Simplicial Homology[draft]

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Abstract goes here

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1 Motivation

One of the main goals when studying (topological) spaces (or in this case simplicial complexes) is to determine whether two spaces are homotopic or not. The first method that is often applied is to check if both spaces are simply connected. A more advanced approach checks not only simply connectedness but also if the fundamental groups π_1 coincide[1].

It turns out that this is a valuable tool for one and two dimensional simplicial complexes. But this method fails for complexes with cells in higher dimensions. It can be shown that the fundamental group actually only depends on the 2-skeleton of the complex [vgl. 1, p. 173].

Even though this method can be generalized to the study of higher homotopy groups π_n, this is often more difficult than necessary since the computation of those groups is far from trivial.

In the following we will explore the concept of simplicial homology groups and how it can be used to classify spaces. We will see that they allow us to tackle the problem of deciding whether two **polyhedra**[2] are homeomorphic, since simplicial homology is only defined on such spaces. However, the concept of simplicial homology can be generalized to singular homology, which can be applied to a broader category of spaces and coincides with simplicial homology if both are defined.

However, singular homology groups will not be covered here. The interested reader might refer to the book of Munkres [3] or Hatcher [2] to learn more about singular homology groups.

When we concern ourselfes with the question how closed paths on the torus are different from those on the sphere, we can notice the following:

Any Jordan curve (i.e. nonselfintersecting, closed path) on a sphere divides the surface into two „regions“ as depicted in Figure 1[3]. The same is not true for the torus. In Figure 2 we see that even tough there are Jordan curves on a torus that bound a „region“of the surface, not all Jordan curves have that property [p. 173f 1, see].

2 Simplicial Homology Groups with Integer Coefficients

We will use the following definition by Munkres [3, p. 26]

Definition 2.1. For a simplex σ we say that two orderings of its vertex set are equivalent if they differ by an even permutation. For a simplex with nonzero dimension, the

[1] If X is a simply connected space then $\pi_1(X) = 0$. Furthermore, the fundamental group is homotopy invariant (up to isomorphism), so homotopic spaces have isomorphic fundamental groups. A more precise formulation of this statement can be found in Hatchers book [2, Prop. 1.18, p. 37]:

If $\phi : X \to Y$ is a homotopy equivalence, then the induces homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.

[2] A polyhedron is the polytope of a simplicial complex.

[3] For a more precise statement see `https://en.wikipedia.org/wiki/Jordan_curve_theorem`

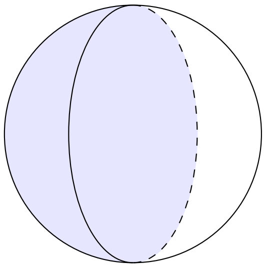


Figure 1: Any closed path on the sphere divides the surface into two regions.

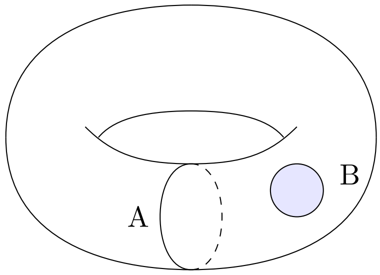
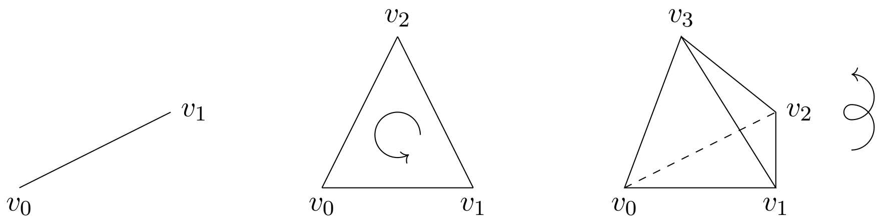


Figure 2: There are closed paths on the torus that do not divide the surface into two regions.



(v_0, v_1) (v_0, v_1, v_2) (v_0, v_1, v_2, v_3)

Figure 3: Indicating orientation with arrows [see 3, p.27]

orderings of the vertices fall into two equivalence classes, called the **orientations** of σ. An **oriented simplex** is a simplex together with an orientation.

Further we will use the same notation as Munkres [3, p. 26]

Notation 2.2. For geometrically independent points $v_0, \ldots, v_p$ we denote the simplex they span with

$$[v_0, \ldots, v_p].$$

For an oriented simplex we will use the notation

$$(v_0, \ldots, v_p).$$

where the orientation is given by this particular ordering.

A choice of orientation is usually depicted with arrows as can be seen in Figure 3

To define simplicial homology groups in any meaningful way, we need a few definitions that allow us to formulate our ideas form section 1 more precise. There are several ways to go about this with various amounts of rigor. In this paper, we will follow the path of Munkres [3, p. 27f].

Definition 2.3. Let K be a simlicial complex. A ***p*-chain** on K is a function c from the set of oriented p-simplices of K to the integers such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.

2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group is denoted $C_p(K)$ and is called the group of (oriented) p-chains of K. If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group.

If σ is an oriented simplex, the elementary chain c_σ corresponding to σ is the function defined as follows:

$$\begin{aligned} c_\sigma(\sigma) &= 1, & \\ c_\sigma(\sigma') &= -1 & &\text{if } \sigma' \text{ is the opposite orientation of } \sigma \\ c_\sigma(\tau) &= 0 & &\text{for all other oriented simplices } \tau. \end{aligned}$$

It is common practice to to abuse the notation here. If clear by context we use the symbol σ not only to denote the (oriented) simplex, but also the corresponding elementary chain.

