

Simplicial Homology

1 Motivation

The main idea of simplicial homology can be summarized as follows:

We want to find a new homotopy invariant as a tool to distinguish topological spaces. With simplicial homology, we assign a succession of abelian Groups to simplicial complexes.

2 Simplicial Homology Groups

Definition 2.1. Let K be a simplicial complex. A p -chain on K is a function c from the set of oriented p -simplices of K to the an abelian Group G such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p -simplices σ .

We add p -chains by adding their values; the resulting group (which is free abelian) is denoted $C_p(K; G)$ and is called the group of (oriented) p -chains of K with coefficients in G .

Definition 2.2. We define the boundary operator

$$\partial_p : C_p(K; G) \rightarrow C_{p-1}(K; G)$$

to be the homomorphism via its action on an elementary chain corresponding to an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol \hat{v}_i means that the vertex v_i is to be deleted from the array.

Lemma 2.3. (C_\bullet, ∂) is a chain complex, i.e. $\partial^2 = 0$

Definition 2.4. We define

$$H_p(K; G) := \ker \partial_p / \operatorname{Im} \partial_{p+1}$$

to be the p -th simplicial homology group of K

Examples 2.5. For the 2-sphere, triangulate the space, compute the boundary of all elementary chains and evaluate the resulting matrices to find:

$$H_0(\mathbb{S}^2) \cong \mathbb{Z} \quad H_1(\mathbb{S}^2) = 0, \quad H_2(\mathbb{S}^2) \cong \mathbb{Z}, \quad H_{n>2}(\mathbb{S}^2) = 0$$

For the Torus T and the Klein Bottle S it is more convenient to reduce the problem to chains that are carried by the boundary of the rectangle associated with the fundamental polygon. This yields

$$\begin{array}{lll}
H_0(S) \cong \mathbb{Z} & H_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}_2 & H_{n>1}(S) = 0 \\
H_0(S; \mathbb{Z}_2) \cong \mathbb{Z}_2 & H_1(S; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & H_{n>1}(S; \mathbb{Z}_2) = 0 \\
H_0(T) \cong \mathbb{Z} & H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} & H_{n>1}(T) = \mathbb{Z} \\
H_0(T; \mathbb{Z}_2) \cong \mathbb{Z}_2 & H_1(T; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & H_{n>1}(T; \mathbb{Z}_2) = 0
\end{array}$$

3 Functoriality Property

Theorem 3.1 (functorility I). ¹ *Simplicial Homology is a functor from the Category of simplicial complexes to abelian groups.*

Theorem 3.2 (functorility II). *Simplicial Homology is a functor from the Category of polyhedra to abelian groups.*

4 Homotopy Invariance of Simplicial Homology

Theorem 4.1. *If two polyhedra K and L are homotopy equivalent, then their homology groups are isomorphic $H_n(X) \cong H_n(Y)$ in all dimensions.*

Exercise 4.2. Compute the simplicial Homology groups of \mathbb{RP}^2 with coefficients in \mathbb{Z} and \mathbb{Z}_2 .

Exercise 4.3. Compute the simplicial homology groups of \mathbb{S}^2 and \mathbb{S}^3 . Formulate a conjecture for \mathbb{S}^n .

Literatur

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¹The statement might not be verbatim from the lecture, but the statement is essentially the same.