Heap Sort

Heapsort

- It is a well-known, traditional sorting algorithm
- Heapsort is *always* O(n log n)
  - Quicksort is usually O(n log n) but in the worst case slows to $O(n^2)$
  - Quicksort is generally faster, but Heapsort is better in time-critical applications

What is a “heap”?

Definitions of heap:

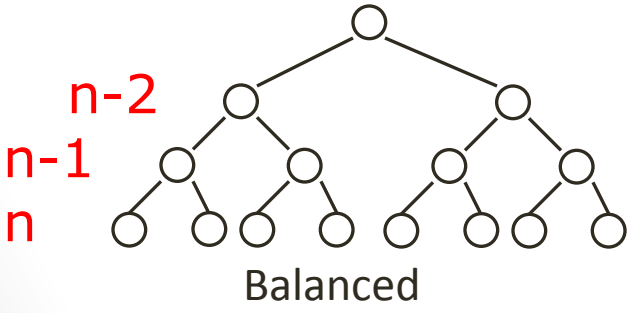
1. A large area of memory from which the programmer can allocate blocks as needed, and deallocate them (or allow them to be garbage collected) when no longer needed
2. A balanced, left-justified binary tree in which no node has a value greater than the value in its parent

These two definitions have little in common

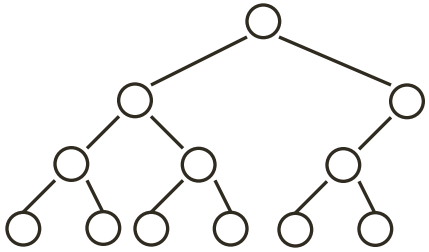
Heap sort uses the second definition

Balanced binary trees

- Concept of Depth:
  - The depth of a node is its distance from the root
  - The depth of a tree is the depth of the deepest node
- A binary tree of depth n is balanced if all the nodes at depths 0 through n-2 have two children

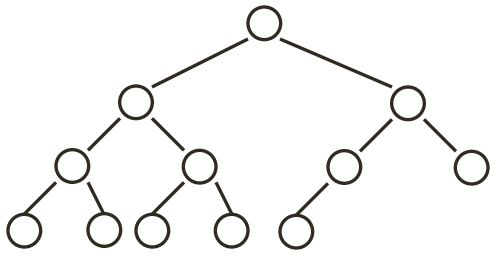


Balanced

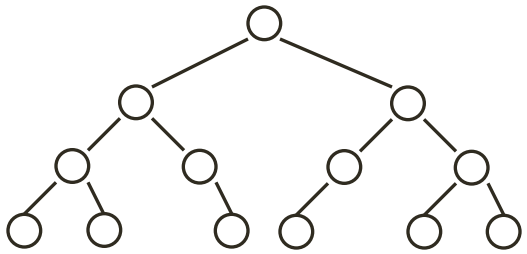

Not balanced

Left-justified binary trees

- A balanced binary tree is left-justified if:
  - all the leaves are at the same depth, or
  - all the leaves at depth `n+1` are to the left of all the nodes at depth `n`



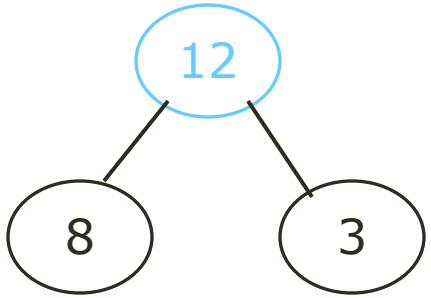
Left-justified



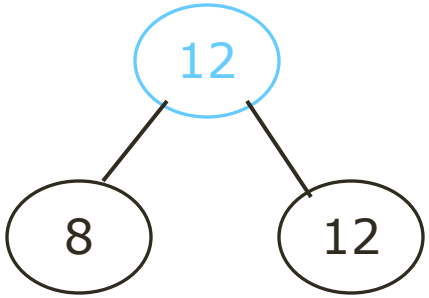
Not left-justified

The heap property

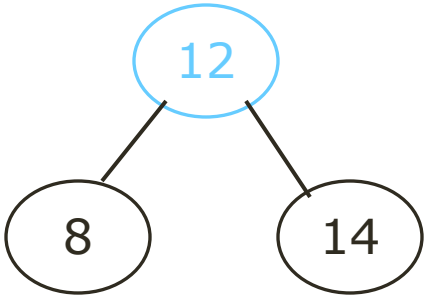
- A node has the heap property if the value in the node is as large as or larger than the values in its children



Blue node has heap property



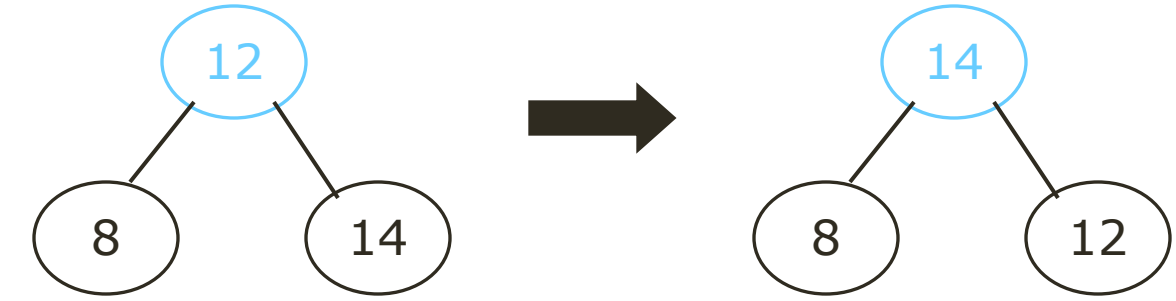
Blue node has heap property



Blue node does not have heap property

- All leaf nodes automatically have the heap property
- A binary tree is a heap if *all* nodes in it have the heap property

