11/01/22

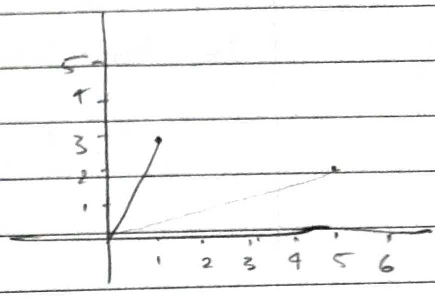
MODULE : 3

v. imp ~~1.~~ SVD
2. PCA
3. Least sq. app.

* Eigen value and Eigen Vector :

Eg. $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$Ax = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$



Two things — (while operating a matrix)
1. stretching
2. Rotation

$Ax = \lambda x$

$x \rightarrow$ Eigenvector.
There's only change in scaling not direction
While operating length changes $\longrightarrow$ eigenvectors.

Def: Let A be a square matrix, the vector x is known as eigen vector if $Ax = \lambda x$, where λ is known as eigen value.

Example — Find an eigen value & eigen vector for the matrix $A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

$\longrightarrow$ Let x be the eigen vector

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$(A - \lambda I)x = 0 \quad \text{—— (1)}$$

$$A - \lambda I = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{bmatrix}$$

For eigen value, consider $|A - \lambda I| = 0$

$$\therefore |A - \lambda I| = (25-\lambda)^2 - 20^2 = 0$$

$$= \cancel{25^2 + \lambda^2 - 50\lambda - 400}$$

$$\Rightarrow (25 - \lambda + 20)(25 - \lambda - 20) = 0$$

$$\Rightarrow (45-\lambda)(5-\lambda) = 0$$

$$\therefore \lambda = 5, 45$$

As X is a eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\therefore$ WKT, $AX = \lambda X \Rightarrow [A - \lambda I]X = 0$

$$\begin{bmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} (25-\lambda)x_1 + 20x_2 &= 0 \\ 20x_1 + (25-\lambda)x_2 &= 0 \end{aligned}\right\} \text{— (A)}$$

Put $\lambda = 45$ in (A), $\left.\begin{aligned} -20x_1 + 20x_2 &= 0 \\ 20x_1 - 20x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 - x_2 = 0$

no. of free variables $= 2 - 1 = 1$

take x_2 as independent and select $x_2 = 1$, $\therefore x_1 = 1$

$\therefore$ eigen vector corresponding to $\lambda = 45$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Put $\lambda = 5$ in (A), $\left.\begin{aligned} 20x_1 + 20x_2 &= 0 \\ 20x_1 + 20x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 + x_2 = 0$

$\therefore$ select x_2 as independent, $x_2 = -1$, $\therefore x_1 = 1$

$\therefore$ Eigen vector corresponding to $\lambda = 5$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

* SVD : Singular Value Decomposition

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$x = r\cos\theta$
$y = r\sin\theta$

Orthogonal matrix

$$= r \begin{pmatrix} & '' & \end{pmatrix} \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$ Dot product 1

$$= r \begin{pmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \end{pmatrix}$$

On operating a Matrix on a vector

Rotational - ~~Matrix that~~ Vector rotates

Stretching - vector stretched

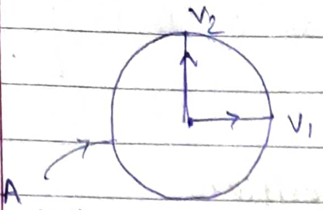
Original -> Rotational + Stretching

Matrix can be decomposed into rotational and stretching matrix in SVD.

