Inner Product Spaces.

27 December 2021 10:32

Linear Algebra.

Subspaces | Transformation | Inner Product Space

$TX = AX$

$[A-\lambda I] = 0$ $\quad$ $|A-\lambda I| = 0$ ✓

$\boxed{AX=0}$ ✓ $\quad$ $\boxed{X=0}$ Trivial soln $\quad$ $[A-\lambda I]X = 0$

$\boxed{X=0}$ ✓

$A \quad \lambda$ $\quad$ $AX = \lambda I$

$|A-\lambda I| = 0$

$|A-\lambda I| = 0$

⊛ Norm:-

A norm $\|\cdot\|$ on a (linear space) vector space X (over the field $\mathbb{K}$ of real or complex numbers) is a function $x \to \|x\|$, $x \in X$ from X to the set $\mathbb{R}$ of all real numbers such that for every $x, y \in X$ and $\alpha \in \mathbb{K}$

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(a) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

(b) $\|\alpha x\| = |\alpha| \, \|x\|$. ✓

(c) $\|x+y\| \leq \|x\| + \|y\|$

$(\sqrt{a^2+b^2+c^2})$

$(x_1+y_1) + (x_2+y_2) = (x_1+x_2, y_1+y_2)$

$(x_1, y_1) + (x_2, y_2) = (x_1+y_2, x_2+y_1)$

Standard norm: $\quad x = (a, b, c)$

(... length)

Standard norm: $x = (a, b, c)$

$\|x\| = \sqrt{a^2+b^2+c^2}$ (standard length)

Inner Product Space:

$a\ (a_1, b_1, c_1)$ $\quad b\ (a_2, b_2, c_2)$

$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

(a, b)

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|a|\,|b|}$$

$|a| = (a_1^2 + b_1^2 + c_1^2)^{1/2}$ $\quad a \cdot b = b \cdot a$

$a \cdot b$ – scalar quantity

symmetric and linear

An inner product space of a vector space X is a map

$$(x, y) \to \langle x, y \rangle \in \mathbb{R}, \quad (x, y) \in X \times X$$

which satisfies the following axioms:

(a) $\langle x, x \rangle \geq 0, \ \forall x \in X$ and

$\langle x, x \rangle = 0 \iff x = 0$

(b) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in X$

(c) $\langle ax, y \rangle = a \langle x, y \rangle, \quad \forall a \in \mathbb{K}, \ \forall x, y \in X$

(d) $\langle x, y \rangle = \langle y, x \rangle$ | $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Real field | Complex field

A vector space X with $\langle \cdot, \cdot \rangle$ inner product is

A vector space X with $\langle \cdot , \cdot \rangle$ inner product is called an inner product space.

eg:- Consider vectors $u = (2,3,5)$ and $v = (1,-4,3)$ in $\mathbb{R}^3$. Then

$$\langle \vec{u}, \vec{v} \rangle = 2-12+15 = 5 \in \mathbb{R}$$

$$\|\vec{u}\| = \sqrt{4+9+25} = \sqrt{38} \in \mathbb{R}$$

(*) Euclidian n-space $\mathbb{R}^n$.

Consider the vector space $\mathbb{R}^n$. The dot product in $\mathbb{R}^n$ is defined by,

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $u = (a_i)$ and $v = (b_i)$

This function defines an inner product on $\mathbb{R}^n$.

$$\|u\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{u \cdot u}$$

$$\|u\|^2 = \langle u \cdot u \rangle \quad \forall \; u \in \mathbb{R}^n$$

(*) Remark:- Frequently the vectors in $\mathbb{R}^n$ will be represented by column vectors, that is by $n \times 1$ column matrices.

In such case, the formula,

$$\langle u, v \rangle = u^T v \in \mathbb{R}$$

defines the usual inner product on $\mathbb{R}^n$.

eg:- Let $\boxed{C[a,b]}$ it consists of all the continuous functions on the interval $[a,b]$. $a \leq t \leq b$

$b \qquad \forall \; f, g \in C[a,b]$

on the interval $[a,b]$.

$$\langle f,g\rangle = \int_a^b f(t)\cdot g(t)\,dt \quad , \quad \forall f,g \in C[a,b]$$

This is the usual inner product on $C[a,b]$.

H.W.

$\Rightarrow$ ⓐ $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ iff $x=0$

Let $f \in C[a,b]$

$$\langle f,f\rangle = \int_a^b \underline{\underline{f^2(t)}}\,dt \geq 0 \quad \forall f \in C[a,b]$$

Let $\int_a^b f^2(t)\,dt = 0 \Rightarrow f(t) = 0$

ⓑ $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

$f,g,h \in C[a,b]$.

$$\therefore \langle f+g,h\rangle = \int_a^b (f(t)+g(t))\,h(t)\,dt$$

$$= \int_a^b f(t)\cdot h(t)\,dt + \int_a^b g(t)\cdot h(t)\,dt$$

$$= \langle f,h\rangle + \langle g,h\rangle$$

ⓒ $\langle ax,y\rangle = a\langle x,y\rangle$

Let $f,g \in C[a,b]$ and $a \in \mathbb{R}$

$$\therefore \langle af,g\rangle = \int_a^b a\,f(t)\cdot g(t)\,dt = a\int_a^b f(t)\cdot g(t)\,dt$$

$$= a\langle f,g\rangle$$

④ $\langle x,y\rangle = \langle y,x\rangle$

Let $f,g \in C[a,b]$

$$\therefore \langle f,g\rangle = \int_a^b f(t)\cdot g(t)\,dt = \int_a^b g(t)\cdot f(t)\,dt$$

$$\boxed{\langle f, g\rangle = \langle g, f\rangle}$$

eg:- Consider $f(t) = 3t - 5$ and $g(t) = t^2$

$C[a,b]$ $[0,1]$

Find $\langle f, g\rangle$.

