z) - \nabla\psi(y),\, x - z \rangle$.
- **Combinations:** $D_{a\psi+b\phi}(x, y) = aD_\psi(x, y) + bD_\phi(x, y)$.
- The Bregman divergence of $D_\psi(\cdot, z)$, that is, $D_{D_\psi(\cdot,z)}(:, z)$ is itself. More generally: $D_{D_\psi(\cdot,y)}(x, z) = D_\psi(x, z)$.
- If ψ is σ-strongly convex, $D_\psi(x, y) \geq \frac{\sigma}{2}\|x - y\|^2$.

Online Learning algorithms: Follow-The-Leader (FTL)

$$x_t = \arg\min_{x \in \mathcal{X}} \sum_{i=1}^{t-1} \ell_i(x).$$

- Works when minimizers move slowly (e.g. squared loss).
- **Fails** badly for adversarial linear losses.

Online Learning algorithms: From FTL to FTRL

- Since FTL can be *unstable*...
- ...stabilize it by adding a *regularizer* ψ_t:

$$x_t \in \operatorname*{arg\,min}_{x \in \mathcal{X}} \Big\{ \sum_{i=1}^{t-1} \ell_i(x) + \psi_t(x) \Big\}.$$

- E.g., choose ψ_t so that $\sum_{i=1}^{t-1} \ell_i + \psi_t$ is *strongly convex*.

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Theorem (FTRL Regret Guarantee)

Suppose each $\ell_t : \mathcal{X} \to \mathbb{R}$ is convex, subdifferentiable, and we use increasing regularizers $\psi_t \stackrel{\text{def}}{=} \frac{1}{\eta_t}\psi(x)$ for ψ being Legendre and 1-strongly convex in $\mathcal{X}$, and $\min_{x \in \mathcal{X}} \psi(x) = 0$. Then, for any $u \in \mathcal{X}$ and $g_t \in \partial \ell_t(x_t)$,

$$\text{Regret}_T(u) \leq \frac{\psi(u)}{\eta_{T+1}} + \sum_{t=1}^{T} \frac{\eta_t \|g_t\|_*^2}{2}.$$

FTRL for Lipschitz Losses

- Suppose ℓ_t is G-Lipschitz w.r.t. $\|\cdot\|$, and $\mathcal{X}$ is compact.
- Use $\psi_t(x) = \frac{1}{\eta_t} D_\psi(x, x_1)$ with ψ 1–strongly convex, $D^2 = \max_{u \in \mathcal{X}} D_\psi(u, x_1)$, and $\eta_t = \dfrac{D}{G}\sqrt{\dfrac{2}{t}}$.

Then, from the theorem, one obtains

$$\text{Regret}_T \leq 3G\sqrt{D^2 T}.$$

Hence $\text{Regret}_T / T = O(1/\sqrt{T})$.
Online-to-batch conversion, for convex Lipschitz optimization over $\mathcal{X}$, gives a point $\bar{x}$ such that

$$f(\bar{x}) - f(x^*) \leq \frac{R_T}{T} \leq \frac{3GD}{\sqrt{T}}.$$

The rate above can be shown to be worse-case optimal up to constant, as long as $T <$ dimension.

Example: Online Subgradient Descent

Take linearized losses $\mathcal{L}_t(x) = \ell_t(x_t) + \langle g_t, x - x_t \rangle$, $g_t \in \partial \ell_t(x_t)$, and

$$\psi_t(x) = \tfrac{\sigma}{2}\|x - x_1\|_2^2.$$

Looser losses lack losing less:

$$\mathrm{Regret}_T^{\ell} \overset{\text{def}}{=} \sum_{t=1}^{T} \ell_t(x_t) - \min_{u \in \mathcal{X}} \sum_{t=1}^{T} \ell_t(u) \le \sum_{t=1}^{T} \mathcal{L}_t(x_t) - \min_{u \in \mathcal{X}} \sum_{t=1}^{T} \mathcal{L}_t(u) \overset{\text{def}}{=} \mathrm{Regret}_T^{\mathcal{L}},$$

Then the FTRL update

$$x_t = \arg\min_x \left\{ \sum_{i=1}^{t-1} \langle g_i, x \rangle + \tfrac{\sigma}{2}\|x - x_1\|^2 \right\} = x_{t-1} - \tfrac{1}{\sigma}\, g_{t-1},$$

recovers subgradient descent.

E.g.: Neg. Entropy-regularized OMD for Experts aka Multiplicative Weights

- Action set $\mathcal{X} = \Delta^{d-1}$, loss $\ell_t(x) = \langle g_t, x \rangle$, $\|g_t\|_\infty \leq G$.
- Use $\psi(x) = \sum_i x_i \ln x_i$ on Δ^{d-1}.
- OMD update in closed form, known as Multiplicative Weights:

$$x_{t+1,i} = \frac{x_{t,i} \exp(-\eta_{t+1} g_{t,i})}{\sum_j x_{t,j} \exp(-\eta_{t+1} g_{t,j})}.$$

- Regret $R_T \leq 3G\sqrt{\ln(d)\, T}$, optimal when $T \to \infty$, up to constants.

Dual View of FTRL

Let $\Psi_t(x) = \psi_t(x) + \mathcal{I}_{\mathcal{X}}(x)$. Then one can show

$$z_1 = 0, \quad z_t = z_{t-1} - g_{t-1}, \quad x_t = \nabla \Psi_t^*(z_t).$$

Equivalently:

$$\hat{x}_t = \nabla \psi_t^*\Big(-\sum_{i=1}^{t-1} g_i\Big), \quad x_t = \operatorname*{arg\,min}_{x \in \mathcal{X}} D_{\psi_t}(x, \hat{x}_t).$$

This is a *lazy* update. The greedy one, aka Online Mirror Descent, often comes w/ similar guarantees:

$$z_1 = 0, \quad z_t = \nabla \psi_t(x_{t-1}) - g_{t-1}, \quad x_t = \nabla \Psi_t^*(z_t).$$

Common regularizers

	Squared ℓ_2	$p \in (0,1)$	$p \in (1,\infty)$	Gen. neg. entropy	$p \in (1,2)$
$\psi(x)$	$\frac{1}{2}\|x\|_2^2$	$-\frac{1}{p}\sum_{i=1}^d x_i^p$	$\frac{1}{p}\|x\|_p^p$	$\sum_{i=1}^d x_i \log x_i - x_i$	$\frac{1}{2}\|x\|_p^2$
$\operatorname{int}(\operatorname{dom}\psi)$	$\mathbb{R}^d$	$\mathbb{R}^d_{>0}$	$\mathbb{R}^d$	$\mathbb{R}^d_{>0}$	$\mathbb{R}^d$
$\nabla_i \psi(x)$	x_i	$-x_i^{p-1}$	$\operatorname{sign}(x_i)\,\lvert x_i\rvert^{p-1}$	$\log x_i$	$\operatorname{sign}(x_i)\,\|x\|_p^{2-p}\lvert x_i\rvert^{p-1}$
$\nabla^2 \psi(x)$	I_d	$(1-p)\operatorname{diag}(1/x_i^{2-p})$	$(p-1)\operatorname{diag}(x_i^{p-2})$	$\operatorname{diag}(1/x_i)$	*(omitted)*
$\psi^*(u)$	$\frac{1}{2}\|u\|_2^2$	$-\frac{1}{p^*}\sum_i(-u_i)^{p^*} + 1_{\mathbb{R}^d_{\le 0}}(u)$	$\frac{1}{p^*}\|u\|_{p^*}^{p^*}$	$\sum_{i=1}^d e^{u_i}$	$\frac{1}{2}\|u\|_{p^*}^2$
$\operatorname{int}(\operatorname{dom}\psi^*)$	$\mathbb{R}^d$	$\mathbb{R}^d_{<0}$	$\mathbb{R}^d$	$\mathbb{R}^d$	$\mathbb{R}^d$
$\nabla_i \psi^*(u)$	u_i	$\lvert u_i\rvert^{p^*-1}$ (p^* s.t. $\frac{1}{p}+\frac{1}{p^*}=1$)	$\operatorname{sign}(u_i)\,\lvert u_i\rvert^{p^*-1}$	e^{u_i}	$\operatorname{sign}(u_i)\,\|u\|_{p^*}^{2-p^*}\lvert u_i\rvert^{p^*-1}$
$D_\psi(x,y)$	$\frac{1}{2}\|x-y\|_2^2$	$\sum_i\left(-\frac{1}{p}x_i^p + \frac{1-p}{p}y_i^p + y_i^{p-1}x_i\right)$	$\leftarrow$similar(omitted)	$\sum_i\left(x_i\log\frac{x_i}{y_i} - x_i + y_i\right)$	*(omitted)*
str. cvx.?	1-wrt $\|\cdot\|_2$	–	$(p-1)$-wrt $\|\cdot\|_p$ on $\|x\|_p \le 1$, $p\in(1,2]$	1-wrt $\|\cdot\|_1$ on $\|x\|_1 \le 1$, $x>0$	$(p-1)$-wrt $\|\cdot\|_p$

