Fakultät für Mathematik

Bachelorarbeit

The Rendezvous Value of Topological Spaces

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Zusammenfassung

1964 veröffentlichte Oliver Alfred Gross seine Abhandlung „The Rendezvous Value of Metric Spaces“[14]. Er beschrieb, wie durch Anwendung wohlbekannter Aussagen der Spieltheorie interessante Ergebnisse in anderen Teilbereichen der Mathematik bewiesen werden können. Er führt dies anhand der „Rendezvous Values“ metrischer Räume vor. Die Hauptaussage ist die folgende:

> Für jeden kompakten, zusammenhängenden nicht-leeren metrischen Raum X existiert eine eindeutige Konstante K mit der Eigenschaft, dass es zu jeder endlichen Familie $A := (x_i)_{i \in I}$ von Punkten in X einen weiteren Punkt p in X gibt, so dass das arithmetische Mittel der Distanzen der Punkte in A zu p gleich K ist.

Wir werden sehen, dass sich diese Aussage auf eine allgemeinere Klasse von Räumen verallgemeinern lässt.

Ziel dieser Arbeit ist es einen Zugang zu diesen und verwandten Aussagen zu schaffen, der nur wenige Kenntnisse voraussetzt, die über die üblichen Aussagen der Analysis III heraus gehen. Es wird lediglich die Kenntnis des Satzes Hahn-Banach benötigt. Ein grundlegendes Verständnis für Banach Räume ist hilfreich für Kapitel 4, aber nicht zwingend erforderlich. In Kapitel 3 werden einige Beispiele vorgestellt und Aussagen zur Berechnung bewiesen.

Contents

1. Prerequisites

1.1. Review of Measure Theory and Topology

In this section we will remind the reader briefly of some definitions and results from measure theory and topology that will be of use later.

Throughout, we will use the following notation.

Notation 1.1. Let Ω be a set.

1. Let $A \subset \Omega$ be a subset. Then A^c denotes the complement of A in Ω.
2. $\mathcal{P}(\Omega)$ shall denote the set of subsets of Ω.

Recall the following definition as can be found in [11].

Definition 1.2. Let Ω be a set. We say $\mathcal{A}$ is a *σ-Algebra on* Ω, if the following statements are true:

1. $\emptyset \in \mathcal{A}$.
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.
3. The union of countably many sets in $\mathcal{A}$ is in $\mathcal{A}$.

Definition 1.3. For $S \subset \mathcal{P}(\Omega)$, we call

$$\langle S \rangle^\sigma := \text{Intersection of all } \sigma\text{-Algebras on } \Omega \text{ that contain } S$$

the σ-Algebra *generated by S*.

Definition 1.4. Let $(X, \mathcal{T})$ be a topological space. We call the σ-Algebra generated by all open sets in X, i.e. $\mathcal{B} = \langle \mathcal{T} \rangle^\sigma$, the *Borel-$\sigma$-Algebra*.

Definition 1.5. Let Ω be a set and $\mathcal{A}$ a σ-Algebra on Ω.

1. A function
$$\mu : \mathbb{A} \to \mathbb{R} \cup \{\pm\infty\}$$
satisfying
   a) $\mu(\emptyset) = 0$
   b) For any pairwise disjoint family $(A_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{A}$
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

   is called a *signed measure on* $(\Omega, \mathcal{A})$.

2. We say μ is a *(positive) measure*, if it is a signed measure with the property
$$\mu(X) \geq 0 \quad \forall X \in \mathcal{A}.$$

3. Let X be a Hausdorff space, $\mathcal{A}$ a σ-algebra with $\mathcal{B} \subset \mathcal{A}$. We say μ is a *(positive) regular measure* if it is a measure such that for all $A \in \mathcal{A}$
$$\mu(A) = \inf\{\mu(U) \colon A \subset U, U \text{ open}\} = \sup\{\mu(K) \colon K \subset A, K \text{ compact}\}.$$
and we say μ is a *signed regular measure* if
$$|\mu|(A) := \sup_{\mathfrak{D}} \sum_{E \in \mathfrak{D}} |\mu(E)|,$$
is a positive regular measure, where the supremum is taken over all decompositions of A into finitely many disjoint sets.

4. We say μ is a *probability measure*, if it is a positive regular measure and
$$\mu(\Omega) = 1.$$

5. We say μ is *finite* if it does not take on the values $\pm\infty$ for any set.

Notation 1.6. Let X be a topological space and $\mathcal{B}(X)$ the Borel-σ-Algebra on X. We write

1. $\mathcal{M}(X) = \{\mu \mid \mu$ is a finite, regular signed Borel measure on $X\}$.
2. $\mathcal{M}^+(X) = \{\mu \mid \mu$ is a finite, regular Borel measure on $X\}$.
3. $\mathcal{M}^1(X) = \{\mu \mid \mu$ is a Borel probability measure on $X\}$.

We will equip $\mathcal{M}^1(X)$ with a suitable topology later (namely the weak*-topology). Note that
$$\mathcal{M}^1(X) \subset \mathcal{M}^+(X) \subset \mathcal{M}(X)$$
In particular the following clearly holds.

Lemma 1.7. *Equipped with the usual operations of addition and scalar multiplication, the set $\mathcal{M}(X)$ is a real vector space, and the spaces $\mathcal{M}^+(X)$ and $\mathcal{M}^1(X)$ are convex subsets of $\mathcal{M}(X)$.*

We now define a special kind of measures, namely those whose support is a singleton.

Definition 1.8. The measures assigning weight one to a single point are called *Dirac measures*, i.e.
$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases},$$
are called *Dirac measures*.

We will see in Theorem 1.20 that Dirac measures are important for approximating more general measures.

We conclude this review of measure theory with a reminder of the following theorem as can be found in [11, Theorem 4.11, 4.13] or [8, S.18]

Theorem 1.9 (Theorem of Carathéodory). *Let X be a set, let μ^* be an outer measure on X, and let $\mathcal{A}_{\mu^*}$ be the collection of all μ^*-measurable subsets of X. Then*

1. *$\mathcal{A}_{\mu^*}$ is a σ-slgebra, and*
2. *the restriction of μ^* to $\mathcal{A}_{\mu^*}$ is a measure on $\mathcal{A}_{\mu^*}$.*

It is well known that any topology induces a concept of what it means for a sequence to converge with respect to that topology. This convergence can be defined as follows.

Definition 1.10. Let $(X, \mathcal{T})$ be a topological space. A sequence $(a_i)_{i\in\mathbb{N}}$ in X *converges to $a \in X$ with respect to the topology $\mathcal{T}$* if for every open neighborhood U of a there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \in U$.

Also, the following connection between closed sets and sequences is well known.

Lemma 1.11. *Let X be a topological space and $A \subset X$ a closed set. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in A, converging to a. Then $a \in A$.*

This gives rise to the question if the converse might be true in the sense that defining what it means for a sequence to converge already uniquely determines the topology on the space. As it turns out, this is generally not the case for non-metrizable spaces. For example, let X be a set with uncountably many points. Then the discrete and the co-countable topology do not coincide, however a sequence converges with respect to the discrete topology if and only if it converges in the co-countable topology.

For a similar statement to hold we need the concept of nets, which are in some sense generalized sequences.

Definition 1.12 ([5], p. 48-50)**.**

1. A *directed set D* is a partially ordered set such that for each $m, n \in D$ there is a $p \in D$ so that $p \geq m$ and $p \geq n$.
2. Let E be a set and D a directed set. A *net on E* is a mapping $f : D \to E$.
3. A net x_a *converges to $a \in E$* if and only if for every neighborhood U of a, there is an a_0 such that $a \geq a_0$ implies $x_a \in U$.
4. We say a net *converges* if it converges to some point in E.

Proposition 1.13. *Let E be a topological space. Then $A \subset E$ is closed if and only if for any net x_a with $x_a \in A$ which converges to $a \in E$, we have $a \in A$.*

Proof. If A is closed then $a \in A$, as a is a cluster point of the set $\{x_a \mid a \in D\}$. Conversely, if A were not closed there would be a cluster point $a \in E \setminus A$. For U any neighborhood of a, choose $x_U \in U \cap A$. Then x_U forms a net in A converging to a. $\square$

We can now define the weak*-topology on $\mathcal{M}(E)$ using nets ([see 5]).

Definition 1.14. We define the weak*-topology on $\mathcal{M}(E)$ to be the unique topology such that for a net (μ_i) in $\mathcal{M}(E)$ converges to a $\mu \in \mathcal{M}(E)$ if and only if

$$\int_E \phi(x)\mu_i(\mathrm{d}x) \to \int_E \phi(x)\mu(\mathrm{d}x) \quad \forall \phi \in \mathcal{K}(E, \mathbb{R}),$$

where $\mathcal{K}(E, \mathbb{R})$ is the set of all continuous functions from E to $\mathbb{R}$ with compact support.

1.2. Functional Analysis

Our preliminary goal in this section will be to show that $\mathcal{M}^1(X)$ is compact, provided that X is a compact Hausdorff space. To show this we need some background in functional analysis, namely the Banach-Alaoglu and the Riesz-Representation Theorems, which are fundamental results from functional analysis.

Theorem 1.15 (Banach-Alaoglu-Theorem,[28])**.** *Let X be a topological (real) vector space and V a neighborhood of* 0*. Let*

$$K := \{\Lambda \in X' \colon |\Lambda x| \leq 1 \forall x \in V\}$$

then K is weak-compact.*

Proof. Since X is a topological vector space, there exists a number $0 < \gamma(x) < \infty$ for every $x \in X$ such that $x \in \gamma(x)V$. Thus, for every $\Lambda \in X'$ we have

$$|\Lambda x| = |\gamma(x)\Lambda(\gamma^{-1}(x) \cdot x)| = \gamma(x) \cdot |\Lambda(\underbrace{\gamma^{-1}(x) \cdot x}_{\in V}) \leq \gamma(x) \cdot 1 = \gamma(x)$$

Let $D_x := [-\gamma(x), \gamma(x)]$ and define P to be the cartesian product of all D_x together with the product topology τ. Since D_x is compact for all $x \in X$, Tychonoff's theorem (Theorem A.3) states that P is compact as well. Since a function $f : X \to \mathbb{R}$ is determined by the relation $x \mapsto f(x)$, we can identify the elements of P with functions $X \to \mathbb{R}$ with $|f(x)| \leq \gamma(x)$. (This identification works as follows: Let $z \in P$. The corresponding function $f_z : X \to \mathbb{R}$ is the function that maps every $x \in X$ to the x-coordinate of z.)

With this identification, we see $K \subset X' \cap P$. Therefore, K inherits the weak* topology as a subspace of X' and the subspace topology τ of P.

Assertion 1. The described topologies coincide on K and

Assertion 2. The subset K is a closed in P.

It is clear that once we proved this, the result of the theorem follows immediately since K is a closed subset of a compact space and thus compact.

Let $\Lambda_0 \in K$ be fixed. Choose $\delta > 0$ and $x_i \in X$ for all $1 \leq i \leq n, n \in \mathbb{N}$. Now define

$$W_1(n, (x_i)_{1 \leq i \leq n}, \delta) := \{\Lambda \in X^* \colon |\Lambda x_i - \Lambda_0 x_i| < \delta \text{ for } 1 \leq i \leq n\}$$

and

$$W_2(n,(x_i)_{1\le i\le n},\delta) := \{f \in P\colon |f(x_i) - \Lambda_0 x_i| < \delta \text{ for } 1 \le i \le n\}$$

Then the collection of all W_1 form a local base for the weak*-topology of X^* at Λ_0. Similarly the collection of all W_2 form a local base for the product topology τ of P at Λ_0. Since $K \subset P \cap X^*$, we find

$$W_1 \cap K = W_2 \cap K,$$

hence the topologies coincide, as stated in Assertion 1.

For the second assertion, let f_0 be an element in the τ-closure of K. Choose $x, y \in X, \alpha, \beta \in \mathbb{R}, \epsilon > 0$. The set of all functions satisfying $|f - f_0| < \epsilon$ at the points x, y and $\alpha x + \beta y$ is a τ-neighborhood of f_0, implying there exists an f with that property in K. This f is linear and thus,

$$\begin{aligned} f_0(\alpha x + \beta y) - \alpha f_0(x) + \beta f_0(y) =& f_0(\alpha x + \beta y) - \alpha f_0(x) + \beta f_0(y) \\ &- \underbrace{f(\alpha x + \beta y) + \alpha f(x) + \beta f(y)}_{=0} \\ =& (f_0 - f)(\alpha x + \beta y) + \alpha(f - f_0)(x) + \beta(f - f_0)(y). \end{aligned}$$

Therefore,

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\epsilon.$$

Since ϵ was arbitrary, we find that the expression above equals 0, thus f_0 is linear.

Now, if $x \in V$ and $\epsilon > 0$, it is analogous to show that $|f(x) - f_0(x)| < \epsilon$. Since $|f(x)| \le 1$ by definition of K, we see that $|f_0(x)| \le 1$. Thus $f_0 \in K$. This shows that K is closed in P. $\square$

The following theorem is a standard result from functional analysis, providing a connection between measures and linear functionals. Note that this theorem holds for both the real and the complex case. However, since the real version is sufficient for our purposes, we will not consider the complex version.

Theorem 1.16 (Riesz Representation Theorem, [31])**.** *Let X be a compact Hausdorff space. Then $C(X)^*$ is isometrically isomorph to $\mathcal{M}(X)$ with the mapping being given by*

$$T\colon M(X) \to C(X)^*, \quad (T\mu)(x) = \int_X x \,\mathrm{d}\mu$$

Proof. It is a well known result from measure theory that continuous functions are Borel measurable, thus the operator T as above is well defined. Furthermore, the inequality

$$\left|\int_X x \mathrm{d}\mu\right| \le \int_X |x| \mathrm{d}|\mu| \le \|x\|_\infty \|\mu\|.$$

implies that $\|T\| \le 1$.

Firstly, we will show that T is an isometry. Let $\mu \in \mathcal{M}(X)$. Consider the Hahn-Jordan decomposition of μ, i.e. write $\mu = \mu_+ - \mu_-$ with positive measures μ_+ and μ_- and

$X = E_+ \cup E_-$ for disjoint Borel sets E_+ and E_- such that for all Borel sets $F \supset E_+$ (resp. E_-) $\mu_-(F) = 0$ (resp. $\mu_+(F) = 0$). Since μ is regular, so are μ_+ and μ_-. Thus, for all $\varepsilon > 0$ there exist compact sets $C_+ \subset E_+$ and $C_- \subset E_-$ with

$$\begin{aligned} \mu(E_+) - \varepsilon &\leq \mu(C_+) \leq \mu(E_+), \\ \mu(E_-) + \varepsilon &\geq \mu(C_-) \geq \mu(E_-). \end{aligned}$$

In particular, C_- and C_+ are disjoint, since E_- and E_+ are disjoint. This implies that the function

$$y(t) = \begin{cases} 1 & \text{if } t \in C_+ \\ -1 & \text{if } t \in C_- \end{cases}$$

is continuous on $C_+ \cup C_-$. The Tietze-Urysohn theorem (Theorem 2.8) now implies, that there is an extension $x \in C(X)$ with $\|x\|_\infty = 1$.