$\Rightarrow$ We have $\langle f, g\rangle = \int_0^1 (3t-5)t^2\,dt$

$$= \int_0^1 (3t^3 - 5t^2)\,dt$$

$$= \frac{3t^4}{4} - \frac{5t^3}{3}\Big|_0^1$$

$$= \frac{3}{4} - \frac{5}{3} = \frac{-11}{12}$$

eg:- Find $\|f\|$ and $\|g\|$

We have, $\|f\|^2 = \langle f, f\rangle$

$$\therefore \|f\|^2 = \int_0^1 (3t-5)(3t-5)\,dt = \int_0^1 (9t^2 - 30t + 25)\,dt$$

$$= 9\frac{t^3}{3} - 30\frac{t^2}{2} + 25t\Big|_0^1$$

$$= 3(1) - 15(1) + 25$$

$\|f\|^2 = 13 \Rightarrow \|f\| = \sqrt{13}$

and $\|g\|^2 = \langle g, g\rangle = \int_0^1 t^4\,dt = \frac{t^5}{5}\Big|_0^1 = \frac{1}{5}$

$$\|g\| = \frac{\sqrt{5}}{5}$$

Matrix space $\quad M = M_{m\times n}$

H.W. $\quad A, B \in M_{m\times n}$

$$\boxed{\langle A, B\rangle = tr(B^T A)}$$

$$\langle A, B\rangle = tr(B^T A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ij}$$

$$\|A\|^2 = \langle A, A\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$$

Angle betn Vectors:-

For any non-zero vectors u, v in an inner product space V, the angle betn u and v where $u, v \in V$, is defined to be the angle θ such that $0 \le \theta \le \pi$ and

$$\boxed{\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}}$$

eg:- Consider vectors $u = (2,3,5)$ and $v = (1,-4,3)$ in $\mathbb{R}^3$. Find the angle betn u and v.

$\Rightarrow \quad \|u\| = \sqrt{4+9+25} = \sqrt{38}$, $\|v\| = \sqrt{1+16+9} = \sqrt{26}$

$\langle u, v\rangle = 2-12+15 = 5.$

$$\therefore \quad \cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{5}{\sqrt{38}\,\sqrt{26}}$$

$P(x) = 2t-5$ and $g(t) = t^2$

eg:- $f(t) = 3t-5$ and $g(t) = t^2$

$\langle f, g \rangle = \frac{-11}{12}$, $\|f\| = \sqrt{13}$

$\|g\| = \frac{\sqrt{5}}{5}$

$\therefore \cos\theta = \dfrac{-11/12}{\sqrt{13}\ \ 1/\sqrt{5}}$

Orthogonality:-

Let X be an I.P.S. The vectors $x, y \in X$ are said to be orthogonal (x is orthogonal to y) iff

$\langle x, y \rangle = 0 = \langle y, x \rangle$

eg:- If $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2) \in \mathbb{R}^2(\mathbb{R})$, let us define $\langle \alpha, \beta \rangle = \langle (a_1, a_2), (b_1, b_2) \rangle$

$= a_1 b_1 - a_2 b_1 - a_1 b_2 + 4 a_2 b_2.$

Verify that the above product is an inner product on $\mathbb{R}^2$.

$\Rightarrow$ (i) $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$

(ii) $\langle \alpha + \beta, r \rangle = \langle \alpha, r \rangle + \langle \beta, r \rangle$

(iii) $\langle x\alpha, \beta \rangle = x \langle \alpha, \beta \rangle$ – [

(iv) $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ – [Symmetry]

Let $\alpha = (a_1, a_2)$

(i) $\therefore \langle \alpha, \alpha \rangle = \langle (a_1, a_2), (a_1, a_2) \rangle$

$= a_1^2 - a_2 a_1 - a_1 a_2 + 4 a_2^2$

$$= \underline{\underline{(a_1-a_2)^2}} + \underline{\underline{3a_2^2}} \geq 0$$

Let $\langle\alpha,\alpha\rangle = 0$

$\Rightarrow (a_1-a_2)^2 + (3a_2)^2 = 0$

$\Rightarrow a_1-a_2 = 0$ and $a_2 = 0$

$\Rightarrow a_1 = a_2 = 0$

$\alpha = (a_1, a_2) = (0,0) = 0$ //

$\therefore \langle\alpha,\alpha\rangle \geq 0$ and $\langle\alpha,\alpha\rangle = 0$ iff $\alpha = 0$

(2) Let α, β and $\gamma \in \mathbb{R}^2(\mathbb{R})$

$\alpha = (a_1, a_2)$, $\beta(b_1, b_2)$ and $\gamma = (c_1, c_2)$

$$\boxed{\langle\alpha+\beta, \gamma\rangle = \langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle}$$

$(\alpha+\beta) = (a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$

$\therefore \langle\alpha+\beta, \gamma\rangle = \langle(a_1+b_1, a_2+b_2), (c_1, c_2)\rangle$

$= (a_1+b_1)c_1 - (a_2+b_2)c_1 - (a_1+b_1)c_2$
$+4(a_2+b_2)c_2$

$= (a_1c_1 - a_2c_1 - a_1c_2 + 4a_2c_2)$
$+ b_1c_1 - b_2c_1 - b_1c_2 + 4b_2c_2$

$= \langle(a_1,a_2),(c_1,c_2)\rangle + \langle(b_1,b_2),(c_1,c_2)\rangle$

$\langle\alpha+\beta,\gamma\rangle = \langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle$