We are now able to formulate the our first result as can be found in Munkres book [3, Lemma 5.1, p. 28]:

Lemma 2.4. *$C_p(K)$ is free abelian; a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains as a basis.*

Proof. It is easy to see that each chain c in $C_p(K)$ can be expressed uniquely as a linear combination of the elementary chains c_{σ_i} of the simplices of K, i.e.

$$c = \sum_i n_i c_{\sigma_i}.$$

Here, the the chain c assigns the value n_i to each simplex σ_i, $-n_i$ for the simplex σ_i with reversed orientation and 0 to every simplex that does not appear in the summation.[4] □

Definition 2.5. We define the boundary operator

$$\partial_p : \quad C_p(K) \to C_{p-1}(K)$$

to be the homomorphism via its action on an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array.

[4] Recall that by definition of the elementary chains we have

$$c_{\sigma_i}(\tau) = \begin{cases} 1 & \text{if } \sigma_i = \tau \\ -1 & \text{if } \tau \text{ is the opposite orientation of } \sigma_i \\ 0 & \text{else} \end{cases}$$

As the name suggests, the result of this homomorphism indeed refers to the (topological) boundary of the simplex once we interpret the sum of two elementary chains as the union of the corresponding simplices as we will see in the several examples, but first we have to check well-definedness. For that, it is necessary to show that

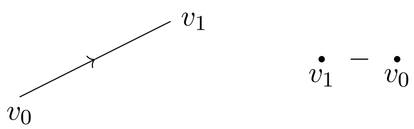
$$\partial_p(-\sigma) = -\partial_p(\sigma)$$

Since the orientation fo a simplex only depends on the *sign* of the permutation, it is sufficient to check for the simple case of exchanging v_0 and v_1:

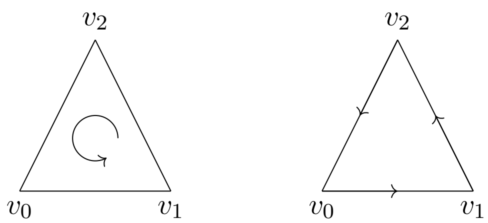
$$\begin{aligned}
&\partial_p(v_0, \dots, v_p) + \partial_p(v_1, v_0, v_2, \dots, v_p) \\
=&(v_1, \dots, v_p) - (v_0, v_2, \dots, v_p) + \sum_{i=2}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) \\
+&(v_0, v_2, \dots, v_p) - (v_1, v_2, \dots, v_p) + \sum_{i=2}^{p} (-1)^i \underbrace{(v_1, v_0, v_2, \dots, \hat{v}_i, \dots, v_p)}_{=-(v_0, v_1, v_2, \dots, \hat{v}_i, \dots, v_p)} = 0
\end{aligned}$$

Examples 2.6.

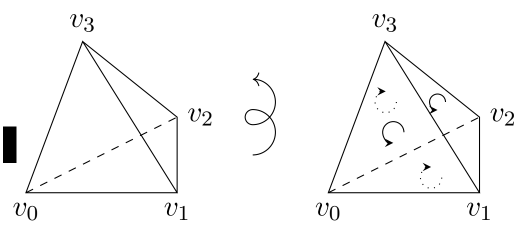
1. $\partial_1(v_0, v_1) = v_1 - v_0$



2. $\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$



3. $\partial_3(v_0, v_1, v_2, v_3) = (v_1, v_2, v_3) - (v_0, v_2, v_3) + (v_0, v_1, v_3) - (v_0, v_1, v_2)$



We will now consider the following diagram. (For the sake of readability, we deleted the dimension subscripts from the boundary operator)

$$C_\bullet : \qquad \cdots \xrightarrow{\partial} C_{p+1} \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots$$

Lemma 2.7. *$C_\bullet$ is a chain complex.*

Proof. We need to show that $\partial_{p-1} \circ \partial_p \equiv 0$. It is not hard to prove this, it just requires a lot of bookkeeping:

Let $\sigma = (v_0, \dots, v_p)$ be an arbitrary p-chain, then

$$\partial_{p-1} \circ \partial_p(\sigma) = \partial_{p-1}\left(\sum_{i=0}^{p}(-1)^i(v_0, \dots, \hat{v}_i, \dots, v_p)\right) = \sum_{i=0}^{p}(-1)^i \partial_{p-1}(v_0, \dots, \hat{v}_i, \dots, v_p)$$
$$= \sum_{i=0}^{p}(-1)^i \sum_{\substack{j=0 \\ j \neq i}}^{p} (-1)^j (v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_p)$$

Each of these term shows up twice with different signs. Thus, everything cancels.

□

A common question for chain complexes is exactness. As we will see later in this chapter, they will only be exact for special cases.

Before we define the simplicial homology groups, we will introduce one more definition:

Definition 2.8.

1. The kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$ is called the group of p-cycles and denoted by $Z_p(K)$.
2. The immage of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is called the group of p-boundaries and is denoted $B_p(K)$.

The previous lemma states that $B_p(K) \subset Z_p(K)$.

Let us remind ourselves of what we wanted to do in section 1. We wanted to consider those curves (*cycles*) that were not the *boundary* of some part of the surface. With this in mind, we can now state the definition of simplical homology groups

Definition 2.9. We define

$$H_p(K) := Z_p(K)/B_p(K) = \ker \partial_p / \operatorname{Im} \partial_{p+1}$$

and call it the **p-th simplicial homology goup of K**.