- Given a node that does not have the heap property, you can give it the heap property by exchanging its value with the value of the larger child



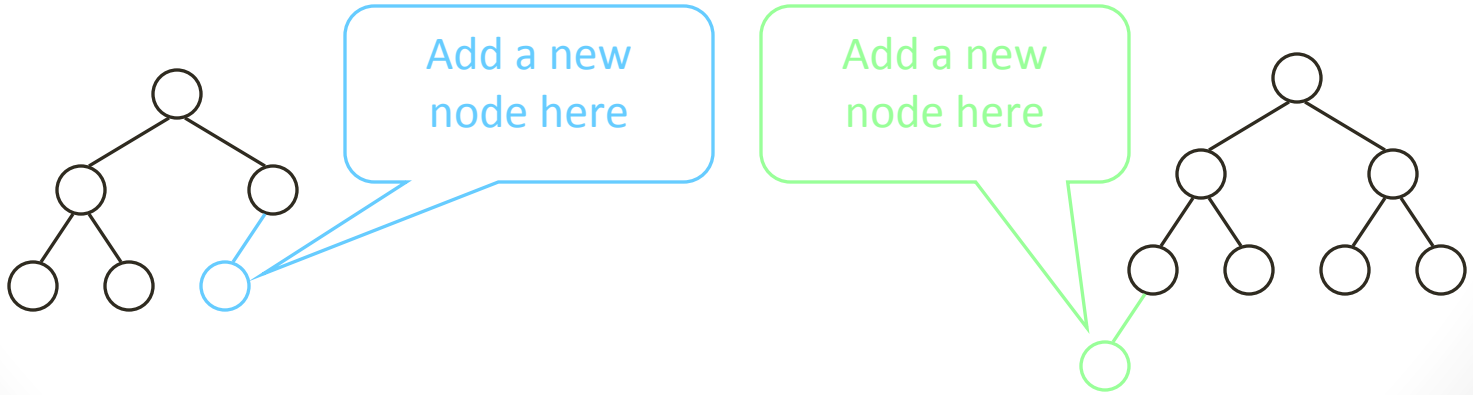
Blue node does not have heap property

Blue node has heap property

- This is sometimes called sifting up
- Notice that the child may have *lost* the heap property

Constructing a heap

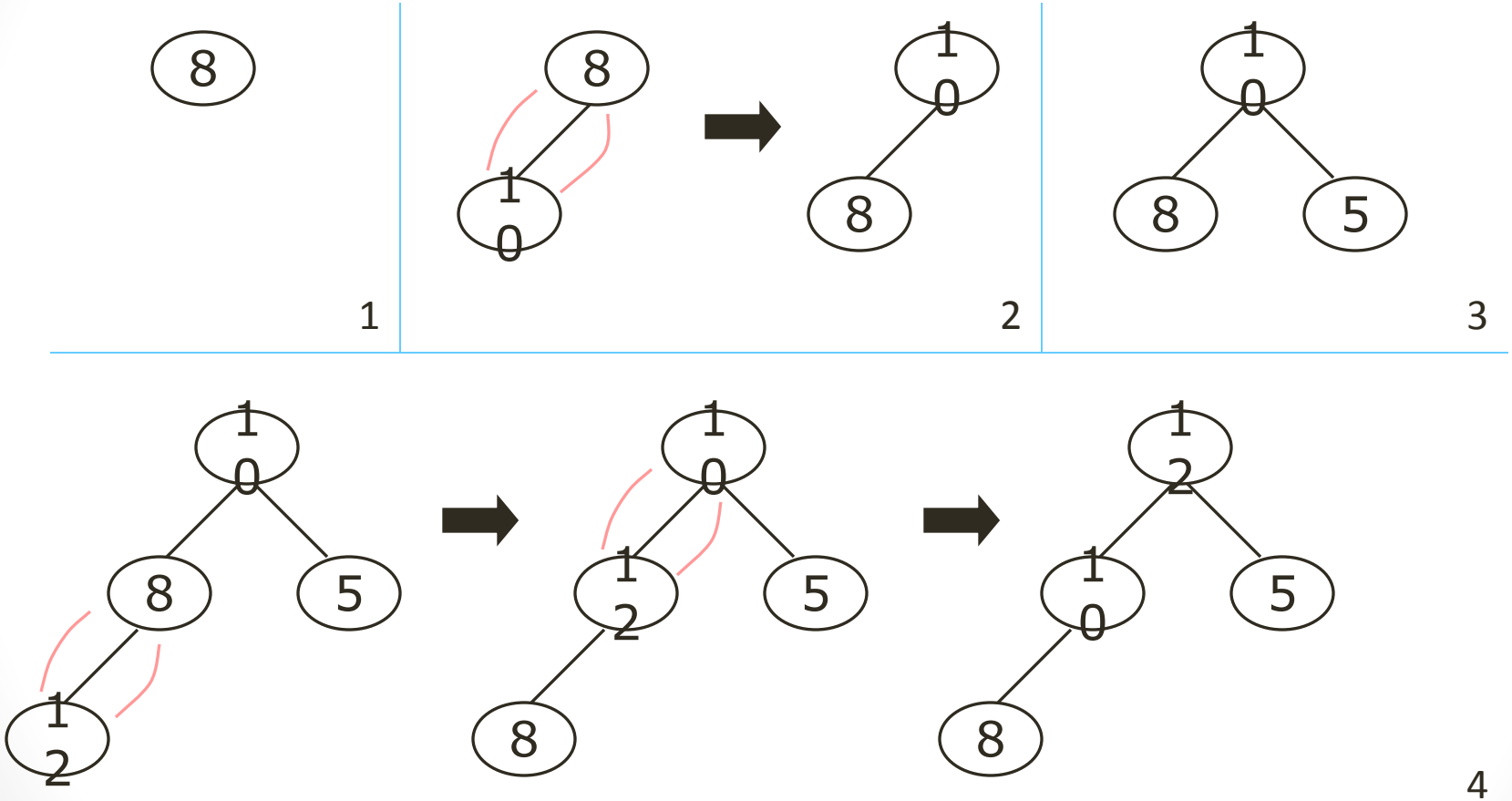
- A tree consisting of a single node is automatically a heap
- We construct a heap by adding nodes one at a time:
  - Add the node just to the right of the rightmost node in the deepest level
  - If the deepest level is full, start a new level
- Examples:



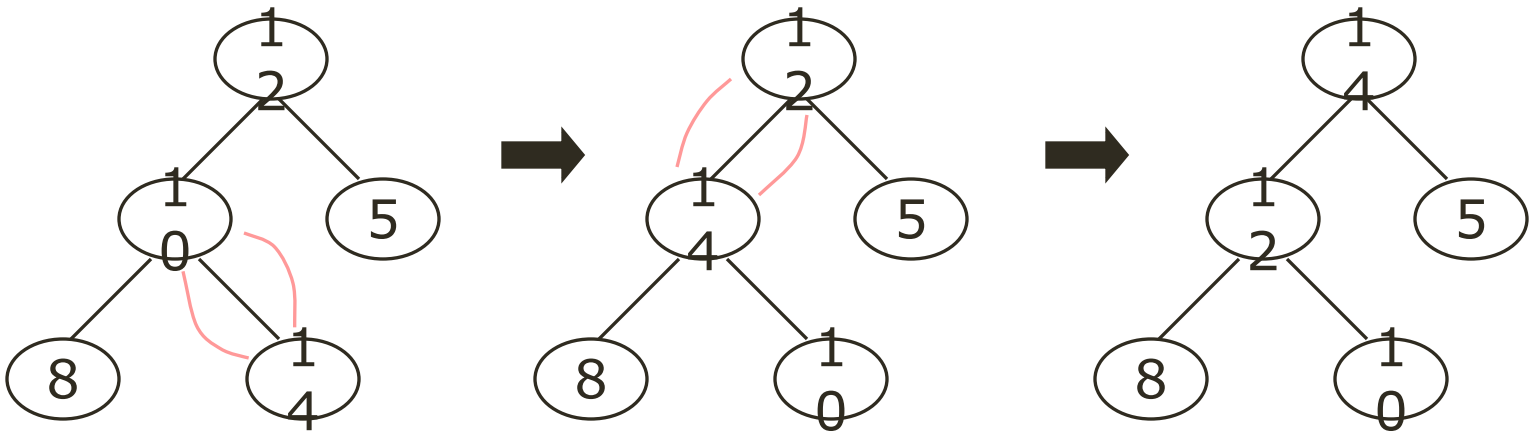
Constructing a heap

- Each time we add a node, we may destroy the heap property of its parent node
- To fix this, we sift up
- But each time we sift up, the value of the topmost node in the sift may increase, and this may destroy the heap property of *its* parent node
- We repeat the sifting up process, moving up in the tree, until either
  - We reach nodes whose values don't need to be swapped (because the parent is *still* larger than both children), or
  - We reach the root