(3) Let $\alpha, \beta \in \mathbb{R}^3$ and $x \in \mathbb{R}$

· $\boxed{\langle x\alpha, \beta\rangle = x\langle\alpha,\beta\rangle}$

$$\therefore \boxed{\langle x\alpha, \beta\rangle = x\langle \alpha, \beta\rangle}$$

$$x\alpha = x(a_1, a_2) = (xa_1, xa_2)$$

$$\therefore \langle x\alpha, \beta\rangle = \langle (xa_1, xa_2), (b_1, b_2)\rangle$$

$$= xa_1b_1 - xa_2b_1 - xa_1b_2 + 4xa_2b_2$$

$$= x(a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2)$$

$$\langle x\alpha, \beta\rangle = x\langle \alpha, \beta\rangle$$

(4) Let $\alpha, \beta \in \mathbb{R}^2(\mathbb{R})$.

$$\langle \alpha, \beta\rangle = \langle \beta, \alpha\rangle$$

$$\langle \alpha, \beta\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2$$

$$= b_1a_1 - b_2a_1 - b_1a_2 + 4b_2a_2$$

$$\boxed{\langle \alpha, \beta\rangle = \langle \beta, \alpha\rangle}$$

Given product on $\mathbb{R}^2(\mathbb{R})$ is an inner product.

H.W.

eg:- Verify that the following is an inner product on $\mathbb{R}^2(\mathbb{R})$, where $u = (x_1, x_2)$ and $v(y_1, y_2)$.

$$f(u,v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2.$$

eg:- Find the value of k so that the following is an inner product on $\mathbb{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$.

$$f(u,v) = x_1y_1 - 3x_1y_2 - 3x_2y_1 + \underline{kx_2y_2}$$

$\Rightarrow$ Let $f(u,v)$ be an inner product on $\mathbb{R}^2$

$\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0 \Rightarrow u = 0$

$\Rightarrow$ Let $f(u,v)$ be an inner p...

$\therefore$ $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0 \Rightarrow u=0$

$$\langle u,u\rangle = x_1^2 - 3x_1x_2 - 3x_2x_1 + kx_2^2$$

$$= (x_1-3x_2)^2 + \underline{(k-9)}x_2^2 \geq 0$$

$$\Rightarrow k-9 \geq 0 \Rightarrow \boxed{k \geq 9}$$

(*) **Orthogonality:-** $V(\mathbb{R})$

$\langle u,v\rangle = 0$ $\quad a \cdot b = 0 \Rightarrow a \perp b$

$\langle v,u\rangle = 0$

$$\boxed{0 \in V \text{ is orthogonal to every } v \in V}$$

$\langle 0,v\rangle = 0$

$\langle 0v, v\rangle = 0\langle v,v\rangle = 0$

Let u is orthogonal to every $v \in V$ then

$u=0$. $\quad u \neq 0 \quad u \in V$

$\langle u,u\rangle = 0 \Rightarrow u=0$

eg:- Consider the vectors $u=(1,1,1)$, $v=(1,2,-3)$ and $w=(1,-4,3)$ in $\mathbb{R}^3$.

$\Rightarrow$ $\langle u,v\rangle = 1+2-3 = 0 \Rightarrow$ u and v are orthogonal to each other.

$\langle u,w\rangle = 1-4+3 = 0 \Rightarrow$ u and w are orthogonal

$\langle v,w\rangle = 1-8-9 = -16 \neq 0$ $\quad$ v and w are not orthogonal.

eg:- Find non-zero vector w that is orthogonal to ... in $\mathbb{R}^3$.

eg:- Find non-zero vector w that is orthogonal
$u = (1,2,1)$ and $v = (2,5,4)$ in $\mathbb{R}^3$.

$\Rightarrow$ Let $w = (x, y, z)$ s.t. it is orthogonal to
$u = (1,2,1)$ and $v = (2,5,4)$.

$$\therefore \left.\begin{aligned} \langle u, w\rangle &= x + 2y + z = 0 \\ \langle v, w\rangle &= 2x + 5y + 4z = 0 \end{aligned}\right\} \qquad \begin{aligned} x + 2y + z &= 0 \\ y + 2z &= 0 \end{aligned}$$

Let $z = 1 \Rightarrow y = -2$ and $x = 3$
$z = t \Rightarrow y = -2t$ and $x = 3t$

$w = (3, -2, 1)$

$$\hat{w} = \frac{w}{\|w\|} = \left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$$

* Orthogonal Complement :-

Let S be the subset of I.P.S. V. The orthogonal complement of S is denoted by $S^{\perp}$ ("S perp") consists of those vectors in V that are orthogonal to every vector $u \in S$. That is

$$S^{\perp} = \{ v \in V : \langle u, v\rangle = 0,\ \forall\ u \in S\}$$

In particular for $u \in V$,

$$u^{\perp} = \{ v \in V : \langle u, v\rangle = 0\}$$

* **Proposition:-** Let S be a subset of Inner product space V. Then $S^{\perp}$ is a subspace of V.

$\Rightarrow$ We have, $S^{\perp} = \{ v \in V : \langle u, v\rangle = 0\ \forall\ u \in S\}$

Consider $\langle 0, u\rangle$, $u \in S$

$\Rightarrow 0 \in S^{\perp}$

Consider $\langle 0, u \rangle$,

$$\langle 0, u \rangle = 0 \quad \Rightarrow \quad 0 \in S^{\perp}$$

Let $a, b \in S^{\perp}$ and k is a scalar.

now, $\langle ka+b, u \rangle = k\langle a, u \rangle + \langle b, u \rangle$, $u \in S$

$$= k(0) + 0 = 0$$

$\Rightarrow S^{\perp}$ is a subspace of V.

