Simplicial Homology

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Abstract. *Simplicial homology is a valuable tool for distinguishing polyhedra. We achieve that by assigning an abelian groups* $H_n(X;G)$ *to the underlying simplicial complex of the polyhedron* $|X|$*. We call these groups the Homology groups of* X *with coefficients in* G*, where* G *is an abelian Group. As we will see, simplicial homology is in fact a functor from the category of simplicial complexes to the category of abelian groups. In addition, we will see that Homology groups only depend on the homotopy type, i.e. two spaces that are homotopy equivalent have isomorphic Homology groups.*

Contents

1 Motivation

One of the main goals when studying (topological) spaces (or in this case simplicial complexes) is to determine whether two spaces are homeomorphic or at least homotopy equivalent (see Definition 5.3) or not. The first approach would be to check if both spaces are simply connected. A more advanced method checks if the fundamental groups π_1 coincide[1].

It turns out that this is a valuable tool for one and two dimensional simplicial complexes. But this method fails for complexes with higher dimensional simplices. For example $\pi_1(\mathbb{S}^n) = 0$ for all $n \geq 2$. It is therefore impossible to show that two spheres of dimension greater or equal 2 are only then homeomorphic if they are of the same dimension. It can be shown that the fundamental group actually only depends on the 2-skeleton of the complex [vgl. 1, p. 173].

Even though this method can be generalized to the study of higher homotopy groups π_n, this is often more difficult than necessary since the computation of those groups is far from trivial.

In the following we will explore the concept of simplicial homology groups and how it can be used to classify spaces. We will see that they allow us to tackle the problem of deciding whether two **polyhedra**[2] are homeomorphic. Note that isomorphic Homology classes do not automatically mean that the spaces are homeomorphic. In fact the torus T and $\mathbb{S}^1 \wedge \mathbb{S}^1$ have the same homology groups in every dimension but they are not homeomorphic ($\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ whereas $\pi_1(\mathbb{S}^1 \wedge \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z}$) .

It is possible to extend the concept of Homology to more general spaces than polyhedra using singular homology, however those will not be covered here. The interested reader might refer to the book of Munkres [4] or Hatcher [3] to learn more about singular homology groups.

While we considered all possible paths when we were talking about the fundamental group we will now only consider those that aren't the boundary of a portion of the surface. A great motivation for this approach is given by Armstrong [1, p. 173f]:

When we concern ourselves with the question how closed paths on the torus are different from those on the sphere, we can notice the following: Any Jordan curve (i.e. nonselfintersecting, closed path) on a sphere divides the surface into two „regions“ as depicted in Figure 1[3]. The same is not true for the torus. In Figure 2 we see that even tough there are Jordan curves on a torus that bound a „region“of the surface, not all Jordan curves have that property [see 1, p. 173f].

[1] The fundamental group $\pi_1(X)$ is the group of all closed paths with the group action being concatenation. If X is a simply connected space then $\pi_1(X) = 0$. Furthermore, the fundamental group is homotopy invariant (up to isomorphism), so homotopic spaces have isomorphic fundamental groups. A more precise formulation of this statement can be found in Hatchers book [3, Prop. 1.18, p. 37]:

If $\phi : X \to Y$ is a homotopy equivalence, then the induces homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.

[2] A polyhedron is the polytope of a simplicial complex.

[3] For a more precise statement see `https://en.wikipedia.org/wiki/Jordan_curve_theorem`

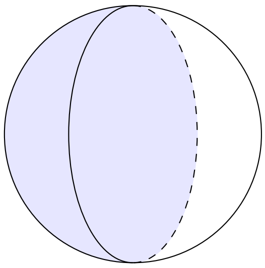


Figure 1: Any closed path on the sphere divides the surface into two regions.

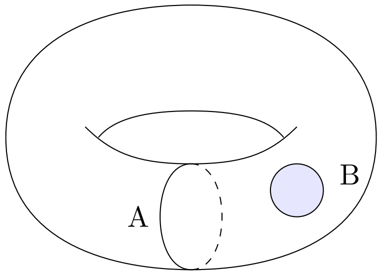


Figure 2: There are closed paths on the torus that do not divide the surface into two regions.

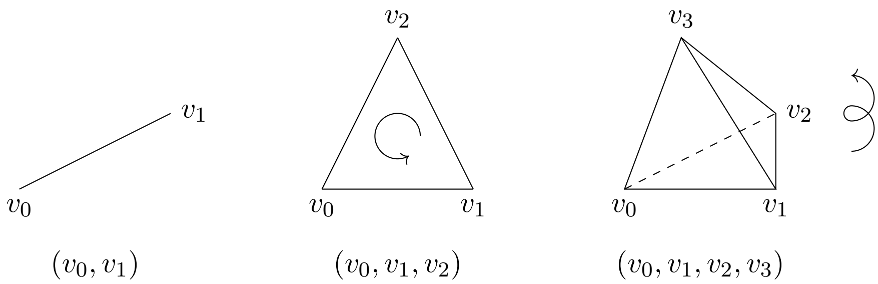


Figure 3: Indicating orientation with arrows [see 4, p.27]

2 Simplicial Homology Groups with Integer Coefficients

Before we get started, let us remind ourselves of some definitions and notations as can be found in Munkres' book [4, p. 26f.]

Definition 2.1. For a simplex σ we say that two orderings of its vertex set are equivalent if they differ by an even permutation. For a simplex with nonzero dimension, the orderings of the vertices fall into two equivalence classes, called the **orientations** of σ. An **oriented simplex** is a simplex together with an orientation.

