Simplicial Homology

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Abstract goes here

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1 Motivation

One of the main goals when studying (topological) spaces (or in this case simplicial complexes) is to determine whether two spaces are homotopic or not. The first method that is often applied is to check if both spaces are simply connected. A more advanced approach checks not only simply connectedness but also if the fundamental groups π_1 coincide[1].

It turns out that this is a valuable tool for one and two dimensional simplicial complexes. But this method fails for complexes with cells in higher dimensions. It can be shown that the fundamental group actually only depends on the 2-skeleton of the complex [vgl. 1, p. 173].

Even though this method can be generalized to the study of higher homotopy groups π_n, this is often more difficult than necessary since the computation of those groups is far from trivial.

In the following we will explore the concept of simplicial homology groups and how it can be used to classify spaces. We will see that they allow us to tackle the problem of deciding whether two **polyhedra**[2] are homeomorphic, since simplicial homology is only defined on such spaces. However, the concept of simplicial homology can be generalized to singular homology, which can be applied to a broader category of spaces and coincides with simplicial homology if both are defined.

However, singular homology groups will not be covered here. The interested reader might refer to the book of Munkres [3] or Hatcher [2] to learn more about singular homology groups.

When we concern ourselfes with the question how closed paths on the torus are different from those on the sphere, we can notice the following:

Any Jordan curve (i.e. nonselfintersecting, closed path) on a sphere divides the surface into two „regions“ as depicted in Figure 1[3]. The same is not true for the torus. In Figure 2 we see that even tough there are Jordan curves on a torus that bound a „region“of the surface, not all Jordan curves have that property [p. 173f 1, see].

2 Simplicial Homology Groups with Integer Coefficients

We will use the following definition by Munkres [3, p. 26]

Definition 2.1. For a simplex σ we say that two orderings of its vertex set are equivalent if they differ by an even permutation. For a simplex with nonzero dimension, the

[1] If X is a simply connected space then $\pi_1(X) = 0$. Furthermore, the fundamental group is homotopy invariant (up to isomorphism), so homotopic spaces have isomorphic fundamental groups. A more precise formulation of this statement can be found in Hatchers book [2, Prop. 1.18, p. 37]:

If $\phi : X \to Y$ is a homotopy equivalence, then the induces homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.

[2] A polyhedron is the polytope of a simplicial complex.

[3] For a more precise statement see `https://en.wikipedia.org/wiki/Jordan_curve_theorem`

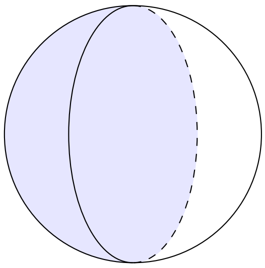


Figure 1: Any closed path on the sphere divides the surface into two regions.

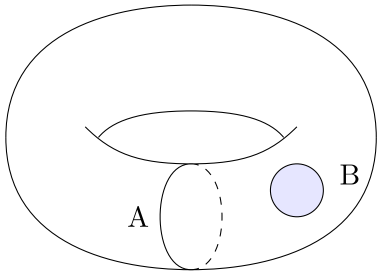


Figure 2: There are closed paths on the torus that do not divide the surface into two regions.

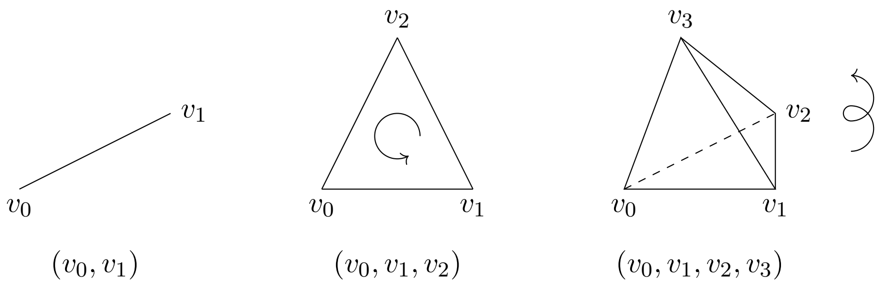


Figure 3: Indicating orientation with arrows [see 3, p.27]

orderings of the vertices fall into two equivalence classes, called the **orientations** of σ. An **oriented simplex** is a simplex together with an orientation.