Remark: Suppose u is a non zero vector in $\mathbb{R}^3$. Then there is geometrical description of $u^{\perp}$. Specifically, $u^{\perp}$ is the plane in $\mathbb{R}^3$ through the origin.

(*) **Remark:-** Let W be the sol^n space of $m \times n$ homogeneous system $Ax = 0$, where $A = [a_{ij}]_{m \times n}$ and $X = [x_i]_{n \times 1}$. W may be viewed as the kernel of the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$. Specifically, each sol^n vector $w = (x_1, x_2, \ldots, x_n)$ is orthogonal to each row of A; and hence W is the orthogonal complement of the row space of A.

Orthogonal sets and Bases

Consider a set $S = \{u_i : i = 1, 2, 3, \ldots, n\}$ of non zero vectors in an inner product space V. S is called an orthogonal set if each pair of vectors in S are orthogonal and S is called orthonormal set if S is orthogonal and each vector in S has unit length. That is:

(i) Orthogonal : $\langle u_i, u_j \rangle = 0$ for $i \neq j$

(ii) Orthonormal : $\langle u_i, u_j \rangle = \begin{cases} 0, & \text{for } i \neq j \\ 1, & \text{for } i = j \end{cases}$

... vectors.

(ii) Orthonormal : $\langle u_i, u_j \rangle = \{ 1 ;$ for $i=j$

Theorem:- Suppose S is an orthogonal set of non zero vectors. Then S is linearly independent.

Let $S = \{u_i, i=1,2,\dots,n\}$ is an orthogonal set,

$$\|u_1+u_2+\dots+u_r\|^2 = \|u_1\|^2+\|u_2\|^2+\dots+\|u_r\|^2$$

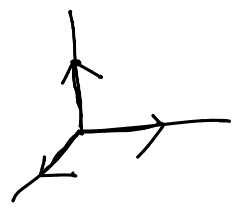
$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\underbrace{\langle u,v\rangle}_{0} + \langle v,v\rangle$$

$$= \|u\|^2 + \|v\|^2$$

Example:- Let $E = \{e_1, e_2, e_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}$ in $\mathbb{R}^3$.

$\langle e_1, e_2\rangle = 0$

$\langle e_2, e_3\rangle = 0 = \langle e_1, e_3\rangle$



$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$

In general, the usual basis of $\mathbb{R}^n$ is orthonormal for every n.

Example:- Let $V = C[-\pi, \pi]$, with I.P. defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt.$$ Then

$\{1, \cos t, \cos 2t, \cos 3t, \dots, \sin t, \sin 2t, \sin 3t, \dots\}$

is orthogonal set.

Orthogonal Basis:-

Let S be the subset of $\mathbb{R}^n$ having exactly n vectors such that,

$\langle u_i, u_j\rangle = 0 \quad \forall\ i \neq j$

n vectors such ...

$\langle u_i, u_j \rangle = 0 \quad \forall\ i \neq j$

$i = j = 1, 2, \cdots, n$

then S forms a orthogonal basis of $\mathbb{R}^n$.

$S = \{u_1, u_2, \cdots, u_n\}$ ✓ V

$v \in V$, $\boxed{v = \sum_{i=1}^{n} x_i u_i}$

Let $S = \{u_1, u_2, u_3\}$, $\mathbb{R}$

$u_1 = (1,2,1)$, $u_2 = (2,1,-4)$, $u_3 = (3,-2,1)$

$$\left.\begin{aligned} \langle u_1, u_2 \rangle &= 2+2-4 = 0 \\ \langle u_1, u_3 \rangle &= 3-4+1 = 0 \\ \langle u_2, u_3 \rangle &= 6-2-4 = 0 \end{aligned}\right\} \text{orthogonal}$$

Hence u_1, u_2, u_3 are linearly independent.

Thus S is an orthogonal basis of $\mathbb{R}^3$.

Let $v = (7,1,9) \in \mathbb{R}^3$

$v = x u_1 + y u_2 + z u_3$

$(7,1,9) = x(1,2,1) + y(2,1,-4) + z(3,-2,1)$

$$\left.\begin{aligned} x+2y+3z &= 7 \\ 2x+y-2z &= 1 \\ x-4y+z &= 9 \end{aligned}\right\} \quad x = 3,\ y = -1 \text{ and } z = 2$$

$\langle v, u_i \rangle = \langle x u_1 + y u_2 + z u_3, u_i \rangle$

$\langle v, u_1 \rangle = x \langle u_1, u_1 \rangle \Rightarrow x = \dfrac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle}$ ✓

$\langle v, u_2 \rangle = y \langle u_2, u_2 \rangle \qquad y = \dfrac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle}$ ✓

$\langle v, u_3 \rangle = z \langle u_3, u_3 \rangle$

$z = \dfrac{\langle v, u_3 \rangle}{\quad}$ ✓

$$\langle v, u_3\rangle = \underline{\underline{z}}\langle u_3, \ldots\rangle$$

$$\underline{\underline{z}} = \frac{\langle v, u_3\rangle}{\langle u_3, u_3\rangle}$$

$$\underline{\underline{x}} = \frac{18}{6} = \underline{\underline{3}} \quad ; \quad y = \frac{-21}{21} = \underline{\underline{-1}}$$

$$z = \frac{28}{14} = \underline{\underline{2}}$$

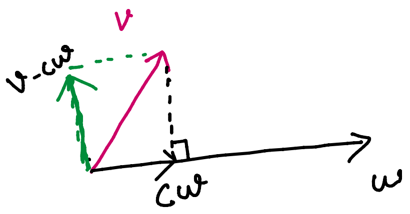
Theorem:- Let $\{u_1, u_2, \ldots, u_n\}$ be an orthogonal basis of V. Then for any $v \in V$,

$$v = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} u_1 + \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} u_2 + \cdots + \frac{\langle v, u_n\rangle}{\langle u_n, u_n\rangle} u_n$$

$$v = \sum_{i=1}^{n} \frac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle} u_i$$

The scalars $k_i = \dfrac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$ is called the Fourier coefficient of v w.r.t. u_i.

