Linear Algebra

December 2, 2021

Problems for Practice

1. Let V be the set of ordered pairs (a, b) of real numbers with addition in V and scaler multiplication on V defined by $(a, b)+(c, d) = (a+c, b+d)$ and $k(a, b) = (ka, 0)$. Determine whether V is a vector space or not?

2. Let V be the set of ordered pairs (a, b) of real numbers. Determine which of the following set is nota Vector space over $\mathbb{R}$.

   (a) $(a, b) + (c, d) = (a + d, b + c)$ and $k(a, b) = (ka, kb)$

   (b) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (a, b)$

   (c) $(a, b) + (c, d) = (0, 0)$ and $k(a, b) = (ka, kb)$

   (d) (a,b)+(c,d)=(ac,bd) and $k(a, b) = (ka, kb)$

3. Let U and W be vector space over a field $\mathbb{R}$. Let V be the set of orderes pairs (u, w) where $u \in U$ and $w \in W$. Show that V is a vector space over $\mathbb{R}$ with addition in V and scaler multiplication on V defined by $(u, w) + (u', w')$ and $k(u, w) = (ku, kw)$.

4. Let V be a vector space of n- square matrices iver a field $\mathbb{R}$. Show that W is a subspace of V if W consist of all matrices $A = [a_{ij}]$ that are:

   (a) symmetric

   (b) triangular

   (c) diagonal

   (d) scaler

5. Let $AX = B$ be a nonhomogeneous system of Linear equations in n unknowns. Show that the solution set is not a subspace of R^n.

6. Consider the vectors $u = (1, 2, 3)$ and $v = (2, 3, 1)$ in R^3.

   (a) Write $w = (1, 3, 8)$ as a linear combination of u and v.

   (b) Write $w = (2, 4, 5)$ as a linear combination of u and v.

   (c) Find k so that $w = (1, k, 4)$ is a linear combination of u and v.

   (d) Find the condition on a, b, c so that $w = (a, b, c)$ is a linear combination of u and v.

7. Write the polynomial $f(t) = at^2 + bt + c$ as a linear combination of the polynomials $p_1 = (t - 1)^2, p_2 = t - 1, p_3 = 1$.

8. Find one vector in R^3 that spans the intersection of U and W where U is the $xy-$ plane and W is the space spanned by the vectors $(1, 1, 1), (1, 2, 3)$.

9. If $S \subseteq T$, then show that $span(S) \subseteq span(T)$.

10. Determine whether the following polynomials u, v, w in $P(t)$ are linearly dependent or independent:

    (a) $u = t^3 - 4t^2 + 3t + 3, v = t^3 + 2t^2 + 4t - 1, w = 2t^3 - t^2 - 3t + 5,$

    (b) $u = t^3 - 5t^2 - 2t + 3, v = t^3 - 4t^2 - 3t + 4, w = 2t^3 - 17t^2 - 7t + 9$

11. Show that the following functions f, g, h are linearly independent: $f = e^t, g(t) = e^{2t}, h(t) = t$

12. Show that $u = (a, b)$ and $v = (c, d)$ in R^2 are linearly dependent if and only if $ad - bc = 0$.

13. Suppose u, v, w are linearly independent vectors then show that $S = u + v - 3w, u + 3v - w, v + w$ are linearly independent.

14. Find a subset of u_1, u_2, u_3, u_4 that gives a basis for $W = span(u_i)$ of R^5 where:

    (a) $u_1 = (1, -2, 1, 3, -1), u_2 = (-2, 4, -2, -6, 2), u_3 = (1, -3, 1, 2, 1), u_4 = (3, -7, 3, 8, -1)$

    (b) $u_1 = (1, 0, 1, 0, 1), u_2 = (1, 1, 2, 1, 0), u_3 = (2, 1, 3, 1, 1), u_4 = (1, 2, 1, 1, 1)$

15. Consider the subspaces $U = (a, b, c, d) : b - 2c + d = 0and$W=(a,b,c,d):a=d,b=2c of R^4. Find a basis ans the dimension of U, W and $U \cap W$.

16. Find a basis and the dimension of the solution space W of the following homogeneous system:
    $x + 2y - z + 3s - 4t = 0; 2x + 4y - 2z - s + 5t = 0; 2x + 4y - 2z + 4s - 2t = 0.$

17. Find a homogeneous system whose solution space is spanned by $(1, -2, 0, 3, -1), (2, -3, 2, 5, -3), (1, -2, 1, 2, -2)$.