Constructing a heap



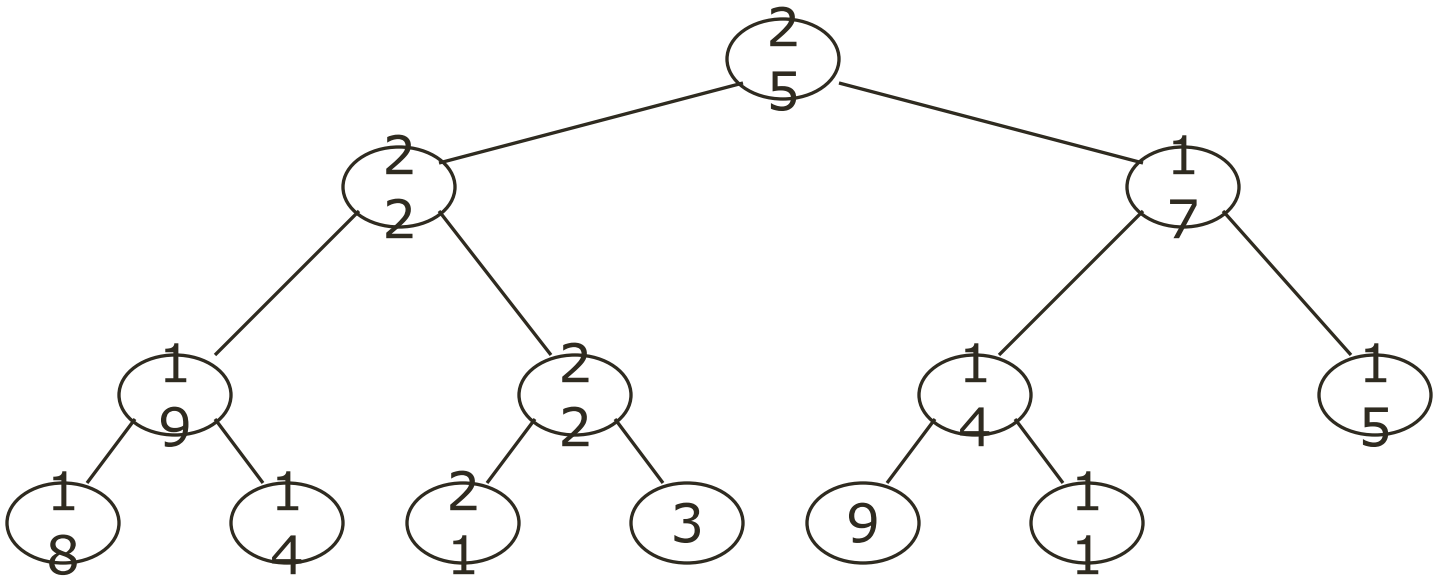
Other children are not affected



- The node containing 8 is not affected because its parent gets larger, not smaller
- The node containing 5 is not affected because its parent gets larger, not smaller

A sample heap

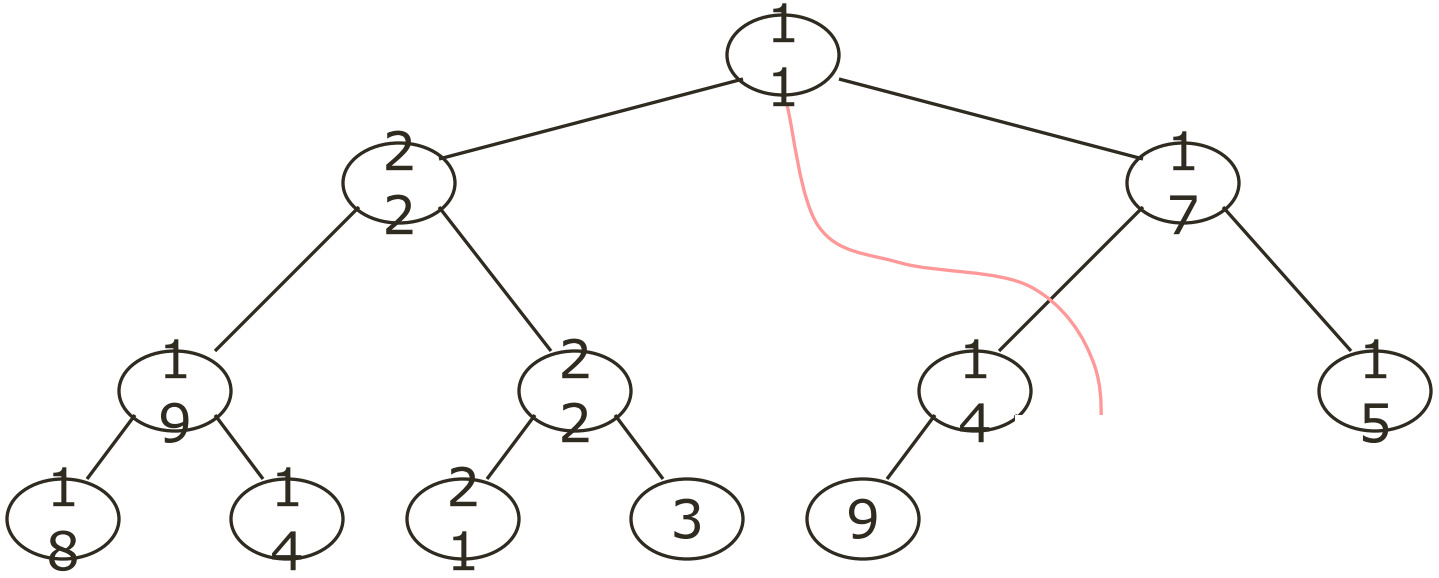
- Here's a sample binary tree after it has been heapified



- Notice that heapified does *not* mean sorted
- Heapifying does *not* change the shape of the binary tree; this binary tree is balanced and left-justified because it started out that way