⊛ Projection:-



$$\langle v - cw, w\rangle = 0$$

$$\downarrow$$

$$\langle v, w\rangle - c\langle w, w\rangle = 0$$

or $\boxed{c = \dfrac{\langle v, w\rangle}{\langle w, w\rangle}}$

$$\boxed{\text{proj}(v, w) = cw = \frac{\langle v, w\rangle}{\langle w, w\rangle} w}$$

Theorem:- Suppose $w_1, w_2, \ldots, w_r$ form an orthogonal set of non-zero vectors in V. Let $v \in V$.

$$\underline{\underline{v'}} = v - \left(\sum_{i=1}^{r} c_i w_i\right)$$

where, $\quad = \langle v, w_i\rangle \quad , \quad i = 1, 2, \ldots, r$

where, $c_i = \dfrac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle}$, $i = 1, 2, \ldots, r$

where v' is orthogonal to w_i.

eg:- Find the Fourier coefficient c and the projection of $v = (1, -2, 3, -4)$ along $w = (1, 2, 1, 2)$ in $\mathbb{R}^4$.

$\Rightarrow$

$$\text{proj}(v, w) = cw = \boxed{\frac{\langle v, w \rangle}{\langle w, w \rangle}}\, w$$

$$\therefore\ c = \frac{\langle v, w \rangle}{\langle w, w \rangle} = \frac{1 - 4 + 3 - 8}{1 + 4 + 1 + 4} = \frac{-8}{10} = \frac{-4}{5}$$

$$\text{proj}(v, w) = cw = \frac{-4}{5}(1, 2, 1, 2)$$

$$= \left(\frac{-4}{5}, \frac{-8}{5}, \frac{-4}{5}, \frac{-8}{5}\right)$$

eg:- Let V be the vector space of polynomials over $\mathbb{R}$ of degree ≤ 2 with inner product defined by, $\langle f, g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$. Find a basis of a subspace W orthogonal to $\boxed{h(t) = 2t + 1}$

$\Rightarrow$ Let $f(t) = at^2 + bt + c$ be orthogonal to $h(t) = 2t + 1$

$$\langle f, h \rangle = \int_0^1 (at^2 + bt + c)(2t + 1)\, dt$$

$$0 = \int_0^1 (2at^3 + 2bt^2 + 2ct + at^2 + bt + c)\, dt$$

$$0 = \frac{2a}{4} + \frac{(2b + a)}{3} + \frac{(2c + b)}{2} + c$$

$$= \frac{a}{2} + \frac{a}{3} + \frac{2b}{3} + \frac{b}{2} + 2c$$

$$\boxed{0 = \frac{5a}{6} + \frac{7b}{6} + 2c}$$

$b = \frac{-5}{}$

Let $a=1$, $c=0$ $\Rightarrow$ $b = \frac{-5}{7}$

$$f(t)\Big|_{a=1,c=0} = t^2 - \frac{5}{7}t$$

Let $a=1$, $b=0$, $c = -5/12$

$$f(t)\Big|_{a=1,b=0} = t^2 - \frac{5}{12}$$

$$\left(t^2 - \frac{5}{7}t \, , \, t^2 - \frac{5}{12} \, , \, 2t+1 \right)$$

This gives a orthogonal basis of subspace W.

eg:- Let $w = (1,-2,-1,3)$ be a vector in $\mathbb{R}^4$. Find

(a) an orthogonal basis for $W^\perp$ ✓

(b) an orthonormal basis for $W^\perp$ ✓

$v = (a,b,c,d)$ $\langle v, w \rangle = 0$

eg:- Let S consists of the following vectors in $\mathbb{R}^4$

$u_1 = (1,1,1,1)$, $u_2 = (1,1,-1,-1)$,

$u_3 = (1,-1,1,-1)$ $u_4 = (1,-1,-1,1)$

(a) Show that S is basis of $\mathbb{R}^4$

(b) Write $v = (1,3,-5,6)$ as a linear combination of u_1, u_2, u_3, u_4.

(c) Normalize S to obtain an orthonormal basis of $\mathbb{R}^4$.

⊛ Gram-Schmidt Orthogonalization Process.

Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis $\{w_1, w_2, \ldots, w_n\}$ of V.

$w_1 = v_1$

giving

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2 - w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

. . . .

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

In otherwords, for $k = 2, 3, \ldots, n$, we define

$$w_k = v_k - c_{k1} w_1 - c_{k2} w_2 - \cdots - c_{k,k-1} w_{k-1}$$

where $c_{ki} = \dfrac{\langle v_k, w_i\rangle}{\langle w_i, w_i\rangle}$

Remark 1:- Each vector w_k is linear combination of v_k and the preceding w's. Hence the each vector w_k is a linear combination of $v_1, v_2, \ldots, v_n$.

Remark 2:- Taking multiples of vectors does not affect the orthogonality, it may be simpler in hand calculations to clear fractions in any new w_k, by multiplying w_k with appropriate scalar.

eg:- Apply the Gram-Smidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by

$v_1 = (1,1,1,1)$, $v_2 = (1,2,4,5)$, $v_3 = (1,-3,-4,-2)$.