For this x the following holds:

$$\begin{aligned} \left|\int_X x \mathrm{d}\mu\right| &= \left|\int_{C_+} \mathrm{d}\mu + \int_{C_-} (-1)\mathrm{d}\mu + \int_{X\setminus(C_+\cup C_-)} x \,\mathrm{d}\mu\right| \\ &\geq \mu(C_+) - \mu(C_-) - \left|\int_{X\setminus(C_+\cup C_-)} x \,\mathrm{d}\mu\right| && \text{(by reverse triangle-inequality)} \\ &\geq \mu(E_+) - \varepsilon - \mu(E_-) - \varepsilon - |\mu|(X \setminus (C_+ \cup C_-)) && (\text{since } \|x\|_\infty = 1) \\ &= |\mu|(X) - 2\varepsilon - (|\mu|(X) - |\mu|(C_+) - |\mu|(C_-)) \\ &= \mu(C_+) - \mu(C_-) - 2\varepsilon && (\text{note } \mu(C_-) = -\mu_-(C_-)) \\ &\geq \mu(E_+) - \varepsilon - \mu(E_-) - \varepsilon - 2\varepsilon \\ &= |\mu|(X) - 4\varepsilon. \end{aligned}$$

Thus, $\|T\mu\| \geq \|\mu\| - 4\varepsilon$ for all $\varepsilon \geq 0$, hence $\|T\mu\| \geq \|\mu\|$. This concludes the proof of isometry since $\|T\mu\| \leq \|\mu\|$.

We will now prove the surjectivity. Let $x' \in C(X)'$ be positive in the sense that if $x(t) \geq 0$ for all t, then $x'(x) \geq 0$. We will now construct a regular Borel measure μ with $T\mu = x'$. Note that we can't use $\mu(E) = x'(\chi_E)$, since χ_E is not continuous in general. Instead, we proceed as follows: Let O be a open subset of X and define

$$\mu^*(O) = \sup\{x'(x) \colon 0 \leq x \leq 1, \overline{\{t \colon x(t) \neq 0\}} \subset O\}$$

and for arbitrary $E \subset X$

$$\mu^*(E) = \inf\{\mu^*(O) \colon E \subset O, O \text{ open}\}.$$

Now, we see that this μ^* is a outer measure, i.e. $\mu^*(E) \geq 0$ for all $E \subset X$, $\mu(\emptyset) = 0$, $\mu^*(E) \leq \mu^*(F)$, if $E \subset F$ and $\mu^*\left(\bigcup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty \mu^*(E_n)$. For open sets V, we see

$$\mu^*(V \cap F) + \mu^*(V^c \cap F) \leq \mu^*(F) \qquad \forall F \subset X \tag{1.1}$$

Now, let $\Sigma_\mu := \{V \subset X \colon V \text{ satisfies (1.1)}\}$. The Extension Theorem of Carathéodory (Theorem 1.9) then implies that Σ_μ is a σ-Algebra and $\mu^* \mid_{\Sigma_\mu}$ is a measure. By construction all open sets are contained in Σ_μ, we have $\Sigma \subset \Sigma_\mu$, thus $\mu := \mu^* \mid_\Sigma$ is a regular (by construction), positive Borel measure and $x'(x) = \int x \mathrm{d}\mu$ for all $x \in C(X)$.

The general case for arbitrary $x' \in C(X)'$ can be reduced to the case of positive functionals. For $x \in C(X)$ define $x_+(t) = \max\{0, x(t)\}, x_-(t) = \max\{0, -x(t)\}$ and for $x \geq 0$ let

$$x'_+(x) = \sup\{x'(y) \colon 0 \leq y \leq x\}$$

and for arbitrary $x \in C(X)$

$$x'_+(x) = x'_+(x_+) - x'_+(x_-).$$

With this definition, x'_+ is linear and continuous. Furthermore, $x'_+(x) \geq 0$ in the sense mentioned above and $x'_- := x'_+ - x' \geq 0$. We can apply essentially the same prove to x'_+ and x'_- as we did to positive functionals and obtain positive regular Borel measures μ_+ and μ_- with

$$x'(x) = x'_+(x) - x'_-(x) = \int_X x \mathrm{d}\mu_+ - \int_X x \mathrm{d}\mu_- \qquad \forall x \in C(X).$$

We can conclude that for $\mu = \mu_+ - \mu_-$ we have $T\mu = x'$.

We now showed that T is a isometry and surjective. This already implies that T is a isometric isomorphism. $\square$

We can now apply the previous two results to show the following theorem.

Theorem 1.17. *Let X be a compact Hausdorff space. Then $\mathcal{M}^1(X)$ is weak*-compact.*

Proof. By Riesz-Representation-Theorem (Theorem 1.16), we can identify $\mathcal{M}(X)$ with $C(X)'$. Clearly, $\mathcal{M}^1(X)$ is a closed subset of $\mathcal{M}(X)$ in the weak*-topology. Furthermore, $\mathcal{M}^1(X)$ is isometrically isomorph to the unit sphere in $C(X)'^{,+}$. The Banach-Alaoglu-Theorem (Theorem 1.15) now states that the unit ball in $C(X)'$ is weak*-compact and since closed subsets of compact sets are compact, the statement follows. $\square$

For the remainder of this section we will consider critical points of spaces, which, in some sense, can be thought of as the „corners“ of a topological space. We will also see the Theorem of Krein-Milman, which states that under certain conditions (see below) we can go back and forth between the actual topological space and its extreme points. We follow the discussion from [31] and finally apply this theory to show that measures with finite support are dense in $\mathcal{M}^1(X)$.

Definition 1.18. Let X be a vector space and $K \subset X$ convex.

1. A set $F \subset K$ is called a side of K, provided F is convex and

$$x_1, x_2 \in K, 0 < \lambda < 1, \lambda x_1 + (1-\lambda)x_2 \in F \Rightarrow x_1, x_2 \in F$$

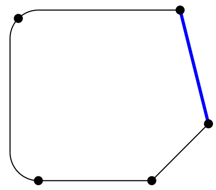


Figure 1.1.: The points mark some extreme values. The blue line is a side.

2. A point $x \in K$ is said to be an extreme point of K, if $\{x\}$ is a side of K. We denote the set of extreme points of K by $\mathrm{ex}K$

The following picture can be found in Werner [31] and illustrates above definition.

Theorem 1.19 (Krein-Milman Theorem)**.** *Let X be a locally convex, non-empty Hausdorff space and let $K \subset X$ be compact, convex and not empty.*

1. $\mathrm{ex}K \neq \emptyset$.
2. $K = \overline{\mathrm{conv}}\,\mathrm{ex}K$.

Proof. 1. Let $\mathcal{F}$ be the set of all closed non-empty sides of K. Since K is itself a closed, non-empty side, we see that $\mathcal{F} \neq \emptyset$. Clearly, $\mathcal{F}$ is inductively ordered in terms of inclusion since the intersection of two closed sides is a closed side, which is nonempty since K is compact[1]. By the lemma of Zorn, there exists a minimal element F_0. Assume for a contradiction that F_0 does not consist of a single point. Applying the Hahn-Banach theorem (Theorem A.5) yields the existence of points $x_0, y_0 \in F_0, x' \in X'$ with
$$\mathrm{Re}x'(x_0) < \mathrm{Re}x'(y_0).$$
Now consider
$$F_1 = \left\{ x \in F_0 \colon \mathrm{Re}x'(x) = \sup_{y \in F_0} \mathrm{Re}x'(y) \right\}.$$
Since F_0 is compact and x' continuous, $F_1 \neq \emptyset$. Furthermore F_1 is closed and a side of F_0. Thus, F_1 is a side of K as well, hence $F_1 \in \mathcal{F}$. However, $x_0 \notin F_1$ implies $F_1 \subsetneq F_0$. This contradicts the minimality of F_0.

2. Let $K_1 := \overline{\mathrm{conv}}\,\mathrm{ex}K$. Then K_1 is compact, convex and not empty. Clearly $K_1 \subset K$. Assume for a contradiction $K_1 \neq K$. Then there exists $x_0 \in K \setminus K_1$. By the Hahn-Banach Theorem (Theorem A.5) there exists $x' \in X', \varepsilon > 0$ with
$$\mathrm{Re}x'(x) \leq \mathrm{Re}x'(x_0) - \varepsilon \quad \forall x \in K_1. \tag{1.2}$$

[1] Recall the following: Let $(X, \mathcal{T})$ be a compact topological space. Then for any family of closed sets $(A_i)_{i \in I}$ in $(X, \mathcal{T})$ with $\bigcap_{i \in I} A_i = \emptyset$, then there exists a finite number of elements $i_1, \ldots, i_n$ in I with $\bigcap_{j=1}^{n} A_{i_j} = \emptyset$

Now let $F := \{x \in K\colon \mathrm{Re}x'(x) = \sup_{y\in X} \mathrm{Re}x'(y)\}$. As before, we see that F is a closed side of K and not empty. By part 1 we know that $\mathrm{ex}F \neq \emptyset$. Furthermore, $\mathrm{ex}F = \mathrm{ex}K \cap F \subset \mathrm{ex}K$. However (1.2) implies

$$(\mathrm{ex}K) \cap F \subset K_1 \cap F = \emptyset.$$

This is a contradiction. □

We will now see an application of the Krein-Milman Theorem, which shows that measures can be approximated arbitrarily good by measures with finite support. The proof is partly due to Chad [4].

Theorem 1.20. *Let X be a compact, non-empty Hausdorff space. Then the measures in $\mathcal{M}^1(X)$ with finite support form a dense subset of $\mathcal{M}^1(X)$.*

Proof. Firstly, note that $C(X)$ is a metric space with the metric that is induced by the variation norm, hence it is Hausdorff. This implies that the dual space $C(X)'$ equipped with the weak*-topology is (completely) Hausdorff. Furthermore, it is clear that $C(X)'$ is convex and therefore also locally convex. Now consider the subspace $K := \{\phi \in C(X)'^{,+}\colon \|\phi\| = 1\}$. Then K is isometrically isomorph to $\mathcal{M}^1(X)$ by Riesz representation theorem and thus compact (see Banach-Alaoglu Theorem 1.15), convex and not empty.

It is now sufficient to show that the extreme points of $\mathcal{M}^1(X)$ are given by $V := \{\delta_x : x \in X\}$, where δ_x is the Dirac measure at point $x \in X$. (Note that the singletons are always Borel-measureable since X is required to be Hausdorff.)

We begin by showing $V \subset \mathrm{ex}X$. Let $x \in X$ and $\mu, \nu \in \mathcal{M}^1(X)$ such that $\delta_x = \lambda\mu + (1-\lambda)$. Now, let $A \in \mathcal{B}(X)$. If $x \in A$, then by definition $\delta_x(A) = 1$. Since $0 \leq \mu, \nu \leq 1$ this implies $\mu(A) = \nu(A) = 1$. If $x \notin A$ then clearly $\delta_x(A) = 0 = \lambda\mu(A) + (1-\lambda)\nu(A)$ and thus $\nu(A) = \mu(A) = 0$. In other words $\mu = \nu = \delta_x$.

For the other inclusion, consider $\mu \in \mathcal{M}^1(X) \setminus V$. Since X is a compact Hausdorff space, the support of μ is measurable and not empty. The support then has to include at least two points x_1 and x_2. Since X is Hausdorff, there exists a open neighborhood U of x_1 with $x_2 \notin U$. Since U is open, it is Borel measurable. Then $0 < \mu(U), \mu(X \setminus U) < 1$. Now define measures $\nu, \lambda \in \mathcal{M}^1(X)$ as follows for $A \in \mathcal{B}(X)$:

$$\nu(A) = \frac{1}{\mu(U)}\mu(A \cap U) \quad \text{and} \quad \lambda(A) = \frac{1}{\mu(X \setminus U)}\mu(A \cap (X \setminus U)).$$

Then for all $A \in \mathcal{B}(X)$,

$$\begin{aligned}\mu(A) &= \mu(A \cap U) + \mu(A \cap (X \setminus U)) \\ &= \mu(U) \cdot \nu(A) + (1 - \mu(U)) \cdot \lambda(A),\end{aligned}$$

thus μ is not a extreme point of $\mathcal{M}^1(X)$.

With part 2 of Theorem 1.19 the statement follows. □

1.3. A (very) Brief Introduction to Game/Decision Theory

The goal of this section is a minimax theorem, which states that in certain situations (see below) the maximum and minimum „operators“ commute, i.e. $\min_x \max_y f(x,y) = \max_x \min_y f(x,y)$.

We will show this by studying the topic of game theory[2]. With this we follow Furguson's decision theoretic approach in [10]. We begin with the definition of what we consider a game.

Definition 1.21. Let Θ and $\mathcal{A}$ be nonempty sets and $L\colon \Theta \times \mathcal{A} \to \mathbb{R}$ a function. We call the triplet $(\Theta, \mathcal{A}, L)$ a *game*. Furthermore, Θ is called the set of *possible states of nature*, $\mathcal{A}$ is called the set of *available actions* and L is called the *loss function*.

To give some motivation for this definition, consider the following example. Let $\Theta = \{1,2\} = \mathcal{A}$ and define $L\colon \{1,2\}^2 \to \mathbb{R}$ by

$$\begin{aligned} L(1,1) &= -2 \qquad & L(1,2) &= 3 \\ L(2,1) &= 3 & L(2,2) &= -4 \end{aligned}$$

Clearly, this satisfies the definition of a game above. Now the triplet $(\Theta, \mathcal{A}, L)$ describes a game (in the non-mathematical sense) of two players (one labeled *nature*, the other *statistician*) where both contestants simultaneously raise either one or two fingers. Now player „nature“ wins, if the total number of raised fingers is even and player „statistician“ wins if it is odd. The winner receives the total amount of raised fingers in dollars from the loser.

We will now couple a game with a random observable, whose distribution depends on the state of nature. On the basis of the outcome of this experiment, the statistician will choose a action.

Definition 1.22. Let $(\Theta, \mathcal{A}, L)$ be a game and X a random variable with distribution P_θ depending on $\theta \in \Theta$. Let $\mathcal{X}$ be the sample space of X and $d : \mathcal{X} \to \mathcal{A}$ a function. Such a function d is called a *nonrandomized decision rule* provided the *risk function*

$$R(\theta, d) = E_\theta L(\theta, d(X)) = \int L(\theta, d(x)) dP_\Theta(x)$$

is defined. The set of all nonrandomized decision rules is denoted by D. We call the triplet (Θ, D, R) a *statistical decision problem* or *statistical game*.

[2] There is a method to achieve the same result without game theory for spaces X for which $C(X)'$ possesses a Schauder basis. This approach uses methods similar to those in [3] by Brumm. Here the Banach fixed point theorem is applied to finite games in order to show the existence of Nash-Equilibria from which the minimax theorem follows. For the general case a similar approach with the Schauder fixed point theorem needs to be applied. However, since the author is unaware of a result stating that such a Schauder base exists for all compact, connected, non-empty Hausdorff spaces, we only describe the game theoretic approach.

It is clear from the definition that d represents an elementary strategy for the statistician. Note that $L(\theta, d(X))$ is now a random quantity. The risk function represents $R(\theta, d(X))$ represents the average loss provided the true state of nature is θ and the statistician chooses actions $d(X)$.

Definition 1.23. Let Z be a random variable with values in D. Any probability distribution δ on the space of nonrandomized decision functions, D, is called a randomized decision fuction or a randomized decision rule, provided the risk function

$$R(\theta, \delta) = ER(\theta, Z)$$

exists and is finite for all $\theta \in \Theta$. The space of all randomized decision rules is denoted by D^*.