Examples 2.10.

1. Lets start with a simple triangulation[5] of $\mathbb{S}^2$ like the hollow tetrahedron (depicted in Figure 4)
$$X = \langle v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3 \rangle.$$
It consists of 4 faces ($f_1 = v_0v_1v_2, f_2 = v_0v_1v_3,\ f_3 = v_0v_2v_3, f_4 = v_1v_2v_3$), 6 edges ($e_1 = v_0v_1, e_2 = v_0v_2, e_3 = v_0v_3,\ e_4 = v_1v_2,\ e_5 = v_1v_3,\ e_6 = v_2v_3$) and

[5] A triangulation of a space T is a simplicial complex X whose geometric realization $|X|$ is homeomorphic to T.

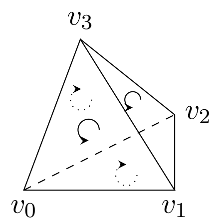


Figure 4: geometric realization of the hollow tetrahedron.

4 vertices (v_0, v_1, v_2, v_3). Therefore, we find the simplicial chain complex to be $\mathbb{Z}\langle f_1, f_2, f_3 f_4\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1, e_2, e_3, e_4, e_5, e_6\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v_0, v_1, v_2, v_3\rangle \xrightarrow{\partial_0} 0$

Computing the boundaries and writing the results in a more user friendly fashion gives

$$
\begin{aligned}
\partial f_1 &= v_1v_2 - v_0v_2 + v_0v_1 = e_1 - e_2 + e_4 \\
\partial f_2 &= v_1v_3 - v_0v_3 + v_0v_1 = e_1 - e_3 + e_5 \\
\partial f_3 &= v_2v_3 - v_0v_3 + v_0v_2 = e_2 - e_3 + e_6 \\
\partial f_4 &= v_2v_3 - v_1v_3 + v_1v_2 = e_4 - e_5 + e_6
\end{aligned}
\qquad
\partial_2 = \begin{array}{c c}
 & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\
\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} &
\begin{bmatrix}
1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

$$
\begin{aligned}
&\partial e_1 = v_1 - v_0, \quad \partial e_2 = v_2 - v_0, \quad \partial e_3 = v_3 - v_0, \\
&\partial e_4 = v_2 - v_1, \quad \partial e_5 = v_3 - v_1, \quad \partial e_6 = v_3 - v_1
\end{aligned}
\qquad
\begin{array}{c c}
 & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} &
\begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

We then find

$$
\begin{aligned}
\ker \partial_2 &= \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4\rangle \\
\ker \partial_1 &= \mathbb{Z}\langle e_1 - e_2 + e_4, e_1 - e_3 + e_5, e_2 - e_3 + e_6\rangle
\end{aligned}
$$

Then we have

$$
\begin{aligned}
H_2(\mathbb{S}^2) &= \ker \partial_2 / \operatorname{Im} \partial_3 = \ker \partial_2 = \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4\rangle \simeq \mathbb{Z} \\
H_1(\mathbb{S}^2) &= \ker \partial_1 / \operatorname{Im} \partial_2 \simeq \mathbb{Z}^3/\mathbb{Z}^3 \simeq 0 \\
H_0(\mathbb{S}^2) &= \ker \partial_0 / \operatorname{Im} \partial_1 \simeq \mathbb{Z}^4/\mathbb{Z}^3 \simeq \mathbb{Z} \\
H_n(\mathbb{S}^2) &= 0 \quad \text{for } n > 2
\end{aligned}
$$

2. The Torus (see Figure 5) is a little more difficult. Unfortunately, the simplest (simplicial) triangulation is depicted in Figure 6. This triangulation would lead to boundary operators in the form of 27×18 and 8×27 matrices respectively. The resulting homology groups are:

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}^2, \quad H_2 = \mathbb{Z}, \quad H_{n>2} = 0$$

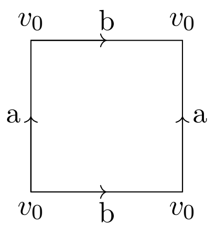


Figure 5: The fundamental polygon of the torus.

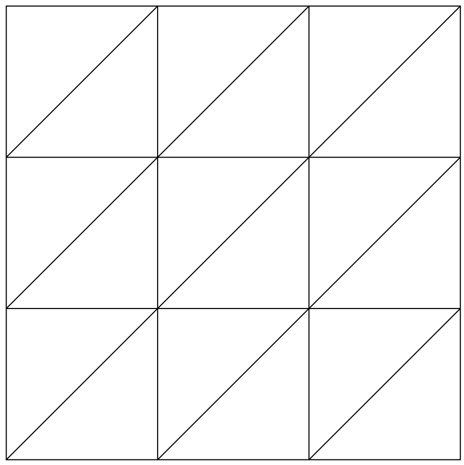
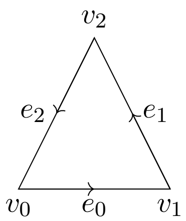


Figure 6: A simplicial triangulation of the torus.

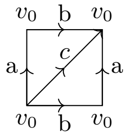
Even tough these aren't hard to compute, it is very tedious and therefore wont be presented here.[6]

We see that the homology groups for the Sphere and the Torus do not agree in every dimension. Since homotopy equivalent spaces have isomorphic homology groups (we will show that later), we find that the sphere cannot be homotopy equivalent to the torus.