- On operating any Orthogonal Matrix on a vector, only Rotational will take place & no stretching
- On operating any Diagonal Matrix on a vector, only Stretching will take place & no rotation

Session 26

13/01/2022



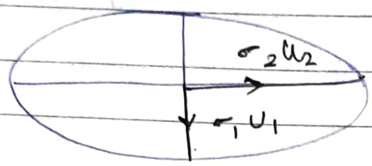
v_1 & v_2 are orthonormal basis

(operating matrix A)

$\Downarrow$

on rotation & stretching

$\boxed{\text{Ellipse}}$



(Jitne angle se v_2 rotate hua utne se hi v_1 bhi hua
$\Rightarrow$ orthonormality maintained)

$v_1, v_2 \longrightarrow u_1, u_2$ $\quad$ σ_1, σ_2 } stretch vectors (singular values)

- On operating A,

$AV_1 \longrightarrow \sigma_1 u_1$

$AV_2 \longrightarrow \sigma_2 u_2$

$u_1, u_2 \in \mathbb{R}^2$ (2-D)

- When A $\rightarrow$ hyper sphere (n-dimension) $\Rightarrow$ hyper ellipse (n-dim)

$$\left.\begin{aligned} AV_1 &\rightarrow \sigma_1 U_1 \\ AV_2 &\rightarrow \sigma_2 U_2 \\ &\vdots \\ AV_n &\rightarrow \sigma_n U_n \end{aligned}\right\} \text{SVD in vector form} \quad \text{I}$$

$$A\begin{bmatrix} V_1, V_2 & \cdots & V_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & U_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \cdots \\ 0 & \sigma_2 & 0 \cdots \\ 0 & \cdots & \sigma_n \end{bmatrix}$$

Example:

- SVD in Vector form :

$$AV_1 \longrightarrow \sigma_1 u_1 \qquad \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$AV_2 \longrightarrow \sigma_2 u_2 \qquad \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} = 3 \begin{bmatrix} f_3 \\ f_4 \end{bmatrix}$$

Eq^{n}s obtained on solving :

$$2e_1 + e_2 = 2f_1$$
$$e_2 = 2f_2$$
$$2e_3 + e_4 = 3f_3$$
$$e_4 = 3f_4$$

- SVD in ______ form :

$$A \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix} = \begin{bmatrix} f_1 & f_3 \\ f_2 & f_4 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 & e_3 \\ e_2 & e_4 \end{bmatrix} = \begin{bmatrix} f_1 & f_3 \\ f_2 & f_4 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2e_1 + e_2 & 2e_3 + e_4 \\ e_2 & e_4 \end{bmatrix} = \begin{bmatrix} 2f_1 & 3f_3 \\ 2f_2 & 3f_4 \end{bmatrix}$$

$$\boxed{A_{m\times n} V_{n\times n} = U_{m\times n} \Sigma_{n\times n}} \rightarrow \text{SVD}$$

$\hookrightarrow$ Orthogonal Matrix ($V^{-1} = V^T$)

To maintain V's orthoganality, we need to make changes in U & diag. matrix -

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Page No.:
Date:
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$$\boxed{A_{m\times n} V_{n\times n} = U_{m\times m} \Sigma_{m\times n}}$$

Multiply by V^{-1}, $A V V^{-1} = U \Sigma V^{-1}$

$$\boxed{A = U \Sigma V^T}$$ reduced SVD

$$\begin{bmatrix} & \end{bmatrix}_{m\times n} = \begin{bmatrix} u_1 & u_2 & \cdots \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots \\ 0 & \sigma_2 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{bmatrix}^T$$

$$= u_1 \sigma_1 v_1^T + u_2 \sigma_2 v_2^T + \cdots$$

Session - 27
17/01/2022

Consider $A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$

$$= (V^T)^T \Sigma^T U^T \, U \Sigma V^T$$

$$= V \Sigma^T U^T U \Sigma V^T \quad (\because U \text{ is an orthogonal matrix})$$

$$= V \Sigma^T \Sigma V^T$$

$$A^T A = V \Sigma^2 V^T$$

$$(A^T A) V = V \Sigma^2 V^T V$$

$$(A^T A) V = V \Sigma^2$$

From the above eqn, it is clear that V_i is an eigenvector corresponding to Σ^2

By vector form of SVD,

V_i is an eigen vector ~~the matrix A^T~~ corresponding to σ_i^2 from the matrix A^TA.

