

Simplicial Homology[draft]

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Abstract goes here

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1 Motivation

One of the main goals when studying (topological) spaces (or in this case simplicial complexes) is to determine whether two spaces are homotopic or not. The first method that is often applied is to check if both spaces are simply connected. A more advanced approach checks not only simply connectedness but also if the fundamental groups π_1 coincide¹.

It turns out that this is a valuable tool for one and two dimensional simplicial complexes. But this method fails for complexes with cells in higher dimensions. It can be shown that the fundamental group actually only depends on the 2-skeleton of the complex [vgl. 1, p. 173].

Even though this method can be generalized to the study of higher homotopy groups π_n , this is often more difficult than necessary since the computation of those groups is far from trivial.

In the following we will explore the concept of simplicial homology groups and how it can be used to classify spaces. We will see that they allow us to tackle the problem of deciding whether two **polyhedra**² are homeomorphic, since simplicial homology is only defined on such spaces. However, the concept of simplicial homology can be generalized to singular homology, which can be applied to a broader category of spaces and coincides with simplicial homology if both are defined.

However, singular homology groups will not be covered here. The interested reader might refer to the book of Munkres [3] or Hatcher [2] to learn more about singular homology groups.

When we concern ourselves with the question how closed paths on the torus are different from those on the sphere, we can notice the following:

Any Jordan curve (i.e. nonselfintersecting, closed path) on a sphere divides the surface into two „regions“ as depicted in Figure 1³. The same is not true for the torus. In Figure 2 we see that even though there are Jordan curves on a torus that bound a „region“ of the surface, not all Jordan curves have that property [p. 173f 1, see].

2 Simplicial Homology Groups with Integer Coefficients

We will use the following definition by Munkres [3, p. 26]

Definition 2.1. For a simplex σ we say that two orderings of its vertex set are equivalent if they differ by an even permutation. For a simplex with nonzero dimension, the

¹If X is a simply connected space then $\pi_1(X) = 0$. Furthermore, the fundamental group is homotopy invariant (up to isomorphism), so homotopic spaces have isomorphic fundamental groups. A more precise formulation of this statement can be found in Hatcher's book [2, Prop. 1.18, p. 37]:

If $\phi : X \rightarrow Y$ is a homotopy equivalence, then it induces a homomorphism $\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$ which is an isomorphism for all $x_0 \in X$.

²A polyhedron is the polytope of a simplicial complex.

³For a more precise statement see https://en.wikipedia.org/wiki/Jordan_curve_theorem

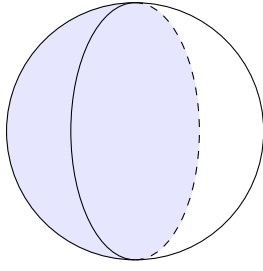


Figure 1: Any closed path on the sphere divides the surface into two regions.

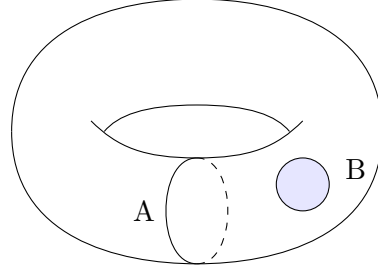


Figure 2: There are closed paths on the torus that do not divide the surface into two regions.

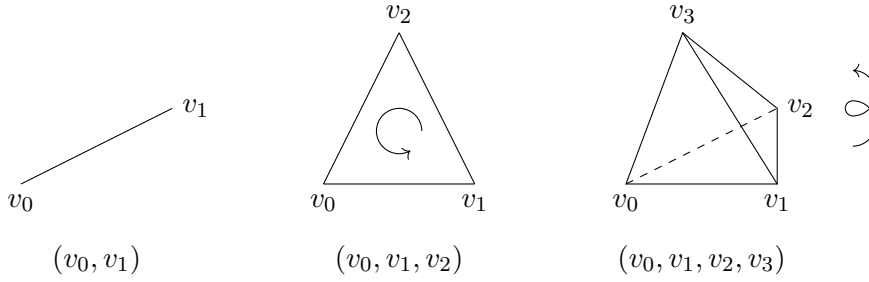


Figure 3: Indicating orientation with arrows [see 3, p.27]

orderings of the vertices fall into two equivalence classes, called the **orientations** of σ . An **oriented simplex** is a simplex together with an orientation.