3. We will now consider the circle. This can be triangulated as an empty triangle, i.e.



[6] By introducing the concept of cellular homology, this can be simplified to 3×2 and 1×3 matrices via the „(non-simplicial) triangulation":



Cellular and simplicial homology actually coincide. However we currently lack the framework for cellular homology

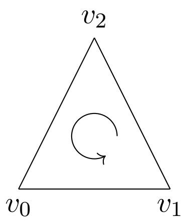
We easily find

$$\partial_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \partial_0 \equiv 0$$

Therefore,

$$H_1(\mathbb{S}^1) \simeq \mathbb{Z}/0 \simeq \mathbb{Z}, \quad H_0(\mathbb{S}) \simeq \mathbb{Z}^3/\mathbb{Z}^2 \simeq \mathbb{Z}$$

4. Now that we have considered the circle, we might also be interested in the 2-ball B^2. The triangulation is very easy this time. It is just one 2-dimensional simplex:



We find that ∂_1 and ∂_0 are the same as before and ∂_2 can be expressed as

$$\partial_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We then find

$$H_2(B^2) = 0, \quad H_1(B^2) = 0, \quad H_0(B^2) = \mathbb{Z}$$

3 Simplicial Homology Groups with $\mathbb{Z}/2\mathbb{Z}$ Coefficients

Let us remind ourselves of the definition for p-chains (Definition 2.3):

> Let K be a simlicial complex. A **p-chain** on K is a function c from the set of oriented p-simplices of K to the integers such that:
>
> 1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
> 2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

There is actually no reason (other than our familiarity with it) to restrict ourselves to the integers here. In fact, the same statements hold true for any abelian group G and it can be helpful to consider other groups than the integers. To denote the change of our underlying group, we use the notation

$$C_n(K;G) \quad \text{and} \quad H_n(K;G)$$

for the group of chains and the homology group of K with coefficients in G.

Since the group of chains with integer coefficients $C_n(K;\mathbb{Z}) = C_n(K)$ merely consists of linear combinations of elementary chains with integer coefficients, there is a neat way to derive $C_n(K;G)$ if $C_n(K;\mathbb{Z})$ is already known. We find that

$$C_n(K;G) \simeq C_n(K;\mathbb{Z}) \otimes_{\mathbb{Z}} G,$$

which leads to many interesting phenomena. The same holds for homology groups if the coefficient ring is a field.

This might be a good time to remember the fundamental theorem of finitely generated abelian groups (such as the homology groups) as it can be found in Munkres' book [see 3, Theorem 4.3, p. 24]:

Theorem 3.1 (The fundamental theorem of finitely generated abelian groups)**.** *Let G be a finitely generated abelian group. Let T be its torsion subgroup.Then there exists a free abelian subgroup H of G having finite rank such that $G = H \oplus T$.*

Since there is no possibility of torsion after taking the tensor product with a field of characteristic 0, we find that for any such field $\mathbb{K}$ the homology group with coefficients in $\mathbb{K}$ is a $\mathbb{K}$-vectorspace.

A special case is the coefficient ring $\mathbb{Z}/2\mathbb{Z}$, since here $-1 = 1$. Hence, we no longer have to worry about orientation. Therefore, this ring is often used if the considered manifolds might or might not be orientable.

Examples 3.2. Inhalt...

4 Functoriality Property

We will now explore how homotopy behaves under continuous functions. We will begin with the following definition and lemma from Munkres' book [3, lemma 12.1, p. 52]:

Definition 4.1. Let $f : K \to L$ be a simplicial map. If $v_0 \dots v_p$ is a simplex of K, then the points $f(v_0) \dots f(v_p)$ span a simplex of L. we define a homomorphism $f_\sharp : C_p(K) \to C_p(L)$ by defining it on oriented simplices as follows:

$$f_\sharp([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_0), \dots, f(v_p) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases}$$

The family $\{f_\sharp\}$ is called the chain map induced by the simplicial map f.

It might be helpful to consider the following diagram:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial} & C_p(K) & \xrightarrow{\partial} & C_{p-1}(K) & \xrightarrow{\partial} & C_{p-2}(K) & \xrightarrow{\partial} & \cdots \\ & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \\ \cdots & \xrightarrow{\partial} & C_p(L) & \xrightarrow{\partial} & C_{p-1}(L) & \xrightarrow{\partial} & C_{p-2}(L) & \xrightarrow{\partial} & \cdots \end{array}$$

This of course raises the question whether this diagram commutes or not.

Lemma 4.2. *The homomorphism $f_\sharp$ commutes with ∂; therefore $f_\sharp$ iduces a homomorphism $f_* : H_p(K) \to H_p(L)$*

Proof. Let τ denote the simplex of L that is spanned by $f(v_0), \dots, f(v_p)$. We need to check

$$\partial f_\sharp([v_0, \dots, v_p]) = f_\sharp(\partial[v_0, \dots, v_p]) \tag{1}$$

Case 1 $\dim \tau = p$**:** Clearly, this requires $f(v_0), \dots, f(v_p)$ to be distinct. Therefore the statement follows directly from the definitions.

Case 2 $\dim \tau \leq p - 2$**:** Here, the left side of (1) vanishes because $f(v_0), \dots, f(v_p)$ are not distinct. The right side also vanishes, since for every i, at least two of the points $f(v_0), \dots, \hat{f}(v_i), \dots, f(v_p)$ are the same.