Further we will use the same notation as Munkres [3, p. 26]

Notation 2.2. For geometrically independent points $v_0, \ldots, v_p$ we denote the simplex they span with

$$[v_0, \ldots, v_p].$$

For an oriented simplex we will use the notation

$$(v_0, \ldots, v_p).$$

where the orientation is given by this particular ordering.

A choice of orientation is usually depicted with arrows as can be seen in Figure 3

To define simplicial homology groups in any meaningful way, we need a few definitions that allow us to formulate our ideas form section 1 more precise. There are several ways to go about this with various amounts of rigor. In this paper, we will follow the path of Munkres [3, p. 27f].

Definition 2.3. Let K be a simlicial complex. A p**-chain** on K is a function c from the set of oriented p-simplices of K to the integers such that:

1. $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.

2. $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group is denoted $C_p(K)$ and is called the group of (oriented) p-chains of K. If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group.

If σ is an oriented simplex, the elementary chain c_σ corresponding to σ is the function defined as follows:

$$\begin{aligned} c_\sigma(\sigma) &= 1, & & \\ c_\sigma(\sigma') &= -1 & & \text{if } \sigma' \text{ is the opposite orientation of } \sigma \\ c_\sigma(\tau) &= 0 & & \text{for all other oriented simplices } \tau. \end{aligned}$$

It is common practice to to abuse the notation here. If clear by context we use the symbol σ not only to denote the (oriented) simplex, but also the corresponding elementary chain.

We are now able to formulate the our first result as can be found in Munkres book [3, Lemma 5.1, p. 28]:

Lemma 2.4. *$C_p(K)$ is free abelian; a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains as a basis.*

Proof. It is easy to see that each chain c in $C_p(K)$ can be expressed uniquely as a linear combination of the elementary chains c_{σ_i} of the simplices of K, i.e.

$$c = \sum_i n_i c_{\sigma_i}.$$

Here, the the chain c assigns the value n_i to each simplex σ_i, $-n_i$ for the simplex σ_i with reversed orientation and 0 to every simplex that does not appear in the summation.[4] □

Definition 2.5. We define the boundary operator

$$\partial_p : \quad C_p(K) \to C_{p-1}(K)$$

to be the homomorphism via its action on an oriented simplex $\sigma = (v_0, \dots, v_p)$:

$$\partial_p \sigma = \partial_p(v_0, \dots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array.

[4]Recall that by definition of the elementary chains we have

$$c_{\sigma_i}(\tau) = \begin{cases} 1 & \text{if } \sigma_i = \tau \\ -1 & \text{if } \tau \text{ is the opposite orientation of } \sigma_i \\ 0 & \text{else} \end{cases}$$

As the name suggests, the result of this homomorphism indeed refers to the (topological) boundary of the simplex once we interpret the sum of two elementary chains as the union of the corresponding simplices as we will see in the several examples, but first we have to check well-definedness. For that, it is necessary to show that

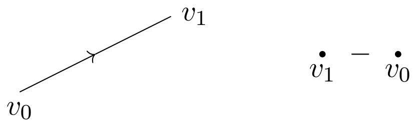
$$\partial_p(-\sigma) = -\partial_p(\sigma)$$

Since the orientation fo a simplex only depends on the *sign* of the permutation, it is sufficient to check for the simple case of exchanging v_0 and v_1:

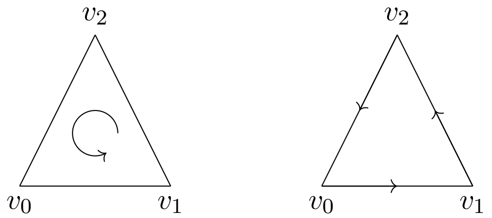
$$\begin{aligned}
&\partial_p(v_0, \dots, v_p) + \partial_p(v_1, v_0, v_2, \dots, v_p) \\
=&(v_1, \dots, v_p) - (v_0, v_2, \dots, v_p) + \sum_{i=2}^{p} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) \\
+&(v_0, v_2, \dots, v_p) - (v_1, v_2, \dots, v_p) + \sum_{i=2}^{p} (-1)^i \underbrace{(v_1, v_0, v_2, \dots, \hat{v}_i, \dots, v_p)}_{=-(v_0, v_1, v_2, \dots, \hat{v}_i, \dots, v_p)} = 0
\end{aligned}$$

Examples 2.6.

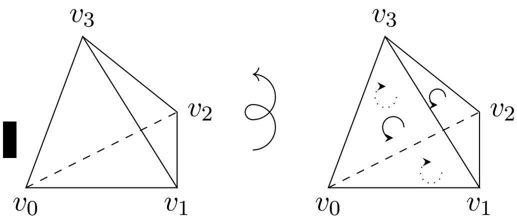
1. $\partial_1(v_0, v_1) = v_1 - v_0$



2. $\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$



3. $\partial_3(v_0, v_1, v_2, v_3) = (v_1, v_2, v_3) - (v_0, v_2, v_3) + (v_0, v_1, v_3) - (v_0, v_1, v_2)$



We will now consider the following diagram. (For the sake of readability, we deleted the dimension subscripts from the boundary operator)

$$C_\bullet : \qquad \cdots \xrightarrow{\partial} C_{p+1} \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots$$

Lemma 2.7. *$C_\bullet$ is a chain complex.*

Proof. We need to show that $\partial_{p-1} \circ \partial_p \equiv 0$. It is not hard to prove this, it just requires a lot of bookkeeping:

Let $\sigma = (v_0, \dots, v_p)$ be an arbitrary p-chain, then

$$\partial_{p-1} \circ \partial_p(\sigma) = \partial_{p-1}\left(\sum_{i=0}^{p}(-1)^i(v_0, \dots, \hat{v}_i, \dots, v_p)\right) = \sum_{i=0}^{p}(-1)^i \partial_{p-1}(v_0, \dots, \hat{v}_i, \dots, v_p)$$
$$= \sum_{i=0}^{p}(-1)^i \sum_{\substack{j=0 \\ j \neq i}}^{p} (-1)^j (v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_p)$$

Each of these term shows up twice with different signs. Thus, everything cancels.

□

A common question for chain complexes is exactness. As we will see later in this chapter, they will only be exact for special cases.

Before we define the simplicial homology groups, we will introduce one more definition:

Definition 2.8.

1. The kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$ is called the group of p-cycles and denoted by $Z_p(K)$.
2. The immage of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ is called the group of p-boundaries and is denoted $B_p(K)$.

The previous lemma states that $B_p(K) \subset Z_p(K)$.

Let us remind ourselves of what we wanted to do in section 1. We wanted to consider those curves (*cycles*) that were not the *boundary* of some part of the surface. With this in mind, we can now state the definition of simplical homology groups

Definition 2.9. We define

$$H_p(K) := Z_p(K)/B_p(K) = \ker \partial_p / \operatorname{Im} \partial_{p+1}$$

and call it the **p-th simplicial homology goup of K**.

Examples 2.10.