Similarly U_i is an eigen vector corresponding to eigen ~~vector~~ values σ_i^2 from the matrix (AA^T)

A & A^T $\rightarrow$ same eigen values as they are transpose of each other

Symmetric Matrix $\longrightarrow$ Real & Distinct Eigen values

Q1. Find the SVD of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

$\rightarrow$ SVD of $A = U \Sigma V^T$

$$\boxed{A_{m \times n} = U_{m \times m} \Sigma_{m \times n} T_{n \times n}}$$

- WKT, V is an orthogonal matrix, whose ~~set~~ col. vectors are eigen vector of matrix A^T

$A^T = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$

~~$AA^TA = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$~~ $= \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

$A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

- $[B - \lambda I] = \begin{bmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{bmatrix}$

$|B - \lambda I| = 0 \Rightarrow (25-\lambda)^2 - 20^2 = 0$

$(25-\lambda-20)(25-\lambda+20) = 0 \Rightarrow \lambda = 5, 45.$

always give σ_1 as the highest value

$\sigma_1^2 = 45$ $\quad\quad$ $\sigma_2^2 = 5$

$\therefore$ Singular values are $\quad \sigma_1 = \sqrt{45}$, $\sigma_2 = \sqrt{5}$

$$\Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

let v_1 be the eigen vector of ~~matrix~~ A^TA corresponding to

$\lambda = 25$

If x is an eigenvector of matrix B { }

$\Rightarrow [B - \lambda I] x = 0$; $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (25-\lambda)v_1 + 20v_2 &= 0 \\ 20v_1 + (25-\lambda)v_2 &= 0 \end{aligned}\right\} (A)$$

Put $\lambda = 45$ in (A)

$$\left.\begin{aligned} -20v_1 + 20v_2 &= 0 \\ 20v_1 - 20v_2 &= 0 \end{aligned}\right\} v_1 - v_2 = 0$$

no. of free variables = no. of variables − num of eqs
$= 2 - 1(=1)$

let v_2 be ~~an indept~~ an independent, $v_2 = 1$, i.e. $v_1 = 1$

$\therefore$ 1st Eigen vector $\quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now put $\lambda = 5$ in (A) $\quad$ $\left.\begin{aligned} 20v_1 + 20v_2 &= 0 \\ 20v_1 + 20v_2 &= 0 \end{aligned}\right\} \Rightarrow v_1 + v_2 = 0$

Put $v_2 = \pm 1$, $\therefore v_1 = -1$

$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

normalize $v_2 = \hat{v_2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

By using vector form of SVD, we can write

$AV_1 = \sigma_1 u_1 \Rightarrow$

$\Rightarrow \boxed{u_1 = \frac{1}{\sigma_1} AV_1}$

$u_1 = \frac{1}{\sqrt{45}} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \frac{1}{\sqrt{45}} \begin{bmatrix} 7/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}$

$= \frac{1}{3\sqrt{10}} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

$\therefore U = \begin{bmatrix} 7/3\sqrt{10} & 1/\sqrt{10} \\ 5/3\sqrt{10} & \sqrt{5}/\sqrt{10} \end{bmatrix}$

* Properties of SVD :

(1) Product of singular values = Det. of Matrix (For sq. matrix)

(2) $\sigma_1^2 + \sigma_2^2 + \dots =$ sum of Sq. of each entry of given matrix

18/07/2022
Session 28: (Lect 21)

H.W. Q2) Find SVD of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

→ WKT, Singular value decomposition of A is $U \Sigma V^T$ where U is the orthogonal matrix of order 3×3 & order of Σ is 3×1 & order of V is 1×1

We know that, V is a matrix whose column's are eigen vectors of matrix A^TA.