Implicit Bias

Theorem (Implicit bias of OMD)

Suppose we have a generalized linear model of the form: $\min_x \mathcal{L}(x) := \sum_{n=1}^N \ell(\langle x, z_n \rangle, y_n)$. and the data is realizable, that is, the minimum loss is 0. Under mild assumptions on the loss (unique finite root property), the iterates x_t of OMD with step sizes so that $x_\infty \overset{\text{def}}{=} \lim_{t\to\infty} x_t$ is a minimizer $\mathcal{L}(x_\infty) = 0$, yield a minimizer satisfying

$$x_\infty = \operatorname*{arg\,min}_{x:\forall n, \langle x, z_n \rangle = y_n} D_\psi(x, x_0).$$

See (Gunasekar et al., 2018).

- If $x_0 = \arg\min_x \psi(x)$, then x_∞ is the solution $\min_{x\in\text{sol}} \psi(x)$ (maximum entropy, minimum Euclidean norm, etc). This implicit bias can benefit generalization.
- Holds because OMD is GD in the dual space & the gradients are in $\text{span}\{z_1, \ldots, z_n\}$.
- Holds even if for stochastic methods using gradients of minibatches.

Problem Adaptivity

$$\text{Regret}_T(u) \leq \frac{\psi(u)}{\eta_{T+1}} + \sum_{t=1}^{T} \frac{\eta_t \|g_t\|_*^2}{2}.$$

Problem Adaptivity

$$\text{Regret}_T(u) \leq \frac{\psi(u)}{\eta} + \sum_{t=1}^{T} \frac{\eta G^2}{2}.$$

Problem Adaptivity

- A small variant with $\eta_{t+1} = \sqrt{\frac{2D^2}{\sum_{i=1}^t \|g_i\|_*^2}}$, where recall $D^2 \stackrel{\text{def}}{=} \max_{u \in \mathcal{X}} D_\psi(u, x_1)$, allows to show

$$R_T(u) \;\leq\; 3\sqrt{D^2 \sum_{t=1}^T \|g_t\|_*^2}\,.$$

- No need to know the future or a uniform bound G.
- Immediately yields $O(1/\sqrt{T})$ optimization rates in G-Lipschitz convex optimization via online-to-batch.
- Other techniques (comparator-adaptive, a.k.a. parameter-free) yield:

$$\text{Regret}_T(u) = \widetilde{O}\left(\sqrt{D_\psi(u, x_1) \sum_{i=1}^t \|g_i\|_*^2}\right),$$

making use of the value of G.

Key Takeaways

- FTRL or Online Mirror Descent generalize subgradient descent to arbitrary geometries via a strongly-convex mirror map ψ.
- Achieves $O(\sqrt{T})$ regret, which leads to $O(1/\sqrt{T})$ Lipschitz convex optimization convergence and adapts easily to $\sum \|g_t\|_*^2$.
- Tight for many problems (e.g. experts, convex Lipschitz minimization,...).
- Forms the basis for accelerated and stochastic algorithmic variants.

Lower Bound for Online Learning: Lipschitz Linear Losses in Compact Sets

Setting. Feasible set $\mathcal{X} \subseteq \mathbb{R}^d$, of diameter D. Losses $\ell_t(x) = \langle g_t, x \rangle$, $\|g_t\| \le G$.

Theorem (OL Lower Bound)

For any algorithm and any $T \ge 1$, there exists a sequence of G-Lipschitz linear losses with

$$\text{Regret}_T \ge \frac{GD}{2\sqrt{2}}\sqrt{T}.$$

Matches the $O(GD\sqrt{T})$ upper bound of FTRL/OMD up to constants.

Proof Sketch: Rademacher Adversary

1. Pick two points $v, w \in \mathcal{X}$ with $\|v - w\| = D$ and direction $z = (v - w)/D$.
2. Let $\varepsilon_t \in \{\pm 1\}$ i.i.d. Rademacher, set $g_t = G\,\varepsilon_t\,z$.
3. Then

$$\sup_{g_1,\dots,g_T} \mathrm{Regret}_T \geq \mathbb{E}\left[\sum_t \langle g_t, x_t\rangle - \min_u \sum_t \langle g_t, u\rangle\right] \;\geq\; \frac{GD}{2}\,\mathbb{E}\left[\left|\sum_t \varepsilon_t\right|\right] \;\geq\; \frac{GD}{2\sqrt{2}}\sqrt{T}.$$

Linearity and Khintchine's inequality give the $\sqrt{T}$ lower bound.

Overview

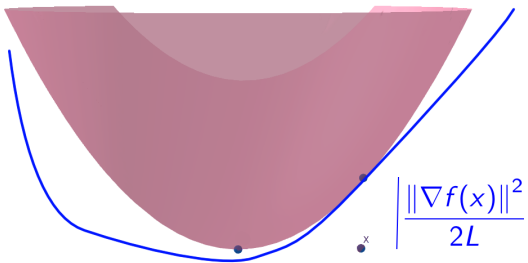
I am looking for PhD students and Postdocs. Tell your friends and/or contact me: David Martínez-Rubio (Madrid, Spain)

Steepest Descent

Smoothness inequality: $f(y) - f(x_t) - \langle \nabla f(x_t), y - x_t \rangle = (y - x_t)^T \nabla^2 f(z)(y - x_t) \leq \frac{L}{2} \|y - x_t\|^2$.
Minimize the upper bound:

$$f(x_{t+1}) - f(x_t) \leq \min_{x \in \mathbb{R}^d} \left\{ \langle \nabla f(x_t), x - x_t \rangle + \frac{L}{2} \|x - x_t\|^2 \right\} = -\frac{1}{2L} \|\nabla f(x_t)\|_*^2.$$

In the Euclidean norm case: $x_{t+1} \leftarrow x_t - \frac{1}{L} \nabla f(x_t)$.



Estimate sequences / Approximate Duality Gap Technique

General strategy to show convergence in convex optimization:

- Define a lower bound on a minimizer: $f(x^*) \geq L_t$. In fact, we could compare to an arbitrary point.
- Upper bound $f(y)$ for y a point you compute: $f(y) \leq U_t$.
- The primal-dual gap is $G_t = U_t - L_t$.
- For increasing weights $A_t > 0$, $A_0 = 0$, bound $A_t G_t - A_{t-1} G_{t-1} \leq E_t$.
- We get

$$f(y) - f(x^*) \leq \frac{A_t G_t}{A_t} \leq \frac{A_0 G_0 + \sum_{i=0}^{t} E_t}{A_t} = \frac{\sum_{i=0}^{t} E_t}{A_t}.$$

- If $E_t \leq 0$, we say that $A_t G_t$ is Lyapunov.

Steepest Descent: Convergence Rate

Theorem (Steepest Descent Convergence)

Let f be convex and L-smooth w.r.t. $\|\cdot\|$ with a minimizer x^*. Steepest Descent satisfies

$$f(x_t) - f(x^*) \leq \frac{2LD_t^2}{t+1},$$

where D_t is the maximum distance from iterates to x^*. They are in the sublevel set.