Further we will use the same notation as Munkres [4, p. 26]

Notation 2.2. For geometrically independent points $v_0, \ldots, v_p$ we denote the simplex they span with

$$[v_0, \ldots, v_p].$$

For an oriented simplex we will use the notation

$$(v_0, \ldots, v_p).$$

where the orientation is given by this particular ordering.

A choice of orientation is usually depicted with arrows as can be seen in Figure 3

To define simplicial homology groups in any meaningful way, we need a few definitions that allow us to formulate our ideas form section 1 more precise. There are several ways to go about this. We will follow the path of Munkres [4, p. 27] here.

Definition 2.3. Let K be a simlicial complex. A ***p*-chain** on K is a function c from the set of oriented p-simplices of K to the integers such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group is denoted $C_p(K)$ and is called the group of (oriented) p-chains of K. If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group.

If σ is an oriented simplex, the elementary chain c_σ corresponding to σ is the function defined as follows:

$$\begin{aligned} c_\sigma(\sigma) &= 1, & \\ c_\sigma(\sigma') &= -1 & \text{if } \sigma' \text{ is the opposite orientation of } \sigma \\ c_\sigma(\tau) &= 0 & \text{for all other oriented simplices } \tau. \end{aligned}$$

It is common practice to to abuse the notation here in the following way: If clear by context we use the symbol σ not only to denote the (oriented) simplex, but also the corresponding elementary chain.

We are now able to formulate the our first result as can be found in Munkres book [4, Lemma 5.1, p. 28]:

Lemma 2.4. *$C_p(K)$ is free abelian.*

Proof. A basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains as a basis. It is easy to see that each chain c in $C_p(K)$ can be expressed uniquely as a linear combination of the elementary chains c_{σ_i} of the simplices of K, i.e.

$$c = \sum_i n_i c_{\sigma_i}.$$

Here, the the chain c assigns the value n_i to each simplex σ_i, $-n_i$ for the simplex σ_i with reversed orientation and 0 to every simplex that does not appear in the summation.[4] □

Some authors such as Armstrong [1, p.176f] or Ferrario [2, p.60] prefer to introduce chains as formal linear combinations of simplices. By virtue of the lemma above, it is clear that the definitions are analog.

[4]Recall that by definition of the elementary chains we have

$$c_{\sigma_i}(\tau) = \begin{cases} 1 & \text{if } \sigma_i = \tau \\ -1 & \text{if } \tau \text{ is the opposite orientation of } \sigma_i \\ 0 & \text{else} \end{cases}$$

Definition 2.5. We define the boundary operator

$$\partial_p : \quad C_p(K) \to C_{p-1}(K)$$

to be the homomorphism via its action on an elementary chain corresponding to an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array.

As the name suggests, the result of this homomorphism indeed refers to the (topological) boundary of the simplex once we interpret the sum of two elementary chains as the union of the corresponding simplices as we will see in several examples 2.6. But first we have to check well-definedness. For that, it is necessary to show that

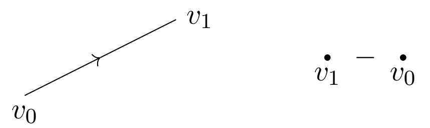
$$\partial_p(-\sigma) = -\partial_p(\sigma)$$

Since the orientation fo a simplex only depends on the *sign* of the permutation, it is sufficient to check for the simple case of exchanging v_0 and v_1:

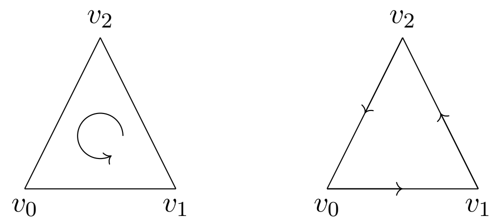
$$\begin{aligned} &\partial_p(v_0, \dots, v_p) + \partial_p(v_1, v_0, v_2, \dots, v_p) \\ =&(v_1, \dots, v_p) - (v_0, v_2, \dots, v_p) + \sum_{i=2}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) \\ +&(v_0, v_2, \dots, v_p) - (v_1, v_2, \dots, v_p) + \sum_{i=2}^{p} (-1)^i \underbrace{(v_1, v_0, v_2, \dots, \hat{v}_i, \dots, v_p)}_{=-(v_0, v_1, v_2, \dots, \hat{v}_i, \dots, v_p)} = 0 \end{aligned}$$

Examples 2.6.

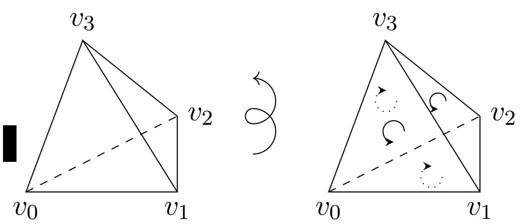
1. $\partial_1(v_0, v_1) = v_1 - v_0$



2. $\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$



3. $\begin{aligned} \partial_3(v_0, v_1, v_2, v_3) &= (v_1, v_2, v_3) - (v_0, v_2, v_3) \\ &+ (v_0, v_1, v_3) - (v_0, v_1, v_2) \end{aligned}$



Recall the following definition of a chain complex as can be found in Ferrario's book [2, p. 65]

Definition 2.7. A chain complex $(\mathcal{C}_\bullet, \partial')$ is a graded abelian group $\{\mathcal{C}_n\}$ together with an endomorphism $\partial' = \{\partial'_n\}$ of degree[5] -1 such that $(\partial')^2 = 0$.