It can also be useful to introduce distributions on the set of possible states of nature.

Definition 1.24. Let T be a random variable with values in Θ and distribution τ. This distribution τ is called a *prior distribution*, provided the *Bayes risk of a decision rule δ with respect to τ*

$$r(\tau, \delta) = ER(T, \delta)$$

exists and there exists a joint distribution of T and X and there is a conditional distribution of T, given X. The latter is called the *posterior distribution of the parameter given the observations*. The space of distributions on Θ satisfying above conditions and is denoted by Θ^*.

Before we can begin with proving the minimax theorem, we need some more definitions.

Definition 1.25. A decision rule δ_0 is said to be *minimax* if

$$\sup_{\Theta} R(\theta, \delta_0) = \inf_{\delta \in \Theta^*} \sup_{\theta} R(\theta, \delta)$$

this value is called *minimax* or *upper value* and denoted by $\overline{V}$.

Definition 1.26. A distribution $\tau_0 \in \Theta^*$ is said to be least favorable if

$$\inf_{\delta} r(\tau_0, \delta) = \sup_{\tau} \inf_{\delta} r(\tau, \delta)$$

this value is called *maximin* or *lower value* and denoted by $\underline{V}$.

Definition 1.27. Suppose that Θ consists of k points $\theta_1, \ldots, \theta_k$ and consider the set S to be called the *risk set* of points of the form

$$(R(\theta_1, \delta), \ldots, R(\theta_k, \delta)) \subset \mathbb{R}^k$$

where δ ranges through D^*. Formally:

$$S = \{(y_1, \ldots, y_k) : y_j = R(\theta_j, \delta) \text{ for some } \delta \in D^*\}.$$

We are now able to prove the Minimax Theorem. We will first consider the case of finite Θ and then generalize the result.

Theorem 1.28 (Minimax Theorem for finite games)**.** *If for a given decision problem (Θ, D, R) with finite $\Theta = \{\theta_1, \dots, \theta_k\}$ the risk set is bounded below, then*

$$\inf_{\delta \in D^*} \sup_{\tau \in \Theta^*} r(\tau, \delta) = \sup_{\tau \in D^*} \inf_{\delta \in D^*} r(\tau, \delta)$$

and there exists a least favorable distribution. [...]

Proof. It is always true that $\underline{V} \leq \overline{V}$, thus, it is sufficient to show $\overline{V} \leq \underline{V}$.

Let V denote the supremum of $\{\alpha : Q_{\alpha \mathbf{1}} \cap S = \emptyset\}$, where $\mathbf{1} = (1, \dots, 1)$ and $Q_c = \{(y_1, \dots, y_k) \colon y_i \leq c_i\}$. Then for every n there exists a rule δ_n such that

$$R(\theta_j, \delta_n) \leq V + \frac{1}{n}$$

for all j.

Hence $r(\tau, \delta_n) \leq V + \frac{1}{n}$ for all τ and $\sup_\tau r(\tau, \delta_n) \leq V + \frac{1}{n}$ for all n, implying that $\overline{V} \leq V$. Now, show $V \leq \underline{V}$

The convex sets $Q^\circ_{V\mathbf{1}}$ and S are disjoint, so there must be a hyperplane $p^t x = c$, which separates $Q^\circ_{V\mathbf{1}}$ and S, say $p^t x \geq c$ for $x \in S$ and $p^t x \leq c$ for $x \in Q^\circ_{V\mathbf{1}}$.

Each coordinate p_j must be nonnegative, for if $p_j < 0$ for some j we may take $x_j \to -\infty$; the other coordinates of x are fixed, $x \in Q^\circ_{V\mathbf{1}}$ s.t. $p^t x \to +\infty$, contradicting $p^t x \leq c$. We may also take $\sum_j p_j = 1$. Thus p may be taken as a prior distribution, τ_0, for nature.

Because $p^t x \leq c$ for all $x \in Q^\circ_{V\mathbf{1}}$, letting $x \to V\mathbf{1}$ implies $V \leq c$. Thus for all δ

$$r(\tau_0, \delta) = \sum_j p_j R(\theta_j, \delta) \geq c \geq V$$

Therefore,

$$\underline{V} = \sup_\tau \inf_\delta r(\tau, \delta) \geq \inf_\delta r(\tau_0, \delta) \geq V$$

also τ_0 is least favorable. $\square$

Before we can begin with the generalization, we need the following definitions.

Definition 1.29. Let (Θ, D, R) be a statistical game.

1. A decision rule δ_1 is said to be *as good as* a rule δ_2, if $R(\theta, \delta_1) \leq R(\theta, \delta_2)$ for all $\theta \in \Theta$.

2. Let $C \subset D^*$ is said to be essentially complete, if, given any rule $\delta \in D^*$, there exists a rule $\delta_0 \in C$ that is as good as δ

As an example for a essentially complete class in D^*, just consider the entire set D^*.

Theorem 1.30 (Minimax Theorem)**.** *Let $C \subset D^*$ be an essentially complete class for the game (Θ, D, R). Assume that there is a topology on C such that*

1. *C is compact and*

2. *$R(\theta, \delta)$ is lower semicontinuous in $\delta \in C$ for all $\theta \in \Theta$. Then the game has a value and the statistician has a minimax strategy.*

Proof. Let $\overline{V} = \inf_\delta \sup_\theta F(\theta, \delta)$ be the upper value of the game (Θ, D, R). If $\overline{V} = +\infty$, any rule is a minimax rule for the statistician. If $\overline{V} \neq \infty$, then, since $\sup_\theta R(\theta, \delta)$ is a lower semicontinuous function defined on a compact set, it achieves its infimum at some point $\delta_0 \in C$; $\sup_\theta R(\theta, \delta_0) = \overline{V}$, which implies that δ_0 is a minimax strategy.

To show that the game has a value, let M be an arbitrary fixed number, $M < \overline{V}$, and let $S_\theta = \{\delta \in C\colon R(\theta, \delta) > M\}$. From the lower semicontinuity of R, each set S_θ is an open subset of C. Furthermore, for each $\delta \in C$ there exists a $\theta \in \Theta$ such that $\delta \in S_\theta$, so that $\{S_\theta\}$ forms an open covering of C. Since C is compact, there exists a finite subcovering $\{S_{\theta_1}, \dots, S_{\theta_k}\}$. Hence

$$\inf_{\delta \in C} \sup_i R(\theta_i, \delta) \geq M.$$

Now let $\Theta_M = \{\theta_1, \dots, \theta_k\}$ and consider the game (Θ_M, C, R). Since every lower semicontinuous function on a compact set achieves its minimum, the risk set is bounded below and hence the first part of Theorem 1.28 is applicable. Therefore, this game has a value $V_M \geq M$ and there exists a least favorable distribution $\{p_1, \dots, p_k\}$ on Θ_M; that is,

$$\inf_{\delta \in C} \sum_{i=1}^{k} p_i F(\theta_i, \delta) = V_M.$$

Since C is essentially complete, we have

$$\inf_{\delta \in D^*} \sum_{i=1}^{k} p_i F(\theta_i, \delta) = V_M \geq M.$$

Since this holds for all $M < \overline{V}$, we find that

$$\underline{V} = \sup_{\tau \in D^*} \inf_{\delta \in D^*} r(\tau, \delta) \geq \overline{V}.$$

Thus, the game has a value. □

2. Rendezvous Value

2.1. Existence and Uniqueness

The following theorem is due to Stadje [29]. The special case of compact, connected, non-empty metric spaces had been formulated by Gross [14], however, it appears that Stadje was unaware of the result from Gross.

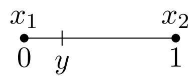
Theorem 2.1 (Gross-Stadje-Theorem)**.** *Let X be a compact, connected, non-empty Hausdorff space and $f : X \times X \to \mathbb{R}$ a continuous, symmetric function. Then there is a uniquely determined constant $a(X, f) \in \mathbb{R}$ such that the following holds:*

$$\forall n \in \mathbb{N} \quad \forall x_1, \dots, x_n \in X \quad \exists y \in X : \frac{1}{n} \sum_{i=1}^{n} f(x_i, y) = a(X, f). \tag{2.1}$$

Before we get started with the proof, let us examine a simple example. More advanced/interesting examples will be examined in a later chapter.

Example 2.2. Let $(X, f) := ([0, 1], d)$ be the unit interval with the usual euclidean metric. Then X is a compact connected hausdorff space and d is a symmetric continuous function from $X \times X$ to $\mathbb{R}$.

We consider the case of $n = 2$ with $x_1 = 0$ and $x_2 = 1$:



It is elementary to see, that in this case $d(x_1, y) + d(x_2, y) \equiv 1$. Thus, if a rendezvous value for $([0, 1], d)$ exists, then it has to be $\frac{1}{2}$.

Proof. We will prove the existence of such a value by giving an explicit formula. However, it will be apparent, that this formula will not be particularly helpful for computing the rendezvous value of arbitrary spaces.

Define

$$a(X, f) := \sup_{\mu \in \mathcal{M}^1} \inf_{\nu \in \mathcal{M}^1} \int\limits_X \int\limits_X f(x, y) \mu(\mathrm{d}x) \nu(\mathrm{d}y).$$

We now need to show that $a(X, f)$ actually satisfies Equation 2.1. As we saw in Theorem 1.17, $\mathcal{M}^1(X)$ equipped with the weak*-topology is compact and the mapping $\mu \mapsto \int_X f(x, y) \mu(\mathrm{d}x)$ is (lower semi-)continuous for each $y \in X$. By minimax theorem (Theorem 1.30) we see

$$a(X, f) = \inf_{\nu \in \mathcal{M}^1} \sup_{\mu \in \mathcal{M}^1} \int\limits_X \int\limits_X f(x, y) \mu(\mathrm{d}x) \nu(\mathrm{d}y).$$

For arbitrary points $x_1, \dots, x_n \in X$, we have

$$\sup_{y \in X} \frac{1}{n} \sum_{i=1}^{n} f(x_i, y) \geq \sup_{y \in X} \inf_{\mu \in \mathcal{M}^1} \int_X \int_X f(x, y)\mu(\mathrm{d}x) = a(X, f), \tag{2.2}$$

$$\inf_{y \in X} \frac{1}{n} \sum_{i=1}^{n} f(x_i, y) \leq \inf_{y \in X} \sup_{\mu \in \mathcal{M}^1} \int_X \int_X f(x, y)\mu(\mathrm{d}x) = a(X, f). \tag{2.3}$$

The mapping $y \overset{\phi}{\mapsto} \frac{1}{n} \sum_{i=1}^{n} f(x_i, y)$ is continuous since f is continuous by assumption. It is generally true, that the image of compact respectively connected spaces under continuous functions is again compact respectively connected. This implies that the image $\mathrm{im}(\phi)$ is a compact interval in $\mathbb{R}$. Now it follows immediately form intermediate value theorem that there exists a $y \in X$ where equality holds in Equation 2.2 and 2.3, i.e.

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i, y) = a(X, f).$$

We shall now direct our attention towards the uniqueness of $a(X, f)$. Suppose there is another constant $a' < a(X, f)$ satisfying Equation 2.1 (The case of $a' > a(X, f)$ is analogous). Choose $\alpha > 0$ such that

$$a' + 3\alpha < \inf_{\nu} \sup_{\mu} \int_X \int_X f(x, y)\nu(\mathrm{d}y)\mu(\mathrm{d}x) = \inf_{y} \sup_{\mu} \int_X f(x, y)\mu(\mathrm{d}x)$$

Since the probability measures with finite support are dense in $M^1(X)$ (see Theorem 1.20) we can find $p_1, \dots, p_n \geq 0, \sum_{i=1}^{n} p_i = 1$ and $x_1, \dots, x_n$, with the property

$$a' + 2\alpha < \inf_{y} \sum_{i=1}^{n} p_i f(x_i, y). \tag{2.4}$$

Since

$$\left| \sum_{i=1}^{n} p_i f(x_i, y) - \sum_{i=1}^{n} p_i' f(x_i, y) \right| < \max_i |p_i - p_i'| \sup_{x,y} |f(x, y)|,$$

the right-hand side of Equation 2.4 depends continuously on $(p_1, \dots, p_n)$. Consider for one moment the case of $p_i \in \mathbb{Q}$. Then there clearly exists $k, m_1, \dots, m_n \in N$ with $p_i = \frac{m_i}{k}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we find that even in the general case there exist $k, m_1, \dots, m_n \in \mathbb{N}$ satisfying

$$a' + \alpha < \inf_{y} \frac{1}{k} \sum_{i=1}^{n} m_i f(x_i, y), \quad \sum_{i=1}^{n} m_i = k. \tag{2.5}$$

For the points

$$\begin{aligned} x_1' = x_2' = \dots = x_{m_1}' &:= x_1, \\ x_{m_1+1}' = \dots = x_{m_2}' &:= x_2, \\ &\vdots \\ x_{k-m_n+1}' = \dots = x_k' &:= x_n, \end{aligned}$$

there is no $y \in X$ such that (2.1) is valid with x_i there replaced by x'_i, n by k and a by a'. Hence $a(X, f)$ is uniquely determined. □

In the following paragraph we shall try to develop an intuition for the conditions under which above theorem holds.

It is quite obvious that X actually has to be connected for this statement to hold, for otherwise consider the space $X = \{1, 2\} \subset \mathbb{R}$ equipped with the subset Topology of $\mathbb{R}$ (this actually coincides with the discrete topology in this case). It is clear, that X is compact and a Hausdorff space but not connected. Let f be given by the euclidean metric. Suppose $a(X, f)$ exists. First, consider the situation, in which only one point is given. Then $y \in \{1, 2\}$. So

$$a(X, f) \in \{0, 1\} =: A.$$

Now consider the case where two points are given, in particular $\{x_1 = 1, x_2 = 2\}$. Again there are only two choices for y, however this gives $a(X, f) = \frac{1}{2}$ regardless of this choice. Since $\frac{1}{2} \notin A$, we see that $a(X, f)$ can't exist.

Compactness is required in general to ensure that $a(X, f)$ is finite for all f satisfying the assumptions.

Note however, that no claim was made whether Theorem 2.1 can be generalized to a broader category of spaces or not. We will actually discuss spheres in Banach spaces in chapter 4, which are only compact if the dimension is finite.

Corollary 2.3. *Let X be a compact, connected, non-empty Hausdorff space and let $C_s(X, \mathbb{R})$ be the set of symmetric continuous functions from X to $\mathbb{R}$ equipped with the uniform topology. The function*

$$\begin{aligned} C_s(X, \mathbb{R}) &\to \mathbb{R} \\ f &\mapsto a(X, f) \end{aligned}$$

is continuous.

Proof. Recall form the proof of Theorem 2.1 we have the formula

$$a(X, f) = \inf_{\nu \in \mathcal{M}^1} \sup_{\mu \in \mathcal{M}^1} \int_X \int_X f(x, y) \nu(\mathrm{d}y) \mu(\mathrm{d}x)$$

Now consider $f, g \in C_s(X, \mathbb{R})$ with $\sup_{x,y} |f(x, y) - g(x, y)| < \varepsilon$ for some $\varepsilon > 0$, then for all $\mu, \nu \in \mathcal{M}^1$:

$$\begin{aligned} \int_X \int_X g(x, y) \mu(\mathrm{d}x) \nu(\mathrm{d}y) - \varepsilon &\leq \int_X \int_X f(x, y) \mu(\mathrm{d}x) \nu(\mathrm{d}y) \\ &\leq \int_X \int_X (x, y) \mu(\mathrm{d}x) \nu(\mathrm{d}y) + \varepsilon. \end{aligned}$$

Applying inf and sup consecutively results in

$$a(X, g) - \varepsilon \leq a(X, f) \leq a(X, g) + \varepsilon,$$

hence the described mapping is continuous with respect to the uniform topology on $C_s(X, \mathbb{R})$. □

2.2. Bounds for the Rendezvous Value

With the existence and uniqueness being proven, it is a natural question to ask which values $a(X, f)$ might take. To examine this we first require the following definition.