For the Euclidean norm (Gradient Descent), updates are non-expansive.
For $a_t > 0$ to be chosen later and for $A_t = \sum_{i=1}^{t} a_i$:

$$A_t f(x^*) \geq \sum_{i=0}^{t} a_i f(x_i) + \sum_{i=0}^{t} a_i \langle \nabla f(x_i), x^* - x_i \rangle =: A_t L_t \qquad A_t f(x_{t+1}) =: A_t U_t$$

Steepest Descent: Convergence Rate

$$
\begin{aligned}
A_t G_t - A_{t-1} G_{t-1} &= A_t (f(x_{t+1}) - f(x_t)) + \cancel{a_t f(x_t)} \\
&\quad - \left(\cancel{a_t f(x_t)} + \sum_{i=0}^{t-1} a_i f(x_i) + a_t \langle \nabla f(x_t), x^* - x_t \rangle + \cancel{\sum_{i=0}^{t-1} a_i (\langle \nabla f(x_i), x^* - x_i \rangle)} \right) \\
&\quad + \left(\cancel{\sum_{i=0}^{t-1} a_i f(x_i)} + \cancel{\sum_{i=0}^{t-1} a_i (\langle \nabla f(x_i), x^* - x_i \rangle)} \right) \\
&\overset{①}{\leq} -\frac{A_t}{2L} \|\nabla f(x_t)\|_*^2 + a_t \|\nabla f(x_t)\|_* D_t \overset{②}{\leq} \cancel{-\frac{A_t}{2L} \|\nabla f(x_t)\|_*^2} + \cancel{\frac{A_t}{2L} \|\nabla f(x_t)\|_*^2} + \frac{L a_t^2}{2 A_t} D_t^2 \overset{\text{def}}{=} E_t,
\end{aligned}
$$

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\end{aligned}
$$

So

$$
f(x_{t+1}) - f(x^*) \leq \frac{\sum_{i=0}^t E_i}{A_t} = \frac{L D_t^2}{2A_t} \sum_{i=0}^t \frac{a_i^2}{A_i} = O\left(\frac{L D_t^2}{t} \right).
$$

Composite Steepest Descent

Goal:

$$\min_{x \in \mathbb{R}^d} f(x) + \psi(x).$$

Algorithm:

$$x_{t+1} \in \arg\min_{x \in \mathbb{R}^d} \left\{ f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \tfrac{L}{2} \|x - x_t\|^2 + \psi(x) \right\}.$$

Encode constraints or regularization with ψ. E.g.: $\mathsf{Ind}_{\mathcal{X}}(x)$, $\|x\|_2^2$, $\|x\|_1$, $\mathsf{KL}(x, x_0)$, etc.

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Theorem (Composite Steepest Descent Convergence)

Let f be convex and L-smooth w.r.t. $\|\cdot\|$ such that $f + \psi$ has a minimizer at x^*. Composite Steepest Descent satisfies

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where D_t is the maximum distance from iterates to x^*. They are in the sublevel set of $f + \psi$.

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where D_t is the maximum distance from iterates to x^*. They are in the sublevel set of $f + \psi$.

Message: If you can optimize a model of f plus ψ exactly, then ψ generally does not affect convergence.

Composite Steepest Descent

Let $\hat{x}_{t+1} \stackrel{\text{def}}{=} \frac{a_t}{A_t} u + \frac{A_{t-1}}{A_t} x_t$ so that $a_t(u - \hat{x}_{t+1}) = A_{t-1}(\hat{x}_{t+1} - x_t)$. Then

$$A_t G_t - A_{t-1} G_{t-1} = A_t(f(x_{t+1}) - f(x_t)) + a_t f(x_t) + A_t \psi(x_{t+1}) - A_{t-1}\psi(x_t)$$

$$- \left(a_t f(x_t) + \sum_{i=0}^{t-1} a_i f(x_i) + a_t \langle \nabla f(x_t), u - x_t \rangle + \sum_{i=0}^{t-1} a_i (\langle \nabla f(x_i), u - x_i \rangle) + A_{t-1}\psi(u) + a_t \psi(u) \right)$$

$$+ \left(\sum_{i=0}^{t-1} a_i f(x_i) + \sum_{i=0}^{t-1} a_i (\langle \nabla f(x_i), u - x_i \rangle) + A_{t-1}\psi(u) \right)$$

$$\stackrel{①}{\leq} \left(A_t \langle \nabla f(x_t), \hat{x}_{t+1} - x_t \rangle + \frac{L a_t^2}{2A_{t-1}} \|u - x_t\|^2 \right) + a_t \langle \nabla f(x_t), u - x_t \rangle + A_t \psi(\hat{x}_{t+1}) - A_{t-1}\psi(x_t) - a_t \psi(u)$$

$$\stackrel{②}{\leq} \frac{L a_t^2}{2A_{t-1}} D_t^2 \stackrel{\text{def}}{=} E_t.$$

Weight Decay as Regularized Optimization

If we have a function f and we want to solve the regularized optimization problem

$$\min_x \left\{ f(x) + \frac{\lambda}{2}\|x\|_2^2 \right\} \quad \text{for some } \lambda > 0,$$

and we sequentially optimize some linear (or quadratic) model of f, like Composite Steepest Descent does, we get

$$x_{t+1} \in \arg\min_{x \in \mathbb{R}^d} \left\{ f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{L}{2}\|x - x_t\|^2 + \frac{\lambda}{2}\psi(x) \right\} = x_t - \frac{1}{L}\nabla f(x_t) - \lambda x_t.$$

This is weight decay!

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This is weight decay!

However, if the model of f is more complex, Composite Optimization $\neq$ adding the weight decay vector to the gradient.

Adam vs AdamW

Algorithm 2 Adam with L_2 regularization and Adam with decoupled weight decay (AdamW)

given $\alpha = 0.001, \beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 10^{-8}, \lambda \in \mathbb{R}$
initialize time step $t \leftarrow 0$, parameter vector $\boldsymbol{\theta}_{t=0} \in \mathbb{R}^n$, first moment vector $\boldsymbol{m}_{t=0} \leftarrow \boldsymbol{0}$, second moment vector $\boldsymbol{v}_{t=0} \leftarrow \boldsymbol{0}$, schedule multiplier $\eta_{t=0} \in \mathbb{R}$
repeat
  $t \leftarrow t + 1$
  $\nabla f_t(\boldsymbol{\theta}_{t-1}) \leftarrow \text{SelectBatch}(\boldsymbol{\theta}_{t-1})$ ▷ select batch and return the corresponding gradient
  $\boldsymbol{g}_t \leftarrow \nabla f_t(\boldsymbol{\theta}_{t-1}) + \lambda \boldsymbol{\theta}_{t-1}$
  $\boldsymbol{m}_t \leftarrow \beta_1 \boldsymbol{m}_{t-1} + (1 - \beta_1)\boldsymbol{g}_t$ ▷ here and below all operations are element-wise
  $\boldsymbol{v}_t \leftarrow \beta_2 \boldsymbol{v}_{t-1} + (1 - \beta_2)\boldsymbol{g}_t^2$
  $\hat{\boldsymbol{m}}_t \leftarrow \boldsymbol{m}_t / (1 - \beta_1^t)$ ▷ β_1 is taken to the power of t
  $\hat{\boldsymbol{v}}_t \leftarrow \boldsymbol{v}_t / (1 - \beta_2^t)$ ▷ β_2 is taken to the power of t
  $\eta_t \leftarrow \text{SetScheduleMultiplier}(t)$ ▷ can be fixed, decay, or also be used for warm restarts
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_{t-1} - \eta_t \left(\alpha \hat{\boldsymbol{m}}_t / (\sqrt{\hat{\boldsymbol{v}}_t} + \epsilon) + \lambda \boldsymbol{\theta}_{t-1} \right)$
until *stopping criterion is met*
return optimized parameters $\boldsymbol{\theta}_t$

Figure: Treating the regularized function as a composite objective (Loshchilov and Hutter, 2017)

Overview

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Convex Theory for the Non-Convex World I: Convex Until Proven Guilty

Let f, possibly non-convex, satisfy gradient and Hessian Lipschitzness, that is:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \text{and} \quad \|\nabla^2 f(x) - \nabla^2 f(y)\| \leq M\|x - y\| \quad \text{for all } x, y.$$

- The fastest provable algorithm known to date for $\|\nabla f(x_T)\| \leq \varepsilon$ runs in $T = O(\varepsilon^{-7/4})$.
- It is a $\varepsilon^{-1/28}$ away from the lower bound (Carmon et al., 2020).
- The algorithms is Nesterov's accelerated gradient descent, with its momentum restarted after some "**guilty of non-convex**" criteria are satisfied.

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- The algorithms is Nesterov's accelerated gradient descent, with its momentum restarted after some "**guilty of non-convex**" criteria are satisfied.
- **Convex algorithm design becomes useful for provably-fast non-convex algorithms!**
- (Carmon et al., 2017, Convex until proven guilty), (Li and Lin, 2023, Removing polylog factors), (Marumo and Takeda, 2024, Parameter-free version).

Convex Theory for the Non-Convex World II: Weakening convexity

Let x^* be a minimizer of f.

