Vector Spaces

29 November 2021 10:30

V

(i) Vector addition

(ii) Scalar multiplication

$u, v \in V$

(i) $u + v \in V$ (Closure property)

(ii) $u + v = v + u$ (Commutative law)

(iii) $(u+v)+w = u+(v+w)$ – (associative law)

(iv) $u + 0 = 0 + u = u$ (Existence additive identity)

(v) $u + (-u) = (-u) + u = 0$ [Existence of additive inverse]

$\alpha, \beta \to$ scalar $\quad u, v \in V$

(i) $\alpha u \in V$, $\boxed{\alpha u + \beta v \in V}$

(ii) $1 \cdot u = u$ – (Existence of multiplicative identity)

(iii) $(\alpha + \beta) u = \alpha u + \beta u$

(iv) $\alpha (u + v) = \alpha u + \alpha v$

(v) $(\alpha \beta) u = \alpha (\beta u)$

(1) Polynomials of degree 2 only. ✗

(2) Polynomials up to degree 2.

Set → Subset

$\boxed{\text{Vector Space}}$ Subspace

$V \subseteq W$

$$x = (x_1, x_2, \ldots, x_n)$$

Let V be the set of n tuples $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ with usual addition and scalar multiplication. with

(i) W consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 = 0$ is a subspace of V?

$u, v \in W$

$$u = (0, x_2, x_3, \ldots, x_n)$$
$$+\; v = (0, y_2, y_3, \ldots, y_n)$$
$$= (0, x_2+y_2, x_3+y_3, \ldots, y_n+x_n)$$

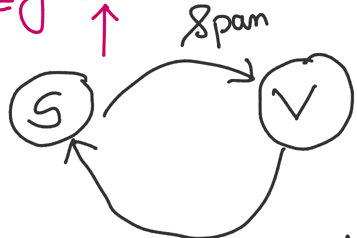
α - scalar and $u \in W$

$$u = (0, u_1, u_2, \ldots, u_n)$$
$$\alpha u = (0, \alpha u_1, \alpha u_2, \ldots, \alpha u_n) \in W \checkmark$$

(ii) W is such that $\boxed{x_1 \geq 0}$ $\qquad (0, 0, 0, \ldots, 0)$

α

Spanning Set :-



Let S be a subset of a vector space V. Suppose that every element in V can be obtained as a linear combination of the elements taken from S. Then S is said to be the spanning set of V.

V - { set of 2×2 real matrices } Vector space

(i) $S \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) $S = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \}$

$S \subseteq V$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\boxed{\alpha_4 = d}$, $\quad \alpha_3 + \alpha_4 = c \Rightarrow \boxed{\alpha_3 = c - d}$

$\alpha_2 + \alpha_3 + \alpha_4 = b$, $\quad \alpha_2 = b - c + d - d$

$\boxed{\alpha_2 = b - c}$

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a \Rightarrow \boxed{\alpha_1 = a - b}$

eg:- Let V be the vector space of all polynomials of degree ≤ 3. Determine whether or not the set $S = \{ t^3, t^2 + t, t^3 + t + 1 \}$ spans V?

Linear Independence and dependence of Vectors

Let V be a vector space. A finite set $\{v_1, v_2, v_3, \dots, v_n\}$ of the elements of V is said to be linearly independent if for the scalars $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

if all α_i's are zero simultaneously.

α_i

$$v_i = -\left(\frac{\alpha_1}{\alpha_i} v_1 + \frac{\alpha_2}{\alpha_i} v_2 + \dots + \frac{\alpha_n}{\alpha_i} v_n\right)$$

eg: Let $v_1 = (1,-1,0)$, $v_2 = (0,1,-1)$, $v_3 = (0,2,1)$ and $v_4 = (1,0,3)$ be elements in $\mathbb{R}^3$. Show that the set of vectors $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

$$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$$

$$\alpha_1(1,-1,0) + \alpha_2(0,1,-1) + \alpha_3(0,2,1) + \alpha_4(1,0,3) = 0$$

$$\alpha_1 + \alpha_4 = 0 \Rightarrow \alpha_1 = -\alpha_4$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 0$$

$$-\alpha_2 + \alpha_3 + 3\alpha_4 = 0$$

$$\boxed{\begin{aligned} \alpha_4 + \alpha_2 + 2\alpha_3 &= 0 \\ 3\alpha_4 + \alpha_3 - \alpha_2 &= 0 \end{aligned}}$$

Here α_4 is arbitrary

Let $\alpha_4 = 1$

$$1 + \alpha_2 + 2\alpha_3 = 0$$

$$3 + \alpha_2 + \alpha_3 = 0$$

$\alpha_1 = 1$

eg:- Let $v_1 = (1,-1,0)$, $v_2 = (0,1,-1)$ and $v_3 = (0,0,1)$ be elements in $\mathbb{R}^3$. Check the vectors are L.I. or L.D.