Definition 2.4. Let X be a compact connected Hausdorff space and $f : X \times X \to \mathbb{R}$ a continuous and symmetric function. We call

$$D(X, f) := \sup_{x,y} |f(x, y)|$$

the *diameter of X with respect to f*.

Note that this definition coincides with the usual one for metric spaces if f is a metric on X.

The following result is due to Cleary, Morris and Yost [7] and is a extension of a statement from Gross [14].

Theorem 2.5. *Let (X, d) be a compact, connected, non-empty Hausdorff space. Let $n \in \mathbb{N}$. Then*

1. *$a(X, d^n) \geq 2^{-n} D(X, d^n)$ and*
2. *$a(X, d^n) < D(X, d^n)$ if X is not a singleton.*

Proof. For the fist part, let x_1 and x_2 be diametrical points in X, i.e. $d(x_1, x_2) = D(X, d)$. Such exist, since X is compact. By the Gross-Stadje-Theorem (Theorem 2.1) we know that there is a $y \in X$ such that

$$\begin{aligned} a(X, d^n) &= \frac{1}{2}\left(d^n(x_1, y) + d^n(x_2, y)\right) \\ &\geq \left(\frac{1}{2}d(x_1, y) + \frac{1}{2}d(x_2, y)\right)^n \\ &\geq \left(\frac{1}{2}d(x_1, x_2)\right)^n = 2^{-n} D(X, d^n). \end{aligned}$$

Thus, $a(X, d^n) \leq 2^{-n} D(X, d^n)$.

For the second part, we only need to show that $a(X, d^n) \neq D(X, d^n)$. Assume for a contradiction that $a(X, d^n) = D(x, d^n)$ and let $x_1, x_2 \in X$ be diametrical points. Then by Theorem 2.1 there is a $x_3 \in X$ satisfying

$$D(X, d^n) = a(X, d^n) = \frac{1}{2}\left(d^n(x_1, x_3) + d^n(x_2, x_3)\right).$$

Since both $d^n(x_1, x_3)$ and $d^n(x_2, x_3)$ are bounded above by $D(X, d^n)$, this can only be the case for $d^n(x_1, x_3) = d^n(x_2, x_3) = D(X, d^n)$. Continuing this process inductively yields a sequence $(x_k)_{k \in \mathbb{N}} \in X$ such that $d^n(x_i, x_j) = D(X, d^n)\delta_{ij}$, where δ_{ij} denotes the Kronecker Delta. It is a sequence in a compact space that does not contain a convergent subsequence, which is not possible. $\square$

Moving forward if will be inconvenient to state the results in terms of the rendezvous value and the diameter. We will simplify this as follows:

Definition 2.6. We call

$$m(X,f) := \frac{a(X,f)}{D(X,f)}$$

the *dispersion constant*, sometimes also referred to as the *magic number*.

Obviously, the dispersion constant $m(X, f)$ can only take values in $[-1, 1]$. We will see in the following section (see Theorem 2.10)that without further assumptions, this bound cannot be sharpened. However, we need some technical results first.

Definition 2.7. A space E is called *normal* provided that any two disjoint closed sets have disjoint neighborhoods.

Theorem 2.8 (Tietze extension theorem and Urysohn's lemma [5])**.** *Let E be a hausdorff space. Then the following statements are equivalent:*

1. *E is normal.*
2. *For any two closed, non-empty, disjoint sets A, B, there is a continuous function $f : E \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.*
3. *For any closed set $A \subset E$ and continuous function $f : A \to [a, b]$, there exists $g : E \to [a, b]$ which is continuous and an extension of f.*

The equivalence (i)$\Leftrightarrow$ (ii) is called Urysohn's lemma and (i)$\Leftrightarrow$(iii) the Tietze extension theorem.

Proof. The implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii) are trivial.

(i)$\Rightarrow$(ii): Let A and B be two closed disjoint subsets of E. Since E is normal, there exists a open set U such that

$$A \subset U \subset \bar{U} \subset E \setminus B.$$

Let D be a countable, dense subset of $[0, 1]$ and repeat above construction inductively (with A replaced by $\bar{U}$), yielding a family of open sets $(U_t)_{t\in D}$ such that if $t < s$ we have

$$A \subset U_t \subset \bar{U}_t \subset U_s \subset \bar{U}_s \subset E \setminus B.$$

Now construct a function $f : X \to [0, 1]$ by $f(x) = 0$ if $x \in \bigcap_{t\in D} U_t$ and $f(x) = \sup\{t : x \notin U_t\}$ otherwise. Note that

$$f^{-1}([0, a)) = \bigcup\{U_t : t < a\}$$

and

$$f^{-1}((a, 1]) = \bigcup\{E \setminus \bar{U}_t : t > a\}$$

implies that f is continuous.

(ii)⇒(iii): Let A be a closed subset of E and $f : A \to [a, b]$ given. Note that we may replace $[0, 1]$ in (ii) by any interval $[a, b]$. In particular, it is sufficient to prove (iii) for the case of $[-1, 1]$. Let $A_0 = \{x \in A : f(x) \leq -\frac{1}{3}\}$ and $B_0 = \{x \in A : f(x) \geq \frac{1}{3}\}$. By (ii) there is a continuous function $g_0 : E \to [-\frac{1}{3}, \frac{1}{3}]$ such that $g_0(A_0) = -\frac{1}{3}$ and $g_0(B_0) = \frac{1}{3}$. Let $f_1 = f - g_0$ and repeat the procedure with f_1. Applying this process inductively yields a sequence

$$f_n = f - (g_0 + g_1 + \cdots + g_{n-1}) = f - s_n$$

such that $\|f_n\|_{\sup} \leq \left(\frac{2}{3}\right)^n$, and $\|g_n\|_{\sup} \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$. Therefore, s_n converges uniformly to a function g on E which agrees with f on A. □

The following statement ensures, that in our usual case, i.e. X being a compact, connected, non-empty Hausdorff space, we may apply the Tietze extension theorem and the Urysohn lemma.

Lemma 2.9. *Every compact Hausdorff space is normal.*

The proof is due to Munkres [23, p. 202].

Proof. We begin by showing that every compact hausdorff space X is regular, i.e. given a closed set A and a point $x \in X$ not in A, there exist disjoint neighborhoods of A and x respectively. Since X is hausdorff, there are disjoint open sets U_y of y and V_y of x. It is clear, that the U_y are a open cover of A. Since A is a closed subset of a compact space, A is compact. Thus, there is a finite selection of points (y_i) such that $U = \bigcup_{i=1}^{n} U_{y_i}$ contains A. Since all U_{y_i} are open, so is U. Similarly, $V = \bigcap_{i=1}^{n} V_{y_i}$ is open as the intersection of finitely many open sets. Further, U and V are disjoint, proving regularity.

We use a similar process to prove normality. Let A and B be disjoint closed subsets of X. Since X is regular, for all $x \in B$ there are neighborhoods U_x of A and V_x of x. Then the V_x form a open cover of B. Since B is compact as a closed subset of a compact space, there is a finite collection (x_i) such that $V = \bigcup_{i=1}^{r} V_{x_i}$ contains B. Define $U := \bigcap_{i=1}^{r} U_{x_i}$, then U and V are disjoint open neighborhoods of A and B respectively, proving regularity. □

We are now able to prove the following statement, which was formulated by Cleary and Morris [6]. However, the proof is based on the argument in [7] by Cleary, Morris and Yost.

Theorem 2.10. *Let X be a compact, connected Hausdorff space with infinitely many points. Then for each real number $m \in [-1, 1]$ there is a $f \in C_s(X, \mathbb{R})$ such that $m(X, f) = m$.*

Proof. We will first examine the case of $m \geq 0$. Choose distinct points $a, b, c \in X$. Define

$$S := (X \times \{c\}) \cup (\{c\} \times X) \cup \{(a, b)\} \cup \{(b, a)\}.$$

Then S is a closed subspace of the compact Hausdorff space $X \times X$. Define $g : S \to [0, 1]$ as $g(x, c) = g(c, x) = \sqrt{m}$ for all $x \in X$ and $g(a, b) = g(b, a) = 1$. Note, that g is continuous. We now apply Tietze's extension theorem, which states that there is a continuous function $\theta : X \times X \to [0, 1]$ such that $g(s) = \theta(s)$, for all $s \in S$. Now set f to be the function from $X \times X$ to $[0, 1]$ with $f(x, y) = \theta(x, y)\theta(y, x)$. Then f is a continuous symmetric function with $f(x, c) = m$ for all $x \in X$. By considering the case $n = 1$ and $x_1 = c$, we immediately see from the definition of the rendezvous value that $a(X, f) = m$. Since $f(a, b) = 1$ it is also clear that $D(X, f) = 1$, yielding $m(X, f) = m$.

In the case of $m < 0$, we can construction an f with $m(X, f) = -m$ analogously. Setting $f' := -f$ gives $m(X, f') = m$. □

The following result is due to Cleary and Morris [6] and shows that the rendezvous value is *not* a topological invariant.

Theorem 2.11. *Let X be a compact, connected, metrizable space with more than one point. Let $m \in [\frac{1}{2}, 1)$, then there exists a compatible metric d such that $m(X, d) = m$.*

Proof. Let ρ be a compatible metric such that $D(X, \rho) = 1$. Choose $a, b \in X$ to be diametrical points and let $c \in X$ such that $\rho(a, c) = \frac{1}{2}$ (c exists by virtue of the intermediate value theorem). Define a metric d by

$$d(x, y) = \min\left\{\rho(x, y), \min\left\{\frac{1}{2}, \rho(c, x)\right\} + \min\left\{\frac{1}{2}, \rho(c, y)\right\}\right\}.$$

We claim that this metric is equivalent to ρ and $m(X, d) = \frac{1}{2}$.

To see that d is a metric, it is easily verified that d is symmetric and $d(x, y) = 0$ if and only if $x = y$, since ρ is a metric. The triangle inequality follows immediately from the following inequality:

$$\rho(c, x) + \rho(c, y) \leq \rho(c, x) + \rho(c, y) - 2\rho(c, z) = \rho(c, x) + \rho(c, z) + \rho(c, y) + \rho(c, z).$$

In order to show that this metric is compatible, i.e. that ρ induces the same topology as d, we only need to show

$$\frac{1}{2}d(x, y) \leq \rho(x, y) \leq d(x, y)$$

Here, the second inequality follows immediately from the definition of ρ. Suppose that $\rho(x, y) = \min\left\{\frac{1}{2}, \rho(c, x)\right\} + \min\left\{\frac{1}{2}, \rho(c, y)\right\}$, i.e. $\rho(x, y) \geq d(x, y)$.

If $\min\left\{\frac{1}{2}, \rho(c, x)\right\} = d(x, c)$ and $\min\left\{\frac{1}{2}, \rho(c, y)\right\} = d(y, c)$, then $\rho(x, c) + \rho(y, c) = d(x, c) + d(y, c) \geq d(x, y)$. Thus, $\rho(x, y) = d(x, y)$. If $\min\left\{\frac{1}{2}, \rho(c, x)\right\} \neq d(x, c)$, then $\min\left\{\frac{1}{2}, \rho(c, x)\right\} = \frac{1}{2} \geq \frac{1}{2}d(x, y)$. Similarly, if $\min\left\{\frac{1}{2}, \rho(c, y)\right\} \neq d(y, c)$, then $\rho(x, y) \geq \frac{1}{2} \geq \frac{1}{2}d(x, y)$. Hence, $\frac{1}{2}d(x, y) \leq \rho(x, y) \leq d(x, y)$.

Now, since $d(x,c) \leq \frac{1}{2}$ for all $x \in X$ clearly $m(X,d) \leq \frac{1}{2}$. With the result from Theorem 2.5 we get $m(X,d) = \frac{1}{2}$.

For $m > \frac{1}{2}$, define for $\lambda \geq 0$ metric

$$d_\lambda := \frac{(\lambda+1)d}{\lambda d + 1}.$$

We need to verify that d_λ is in fact a metric. Symmetry and identity of indiscernibles are obvious, since d is a metric. The proof of the triangle inequality is not hard, but tedious:

Firstly, notice that for $x, y, n \in \mathbb{R}$

$$\frac{x+n}{y+n} - \frac{x}{y} = \frac{n(y-x)}{y(y+n)}.$$

This enables us to find conditions under which $\frac{x+n}{y+n} \geq \frac{x}{y}$ (or $\leq$ respectively). Consider

$$\begin{aligned}
\frac{(\lambda+1)d(x,z)}{\lambda d(x,z)+1} + \frac{(\lambda+1)d(z,y)}{\lambda d(z,y)+1} &= \frac{2\lambda(\lambda+1)d(z,y)d(z,x) + (\lambda+1)(d(x,z)+d(y,z)}{\lambda^2 d(x,z)d(y,z) + \lambda(d(x,z)+d(y,z)) + 1} \\
&\geq \frac{2\lambda(\lambda+1)d(z,y)d(z,x) + d(x,z) + d(z,y) + \lambda d(x,y)}{\lambda^2 d(x,z)d(z,y) + \lambda d(x,y) + 1} \\
&\geq \frac{2\lambda(\lambda+1)d(z,x)d(z,y) + (\lambda+1)d(x,y)}{\lambda^2 d(x,z)d(y,z) + \lambda d(x,y) + 1} \\
&\geq \frac{\lambda^2 d(x,z)d(z,y) + (\lambda+1)d(x,y)}{\lambda^2 d(x,z)d(z,y) + \lambda d(x,y) + 1} \\
&\geq \frac{(\lambda+1)d(x,y)}{\lambda d(x,y)+1}.
\end{aligned}$$

Since $d_\lambda(a,b) = 1$ for all $a, b \in X$ for which $d(x,y) = 1$, we have $D(X, d_\lambda) = 1$ for all λ. Further, if $x \neq y$, then $d_\lambda(x,y) \to 1$ as $\lambda \to \infty$.