- Polyak Lojasiewicz (PL): $(f(x) - f(x^*)) \geq \mu \|\nabla f(x)\|$.
  - If NNs are very wide, they $\approx$ Neural Tangent Kernel, and are PL-like. (Du et al., 2019; Allen-Zhu et al., 2019).
- For $\gamma \in (0, 1]$, γ-quasar strongly convex: $f(x^*) \geq f(x) + \frac{1}{\gamma}\langle \nabla f(x), x^* - x \rangle + \frac{\mu}{2}\|x - x^*\|^2$.
  - DNNs landscapes along wrt its trajectory (Tran et al., 2024).
- Weak-convexity $\nabla^2 f \succeq -\sigma I$.
- Many more: Quadratic growth, restricted secant condition, one-point strong convexity, variational coherence, quasi-convexity, pseudoconvexity, invexity... Cf. (Hinder et al., 2020).

- Metric of optimality?
- Convex minimization: $f(x_t) - f(x^*)$ or $\|\nabla f(x_t)\|$.
- Non convex smooth optimization: $\mathbb{E}\left[\nabla f(x_i)_2^2\right]$.
- Non-convex non-smooth optimization? I.e., Deep NNs with ReLUs.

Definition. Point x is a (δ, ε)-stationary point of a.e. differentiable f if $\exists y_1, \ldots, y_n \in B(x, \delta)$ s.t.:

$$\frac{1}{n}\sum_{i=1}^{n} y_i = x, \quad \text{and} \quad \left\|\frac{1}{n}\sum_{i=1}^{n} \nabla F(y_i)\right\| \leq \varepsilon,$$

cf. (Zhang, Lin, et al., 2020).

Under G-Lipschitzness and mild function regularity:

$$f(y) - f(x) = \int_0^1 \langle \nabla f(x + t(y - x)), y - x \rangle \mathrm{d}t.$$

- (Cutkosky et al., 2023; Zhang and Cutkosky, 2024) designed reductions where an OCO or a restarted OCO is fed with linear losses $\ell_t(x) = \langle \nabla f(x_t), x \rangle$ and decides the updates of the algorithm.
- Step sizes that decide the next point are random variables.
- Achieves worse-case optimality $O(\delta/\varepsilon^3)$ when $\varepsilon = O(\delta)$, with a very general reduction.
- Promising direction for understanding current methods.

Convex Theory for the Non-Convex World IV: Empirical similar behaviors

The Surprising Agreement between Convex Optimization Theory and Learning-Rate Scheduling (Schaipp et al., 2025).

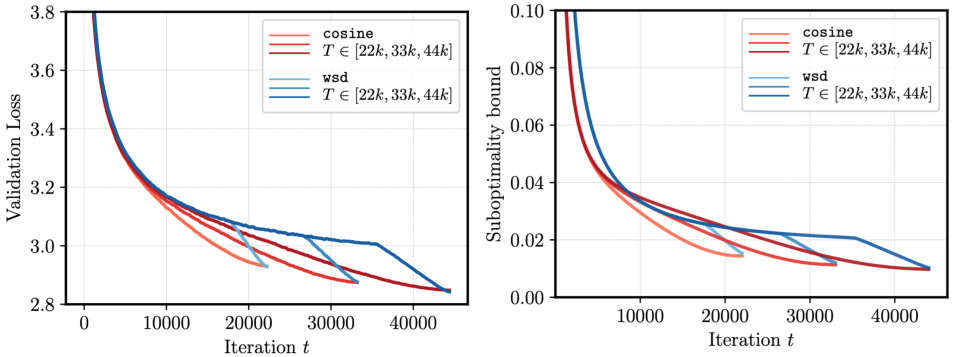

Figure 1. Real loss curves **(left)** and theoretical bound **(right)**

Convex Theory for the Non-Convex World IV: Empirical similar behaviors

Prediction of loss decay by (Defazio et al., 2023), depending on learning rates. One of several works in this direction.

$$\mathbb{E}[f(x_T) - f(x_\star)] \leq \frac{1}{2\gamma \sum_{t=1}^{T} \eta_t} \left[D^2 + \gamma^2 G^2 \sum_{t=1}^{T} \eta_t^2 \right] + \frac{\gamma G^2}{2} \sum_{k=1}^{T-1} \left(\frac{\eta_k}{\left(\sum_{t=k+1}^{T} \eta_t \right) \left(\sum_{t=k}^{T} \eta_t \right)} \sum_{t=k}^{T} \eta_t^2 \right).$$

Convex Theory for the Non-Convex World IV: Empirical similar behaviors

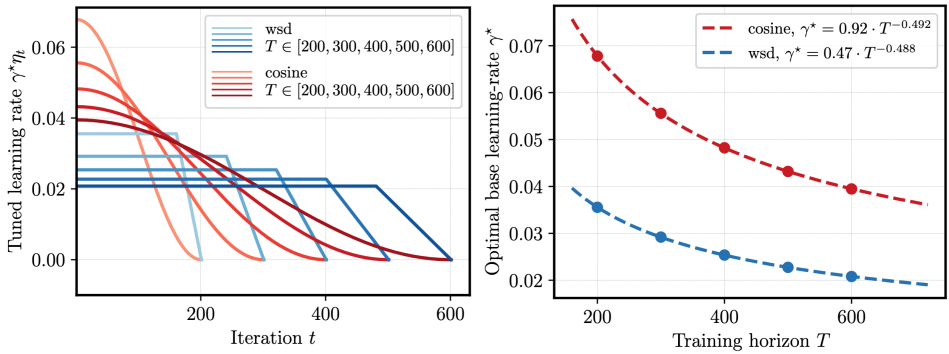

Figure: Left: Cosine & ctant + cooldown schedules. Right: Optimal initial learning rate. Theory predicts one should be $\approx$ double than the other one.

Overview

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Frank–Wolfe (a.k.a. Conditional Gradients)

- **Goal:**

$$\min_{x \in \mathcal{X}} f(x),$$

where f is L-smooth, $\mathcal{X}$ is compact and convex. (We can also phrase the composite problem).

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- **Goal:**

$$\min_{x \in \mathcal{X}} f(x),$$

where f is L-smooth, $\mathcal{X}$ is compact and convex. (We can also phrase the composite problem).

- Projection $\arg\min_{x \in \mathcal{X}} \|x - y\|^2$ may be expensive; linear minimization $\arg\min\{\langle \nabla f(x), v \rangle + \psi(v)\}$ can be much cheaper.

Set	**Linear minimization**	**Projection**
ℓ_p-ball, $p \notin \{1, 2, \infty\}$	$O(d)$	$\widetilde{O}(d\|y - x^*\|^2/\hat{\varepsilon}^2)$
Nuclear norm ball aka Spectrahedron	$O(\mathrm{nnz}(Y)\log(m+n)\sqrt{\sigma_1/\hat{\varepsilon}})$	$O(mn\min\{m,n\})$
Flow polytope	$O(m+n)$	$\widetilde{O}(m^3 n + n^2)$
Birkhoff polytope	$O(d^3)$	$O(d^2 d_z/\hat{\varepsilon}^2)$

Frank–Wolfe (a.k.a. Conditional Gradients)

$$
\begin{aligned}
v_t &\in \arg\min_v \{\langle \nabla f(x_t), v\rangle + \psi(v)\},\\
x_{t+1} &= \tfrac{t}{t+2} x_t + \tfrac{2}{t+2} v_t.
\end{aligned}
$$

Convergence of Frank–Wolfe

Theorem (Frank-Wolfe Convergence)

If f is L-smooth w.r.t. $\|\cdot\|$ on $\operatorname{dom}\psi$, and x^* is a minimizer, then after t steps of the FW algorithm:

$$f(x_t) + \psi(x_t) - (f(x^*) + \psi(x^*)) \leq \frac{2L\,D^2}{t+1},$$

where $D = \max_{0 \leq i \leq t} \|v_i - x_i\|$ is the "step-size diameter" (e.g. diameter of the domain).

- No need to know t in advance.
- Each iteration calls one Linear Minimization Oracle.

Proof Sketch of FW Rate

1. Maintain a similar "dual" lower bound but substituting x^* by the minimizer v_t:

$$(f+\psi)(x^*) \geq \frac{1}{A_t}\sum_{i=0}^{t} a_i\big(f(x_i) + \langle \nabla f(x_i), x^* - x_i\rangle + \psi(x^*)\big).$$
$$\geq \frac{1}{A_t}\sum_{i=0}^{t} a_i\big(f(x_i) + \langle \nabla f(x_i), v_i - x_i\rangle + \psi(v_i)\big) =: L_t.$$

2. For $G_t \stackrel{\text{def}}{=} (f(x_{t+1}) + \psi(x_{t+1})) - L_t$, show

$$A_t G_t - A_{t-1} G_{t-1} \leq \frac{L\, a_t^2}{2A_t}\|v_t - x_t\|^2.$$

3. Choose $a_t = 2t+2$, $A_t = (t+1)(t+2)$.
4. Similarly as before, it telescopes to $O(1/t)$.

Remarks and Variants

- **Short-steps**: line-search on the upper bound given by smoothness.
- **Affine-invariant**: best norm and scaling automatically.
- **Lower bound**: any LMO-based method on $L\|x\|^2$ over the unit simplex needs $\Omega(LD^2/\varepsilon)$ queries when $t \leq d$.
- **Better rates for better sets**: Polytopes, strongly convex sets...
- **Applications**: Approximate Carathéodory, matrix completion (nuclear ball regularization), flow polytope, etc.

