Inner Product Spaces.

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Linear Algebra.

- Subspaces
- Transformation
  - $TX = AX$
- Inner Product Space

$[A - \lambda I] = 0$

$|A - \lambda I| = 0$

$Ax = 0$

$x = 0$

$\boxed{x = 0}$ Trivial solⁿ

$[A - \lambda I]\, X = 0$

$AX = \lambda X$

$A \quad \lambda$

$|A - \lambda I| = 0$

$|A - \lambda I| = 0$

* Norm:-

A norm $\|\cdot\|$ on a (linear space) vector space X (over the field $\mathbb{K}$ of real or complex numbers) is a function

$$x \to \|x\|, \quad x \in X$$

from X to the set $\mathbb{R}$ of all real numbers such that for every $x, y \in X$ and $\alpha \in \mathbb{K}$

(a) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

(b) $\|\alpha x\| = |\alpha|\,\|x\|$.

$(\sqrt{a^2+b^2+c^2})$

2

4

(b) $\|\alpha x\| = |\alpha| \, \|x\|$.

(c) $\|x+y\| \leq \|x\| + \|y\|$

$(\sqrt{a^2+b^2+c^2})$

$(x_1+y_1) + (x_2+y_2) = (x_1+x_2, y_1+y_2)$

$(x_1, y_1) + (x_2, y_2) = (x_1+y_2, x_2+y_1)$

Standard norm: $x = (a, b, c)$

$\|x\| = \sqrt{a^2+b^2+c^2}$ (standard length)

Inner Product Space:

$a \, (a_1, b_1, c_1)$ $\quad b \, (a_2, b_2, c_2)$

$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\langle a, b \rangle$

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|a| \, |b|}$$

$|a| = (a_1^2 + b_1^2 + c_1^2)^{1/2}$ $\quad a \cdot b = b \cdot a$

$a \cdot b$ – scalar quantity

symmetric

and linear

An inner product space of a vector space X is a map

$\langle \, , \, \rangle : \langle x, y \rangle \in \mathbb{R}, \quad (x, y) \in X \times X$

X is a map

$$(x,y) \to \langle x,y \rangle \in \mathbb{R}, \quad (x,y) \in X \times X$$

which satisfies the following axioms:

(a) $\langle x,x \rangle \geq 0, \ \forall\, x \in X$ and

$$\langle x,x \rangle = 0 \iff x = 0$$

(b) $\langle x+y, z \rangle = \langle x,z \rangle + \langle y,z \rangle \quad \forall\, x,y,z \in X$

(c) $\langle ax, y \rangle = a\langle x,y \rangle, \quad \forall\, a \in \mathbb{K}, \ \forall\, x,y \in X$

(d) $\langle x,y \rangle = \langle y,x \rangle$ ✓ | $\langle x,y \rangle = \overline{\langle y,x \rangle}$

Real field | Complex field

A vector space X with $\langle \cdot,\cdot \rangle$ inner product is called an inner product space.

eg: Consider vectors $u = (2,3,5)$ and $v = (1,-4,3)$ in $\mathbb{R}^3$. Then

$$\langle \vec{u}, \vec{v} \rangle = 1 - 12 + 15 = 5, \in \mathbb{R}$$

$$\|\vec{u}\| = \sqrt{4+9+25} = \sqrt{38} \in \mathbb{R}$$

(*) Euclidian n-space $\mathbb{R}^n$.

Consider the vector space $\mathbb{R}^n$. The dot product in $\mathbb{R}^n$ is

Consider the . . .

defined by,

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $u = (a_i)$ and $v = (b_i)$

This function defines an inner product on $\mathbb{R}^n$.

$$\|u\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{u \cdot u}$$

$$\|u\|^2 = \langle u \cdot u \rangle \quad \forall\, u \in \mathbb{R}^n$$

(*) Remark :- Frequently the vectors in $\mathbb{R}^n$ will be represented by column vectors, that is by $n \times 1$ column matrices.