$\Rightarrow$ (1) First let, $v_1 = w_1$.

$\therefore w_1 = v_1 = (1,1,1,1)$.

$\therefore\ w_1 = v_1 = (1,1,1,1).$

② Compute $w_2 = v_2 - \dfrac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$$= (1,2,4,5) - \frac{(1+2+4+5)}{1+1+1+1}(1,1,1,1)$$

$$= (1,2,4,5) - \frac{12}{4}(1,1,1,1)$$

$$\boxed{w_2 = (-2,-1,1,2)}$$

③ Compute w_3

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= (1,-3,-4,-2) - \frac{(1-3-4-2)}{4}(1,1,1,1) - \frac{(-2+3-4-4)}{4+1+1+4}(-2,-1,1,2)$$

$$w_3' = (1,-3,-4,-2) + 2(1,1,1,1) + \frac{7}{10}(-2,-1,1,2)$$

$$= (3,-1,-2,0) + \left(-\frac{7}{5}, -\frac{7}{10}, \frac{7}{10}, \frac{7}{5}\right)$$

$$= \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}\right)$$

$$\boxed{w_3 = (16,-17,-13,14)}$$

$\hat{w}_1 = \frac{1}{2}(1,1,1,1)$

$\hat{w}_2 = \frac{1}{\sqrt{10}}(-2,-1,1,2)$

$\hat{w}_3 = \dfrac{1}{\sqrt{16^2+17^2+13^2+14^2}}(16,-17,-13,14)$

The w_1, w_2, w_3 are the orthogonal basis of given subspace U corresponding to v_1, v_2, v_3.

eg:- Let V be the vector space of polynomials $f(t)$ with inner product $\langle f,g\rangle = \int_{-1}^{1} f(t)\cdot g(t)\,dt$. Apply the Gram-Smidt orthogonalization basis $\{f_0, f_1, f_2, f_3\}$ to find an orthogonal basis ... t^2, $t^3\}$

$\langle f,g\rangle = \int_{-1} f(t)\cdot g(t)$ ... orthogonal basis

process to $\{1, t, t^2, t^3\}$ to find an orthogonal basis

with integer coefficient for $P_3(t)$.

v_1, v_2, v_3

$w_1, w_2, \ldots$

$w_1 = v_1$

$$\int_{-1}^{1} t^n\,dt = \begin{cases} \frac{2}{n+1}, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow$ (1) Let $f_0 = 1$.

(2) Compute $f_1 = t - \frac{\langle t, f_0\rangle}{\langle f_0, f_0\rangle} f_0$

$$t - \frac{\langle t, f_0\rangle}{\langle f_0, f_0\rangle} f_0 = t - \frac{\int_{-1}^{1} t\,dt}{\int_{-1}^{1} 1\,dt}(1)$$

$$\boxed{f_1 = t}$$

(3) Compute, $f_2 = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\overset{0}{\overset{\parallel}{\langle t^2, t\rangle}}}{\langle t, t\rangle} t$

$$\therefore\ t^2 - \frac{\int_{-1}^{1} t^2\,dt}{\int_{-1}^{1} dt} = t^2 - \frac{2/3}{2} = t^2 - 1/3$$

Set $\boxed{f_2 = 3t^2 - 1}$

(4) Compute f_3.

$$f_3 = t^3 - \frac{\overset{0}{\overset{\parallel}{\langle t^3, 1\rangle}}}{\langle 1, 1\rangle}(1) - \frac{\langle t^3, t\rangle}{\langle t, t\rangle}(t) - \frac{\overset{0}{\overset{\parallel}{\langle t^3, 3t^2 - 1\rangle}}}{\langle 3t^2-1, 3t^2-1\rangle}(3t^2 - 1)$$

$$f_3 = t^3 - \frac{2/5}{2/3}(t) = t^3 - \frac{3}{5}t$$

Set $\boxed{f_3 = 5t^3 - 3t}$

$\therefore$ required orthogonal basis.

set $\boxed{t_3 = 0.}$

$\therefore \{1, t, (3t^2-1), (5t^3-3t)\}$ is the required orthogonal basis.

⊛ Orthogonal Matrices

A real matrix A is orthogonal if it is non-singular and $A^T = A^{-1}$.

In other words,

$AA^T = I = A^TA$. **Unitary Matrix.**

$|AA^T| = |I| \Rightarrow |A||A^T| = 1$

$|A|^2 = 1 \Rightarrow \boxed{|A| = \pm 1}$

$$\underset{A}{\begin{bmatrix} \cdots a_1 \cdots \\ \cdots a_2 \cdots \\ \vdots \\ \cdots a_n \cdots \end{bmatrix}} \underset{A^T}{\begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix}} = \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & & \\ 0 & & & 1 \end{bmatrix}$$

$\langle a_i, a_i \rangle = 1, \; i = j$

$\langle a_i, a_j \rangle = 0, \; i \neq j = 1, 2, \ldots, n$

$\langle a_i, a_j \rangle = 1$, if $i = j$

$= 0$, if $i \neq j$

Theorem:- Let A be a real matrix.