Example: Approximate Carathéodory

- Given $\hat{x} \in \mathcal{P} \stackrel{\text{def}}{=} \operatorname{conv}\{V\}$, find $\tilde{x}$ that is a convex combination of $\leq k$ vertices with $\|\hat{x} - \tilde{x}\| \leq \varepsilon$.
- Set $f(x) = \|x - \hat{x}\|^2$, $\psi = \mathcal{I}_{\mathcal{P}}$.
- Frank–Wolfe finds $\tilde{x} \in \mathcal{P}$ with $f(\tilde{x}) \leq \varepsilon^2$ in $O(D^2/\varepsilon^2)$ iterations $\implies \|\hat{x} - \tilde{x}\| \leq \varepsilon$. Each uses one linear minimization over V.
- Improves the classical $d + 1$ bound when $d \gg D^2/\varepsilon^2$.

Nesterov's Accelerated Gradient Descent (AGD) Methods

- Optimal first-order method for the minimization of Euclidean convex (resp. μ-strongly convex) and L-smooth functions.

	$\mu > 0$ $[\kappa \stackrel{\text{def}}{=} L/\mu]$	$\mu = 0$
Accelerated Gradient Descent	$O(\sqrt{\kappa} \log 1/\varepsilon)$	$O(\sqrt{L/\varepsilon})$
Gradient Descent	$O(\kappa \log 1/\varepsilon)$	$O(L/\varepsilon)$

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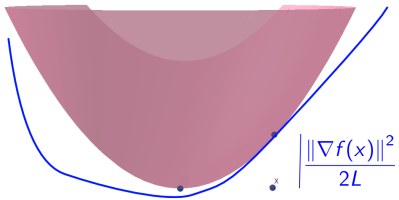

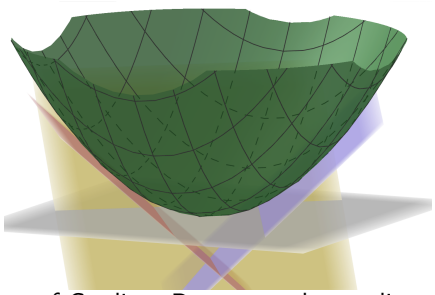

Accelerated Gradient Descent can be seen as a combination of Gradient Descent and an online learning algorithm that have, respectively, progress and instantaneous regret that are proportional to each other (proportional to $\|\nabla f(x)\|^2$ in the unconstrained case).

AGD: Algorithm (Sketch)

1. Maintain three sequences $x_t, z_t, y_t \in \mathcal{X}$ for optimizing $f + \phi$.
2. Weights $a_t = \frac{t}{2L}$, cumulative $A_t = \sum_{i=1}^{t} a_i = \frac{t(t+1)}{4L}$.
3. *Coupling step:*
$$x_t = \frac{A_{t-1}}{A_t} y_{t-1} + \frac{a_t}{A_t} z_{t-1}.$$
4. *Online subproblem (FTRL):*
$$z_t = \arg\min_{z \in \mathcal{X}} \Big\{ \sum_{i=1}^{t} a_i \langle \nabla f(x_i), z \rangle + A_t \phi(z) + \psi(z) \Big\}.$$
5. *Descent step:*
$$y_t = \arg\min_{y \in \mathcal{X}} \Big\{ Q_{f,x_t}(y) + \phi(y) \Big\},$$
where $Q_{f,x}(y) = f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2$.

Pseudocode in the Euclidean case

First version. (Nesterov, 1983)

$$x_{k+1} \leftarrow y_k + (a_k - 1)(y_k - y_{k-1})/a_{k+1}$$
$$y_{k+1} \leftarrow \arg\min_{y \in \mathcal{X}} \left\{ f(x_{k+1}) + \langle \nabla f(x_{k+1}), y - x_{k+1} \rangle + \frac{L}{2} \|y - x_{k+1}\|^2 \right\}$$

We have that $y_{k+1} = x_{k+1} - \frac{1}{L} \nabla f(x_{k+1})$ if the problem is unconstrained. If it's constrained gradients are queried possible outside of $\mathcal{X}$. Undesirable.

Improved version. (Nesterov, 1998)

$$a_{k+1} = (k+2)/L$$
$$\tau_k = 1/(k+2)$$
$$x_{k+1} \leftarrow \tau_k z_k + (1 - \tau_k) y_k$$
$$z_{k+1} \leftarrow \arg\min_{z \in \mathcal{X}} \left\{ \sum_{i=0}^{k} a_{i+1} \langle \nabla f(x_{i+1}), z \rangle + D_\psi(z, x_0) \right\}$$
$$y_{k+1} \leftarrow \tau_k z_{k+1} + (1 - \tau_k) y_k$$

Optimism and Acceleration

- Combine two ideas:

  1. *Anytime online-to-batch*: Last-iterate guarantees for optimization.

  2. *Optimistic FTRL*: use "hints" to reduce regret when losses vary smoothly.

- Leads to a simple accelerated method in the smooth convex case.

Optimistic FTRL (OFTRL)

Suppose $\psi_t = \frac{1}{\eta_t}\psi$ for a 1-strongly convex regularizer ψ with minimizer at x_1. *Given hints* $\tilde{\ell}_t$, *play*

$$x_t = \operatorname*{arg\,min}_{x \in \mathcal{X}} \Big\{ \sum_{i=1}^{t-1} \ell_i(x) + \tilde{\ell}_t(x) + \psi_t(x) \Big\}.$$

If $\ell_t - \tilde{\ell}_t$ is convex and $g_t - \tilde{g}_t \in \partial(\ell_t - \tilde{\ell}_t)(x_t)$, then

$$R_T(u) \;\le\; \frac{\psi(u)}{\eta_T} + \sum_{t=1}^{T} \left(\langle g_t - \tilde{g}_t, x_t - x_{t+1} \rangle - \frac{1}{2\eta_t} \|x_t - x_{t+1}\|^2 \right) \le \frac{\psi(u)}{\eta_T} + \sum_{t=1}^{T} \frac{\eta_t}{2} \|g_t - \tilde{g}_t\|_*^2.$$

Hints that predict the next gradient well, sharply reduce regret. But in the Lipschitz case, if predictions are quite off, we have a similar worse-case performance.
For acceleration, choose $\tilde{\ell}_t \equiv \langle \nabla f(x_{t-1}), \cdot \rangle$ given by the previous gradient.

Optimistic Acceleration Guarantee

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth and convex with a minimizer x^*, the output of Anytime OFTRL with hint $\langle \nabla f(x_{t-1}), \cdot \rangle$ satisfies

$$f(\bar{x}_T) - f(x^*) = O\left(\frac{L D_\psi(u, x_0)}{T^2}\right).$$

Key steps:

- Apply anytime online-to-batch to relate $f(\bar{x}_T) - f(x^*)$ to regret and negative Bregman terms.
- Bound regret of OFTRL with hint-errors $\|\nabla f(\bar{x}_t) - \nabla f(\bar{x}_{t-1})\|_*^2$.
- Use smoothness to absorb these into the negative Bregman terms.
- Choose $a_t^2 \leq L\, A_{t-1}$ so all error terms cancel.