1. Lets start with a simple triangulation[5] of $\mathbb{S}^2$ like the hollow tetrahedron (depicted in Figure 4)
$$X = \langle v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3 \rangle.$$

   It consists of 4 faces ($f_1 = v_0v_1v_2, f_2 = v_0v_1v_3,\ f_3 = v_0v_2v_3, f_4 = v_1v_2v_3$), 6 edges ($e_1 = v_0v_1, e_2 = v_0v_2, e_3 = v_0v_3,\ e_4 = v_1v_2,\ e_5 = v_1v_3,\ e_6 = v_2v_3$) and

[5] A triangulation of a space T is a simplicial complex X whose geometric realization $|X|$ is homeomorphic to T.

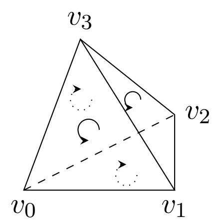


Figure 4: geometric realization of the hollow tetrahedron.

4 vertices (v_0, v_1, v_2, v_3). Therefore, we find the simplicial chain complex to be $\mathbb{Z}\langle f_1, f_2, f_3 f_4\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1, e_2, e_3, e_4, e_5, e_6\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v_0, v_1, v_2, v_3\rangle \xrightarrow{\partial_0} 0$

Computing the boundaries and writing the results in a more user friendly fashion gives

$$\begin{aligned}
\partial f_1 &= v_1v_2 - v_0v_2 + v_0v_1 = e_1 - e_2 + e_4\\
\partial f_2 &= v_1v_3 - v_0v_3 + v_0v_1 = e_1 - e_3 + e_5\\
\partial f_3 &= v_2v_3 - v_0v_3 + v_0v_2 = e_2 - e_3 + e_6\\
\partial f_4 &= v_2v_3 - v_1v_3 + v_1v_2 = e_4 - e_5 + e_6
\end{aligned}
\qquad
\partial_2 = \begin{array}{c} \\ e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6 \end{array}
\begin{array}{c}
\begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array}\\
\begin{bmatrix}
1 & 1 & 0 & 0\\
-1 & 0 & 1 & 0\\
0 & -1 & -1 & 0\\
1 & 0 & 0 & 1\\
0 & 1 & 0 & -1\\
0 & 0 & 1 & 1
\end{bmatrix}
\end{array}$$

$$\begin{aligned}
&\partial e_1 = v_1 - v_0, \quad \partial e_2 = v_2 - v_0, \quad \partial e_3 = v_3 - v_0,\\
&\partial e_4 = v_2 - v_1, \quad \partial e_5 = v_3 - v_1, \quad \partial e_6 = v_3 - v_1
\end{aligned}
\qquad
\begin{array}{c} \\ v_0\\ v_1\\ v_2\\ v_3 \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array}\\
\begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & -1 & 0\\
0 & 1 & 0 & 1 & 0 & -1\\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}$$

We then find

$$\begin{aligned}
\ker \partial_2 &= \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4\rangle\\
\ker \partial_1 &= \mathbb{Z}\langle e_1 - e_2 + e_4, e_1 - e_3 + e_5, e_2 - e_3 + e_6\rangle
\end{aligned}$$

Then we have

$$\begin{aligned}
H_2(\mathbb{S}^2) &= \ker \partial_2/\operatorname{Im} \partial_3 = \ker \partial_2 = \mathbb{Z}\langle f_1 - f_2 + f_3 - f_4\rangle \simeq \mathbb{Z}\\
H_1(\mathbb{S}^2) &= \ker \partial_1/\operatorname{Im} \partial_2 \simeq \mathbb{Z}^3/\mathbb{Z}^3 \simeq 0\\
H_0(\mathbb{S}^2) &= \ker \partial_0/\operatorname{Im} \partial_1 \simeq \mathbb{Z}^4/\mathbb{Z}^4 \simeq 0\\
H_n(\mathbb{S}^2) &= 0 \quad \text{for } n > 2
\end{aligned}$$

3 Simplicial Homology Groups with $\mathbb{Z}/2\mathbb{Z}$ Coefficients

4 Functoriality Property

5 Homotopy Invariance of Simplicial Homology

References

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