We now claim that the dispersion constant $m(X, d_\lambda)$ depends continuously on λ. To see that, we only need to show that d depends continuously on λ with respect to the uniform topology on $C_S(X)$ and then apply Corollary 2.3. We will prove this with a standard ε-δ-proof:

Let $\varepsilon > 0$ and and $\lambda_0 \geq 0$. Set

$$\delta := \frac{\varepsilon}{2 \cdot \sup\limits_{x,y \in X} (d^2(x,y) + d(x,y)} > 0.$$

Now, let $|\lambda - \lambda_0| < \delta$. Then

$$\begin{aligned}
|d_\lambda - d_{\lambda_0}| =& \left| \frac{(\lambda+1)d(x,y)}{\lambda d(x,y)+1} - \frac{(\lambda_0+1)d(x,y)}{\lambda_0 d(x,y)+1} \right| \\
=& \left| \frac{(\lambda+1)\lambda_0 d^2(x,y) + (\lambda+1)d(x,y)}{\lambda\lambda_0 d^2(x,y) + \lambda d(x,y) + \lambda_0 d(x,y) + 1} \right. \\
& \qquad \left. - \frac{(\lambda_0+1)\lambda d^2(x,y) + (\lambda_0+1)d(x,y)}{\lambda\lambda_0 d^2(x,y) + \lambda d(x,y) + \lambda_0 d(x,y) + 1} \right| \\
\leq& \frac{d^2(x,y)|\lambda_0 - \lambda| + d(x,y)|\lambda - \lambda_0|}{|\lambda\lambda_0 d^2(x,y) + \lambda d(x,y) + \lambda_0 d(x,y) + 1|} \\
\leq& \underbrace{\frac{(d^2(x,y) + d(x,y))\delta}{|\lambda\lambda_0 d^2(x,y) + \lambda d(x,y) + \lambda_0 d(x,y) + 1|}}_{>1} \\
\leq& (d^2(x,y) + d(x,y))\delta.
\end{aligned}$$

Therefore,

$$\|d_\lambda - d_{\lambda_0}\|_\infty < \epsilon.$$

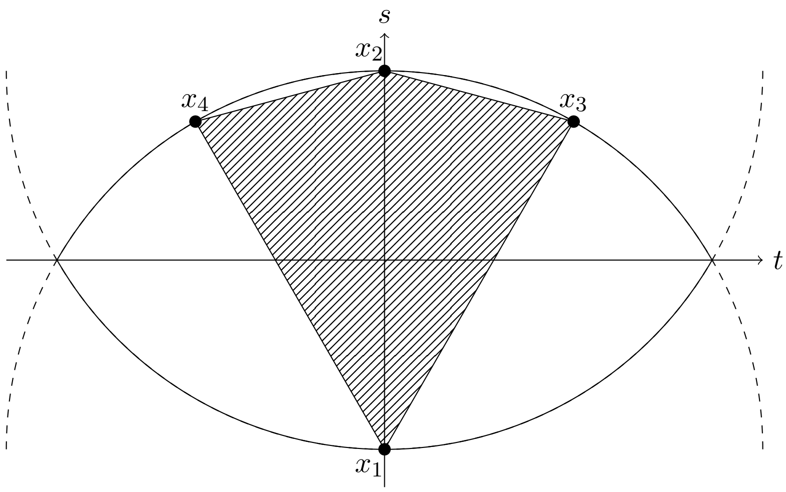
Clearly $m(X, d_\lambda) \to 1$ as $\lambda \to \infty$. Since $m(X, d_0) = \frac{1}{2}$, the intermediate value theorem yields the existence of a $\lambda \in [0, \infty)$ with $m(X, d_\lambda) = m$. $\square$

Stadje [29] found a bound for compact convex subspaces of the euclidean 2-space. He further stated, that the same bound holds for higher dimensions. J. Strantzen [30] and E. Szekeres and G. Szekeres (according to Cleary, Morris and Yost [7], quoting private communication), however, independently proved that this is not correct. Stranzen gives the correct bound, which we will discuss after some preliminary work.

Theorem 2.12. *Let d be the euclidean metric on $\mathbb{R}^2$ and $X \subset \mathbb{R}^2$ be a compact convex set. Then*

$$a(X,d) \leq \frac{1}{2}\sqrt{5 - 2\sqrt{3}}D(X,d).$$

Proof. If X consists of a single point, then the statement is trivial. Consider the case where X consists of at least two points. Choose $x_1, x_2 \in X$ to be diametrical and let $\frac{1}{2}(x_1 + x_2)$ be the origin of a coordinate system as pictured below.



For the purpose of finding an upper bound for $a(X, d)$ we may restrict ourselves to dirac measures of points on the t-axis. Let $B((t, 0), a)$ be the ball with center $(t, 0)$ and radius a. Clearly, if $X \subset B((t, 0), a)$ then $a \geq a(X, d)$, since

$$\int_X d(x, (t, 0))\mu(\mathrm{d}x) \leq a \quad \forall \mu \in \mathcal{M}^1(X).$$

Let $(t', s') = \arg \max_{(t,s) \in X} (|t|)$. Without loss of generality, we can assume $t', s' > 0$. Again, $X \subset B((t', s'), D)$. The goal is now to choose a such that $B(x_1, D) \cap B(x_2, D) \cap B((t', s')) \subset B((t, 0), a)$ for some t.

Let x_3 and x_4 be the points on the upper arc which have the same s-coordinate and distance $D(X, d)$. A simple geometrical argument yields, that

$$x_3 = \left(\frac{D(X, d)}{2}, D(X, d) \cos\left(\frac{\pi}{6}\right) - \frac{D(X, d)}{2} \right) = \frac{D(X, d)}{2}(1, \sqrt{3} - 1)$$
$$x_4 = \frac{D(X, d)}{2}(-1, \sqrt{3} - 1)$$

The distance from x_3 (and x_4) to the origin is easily computed to be $\frac{D(X,d)}{2}\sqrt{5 - 2\sqrt{3}}$. The (filled, 2-dimensional) quadrangle spanned by x_1, x_2, x_3, x_4 (shaded in the picture above) cannot be contained in a ball with center on the t-axis and radius $a \leq \frac{D(X,d)}{2}\sqrt{5 - 2\sqrt{3}}$. Further, this radius suffices for all possible (s', t') by symmetry. A interactive GeoGebra worksheet to this is available on: `www.geogebra.org/u/janzwank` $\square$

To find the upper bound for the higher dimensional case, we fist need to investigate surfaces of smallest radius enclosing bounded subsets of n-dimensional euclidean space, following Blumenthal's approach [2]. For this purpose, we need the following result from Helly [15] first.

Lemma 2.13. *For any collection of closed convex subsets $\mathcal{K} := \{K_i\}_{i\in I}$ of some $\mathbb{R}^n$ for which each subcollection $\{K_{i_1}, \dots, K_{i_{n+1}}\}$ of $n+1$ sets has at least one common point, i.e. $\bigcap\limits_{k=1}^{n+1} K_{i_k} \neq \emptyset$, there exists a point in the intersection of all sets, i.e. $\bigcap\limits_{i\in I} K_i \neq \emptyset$.*

Proof. For $n = 1$ this is easy to see. Note that in dimension one closed convex subsets of $\mathbb{R}^1$ are essentially line segments (whose endpoint might be $\pm\infty$). Each pair of two such line segments has at lest one common point. Let

$$a_i = \inf K_i \quad \text{and} \quad b_i = \sup K_i,$$

then clearly

$$\sup a_i \leq \inf b_i.$$

Thus, there exists a point $p \in \mathbb{R}^n$ with $p \in \bigcap\limits_{i\in I} K_i$.

We will now apply a proof by induction. Suppose above statement holds for dimensional less than $n-1$ and for the special case of dimension n for $|I| \leq p$ sets, where $p > n+1$ (the case $p = n+1$ is trivial).

Let $K_1, \dots K_p$ be p closed convex subsets of $\mathbb{R}^n$. Let K be another closed convex subset such that $K_1, \dots, K_p, K$ satisfy the requirements of the lemma. Let $G = \bigcap_{i=1}^p K_i \neq \emptyset$. G itself is a closed convex subset of $\mathbb{R}^n$. Suppose for a contradiction that our statement is incorrect, then it must be possible to choose K such that K has a common point with each subcollection of n sets of $K_1, \dots, K_p$, and simultaneously $K \cap G = \emptyset$. In this case there exists a hypersurface H separating G and K with $H \cap G = H \cap K = \emptyset$. Now consider the intersections of each pairing of $n+1$ sets in $\mathcal{K}$. Then these sets all contain G and at least one point of K. Thus, all such sets must have at least one common point with H. This means that for

$$k_i = K_i \cap H$$

each collection of n such k_i has at least one common point in H. By induction hypothesis we have $\bigcap_{i\in I} k_i \neq 0$. But $\bigcap_{i\in I} k_i \subset G$ and thus we get a contradiction to $H \cap G = \emptyset$. Hence, K and G have at least one common point. This completes the proof for finitely many sets, i.e. $|\mathcal{K}| \neq \infty$.

The statement for countably many sets follows by successively adding more sets and exploiting that all sets in $\mathcal{K}$ are closed. Now, consider the case of I being uncountable. For this, choose any countable subset N' of I. Denote the (nonempty) set of common points as G'. Let r' be its dimension. Let r denote the smallest possible value of r'.

Further, each G' has a capacity V'. Let $V := \inf V'$, where the infimum is taken over all G' of dimension r.

Let $N_1, N_2, \dots$ be a sequence of countable subsets of I, with common sets $G_1, G_2, \dots$ such that the corresponding sequence $V_1, V_2, \dots$ converges to V. Now, let $N = \bigcup_{k=1}^\infty N_k$. Then N is again countable and the common sets G has dimension r and capacity V. We now claim that G is contained in all sets in $\mathcal{K}$, for if it wasn't contained in some set $K \in \mathcal{K}$, then the sequence consisting of K and $\{K_i\}_{i\in K}$ would yield a capacity less than V, which contradicts the definition of V. $\square$

Definition 2.14. Let $S_{n-1,r}$ denote the ($n-1$-dimensional) spherical surface (i.e. the boundary of the n-dimensional ball) of radius r in E_n (the euclidean n-space). A given spherical surface *encloses* $M \subset E_n$ if M is a subset of the closed ball whose boundary is given by $S_{n-1,r}$. If there exists a smallest radius $\rho(M)$ for which $S_{n-1,\rho(M)}$ encloses M, we call $\rho(M)$ the *Chebyshev-out radius of* M and the center $c(M) \subset M$ of this sphere the *Chebyshev center of* M.

The following lemma reduces the problem to the case of finitely many points.

Lemma 2.15. *If each set of $n+1$ points of a subset M of E_n is enclosable by $S_{n-1,r}$ of given radius r, then M is itself enclosable by this $S_{n-1,r}$.*

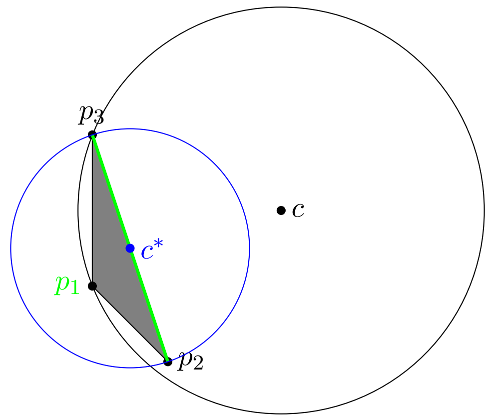
Proof. Let $\overline{B^{(n)}(x,r)}$ denote the n-dimensional ball with radius r and center x. Consider the family $\{\overline{B^{(n)}(x,r)}\}_{x \in M}$. Since each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ we see that each choice of $n+1$ balls has non-empty intersection. With a lemma of Helly (Lemma 2.13) we find that there is a point p common to all such balls. Thus, $M \subset \overline{B^{(n)}(p,r)}$. □

Lemma 2.16. *Let $P = \{p_1, \dots, p_{n+1}\}$ be a set of $n+1$ points of E_n with diameter $D > 0$. There exists a positive number r such that P is enclosable by $S_{n-1,r}$ and not enclosable by any S_{n-1,r^*} with $r^* < r$.*

Proof. This statement is an immediate consequence of P being compact in E_n. □

Lemma 2.17. *In the situation of the previous lemma, the center c of $S_{n-1,r}$ is a point of the simplex whose vertices are the points of P.*

Proof. Assume for a contradiction that c is not contained in this simplex. Then a face of codimension one separates c from the vertex opposite of this hyperplane. Since the spherical surface S_{n-1,r^*}, erected on the intersection of this hyperplane with $S_{n-1,r}$ (the center of this spherical surface is the orthogonal projection of c on $x_n = 0$) encloses P with $r^* < r$, we get a contradiction. For $n = 2$ this is depicted below. □



Lemma 2.18. *Let P, c and $S_{n-1,r}$ as above. If a point of P is not on $S_{n-1,r}$, then c lies in the face of the simplex opposite this point.*

Proof. Assume for a contradiction that there exists a point $p_j \in P$ which is not in $S_{n-1,r}$ and select a Cartesian coordinate system so that the $n-1$-dimensional hyperplane determined by the points $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}$ have vanishing n-th coordinate x_n and the n-th coordinate of p_j, denoted by $x_n^{(j)}$ is positive. The previous lemma implies that c_n (the n-th coordinate of c) is positive.

Let $0 < t < \min\left\{\frac{|S_{n-1,r}(p_j)|}{2x_n^{(j)}}, c_n\right\}$, where $S_{n-1,r}(x) := (x_1 - c_1)^2 + \dots + (x_n - c_n)^2 - r^2$. Consider S_{n-1,r^*} satisfying the equation

$$S_{n-1,r} + 2tx_n = 0$$

The left-hand-side of this equation is always less or equal to zero for each of the points $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}$ since $S_{n-1,r}$ encloses these points and each of them are in the plane $x_n = 0$. Furthermore, we chose t such that

$$S_{n-1,r}(p_j) + 2tx_n^{(j)} < S_{n-1,r}(p_j) + |S_{n-1,r}(p_j)| = 0.$$

Thus S_{n-1,r^*} encloses all of the $n+1$ points of P. However, this is impossible, since S_{n-1,r^*} is a linear combination of these loci, and since $0 < t < c_n$ the center $(c_1, c_2, \dots, c_{n-1}, c_n - t)$ of S_{n-1,r^*} is closer to the plane $x_n = 0$ than it is to the center c of $S_{n-1,r}$. Hence $r^* < r$ contradicting the minimality of r. □

Lemma 2.19. *Let P be a set of $n+1$ points of E_n, not of E_{n-1} of diameter D. If $S_{n-1,r}$ is an $(n-1)$-dimensional spherical surface of smallest radius r enclosing P, then $r \le D\sqrt{\frac{n}{2n+2}}$.*

Proof. Let $P_1, P_2, \dots, P_{n+1}$ be the vectors corresponding to the points $p_1, p_2, \dots, p_{n+1}$ of P and let C be the corresponding vector to the center c. By virtue of lemma 2.17 there exist $k_1, \dots, k_{n+1} \in \mathbb{R}_0^+$ with

$$\sum_{i=1}^{n+1} k_i = 1 \qquad \text{and} \qquad C = k_1P_1 + k_2P_2 + \dots + k_{n+1}P_{n+1} \tag{2.6}$$

Without loss of generality, we can assume $k_{n+1} \ge k_i$ for all $1 \le i \le n+1$ by relabeling if necessary. Then $k_{n+1} > 0$, thus c is not contained in the face opposite of p_{n+1} and by virtue of lemma 2.18 p_{n+1} is on $S_{n-1,r}$.