Generalizations

- Similar ideas can be used in the case of having an unbiased gradient oracle $x \mapsto g_x$ such that $\mathbb{E}[g_x] = \nabla f(x)$.
- Example. For $f(x) \stackrel{\text{def}}{=} \sum_{i=1}^n f_i(x)$, we can take a minibatch:

$$g_x \stackrel{\text{def}}{=} \sum_{i \in \mathcal{B}} \nabla f_i(x),$$

  where $\mathcal{B}$ is a subset of fixed size chosen uniformly at random among all such subsets.
- If we have bounded variance, one can achieve:

$$\mathbb{E}[f(x_T) - f(x^*)] = O\left(\frac{LR^2}{T^2} + \frac{D\sigma}{\sqrt{T}}\right).$$

- In the finite-sum case one has:

$$O\left(\sqrt{\frac{nLR^2}{\varepsilon}} + n \log\log(n)\right) \text{ for stochastic algs vs } O\left(n\sqrt{\frac{LR^2}{\varepsilon}}\right) \text{ for deterministic algs.}$$

Generalizations

- Accelerated Sparse PageRank (Martínez-Rubio, Wirth, et al., 2023).
- Accelerated Packing Proportional Fair Allocations (Criado et al., 2022).
- Accelerated Riemannian Optimization (Martínez-Rubio, 2022; Martínez-Rubio and Pokutta, 2023).
- Accelerated Non-Euclidean High-Order Smooth Convex Optimization (Contreras et al., 2024).
- Black-box Accelerated Stable Algorithms (Vary et al., 2024).

Overview

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Scale Invariance of Optimization Methods

- **Scale Invariance:** An algorithm $\mathcal{A}$ is *scale invariant* if solving

$$\min_x^{\mathcal{A}} f(x) \quad \text{is equivalent to} \quad \min_{\hat{x}}^{\mathcal{A}} af(b\hat{x} + c) + d,$$

for any $a, b > 0$, $c \in \mathbb{R}^d$, $d \in \mathbb{R}$. That is, the iterates of the algorithm are equivalent.

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- **Feature scale invariance:** For any vector λ, this is satisfied for algorithms such that $\min^{\mathcal{A}} f(x) \leftrightarrow \min^{\mathcal{A}} f(\text{diag}(\lambda)x)$.

  - Essentially satisfied by RMSProp, Adagrad, Adam, Lion...

Scale Invariance of Optimization Methods

(Kingma and Ba, 2014, Adam) is feature scale invariant.

$$g_t \leftarrow \nabla f_t(x_{t-1})$$

$$m_t \leftarrow \beta_1 m_{t-1} + (1-\beta_1) g_t; \quad \hat{m}_t \leftarrow m_t/(1-\beta_1^t)$$

$$v_t \leftarrow \beta_2 v_{t-1} + (1-\beta_2) g_t^2; \quad \hat{v}_t \leftarrow v_t/(1-\beta_2^t)$$

$$x_t \leftarrow x_{t-1} - \eta \cdot \hat{m}_t/(\sqrt{\hat{v}_t} + \varepsilon)$$

Assigning Units for Scale Invariant Algorithms

Distance $\|x-y\|$: $[\|x-y\|] = \mathrm{u}_x$.

Function gap $f(x)-f(x^*)$: $[\cdot] = \mathrm{u}_f$.

Gradient $\nabla f(x)$: $[\nabla f] = \mathrm{u}_f/\mathrm{u}_x$.

Hessian $\nabla^2 f(x)$: $[\nabla f] = \mathrm{u}_f/{\mathrm{u}_x}^2$.

Smoothness $\nabla^2 f \preceq L\, I$: $[L] = \mathrm{u}_f/\mathrm{u}_x^2$.

Lipschitzness $|f(x)-f(y)| \leq G\|x-y\|$: $[G] = \mathrm{u}_f/\mathrm{u}_x$.

- **Check both sides of your bounds have the same units!** Nice for spotting errors!
  - $f(x_t)-f(x^*) = O(\frac{LR^2}{T})$, $x_{t+1} = x_t - \eta(\nabla^2 f(x_t))^{-1}\nabla f(x_t)$,
  - $\|x_t - x^*\| \leq 3R + 4R^2$, for $R = \|x_0 - x^*\|$ the initial distance. Possibly wrong!
  - Claim: An algorithm on a smooth convex problem satisfies $\|\nabla f(x_t)\| \leq \frac{LR^2}{\varepsilon}$. Wrong! If you scale you get as fast as you want, breaking lower bounds!
- **Sometimes in papers people assume wlog some constants are 1. Hard to keep track of units, try not to do that.**

A Cautionary Example

Suppose one proves

$$T = O\left(\max\{\frac{1}{L}, \frac{L}{\mu}\} \log\frac{LR^2}{\varepsilon}\right).$$

- Inside the log: $\frac{LR^2}{\varepsilon}$ is unitless, that is, $[\frac{LR^2}{\varepsilon}] = \mathrm{u}_x^0\mathrm{u}_f^0$.
- But $\frac{1}{L}$ has units $\mathrm{u}_x^2/\mathrm{u}_f$, while $\frac{L}{\mu}$ is unitless.
- Rate can be arbitrary slow if we optimize the rescaled $af(x)$, when a is small! Because $L \to 0$.
- Scaling with large a results in $T = O(\frac{L}{\mu} \log \frac{LR^2}{\varepsilon})$. Probably step sizes were wrong to begin with.

Overview

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Reduction from Strongly Convex $\to$ Convex: Restarts

Suppose f is μ–strongly convex and an algorithm satisfies ① below

$$\tfrac{\mu}{2}\|x_t - x^\star\|^2 \;\le\; f(x_t) - f(x^\star) \overset{①}{\le} c_t\,\|x_0 - x^\star\|^2, \quad c_t \to_{t\to\infty} 0.$$

Then for some t with $c_t \le \frac{\mu}{8}$, $\|x_t - x^\star\| \le \frac{1}{2}\|x_0 - x^\star\|$. Restart to halve the distance each stage. Restarts work well in several applications (Roulet and d'Aspremont, 2017).

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Rates of some algorithms for L-convex functions:

- **Gradient descent** converges as $O(\frac{1}{t})$ so it needs $t = O(L/\mu)$ to halve the distance to x^*. $\implies$ overall $O\big(\frac{L}{\mu}\log\frac{1}{\varepsilon}\big)$ steps.
- **Accelerated GD** needs $t = O(\sqrt{L/\mu})$ per halving. $\implies$ $O\big(\sqrt{\frac{L}{\mu}}\log\frac{1}{\varepsilon}\big)$.

These *linear convergence* rates.

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Then for some t with $c_t \le \frac{\mu}{8}$, $\|x_t - x^\star\| \le \frac{1}{2}\|x_0 - x^\star\|$. Restart to halve the distance each stage. Restarts work well in several applications (Roulet and d'Aspremont, 2017).

- **Subgradient Descent (G-Lipschitz)** converges as $O(G\|x_0^{(t)} - x^*\|/\sqrt{T})$, so we restart every $O(G^2/(\|x_0^{(t)} - x^*\|^2\mu^2))$ we can halve the squared distance to x^* until it's $\le \frac{\mu}{\varepsilon}$ computing the following number of gradients.

$$O\left(\frac{G^2}{\mu^2}\left(\frac{1}{D^2} + \frac{4}{D^2} + \frac{16}{D^2} + \cdots + \frac{\mu}{\varepsilon}\right)\right) = O\left(\frac{G^2}{\mu^2}\cdot\frac{\mu}{\varepsilon}\sum_{i=0}^{\infty} 2^{-2i}\right) = O\left(\frac{G^2}{\mu\varepsilon}\right),$$

  which is optimal up to constants.

Last iteration as expensive as the rest. **Often the case:** with $\frac{1}{\varepsilon}$ convergence rates: to 2ε takes $\frac{1}{2\varepsilon}$, but from 2ε to ε takes the same $\frac{1}{\varepsilon} - \frac{1}{2\varepsilon}$.

Reductions from Convex $\to$ Strongly Convex: Regularization

- To solve $\min f(x)$ with minimizer $x^\star$, add $r(x) \geq 0$ s.t. $r(x^\star) \leq \frac{\varepsilon}{2}$, and approximately solve $\min_x \{f(x) + r(x)\}$.
- If $\hat{x}$ satisfies $(f + r)(\hat{x}) \leq \min(f + r) + \frac{\varepsilon}{2}$, then $f(\hat{x}) \leq f(x^\star) + \varepsilon$.
- Choose e.g. $r(x) = \frac{\varepsilon}{2D^2}\|x - x_0\|^2$.
- **Smooth case:** $f + r$ is $(L + \frac{\varepsilon}{D^2})$–smooth, $(\frac{\varepsilon}{D^2})$–strongly convex. Accelerated methods yield $\tilde{O}(\sqrt{LD^2/\varepsilon})$ oracle calls.
- **Lipschitz case:** $f + r$ is $(\frac{\varepsilon}{D^2})$-strongly convex, and we obtain rates of $\tilde{O}(G^2D^2/\varepsilon^2)$ from an optimal algorithm that runs in $O(G^2/(\mu\varepsilon))$.