Removing the root

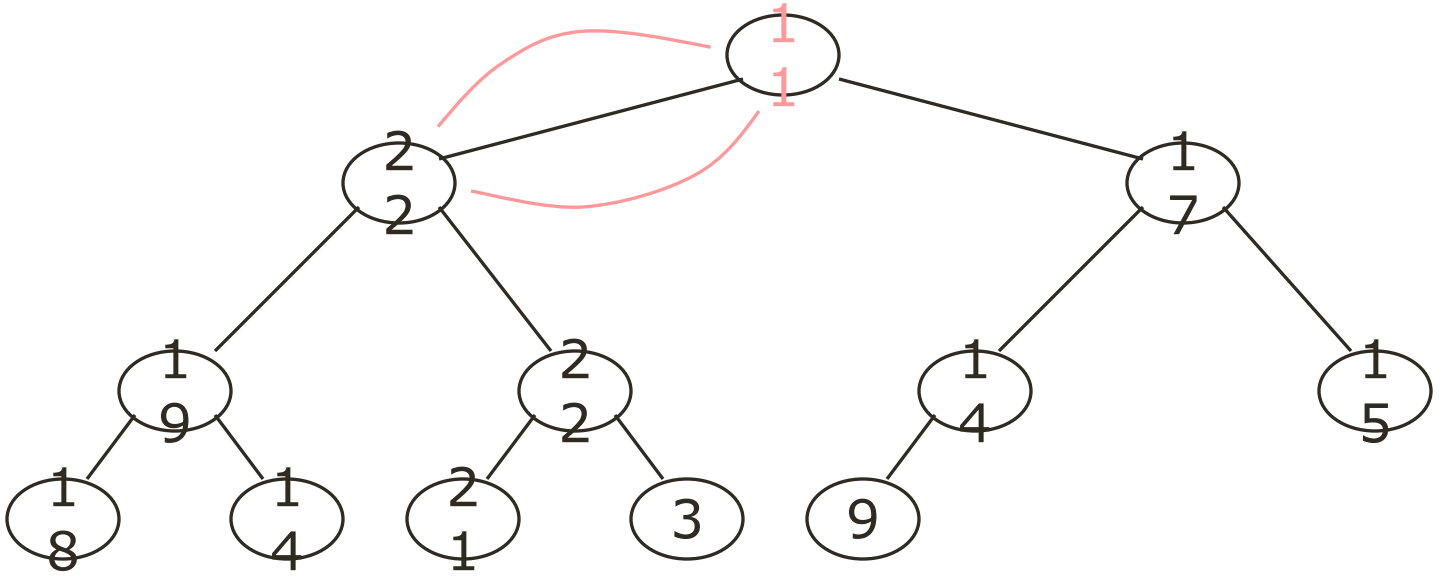
- Notice that the largest number is now in the root
- Suppose we *discard* the root:



- How can we fix the binary tree so it is once again *balanced and left-justified?*
- Solution: remove the rightmost leaf at the deepest level and use it for the new root

The reHeap method

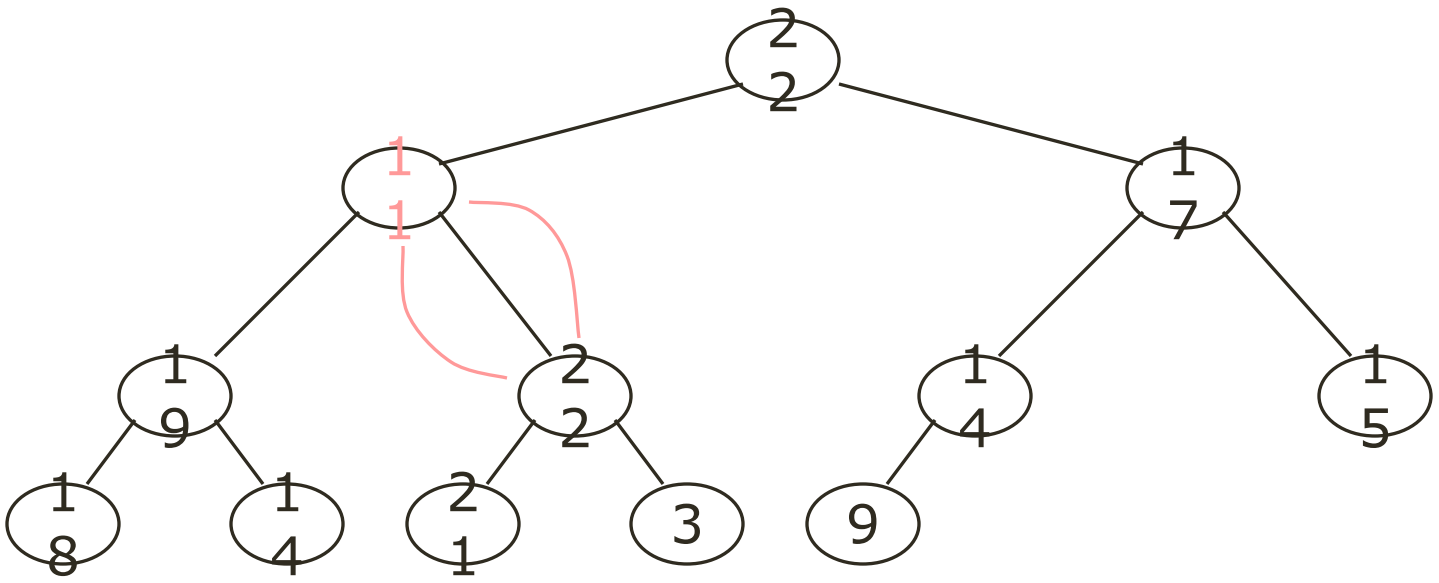
- Our tree is balanced and left-justified, but no longer a heap
- However, *only the root* lacks the heap property



- We can siftUp() the root
- After doing this, one and only one of its children may have lost the heap property