Lemma 2.8. *$(C_\bullet, \partial)$ is a chain complex.*

Proof. We need to show that $\partial_{p-1} \circ \partial_p \equiv 0$. It is not hard to prove this, it just requires some bookkeeping:

Let $\sigma = (v_0, \dots, v_p)$ be an arbitrary p-chain, then

$$\begin{aligned}\partial_{p-1} \circ \partial_p(\sigma) &= \partial_{p-1}\left(\sum_{i=0}^{p}(-1)^i(v_0, \dots, \hat{v}_i, \dots, v_p)\right) = \sum_{i=0}^{p}(-1)^i \partial_{p-1}(v_0, \dots, \hat{v}_i, \dots, v_p) \\ &= \sum_{i=0}^{p}(-1)^i \sum_{\substack{j=0 \\ j \neq i}}^{p} (-1)^j (v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_p)\end{aligned}$$

Each of these term shows up twice with different signs. Thus, everything cancels.

□

A common question for chain complexes is exactness. As we will see later in this chapter, $(C_\bullet, \partial)$ will only be exact for very special cases.

Before we define the simplicial homology groups, we will introduce one more definition:

Definition 2.9.

1. The kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$ is called the group of p-cycles and denoted by $Z_p(K)$.
2. The immage of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is called the group of p-boundaries and is denoted $B_p(K)$.

The previous lemma states that $B_p(K) \subset Z_p(K)$.

Let us remind ourselves of what we wanted to do in section 1. We wanted to consider those closed curves (*cycles*) that were not the *boundary* of some part of the surface. With this in mind, we can now state the definition of simplical homology groups

Definition 2.10. We define

$$H_p(K) := Z_p(K)/B_p(K) = \ker \partial_p / \operatorname{Im} \partial_{p+1}$$

and call it the **p-th simplicial homology goup of K**.

Examples 2.11.

[5] A graded abelian group G is a succession $\{G_i \mid i \in \mathbb{Z}\}$. For graded abelian groups G and K, a morphism $g : G \to K$ is said to have degree d if $f(G_n) \subset K_{n+d}$

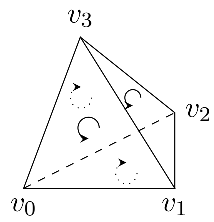


Figure 4: geometric realization of the hollow tetrahedron.

1. Lets start with a simple triangulation[6] of $\mathbb{S}^2$ like the hollow tetrahedron (depicted in Figure 4)

$$X = \langle v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3 \rangle.$$

It consists of four faces ($f_1 = v_0v_1v_2, f_2 = v_0v_1v_3,\ f_3 = v_0v_2v_3, f_4 = v_1v_2v_3$), six edges ($e_1 = v_0v_1, e_2 = v_0v_2, e_3 = v_0v_3,\ e_4 = v_1v_2,\ e_5 = v_1v_3,\ e_6 = v_2v_3$) and four vertices ($v_0, v_1, v_2, v_3$). Therefore, we find the simplicial chain complex to be

$$\mathbb{Z}\langle f_1, f_2, f_3 f_4 \rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v_0, v_1, v_2, v_3 \rangle \xrightarrow{\partial_0} 0$$

Computing the boundaries and writing the results in a more user friendly fashion gives

$$\begin{aligned} \partial f_1 &= v_1v_2 - v_0v_2 + v_0v_1 = e_1 - e_2 + e_4 \\ \partial f_2 &= v_1v_3 - v_0v_3 + v_0v_1 = e_1 - e_3 + e_5 \\ \partial f_3 &= v_2v_3 - v_0v_3 + v_0v_2 = e_2 - e_3 + e_6 \\ \partial f_4 &= v_2v_3 - v_1v_3 + v_1v_2 = e_4 - e_5 + e_6 \end{aligned} \qquad \partial_2 = \begin{array}{c c} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{aligned} &\partial e_1 = v_1 - v_0, \quad \partial e_2 = v_2 - v_0, \quad \partial e_3 = v_3 - v_0, \\ &\partial e_4 = v_2 - v_1, \quad \partial e_5 = v_3 - v_1, \quad \partial e_6 = v_3 - v_1 \end{aligned} \qquad \begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

We then find

$$\begin{aligned} \ker \partial_2 &= \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4 \rangle \\ \ker \partial_1 &= \mathbb{Z}\langle e_1 - e_2 + e_4, e_1 - e_3 + e_5, e_2 - e_3 + e_6 \rangle \end{aligned}$$

[6] A triangulation of a space T is a simplicial complex X whose geometric realization $|X|$ is homeomorphic to T.

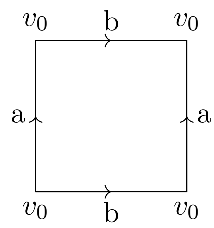


Figure 5: The fundamental polygon of the torus.

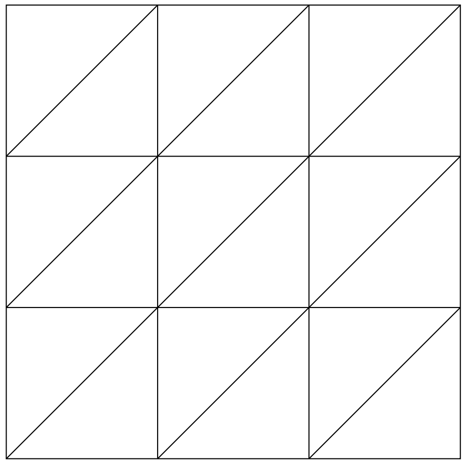


Figure 6: A simplicial triangulation of the torus.