(a) A is orthonal ✓

(b) The rows of A forms an orthonormal set ✓

(c) The columns of A forms an orthonoral set. ✓

eg:- $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ $\quad |A| = 1$

(2) $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$, $\quad A^{-1} = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

⊛ Positive definite matrices

⊛ Positive definite matrices

Let A be a real symmetric matrix (i.e. $A = A^T$). A is said to be positive definite, if for every non-zero vector u in $\mathbb{R}^n$

$$\langle u, Au\rangle = u^T A u > 0$$

Let A be a 2×2 symmetric matrix i.e $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is positive definite if the diagonal entries a and d are positive and the determinant $ad - b^2 > 0$.

eg:- Check the given matrices are positive definite or not,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

$|A| = 4 - 9 = -5 < 0 \rightarrow A$ is not positive definite

any the diagonal element is not positive $\rightarrow B$ is not positive definite

$|C| = 5 - 4 = 1 > 0$.

$\therefore C$ is positive definite.

$= \times =$

$\underline{Ax = 0}$ $\qquad \underline{x = 0} \rightarrow$ trivial sol^n $\qquad |A| \neq 0$

$|A| = 0$

$\boxed{Ax = \lambda x} \Rightarrow \boxed{[A - \lambda I]x = 0}$

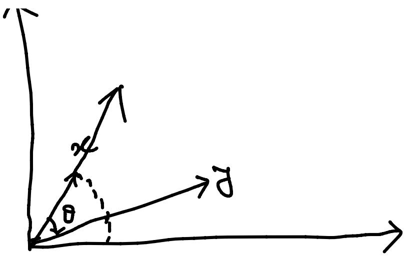
$|A - \lambda I| = 0$

$Ax = \lambda x$

⊛ Singular Value Decomposition (SVD)

$x \quad A \; : \quad \underline{Ax = y}$

$x \curvearrowleft A$: $\underline{\underline{Ax = y}}$

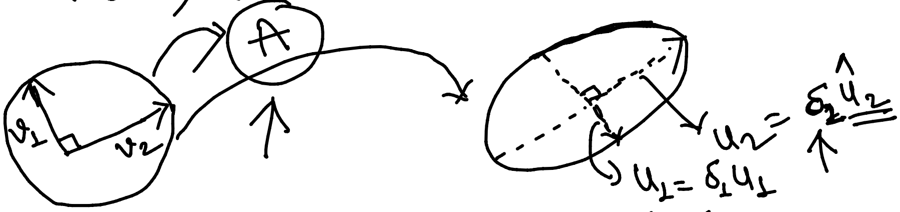


$x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}_{2\times 1}, \quad A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}_{2\times 2}$

$Ax = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ $\quad \alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \to$ stretching

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

V



$\hat{u}_1, \hat{u}_2 \to$ unit vectors

$\sigma_1, \sigma_2 \to$ stretching
↳ singular values

$A v_i = \sigma_j \hat{u}_j$, $j = 1, 2, \cdots, n$ $\qquad Ax = \lambda x$

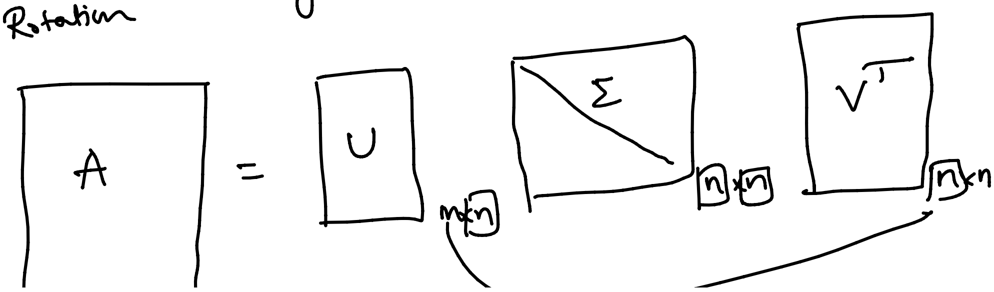
$[A]_{m\times n} [v_1 \; v_2 \cdots v_n]_{n\times n} = [u_1 \; u_2 \cdots u_n]_{m\times n} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \; \sigma_n \end{bmatrix}_{n\times n}$

$AV = U\Sigma \quad \Big| \quad V^T = V^{-1}$

$\boxed{A = U \Sigma V^T} \to$ singular value decomposition

$U \to$ (Orthogonal) $\quad \Sigma$ (diagonal) $\quad V^T$ (orthogonal)

$A = \underbrace{U}_{\text{Rotation}} \underbrace{\Sigma}_{\text{stretching}} \underbrace{V^T}_{\text{Rotation}}$ $\qquad \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$



Th^m:- (SVD): Every matrix $A \in \mathbb{C}^{m\times n}$ has a singular value decomposition.

⊛ Singular values σ_j are uniquely determined

⊛ $\{u_i\}$ and $\{v_j\}$ are also unique

We have, $A = U\Sigma V^T$, $A^T = V\Sigma^T U^T$

$$AA^T = (U\Sigma V^T)(V\Sigma^T U^T)$$

$$= U\Sigma\Sigma^T U^T \quad (\because V^T = V^{-1} \text{ (orthogonal)})$$

$$AA^T U = U\Sigma^2 U^T U = U\Sigma^2$$

$$\boxed{AA^T U = U\Sigma^2} \rightarrow \text{Eigen value problem}$$

$$\boxed{BX = X\lambda}$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3$

$AA^T \rightarrow$ symmetric

$$\Sigma = \begin{bmatrix} \lambda_1^{1/2} & & 0 \\ & \lambda_2^{1/2} & \\ 0 & & \lambda_3^{1/2} \end{bmatrix}$$

stretching

$X = [x_1 \; x_2 \; x_3] = U$

Now, take

$$A^TA = (V\Sigma^T U^T)(U\Sigma V^T)$$

$$= V\Sigma^T\Sigma V^T$$

$$\boxed{A^TA = V\Sigma^2 V^T} \rightarrow$$

$$\boxed{A^TAV = V\Sigma^2} \rightarrow \text{Eigen value problem}$$

$DV = \lambda x$

Σ

$$\boxed{A = U\Sigma V^T}$$

Working Rule:-

① Let A matrix is given. Find A^T

... singular vectors

Working Rule:

① Let A matrix is given. Find A^T

② Find $\underline{AA^T}$ and find eigen values and eigen vectors of AA^T.