$\therefore A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = [1+4+4] = [9]$

For getting eigen values of A^TA; consider $[A^TA - \lambda I]$
$= [9 - \lambda]$

$\therefore$ for eigen value, consider $|9-\lambda| = 0$

$\Rightarrow \lambda = 9 = \sigma_1^2$

$\therefore$ singular value is $\sigma_1 = 3$

Let x be an eigen vector corresponding to matrix A^TA & eigen value 9

$\therefore [A^TA - \lambda I]x = 0$

$\Rightarrow [9-\lambda]x_1 = 0$

$\Rightarrow$ put $\lambda = 9$, $(9-9)x_1 = 0$

Select $x_1 = 1$

$v_1 = [1]$

For finding U_1, U_2, U_3 use vectors from SVD

$AV_1 = \sigma_1 U_1 \Rightarrow u_1 = \frac{1}{\sigma_1} AV_1$

$$u_1 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}[1]$$

$$U_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad U_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \quad U_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

(Q3) Find SVD of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$\longrightarrow$ $\underset{2\times 3}{A} = \underset{2\times 2}{U} \underset{2\times 3}{\Sigma} \underset{3\times 3}{V^T}$

Order of matrix $A^TA \to 3\times3$, order of matrix $AA^T \to 2\times2$

Since order of AA^T is smaller than A^TA,

$\therefore$ we will find singular values by using matrix AA^T.

$$AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\therefore$ Characteristic matrix is $[AA^T - \lambda I] = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

Char. eq^n is $|AA^T - \lambda I| = 0$

$$\therefore \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)^2 - 1 = 0 \quad \Rightarrow (2-\lambda-1)(2-\lambda+1) = 0$

$(1-\lambda)(3-\lambda) = 0$

$\lambda = 1, 3$

$\therefore$ Singular values are $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$, $\sigma_3 = 0$

WKT, V is orthogonal matrix whose column's are eigen vectors of the matrix A^TA & eigen values are 3, 1, 0.

$$A^TA = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Let x be an eigen vector

$$\therefore [A^TA - \lambda I]x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} (1-\lambda)x_1 - x_2 &= 0 \\ -x_1 + (2-\lambda)x_2 - x_3 &= 0 \\ -x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} (A)$$

- put $\lambda = 3$ in (A)

$$-2x_1 - x_2 = 0 \quad —(1)$$
$$-x_1 - x_2 - x_3 = 0 \quad —(2)$$
$$-x_2 - 2x_3 = 0 \quad —(3)$$

* Here all eqns are distinct

(1) − (2) ⇒ $-2x_1 + 2x_3 = 0$ ⇒ $x_1 - x_3 = 0$ —(4)

& $-x_1 - x_2 - x_3 = 0$ —(5)

(4)+(5) ⇒ $-x_2 - 2x_3 = 0$

One eqn two variables

∴ put $x_3 = 1$, $x_2 = -2$, ∴ $x_1 = 1$

$$\therefore x = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

- put $\lambda = 1$ in (A)

$$x_2 = 0$$
$$-x_1 - x_3 = 0 \Rightarrow x_1 + x_3 = 0$$

choose $x_3 = 1$; ∴ $x_1 = -1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

- For V_3, put $\lambda = 0$ in (A)

$$x_1 - x_2 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$
$$-x_2 + x_3 = 0$$

* All 3 eqns are distinct

$(1)-(3) \Rightarrow x_1 - x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$2x_2 - 2x_3 = 0$

$\Rightarrow$ on adding both eqn

put $x_1 = 1, x_2 = 1, x_3 = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

By using SVD in vector form, we're $Av_1 = \sigma_1 u_1$

- $u_1 = \frac{1}{\sigma_1} Av_1$

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$$

- For v_2, consider $Av_2 = \sigma_2 u_2$

$\sigma_2 = 1$

$$\therefore u_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \boxed{U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix}$$

20/01/2022 Lect 22 (Session 29)

Least Square Method -

(I) Consider a linear system of eq^ns
$AX = b$; where matrix A is a sq matrix of order 'n' & it is non-singular matrix

$\therefore$ A inverse exist
$\therefore$ Solution of given system of eq^n is $X = A^{-1} b$

(II) If A is a 'm×n' matrix $(m > n)$ { means more eq^ns than unknowns
This type of system of eq^ns are known as ~~overdeter~~ overdetermined system.