$\Rightarrow \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$

$\boxed{r=3}$

$S \subseteq V$ $\qquad S = \{u_1, u_2, \ldots, u_n\}$

$\boxed{= \{w, u_1, u_2, \ldots, u_n\}}$

$\underline{a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n} = v \in V$

$\forall\ v \in V$

S is spanning set

S spans V

$\mathbb{R}^3$ $\quad S = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$

$S = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$

$u_1, u_2, \ldots, u_n \rightarrow$ spans V

$w, u_1, u_2, \ldots, u_n \rightarrow$ Spans V $\qquad w$ span V

S

$S \subset W$

Remark:- Suppose $u_1, u_2, \ldots, u_n$ span V. Then for any vector w the set $w, u_1, u_2, \ldots, u_n$ also span V

$u_1, u_2, \dots, u_n$ $\quad$ u_i's $\quad$ $i = 1, 2, \dots, n$

u_k

② Suppose $\boxed{u_1, u_2, \dots, u_n}$ spans V and suppose u_k is a linear combination of some of the u_i's.
Then u_i's without u_k also span V.

$$u_k = \boxed{a_1 u_1 + a_2 u_2 + \dots + a_{k-1} u_k}$$

e.g. Suppose we want to express $v = (3, 7, -4)$ in $\mathbb{R}^3$ as a linear combination of the vectors
$u_1 = (1, 2, 3)$, $u_2 = (2, 3, 7)$ and $u_3 = (3, 5, 6)$

$\Rightarrow$ We seek for scalars x, y, z such that

$$v = x u_1 + y u_2 + z u_3$$

$$\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + z \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$$

$$\left.\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + 5z &= 7 \\ 3x + 7y + 6z &= -4 \end{aligned}\right\}$$
Reducing the system to echelon form

$$\begin{aligned} x + 2y + 3z &= 3 \\ -y - z &= 1 \\ y - 3z &= -13 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} x + 2y + 3z &= 3 \\ -y - z &= 1 \\ -4z &= -12 \end{aligned}$$

$\boxed{z = 3}$ $\quad$ $\boxed{y = -4}$ $\quad$ $\boxed{x = 2}$ ✓

$$\boxed{v = 2u_1 - 4u_2 + 3u_3}$$

⊛ Linear Span, Row space of a Matrix

Suppose $u_1, u_2, \dots, u_n$ are any vectors in a vector space V. Any vector of the form,

$\boxed{a_1u_1 + a_2u_2 + \dots + a_nu_n}$, where the a_i's are scalars

is called a linear combination of $u_1, u_2, \dots, u_n$.

The collection of all such combinations, denoted by: $\text{Span}(u_1, u_2, \dots, u_n)$ or $\text{span}(u_i)$ is a linear span of $u_1, u_2, \dots, u_n$.

$$0 = 0u_1 + 0u_2 + \dots + 0u_n$$

Clearly the zero vector 0 belongs to $\text{span}(u_i)$.

Let v and v' belongs to $\text{span}(u_i)$

$\therefore$ $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$ and

$v' = b_1u_1 + b_2u_2 + \dots + b_nu_n$

Let k is some another scalar

$\therefore$ $kv + v' = ka_1u_1 + ka_2u_2 + \dots + ka_nu_n + b_1u_1 + b_2u_2 + \dots + b_nu_n$

$kv + v' = (ka_1 + b_1)u_1 + (ka_2 + b_2)u_2 + \dots + (ka_n + b_n)u_n$

$\Rightarrow$ $\forall\, v, v' \in \text{Span}(u_i)$, $kv + v' \in \text{Span}(u_i)$

$\therefore$ $\text{Span}(u_i)$ is a subspace of V.

More generally, for any subset S of V, $\text{Span}(S)$ consists of all linear combinations of vectors in S

Span(S) consists of all linear combinations of vectors in S.

When $S = \phi$, span(S) = $\{0\}$

Thus in particular S is a spanning set of span(S).

Theorem:- Let S be a subset of a vector space V.

