Sequent Calculus Paper

Ismaiel Sabet 900221277
Jana Elfeky 900215017
Noor Emam 900222081

Formal and Mathematical Logic
Department of Mathematics and Acturial Science

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1 Abstract

Following our discussion of propositional logic in the course, we proceeded to explore proof systems. We declared our proof systems to be comprised of three features: A formal language, deduction rules, and axioms [Sin]. We then used a Hilbert-style proof system to syntactically prove formulas, and to show that certain features regarding the proof system hold, such as Soundness and Completeness. However, the Hilbert-style proof system is not the only proof system out there, and in this paper, we formally introduce Sequent Calculus as another viable proof system. We will begin by defining the fundamental building blocks of this proof system i.e. the formal language, followed by the axioms and rules of inferences that help us generate new formulas. We then conclude with proving metatheorems about the proof system and comparing it with the Hilbert-style, along with highlighting potential applications of such a system.

2 The Foundations of Sequent Calculus

In this paper, we will focus on the Sequent Calculus applied to propositional logic. Therefore, our formal language will be the language used by propositional logic as defined in the course, and we will keep the same definition of formulaic construction. Moreover, like with the Hilbert-style system we previously studied, we will be working on Sequent Calculus with propositional formulas belonging to the set $\mathcal{F}^*$, which is the set of all formulas whose connectives are among $\neg$ and $\rightarrow$.

We need to show the syntactic notion of how a formula φ can be derived from a set of propositional formulas Γ using derivations of the Sequent Calculus proof system we are exploring. A derivation in Sequent Calculus is defined to be a finite sequence of **sequents**. Now, it remains to define what a sequent is.

Definition 2.1. *A **sequent** is a syntactic object of the form $\varphi_1, \varphi_2, ..., \varphi_n \vdash_{SQ} \psi_1, \psi_2, ..., \psi_m$, for every φ_i, where $1 \leq i \leq n$, and every ψj, where $1 \leq j \leq m$, is defined to be a propositional formula. The formulas on the left of the $\vdash$ symbol are called the antecedent and the formulas on the right of it are called the succedent [Pro24a]. We can think of the antecedent as a set Γ and the succedent as a set Δ for ease.*

Semantically, sequents express that whenever $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n$ is true, then $\psi_1 \vee \psi_2 \vee ... \vee \psi_m$ is true, for the antecedent $\varphi_1, \varphi_2, ..., \varphi_n$ and the succedent $\psi_1, \psi_2, ..., \psi_m$. Informally, this means that for a sequent $\Gamma \vdash \Delta$, whenever each formula in Γ is true, at least one of the formulas in Δ must be true[Pro24a].

The last sequent in a derivation in SQ (Sequent Calculus) is in the form $\Gamma \vdash_{\mathrm{SQ}} \Delta$, where Δ is the set including the formula we were initially deriving, similar to derivations in the Hilbert-style proof system.

Each step of the derivation can be one of the following:

(i) a sequent in the form $\Gamma \vdash_{\mathrm{SQ}} \Delta$, such that $\Delta \subseteq \Gamma$, which we will call assumptions;
(ii) a sequent obtained by applying any of the rules of inference of the system to sequents that were already derived.

Definition 2.2. *A theorem in Sequent Calculus is defined to be the antecedent, Δ, where $\vdash_{SQ} \Delta$ is the conclusion of a valid derivation[Pro24a].*

Now, we must introduce the rules of inference of Sequent Calculus.

As for axioms, there are none, which makes it very different from the Hilbert-style system in this sense.

Now, for the rules of inference, we have the following applications of **Backwards Deduction** , which is a technique that allows us to take any sequent and reduces it to sequent(s) containing no logical connectives.
The way the rules are written mean that the sequent at the bottom of the bar is reduced to the sequent(s) at the top, which ultimately means that the above is true if and only if the bottom is true. Also, the first character to the left of the rule indicates the applied logical connective found and the next indicates to which side it was found:

(¬l)

$$\frac{\Gamma \vdash_{\mathrm{SQ}} \varphi, \Delta}{\Gamma, \neg\varphi \vdash_{\mathrm{SQ}} \Delta}$$

(¬r)

$$\frac{\Gamma, \varphi \vdash_{\mathrm{SQ}} \Delta}{\Gamma \vdash_{\mathrm{SQ}} \neg\varphi, \Delta}$$

(→l)

$$\frac{\Gamma \vdash_{\mathrm{SQ}} \varphi, \Delta \quad \Gamma, \psi \vdash_{\mathrm{SQ}} \Delta}{\Gamma, (\varphi \rightarrow \psi) \vdash_{\mathrm{SQ}} \Delta}$$

(→r)

$$\frac{\varphi, \Gamma \vdash_{\mathrm{SQ}} \psi, \Delta}{\Gamma \vdash_{\mathrm{SQ}} (\varphi \rightarrow \psi), \Delta}$$

[Pro24a]

Note that: The ($\rightarrow$r) rule can also be called the **Deduction Theorem** [Koe].

Using these inference rules, we can derive two other rules which we will take for granted, as their derivations are very long.