Since the constants $k_1, \dots, k_{n+1}$ are invariant under translation, we can translate the origin of the coordinate system to p_{n+1} and have $P_i^T \cdot (2C - P_i) = 0$ for each index i with $k_i > 0$, for if k_i is positive then $p_i \in S_{n-1,r}$ by the same argument as above. Therefore for each such index i, we have $2P_i^T \cdot C = P_i^T \cdot P_i$. Since k_i are either positive or zero, we see that the equality $2k_iP_i^T \cdot C = k_i(P_i^T \cdot P_i)$ holds for all indices $1 \le i \le n+1$. We get

$$\sum_{i=1}^{n}(k_iP_i^T) \cdot C = \frac{1}{2}\sum_{i=1}^{n} k_i(P_i^T \cdot P_i) = \frac{1}{2}\sum_{i=1}^{n} k_i d_i^2,$$

where d_i denotes the length of P_i. With (2.6) and $P_{n+1} = 0$ we conclude

$$r^2 = C^T \cdot C \leq \frac{1}{2} D^2 \sum_{i=1}^{n} k_i.$$

With $k_{n+1} \geq k_i$ for all $1 \leq i \leq n+1$ and $\sum_{i=1}^{n} k_i = 1 - k_{n+1}$, so $k_{n+1} > \frac{1}{n+1}$ and we find

$$r^2 \leq \frac{n}{2(n+1)} D^2.$$

□

Theorem 2.20. *Let D be the diameter of the bounded set M (containing more than a single point) of the n-dimensional euclidean space E_n. Then*

1. *there exists a unique smallest spherical surface $S_{n-1,r}$ enclosing M and*
2. $r \leq D\sqrt{\frac{n}{2(n+1)}}$.

Proof. We apply a inductive proof. The statement is trivial for $n = 1$. Assume the assertion holds for every positive integer $k < n$. We consider two cases:

Case 1. $M \subset E_k$**,** $1 \leq k < n$**.** By induction hypothesis there exists a unique smallest $S_{k-1,r}$ enclosing M and $r \leq D\sqrt{\frac{k}{2(k+1)}} < D\sqrt{\frac{n}{2(n+1)}}$, since $\frac{\mathrm{d}}{\mathrm{d}k} \frac{k}{2(k+1)} = \frac{1}{2(k+1)^2} > 0$. It is clear that $S_{n-1,r}$ satisfies the requirements of the theorem.

Case 2. $M \not\subset E_k$ $\forall k < n$**.** Consider the set $\{P\}$ of all sets of $n+1$ points of M is not empty, and by lemma 2.16 there is a smallest $S_{n-1,r(P)}$ enclosing each P of $\{P\}$. Let $r := \sup\limits_{P \in \{P\}} r(P)$. Since $0 < r(P) < D, P \in \{P\}, r$ is a positive (finite) number.

Assertion. The set M is enclosable by $S_{n-1,r}$ and by no spherical surface of smaller radius.

Since $r \geq r(P)$ for all $P \subset M$ each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ and hence by lemma 2.15, M is enclosable by S_{n-1,r^*}. Assume $r^* < r$, then we get the existence of subset P of $n+1$ points of M with $r(P) > r^*$, by definition of r; thereby the smallest spherical surface enclosing this subset P has a radius exceeding r^*. Thus, this subset and thereby M is not enclosable by an S_{n-1,r^*}.

For the uniqueness part of the statement consider the following: Let $S_{n-1,r}(p)$ be a $(n-1)$-dimensional spherical surface of smallest radius r enclosing M with center p. Suppose that $S_{n-1,r}(q)$ is another spherical surface enclosing M. Then M is contained in the intersection of the corresponding n-balls of radius r. Consequently, M is enclosable by a S_{n-1,r^*} with $r^* < r$. This proves uniqueness of both the radius as well as the center of $S_{n-1,r}$.

Statement 2 follows from lemma 2.19. □

With this preliminary work, we are now able to understand the proof for the following statement by Strantzen [30].

Theorem 2.21. *Let X be a bounded, closed, convex, non-empty subset of $\mathbb{R}^n$ and D be the diameter of X. Then $a(X,d) \le D\sqrt{\frac{n}{2n+2}}$.*

Proof. The statement is trivial if X is just one point. If X contains at least two points, we need only apply Theorem 2.20 to find a ball B of smallest radius $r \le D\sqrt{\frac{n}{2(n+1)}}$. Let b be the center of B. It is a immediate consequence of lemma 2.15 and lemma 2.17 that $b \in X$. Now apply the Gross-Stadje-Theorem (Theorem 2.1) to the case of $n = 1$ and $x_1 = b$. Since $d(b,y) \le r$ for all $y \in X$ we find $a(X,d) \le r \le D\sqrt{\frac{n}{2(n+1)}}$. □

Strantzen further proves that this bound is optimal by computing the rendezvous-value of the k-skeleton of a regular n-simplex of diameter D to be

$$a(\sigma^k, d) = \frac{D}{(n+1)\sqrt{2k+2}}((k+1)\sqrt{k} + (n-k)\sqrt{k+2}).$$

For the case $k = n$ we see $a(\sigma^k, d) = d\sqrt{\frac{n}{(2n+2)}}$. Note that for $n \ge 4$ we have

$$\sqrt{\frac{n}{2n+2}} \ge \frac{1}{2}\sqrt{5 - 2\sqrt{3}},$$

proving that Stadje's bound is in fact erroneous for $n \ge 4$. For details, the reader is referred to Strantzen's paper [30].

According to Cleary, Morris and Yost [7] (quoting private communication) Esther and George Szekeres proved the following closely related result.

Theorem 2.22. *Let X be a compact, convex subset of some normed space, with d being the metric given by the norm. Then $a(X,d) = \rho(X)$, where $\rho(X)$ is the Chebyshev-out radius.*

Proof. Let $c(X) \in X$ be the Chebyshev center of X. Following the same logic as in the previous statement, we see $a(X,d) \le \rho(X)$. To see the other inequality, choose $x_1, \dots, x_n \in X$ and define

$$\Theta\colon X \to \mathbb{R}, \quad x \mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, d_i).$$

Since X is connected, we only need to show $\Theta(x) \ge \rho(X)$ for some $x \in X$.

Let $x_0 = \frac{1}{n}\sum_{i=1}^{n} x_i \in X$, and choose $y \in X$ such that $d(y, x_0) \ge \rho(X)$. The existence of such y follows form the definition of $\rho(X)$. Then

$$\rho(X) \le \|c - y\| \le \frac{1}{n}\sum_{i=1}^{n} \|x_i - y\| = \Theta(y).$$ □

In general there is no inequality relation for subsets in the sense of $a(X,d) \stackrel{\le}{\ge} a(Y,d)$ whenever $X \subset Y$. However, the following result due to Yost [36] holds true.

Theorem 2.23. *Let X be a compact, connected, non-empty subset of a normed space. Let Y be a closed, connected subset of X, and suppose that the convex hull of Y contains X. Then $m(X, d) \leq m(Y, d)$, where d is the metric induced by the norm.*

Proof. Since $Y \subset X \subset \operatorname{conv}(Y)$ it is clear that $D(X, d) = D(Y, d)$. Thus, we only need to show $a(X, d) \leq a(Y, d)$. Let F denote any finite collection of points $x_1, \ldots, x_n$ ($n \in \mathbb{N}$) in Y (and therefore in X). Then by the Gross-Stadje-Theorem (Theorem 2.1) there exists $x \in X$ satisfying

$$a(X, d) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, x).$$

Since $X \subset \operatorname{conv}(Y)$ there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_0^+, \sum_{j=1}^{k} \lambda_j = 1$ and $y_1, \ldots y_k \in Y$ with $x = \sum_{j=1}^{k} \lambda_j y_j$. Thus,

$$a(X, d) = \frac{1}{n} \sum_{i=1}^{n} d\left(\sum_{j=1}^{k} y_j, x_i\right) \leq \sum_{j=1}^{k} \frac{\lambda_j}{n} \sum_{i=1}^{n} d(y_j, x_i).$$

It is now clear that there is at least one j with $a(X, d) \leq \frac{1}{n} \sum_{i=1}^{n} d(y_j, x_i)$ and therefore $a(X, d) \leq a(Y, d)$. □

The following result is an immediate consequence of this statement and due to Nickolas and Yost [27]

Theorem 2.24. *Let X be a compact connected subset of $\mathbb{R}^n$. Then*

$$m(X, d) \leq \sqrt{\frac{2n}{n+1}} a(S^{n-1}, d),$$

where d denotes the euclidean metric.

Proof. Without loss of generality, assume that $D(X, d) = 1$. Theorem 2.20 implies that $X \subset \operatorname{conv}(S_\rho^{n-1})$, where $\rho(X)$ denotes the Chebyshev-out radius and thus $\rho(X) \leq \sqrt{\frac{n}{2(n+1)}}$. With the result from the previous theorem we see $a(X, d) \leq \sqrt{\frac{2n}{n+1}} a(S^{n-1})$. □

In the next chapter we will discuss the computation of the rendezvous value of several spaces, including spheres.

3. Computations

This chapter is dedicated to computations of rendezvous values for various spaces and related results.

We begin with an easy example

Example 3.1. Consider the m-dimensional closed Ball $B := \overline{B(O, \frac{1}{2})}$ with center at O and radius $\frac{1}{2}$ in Euclidean m-space. Clearly B is a compact connected Hausdorff space, therefore the Gross-Stadje-Theorem (Theorem 2.1) is applicable and there exists a unique rendezvous value $a(B, d)$ where d is the usual Euclidean metric.

Firstly, let us examine the case of $n = 1$ and $x_1 = O$. In this situation it is apparent that the rendezvous value is at most $\frac{1}{2}$.

Secondly, consider the case $n = 2$ with $x_1, x_2 \in \partial B$ being antipodal points. Note that $d(x_1, x_2) = 1$ and by triangle inequality we have for all $y \in B$:

$$1 = d(x_1, x_2) \leq d(x_1, y) + d(x_2, y)$$

Multiplying both sides with $\frac{1}{2}$ gives

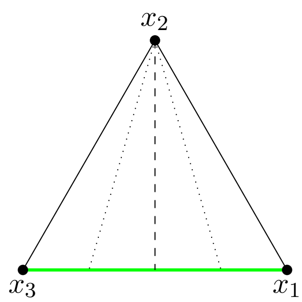
$$\frac{1}{2} \leq a(B, d).$$

Together with the previous result we conclude:

$$a(B, d) \leq \frac{1}{2} \leq a(B, d) \quad \Rightarrow \quad a(B, d) = \frac{1}{2}$$

The following example can also be found in [7]. The methods used here can be applied for many spaces.

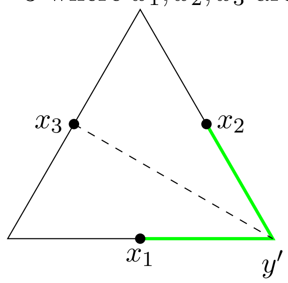
Example 3.2. Let $X \subset \mathbb{R}^2$ be an equilateral triangle (the 1-dimensional polygon) with side length ℓ (with the usual subspace topology) and let d be the Euclidean metric as before.



First, consider the case $n = 3$ and let x_1, x_2, x_3 be the vertices of the triangle. By symmetry we can assume that y is contained in the edge between x_1 and x_3 (here marked green). It is easily verified that $d(x_1, y) + d(x_3, y)$ is independent from y as long as it is contained in the edge joining those two vertices. Furthermore, it is clear, that the minimum of $d(x_2, y)$ is obtained for $y' = (0.5, 0)$. We therefore find:

$$a(X, d) \geq \frac{1}{3} \cdot (d(x_1, y') + d(x_2, y') + d(x_3, y')) = \frac{\ell}{3} \cdot \left(\frac{1}{2} + \sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\right) = \ell \left(\frac{2 + \sqrt{3}}{6}\right).$$

Now, consider the case $n = 3$ where x_1, x_2, x_3 are the midpoints of the edges.



Again, by symmetry we see that we can restrict y to be the "corner" between x_1 and x_2 (marked green again). It is easy to see that the vertex between x_1 and x_2 , denoted by y' from here on, is the point that is the furthest away from x_1, x_2 and x_3 simultaneously. Therefore, we find

$$a(X, d) \leq \frac{\ell}{3}(d(x_1, y') + d(x_2, y') + d(x_3, y')) = \frac{\ell}{3}\left(\frac{1}{2} + \frac{1}{2} + \sin\left(\frac{\pi}{3}\right)\right) = \ell \cdot \left(\frac{2 + \sqrt{3}}{6}\right).$$

With above result we see that equality holds.

According to Cleary, Morris and Yost [7], Cleary computed the rendezvous value for regular n-gons in euclidean 2-space with diameter 1 to be

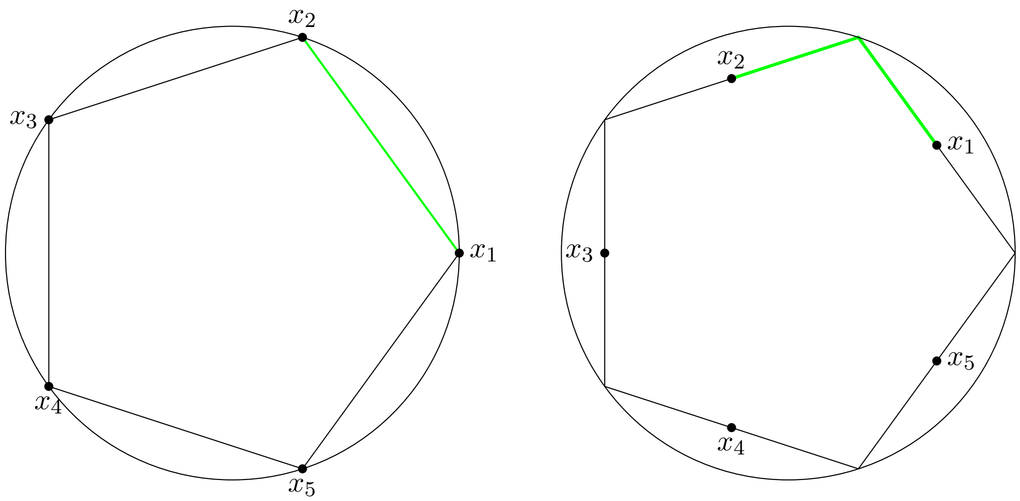
$$a(X_n, d) = \frac{1}{2n} \sum_{k=0}^{n-1} \left[\frac{3}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{2k\pi}{n}\right) - \cos\left(\frac{2(k-1)\pi}{n}\right)\right]^{\frac{1}{2}} \quad \text{,when } n \text{ even}$$

$$a(X_n, d) = \frac{1}{n} \sum_{k=0}^{n-1} \left[\frac{\frac{3}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{2k\pi}{n}\right) - \cos\left(\frac{2(k-1)\pi}{n}\right)}{2 - 2\cos\left(\frac{(n-1)\pi}{n}\right)}\right]^{\frac{1}{2}} \quad \text{,when } n \text{ odd}$$

Unfortunately, the original paper seems to have been lost, according to the La Trobe University Library (private communication).

However, a similar computation can be included in this paper from which Cleary's result follows easily:

Example 3.3. Let X_n be the regular n-gon in euclidean 2-space with vertices on the unit circle. Consider the case $k = n$ and x_i being the vertices of the n-gon.