(i) The span(S) is a subspace of V that contains S.

(ii) If W is subspace of V containing S, then span(S) $\subseteq$ W

$\Rightarrow$ i.e. span(S) is the smallest subspace of V containing S.

eg:- Let u be any non-zero vector in $\mathbb{R}^3$. Then span(u) consists of all scalar multiples of u. Geometrically, span(u) is a line through origine and the end points of u.

eg:- Let u and v be vectors in $\mathbb{R}^3$ that are not multiples of each other. Then span(u,v) is the plane through origine and the end points of u & v

eg:- Consider the vector $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ in $\mathbb{R}^3$. Every vector in $\mathbb{R}^3$ is a linear combination of e_1, e_2 and e_3.

$$\text{Span}(e_1, e_2, e_3) = \mathbb{R}^3$$

(*) Row Space of a matrix

Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix over a field K. The rows of A, over a field K. The rows of A,

$R_1 = (a_{11}, a_{12}, \cdots, a_{1n})$, $R_2 = (a_{21}, a_{22}, \cdots, a_{2n})$, $\cdots$, $R_m = (a_{m1}, a_{m2}, \cdots, a_{mn})$ may be viewed as vectors in K^n hence they span a subspace of K^n called the row space of A and is denoted by rowsp(A).

i.e. $\text{rowsp}(A) = \text{span}(R_1, R_2, \cdots, R_m)$

Similarly, the columns of A may be viewed as vectors in K^m hence they span a subspace of K^m called the column space of A, denoted by colsp(A).

$$\text{colsp}(A) = \text{span}(C_1, C_2, \cdots, C_n)$$

Observe that, $\text{colsp}(A) = \text{rowsp}(A^T)$

=×=

⊛ Basis and Dimension

Defn:- A set $S = \{u_1, u_2, \cdots, u_n\}$ of vectors is a basis of vector space V if it has the following properties

(i) S is linearly independent

(ii) S span V

Theorem:- Let V be a vector space such that one basis has m elements and another basis has n elements. Then $m = n$

The number of elements in a basis (if basis is finite) is the dimension of a vector space

The number of elements ... finite) is the dimension of a vector space

$\mathbb{R}^2$ $\quad e_1\ (1,0)\quad e_2\ (0,1) \rightarrow$ Standard basis

$\mathbb{R}^n$ $\quad \underset{e_1}{(1,0,0,\cdots,0)}, \underset{e_2}{(0,1,0,\cdots,0)}, \cdots, \underset{e_n}{(0,0,\cdots,0,1)}$

n

A vector space V is said to be of finite dimensions n or n-dimensional if S (basis of V) has n elements.

$$\boxed{\dim V = n}$$

eg:- If V consists of all $m\times n$ matrices, then

$$E_{rs} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad r = 1,2,\cdots,m \text{ and } s = 1,2,\cdots,n$$

where 1 is located in the (r,s) location i.e. in the r^{th} row and s^{th} column is called standard basis.

Consider $M_{2\times3}$ i.e. vector space $M_{2\times3}$ of all 2×3 matrices or K:

Six elements,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\dim M_{r\times s} = r\times s}$$

eg:- Vector Space $P_n(t)$ of all polynomials of degree $\leq n$.

The set $S = \{1, t, t^2, \cdots, t^n\}$ of $\underline{(n+1)}$ polynomials

... basis of $P_n(t)$.

The set ...

is a basis of $P_n(t)$.

$\boxed{\therefore \dim P_n(t) = n+1}$

$\boxed{f(t),\ g(t),\ h(t)}$ — W

$$\boxed{\begin{vmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{vmatrix}} \begin{cases} = 0 \to \text{L.D.} \\ \neq 0 \to \text{L.I} \end{cases}$$

$$\begin{array}{cc} f(t) & g(t) \end{array}$$

$$W = \boxed{\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}} \begin{cases} = 0 & \text{L.D} \\ \neq 0 & \text{L.I} \end{cases}$$

Wronskian

eg:- Vector Space $P(t)$ of all polynomials

$S = \{ f_1(t), f_2(t), \ldots, \boxed{f_m(t)} \}$

$\boxed{S = \{1, t, t^2, \ldots, t^n, \ldots\}}$

$P(t)$ is infinite dimensional vector space

Theorem:-

(*) Let V be a vector space of dimension n. Let $v_1, v_2, \ldots, v_n$ be the linearly independent elements of V. [Then, every other element of V can be represented as linear combination of these elements.] Further this representation is unique.