Then we have

$$
\begin{aligned}
H_2(\mathbb{S}^2) &= \ker \partial_2 / \operatorname{Im} \partial_3 = \ker \partial_2 = \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4 \rangle \simeq \mathbb{Z} \\
H_1(\mathbb{S}^2) &= \ker \partial_1 / \operatorname{Im} \partial_2 \simeq \mathbb{Z}^3 / \mathbb{Z}^3 \simeq 0 \\
H_0(\mathbb{S}^2) &= \ker \partial_0 / \operatorname{Im} \partial_1 \simeq \mathbb{Z}^4 / \mathbb{Z}^3 \simeq \mathbb{Z} \\
H_n(\mathbb{S}^2) &= 0 \quad \text{for } n > 2
\end{aligned}
$$

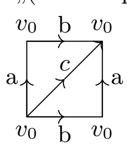
2. The Torus (see Figure 5) is a little more difficult. Unfortunately, the simplest (simplicial) triangulation is depicted in Figure 6. This triangulation would lead to boundary operators in the form of 27×18 and 8×27 matrices respectively. The resulting homology groups are:

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}^2, \quad H_2 = \mathbb{Z}, \quad H_{n>2} = 0$$

Even tough these aren't hard to compute, it is very tedious and therefore wont be presented here.[7]

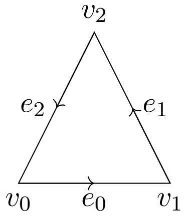
We see that the homology groups for the Sphere and the Torus do not agree in every dimension. Since homotopy equivalent spaces have isomorphic homology

[7] By introducing the concept of cellular homology, this can be simplified to 3×2 and 1×3 matrices via the „(non-simplicial) triangulation":



Cellular and simplicial homology actually coincide. However we currently lack the framework for cellular homology groups (we will show that later), we find that the sphere cannot be homotopy equivalent to the torus.

3. We will now consider the circle. This can be triangulated as an empty triangle, i.e.



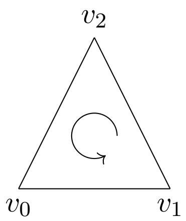
We easily find

$$\partial_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \partial_0 \equiv 0$$

Therefore,

$$H_1(\mathbb{S}^1) \simeq \mathbb{Z}/0 \simeq \mathbb{Z}, \quad H_0(\mathbb{S}) \simeq \mathbb{Z}^3/\mathbb{Z}^2 \simeq \mathbb{Z}$$

4. Now that we have considered the circle, we might also be interested in the 2-ball B^2. The triangulation is very easy this time. It is just one 2-dimensional simplex:



We find that ∂_1 and ∂_0 are the same as before and ∂_2 can be expressed as

$$\partial_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We then find

$$H_2(B^2) = 0, \quad H_1(B^2) = 0, \quad H_0(B^2) = \mathbb{Z}$$

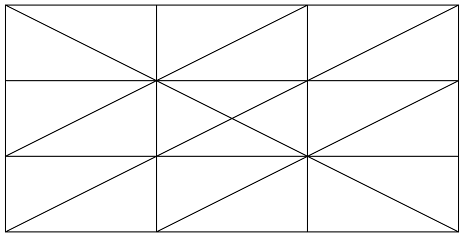


Figure 7: Complex whose underlying space is a rectangle.

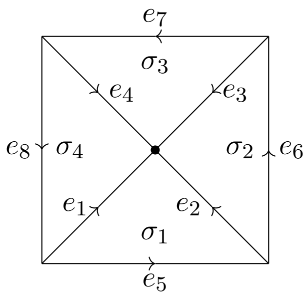


Figure 8: Structure of center. 2-simplices are oriented counterclockwise

Lemma 2.12. *Let L be the complex depicted in figure 7, whose underlying space is a rectangle. Let* BdL *denote the complex whose space is the boundary of the rectangle. Orient each 2-simplex σ_i of L counterclockwise. Orient the 1-simplices arbitrarily. Then:*

1. *Every 1-cycle of L is homologous to a 1-cycle carried by* BdL*.*
2. *If d is a 2-chain of L and if ∂d is carried by* BdL*, then d is a multiple of the chain $\sum \sigma_i$.*

Proof. We will prove the second statement fist. If σ_i and σ_j have an edge e in common, then ∂d must have value 0 on e. It follows that d must have the same value σ_i as it does in σ_j. Continuing this process, we see that d has the same value on every oriented 2-simplex σ_i.

For the second part, let c be be a 1-chain of L. We will „push it off " the 1-simplices in the following way:

We concentrate on the center at first. There we notice the structure as depicted in figure 8.

Let c denote a 1-chain with value a on e_1. Then c is clearly homologous to the chain $c_1 = c + \partial(a\sigma_1)$. The resulting 1-chain c_1 vanishes on e_1. We have „pushed it off" e_1. We can apply this multiple times for other edges until we arrive at a chain c_2 which is carried by the subcomplex depicted in figure 9

Since our original c is a cycle, c_2 must be carried by BdL, because otherwise ∂c_2 would have non-zero coefficients on one or more of the vertices $v_1, \ldots, v_5$. $\square$

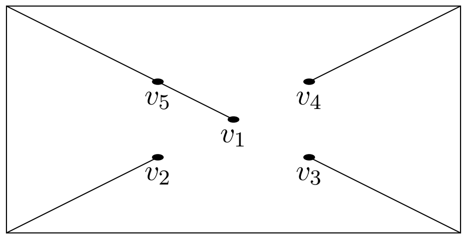


Figure 9: Subcomplex that carries c_2.