In such case, the formula,

$$\langle u, v \rangle = u^T v \in \mathbb{R}$$

defines the usual inner product on $\mathbb{R}^n$.

eg :- Let $\boxed{C[a,b]}$ it consists of all the continuous functions on the interval $[a,b]$. $a \le t \le b$

$$\langle f, g \rangle = \int_a^b f(t) \cdot g(t)\, dt \quad , \quad \forall\, f, g \in C[a,b]$$

This is the usual inner product on $C[a,b]$.

H.W.

$\Rightarrow$ (a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

$\Rightarrow$ (a) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ DU

Let $f \in C[a,b]$

$\langle f, f\rangle = \int_a^b \underline{\underline{f^2(t)}}\, dt \geq 0 \quad \forall f \in C[a,b]$

Let $\int_a^b f^2(t)\, dt = 0 \Rightarrow f(t) = 0$

(b) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

$f, g, h \in C[a,b]$.

$\therefore \langle f+g, h\rangle = \int_a^b (f(t)+g(t))\, h(t)\, dt$

$= \int_a^b f(t)\cdot h(t)\, dt + \int_a^b g(t)\cdot h(t)\, dt$

$= \langle f, h\rangle + \langle g, h\rangle$

(c) $\langle ax, y\rangle = a\langle x, y\rangle$

Let $f, g \in C[a,b]$ and $a \in \mathbb{R}$

$\therefore \langle af, g\rangle = \int_a^b a\, f(t)\cdot g(t)\, dt = a\int_a^b f(t)\cdot g(t)\, dt$

$= a\langle f, g\rangle$

(4) $\langle x, y\rangle = \langle y, x\rangle$

Let $f, g \in C[a,b]$

$\int^b f(t)\cdot g(t)\, dt = \int^b g(t)\cdot f(t)\, dt$

Let $f, g \in C[a,b]$

$$\therefore \langle f, g \rangle = \int_a^b f(t) \cdot g(t)\, dt = \int_a^b g(t) \cdot f(t)\, dt$$

$$\boxed{\langle f, g \rangle = \langle g, f \rangle}$$

eg:- Consider $f(t) = 3t - 5$ and $g(t) = t^2$

$C[a,b]$ $\quad [0,1]$

Find $\langle f, g \rangle$.

$\Rightarrow$ We have $\langle f, g \rangle = \int_0^1 (3t-5)t^2\, dt$

$$= \int_0^1 (3t^3 - 5t^2)\, dt \checkmark$$

$$= \left. \frac{3t^4}{4} - \frac{5t^3}{3} \right|_0^1$$

$$= \frac{3}{4} - \frac{5}{3} = \underline{\frac{-11}{12}}$$

eg:- Find $\|f\|$ and $\|g\|$

We have, $\|f\|^2 = \langle f, f \rangle$

$$\|f\|^2 = \int_0^1 (3t-5)(3t-5)\, dt = \int_0^1 (9t^2 - 30t + 25)\, dt$$

$$\therefore \|f\|^2 = \int_0^1 (3t-5)(3t-5)\,dt = \int_0^1 (\ldots$$

$$= 9\frac{t^3}{3} - 30\frac{t^2}{2} + 25t \Big|_0^1$$

$$= 3(1) - 15(1) + 25$$

$$\|f\|^2 = 13 \Rightarrow \|f\| = \sqrt{13}$$

and $\|g\|^2 = \langle g, g\rangle = \int_0^1 t^4\,dt = \frac{t^5}{5}\Big|_0^1 = \frac{1}{5}$

$$\|g\| = \frac{\sqrt{5}}{5}$$

Matrix Space $V = M_{m\times n}$

$A, B \in M_{m\times n}$

A.k.l. $\boxed{\langle A, B\rangle = \operatorname{tr}(B^T A)}$

$$\langle A, B\rangle = \operatorname{tr}(B^T A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ij}$$

$$\|A\|^2 = \langle A, A\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$$

Angle betn Vectors:-

vectors u, v in an

Angle bet^n vectors:-

For any non-zero vectors u, v in an inner product space V, the angle bet^n u and v where $u, v \in V$, is defined to be the angle θ such that $0 \leq \theta \leq \pi$ and

$$\boxed{\cos\theta = \frac{\langle u, v\rangle}{\|u\| \, \|v\|}}$$

eg:- Consider vectors $u = (2, 3, 5)$ and $v = (1, -4, 3)$ in $\mathbb{R}^3$. Find the angle bet^n u and v.