Case 3 $\dim \tau = p - 1$**:** We will assume the following ordering of vertices:

$$f(v_0) = f(v_1), \qquad \text{and} \qquad f(v_1), \dots, f(v_p) \text{ are distinct.} \tag{2}$$

Similarly, to the previous case, the left side of (1) vanishes and the right side reads

$$[f(v_1)f(v_2), \dots, f(v_p)] - [f(v_0), f(v_2), ..., f(v_p)].$$

By (2) this also vanishes.

To show the second part of the lemma, we need to understand how $f_\sharp$ acts on boundaries and cycles. Let c be an arbitrary cycle of K, i.e. $\partial c = 0$. Then by (1), we find $\partial f_\sharp(c) = f_\sharp(\partial c) = f_\sharp(0) = 0$.

Now, let b be an arbitrary boundary, i.e. there exists some d with $b = \partial d$. Then, $f_\sharp(b) = f_\sharp(\partial d) = \partial f_\sharp(d)$.

We have thereby shown that $f_\sharp$ maps boundaries (cycles) of K to boundaries (cycles) of L. □

With this in mind we can now state the functional properties of these homomorphisms. We do this by following Munkres' book [3, Theorem 12.2, p. 53] once more.

Theorem 4.3 (functorial properties I)**.**

1. *Let $id : K \to K$ be the identity simplicial map. Then $id_* : H_p(K) \to H_p(K)$ is the identity homomorphism.*
2. *Let $f : K \to L$ and $g : L \to M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$; That is, the following diagram commutes:*

$$\begin{tikzcd} H_p(K) \arrow[rr, "(g\circ f)_*"] \arrow[dr, "f_*"'] & & H_p(M) \\ & H_p(L) \arrow[ur, "g_*"'] & \end{tikzcd}$$

Proof. The first part of the Theorem is pretty obvious. The second part follows immediately form $(g_\sharp \circ f_\sharp) = (g \circ f)_\sharp$. To see this, we shall consider vertices $v_0, \ldots, v_p$ that span a simplex in K. Then by definition

$$f_\sharp([v_0, \ldots, v_p]) = \begin{cases} [f(v_0), \ldots, f(v_p)], & \text{if } f(v_0), \ldots, f(v_p) \text{ are distinct,} \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} g_\sharp \circ f_\sharp([v_0, \ldots, v_p]) &= \begin{cases} [g \circ f(v_0), \ldots, g \circ f(v_p)], & \text{if } g \circ f(v_0), \ldots, g \circ f(v_p) \text{ are distinct,} \\ 0, & \text{if } g \circ f(v_0), \ldots, g \circ f(v_p) \text{ aren't distinct} \\ 0, & \text{if } f_\sharp([v_0, \ldots, v_p]) = 0 \end{cases} \\ &= \begin{cases} [g \circ f(v_0), \ldots, g \circ f(v_p)], & \text{if } g \circ f(v_0), \ldots, g \circ f(v_p) \text{ are distinct,} \\ 0, & \text{otherwise} \end{cases} \\ &= (g \circ f)_\sharp([v_0, \ldots, v_p]) \end{aligned}$$

□

It follows immediately that

Corollary 4.4. *Let $f : K \to L$ be a bijective simplicial map such that its inverse is also a simplicial map. Then*

$$\forall p \geq 0 : \quad H_p(K) = H_p(L)$$

□

This already gives the main Idea for the next section.

Before we can get started with that we need some preliminary definitions and results:

Definition 4.5. Given two simplicial maps $f, g : K \to L$, these maps are said to be contiguos if for each simplex $v_0 \ldots v_p$ of K, the points

$$f(v_0), \ldots, f(v_p), g(v_0), \ldots, g(v_p)$$

span a simplex of L.

Definition 4.6. Let $f, g : K \to L$ be simplicial maps. Suppose that for each p, one has a homomorphism

$$D : C_p(K) \to C_{p+1}(L)$$

satisfying the equation

$$\partial D + D\partial = g_\sharp - f_\sharp.$$

Then D is said to be a chain homotopy between $f_\sharp$ and $g_\sharp$. Note that the diagramm

$$\begin{tikzcd}
& C_{p+1}(L) \arrow[d, "\partial"] \\
C_p(K) \arrow[ur, "D_p"] \arrow[r, shift left, "(g_\sharp)_p"] \arrow[r, shift right, "(f_\sharp)_p"'] \arrow[d, "\partial"] & C_p(L) \\
C_{p-1}(K) \arrow[ur, "D_{p-1}"'] &
\end{tikzcd}$$

is not commutative.

Theorem 4.7. *If there is a chain homotopy between $f_\sharp$ and $g_\sharp$ then the induced homomorphisms f_* and g_* are equal.*

Proof. Let c be a p-cycle of K. Then

$$g_\sharp(c) - f_\sharp(c) = \partial Dz + D\partial z = \partial Dz + 0.$$

That means that $f_\sharp$ and $g_\sharp$ differ only by a boundary. Therefore they are in the same homology class, i.e. $f_*(c) = g_*(c)$. $\square$

Theorem 4.8. *If $f, g : K \to L$ are contiguous simplicial maps, then there is a chain homotopy between $f_\sharp$ and $g_\sharp$.*

Proof. Let $\sigma = (v_0, \dots, v_p) \subset K$ and $L(\sigma) = (f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)) \subset L$. Then

1. $L(\sigma)$ is a nonempty, and $\tilde{H}_n(L(\sigma)) = 0$ for $n \geq 0$.
2. If s is a face of σ, then $L(s) \subset L(\sigma)$.
3. For each oriented simplex σ, the chains $f_\sharp(\sigma)$ and $g_\sharp(\sigma)$ are zero on every simplex not in $L(\sigma)$.