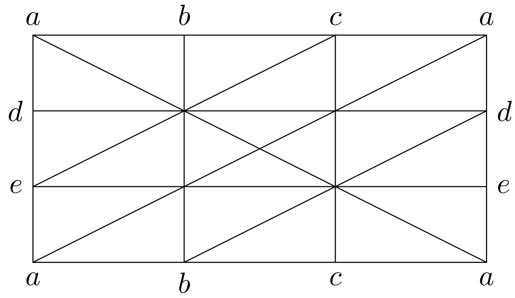


Figure 10: Labeled Complex whose underlying space is a torus.

Theorem 2.13. *Let T denote the complex represented by the labeled rectangle L of Figure 10. Its underlying space is the torus. Then:*

$$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \textit{and} \quad H_2(T) \cong \mathbb{Z}.$$

Orient each 2-simplex of L counterclockwise; use the induced orientation of the 2-simplices of T; Let γ denote their sum. Let

$$\begin{aligned} w_1 =& [a,b] + [b,c] + [c,a], \\ z_1 =& [a,d] + [d,e] + [e,a]. \end{aligned}$$

Then γ generates $H_2(T)$ and w_1 and z_1 represent a basis for $H_1(T)$.

Proof. Inhalt... □

Theorem 2.14. *let S denote the complex represented by the labelled rectangle of figure* **??***; its underlying space is the Klein bottle. Then*

$$H_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad \textit{and} \quad H_2(S) = 0.$$

The torsion element of $H_1(S)$ is represented by the chain z_1, and a generator for the group $H_1(S)$ modulo torsion is represented by w_1, where

$$\begin{aligned} w_1 &= [a,b] + [b,c] + [c,a] \\ z_1 &= [a,d] + [d,e] + [e,a]. \end{aligned}$$

3 Simplicial Homology Groups with $\mathbb{Z}/2\mathbb{Z}$ Coefficients

Let us remind ourselves of the definition for p-chains (Definition 2.3):

> Let K be a simlicial complex. A **p-chain** on K is a function c from the set of oriented p-simplices of K to the integers such that:
>
> 1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
> 2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

There is actually no reason (other than our familiarity with it) to restrict ourselves to the integers here. In fact, the same statements hold true for any abelian group G and it can be helpful to consider other groups than the integers. To denote the change of our underlying group, we use the notation

$$C_n(K; G) \quad \text{and} \quad H_n(K; G)$$

for the group of chains and the homology group of K with coefficients in G.

Since the group of chains with integer coefficients $C_n(K; \mathbb{Z}) = C_n(K)$ merely consists of linear combinations of elementary chains with integer coefficients, there is a neat way to derive $C_n(K; G)$ if $C_n(K; \mathbb{Z})$ is already known. We find that

$$C_n(K; G) \simeq C_n(K; \mathbb{Z}) \otimes_{\mathbb{Z}} G,$$

which leads to many interesting phenomena. The same holds for homology groups if the coefficient ring is a field.

A special case is the coefficient ring $\mathbb{Z}/2\mathbb{Z}$, since here $-1 = 1$. Hence, we no longer have to worry about orientation. Therefore, this ring is often used if the considered manifolds might or might not be orientable.

Examples 3.1. Inhalt...

4 Functoriality Property

We will now explore how homotopy behaves under continuous functions. We will begin with the following definition and lemma from Munkres' book [4, lemma 12.1, p. 62]:

Definition 4.1. Let $f : K \to L$ be a simplicial map. If $v_0 \dots v_p$ is a simplex of K, then the points $f(v_0) \dots f(v_p)$ span a simplex of L. we define a homomorphism $f_\sharp : C_p(K) \to C_p(L)$ by defining it on oriented simplices as follows:

$$f_\sharp([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_0), \dots, f(v_p) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases}$$

The family $\{f_\sharp\}$ is called the chain map induced by the simplicial map f.

This gives rise to the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & C_p(K) & \xrightarrow{\partial} & C_{p-1}(K) & \xrightarrow{\partial} & C_{p-2}(K) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \\
\cdots & \xrightarrow{\partial} & C_p(L) & \xrightarrow{\partial} & C_{p-1}(L) & \xrightarrow{\partial} & C_{p-2}(L) & \xrightarrow{\partial} & \cdots
\end{array}$$

This of course raises the question whether this diagram commutes or not.

Lemma 4.2. *The homomorphism $f_\sharp$ commutes with ∂; therefore $f_\sharp$ induces a homomorphism $f_* : H_p(K) \to H_p(L)$*

Proof. Let τ denote the simplex of L that is spanned by $f(v_0), \dots, f(v_p)$. We need to check

$$\partial f_\sharp([v_0, \dots, v_p]) = f_\sharp(\partial[v_0, \dots, v_p]) \tag{1}$$

Case 1 $\dim \tau = p$**:** Clearly, this requires $f(v_0), \dots, f(v_p)$ to be distinct. Therefore the statement follows directly from the definitions.

Case 2 $\dim \tau \leq p - 2$**:** Here, the left side of (1) vanishes because $f(v_0), \dots, f(v_p)$ are not distinct. The right side also vanishes, since for every i, at least two of the points $f(v_0), \dots, \hat{f}(v_i), \dots, f(v_p)$ are the same.