Further we will use the same notation as Munkres [3, p. 26]

Notation 2.2. For geometrically independent points v_0, \dots, v_p we denote the simplex they span with

$$[v_0, \dots, v_p].$$

For an oriented simplex we will use the notation

$$(v_0, \dots, v_p).$$

where the orientation is given by this particular ordering.

A choice of orientation is usually depicted with arrows as can be seen in Figure 3

To define simplicial homology groups in any meaningful way, we need a few definitions that allow us to formulate our ideas from section 1 more precise. There are several ways to go about this with various amounts of rigor. In this paper, we will follow the path of Munkres [3, p. 27f].

Definition 2.3. Let K be a simplicial complex. A **p -chain** on K is a function c from the set of oriented p -simplices of K to the integers such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p -simplices σ .

We add p -chains by adding their values; the resulting group is denoted $C_p(K)$ and is called the group of (oriented) p -chains of K . If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group.

If σ is an oriented simplex, the elementary chain c_σ corresponding to σ is the function defined as follows:

$$\begin{aligned} c_\sigma(\sigma) &= 1, \\ c_\sigma(\sigma') &= -1 && \text{if } \sigma' \text{ is the opposite orientation of } \sigma \\ c_\sigma(\tau) &= 0 && \text{for all other oriented simplices } \tau. \end{aligned}$$

It is common practice to to abuse the notation here. If clear by context we use the symbol σ not only to denote the (oriented) simplex, but also the corresponding elementary chain.

We are now able to formulate the our first result as can be found in Munkres book [3, Lemma 5.1, p. 28]:

Lemma 2.4. *$C_p(K)$ is free abelian; a basis for $C_p(K)$ can be obtained by orienting each p -simplex and using the corresponding elementary chains as a basis.*

Proof. It is easy to see that each chain c in $C_p(K)$ can be expressed uniquely as a linear combination of the elementary chains c_{σ_i} of the simplices of K , i.e.

$$c = \sum_i n_i c_{\sigma_i}.$$

Here, the the chain c assigns the value n_i to each simplex σ_i , $-n_i$ for the simplex σ_i with reversed orientation and 0 to every simplex that does not appear in the summation.⁴ \square

Definition 2.5. We define the boundary operator

$$\partial_p : C_p(K) \rightarrow C_{p-1}(K)$$

to be the homomorphism via its action on an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol \hat{v}_i means that the vertex v_i is to be deleted from the array.

⁴Recall that by definition of the elementary chains we have

$$c_{\sigma_i}(\tau) = \begin{cases} 1 & \text{if } \sigma_i = \tau \\ -1 & \text{if } \tau \text{ is the opposite orientation of } \sigma_i \\ 0 & \text{else} \end{cases}$$

As the name suggests, the result of this homomorphism indeed refers to the (topological) boundary of the simplex once we interpret the sum of two elementary chains as the union of the corresponding simplices as we will see in the several examples, but first we have to check well-definedness. For that, it is necessary to show that

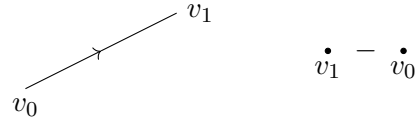
$$\partial_p(-\sigma) = -\partial_p(\sigma)$$

Since the orientation for a simplex only depends on the *sign* of the permutation, it is sufficient to check for the simple case of exchanging v_0 and v_1 :