$\{v_1, v_2, \ldots, v_n\}$ will span V

$\{v_1, v_2, \dots, v_n\}$ with ...

This forms a basis of V.

Remarks

(*) A set of $(n+1)$ vectors in $\mathbb{R}^n$ is linearly dependent!

(*) A set of vectors containing 0 as one of it's elements is linearly dependent as 0 is the linear combination of any set of vectors.

Theorem:- Let V be an n-dimentional vector space. If $\boxed{v_1, v_2, \dots, v_k}$, $k < n$ are L.I. elements of V, then $\exists$ elements $v_{k+1}, v_{k+2}, \dots, v_n$ such that $\{v_1, v_2, \dots, v_{k+1}, v_{k+2}, \dots v_n\}$ is basis of V.

(*) There can be many basis for the same vector space.

eg:. Consider the vector space $\mathbb{R}^3$. Each of the following set of vectors (a, b, c)

(i) $[1, -1, 0]$, $[0, 1, -1]$, $[0, 0, 1]$

(ii) $[1, -1, 0]$, $[0, 0, 1]$, $[1, 2, 3]$

(iii) $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ standard basis.

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

are linearly independent and therefore form a basis in $\mathbb{R}^3$.

... $\{u, v, w\}$

a basis in $\mathbb{R}^3$. U

eg:- Determine the following set of vectors $\{u,v,w\}$ form a basis in $\mathbb{R}^3$, where

(i) $u=(2,2,0)$, $v=(3,0,2)$, $w=(2,-2,2)$

(ii) $u=(0,1,-1)$, $v=(-1,0,-1)$, $w=(3,1,3)$.

$\Rightarrow$ (i) Linear combination of u,v,w is

$$\alpha_1 u+\alpha_2 v+\alpha_3 w=0$$

$$\alpha_1(2,2,0)+\alpha_2(3,0,2)+\alpha_3(2,-2,2)=0$$

$$2\alpha_1+3\alpha_2+2\alpha_3=0, \quad 2\alpha_1-2\alpha_3=0 \Rightarrow \alpha_1=\alpha_3$$

$$2\alpha_2+2\alpha_3=0 \Rightarrow \alpha_2=-\alpha_3$$

$$\alpha_1=-\alpha_2=\alpha_3$$

$$2\alpha_1-3\alpha_1+2\alpha_1=0 \Rightarrow \boxed{\alpha_1=0}$$

$$\boxed{\alpha_1=-\alpha_2=\alpha_3=0}$$

$\Rightarrow \{u,v,w\}$ is L.I. Therefore it forms a basis of $\mathbb{R}^3$

eg(2): Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$. Hence find its basis.

$\Rightarrow$ The dimension of the set is ≤ 4. If it is 4 then the only soln of the vector equation

$$a(1,0,0,0)+b(0,1,0,0)+c(1,2,0,1)+d(0,0,0,1)=0$$

should be $a=b=c=d=0$.

Comparing, we get the system of equation

$$a+c=0, \quad b+2c=0, \quad c+d=0$$

soln Therefore the given set

$a+c=0, \quad b+2c=0, \quad c+d=0$

Here we get the non zero sol^n. Therefore the given set of vectors are linearly dependent. Hence the dimension of the set is less than 4.

Now consider any three elements of the set.

say $(1,0,0,0)$, $(0,1,0,0)$, $(1,2,0,1)$.

Consider the vector eq^n

$a(1,0,0,0) + b(0,1,0,0) + c(1,2,0,1) = 0$

$a+c=0, \quad b+2c=0, \quad c=0$

$\Rightarrow \boxed{a=b=c=0}$

Hence these three elements are linearly independent.

Therefore the dimension of the subspace spaned by $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$ is 3.

$$\boxed{(0,0,0,1) = -1(1,0,0,0) - 2(0,1,0,0) + 1(1,2,0,1)}$$

eg:: Let $\boxed{U = \{(a,b,c,d), \text{ such that } a+c+d=0, \ b+d=0\}}$ be a subspace of $\mathbb{R}^4$. Find the dimension and the basis of the subspace.

$\Rightarrow$ U satisfies clossure property. From the given equation we have,

$a=-c-d$ and $b=-d$

We have two free parameters, say c & d.

Therefore the dimension of the given subspace is 2.