We will now construct the chain homotopy $D : C_p(K) \to C_{p+1}(L)$ by induction on p. For each σ, the chain $D\sigma$ will vanish on any simplex not contained in $L(\sigma)$.

Let $p = 0$; let v be a vertex of K. Because $f_\sharp$ and $g_\sharp$ preserve augmentation,

$$\epsilon(g_\sharp(v) - f_\sharp(v)) = 1 - 1 = 0.$$

Thus $g_\sharp(v) - f_\sharp(v)$ represents an element of the reduced homology group $\tilde{H}_0(L(v))$. Because this group vanishes, we can choose a 1-chain Dv of L carried by the subcomplex $L(v)$ such that

$$\partial(Dv) = g_\sharp(v) - f_\sharp(v)$$

Then $\partial Dv + D\partial v = \partial Dv + 0 = g_\sharp(v) - f_\sharp(v)$, as desired. Define D in this way for each vertex of K.

Now suppose D is defined in dimensions less than p, such that for each oriented simplex s of dimension less than p, the chain Ds is carried by $L(s)$, and such that

$$\partial Ds + D\partial s = g_\sharp(s) - f_\sharp(s)$$

Let σ be an oriented simplex of dimension p. We wish to define $D\sigma$ so that $\partial(D\sigma)$ equals the chain

$$c = g_\sharp(\sigma) - f_\sharp(\sigma) - D\partial\sigma$$

Note that c is a well-defined chain; $D\partial\sigma$ is defined because $\partial\sigma$ has dimension $p-1$. Furthermore, c is a cycle, for we compute

$$\begin{aligned}\partial c &= \partial g_\sharp(\sigma) - \partial f_\sharp(\sigma) - \partial D(\partial\sigma)\\ &= \partial g_\sharp(\sigma) - \partial f_\sharp(\sigma) - [g_\sharp(\partial\sigma) - f_\sharp(\partial\sigma) - D\partial(\partial\sigma)]\end{aligned}$$

applying the induction hypothesis to the $p-1$ chain $\partial\sigma$. Using the fact that $\partial \circ \partial = 0$, we see that $\partial c = 0$.

Finally, we note that c is carried by $L(\sigma)$, and since $\tilde{H}_p(L(\sigma)) = 0$, we can choose a $p+1$ chain $D\sigma$ carried by $L(\sigma)$ such that

$$\partial D\sigma = c = g_\sharp(\sigma) - f_\sharp(\sigma) - D\partial\sigma$$

We then define $D(-\sigma) = -D(\sigma)$. We repeat this process for each p-simplex σ of K; then we have the required chain homotopy D in dimension p. The theorem follows. □

Let us remind ourselves of the general simplicial approximation theorem, as it can be found in Munkres' book [3, Thm. 16.5, p. 85]:

Theorem 4.9 (The general simplicial approximation theorem)**.** *Let K and L be complexes; let $h : |K| \to |L|$ be a continuous map. There exists a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$.*

We will further need the following definition again for Munkres' book [3, p. 100]:

Definition 4.10. Let K and L be simplicial complexes; let $h : |K| \to |L|$ be a continuous map. Choose a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$ (such exists by virtue of the previous theorem). Let $\lambda : C_\bullet(K) \to C_\bullet(K')$ be the subdivision operator. We define the homomorphism induced by h

$$h_* : H_p(K) \to H_p(L)$$

via

$$h_* = f_* \circ \lambda_*.$$

Since this definition involves several choices, we need to make sure, that the resulting homomorphism isn't dependent on these choices. We do this in two steps. Firstly, notice that for some choice of a subdivision K' two simplicial approximations, then these are contiguous and therefore induce the same homomorphism $f_* \equiv g_*$ (see Theorems 4.7 and 4.8).

Let $\iota : K' \to K$ be a simplicial approximation to the identity map of K, then $\lambda_* = (g_*)^{-1}$, hence

$$h_* = f_* \circ (g_*)^{-1}$$

Secondly, we will now use this to show that h_* doesn't depend on the choice of subdivision K' either. Let K'' be another subdivision of K such that h has a simplicial approximation mapping $K\prime\prime$ to L. We will see that the induced homomorphism h_* does not change if we use substitute K with K'' in its definition.

We will first explore the special case that $|K|$ has a simplicial approximation $k : K'' \to K'$

$$\begin{tikzcd} & & K \\ K'' \arrow[r, "k"] & K' \arrow[ur, "g"] \arrow[dr, "f"] & \\ & & L \end{tikzcd}$$

Let $h*'$ be the induced homomorphism defined via K''. We can then write

$$h'_*(f \circ k)_* \circ (g \circ k)_*^{-1} = (f_* \circ k_*) \circ (g_* \circ k_*)^{-1} = f \circ g_*^{-1} = g_* \circ \lambda_* = h_*$$

We generalize this method by considering a subdivision $K\prime\prime\prime$ of K such that the identity map has simplicial approximations

$$k_1 : K''' \to K' \quad \text{and} \quad k_2 : K''' \to K''.$$

We can now use this subdivision to construct a homomorphism h''_*. By our previous work, we already know that

$$h'_* = h''_* = h_*$$

We are now able to generalize Theorem 4.3

Theorem 4.11 (functorial properties II)**.** *The identity map id* : $|K| \to |K|$ *induces the identity homomorphism* $i_* : H_p(K) \to H_p(K)$. *If* $h : |K| \to |L|$ *and* $k : |L| \to |M|$ *are continuous maps, then* $(k \circ h)_* = k_* \circ h_*$.