Similar to the example of the equilateral triangle, we can restrict y to be on the edge connecting x_1 and x_2. Evidently,

$$\frac{1}{k}\sum_{i=1}^{k} d(x_i, y)$$

is minimized, if y is the midpoint of this edge. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ we find that this minimizer is given by

$$y_{min} = \frac{1}{2}(\zeta_n + 1),$$

where $\zeta_n = \exp\left(\frac{2\pi i}{n}\right)$ is the n-th root of unity in $\mathbb{C}$. We find that

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} d(x_j, y) \leq \frac{1}{n}\sum_{j=1}^{n}\left|\frac{1}{2}(\zeta_n+1)-\zeta_n^j\right| &= \frac{1}{n}\sum_{j=1}^{n}\left|\frac{1}{2}(\exp\left(\frac{2\pi i}{n}\right)+1)-\exp\left(\frac{2\pi ij}{n}\right)\right| \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\left(\frac{1}{2}\left(1+\exp\left(\frac{2\pi i}{n}\right)\right)-\exp\left(\frac{2\pi ij}{n}\right)\right)\cdot\right. \\
&\qquad \left.\cdot\left(\frac{1}{2}\left(1+\exp\left(\frac{2\pi i}{n}\right)\right)-\exp\left(\frac{2\pi ij}{n}\right)\right)\right]^{\frac{1}{2}} \\
&= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{4}\left(1+\exp\left(\frac{2\pi i}{n}\right)\right)\left(1+\exp\left(\frac{2\pi i}{n}\right)\right)\right. \\
&\qquad -\frac{1}{2}\left(1+\exp\left(\frac{2\pi i}{n}\right)\right)\exp\left(\frac{2\pi ij}{n}\right) \\
&\qquad \left.-\frac{1}{2}\exp\left(\frac{2\pi i}{n}\right)\left(1+\exp\left(\frac{2\pi ij}{n}\right)\right)+1\right]^{\frac{1}{2}}
\end{aligned}$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{4}\left(2+2\cos\left(\frac{2\pi}{n}\right)\right) - \frac{1}{2}\exp\left(\frac{2\pi i}{n}\right)\right.$$
$$\left. - \frac{1}{2}\exp\left(\frac{2\pi i(j-1)}{n}\right) - \frac{1}{2}\exp\left(\frac{2\pi ij}{n}\right) - \frac{1}{2}\exp\left(\frac{2\pi i(j-1)}{n}\right) + 1\right]^{\frac{1}{2}}$$
$$= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{3}{2} + \frac{\cos\left(\frac{2\pi}{n}\right)}{2} - \cos\left(\frac{2\pi j}{n}\right) - \cos(\frac{2\pi(j-1)}{n})\right]^{\frac{1}{2}} := \alpha.$$

To find a upper bound for $a(X_n, d)$, consider the case of $k = n$ and x_j being the midpoints of the edges. It is easy to see that we would need to compute the same distances again, thus we can conclude that

$$\alpha \geq \frac{1}{n}\sum_{j=1}^{k} d(x_j, y),$$

yielding

$$a(X_n, d) = \alpha.$$

Morris and Nickolas [22] found the following method for computing the rendezvous number:

Theorem 3.4. *Let (X, d) be a compact, connected, non-empty metric space. If there is a Borel probability measure μ_0 on X such that the integral $\int_X d(x,y)\mu_0(\mathrm{d}x)$ is independent of the choice of $y \in X$, then the rendezvous number $a(X, d)$ is equal to $\int_X d(x,y)\mu_0(\mathrm{d}x)$ for any $y \in X$.*

Proof. Fix one element e in X and let $\nu \in \mathcal{M}^1(X)$ be arbitrary. Then

$$\begin{aligned}\int_X\int_X d(x,y)\nu(\mathrm{d}y)\mu_0(\mathrm{d}x) &= \int_X\int_X d(x,y)\mu_0(\mathrm{d}x)\nu(\mathrm{d}y)\\ &= \int_X\int_X d(x,e)\mu_0(\mathrm{d}x)\nu(\mathrm{d}y)\\ &= \left(\int_X d(x,e)\mu_0(\mathrm{d}x)\right)\cdot\left(\int_X \nu(\mathrm{d}y)\right)\\ &= \int_X d(x,e)\mu_0(\mathrm{d}x).\end{aligned}$$

Since $\nu \in \mathcal{M}^1$ was arbitrary and d is symmetric, this immediately implies that for any $\mu \in \mathcal{M}^1(X)$

$$\int_X d(x,e) = \int_X\int_X d(x,y)\mu(\mathrm{d}x)\mu_0(\mathrm{d}y) \geq \inf_{\nu\in\mathcal{M}^1}\int_X\int_X d(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y).$$

Thus,

$$\sup_{\mu\in\mathcal{M}^1}\inf_{\nu\in\mathcal{M}^1}\int_X\int_X d(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \leq \int_X d(x,e)\mu_0(\mathrm{d}x).$$

However, by our previous calculation,

$$\int_X d(x,e)\mu_0(\mathrm{d}x) = \inf_{\mu \in \mathcal{M}^1} \int_X \int_X d(x,y)\mu_0(\mathrm{d}x)\nu(\mathrm{d}y)$$
$$\leq \sup_{\mu \in \mathcal{M}^1} \inf_{\nu \in \mathcal{M}^1} \int_X \int_X d(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y).$$

We conclude,

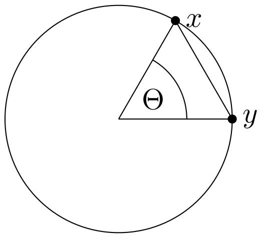
$$\int_X d(x,e)\mu_0(\mathrm{d}x) = \sup_{\mu \in \mathcal{M}^1} \inf_{\nu \in \mathcal{M}^1} \int_X \int_X d(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) = a(X,d). \qquad \square$$

Before we continue with a closely related result, we will use this statement to compute the rendezvous value of the circle as described by Morris and Nickolas [22].

Example 3.5. Let $X \subset \mathbb{R}^2$ be a 1-sphere with unite diameter and center at the origin. Let μ_0 be the normed Lebesgue-measure. Then it is clear that $\int_X d(x,y)\mu_0(\mathrm{d}x)$ does not depend on the choice of y by symmetry of the sphere. Thus, we may choose $y = \left(\frac{1}{2}, 0\right)$. We can express every point in X as $x = \left(\frac{1}{2}\cos(\Theta), \frac{1}{2}\sin(\Theta)\right)$. Furthermore,

$$\begin{aligned} d(x,y) &= \left\| \frac{1}{2}\cos(\Theta) - \frac{1}{2}, \frac{1}{2}\sin(\Theta) \right\| \\ &= \frac{1}{2}\sqrt{1 - 2\cos(\Theta) + \cos^2(\Theta) + \sin^2(\Theta)} \\ &= \frac{1}{2}\sqrt{2 - 2\cos(\Theta)} \\ &= \sin\left(\frac{x}{2}\right) \end{aligned}$$

The situation is depicted in the picture below.



With the statement of the theorem we get

$$a(X,d) = \int_X d(x,y)\mu_0(\mathrm{d}x) = \frac{1}{2\pi}\int_0^{2\pi} \sin\left(\frac{\Theta}{2}\right) \mathrm{d}\Theta = \frac{2}{\pi}.$$

This result can also be obtained by approximating the circle by regular n-gons with $n \to \infty$, see for example [7] by Cleary, Morris and Yost. In Appendix B this is exemplified with a short Python3 script, which calculates the rendezvous value for n-gons of unit diameter as well as the error.

The following lemma extend the result of the previous theorem and is also due to Morris and Nickolas [22].

Lemma 3.6. *Let (X, d) be a compact, connected, non-empty metric space and let $\mu_0 \in \mathcal{M}^1(X)$. If for each pair of points y and e in X there exists an isometry $T : X \to X$ such that $T(y) = e$ and $\mu_0(A) = \mu_0(T(A))$ for all Borel sets A, then $\int_X d(x, y)\mu_0(\mathrm{d}x)$ is independent of y.*

Proof. Let $y, e \in X$ and let T be an isometry as described above. Then

$$\begin{aligned}\int_X d(x,y)\mu_0(dx) &= \int_X d(T(x),T(y))\mu_0(\mathrm{d}x)\\ &= \int_X d(T(x),e)\mu_0(\mathrm{d}x)\\ &= \int_X d(x,e)\mu_0(\mathrm{d}T^{-1}x)\\ &= \int_X d(x,e)\mu_0(\mathrm{d}x).\end{aligned}$$

Since there exists such T for all pairs y, e the expression above does not depend on y. □

Corollary 3.7. *Let S^n be the n-dimensional sphere in $\mathbb{R}^{n+1}$, and let d be the inherited metric on S^n. Let μ_0 be the normalized Lebesgue measure on S^n. Then*

$$a(S^n, d) = \int_{S^n} d(x, y)\mu_0(\mathrm{d}x)$$

for any $y \in S^n$.

According to Morris and Nickolas, this integral can be expressed as follows: Let S^n be the n-sphere with radius $\frac{1}{2}$ then

$$a(S^n, d) = \frac{2^n \left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}{\sqrt{\pi}\Gamma\left(\frac{2n+1}{2}\right)}.$$

The detailed computation was displayed by Chad [4].

4. Further Results

In this chapter we will give an short overview of connected work that has been done over the years.

Many people have been interested in the topic of Average distance properties in Banach spaces (see for example Wolf [35, 34, 32, 33], Lin [21], Baronti, Casini and Papini [1], Hinrichs [16, 17, 18], García-Vázquez [13, 12], Farkas and Révész [9]). However, in this section, we will only touch on some basic results.

Definition 4.1. We say a Banach space E has the average distance property if the statement of the Gross-Stadje-Theorem (Theorem 2.1) holds for the unit sphere in E.

Note that the spheres in Banach spaces are compact with respect to the norm of the Banach space only if the dimension of the space is finite. Thus, the statements of Gross [14] and Stadje [29] are not applicable for the infinite dimensional case [vgl. 35].

Notation 4.2. For $n \in \mathbb{N}$, $1 \leq p \leq \infty$

1. let $\ell^p(n)$ denote $\mathbb{R}^n$ with the usual p-norm,
2. let ℓ^p denote the sequence space equipped with the p-norm,
3. let $c_0 \subset \ell^p$ denote the subspace of all sequences which are tending to zero,
4. $(\cdot \mid \cdot)$ denotes the usual inner product in $\mathbb{R}^n$ and ℓ^2,
5. e_i $(i \geq 1)$ denotes the canonical unit vector in all Banach spaces mentioned above.

We further write $a(E) := a(E, d)$, where d is the metric induced by the norm of the Banach space.

Before we give an examples, we need to consider the following lemma from Wolf [35].

Lemma 4.3. *Let* $x = (\alpha_1, \ldots, \alpha_n) \in \ell^1(n)$, $\max_{1 \leq i \leq n} |\alpha_i| \leq 1$. *Then*

$$\frac{1}{2n} \sum_{i=1}^{n} \|x - e_i\| + \|x + e_i\| = 1 + \frac{n-1}{n} \|x\|.$$

Proof. This statement follows form the fact that for all $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$ we have

$$|\alpha - 1| + |\alpha + 1| = 2.$$

Thus,

$$\begin{aligned}
&\frac{1}{2n}\sum_{i=1}^{n}\|x-e_i\|+\|x+e_i\| = \\
=&\frac{1}{2n}(\underbrace{|x_1-1|+|x_1+1|}_{2}+\underbrace{2|x_2|+2|x_3|+\cdots+2|x_n|}_{2\|x\|-2|x_1|} \\
&\quad+\underbrace{|x_2-1|+|x_2+1|}_{2}+\underbrace{2|x_1|+2|x_3|+\cdots+2|x_n|}_{2\|x\|-2|x_2|} \\
&\qquad\qquad\vdots \\
&\quad+\underbrace{|x_n-1|+|x_n+1|}_{2}+\underbrace{2|x_1|+2|x_2|+\cdots+2|x_{n-1}|}_{2\|x\|-2|x_n|}) \\
=&\frac{2n}{2n}+\frac{1}{n}(n\|x\|-|x_1|-|x_2|-\cdots-|x_n|) \\
=&1+\frac{n-1}{n}\|x\|
\end{aligned}$$

□

We begin with a finite dimensional problem.

Lemma 4.4. *For all $n \geq 2$, $a(\ell^{(n)}1) = 2 - \frac{1}{n}$ and $a(\ell^{\infty}(n)) = \frac{3}{2}$, where d is induced by the respective norms.*

Proof. In the case of $\ell^1(n)$ let $x \in \ell^1(n)$ with $\|x\| = 1$. Then by above lemma we find

$$\frac{1}{2n}\sum_{i=1}^{n}\|x-e_i\|+\|x+e_i\| = 2-\frac{1}{n}.$$

Thus, by our statements for compact, connected, non-empty Hausdorff spaces, we find $a(\ell^1(n)) = 2 - \frac{1}{n}$.

In the case of $\ell^{\infty}(n)$ let $x = (\alpha_1, \ldots, \alpha_n)$ with $\|x\| = 1$. Then

$$\|x-e_1\|+\|x+e_1\| \leq \max(|\alpha_1-1|,1)+\max(|\alpha_1+1,1) \leq 3.$$

Thus,

$$\frac{1}{2}(\|x-e_1\|+\|x+e_1\|) \leq \frac{3}{2}.$$

Now let $b_1 = (1,1,\ldots,1)$ and $b_2 = (-1,1,\ldots,1)$ and assume without loss of generality that $\alpha_1 = 1$ or $\alpha_2 = 1$. If the former is the case then

$$\|x-b_1\|+\|x+b_1\|+\|x-b_2\|+\|x+b_2\| \geq |\alpha_2-1|+2+2+|\alpha_2+1| = 6,$$

and if the latter is the case then

$$\|x-b_1\|+\|x+b_1\|+\|x-b_2\|+\|x+b_2\| \geq |\alpha_1-1|+2+|\alpha_1+1|+2 = 6$$

and therefore,

$$\frac{1}{4}(\|x-b_1\|+\|x+b_1\|+\|x-b_2\|+\|x+b_2\|) \geq \frac{3}{2}.$$

Hence, $a(\ell^\infty(n), d) = \frac{3}{2}$. □

We will now advance our theory to the infinite dimensional case again following the discussion by Wolf [35]. Recall that we can't apply our Gross-Stadje-Theorem (Theorem 2.1), since the unit sphere is not compact.

Theorem 4.5. *The Hilbert space ℓ^2 has the average distance property with rendezvous value $a(\ell^2) = \sqrt{2}$.*

Proof. Let S denote the unit sphere in ℓ^2. Now let $x_1, \dots, x_n \in S$ and choose $x \in S$ such that x is in the orthogonal complement of the space spanned by $x_1, \dots, x_n$. Then

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i - x\| = \sqrt{2},$$

since

$$\|x_i - x\| = \sqrt{(x_i - x \mid x_i - x)} = \sqrt{(x \mid x) - (x_i \mid x) - (x \mid x_i) + (x, x)} = \sqrt{2}$$

In order to show that this is in fact the rendezvous value, we need to show uniqueness. Therefore, let α be a positive real number with the desired property. Choose $x_1 \in S$. Then similar to lemma 4.3

$$\|x_1 - x\| + \|x_1 + x\| \leq \sqrt{2}\sqrt{\|x_1 - x\|^2 + \|x_1 + x\|^2} = 2\sqrt{2}.$$

for all $x \in S$ we have $\alpha \leq \sqrt{2}$.

To verity the other inequality, let $x \in S$. Then

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\|x - e_i\| &= \frac{\sqrt{2}}{n}\sum_{i=1}^{n}\sqrt{1-(x \mid e_i)} \geq \frac{\sqrt{2}}{n}\sum_{i=1}^{n}\sqrt{1-|(x \mid e_i)|} \\ &\geq \frac{\sqrt{2}}{n}\sum_{i=1}^{n}(1-|(x \mid e_i)|) \geq \sqrt{2} - \frac{\sqrt{2}}{\sqrt{n}}\end{aligned}$$

for all $n \in \mathbb{N}$ and thus $\alpha \geq \sqrt{2}$. □

We will now see that not all Banach spaces have the average distance property as described by Wolf [35].