Case 3 $\dim \tau = p - 1$**:** We will assume the following ordering of vertices:

$$f(v_0) = f(v_1), \qquad \text{and} \qquad f(v_1), \dots, f(v_p) \text{ are distinct.} \tag{2}$$

Similarly, to the previous case, the left side of (1) vanishes and the right side reads

$$[f(v_1)f(v_2), \dots, f(v_p)] - [f(v_0), f(v_2), ..., f(v_p)].$$

By (2) this also vanishes.

To show the second part of the lemma, we need to understand how $f_\sharp$ acts on boundaries and cycles. Let c be an arbitrary cycle of K, i.e. $\partial c = 0$. Then by (1), we find $\partial f_\sharp(c) = f_\sharp(\partial c) = f_\sharp(0) = 0$.

Now, let b be an arbitrary boundary, i.e. there exists some d with $b = \partial d$. Then, $f_\sharp(b) = f_\sharp(\partial d) = \partial f_\sharp(d)$.

We have thereby shown that $f_\sharp$ maps boundaries (cycles) of K to boundaries (cycles) of L. $\square$

With this in mind we can now state the functional properties of these homomorphisms. We do this by following Munkres' book [4, Theorem 12.2, p. 53] once more.

Theorem 4.3 (functorial properties I)**.**

1. *Let $id : K \to K$ be the identity simplicial map. Then $id_* : H_p(K) \to H_p(K)$ is the identity homomorphism.*

2. *Let $f : K \to L$ and $g : L \to M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$; That is, the following diagram commutes:*

$$\begin{tikzcd}
H_p(K) \arrow[rr, "(g\circ f)_*"] \arrow[dr, "f_*"] & & H_p(M) \\
& H_p(L) \arrow[ur, "g_*"] &
\end{tikzcd}$$

Proof. The first part of the Theorem is pretty obvious. The second part follows immediately form $(g_\sharp \circ f_\sharp) = (g \circ f)_\sharp$. To see this, we shall consider vertices $v_0, \ldots, v_p$ that span a simplex in K. Then by definition

$$f_\sharp([v_0, \ldots, v_p]) = \begin{cases} [f(v_0), \ldots, f(v_p)], & \text{if } f(v_0), \ldots, f(v_p) \text{ are distinct,} \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} g_\sharp \circ f_\sharp([v_0, \ldots, v_p]) &= \begin{cases} [g \circ f(v_0), \ldots, g \circ f(v_p)], & \text{if } g \circ f(v_0), \ldots, g \circ f(v_p) \text{ are distinct,} \\ 0, & \text{if } g \circ f(v_0), \ldots, g \circ f(v_p) \text{ aren't distinct} \\ 0, & \text{if } f_\sharp([v_0, \ldots, v_p]) = 0 \end{cases} \\ &= \begin{cases} [g \circ f(v_0), \ldots, g \circ f(v_p)], & \text{if } g \circ f(v_0), \ldots, g \circ f(v_p) \text{ are distinct,} \\ 0, & \text{otherwise} \end{cases} \\ &= (g \circ f)_\sharp([v_0, \ldots, v_p]) \end{aligned}$$

□

Corollary 4.4. *Let $f : K \to L$ be a bijective simplicial map such that its inverse is also a simplicial map. Then*

$$\forall p \geq 0 : \quad H_p(K) = H_p(L)$$

□

This already gives the main Idea for the next section. Before we can get started with that we need some preliminary definitions and results:

Definition 4.5. Let $f, g : K \to L$ be simplicial maps. Suppose that for each p, one has a homomorphism

$$D : C_p(K) \to C_{p+1}(L)$$

satisfying the equation

$$\partial D + D\partial = g_\sharp - f_\sharp.$$

Then D is said to be a chain homotopy between $f_\sharp$ and $g_\sharp$. Note that the diagramm

$$\begin{tikzcd}
 & C_{p+1}(L) \arrow[d, "\partial"] \\
C_p(K) \arrow[ur, "D_p"] \arrow[r, shift left, "(g_\sharp)_p"] \arrow[r, shift right, "(f_\sharp)_p"'] \arrow[d, "\partial"] & C_p(L) \\
C_{p-1}(K) \arrow[ur, "D_{p-1}"'] &
\end{tikzcd}$$

is not commutative.

Theorem 4.6. *If there is a chain homotopy between $f_\sharp$ and $g_\sharp$ then the induced homomorphisms f_* and g_* are equal.*

Proof. Let c be a p-cycle of K. Then

$$g_\sharp(c) - f_\sharp(c) = \partial Dz + D\partial z = \partial Dz + 0.$$

That means that $f_\sharp$ and $g_\sharp$ differ only by a boundary. Therefore they are in the same homology class, i.e. $f_*(c) = g_*(c)$. □

Definition 4.7. Given two simplicial maps $f, g : K \to L$, these maps are said to be contiguos if for each simplex $v_0 \dots v_p$ of K, the points

$$f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$$

span a simplex of L.

Theorem 4.8. *If $f, g : K \to L$ are contiguous simplicial maps, then there is a chain homotopy between $f_\sharp$ and $g_\sharp$.*

Proof. Let $\sigma = (v_0, \dots, v_p) \subset K$ and $L(\sigma) = (f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)) \subset L$. Then

1. $L(\sigma)$ is a nonempty, and $\tilde{H}_n(L(\sigma)) = 0$ for $n \geq 0$.
2. If s is a face of σ, then $L(s) \subset L(\sigma)$.
3. For each oriented simplex σ, the chains $f_\sharp(\sigma)$ and $g_\sharp(\sigma)$ are zero on every simplex not in $L(\sigma)$.