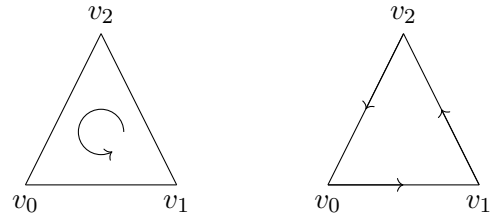
$$\begin{aligned} & \partial_p(v_0, \dots, v_p) + \partial_p(v_1, v_0, v_2, \dots, v_p) \\ &= (v_1, \dots, v_p) - (v_0, v_2, \dots, v_p) + \sum_{i=2}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) \\ &+ (v_0, v_2, \dots, v_p) - (v_1, v_2, \dots, v_p) + \sum_{i=2}^p (-1)^i \underbrace{(v_1, v_0, v_2, \dots, \hat{v}_i, \dots, v_p)}_{=-(v_0, v_1, v_2, \dots, \hat{v}_i, \dots, v_p)} = 0 \end{aligned}$$

Examples 2.6.

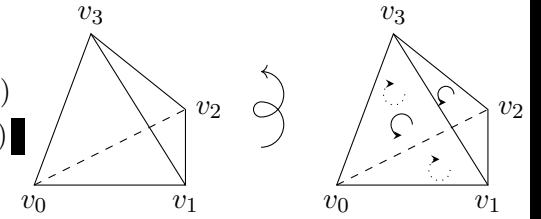
1. $\partial_1(v_0, v_1) = v_1 - v_0$



2. $\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$



3. $\partial_3(v_0, v_1, v_2, v_3) = (v_1, v_2, v_3) - (v_0, v_2, v_3) + (v_0, v_1, v_3) - (v_0, v_1, v_2)$



We will now consider the following diagram. (For the sake of readability, we deleted the dimension subscripts from the boundary operator)

$$C_\bullet: \quad \cdots \xrightarrow{\partial} C_{p+1} \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots$$

Lemma 2.7. C_\bullet is a chain complex.

Proof. We need to show that $\partial_{p-1} \circ \partial_p \equiv 0$. It is not hard to prove this, it just requires a lot of bookkeeping:

Let $\sigma = (v_0, \dots, v_p)$ be an arbitrary p -chain, then

$$\begin{aligned} \partial_{p-1} \circ \partial_p(\sigma) &= \partial_{p-1} \left(\sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) \right) = \sum_{i=0}^p (-1)^i \partial_{p-1}(v_0, \dots, \hat{v}_i, \dots, v_p) \\ &= \sum_{i=0}^p (-1)^i \sum_{\substack{j=0 \\ j \neq i}}^p (-1)^j (v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_p) \end{aligned}$$

Each of these term shows up twice with different signs. Thus, everything cancels. \square

A common question for chain complexes is exactness. As we will see later in this chapter, they will only be exact for special cases.

Before we define the simplicial homology groups, we will introduce one more definition:

Definition 2.8.

1. The kernel of $\partial_p : C_p(K) \rightarrow C_{p-1}(K)$ is called the group of p -cycles and denoted by $Z_p(K)$.
2. The image of $\partial_{p+1} : C_{p+1}(K) \rightarrow C_p(K)$ is called the group of p -boundaries and is denoted $B_p(K)$.

The previous lemma states that $B_p(K) \subset Z_p(K)$.

Let us remind ourselves of what we wanted to do in section 1. We wanted to consider those curves (*cycles*) that were not the *boundary* of some part of the surface. With this in mind, we can now state the definition of simplicial homology groups

Definition 2.9. We define

$$H_p(K) := Z_p(K) / B_p(K) = \ker \partial_p / \text{Im } \partial_{p+1}$$

and call it the **p -th simplicial homology group of K** .

Examples 2.10.

1. Lets start with a simple triangulation⁵ of \mathbb{S}^2 like the hollow tetrahedron (depicted in Figure 4)

$$X = \langle v_0 v_1 v_2, v_0 v_1 v_3, v_0 v_2 v_3, v_1 v_2 v_3 \rangle.$$

It consists of 4 faces ($f_1 = v_0 v_1 v_2, f_2 = v_0 v_1 v_3, f_3 = v_0 v_2 v_3, f_4 = v_1 v_2 v_3$), 6 edges ($e_1 = v_0 v_1, e_2 = v_0 v_2, e_3 = v_0 v_3, e_4 = v_1 v_2, e_5 = v_1 v_3, e_6 = v_2 v_3$) and

⁵A triangulation of a space T is a simplicial complex X whose geometric realization $|X|$ is homeomorphic to T .