The reHeap method

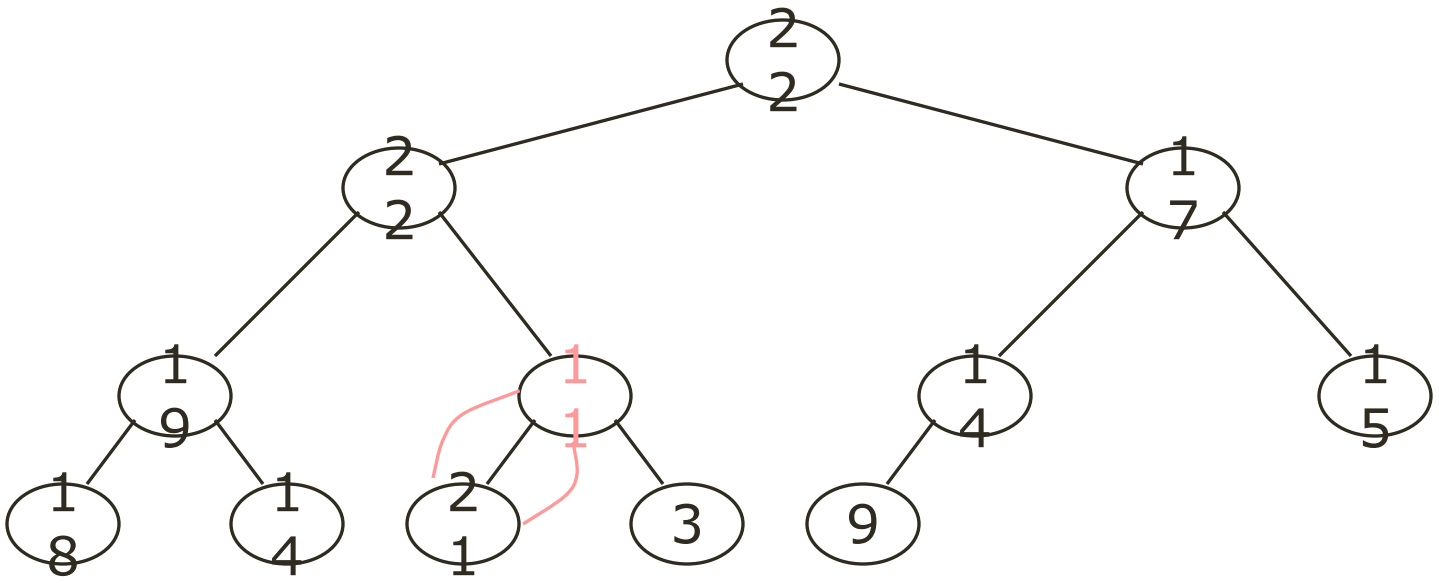
- Now the left child of the root (still the number 11) lacks the heap property



- We can siftUp() this node
- After doing this, one and only one of its children may have lost the heap property

The reHeap method

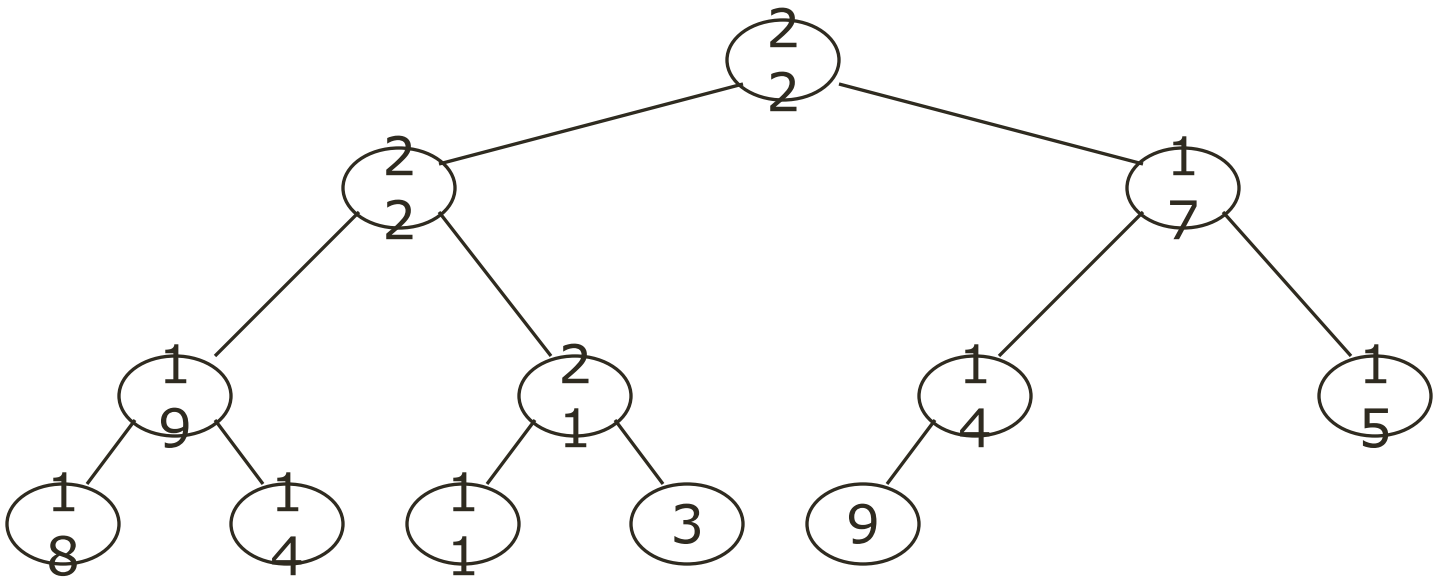
- Now the right child of the left child of the root (still the number 11) lacks the heap property:



- We can siftUp() this node
- After doing this, one and only one of its children may have lost the heap property —but it doesn't, because it's a leaf