Proof. The first part is elementary. For the second part, choose simplicial approximations as follows:

$f_0 : L' \to M$	simplicial approximation to k
$g_0 : L' \to L$	simplicial approximation to the identity on $\lvert L\rvert$
$f_1 : K' \to L'$	simplicial approximation to h
$g_1 : K' \to K$	simplicial approximation to the identity on $\lvert K\rvert$

$$|K| \xrightarrow{h} |L| \xrightarrow{k} |M|$$

$$\begin{tikzcd} K & L & M \\ & L' \arrow[u, "g_0"] \arrow[ur, "f_0"] & \\ K' \arrow[uu, "g_1"] \arrow[ur, "f_1"] & & \end{tikzcd}$$

With these choices, $f_0 \circ f_1$ is a simplicial approximation to $k \circ h$, hence

$$(k \circ h)_* = (f_0 \circ f_1)_* \circ (g_1)_*^{-1}$$

Since $g_0 \circ f_1$ is a simplicial approximation to h, have

$$h_* = (g_0 \circ f_1)_* \circ (g_1)_*^{-1} \quad \text{and} \quad k_* = (f_0)_* \circ (g_0)_*^{-1}$$

Combining these results and applying Theorem 4.3 we obtain

$$(k \circ h)_* = k_* \circ h_*$$

as desired.

□

5 Homotopy Invariance of Simplicial Homology

We have arrived at what is probably the most important chapter of this paper. In section 1 we already mentioned that we can use (simplicial) homology groups to show that two spaces are not homotopic (or homeomorphic). However, so far we never explained why non-isomorphic homology groups imply that. We need to show that homology groups are topological invariants, i.e. that homotopic spaces have isomorphic homology groups.

We already saw a very special case in Corollary 4.4. As we will see shortly, the main idea is essentially the same for a more general statement

Corollary 5.1. *Let $f : |K| \to |L|$ be a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism.*

□

We have discovered the topological invariance of homology groups. However, there is an even stronger statement to be made here. As we will see in the following, homology groups are not only invariant under homeomorphisms, but only depend on the homotopy type.

Recall the definition of homotopy, taken from Munkres [3, p. 94]:

Definition 5.2. If X and Y are topological spaces, two cocntinuous maps $h, k : X \to Y$ are said to be homotopic if there is a continuous map

$$F : X \times I \to Y$$

such that $F(x, 0) = h(x)$ and $F(x, 1) = k(x)$ for all $x \in X$. If h and k are homotopic, we write $h \simeq k$. The map F is called a homotopy of h to k.

We will show that for homotopic maps h and k, the induced homomorphisms coincide $h_* = k_*$. For that we will need the following results

Lemma 5.3. *the topology of the product space $|K| \times I$ is coherent with the subspaces $\sigma \times I$, for $\sigma \in K$.*

Lemma 5.4. *If K is a comoplex, then $|K| \times I$ is the polytope of a complex M, such that each set $\sigma \times I$ is the polytope of a subcomplex of M, and the sets $\sigma \times 0$ and $\sigma \times 1$ are simplices of M, for each simplex σ of K.*

Proof. There exists a J such that $|K| \subset \mathbb{E}^J$, so $|K| \times I \subset \mathbb{E}^J \times \mathbb{R}$. We will use a inductive method to subdivide $|K| \times I$ into simplices.

For $p \geq 0$ let

$$X_p := (|K| \times 0) \cup (|K| \times 1) \cup (|K^{(p)}| \times I)$$

For $p = 0$ this space consists of all simplices of the form $\sigma \times 0$, $\sigma \times 1$ ($\sigma \in K$) and all 1-simplices of the form $v \times I$ (v a vertex of K) and the corresponding vertices. We denote the complex that arises as the union of those simplices as M_0. The polytope of M_0 is X_0.

Suppose M_{p-1} is a complex whose plytope is X_{p-1}, such that for every simplex s in K of dimension less than p the set $s \times I$ is the polytope of a subcomplex of M_{p-1}. Consider a set $\sigma \times I$ with $\dim \sigma = p$ and let

$$\begin{aligned} \operatorname{Bd}(\sigma \times I) &= (\sigma \times I) \setminus (\operatorname{Int} \sigma \times \operatorname{Int} I) \\ &= ((\operatorname{Bd} \sigma) \times I) \cup (\sigma \times 0) \cup (\sigma \times 1). \end{aligned}$$

We know that $\operatorname{Bd} \sigma$ is the union fo simplices s of K of dimension $p-1$, therefore there exists a subcomplex $M_\sigma \subset M_{p-1}$ such that $\operatorname{Bd}(\sigma \times I)$ is the polytope of M_σ. Since $\operatorname{Bd}(\sigma \times I)$ is compact, M_σ is finite.

Define w_σ to be the point $(\hat{\sigma}, \frac{1}{2}) \in \sigma \times I$. Then the cone[7] $w * M_\sigma$ is a complex whose polytope is $\sigma \times I$. The intersection of $|w_\sigma * M_\sigma|$ and $|M_{p-1}|$ is the polytope of a subcomplex of each of them.