Overview

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Lower Bound Techniques: Hard functions

- The simplest hard function for the Euclidean Lipschitz convex class for $x_0 = 0$:

$$x \mapsto \max_{i \in [d]} \left\{ x_i - \frac{i}{d} \right\} \text{ for } x \in B(0,1).$$

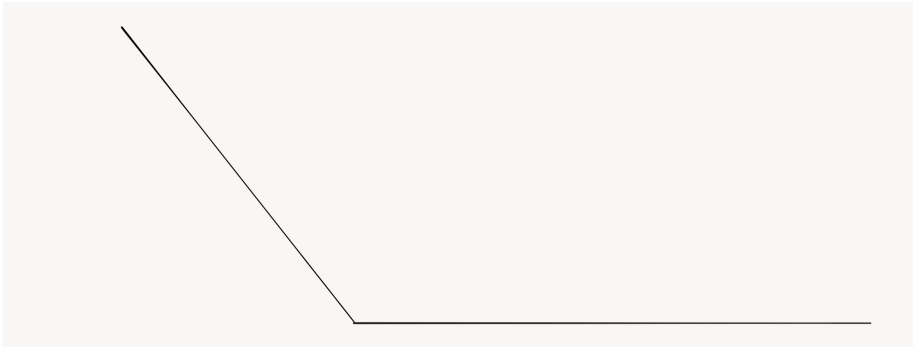
- If we have a point $x = (x_1, \ldots, x_k, 0, \ldots, 0)$ we only observe k of the linear functions!
- We need to observe them all to find the minimizer. We need many to approximate it.
- Leads to $T = \Omega(\frac{G^2 D^2}{\varepsilon^2})$ as long as this is $< d$.

Moreau Envelope: Smoothing the Hard Instance

$$M_{\lambda,f}(x) \stackrel{\text{def}}{=} \min_y \left\{ f(y) + \frac{1}{2\lambda}\|y-x\|_2^2 \right\}; \quad \text{prox}_{\lambda,f}(x) \stackrel{\text{def}}{=} \arg\min_y \left\{ f(y) + \frac{1}{2\lambda}\|y-x\|_2^2 \right\}.$$

By optimality $\nabla_y \left(f(y) + \frac{1}{2\lambda}\|y-x\|_2^2 \right) (\text{prox}(x)) = 0$, so

$\text{prox}(x) = x - \lambda \nabla f(\text{prox}(x))$, i.e., implicit Gradient Descent. And minimizers are preserved.



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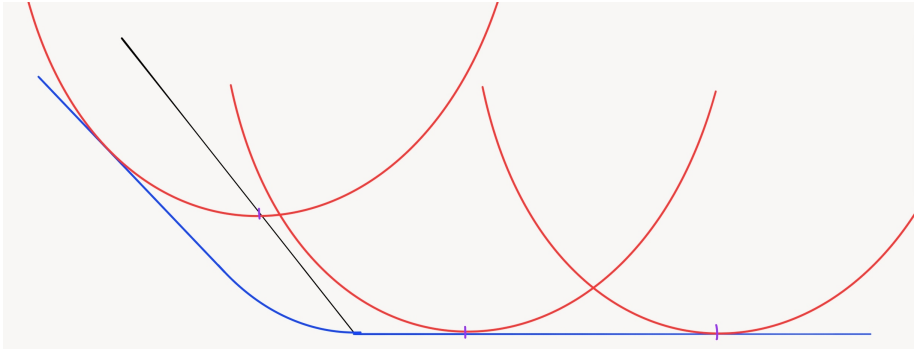


Moreau Envelope: Smoothing the Hard Instance

Goal. Lower bound $T = \Omega(\sqrt{L}R/\sqrt{\varepsilon})$ for L-smooth f on $\|x\| \le R$.
Properties:

1. Well-defined & *jointly* strongly convex $\implies$ unique prox.
2. $M_{f,\lambda}$ is convex (marginalization of a jointly convex function).
3. *Surrogate:*
$$M_{f,\lambda}(x) \le f(x), \quad M_{f,\lambda}(x^\star) = f(x^\star) \implies \mathrm{prox}(x^\star) = x^\star.$$
4. $\nabla M_{f,\lambda}(x) = \frac{1}{\lambda}\left(x - \mathrm{prox}_{f,\lambda}(x)\right)$.
5. $M_{f,\lambda}(x)$ is $\frac{1}{\lambda}$-smooth.
6. Local if f is G-Lipschitz: depends only on f near x (radius $O(\lambda G)$).
7. Close to f if f is G-Lipschitz: $|M_{f,\lambda}(x) - f(x)| \le G^2\lambda$.

Apply it to the hard Lipschitz instance and obtain a hard smooth instance.

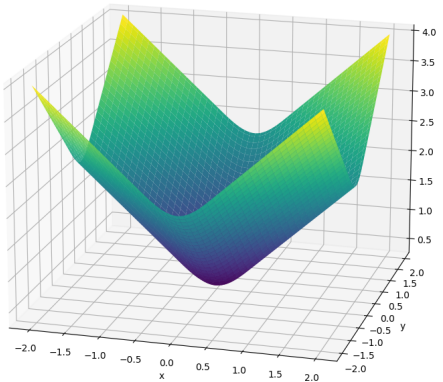
Lower Bound Techniques: Randomized Smoothing

- Randomized Smoothing of G-Lipschitz f wrt $\|\cdot\|$:

$$S_\beta[f](x) \stackrel{\text{def}}{=} \mathbb{E}_{v\sim\nu_{B_{\|\cdot\|}(\mathbf{o},\beta)}}[f(x+v)]$$

- $S_\beta[f](x)$ is also G-Lipschitz and its gradient is $\frac{dG}{\beta}$-Lipschitz wrt $\|\cdot\|$.
- $|S_\beta[f](x) - f(x)| \leq \beta G$.
- Since $S_\beta[f](x)$ or $M_{\lambda,f}$ depends on f locally it will preserve local hardness.

Figure: A smoothing of $x \mapsto \|x\|_1$ wrt $\|\cdot\|_1$



Lower Bound Techniques: Softmax and our Final Hard Function

- Let $A(x) = (\langle a^{(1)}, x\rangle, \ldots, \langle a^{(d)}, x\rangle)$, with $\|a^{(i)}\|_* \leq 1$.
- Define the softmax function as $\mathrm{smax}_\mu(x) \stackrel{\mathrm{def}}{=} \mu \ln\left(\sum_{j=1}^{d} \exp(x_i/\mu)\right)$.

Lower Bound Techniques: Softmax and our Final Hard Function

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- $\mathrm{smax}_\mu(Ax)$ is 1-Lipschitz wrt $\|\cdot\|$.
- $\nabla^q \mathrm{smax}_\mu$ is $\widetilde{O}_q(\frac{1}{\mu^q})$-Lipschitz wrt $\|\cdot\|$.

Lower Bound Techniques: Softmax and our Final Hard Function

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- $\mathrm{smax}_\mu(Ax)$ is 1-Lipschitz wrt $\|\cdot\|$.
- $\nabla^q \mathrm{smax}_\mu$ is $\widetilde{O}_q(\frac{1}{\mu^q})$-Lipschitz wrt $\|\cdot\|$.
- $f_i \equiv$ softmax of $(Ax)_1 - \gamma, \ldots, (Ax)_i - i\gamma$, for some $\gamma > 0$, up to some shifts.
- $h(x) \stackrel{\text{def}}{=} \max_{i\in[T]} f_i(x)$.
- Hard function $g(x) = (S_{\beta/2^q} \circ S_{\beta/2^{q-1}} \circ \cdots \circ S_{\beta/2})(h)$.

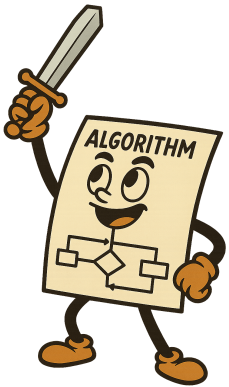
Convergence Results

Suppose convexity and $\|\nabla^q f(x) - \nabla^q f(y)\|_* \leq L_q \|x - y\|$.
Initial distance: $R_p \overset{\text{def}}{=} \|x_0 - x^*\|_p$; Accuracy: ε;
$m \overset{\text{def}}{=} \max\{2, p\}$. $\widetilde{O}_p(\cdot)$: big-O notation up to log factors and constants on p.
(Contreras et al., 2024).