$\Rightarrow$ $\|u\| = \sqrt{4+9+25} = \sqrt{38}$, $\|v\| = \sqrt{1+16+9} = \sqrt{26}$

$\langle u, v\rangle = 2 - 12 + 15 = 5$.

$$\therefore \quad \cos\theta = \frac{\langle u, v\rangle}{\|u\| \; \|v\|} = \frac{5}{\sqrt{38}\,\sqrt{26}}$$

eg:- $f(t) = 3t - 5$ and $g(t) = t^2$

$\langle f, g\rangle = \frac{-11}{12}$, $\|f\| = \sqrt{13}$

$\|g\| = \frac{\sqrt{5}}{5}$

$$\therefore \quad \cos\theta = \frac{-11/12}{\sqrt{13} \; 1/\sqrt{5}}$$

✱ Orthogonality :-

Let X be an I.P.S. The vectors $x, y \in X$ are said to be orthogonal (x is orthogonal to y) iff

$$\langle x, y\rangle = 0 = \langle y, x\rangle$$

eg:- If $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2) \in \mathbb{R}^2(\mathbb{R})$, let us define $\langle \alpha, \beta\rangle = \langle (a_1, a_2), (b_1, b_2)\rangle$

$$= a_1 b_1 - a_2 b_1 - a_1 b_2 + 4 a_2 b_2.$$

Verify that the above product is an inner product on $\mathbb{R}^2$.

$\Rightarrow$

(i) $\langle \alpha, \alpha\rangle \geq 0$ and $\langle \alpha, \alpha\rangle = 0$ iff $\alpha = 0$ ✓

(ii) $\langle \alpha+\beta, r\rangle = \langle \alpha, r\rangle + \langle \beta, r\rangle$

(iii) $\langle x\alpha, \beta\rangle = x\langle \alpha, \beta\rangle$ – [

(iv) $\langle \alpha, \beta\rangle = \langle \beta, \alpha\rangle$ – [Symmetry]

Let $\alpha = (a_1, a_2)$

(1) $\therefore \langle \alpha, \alpha\rangle = \langle (a_1, a_2), (a_1, a_2)\rangle$

$$= a_1^2 - a_2 a_1 - a_1 a_2 + 4a_2^2$$

$$= (a_1 - a_2)^2 + 3a_2^2 \geq 0$$

Let $\langle \alpha, \alpha \rangle = 0$

$\Rightarrow (a_1 - a_2)^2 + (3a_2)^2 = 0$

$\Rightarrow a_1 - a_2 = 0$ and $a_2 = 0$

$\Rightarrow a_1 = a_2 = 0$

$\alpha = (a_1, a_2) = (0, 0) = 0$

$\therefore \langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$

(2) Let α, β and $\gamma \in \mathbb{R}^2(\mathbb{R})$

$\alpha = (a_1, a_2)$, $\beta(b_1, b_2)$ and $\gamma = (c_1, c_2)$

$$\boxed{\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle}$$

$(\alpha + \beta) = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

$\therefore \langle \alpha + \beta, \gamma \rangle = \langle (a_1 + b_1, a_2 + b_2), (c_1, c_2) \rangle$

$= (a_1 + b_1) c_1 - (a_2 + b_2) c_1 - (a_1 + b_1) c_2$

$+ 4(a_2 + b_2) c_2$

$= (a_1 c_1 - a_2 c_1 - a_1 c_2 + 4 a_2 c_2)$

$+ b_1 c_1 - b_2 c_1 - b_1 c_2 + 4 b_2 c_2$

$= \langle (a_1, a_2), (c_1, c_2) \rangle + \langle (b_1, b_2), (c_1, c_2) \rangle$

$\langle \alpha+\beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

③ Let $\alpha, \beta \in \mathbb{R}^3$ and $x \in \mathbb{R}$

$\therefore \boxed{\langle x\alpha, \beta \rangle = x\langle \alpha, \beta \rangle}$

$x\alpha = x(a_1, a_2) = (xa_1, xa_2)$

$\therefore \langle x\alpha, \beta \rangle = \langle (xa_1, xa_2), (b_1, b_2) \rangle$

$= xa_1b_1 - xa_2b_1 - xa_1b_2 + 4xa_2b_2$

$= x(a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2)$

$\langle x\alpha, \beta \rangle = x\langle \alpha, \beta \rangle$

④ Let $\alpha, \beta \in \mathbb{R}^2(\mathbb{R})$.