Choosing $c=1, d=1$ and $c=1, d=0$

we may write basis as,

$\{(-2,-1,0,1), (-1,0,1,0)\}$

we may $\{(-2,-1,0,1),\ (-1,0,1,0)\}$

H.W. Determine whether or not each of the following form a basis of $\mathbb{R}^3$

ⓐ $(1,1,1),\ (1,0,1)$; ⓑ $(1,2,3),\ (1,3,5),\ (1,0,1),\ (2,3,0)$

ⓒ $(1,1,2),(1,2,3),\ (2,-1,1)$; ✓

ⓓ $(1,1,2),\ (1,2,5),\ (5,3,4)$

eg:- Determine whether $(1,1,1,1),\ (1,2,3,2),\ (2,5,6,4),\ (2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not, find the dimension of the subspace they span.

$\Rightarrow$ Form the matrix whose rows are the given vectors, and reduce it to row echelon form

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form has a zero row. Hence the four vectors are linearly dependent and do not form a basis of $\mathbb{R}^4$. Since the echelon matrix has three non-zero rows, the four vectors span a subspace of dimension **3**.

… $u_2 = (2,2,3,4)\}$ to a basis

eg:- Extend $\{u_1 = (1,1,1,1),\ u_2 = (2,2,3,4)\}$ to a basis of $\mathbb{R}^4$.

⇒ First form the matrix with rows u_1 and u_2 and reduce to echelon form. $(1,0,0,0)$ e_1 $(0,0,1,0)$ e_3 e_1 e_4

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

e_2 e_4

u_1 and u_2 spans the same set of vectors spanned by $(1,1,1,1)$ and $(0,0,1,2)$. Let $u_3 = (0,1,0,0)$ and $u_4 = (0,0,0,1)$. Then u_1, u_2, u_3 and u_4 form a matrix in echelon form without zero rows. Therefore they are linearly independent and they form a basis of $\mathbb{R}^4$.

Hi. Linear Algebra by

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3. Higher Engineering Mathematics by Jain Ringer Jain
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eg:- Let W be the subspace of $\mathbb{R}^5$ spanned by $u_1 = (1,2,-1,3,4)$, $u_2 = (2,4,-2,6,8)$, $u_3 = (1,3,2,2,6)$, $u_4 = (1,4,5,1,8)$, $u_5 = (2,7,3,3,9)$. Find a subset of the vectors that form a basis of W.

⇒ Let,

$$M = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \end{bmatrix}$$

$\Rightarrow$ Let, $M = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix} \end{array} \begin{array}{l} u_1 \\ \\ u_3 \\ \\ u_5 \end{array}$$

The rows 1st, 3rd and 5th are non-zero, thus u_1, u_3 and u_5 are linearly independent and form basis of W. and $\dim(W) = 3$

$$\mathcal{M} = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccccc} u_1 & & u_3 & & u_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The pivot positions are in columns C_1, C_3, C_5.

Hence the corresponding vectors u_1, u_3, u_5 forms a basis of W and $\dim(W) = 3$.

eg:- Let V be the vector space of 2×2 matrices over K. Let W be the subspace of symmetric matrices. Show that $\dim(W) = 3$, by finding a basis of W.

$\Rightarrow$ Let $A \in W$ $\Rightarrow$ $A = A^T$ i.e. $A = [a_{ij}]$ and $a_{ij} = a_{ji}$ $\forall$ $i \& j$

$\Rightarrow$ Let $A \in W$ ...

Thus, $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

Setting (i) $a=1$, $b=0$, $d=0$ (ii) $a=0$, $b=1$ and $d=0$
(iii) $a=0$, $b=0$ and $d=1$. we get

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$S = \{E_1, E_2, E_3\}$

(a) The above matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = aE_1 + bE_2 + dE_3$

Thus S spans W.

(b) Suppose $xE_1 + yE_2 + zE = 0$, where x, y and z are unknowns.

$$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow x=0,\ y=0\ \&\ z=0$

$\Rightarrow$ S is linearly independent

$\Rightarrow$ Therefore, S is a basis of W and $\dim(W) = 3$.

* Linear Transformation (mapping) (function)

Let A and B be two arbitrary sets. A rule that assigns to elements of A exactly one element (unique) of B is called a function or a mapping or a transformation.

$$T : A \to B$$

The set A is called the domain

For $\forall\ a \in A$, we get unique $b \in B$. We write $T(a) = b$ or $b = T(a)$ and b is called the ... the transformation T;

$T(a) = b$ or $b = T(a)$ and ...

image of a under the transformation T.