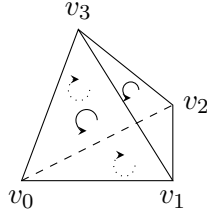


Figure 4: geometric realization of the hollow tetrahedron.

4 vertices (v_0, v_1, v_2, v_3) . Therefore, we find the simplicial chain complex to be

$$\mathbb{Z}\langle f_1, f_2, f_3, f_4 \rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v_0, v_1, v_2, v_3 \rangle \xrightarrow{\partial_0} 0$$

Computing the boundaries and writing the results in a more user friendly fashion gives

$$\begin{aligned} \partial f_1 &= v_1 v_2 - v_0 v_2 + v_0 v_1 = e_1 - e_2 + e_4 \\ \partial f_2 &= v_1 v_3 - v_0 v_3 + v_0 v_1 = e_1 - e_3 + e_5 \\ \partial f_3 &= v_2 v_3 - v_0 v_3 + v_0 v_2 = e_2 - e_3 + e_6 \\ \partial f_4 &= v_2 v_3 - v_1 v_3 + v_1 v_2 = e_4 - e_5 + e_6 \end{aligned} \quad \partial_2 = \begin{matrix} & \begin{matrix} f_1 & f_2 & f_3 & f_4 \end{matrix} \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$\begin{aligned} \partial e_1 &= v_1 - v_0, & \partial e_2 &= v_2 - v_0, & \partial e_3 &= v_3 - v_0, \\ \partial e_4 &= v_2 - v_1, & \partial e_5 &= v_3 - v_1, & \partial e_6 &= v_3 - v_1 \end{aligned} \quad \partial_1 = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

We then find

$$\begin{aligned} \ker \partial_2 &= \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4 \rangle \\ \ker \partial_1 &= \mathbb{Z}\langle e_1 - e_2 + e_4, e_1 - e_3 + e_5, e_2 - e_3 + e_6 \rangle \end{aligned}$$

Then we have

$$\begin{aligned} H_2(\mathbb{S}^2) &= \ker \partial_2 / \text{Im } \partial_3 = \ker \partial_2 = \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4 \rangle \simeq \mathbb{Z} \\ H_1(\mathbb{S}^2) &= \ker \partial_1 / \text{Im } \partial_2 \simeq \mathbb{Z}^3 / \mathbb{Z}^3 \simeq 0 \\ H_0(\mathbb{S}^2) &= \ker \partial_0 / \text{Im } \partial_1 \simeq \mathbb{Z}^4 / \mathbb{Z}^3 \simeq \mathbb{Z} \\ H_n(\mathbb{S}^2) &= 0 \quad \text{for } n > 2 \end{aligned}$$

2. The Torus (see Figure 5) is a little more difficult. Unfortunately, the simplest (simplicial) triangulation is depicted in Figure 6. This triangulation would lead to boundary operators in the form of 27×18 and 8×27 matrices respectively. The resulting homology groups are:

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}^2, \quad H_2 = \mathbb{Z}, \quad H_{n>2} = 0$$

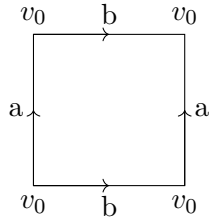


Figure 5: The fundamental polygon of the torus.

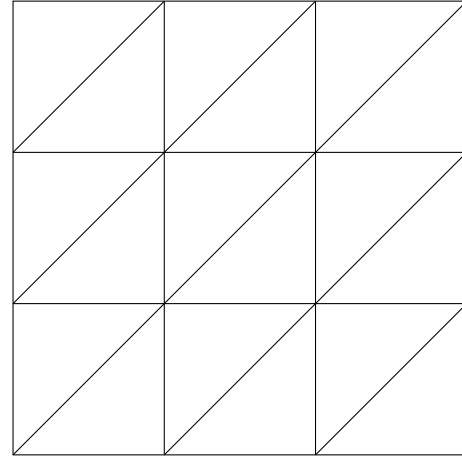
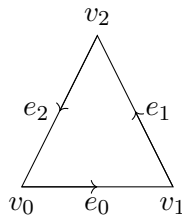


Figure 6: A simplicial triangulation of the torus.

