Simplicial Homology

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1 Motivation

The main idea of simplicial homology can be summarized as follows:

We want to find a new homotopy invariant as a tool to distinguish topological spaces. With simplicial homology, we assign a succession of abelian Groups to simplicial complexes.

2 Simplicial Homology Groups

Definition 2.1. Let K be a simplicial complex. A p-chain on K is a function c from the set of oriented p-simplices of K to the an abelian Group G such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group (which is free abelian) is denoted $C_p(K; G)$ and is called the group of (oriented) p-chains of K with coefficients in G.

Definition 2.2. Let K be a simplicial complex and G an abelian group. We define the boundary operator

$$\partial_p : C_p(K; G) \to C_{p-1}(K; G)$$

to be the linear map uniquely defined by its action on an elementary chain corresponding to an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the vertex set.

Lemma 2.3. *Let K be a simplicial complex and G an abelian group then $(C_p(K; G), \partial_p)_{p \in \mathbb{Z}}$ is a chain complex, i.e. $\partial^2 = 0$*

Definition 2.4. We define

$$\boxed{H_p(K; G) := \ker\partial_p / \mathrm{Im}\partial_{p+1}}$$

to be the **p-th simplicial homology group of K**

Examples 2.5. For the 1-sphere, triangulate the space, compute the boundary of all elementary chains and evaluate the resulting matrices to find:

$$H_0(\mathbb{S}^1) \cong \mathbb{Z} \quad H_1(\mathbb{S}^1) = \mathbb{Z}, \quad H_{n>1}(\mathbb{S}^1) = 0$$

For the Torus T and the Klein Bottle S it is more convenient to reduce the problem to chains that are carried by the boundary of the rectangle associated with the fundamental polygon. This yields

$$\begin{array}{cccc}
H_0(S) \cong \mathbb{Z} & H_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}_2 & H_{n>1}(S) = 0 & \\
H_0(S; \mathbb{Z}_2) \cong \mathbb{Z}_2 & H_1(S; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & H_2(S; \mathbb{Z}_2) = \mathbb{Z}_2 & H_{n>2}(S; \mathbb{Z}_2) = 0 \\
H_0(T) \cong \mathbb{Z} & H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} & H_2(T) = \mathbb{Z} & H_{n>2}(T) = 0 \\
H_0(T; \mathbb{Z}_2) \cong \mathbb{Z}_2 & H_1(T; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & H_2(T; \mathbb{Z}_2) = \mathbb{Z}_2 & H_{n>2}(T; \mathbb{Z}_2) = \mathbb{Z}_2
\end{array}$$

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4 Functoriality Property

Definition 4.1. Let $f : K \to L$ be a simplicial map. If $[v_0 \dots v_p]$ is a simplex of K, then the points $f(v_0) \dots f(v_p)$ span a simplex of L. We construct a homomorphism $f_\sharp : C_p(K) \to C_p(L)$ by defining it on oriented simplices as follows:

$$f_\sharp([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_0), \dots, f(v_p) \text{ are distinct} \\ 0, & \text{otherwise.} \end{cases}$$

The family $(f_\sharp)_{p \in \mathbb{Z}}$ is called the *chain map induced by the simplicial map f*.

Definition 4.2. Let K and L be simplicial complexes; let $h : |K| \to |L|$ be a continuous map. Choose a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$. Let $\lambda : \mathcal{C}(K) \to \mathcal{C}(K')$ be the subdivision operator.

We define the *homomorphism induced by h*

$$h_* : H_p(K) \to H_p(L) \quad \text{via} \quad h_* = f_* \circ \lambda_*.$$

Theorem 4.3 (functioriality II)**.** *The identity map $id : |K| \to |K|$ induces the identity homomorphism $i_* : H_p(K) \to H_p(K)$. If $h : |K| \to |L|$ and $k : |L| \to |M|$ are continuous maps, then $(k \circ h)_* = k_* \circ h_*$.*

5 Homotopy Invariance of Simplical Homology

Theorem 5.1. *Let $f : |K| \to |L|$ be a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism.*

Theorem 5.2. *Let K, L be simplicial complexes. If $f, g : |K| \to |L|$ are homotopic maps then $f_* = g_* : H_q(K) \to H_q(L)$ for all q*

Theorem 5.3. *If two polyhedra K and L are homotopy equivalent, then their homology groups are isomorphic $H_n(X) \cong H_n(Y)$ in all dimensions.*

Definition 5.4. Given two simplicial maps $f, g : K \to L$, these maps are said to be *contiguous* if for each simplex $v_0 \dots v_p$ of K, the points

$$f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$$

span a simplex of L.

Theorem 5.5. *If $f, g : K \to L$ are contiguous simplicial maps, then they induce the same chain maps $f_\sharp$ and $g_\sharp$.*

Theorem 5.6. *If $f, g : |K| \to |L|$ are homotopic maps we can find a barycentric subdivision K^m and a sequence of simplicial maps $s_1, \dots, s_n : |K^m| \to |L|$ such that s_1 simplicially approximates f while s_n simplicially approximates g, and each pair s_i, s_{i+1} is contiguous.*

Exercise 1. Compute the simplicial Homology groups of $\mathbb{R}\mathrm{P}^2$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$.

Exercise 2. Compute the simplicial homology groups of $\mathbb{S}^2$ and $\mathbb{S}^3$. Formulate a conjecture for $\mathbb{S}^n$.

Literatur

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