Let V and W be two vector spaces and T is transformation from V to W.

$$T: V \to W$$

The T is said to be linear transformation or linear mapping, if it satisfies the following properties.

(i) For every scalar α and every element v in V

$$T(\alpha v) = \alpha T(v)$$

(ii) For any two elements v_1 and v_2 in V

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$\boxed{T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)}$$

$$T: V \to W$$

Let V be a vector space of dimension n. and let $\{v_1, v_2, \dots, v_n\}$ be its basis.

$$v = \sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, \quad v \in V$$

where α_i's are scalars, not all zero simultaneous.

$$T(v) = T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$$

$$= \alpha_1 \underline{T(v_1)} + \alpha_2 \underline{T(v_2)} + \dots + \alpha_n \underline{T(v_n)}$$

Thus, a linear transformation is completely determined by its action on the basis of a vector space.

$$T(0v) = 0T(v) = \underline{\underline{0}}$$

$$T(0v) = 0T(v) = 0$$

Therefore, the zero element in V is mapped into zero element in W by the linear transformation T.

$$T : V \to W$$

$$\text{ran}(T) = \{ w \in W : T(v) = w,\ v \in V \} \text{ Range of } T$$

The collection of all elements $w = T(v)$ is called the range of T. We denote it as $\text{ran}(T)$.

* Show that $\text{ran}(T)$ is a subspace of W.

Kernel of Transformation (T) $\quad$ $\ker(T)$

$$\ker(T) = \{ v \in V \mid T(v) = 0 \}$$

The set of all elements of V (Domain) that are mapped into the zero element by linear transformation T and is called the kernel or the null-space of T and is denoted by $\ker(T)$.

eg:- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection of a vector v into the plane (xy-plane).

$$(x, y, z) \in \mathbb{R}^3, \quad T(x, y, z) = (x, y, 0)$$

$$\Rightarrow \underset{v}{(x_1, y_1, z_1)}, \underset{w}{(x_2, y_2, z_2)} \in \mathbb{R}^3, \quad \alpha \in F$$

$$\alpha v + w = (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2)$$

$$T(\alpha v + w) = T(\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2)$$

$$= (\alpha x_1 + x_2, \alpha y_1 + y_2, 0)$$

$= (\alpha x_1, \alpha y_1, 0) + (x_2, y_2, 0)$

$$\boxed{T(\alpha v + w) = \alpha T(v) + T(w)}$$

$Im(T) = ran(T) = \{(a,b,c) : c = 0\} = xy$- plane

$ker(T) = \{(a,b,c) : a = 0, b = 0\} = z$- axis

eg:- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(x,y) = (x+1, y+2)$. Check whether T is a linear Transformation or not.

$\Rightarrow$ $T(0,0) = (0+1, 0+2) = (1,2) \neq 0$

$D : V \to W$, $D(\alpha u + v) = \alpha Du + Dv$

$I : V \to W$, $I(\alpha u + v) = \alpha \int u\,dx + \int v\,dx$

⊛ Zero mapping

Let $T : V \to W$ be the mapping that assigns the zero vector $0 \in W$ to every vector $v \in V$.

$T(v) = 0, \quad \forall\, v \in V$

⊛ Identity mapping:

$T : V \to W$ is said to be identity mapping if it maps each $v \in V$ into itself. i.e.

$T(v) = v$. I

$T(av + u) = av + u = aT(v) + T(u)$

⊛ Rank Nullity Theorem:-

... be a linear transformation. If T has ...

⊛ Rank Nullity Theorem:

Let $T : V \to W$ be a linear transformation. If T has rank (r) $[\dim \operatorname{ran}(T) = r]$ and dimension of V is n, then the nullity of T is $n - r$, that is,

$$\underline{\underline{\text{rank}(T)}} + \underline{\underline{\text{nullity}}} = n = \dim(V).$$

⊛ Kernel and Image of Matrix Mapping

Matrix Mapping: Let $A : \mathbb{R}^m \to \mathbb{R}^n$.

$$\underline{\underline{A_{n\times m} X_{m\times 1} \to Y_{n\times 1}}}$$

$$A : \mathbb{R}^4 \to \mathbb{R}^3$$

Consider a 3×4 matrix A the usual basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$,

$$e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}_{3\times 4}$$

$$Ae_1 = [a_1, b_1, c_1]^T, \quad Ae_2 = [a_2, b_2, c_2]^T,$$

$$Ae_3 = [a_3, b_3, c_3]^T, \quad Ae_4 = [a_4, b_4, c_4]^T$$

⊛ Thus the image of A (matrix mapping) is precisely the column space of A.