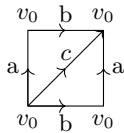
Even though these aren't hard to compute, it is very tedious and therefore won't be presented here. ⁶.

We see that the homology groups for the Sphere and the Torus do not agree in every dimension. Since homotopy equivalent spaces have isomorphic homology groups (we will show that later), we find that the sphere cannot be homotopy equivalent to the torus.

3. We will now consider the circle. This can be triangulated as an empty triangle, i.e.



⁶By introducing the concept of cellular homology, this can be simplified to 3×2 and 1×3 matrices via the „(non-simplicial) triangulation“:



Cellular and simplicial homology actually coincide. However we currently lack the framework for cellular homology

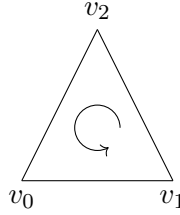
We easily find

$$\partial_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \partial_0 \equiv 0$$

Therefore,

$$H_1(\mathbb{S}^1) \simeq \mathbb{Z}/0 \simeq \mathbb{Z}, \quad H_0(\mathbb{S}) \simeq \mathbb{Z}^3/\mathbb{Z}^2 \simeq \mathbb{Z}$$

4. Now that we have considered the circle, we might also be interested in the 2-ball B^2 . The triangulation is very easy this time. It is just one 2-dimensional simplex:



We find that ∂_1 and ∂_0 are the same as before and ∂_2 can be expressed as

$$\partial_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We then find

$$H_2(B^2) = 0, \quad H_1(B^2) = 0, \quad H_0(B^2) = \mathbb{Z}$$

3 Simplicial Homology Groups with $\mathbb{Z}/2\mathbb{Z}$ Coefficients

Let us remind ourselves of the definition for p -chains (Definition 2.3):

Let K be a simplicial complex. A **p -chain** on K is a function c from the set of oriented p -simplices of K to the integers such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p -simplices σ .

There is actually no reason (other than our familiarity with it) to restrict ourselves to the integers here. In fact, the same statements hold true for any abelian group G and it can be helpful to consider other groups than the integers. To denote the change of our underlying group, we use the notation

$$C_n(K; G) \quad \text{and} \quad H_n(K; G)$$

for the group of chains and the homology group of K with coefficients in G .

Since the group of chains with integer coefficients $C_n(K; \mathbb{Z}) = C_n(K)$ merely consists of linear combinations of elementary chains with integer coefficients, there is a neat way to derive $C_n(K; G)$ if $C_n(K; \mathbb{Z})$ is already known. We find that

$$C_n(K; G) \simeq C_n(K; \mathbb{Z}) \otimes_{\mathbb{Z}} G,$$

which leads to many interesting phenomena. The same holds for homology groups if the coefficient ring is a field.

This might be a good time to remember the fundamental theorem of finitely generated abelian groups (such as the homology groups) as it can be found in Munkres' book [see 3, Theorem 4.3, p. 24]:

Theorem 3.1 (The fundamental theorem of finitely generated abelian groups). *Let G be a finitely generated abelian group. Let T be its torsion subgroup. Then there exists a free abelian subgroup H of G having finite rank such that $G = H \oplus T$.*

Since there is no possibility of torsion after taking the tensor product with a field of characteristic 0, we find that for any such field \mathbb{K} the homology group with coefficients in \mathbb{K} is a \mathbb{K} -vector space.

A special case is the coefficient ring $\mathbb{Z}/2\mathbb{Z}$, since here $-1 = 1$. Hence, we no longer have to worry about orientation. Therefore, this ring is often used if the considered manifolds might or might not be orientable.

Examples 3.2. Inhalt...