Now, let M_p be the union of M_{p-1} and the cones $w_\sigma * M_\sigma$ for all p-simplices of K and define M to be the union of the M_p for all p.

We already know that the polytope of M is the same as $|K| \times I$ as *sets*, however, we still need to show that they are the same as *topological spaces*.

Lemma 5.3 already stated that the topology of $|K| \times I$ is coherent with that of $\sigma \times I$ for $\sigma \in K$. Similarly, the topology of $|M|$ is coherent with the subspaces s, for $s \in M$. Let C be closed in $|K| \times I$, then $C \cap (\sigma \times I)$ is closed in $\sigma \times I$. If s is a simplex of M lying in $\sigma \times I$, then s is a subpace of $\sigma \times I$. Hence $C \cap s$ is closed in s. It follows that C is closed in $|M|$.

Conversely, if C is closed in $|M|$, then $C \cap s$ is closed in s for each $s \in M$. Because $\sigma \times I$ is a finite union of simplices s of M, the set $C \cap (\sigma \times I)$ is closed in $\sigma \times I$. Thus σ is closed in $|K| \times I$. □

[7] Suppose that K is a complex in $\mathbb{E}^J$, and w is a point of $\mathbb{E}^J$ such that each ray emanating from w intersects $|K|$ in at most one point. We define the cone on K with vertex w to be the collection of all simplices of the form $wa_0 \dots a_p$, where $a_0 \dots a_p$ is a simplex of K, along with all faces of such simplices. We denote this collection $w * K$.[3, p. 44]

Theorem 5.5. *IF $h, k : |K| \to |L|$ are homotopic, then $h_*, k_* : H_p(K) \to H_p(L)$ are equal.*

Before we can tend to the proof, we need more framework.

Definition 5.6. Let K and L be simplicial complexes. An acyclic carrier from K to L is a function Φ that assigns to each simplex σ of K, a subcomplex $\Phi(\sigma)$ fo L such that:

1. $\Phi(\sigma)$ is nonempty and acyclic.

2. If s is a face of σ, then $\Phi(s) \subset \Phi(\sigma)$

If $f : C_p(K) \to C_p(L)$ is a homomorphism, we say that f is carried by ϕ if for each oriented p-simplex σ of K, the chain $f(\sigma)$ is carried by the subcomplex $\Phi(\sigma)$ of L.

Theorem 5.7 (Acyclic carrier theorem, geometric version)**.** *Let Φ be an acyclic carrier from K to L.*

1. *If ϕ and ψ are two augmentation-preserving chain maps from $C_\bullet(K)$ to $C_\bullet(L)$ that are carried by Φ, there exists a chain homotopy D of ϕ to ψ that is also carried by Φ.*

2. *There exists an augmentation-preserving chain map from $C_\bullet(K)$ to $C_\bullet(L)$ is carried by Φ.*

We only need the first part of this theorem. The second part is only stated for the sake of completeness. Therefore we will only refer the reader to Munkres' book for a proof of the second part. The proof of the first part is essentially the same as for Theorem 4.8

Proof. Let K be a complex. Let M be a complex whose underlying space is $|K| \times I$, such that for each $\sigma \in K$, both $\sigma \times 0$ and $\sigma \times 1$ are simplices of M, and $\sigma \times I$ is the polytope of a subcomplex of M.

Let $F : |K| \times I \to |L|$ be the homotopy of h to k. Let $i, j : |K| \to |K| \times I$ be the maps $i(x) = (x, 0)$ and $j(x) = (x, 1)$. Then i and j are simplicial maps of K into the complex M; furthermore,

$$F \circ i = h \quad \text{and} \quad F \circ j = k.$$

We assert that the chain maps $i_\sharp$ and $j_\sharp$ are chain homotopic. Consider the function Φ assigning, to each simpley σ of K, the subcomplex of M whose polytope is $\sigma \times I$. Now the space $\sigma \times I$ is acyclic because it is homoeomorphic to a closed ball. And if $s < \sigma$,then $s \times I \subset \sigma \times I$, so $\Phi(s)$ is a subcomplex of $\Phi(\sigma)$. Therefore, Φ is an acyclic carrier from K to M. Furthermore, it carries both $i_\sharp$ and $j_\sharp$, for both $i(\sigma) = \sigma \times 0$ and $j(\sigma) = \sigma \times 1$ belong to $\Phi(\sigma)$. It follows form Theorem 5.7 that $i_\sharp$ adn $j_\sharp$ are chain homotopic. By virtue of Theorem 4.7 it follow that $i_* = j_*$, so

$$h_* = F_* \circ i_* = F_* \circ j_* = k_*$$

□

Definition 5.8. Two spaces X and Y are said to be homotopy equivalent, or to have the same homotopy type, if there are maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

such that $g \circ f \simeq i_X$ and $f \circ g \simeq i_Y$. The maps f and g are often called homotopy equivalences, and g is said to be the homotopy inverse to f.

Corollary 5.9. *If two spaces X and Y are homotopy equivalent, then their homology groups are isomorphic $H_n(X) \cong H_n(Y)$ in all dimensions.*

□

References

[1] M.A. Armstrong. *Basic Topology.* Undergraduate Texts in Mathematics. Springer New York, 1983. ISBN: 9780387908397.

[2] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002. ISBN: 9780521795401. URL: `http://pi.math.cornell.edu/~hatcher/AT/AT.pdf`.

[3] J.R. Munkres. *Elements Of Algebraic Topology.* CRC Press, 1984. ISBN: 9780201045864.