The kernel of A is consists of all vectors $v \in \mathbb{R}^4$ for which $\underline{\underline{Av = 0}}$. This means that the kernel of A is the solution space of the homogeneous system $\boxed{Ax = 0}$, null space of A.

the solution space of ...
called the null space of A.

⊛ Let A be any $m \times n$ matrix over field $\mathbb{R}$ viewed as a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. Then

$\ker(A) = $ null space (A) and $\mathrm{Im}(A) = \mathrm{col\,sp}(A)$.

eg:- Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by

$$F(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$$

Find the basis and the image of F.

$\Rightarrow$ We have, $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 (0,0,0,1)$ standard basis in $\mathbb{R}^4$.

$\therefore Fe_1 = F(1,0,0,0) = (1,2,3)$, $Fe_2 = F(0,1,0,0) = (-1,-2,-3)$,

$Fe_3 = F(0,0,1,0) = (1,3,4)$, $Fe_4 = F(0,0,0,1) = (1,4,5)$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $(1,2,3)$, $(0,1,1)$ form a basis of $\mathrm{Im}(F)$.

$\therefore \mathrm{Dim}(\mathrm{Im}\,F) = 2 = \mathrm{ran}(F)$.

(b) Find basis and dimension of the kernel of (F).

Set $\{ \underline{v} \in V \text{ s.t. } F(v) = 0 \}$.

Let $v = (x,y,z,t)$

$\therefore (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$

Let $v = (x, y, z, t)$

$\therefore F(v) = F(x, y, z, t) = (x - y + z + t,\ 2x - 2y + 3z + 4t,\ 3x - 3y + 4z + 5t)$

$= (0, 0, 0)$

$$\begin{aligned} x - y + z + t &= 0 \\ 2x - 2y + 3z + 4t &= 0 \\ 3x - 3y + 4z + 5t &= 0 \end{aligned} \sim \begin{aligned} x - y + z + t &= 0 \\ z + 2t &= 0 \\ z + 2t &= 0 \end{aligned}$$

$Ax -$

$$\sim \left.\begin{aligned} x - y + z + t &= 0 \\ z + 2t &= 0 \end{aligned}\right\}$$

The Free variables are y and t.

Hence $\dim(\ker F) = 2$ or

nullity$(F) = 2$

① $y = 0,\ t = 1$, $(1, 0, -2, 1)$

② $y = 1,\ t = 0$, $(1, 1, 0, 0)$

$\therefore (1, 0, -2, 1)$ and $(1, 1, 0, 0)$ forms a basis of $\ker(F)$.

Eg: For the set of vectors $x_1 = (1, 3)^T$, $x_2 = (4, 6)^T$, are in $\mathbb{R}^2$, Find the matrix of linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, such that,

$$T x_1 = (-2, 2, -7)^T$$

$$T x_2 = (-2, -4, -10)^T.$$

$\Rightarrow$ Let $A_{3\times2} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$ be a matrix of linear transformation.

$\Rightarrow$ Let $A_{3\times 2} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$ be a matrix of transformation. ✓

$T = AX$

$\therefore \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -7 \end{bmatrix}$

and $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -10 \end{bmatrix}$

$a_1 + 3b_1 = -2$
$a_2 + 3b_2 = 2$
$a_3 + 3b_3 = -7$
$4a_1 + 6b_1 = -2$
$2a_1 + 3b_1 = -1$
$2a_2 + 3b_2 = -2$
$2a_3 + 3b_3 = -5$

$A = \begin{bmatrix} 1 & -1 \\ -4 & 2 \\ 2 & -3 \end{bmatrix}$

This is the required matrix of linear transformation.

eg:- Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping for which $F(1,2) = (2,3)$ and $F(0,1) = (1,4)$. Find a formula for F.

$F(a,b) = ?$

$\Rightarrow$ Let $(a,b) \in \mathbb{R}^2$,

$(a,b) = x(1,2) + y(0,1)$
$(a,b) = (x, 2x+y)$

$F(a,b) = ?$

$F(a,b) = x\,F(1,2) + y\,F(0,1)$

$F(a,b) = x(2,3) + y(1,4)$

H.W.

eg:- Let $G: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by $G(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$. Find a $\boxed{\text{basis}}$ and the dimensions of

ⓐ the image of G ✓ $\text{ran}(G) = 2$

ⓑ the kernel of G ✓ $\ker(G) \to$ nullity of $G = 1$.

eg:- Consider the linear map $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $G(x,y) = (3x+5y, 2x+3y)$ and let S be the unit circle in $\mathbb{R}^2$.