Theorem 4.6. *The Banach spaces c_0 and ℓ^1 do not have the average distance property.*

Proof. We begin with the space c_0. We will prove the statement by showing that the rendezvous value is not unique. For $k \geq 1$ let P_k denote the canonical projection onto the subspace spanned by $e_1, \dots, e_k$.

Let $x_1, \dots, x_n \in S$ and choose $k_0 \geq 2$ such that $\|P_{k_0} x_i\| = 1$ for $i \in \{1, \dots, n\}$. We can now apply our result for $\ell^\infty(k_0)$ from lemma 4.4 to see that there exists $x \in S$ such that $(E - P_{k_0})x = 0$ and $\frac{1}{n}\sum_{i=1}^n \|P_{k_0} x_i - x\| = \frac{3}{2}$ and therefore

$$\frac{1}{n}\sum_{i=1}^n \|x_i - x\| \geq \frac{1}{n}\sum_{i=1}^n \|P_{k_0}(x_i - x)\| = \frac{1}{n}\sum_{i=1}^n \|P_{k_0} x_i - x\| = \frac{3}{2}.$$

Now, let $\varepsilon > 0$ and choose $k_1 \in \mathbb{N}$ such that

$$|(x_i \mid e_{k_1})| \leq \varepsilon \quad \text{for} \quad i = 1, 2, \dots, n.$$

Then

$$\frac{1}{n}\sum_{i=1}^n \|x_i - e_{k_1}\| \leq \frac{1}{n}\sum_{i=1}^n \max(|(x_i \mid e_{k_1}) - 1|, 1) < 1 + \varepsilon.$$

Thus, by Intermediate Value Theorem, we see that every real number in the interval $(1, \frac{3}{2}]$ satisfies the property of the rendezvous value, which contradicts the uniqueness.

Now consider the case of the space ℓ^1. Let $x = (\alpha_1, \alpha_2, \dots) \in S$. with lemma 4.3 we have

$$\frac{1}{2n}\sum_{i=1}^n \|x - e_i\| + \|x + e_i\| = 2 - \frac{1}{n}\sum_{i=1}^n |\alpha_i| \geq 2 - \frac{1}{n} \quad \forall n \in \mathbb{N}$$

Thus, if there was a constant with the property of the rendezvous constant, it would have to be equal to 2. However, that is not possible. To see this, consider $x_1 = (\beta_1, \beta_2, \dots) \in S$ with $\beta_i > 0$ for $i \in \mathbb{N}$. Under our assumption, there would have to be $x = (\alpha_1, \alpha_2, \dots) \in S$ with $\frac{1}{2}(\|x - x_1\| + \|x + x_1\|) = 2$. Then $\|x - x_1\| = \|x + x_1\| = \|x\| + \|x_1\|$ and therefore $|\alpha_i - \beta_i| = |\alpha_i + \beta_i| = |\alpha_i| + |\beta_i|$ for $i \in \mathbb{N}$. Since $\beta_i \geq 0$ for all $i \in \mathbb{N}$ we have $\alpha_i = 0$ for all $i \in \mathbb{N}$, which is a contradiction. □

Note that we just showed that a Banach space can have no, multiple or one unique real number satisfying the property of a rendezvous value.

Theorem 4.7. *[35] The Banach space ℓ^∞ has the average distance property with rendezvous number $a(\ell^\infty) = \frac{3}{2}$.*

Proof. Let $x_1, \dots, x_n \in S$ and P_k as before. Without loss of generality we can assume that the elements are labeled in a way such that there is a s such that elements in the subset $\{x_1, \dots, x_s\}$ satisfy $\|P_{k_0}\| = 1$ for all $1 \leq i \leq s$ and some $k_0 \geq 2$.

Similar to the previous proof we may now apply our result from Lemma 4.4 to find the existence of a $y \in S$ with $(E - P_{k_0})y = 0$ and $\frac{1}{s}\sum_{i=1}^s \|P_{k_0} x_i - y\| = \frac{3}{2}$. For the other elements, choose $k_0 < a_{s+1} < a_{s+2} < \dots < a_n$ with $|(x_i \mid e_{a_i})| \geq \frac{1}{2}$. Now, let

$x = y + \sum_{i=s+1}^{n} -\mathrm{sgn}(x_i \mid e_{a_i})e_{a_i}$. By construction, $x \in S$ and $P_{k_0}x = y$. Thus,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\|x_i - x\| \geq& \frac{1}{n}\left(\sum_{i=1}^{s}\|P_{k_0}x_i - y\| + \sum_{i=s+1}^{n}\|x_i - x\|\right)\\ \geq& \frac{1}{n}\left(\frac{3}{2}s + \sum_{i=s+1}^{n}|(x_i - x \mid e_{a_i})|\right)\\ \geq& \frac{s}{n}\cdot\frac{3}{2} + \frac{n-s}{n}\cdot\frac{3}{2} = \frac{3}{2}.\end{aligned}$$

To verify the other inequality, we only need to see that

$$\min\left(\frac{1}{n}\sum_{i=1}^{n}\|d_i - e_1\|, \frac{1}{n}\sum_{i=1}^{n}\|x_i + e_1\|\right) \leq \frac{3}{2}.$$

Intermediate Value Theorem guarantees once more that $\frac{3}{2}$ actually satisfies the desired properties. The proof of uniqueness is analogous to that of lemma 4.4 with $b_1 = \sum_{i=1}^{\infty} e_i, b_2 = -e_1 + \sum_{i=2}^{\infty} e_i$. □

This result showcases that it is not necessary for a Banach space to be separable in order to have the average distance property.

Lin [21] found that $L^p(0,1)$ and ℓ^p do not have the average distance property if $1 \leq p < 2$. Later Hinrichs first showed in [17]that $L^p(0,1)$ and ℓ^p do not have the average distance property for $3 \leq p < \infty$ and later showed in joint work with Wenzel [18] that the same holds for $2 < p < \infty$, thereby completing the study of average distance properties of L^p- and ℓ^p-spaces.

Lin [21] further found that if K is a normed space, then $C(K)$ has the average distance property if and only if K contains at least one isolated point, in which case the rendezvous constant is equal to $\frac{3}{2}$.

García-Vázquez and Villa [13] were able to compute the rendezvous values of $\ell^\infty(n)(\mathbb{C})$ and $\ell^\infty(\mathbb{C})$ to be $\frac{1}{3} + \frac{2\sqrt{3}}{\pi}$ and furthermore expressed the rendezvous value of $\ell^1(n)$ in terms of the complete elliptic integral function.

Wolf [34] found that for real N-dimensional ($N \geq 2$) Banach spaces X with 1-unconditional basis, the inequality $a(X) \leq 2 - \frac{1}{N}$ holds. In [33] he further showed that the same bound holds for real quasihypermetric Banach spaces of finite dimension. Later Hinrichs [16] was able to show that the restriction to qusihypermetric Banach spaces is not necessary, using methods involving the John's Ellipsoid. García-Vázquez and Villa [12] showed that this can be generalized to the complex case if the dimension is greater than 3.

Kokkendorf [19] related the study of rendezvous values to the concept of curvature. And Kulshestha, Sag and Yang [20] studied polygons in M-spaces with non-positive curvature. (A M-space is a metric space in which there exists for each pair of distinct points $x, y \in X$ and all $\alpha \in (0,1)$ a point $z(\alpha) \in X$ such that $d(x, z(\alpha)) = \alpha d(x,y)$ and $d(y, \alpha(z)) = (1-\alpha)d(x,y)$.)

A concept which is related to the rendezvous value of a topological space is that of average distances, more precisely studying the constants

$$
\begin{aligned}
M(X,d) &:= \sup \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_i, x_j) = \sup_{\mu \in \mathcal{M}^1} \int_X \int_X d(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\
\overline{M}(X,d) &:= \sup \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j d(x_i, x_j),
\end{aligned}
$$

where for the former the supremum is taken over all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ and in the latter case over all $n \in \mathbb{N}$, $x_1, \ldots, x_n$ and $w_1, \ldots, w_n$ with $\sum_{i=1}^{n} w_i = 1$. These values have been of particular interest for spaces with the quasihypermetric condition. The interested reader is referred to the work of Nickolas and Wolf [24, 25, 26] as well as the thesis by Chad [4].

A. Technical Results

In this chapter, the reader can find the more technical results, which were necessary to prove the desired theorems but of no further interest at the time. We start with preliminary results that was used in the proof of the Banach-Alaoglu-Theorem.

Definition A.1. Let (X, τ) be a topological space. We say a collection $\mathscr{S}$ of open subsets is called a *subbase* of the topology τ if the collection of all finite intersections of members of $\mathscr{S}$ forms a base for τ. A *$\mathscr{S}$-cover* of X is a cover of X whose elements are all contained in $\mathscr{S}$.

Compactness is usually defined as the property that every cover has a finite subcollection that is still a cover. The following statement [see 28] shows that it is sufficient to verify this for $\mathscr{S}$-covers.

Theorem A.2 (Alexander's Subbase Theorem)**.** *If $\mathscr{S}$ is a subbase for the topology of a space X and if every $\mathscr{S}$-cover of X has a finite subcover, then X is compact.*

Proof. Assume for a contradiction that X is not compact. We will show that X has a $\mathscr{S}$-cover $\tilde{\Gamma}$ that does not have a finite subcover.

We will denote the collection of all subcovers of X that do not have a finite subcover with $\mathfrak{P}$. Since X is assumed not to be compact, $\mathfrak{P} \neq \emptyset$. The inclusion relation induces a partial order on $\mathfrak{P}$. Let Ω denote the maximal[1] totally ordered subcollection of $\mathfrak{P}$ and Γ shall be defined to be the union of all members of Ω.

Assertion 1. Γ is an open cover of X.

Assertion 2. Γ has no finite subcover, but

Assertion 3. $\Gamma \cup \{V\}$ has a finite subcover for every open $V \notin \Gamma$.

Assertion 1 is obvious by construction of Γ. We defined Ω to be totally ordered by inclusion. This implies that any subcover of Γ is already contained in some element of Ω. Therefore Γ cannot have a finite subcover, proving Assertion 2. Assertion 3 is a direct consequence of the maximality of Ω.

Now, consider $\tilde{\Gamma} := \Gamma \cap \mathscr{S}$. Assertion 2 provides that $\tilde{\Gamma}$ cannot have a finite subcover. Suppose $\tilde{\Gamma}$ is not a cover of X, i.e. there exists a $x \in X \setminus \tilde{\Gamma}$. Assertion 1 tells us that there is some $W \in \Gamma$ such that $x \in W$. By the subbase property of $\mathscr{S}$, we know that there are sets $V_1, \ldots, V_n \in \mathscr{S}$ such that $x \in \bigcap_{i=1}^{n} V_i \subset W$. Since $x \in V_i$ for all $i \in \{1, \ldots, n\}$, we conclude that $V_i \notin \Gamma$ for all $i \in \{1, \ldots, n\}$.

[1] To see that such exists we need to apply Zorn's Lemma.

Applying Assertion 3 we find that there are sets $Y_1, \dots, Y_n$, each finite union of members of Γ such that $X = V_i \cup Y_i$ for all $1 \leq i \leq n$, implying

$$X = Y_1 \cup \cdots \cup Y_n \cup \bigcap_{i=1}^{n} V_i \subset Y_1 \cup \cdots \cup Y_n \cup W.$$

This is a contradiction to Assertion 2. □

It is well known that the caresian product of two compact spaces is compact with respect to the usual product topology. The following theorem [see 28] states that this is the case for products of arbitrarily many compact spaces.

Theorem A.3 (Tychonoff's Theorem)**.** *Let X_α be a collection of arbitrary many nonempty compact spaces and let X be the cartesian product of all X_α. Then X is compact.*

Proof. Let $\pi_\alpha : X \to X_\alpha$ be the projection onto the X_α-coordinate. The topology on X is defined as the initial topology with respect to $(\pi_\alpha)_\alpha$, i.e. the coarsest under which all π_α are continuous.

Consider the set $W_\alpha := \{V_\alpha \colon V_\alpha \subset X_\alpha \text{ open}\}$ of open subsets. We define $\mathscr{S}_\alpha$ to be the collection of all sets $\pi_\alpha^{-1}(V_\alpha)$, where $V_\alpha \in W_\alpha$. Now, let $\mathscr{S}$ denote the union of all the $\mathscr{S}_\alpha$, then $\mathscr{S}$ is a subbase of the topology.

Let Γ be any given $\mathscr{S}$-cover of X. Define $\Gamma_\alpha = \Gamma \cap \mathscr{S}_\alpha$. Assume for a contradiction that Γ_α does not cover X for any choice of α. Then, for each α there is a point x_α in X_α such that Γ_α covers no point of the set $\pi_\alpha^{-1}(x_\alpha)$, and if $x \in X$ is chosen so that $\pi_\alpha(x) = x_\alpha$, then x is not covered by Γ. However, Γ is a cover of X, thus at least one of the Γ_α has to be a cover of X. Since X_α is compact, there is a finite subcollection of Γ_α, that covers X. Since $\Gamma_\alpha \subset \Gamma$, we get that Γ has a finite subcover. Applying Alexander's Subbase Theorem (Theorem A.2) yields, that X is compact. □

The following two statements are fundamental theorems of functional analysis. A detailed proof can be found in Werner [31].

Theorem A.4 (Hahn-Banach Theorem)**.** *Let X be a complex vector space and let U be a sub vector space of X. Furthermore, let $p : X \to \mathbb{R}$ be sublinear and $\ell : U \to \mathbb{C}$ linear with*

$$\mathrm{Re}\ell(x) \leq p(x) \quad \forall x \in U.$$

Then there exists a linear extension $L : X \to \mathbb{C}, L\mid_U = \ell$ with

$$\mathrm{Re}L(x) \leq p(x) \quad \forall x \in X.$$

Corollary A.5 (Banach separation theorem)**.** *Let X be a normed space, $V_1, V_2 \subset X$ convex and V_1 open. Let $V_1 \cap V_2 = \emptyset$. Then there exists $x' \in X'$ with*

$$\mathrm{Re}x'(v_1) < \mathrm{Re}x'(v_2) \quad \forall v_1 \in V_1, v_2 \in V_2.$$

B. Sourcecode

The following program computes the rendezvous value of n-gons with unit diameter. It was created using Python 3.7. Computations with increasing n indicate that the rendezvous value of n-gons tends to that of a circle, as expected.

```
import math as m
n=3
a_prev=0
while n<500:
    n=n+1
    k=0
    a=0
    while k<n:
        k+=1
        if n%2==0:
            a+=1/(2*n)*m.sqrt(1.5+0.5*m.cos(2*m.pi/n)-m.cos(2*k*m.pi/n) \
                -m.cos(2*(k-1)*m.pi/n))
        else:
            a+=1/(n)*m.sqrt((1.5+0.5*m.cos(2*m.pi/n)-m.cos(2*k*m.pi/n) \
                -m.cos(2*(k-1)*m.pi/n))/(2-2*m.cos((n-1)*m.pi/n)))
    adjustment=a-a_prev
    a_prev=a
    error=(2/m.pi)-a
    print("Result (n,a):", n,a, "\tadjustment:",\
        adjustment,"\terror:",error)
```

The computations show for example that if we approximate the circle with radius $\frac{1}{2}$, then the error for the rendezvous value is approximately $5.23576764943634e-06$.

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