Algorithm	$p \in [1, \infty)$	$p = \infty$
Accelerated ($q < \infty$)	$\widetilde{O}_{q,p}\left(\left(\frac{LR_p^{q+\mathbf{1}}}{\varepsilon}\right)^{\frac{m}{(m+\mathbf{1})(q+\mathbf{1})-m}}\right)$	–
Unaccelerated ($q < \infty$)	$\widetilde{O}_q\left(\left(\frac{LR_p^{q+\mathbf{1}}}{\varepsilon}\right)^{\frac{\mathbf{1}}{q}}\right)$	

Corresponding Lower Bounds

If f convex and $\|\nabla^q f(x) - \nabla^q f(y)\|_* \leq L_q \|x - y\|$ (Contreras et al., 2024):



Algorithms	Lower Bounds
q-th order oracle	Local oracle
Deterministic	Possibly random algs
Single-call per round	$\text{poly}(d)$ parallel queries
Any norm	Any norm
$\|\cdot\|_p$-setting: they match up to log factors	

Overview

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Optimizers in Deep Learning 0

- **The following is an incomplete overview.** There are many more important methods.
- We will use $f_t \stackrel{\text{def}}{=} \frac{1}{m} \sum_{i=1}^{m} f^{(j_i)}$ to represent a minibatch.
- We will use capital letters $G_t \leftarrow \nabla f_t$ to represent the gradient of f_t grouped as a collection of matrices, one per-layer.

Optimizers in Deep Learning 1

(Robbins and Monro, 1951, SGD)

$$g_t \leftarrow x_{t-1} - \eta_t \nabla f_t(x_{t-1})$$

Simple, effective.

(McMahan and Streeter, 2010; Duchi et al., 2011, AdaGrad). For a minibatch f_t:

$$
\begin{aligned}
g_t &\leftarrow \nabla f_t(x_{t-1}) \\
v_t &\leftarrow v_{t-1} + g_t^2 \\
x_t &\leftarrow x_{t-1} - \eta \frac{g_t}{\sqrt{v_t} + \varepsilon}
\end{aligned}
$$

Theorem (AdaGrad's Guarantee)

$$
\text{Regret}_T \leq \max_{t \leq T} \|x_t - x^*\|_\infty d^{1/2} \sqrt{\inf \left\{ \sum_{t=1}^{T} \langle g_t, \text{diag}(s)^{-1} g_t \rangle : s \succcurlyeq 0, \langle 1, s \rangle \leq d \right\}},
$$

where x^* is the best fixed action defining the regret.

Optimizers in Deep Learning 3

(Tieleman and Hinton, 2012, RMSProp)

- Version of Adagrad but with v_t accumulating with exponential average (forget distant past).
- β is chosen to be close to 1.

$$
\begin{aligned}
g_t &\leftarrow \nabla_\theta f_t(\theta_{t-1}) \\
v_t &\leftarrow \beta v_{t-1} + (1-\beta) g_t^2 \\
\theta_t &\leftarrow \theta_{t-1} - \eta \frac{g_t}{\sqrt{v_t} + \varepsilon}
\end{aligned}
$$

Optimizers in Deep Learning 4

(Kingma and Ba, 2014, Adam)

- Mutation of Adagrad and RMSProp to incorporate momentum in the numerator.
- It also contains a bias correction factor.
- Only cited 221,892 times (by night August 7, 2025). Only 222,029 in the morning of August 8.

$$
\begin{aligned}
g_t &\leftarrow \nabla f_t(x_{t-1}) \\
m_t &\leftarrow \beta_1 m_{t-1} + (1-\beta_1) g_t; \quad \hat{m}_t \leftarrow m_t/(1-\beta_1^t) \\
v_t &\leftarrow \beta_2 v_{t-1} + (1-\beta_2) g_t^2; \quad \hat{v}_t \leftarrow v_t/(1-\beta_2^t) \\
x_t &\leftarrow x_{t-1} - \eta \cdot \hat{m}_t/(\sqrt{\hat{v}_t} + \varepsilon)
\end{aligned}
$$

- Coordinatewise scale invariance.

Optimizers in Deep Learning 5

(Loshchilov and Hutter, 2017, AdamW) decoupling momentum from weight decay (this is like proximal gradient). Leads to better regularization.

Algorithm 2 Adam with L_2 regularization and Adam with decoupled weight decay (AdamW)

given $\alpha = 0.001, \beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 10^{-8}, \lambda \in \mathbb{R}$
initialize time step $t \leftarrow 0$, parameter vector $\boldsymbol{\theta}_{t=0} \in \mathbb{R}^n$, first moment vector $\boldsymbol{m}_{t=0} \leftarrow \boldsymbol{0}$, second moment vector $\boldsymbol{v}_{t=0} \leftarrow \boldsymbol{0}$, schedule multiplier $\eta_{t=0} \in \mathbb{R}$
repeat
  $t \leftarrow t + 1$
  $\nabla f_t(\boldsymbol{\theta}_{t-1}) \leftarrow \text{SelectBatch}(\boldsymbol{\theta}_{t-1})$ ▷ select batch and return the corresponding gradient
  $\boldsymbol{g}_t \leftarrow \nabla f_t(\boldsymbol{\theta}_{t-1}) + \lambda \boldsymbol{\theta}_{t-1}$
  $\boldsymbol{m}_t \leftarrow \beta_1 \boldsymbol{m}_{t-1} + (1 - \beta_1)\boldsymbol{g}_t$ ▷ here and below all operations are element-wise
  $\boldsymbol{v}_t \leftarrow \beta_2 \boldsymbol{v}_{t-1} + (1 - \beta_2)\boldsymbol{g}_t^2$
  $\hat{\boldsymbol{m}}_t \leftarrow \boldsymbol{m}_t / (1 - \beta_1^t)$ ▷ β_1 is taken to the power of t
  $\hat{\boldsymbol{v}}_t \leftarrow \boldsymbol{v}_t / (1 - \beta_2^t)$ ▷ β_2 is taken to the power of t
  $\eta_t \leftarrow \text{SetScheduleMultiplier}(t)$ ▷ can be fixed, decay, or also be used for warm restarts
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_{t-1} - \eta_t \left(\alpha \hat{\boldsymbol{m}}_t / (\sqrt{\hat{\boldsymbol{v}}_t} + \epsilon) + \lambda \boldsymbol{\theta}_{t-1} \right)$
until *stopping criterion is met*
return optimized parameters $\boldsymbol{\theta}_t$

(Gupta et al., 2018, SHAMPOO). This is a of normalizing the per-layer gradient matrices. Normalized subgradient descent $x_t \leftarrow x_t - \frac{g_t}{\|g_t\|}$ was used as early as in (Nesterov, 1998, Theorem 3.2.2) to obtain rates for convex non-smooth methods without knowledge of the (local) Lipschitz constant.

$$
\begin{aligned}
G_t &\leftarrow \nabla f_t(W_t) \in \mathbb{R}^{m \times n} \\
L_t &\leftarrow L_{t-1} + G_t \, G_t^\top \\
R_t &\leftarrow R_{t-1} + G_t^\top \, G_t \\
W_{t+1} &\leftarrow W_t - \eta \, L_t^{-\frac{1}{4}} \, G_t \, R_t^{-\frac{1}{4}}
\end{aligned}
$$

Lion (Chen et al., 2023)
More direct normalized gradient descent, with momentum.

$$
\begin{aligned}
g_t &\leftarrow \nabla f_t(x_{t-1}) \\
x_t &\leftarrow x_{t-1} - \eta \cdot \operatorname{sign}(\beta_2 m_{t-1} + (1-\beta_2) g_t) \\
m_t &\leftarrow \beta_1 m_{t-1} + (1-\beta_1) g_t
\end{aligned}
$$

Optimizers in Deep Learning 8

(Jordan, 2025, Muon)

- Muon is an optimizer by layers, where G_t represents a collection of 2D gradients. And starts with a momentum parameter $B_0 \leftarrow 0$.
- The update is approximately orthogonalized before being used.

$$
\begin{aligned}
G_t &\leftarrow \nabla f_t(x_{t-1}) \\
B_t &\leftarrow \mu B_{t-1} + G_t \\
O_t &\leftarrow \text{NewtonSchultz5}(B_t) \\
\theta_t &\leftarrow \theta_{t-1} - \eta O_t
\end{aligned}
$$

- Layer orthogonalization explored before by, e.g. (Arjovsky et al., 2016; Mhammedi et al., 2017; Maduranga et al., 2019; Lezcano-Casado and Martínez-Rubio, 2019), in the context of RNNs for preventing vanishing or exploding gradients for long sequences.
- Current models are very deep, and layer orthogonalization also becomes a very useful tool. It also acts as a normalization step. The algorithm is scale invariant.

Thank you!

Questions?

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