18. Determine whether each of the following is a basis of vector space $P_n(t)$:

    (a) $1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3, ...1 + t + t^2 + ... + t^{n-1} + t^n$

    (b) $1 + t, t + t^2, t^2 + t^3, ..., t^{n-1}, t^n$

19. Find a basis and the dimension of the subspace W of $P(t)$ spanned by

    (a) $u = t^3 + 2t^2 - 2t + 1, v = t^3 + 3t^2 + -3t + 4, v = 2t^3 + t^2 - 7t - 7.$

    (b) $u = t^3 + t^2 - 3t + 2, v = 2t^3 + t^2 + t - 4, w = 4t^3 + 3t^2 - 5t + 2$

20. Find a basis and the dimension of the subspace W of $V = M_{2,2}$ spanned by $A = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, D = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$

21. Construct a subset of the X-Y plane R^2 that is

    a) closed under vector addition and subtraction, but not scalar multiplication.

    b) closed under scalar multiplication but not under vector addition.

22. Which of the following are subspaces of R^∞?

    a) All sequences like $(1, 0, 1, 0, ...)$ that include infinitely many zeros.

    b) All sequences $(x_1, x_2, x_3, ...)$ with $x_j = 0$ from some point onward.

    c) All decreasing sequences: $x_{j+1} \leq x_j$ for each j.

23. Describe the column space and the nullspace of the matrices:
    $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

24. Let P be the plane in 3-space with equation $x+2y+z=6$. What is the equation of the plane P_0 through origin parallel to P? Are P and P_0 subspace of R^3?

25. a) Describe a subspace of M that contains $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ but not

    $B=\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}$.

    b) If a subspace of M contains A and B, must it contain I?

    c) Describe a subspace of M that contains no nonzero diagonal matrices.

26. The four types of subspaces of R^3 are planes, lines, R^3 itself, or Z contains only $(0,0,0)$.

    a) Describe the three types of subspaces of R^2.

    b) Describe the five types of subspaces of R^4.

27. Let P be the plane in R^3 with euqation $x+y-2z=4$. The origin $(0,0,0)$ is not in P. Find two vectors in P and check that their sum is not in P.

28. If we add an extra column b to a matrix A, then the column space gets larger unless .... .Give an example in which the column space gets larger and an example in which it doen't . Why is $Ax=b$ sikvable exactly when the column space doesn't get larger by including b?

29. If A is any 8 by 8 invertible matrix, then its column space is .... Why?

30. True or false (with counterexample if false)?

    a) The vector b that are not in the coulumn space form a subspace.

    b) If coulmn space contains only the zero vector, then A is the zero matrix.

    c) the column space of 2A equals the column space of A.

    d) the coulmn space of A-I equals the column space of A.

31. Why isn't R^2 a subspace of R^3?

32. Find the row echelon form and hence the rank of following matrices:

    a) the 3 by 4 matrix of all 1s.

    b) the 4 by 4 matrix with $a_{ij}=(-1)^{ij}$.

    c) the 3 by 4 matrix with $a_{ij}=(-1)^{j}$.

33. What conditions on b_1, b_2, b_3, b_4 make each system solvable? Solve for x:

    a) $\begin{bmatrix}1 & 2\\ 2 & 4\\ 2 & 5\\ 3 & 9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\\ b_4\end{bmatrix}$. b) $\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 2 & 5 & 7\\ 3 & 9 & 12\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\\ b_4\end{bmatrix}$.

34. The complete solution of $Ax=\begin{bmatrix}1\\ 3\end{bmatrix}$ is $x=\begin{bmatrix}1\\ 0\end{bmatrix}+\text{c}\begin{bmatrix}0\\ 1\end{bmatrix}$. Find A.

35. True or false (Give reason if true, or counterexample if false)

    a) A square matrix has no free variables.

    b) an invertible matrix has no free variables.

    an m by n matrix has no more than n pivot variables.

    d) an m by n matrix has no more than m pivot variables.

36. Decide the dependence or independence of :

a) the vectors $(1, 3, 2), (2, 1, 3)$ and $(3, 2, 1)$.

b) the vectors $(1, -3, 2), (2, 1, -3)$ and $(-3, 2, 1)$

37. Suppose v_1, v_2, v_3 and v_4 are vectors in R^3.

a) These four vectors are dependent becouse ....

b) the two vectors v_1 and v_2 will be dependent if ....

c) the vector v_1 and $(0, 0, 0)$ are dependent because ....

38. Find the basis for each of these subspaces of R^4 :

a) All vectors whose components are equal.

all vectors whose components add to zero.

c) all vecors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

d) the column space and null space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.