4 Functoriality Property

We will now explore how homotopy behaves under continuous functions. We will begin with the following definition and lemma from Munkres' book [3, lemma 12.1, p. 52]:

Definition 4.1. Let $f : K \rightarrow L$ be a simplicial map. If $v_0 \dots v_p$ is a simplex of K , then the points $f(v_0) \dots f(v_p)$ span a simplex of L . we define a homomorphism $f_{\#} : C_p(K) \rightarrow C_p(L)$ by defining it on oriented simplices as follows:

$$f_{\#}([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_0), \dots, f(v_p) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases}$$

The family $\{f_{\#}\}$ is called the chain map induced by the simplicial map f .

It might be helpful to consider the following diagram:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & C_p(K) & \xrightarrow{\partial} & C_{p-1}(K) & \xrightarrow{\partial} & C_{p-2}(K) & \xrightarrow{\partial} & \dots \\ & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & & \\ \dots & \xrightarrow{\partial} & C_p(L) & \xrightarrow{\partial} & C_{p-1}(L) & \xrightarrow{\partial} & C_{p-2}(L) & \xrightarrow{\partial} & \dots \end{array}$$

This of course raises the question whether this diagram commutes or not.

Lemma 4.2. *The homomorphism $f_{\#}$ commutes with ∂ ; therefore $f_{\#}$ induces a homomorphism $f_* : H_p(K) \rightarrow H_p(L)$*

Proof. Let τ denote the simplex of L that is spanned by $f(v_0), \dots, f(v_p)$. We need to check

$$\partial f_{\#}([v_0, \dots, v_p]) = f_{\#}(\partial[v_0, \dots, v_p]) \quad (1)$$

Case 1 $\dim \tau = p$: Clearly, this requires $f(v_0), \dots, f(v_p)$ to be distinct. Therefore the statement follows directly from the definitions.

Case 2 $\dim \tau \leq p - 2$: Here, the left side of (1) vanishes because $f(v_0), \dots, f(v_p)$ are not distinct. The right side also vanishes, since for every i , at least two of the points $f(v_0), \dots, \hat{f}(v_i), \dots, f(v_p)$ are the same.

Case 3 $\dim \tau = p - 1$: We will assume the following ordering of vertices:

$$f(v_0) = f(v_1), \quad \text{and} \quad f(v_1), \dots, f(v_p) \text{ are distinct.} \quad (2)$$

Similarly, to the previous case, the left side of (1) vanishes and the right side reads

$$[f(v_1)f(v_2), \dots, f(v_p)] - [f(v_0), f(v_2), \dots, f(v_p)].$$

By (2) this also vanishes.

To show the second part of the lemma, we need to understand how $f_{\#}$ acts on boundaries and cycles. Let c be an arbitrary cycle of K , i.e. $\partial c = 0$. Then by (1), we find $\partial f_{\#}(c) = f_{\#}(\partial c) = f_{\#}(0) = 0$.

Now, let b be an arbitrary boundary, i.e. there exists some d with $b = \partial d$. Then, $f_{\#}(b) = f_{\#}(\partial d) = \partial f_{\#}(d)$.

We have thereby shown that $f_{\#}$ maps boundaries (cycles) of K to boundaries (cycles) of L . \square

With this in mind we can now state the functional properties of these homomorphisms. We do this by following Munkres' book [3, Theorem 12.2, p. 53] once more.

Theorem 4.3 (functorial properties).

1. *Let $id : K \rightarrow K$ be the identity simplicial map. Then $id_* : H_p(K) \rightarrow H_p(K)$ is the identity homomorphism.*
2. *Let $f : K \rightarrow L$ and $g : L \rightarrow M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$; That is, the following diagram commutes:*

$$\begin{array}{ccc} H_p(K) & \xrightarrow{(g \circ f)_*} & H_p(M) \\ & \searrow f_* \quad \nearrow g_* & \\ & H_p(L) & \end{array}$$

Proof. Inhalt... \square

5 Homotopy Invariance of Simplicial Homology

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