Here we have to find out approximate solution which minimize $\|AX - b\|^2$

$AX = b \to$ Norm
$\|AX - b\| \to$ min
$\|AX - b\|^2 \to$ min

$x_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\|x_1\| = \sqrt{x_1^2 + x_2^2}$
$\|x_1\|^2 = x_1^2 + x_2^2$
$= [x_1 \; x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$= x^T x$

Let $E = \|AX - b\|^2$
$= (AX - b)^T (AX - b)$
$= ((AX)^T - b^T)(AX - b)$
$= (AX)^T (AX) - b^T (AX) - (AX)^T b + b^T b$
$E = X^T A^T A X - b^T (AX) - (AX)^T b + b^T b$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
$x^T y = y^T x$
$[x_1 \; x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [y_1 \; y_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\Rightarrow x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2$

WKT, if E is minimum at some vector X then at that vector $\frac{\partial E}{\partial X} = 0$

$E = X^T A^T A X - 2 (AX)^T b + b^T b$

But at point of extremum $\frac{\partial E}{\partial x} = 0$

$2 A^T A X - 2 A^T b = 0$
$A^T A X - A^T b = 0$

$\therefore \quad A^TAx = A^Tb$

$\therefore x = (A^TA)^{-1} A^T b$ (provided $(A^TA)^{-1}$ exist)

let $\boxed{A^+ = (A^TA)^{-1} A^T}$

$\therefore \boxed{X = A^+ b}$ is called least square approximation of overdetermined system.

— this A^+ is known as Pseudoinverse of A.

Session-30
24/01/2022

Example – Fit the lines for the following data points:
(1, 2), (2, 3) and (3, 5)

(1, 2)
(2, 3)
(3, 5)

$\Rightarrow$ let $y = mx + c$ be a line which is best fit for given data

$$\left.\begin{aligned} 2 &= m + c \\ 3 &= 2m + c \\ 5 &= 3m + c \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}; \quad x = \begin{bmatrix} m \\ c \end{bmatrix}; \quad b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Clearly, this is an overdetermined linear system of eq^{n}s

$\therefore x = ((A^TA)^{-1} A^T) b$ —— (1)

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}; \quad \therefore A^TA = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}_{2\times 2}$$

Page No.:
Date:

(1) $\Rightarrow X = \frac{1}{6}\begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

$= \frac{\cancel{3}}{\cancel{6}}\begin{bmatrix} 1/2 & -1 \\ -1 & 7/3 \end{bmatrix}\begin{bmatrix} 23 \\ 10 \end{bmatrix}$

$= \begin{bmatrix} 23/2 - 10 \\ -23 + 70/3 \end{bmatrix}$

$\therefore X = \begin{bmatrix} 3/2 \\ 1/3 \end{bmatrix}$

$\therefore m = 3/2 \ ; \ c = 1/3$

$\therefore$ Best fit line is $y = \frac{3}{2}x + \frac{1}{3}$

(2) Solve $\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$ $\quad X = ((A^TA)^{-1}A^T)b$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix}$ $\quad X = \begin{bmatrix} m \\ c \end{bmatrix}$ $\quad b = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 7 & 29 \end{bmatrix}$

58-49

9

$X = \frac{1}{9}\begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$

$$X = \frac{1}{9}\begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 23 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 29\times 7 - 7\times 23 \\ -49 + 46 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 42 \\ -3 \end{bmatrix} = \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 23 \end{bmatrix} = \begin{bmatrix} 29(7) - 23(7) \\ -49 + 46 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6\times 7 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix}$$

∴ Best fit line is $y = \_\,x + \_$

III Consider a system $AX = b$ where A is $m\times n$ matrix and $m<n$. Such system of eq^ns are known as under-determined system of eq^ns.

Eg. $2x + 4y + 5z = 6$, $x + y + z = 3$ } ∞ sol^ns.