Write $\Sigma = \begin{bmatrix} \lambda_1^{1/2} & & & \\ & \lambda_2^{1/2} & & \\ & & \ddots & \\ & & & \\ & 0 & & \end{bmatrix}$ $n\times n$, $m\times n$, $m-n$

Let $x_1, x_2, \ldots, x_n$ forms an orthonormal set

$U = [x_1 \; x_2 \; \cdots \; x_n]$ orthogonal matrix

$U_{m\times n}$

Similarly get V. $\boxed{A = U\Sigma V^T}$ ✓

$Ax = \lambda x \quad \Rightarrow (A-\lambda I)x = 0$

$|A-\lambda I| = 0 \rightarrow$ characteristic eqn

If A is $\underline{2\times 2}$ $\boxed{\lambda^2 - S_1\lambda + |A| = 0}$ ✓ $S_1 \rightarrow$ trace of the matrix

✓ $\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ $\boxed{\lambda^2 - 7\lambda + 10 = 0}$ $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$

$\lambda_1 \times \lambda_2 = |A|$

$\lambda_1 + \lambda_2 = \text{trace}(A)$ $S_1 \rightarrow$ trace of the matrix

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

S_2 = sum of minors of diagonal elements

$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$

eg: Find the singular value decomposition of

$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix};$$

characteristic eqn of A^TA is given by $|A - \lambda I| = 0$

$$\lambda^2 - S_1 \lambda + |A| = 0 \quad \Rightarrow \quad \lambda^2 - 18\lambda = 0$$

$$\lambda(\lambda - 18) = 0 \quad \Rightarrow \quad \lambda = 18, 0$$

$$\lambda_1 = 18, \quad \lambda_2 = 0$$

for $\lambda_1 = 18$

$$\begin{bmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = -x_2, \quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly for $\lambda_2 = 0$

$$\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2, \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let v_1 and v_2 be the orthonormal vectors

$$\therefore v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}; \quad v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore V^T = [v_1 \; v_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A_{m\times n} = U_{m\times m} \boxed{\Sigma_{m\times n}} V^T_{n\times n}$$

$$\therefore \sigma_1 = \sqrt{18} = 3\sqrt{2} \quad \& \quad \sigma_2 = 0$$

$$\begin{bmatrix} 3\sqrt{2} & 0 \end{bmatrix}$$

$\therefore \sigma_1 = \sqrt{18} = 3\sqrt{2}$ & $\sigma_2 = 0$

$$\therefore \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

To construct U. $U = [u_1 \ u_2 \ u_3]$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} \sqrt{2} \\ -2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

u_2 does not exists as $\sigma_2 = 0$

Now, u_2 and u_3 are orthogonal to u_1

i.e. we need to find u_2 and u_3 such that

$\langle u_1, u_2 \rangle = 0$ & $\langle u_1, u_3 \rangle = 0$

i.e. each vector must satisfy $u_1^T x = 0$

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_1 - 2x_2 + 2x_3 = 0$

$\therefore x_1 = 2x_2 - 2x_3$ ($\because x_2$ & x_3 are free variables)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Clearly. $w_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ & $w_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

are linearly independent.

Using Gram - Schmidth orthogonalization.

$$u_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} \cdots \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_3' = w_3 - \frac{\langle u_2, w_3\rangle}{\langle w_2, w_3\rangle} u_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3' = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

Let,

$$u_3 = \frac{u_3'}{\|u_3'\|} \qquad \Bigg| \quad \|u_3'\| = \sqrt{\frac{4}{25} + \frac{16}{25} + 1} = \sqrt{\frac{45}{25}}$$

$$\frac{\sqrt{45}}{5}$$

$$u_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$\boxed{\therefore\ A = U\Sigma V^T} \qquad \underline{\underline{A_{3\times 2}}} = U_{3\times 3}\, \Sigma_{\boxed{3\times 2}}\, V^T_{2\times 2}$$

eg:: Factorize the matrix A into $U\Sigma V^T$ using S.V.D.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$\Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Characteristic eq^n of A^TA is given by

$$|A^TA - \lambda I| = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| \ldots$

$S_1 = \text{trace} = 4, \quad S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$|A| = 1 - 1 = 0$

$\lambda^3 - 4\lambda^2 + 3\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4\lambda + 3) = 0$

$\lambda(\lambda-1)(\lambda-3) = 0 \Rightarrow \boxed{\lambda = 3, 1, 0}$

$\lambda_1 = 3 \Rightarrow \sigma_1 = \sqrt{3}, \quad \lambda_2 = 1, \quad \sigma_2 = 1, \quad \lambda_3 = 0, \quad \sigma_3 = 0$

$$\Sigma_{2\times 3} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 3$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\frac{x_1}{1} = \frac{-x_2}{-2} = \frac{x_3}{1}$$

$$v_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

for $\lambda_2 = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{1} = \frac{-x_2}{0} = \frac{x_3}{-1}$$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

Similarly for $\lambda = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\therefore V^T = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Construct U.

$$u_1 = \frac{A v_1}{\sigma_1}$$

$$A v_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = u_1$$

$$\frac{A v_1}{\sqrt{3}} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = u_1$$

$$u_2 = \frac{A v_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}}{1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\therefore \boxed{A = U \Sigma V^T}$$