(a) Find the preimage of S or $G^{-1}(S)$

$\Rightarrow$ $\boxed{\text{Let } (x,y) \in S \quad \Rightarrow \quad x^2+y^2=1}$

$$G(x,y) = (3x+5y, 2x+3y)$$

$$\{(x,y) \in \mathbb{R}^2 \text{ s.t. } G(x,y) = (3x+5y, 2x+3y), \; (3x+5y)^2 + (2x+3y)^2 = 1\}$$

$$9x^2+30xy+25y^2 + 4x^2+12xy+9y^2 = 1$$

$$\boxed{13x^2+42xy+34y^2=1}$$ Preimage of S.

$$\text{Pre}(S) = \{(x,y) \in \mathbb{R}^2 : 13x^2+42xy+34y^2=1\}$$

find Image of $G(S)$.

⊛ Matrix Representation of a linear transformation

$T : V \to W$; $T : \mathbb{R}^n \to \mathbb{R}^m$

(n) (m)

$A_{m \times n}$

Let V and W be respectively, n-dimentional and m-dimentional vectors spaces over the same field ($\mathbb{F}$).

Let T be a linear transformation such that

$T : V \to W.$

Let $X = \{v_1, v_2, \cdots, v_n\}$; $Y = \{w_1, w_2, \cdots, w_n\}$

be the ordered basis of V and W respectively.

Let $v \in V$ and $w \in W$.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where α_i's are scalars not all zero

$w = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n$, where β_i's are scalars not all zero

$$T(v) = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$w = \alpha_1 \underline{T(v_1)} + \alpha_2 \underline{T(v_2)} + \cdots + \alpha_n \underline{T(v_n)}$$

Since every element $T(v_i)$, $i = 1, 2, \cdots, n$ is in W, it can be written as linear combination of the basis vectors $w_1, w_2, \cdots, w_m$.

That is $\exists$ scalars a_{ij} , $i = 1, 2, \cdots, n$, $j = 1, 2, 3, \cdots, m$

$$\underline{T(v_i)} = a_{1i} w_1 + a_{2i} w_2 + \cdots + a_{mi} w_m$$

$$= [w_1, w_2, \cdots, w_m]_{1 \times m} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$\boxed{T\underline{X} = \underline{YA}}$$

where A is the $m \times n$ matrix.

The $m \times n$ matrix A is called the matrix representation of T.

⊛ For a given ordered basis X and Y of vector spaces V and W respectively, and a linear transformation $T : V \to W$, the matrix $\underline{A}$ is unique.

the matrix $\underline{\underline{A}}$ is unique.

eg:: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ (V, W) be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}.$$

Determine the matrix of the linear transformation T, with respect to the ordered basis,

(i) Standard basis.

$\Rightarrow$ We have, $X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$

and $Y = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$

we have, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$T(v_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} 0$$

$$T(v_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1)$$

$$T(v_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-1)$$

$$TX = YA$$

$$T\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$TX = Y\underline{\underline{A}}$$

Therefore, the matrix of the linear transformation with respect to the standard basis is given by with respect to the standa...

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

(ii) $X = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^3$, $Y = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

$$T(v_1) = T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(2) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-2)$$

$$T(v_2) = T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-2)$$

$$T(v_3) = T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0)$$

$$TX = YA$$

$$\beta = A\alpha$$
$m\times 1$ $m\times n$ $n\times 1$

$$\boxed{\therefore A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & -2 & 0 \end{bmatrix}}$$

H.W. $X = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$; $Y = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$T : V \to W$ $\quad \forall v \in V$

Im $\quad T(v) = w \in W$

Rank: $\operatorname{Im}(T) = \{ w \in W \text{ s.t. } T(v) = w,\ v \in V \}$

Nullity $\ker(T) = \{ v \in V \text{ s.t. } T(v) = 0 \in W \}$

$Tv = 0$ $\qquad Ax = 0$

$T : V \to V$

$\boxed{T : \underline{\underline{V}} \to \underline{\underline{K}}}$ $\quad$ $T : \underline{\underline{V}} \to \underline{\underline{V}}$