$\|x - 0\| = \|x\|^2$

This system will have infinite solutions.
Now, we want to pick one of these sol^ns by finding smaller one.

i.e., Here we've to minimize $\|x\|^2$ subject to $Ax = b$

$F = f + \lambda g$ By using method of Langrange multiplier, we have to minimize

$$E = \|x\|^2 + \lambda^T(b - Ax)$$
$$= \|x\|^2 + (b - Ax)^T\lambda$$

WKT, the condⁿ for extrema is $\frac{\partial E}{\partial x} = 0$

$2X - A^T\lambda = 0$ —— (1)

premultiply by A

$2AX - (AA^T)\lambda = 0$

But $AX = b$

$\therefore \Rightarrow 2b - (AA^T)\lambda = 0$

$\Rightarrow \lambda = 2(AA^T)^{-1} b$

put λ in (1),

$2X - A^T(2(AA^T)^{-1} b) = 0$

$X = A^T(AA^T)^{-1} b$

let $\boxed{\begin{array}{l} A^+ = A^T(AA^T)^{-1} \\ \therefore X = A^+ b \end{array}}$

This A^+ is known as pseudo-inverse of matrix

Example - Find the least sq. approximation of linear system

$2x_1 + x_2 + x_3 = 4$

$2x_1 - x_2 + x_3 = 2$

→ The matrix form is $\begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

WKT, LSA solⁿ is $X = A^+B$

Since System is under-determined

$$A^+ = A^T(AA^T)^{-1}$$

$$\therefore AA^T = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$$

$$\therefore (AA^T)^{-1} = \frac{1}{20}\begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}$$

$$X = \frac{1}{20}\begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 4 & 4 \\ 10 & -10 \\ 2 & 2 \end{bmatrix}_{3\times 2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.2 \\ 1 \\ 0.6 \end{bmatrix}$$

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* WKT, whenever system is overdetermined then sol^n is given by

$X = A^+b$ where $A^+ = (A^TA)^{-1}A^T$

But, if $(A^TA)^{-1}$ does not exist then we can't find x.

then Any linear system can be written as
$Ax = b$ —— (matrix form) where A is matrix of order $m\times n$ $\{m > n\}$

Suppose $X = A^+b$

To get A^+, we will use SVD

By SVD, we can write

$$\boxed{\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^T}_{n\times n}}$$; V & U are orthogonal matrix

$$\therefore \quad A^{-1} = (U\Sigma V^T)^{-1}$$
$$= (V^T)^{-1}(\Sigma)^{-1}U^{-1}$$

But $V^T = V^{-1}$, $A^{-1} = V(\Sigma)^{-1}U^{-1}$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & & & \\ \vdots & & \ddots & & \\ & & & & \frac{1}{\sigma_r} \end{bmatrix}$$

not imp for exam

Example - Fit a line through data points $(1,2)$, $(2,3)$ & $(3,5)$

$\Rightarrow$ Let the eq^n of line

$y = mx + c$

$2 = m + c$
$3 = 2m + c$
$5 = 3m + c$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

SVD of $A = U\Sigma V^T$

$$U = \begin{bmatrix} -0.3231 & 0.8538 & 0.4082 \\ -0.5475 & 0.1832 & -0.8165 \\ -0.7719 & -0.4873 & 0.4082 \end{bmatrix}$$

$$S = \begin{bmatrix} 4.0791 & 0 \\ 0 & 0.6005 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.9573 & -0.4027 \\ -0.4027 & 0.9153 \end{bmatrix}$$

$$X = V \Sigma^{+} V^{T}$$

$$= \begin{bmatrix} -0.9573 & -0.4027 \\ -0.4027 & 0.9153 \end{bmatrix} \begin{bmatrix} \frac{1}{4.0791} & 0 \\ 0 & \frac{1}{0.6005} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.3231 & -0.5475 & -0.7779 \\ 0.8539 & 0.1832 & -0.4875 \\ 0.4082 & -0.8165 & 0.4082 \end{bmatrix}$$