We will now construct the chain homotopy $D : C_p(K) \to C_{p+1}(L)$ by induction on p. For each σ, the chain $D\sigma$ will vanish on any simplex not contained in $L(\sigma)$.

Let $p = 0$; let v be a vertex of K. Because $f_\sharp$ and $g_\sharp$ preserve augmentation,

$$\epsilon(g_\sharp(v) - f_\sharp(v)) = 1 - 1 = 0.$$

Thus $g_\sharp(v) - f_\sharp(v)$ represents an element of the reduced homology group $\tilde{H}_0(L(v))$. Because this group vanishes, we can choose a 1-chain Dv of L carried by the subcomplex $L(v)$ such that

$$\partial(Dv) = g_\sharp(v) - f_\sharp(v)$$

Then $\partial Dv + D\partial v = \partial Dv + 0 = g_\sharp(v) - f_\sharp(v)$, as desired. Define D in this way for each vertex of K.

Now suppose D is defined in dimensions less than p, such that for each oriented simplex s of dimension less than p, the chain Ds is carried by $L(s)$, and such that

$$\partial Ds + D\partial s = g_\sharp(s) - f_\sharp(s)$$

Let σ be an oriented simplex of dimension p. We wish to define $D\sigma$ so that $\partial(D\sigma)$ equals the chain

$$c = g_\sharp(\sigma) - f_\sharp(\sigma) - D\partial\sigma$$

Note that c is a well-defined chain; $D\partial\sigma$ is defined because $\partial\sigma$ has dimension $p-1$. Furthermore, c is a cycle, for we compute

$$\begin{aligned}\partial c &= \partial g_\sharp(\sigma) - \partial f_\sharp(\sigma) - \partial D(\partial\sigma) \\ &= \partial g_\sharp(\sigma) - \partial f_\sharp(\sigma) - [g_\sharp(\partial\sigma) - f_\sharp(\partial\sigma) - D\partial(\partial\sigma)]\end{aligned}$$

applying the induction hypothesis to the $p-1$ chain $\partial\sigma$. Using the fact that $\partial \circ \partial = 0$, we see that $\partial c = 0$.

Finally, we note that c is carried by $L(\sigma)$, and since $\tilde{H}_p(L(\sigma)) = 0$, we can choose a $p+1$ chain $D\sigma$ carried by $L(\sigma)$ such that

$$\partial D\sigma = c = g_\sharp(\sigma) - f_\sharp(\sigma) - D\partial\sigma$$

We then define $D(-\sigma) = -D(\sigma)$. We repeat this process for each p-simplex σ of K; then we have the required chain homotopy D in dimension p. The theorem follows. $\square$

Let us remind ourselves of the general simplicial approximation theorem, as it can be found in Munkres' book [4, Thm. 16.5, p. 85]:

Theorem 4.9 (The general simplicial approximation theorem). *Let K and L be complexes; let $h : |K| \to |L|$ be a continuous map. There exists a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$.*

We will further need the following definition again for Munkres' book [4, p. 100]:

Definition 4.10. Let K and L be simplicial complexes; let $h : |K| \to |L|$ be a continuous map. Choose a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$ (such exists by virtue of the previous theorem). Let $\lambda : C_\bullet(K) \to C_\bullet(K')$ be the subdivision operator. We define the homomorphism induced by h

$$h_* : H_p(K) \to H_p(L)$$

via

$$h_* = f_* \circ \lambda_*.$$

Since this definition involves several choices, we need to make sure, that the resulting homomorphism isn't dependent on these choices. We do this in two steps. Firstly, notice that for some choice of a subdivision K' two simplicial approximations, then these are contiguous and therefore induce the same homomorphism $f_* \equiv g_*$ (see Theorems 4.6 and 4.8).

Let $\iota : K' \to K$ be a simplicial approximation to the identity map of K, then $\lambda_* = (g_*)^{-1}$, hence

$$h_* = f_* \circ (g_*)^{-1}$$

Secondly, we will now use this to show that h_* doesn't depend on the choice of subdivision K' either. Let K'' be another subdivision of K such that h has a simplicial approximation mapping K'' to L. We will see that the induced homomorphism h_* does not change if we use substitute K with K'' in its definition.

We will first explore the special case that $|K|$ has a simplicial approximation $k : K'' \to K'$

$$\begin{tikzcd} & & K \\ K'' \arrow[r, "k"] & K' \arrow[ur, "g"] \arrow[dr, "f"] & \\ & & L \end{tikzcd}$$

Let $h*'$ be the induced homomorphism defined via K''. We can then write

$$h'_*(f \circ k)_* \circ (g \circ k)_*^{-1} = (f_* \circ k_*) \circ (g_* \circ k_*)^{-1} = f \circ g_*^{-1} = g_* \circ \lambda_* = h_*$$

We generalize this method by considering a subdivision K''' of K such that the identity map has simplicial approximations

$$k_1 : K''' \to K' \quad \text{and} \quad k_2 : K''' \to K''.$$

We can now use this subdivision to construct a homomorphism h''_*. By our previous work, we already know that

$$h'_* = h''_* = h_*$$

We are now able to generalize Theorem 4.3

Theorem 4.11 (functorial properties II)**.** *The identity map* $id : |K| \to |K|$ *induces the identity homomorphism* $i_* : H_p(K) \to H_p(K)$*. If* $h : |K| \to |L|$ *and* $k : |L| \to |M|$ *are continuous maps, then* $(k \circ h)_* = k_* \circ h_*$*.*