The reHeap method

- Our tree is once again a heap, because every node in it has the heap property



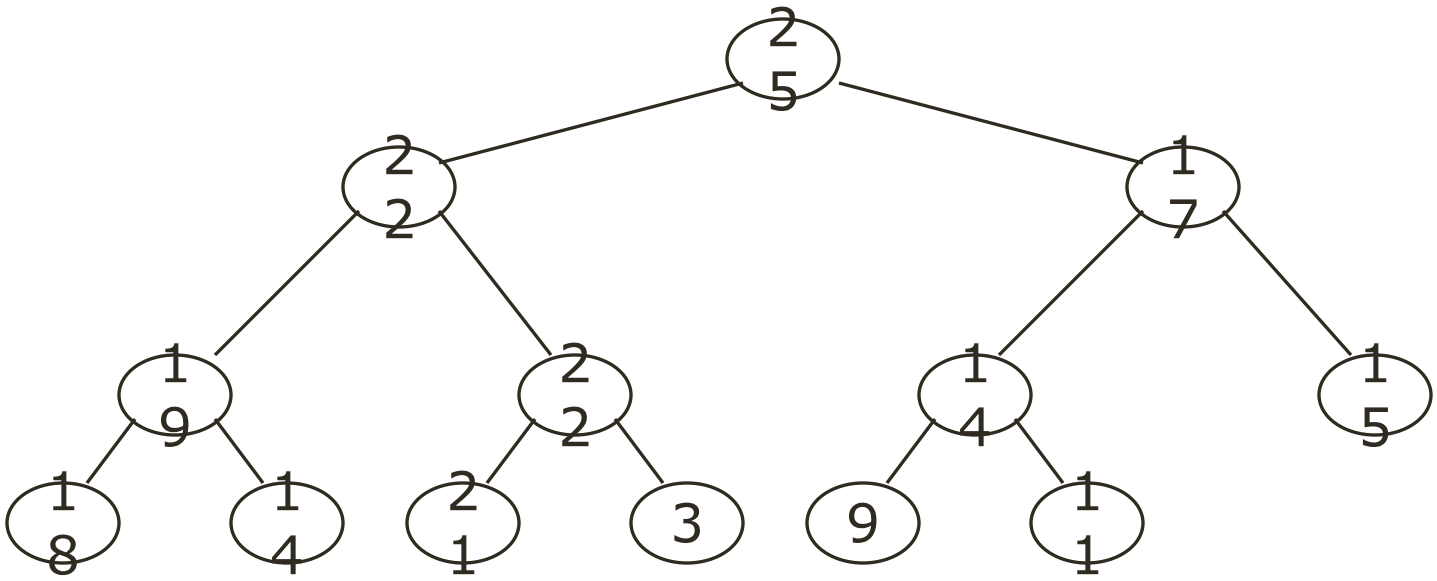
- Once again, the largest (or *a* largest) value is in the root
- We can repeat this process until the tree becomes empty
- This produces a sequence of values in order largest to smallest

Sorting

- What do heaps have to do with sorting an array?
- Here's the neat part:
  - Because the binary tree is *balanced* and *left justified,* it can be represented as an array
  - All our operations on binary trees can be represented as operations on *arrays*
  - To sort:

```
heapify
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

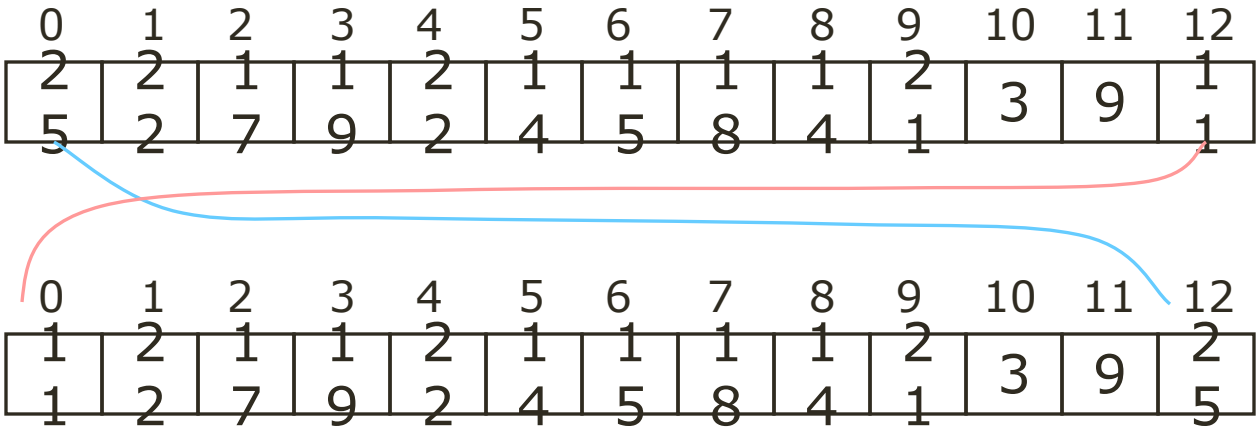
Mapping into an array



- Notice:
  - The left child of index i is at index 2*i+1
  - The right child of index i is at index 2*i+2
  - Example: the children of node 3 (19) are 7 (18) and 8 (14)

Removing and replacing the root

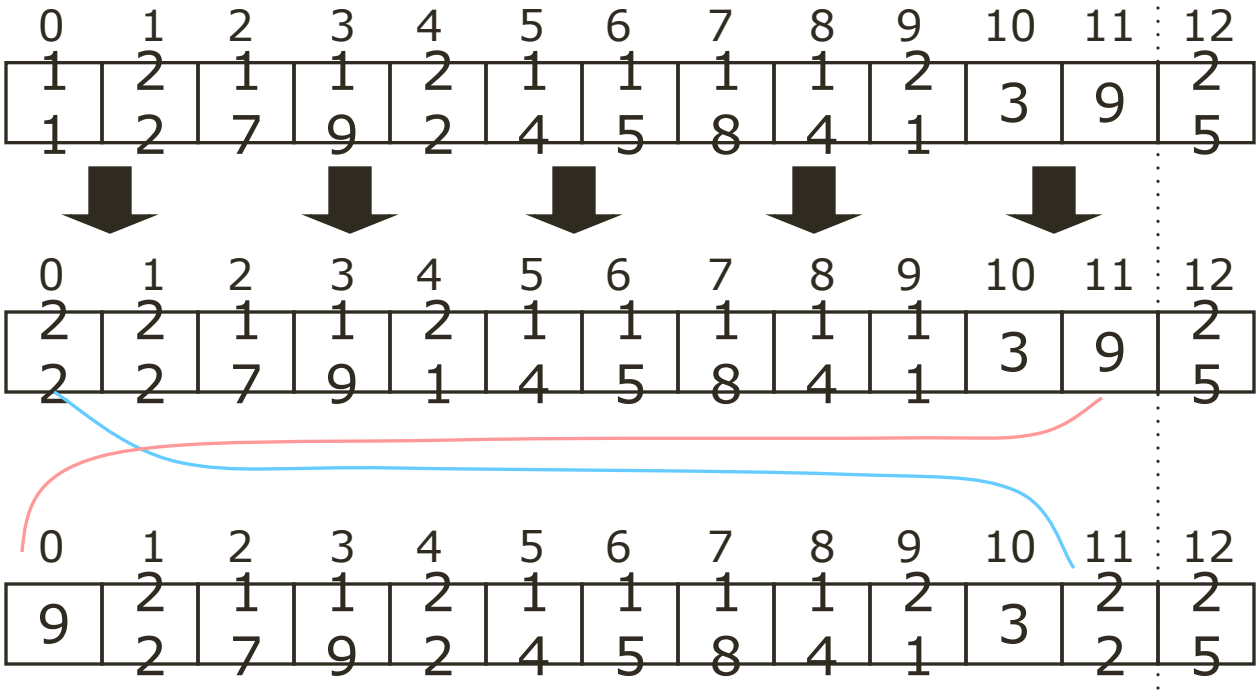
- The “root” is the first element in the array
- The “rightmost node at the deepest level” is the last element
- Swap them...



- ...And pretend that the last element in the array no longer exists—that is, the “last index” is 11 (9)

Reheap and repeat

- Reheap the root node (index 0, containing 11)...



- ...And again, remove and replace the root node
- Remember, though, that the “last” array index is changed
- Repeat until the last becomes first, and the array is sorted!

Analysis

- Here's how the algorithm starts:

  heapify the array ;

- Heapifying the array: we add each of n nodes
  - Each node has to be sifted up, possibly as far as the root
    - Since the binary tree is perfectly balanced, sifting up a single node takes O(log n) time
  - Since we do this n times, insertion takes n*O(log n) time, that is, O(n log n) time

Analysis

- Here's the rest of the algorithm:

```
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

- We do the while loop n times (actually, $n-1$ times), because we remove one of the n nodes each time
- Removing and replacing the root takes $O(1)$ time
- Therefore, the total time is n times
- However the reheap method takes long time

Analysis

- To reheap the root node, we have to follow *one path* from the root to a leaf node (and we might stop before we reach a leaf)
- The binary tree is perfectly balanced. Therefore, this path is $O(\log n)$ long
  - And we only do $O(1)$ operations at each node
  - Therefore, reheaping takes $O(\log n)$ times
- Since we reheap inside a while loop that we do n times, the total time for the while loop is $n*O(\log n)$, or $O(n \log n)$

Analysis

- Here's the algorithm again:

```
heapify the array;
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

- We have seen that heapifying takes $O(n \log n)$ time
- The while loop takes $O(n \log n)$ time
- The total time is therefore $O(n \log n) + O(n \log n)$
- This is the same as $O(n \log n)$ time

Example…

- Sort the number using Heap Sort

6, 5, 3, 1, 8, 7, 2, 4

Algorithm HeapSort

HEAPSORT(ARR, N)

Step 1: [Build Heap H]

Repeat for I = 0 to N-1

CALL heapify(Array, I)

[END OF LOOP]

Step 2: (Repeatedly delete the root element)

Repeat while N>0

swap(root, N-1)

CALL heapify(array, N-1)

SET N = N - 1

[END OF LOOP]

Step 3: END

Algorithm

```
heapify(array, i)
    Root = array[i]
   Largest = largest( array[i] , array [2*i + 1],  array[2*i+2])
   if(Root != Largest)
        Swap(Root, Largest)
```

Heapify function

```
void heapify(int arr[], int n, int i) {
 int largest = i;
int l = 2*i + 1;
 int r = 2*i + 2;
// if left child is larger than root
 if (l < n && arr[l] > arr[largest])
largest = l;
 // if right child is larger than largest so far
 if (r < n && arr[r] > arr[largest])
 largest = r;
// if largest is not root
 if (largest != i) {
swap(arr[i], arr[largest]);
// recursively heapify the affected sub-tree
heapify(arr, n, largest);  }  }
```

```
void heapSort(int arr[], int n) {
// build heap (rearrange array)
 for (int i = n / 2 - 1; i >= 0; i--)
heapify(arr, n, i);
// one by one extract an element from heap
for (int i=n-1; i>=0; i--) {
// move current root to end
swap(arr[0], arr[i]);
// call heapify on the reduced heap
heapify(arr, i, 0);
 }
 }
```