$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$

$\langle \alpha, \beta \rangle = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2$

$= b_1a_1 - b_2a_1 - b_1a_2 + 4b_2a_2$

$\boxed{\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle}$

Given product on $\mathbb{R}^2(\mathbb{R})$ is an Inner product.

H.W.

eg:- Verify that the following is an inner product on $\mathbb{R}^2(\mathbb{R})$ where $u = (x_1, x_2)$ and $v(y_1, y_2)$.

eg: verify ... $u=(x_1,x_2)$ and $v(y_1,y_2)$.

$\mathbb{R}^2(\mathbb{R})$, where

$$f(u,v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2.$$

eg:- Find the value of k so that the following is an inner product on $\mathbb{R}^2$ where $u=(x_1,x_2)$ and $v=(y_1,y_2)$.

$$f(u,v) = x_1y_1 - 3x_1y_2 - 3x_2y_1 + kx_2y_2$$

$\Rightarrow$ Let $f(u,v)$ be an inner product on $\mathbb{R}^2$

$\therefore$ $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0 \Rightarrow u=0$

$$\langle u,u\rangle = x_1^2 - 3x_1x_2 - 3x_2x_1 + kx_2^2$$

$$= (x_1-3x_2)^2 + (k-9)x_2^2 \geq 0$$

$$\Rightarrow \quad k-9 \geq 0 \Rightarrow \boxed{k \geq 9}$$

(*) Orthogonality:- $V(\mathbb{R})$

$\langle u,v\rangle = 0$ $\qquad a\cdot b = 0 \Rightarrow a \perp b$

$\langle v,u\rangle = 0$

$\boxed{0 \in V \text{ is orthogonal to every } v \in V}$

$\langle 0,v\rangle = 0$

$\langle 0, v\rangle = 0$

$\langle 0v, v\rangle = 0\langle v, v\rangle = 0$ //

Let u is orthogonal to every $\underline{\underline{v \in V}}$ then

$\underline{\underline{u = 0}}$. $u \neq 0$ $u \in V$

$\langle u, u\rangle = 0 \Rightarrow \underline{\underline{u = 0}}$

eg:- Consider the vectors $u = (1,1,1)$, $v = (1,2,-3)$ and $w = (1,-4,3)$ in $\mathbb{R}^3$.

$\Rightarrow$ $\langle u, v\rangle = 1+2-3 = 0 \Rightarrow$ u and v are orthogonal to each other.

$\langle u, w\rangle = 1-4+3 = 0 \Rightarrow$ u and w are orthogonal

$\langle v, w\rangle = 1-8-9 = -16 \neq 0$ v and w are not orthogonal.

eg:- Find non-zero vector w that is orthogonal to $u = (1,2,1)$ and $v = (2,5,4)$ in $\mathbb{R}^3$.

$\Rightarrow$ Let $w = (x,y,z)$ s.t. it is orthogonal to $u = (1,2,1)$ and $v = (2,5,4)$.

$$\therefore \left.\begin{aligned} \langle u, w\rangle &= x+2y+z = 0 \\ \langle v, w\rangle &= 2x+5y+4z = 0 \end{aligned}\right\} \quad \begin{aligned} x+2y+z &= 0 \\ y+2z &= 0 \end{aligned}$$

$\therefore$ $z = -2$ and $x = 3$

Let $z = 1 \Rightarrow y = -2$ and $x = 3$

$z = t \Rightarrow y = -2t$ and $x = 3t$

$$w = (3, -2, 1)$$

$$\hat{w} = \frac{w}{\|w\|} = \left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right)$$