Simplicial Homology

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1 Motivation

The main idea of simplicial homology can be summarized as follows:

We want to find a new homotopy invariant as a tool to distinguish topological spaces. With simplicial homology, we assign a succession of abelian Groups to simplicial complexes.

2 Simplicial Homology Groups

Definition 2.1. Let K be a simplicial complex. A p-chain on K is a function c from the set of oriented p-simplices of K to the an abelian Group G such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group (which is free abelian) is denoted $C_p(K;G)$ and is called the group of (oriented) p-chains of K with coefficients in G.

Definition 2.2. We defnie the boundary operator

$$\partial_p : C_p(K;G) \to C_{p-1}(K;G)$$

to be the homomorphism via its action on an elementary chain corresponding to an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array.

Lemma 2.3. *$(C_\bullet, \partial)$ is a chain complex, i.e.* $\partial^2 = 0$

Definition 2.4. We define

$$\boxed{H_p(K;G) := \ker\partial_p / \mathrm{Im}\partial_{p+1}}$$

to be the **p-th simplicia homology group of K**

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Examples 2.5. For the 2-sphere, triangulate the space, compute the boundary of all elementary chains and evaluate the resulting matrices to find:

$$H_0(\mathbb{S}^2) \cong \mathbb{Z} \quad H_1(\mathbb{S}^2) = 0, \quad H_2(\mathbb{S}^2) \cong \mathbb{Z}, \quad H_{n>2}(\mathbb{S}^2) = 0$$

For the Torus T and the Klein Bottle S it is more convenient to reduce the problem to chains that are carried by the boundary of the rectangle associated with the fundamental polygon. This yields

$$\begin{array}{rrr} H_0(S) \cong \mathbb{Z} & H_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}_2 & H_{n>1}(S) = 0 \\ H_0(S;\mathbb{Z}_2) \cong \mathbb{Z}_2 & H_1(S;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & H_{n>1}(S;\mathbb{Z}_2) = 0 \\ H_0(T) \cong \mathbb{Z} & H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} & H_{n>1}(T) = \mathbb{Z} \\ H_0(T;\mathbb{Z}_2) \cong \mathbb{Z}_2 & H_1(S;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & H_{n>1}(S;\mathbb{Z}_2) = 0 \end{array}$$

3 Functoriality Property

Theorem 3.1 (functorility I)**.** [1] *Simplicial Homology is a functor from the Category of simplicial complexes to abelian groups.*

Theorem 3.2 (functioriality II)**.** *Simplicial Homology is a functor from the Category of polyhedra to abelian groups.*

4 Homotopy Invariance of Simplical Homology

Theorem 4.1. *If two polyhedra K and L are homotopy equivalent, then their homology groups are isomorphic $H_n(X) \cong H_n(Y)$ in all dimensions.*

Aufgabe 4.2. Compute the Homology groups of $\mathbb{R}\mathrm{P}^2$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$.

Aufgabe 4.3. Prove that the only non-trivial simplicial homology groups of $\mathbb{S}^n$ are $H_0(\mathbb{S}) \cong H_n(\mathbb{S}) \cong \mathbb{Z}$.

Literatur

[1] M.A. Armstrong, *Basic Topology*, korrigierter Nachdruck des Originals von 1979, Undergraduate Texts in Mathematics, Springer, 1983.

[2] A. Hatcher. *Algebraic Topology*, Cambridge University Press, 2002. Online verfügbar unter http://www.math.cornell.edu/~hatcher/.

[3] J.R. Munkres. *Elements of algebraic topology*, Addison-Wesley, 1984.

[4] D.L. Ferrario and R.A. Piccinini. *Simlicial Structures in Topology*, CMS Books in Mathematics, Springer New York, 2010. ISBN:9781441972361

[1] The statement might not be verbatim from the lecture, but the statement is essentially the same.