Proof. The first part is elementary. For the second part, choose simplicial approximations as follows:

$f_0 : L' \to M$	simplicial approximation to k
$g_0 : L' \to L$	simplicial approximation to the identity on $\lvert L\rvert$
$f_1 : K' \to L'$	simplicial approximation to h
$g_1 : K' \to K$	simplicial approximation to the identity on $\lvert K\rvert$

$$|K| \xrightarrow{h} |L| \xrightarrow{k} |M|$$

$$\begin{tikzcd} K & L & M \\ & L' \arrow[u, "g_0"] \arrow[ur, "f_0"] & \\ K' \arrow[uu, "g_1"] \arrow[ur, "f_1"] & & \end{tikzcd}$$

With these choices, $f_0 \circ f_1$ is a simplicial approximation to $k \circ h$, hence

$$(k \circ h)_* = (f_0 \circ f_1)_* \circ (g_1)_*^{-1}$$

Since $g_0 \circ f_1$ is a simplicial approximation to h, have

$$h_* = (g_0 \circ f_1)_* \circ (g_1)_*^{-1} \quad \text{and} \quad k_* = (f_0)_* \circ (g_0)_*^{-1}$$

Combining these results and applying Theorem 4.3 we obtain

$$(k \circ h)_* = k_* \circ h_*$$

as desired.

□

5 Homotopy Invariance of Simplicial Homology

We have arrived at what is probably the most important chapter of this paper. In section 1 we already mentioned that we can use (simplicial) homology groups to show that two spaces are not homotopic (or homeomorphic). However, so far we never explained why non-isomorphic homology groups imply that. We need to show that homology groups are topological invariants, i.e. that homotopic spaces have isomorphic homology groups.

We already saw a very special case in Corollary 4.4. As we will see shortly, the main idea is essentially the same for a more general statement

Corollary 5.1. *Let $f : |K| \to |L|$ be a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism.*

□

We have discovered the topological invariance of homology groups. However, there is an even stronger statement to be made here. As we will see in the following, homology groups are not only invariant under homeomorphisms, but only depend on the homotopy type. Note that we will only give a sketch of the necessary proof here. For the complete discussion the reader may refer to Munkers' book [4, §19].

Recall the definition of homotopy, taken from Munkres [4, p. 94]:

Definition 5.2. If X and Y are topological spaces, two continuous maps $h, k : X \to Y$ are said to be homotopic if there is a continuous map

$$F : X \times I \to Y$$

such that $F(x, 0) = h(x)$ and $F(x, 1) = k(x)$ for all $x \in X$. If h and k are homotopic, we write $h \simeq k$. The map F is called a homotopy of h to k.

Definition 5.3. Two spaces X and Y are said to be homotopy equivalent, or to have the same homotopy type, if there are maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

such that $g \circ f \simeq i_X$ and $f \circ g \simeq i_Y$. The maps f and g are often called homotopy equivalences, and g is said to be the homotopy inverse to f.

The main task is to prove the following statement:

Theorem 5.4. *If $f, g : |K| \to |L|$ are homotopic maps then $f_* = g_* : H_q(K) \to H_q(L)$ for all q*

Once we have obtained this result, the homotopy invariance follows easily [see 1, Theorem 8.11]:

Corollary 5.5. *If two polyhedra K and L are homotopy equivalent, then their homology groups are isomorphic $H_n(X) \cong H_n(Y)$ in all dimensions.*

Proof. Let f, g denote the maps as described in the definition of homotopy equivalence, i.e. let g be the homotopy inverse of f. Consider the following Diagram:

$$H_q(K) \xrightarrow{f_*} H_q(L) \xrightarrow{g_*} H_q(K)$$

$$H_q(L) \xrightarrow{g_*} H_q(K) \xrightarrow{f_*} H_q(L)$$

Then

$$f_* \circ g_* = (\underbrace{f \circ g}_{\simeq i_K})_* = (i_K)_* = (i_*)_{H_q(K)}$$

and

$$g_* \circ f_* = (\underbrace{g \circ f}_{\simeq i_L})_* = (i_L)_* = (i_*)_{H_q(L)}$$

Therefore, f_* and g_* are isomorphisms and $f_* = (g_*)^{-1}$. ☐

With the results form the previous chapter we can conclude that for any compact triangulable space, we can choose a triangulation $t : |K| \to X$ without loss of generality (i.e. it doesn't matter which triangulation we choose up to isomorphism) and use it to define the homology groups of X as $H_q(X) := H_q(K)$.

We will state the next two results without proof:

1. If $s, t : |K| \to |L|$ are „close" simplicial maps, in the sense that for each simplex A of K we can find a simiplex B in L such that both $s(A)$ and $t(A)$ are faces of B, then $s_* = t_* : H_q(K) \to H_q(L)$ for all q.

2. If $f, g : |K| \to |L|$ are homotopic maps we can find a barycentric subdivision K^m and a sequence of simplicial maps $s_1, \dots, s_n : |K^m| \to |L|$ such that s_1 simplicially approximates f, s, n simplicially approximates g, and each pair s_i, s_{i+1} are close in the sense above.

Proof of Theorem 5.4. Let λ_* be the homomorphism induced by the subdivision operator. With the results above, we find

$$f_* = s_{1*}\lambda_* = s_{2*}\lambda_* = \cdots = s_{n*}\lambda_* = g_*$$

□

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