These rules are as follows:
1. **Modus Ponens:** from $\Delta \vdash_{\mathrm{SQ}} \psi$ and $\Delta' \vdash_{\mathrm{SQ}} (\psi \rightarrow \chi)$, we have $\Delta, \Delta' \vdash_{\mathrm{SQ}} \chi$ [Koe].

2. **Proof by Contradiction:** from $\Delta, \neg\psi \vdash_{\mathrm{SQ}} \chi$ and $\Delta', \neg\psi \vdash_{\mathrm{SQ}} \neg\chi$, we have $\Delta, \Delta' \vdash_{\mathrm{SQ}} \psi$. [Koe]

Below are examples of derivations in the Sequent Calculus proof system:

Show that: $\vdash_{\mathrm{SQ}} (\phi \rightarrow (\psi \rightarrow \phi))$

1.	$\phi, \psi \vdash_{\mathrm{SQ}} \phi$	(Assumption)
2.	$\phi \vdash_{\mathrm{SQ}} (\psi \rightarrow \phi)$	($\rightarrow$r)
3.	$\vdash_{\mathrm{SQ}} (\phi \rightarrow (\psi \rightarrow \phi))$	($\rightarrow$r)

Show that: $\vdash_{\mathrm{SQ}} ((\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi))$

1.	$\neg\psi \vdash_{\mathrm{SQ}} \neg\psi$	(Assumption)
2.	$(\neg\psi \rightarrow \neg\varphi) \vdash_{\mathrm{SQ}} (\neg\psi \rightarrow \neg\varphi)$	(Assumption)
3.	$(\neg\psi \rightarrow \neg\varphi), \neg\psi \vdash_{\mathrm{SQ}} \neg\varphi$	(MP 1,2)
4.	$\varphi, \neg\psi \vdash_{\mathrm{SQ}} \varphi$	(Assumption)
5.	$(\neg\psi \rightarrow \neg\varphi), \varphi \vdash_{\mathrm{SQ}} \psi$	(PC 3,4)
6.	$(\neg\varphi \rightarrow \neg\psi) \vdash_{\mathrm{SQ}} (\varphi \rightarrow \psi)$	($\rightarrow$r)
7.	$\vdash_{\mathrm{SQ}} ((\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi))$	($\rightarrow$r)

[Koe]

Theorem 2.3. *(Soundness theorem for SQ). Let Γ be a set of propositional* *formulas and let ψ be any propositional formula. If Γ proves ψ, then Γ logically implies ψ. In symbols, if $\Gamma \vdash_{SQ} \psi$, then $\Gamma \models \psi$.*

Proof. We will prove this using strong induction on the lengths of the derivation. Let Γ, Δ be sets of propositional formulas, φ, ψ be any propositional formulas. First, we assume that $\Gamma \vdash_{\mathrm{SQ}} \varphi$, meaning there exists a derivation of φ from Γ, which we will denoted by the sequence of sequents $a_1, a_2, ...a_n$ where n is the length of the derivation.

For our base case, we will consider derivations of length 1. In Sequent Calculus, the only derivation we could have of length 1 is when $\varphi \in \Gamma$, i.e. it is an assumption. Take δ to be a truth assignment which satisfies Γ, meaning that $\delta[\psi] = 1$ for every $\psi \in \Gamma$. Therefore, given that δ satisfies Γ, we know that $\delta[\varphi] = 1$, meaning that $\Gamma \models \varphi$.

Now, we assume that if $\Gamma \vdash_{\mathrm{SQ}} \varphi$, then $\Gamma \models \varphi$ for derivations of lengths m, where $1 \leq m \leq n$. We want to show that if $\Gamma \vdash_{\mathrm{SQ}} \psi$, then $\Gamma \models \varphi$ for derivations of lengths $n+1$. We know that the sequent at the end of any derivation of this length is either an assumption or inferred by one of the inference rules.

If the sequent is an assumption, then by the same logic as the base case, we are done. Otherwise, it is inferred by the rules we established. Now, we will show that all applications of the rules of inference will satisfy soundness.

The first case is when the last sequent is inferred from $(\neg \mathrm{l})$: then the premise of the last sequent is in the form $\Gamma \vdash_{\mathrm{SQ}} \varphi, \Delta$ and the conclusion is $\Gamma, \neg\varphi \vdash_{\mathrm{SQ}} \Delta$ i.e., the derivation ends in $\Gamma, \neg\varphi \vdash_{\mathrm{SQ}} \Delta$.
The induction hypothesis tells us that $\Gamma \models \varphi, \Delta$ as the derivation of $\Gamma \vdash_{\mathrm{SQ}} \varphi, \Delta$ is of length at most n. Since we know that $\Gamma \models \varphi, \Delta$, then for every truth assignment δ, either (a) for some $\psi \in \Gamma$, $\delta[\psi] = 0$, so in this case Γ is not satisfied, or (b) for some $\chi \in \Delta$, $\delta[\chi] = 1$, meaning that Δ is satified, or (c) $\delta[\varphi] = 1$. We want to show that $\Gamma, \neg\varphi \models \Delta$. Let δ be a truth assignment. If (a) holds, then there is $\psi \in \Gamma$ so that $\delta[\psi] = 0$, meaning that Γ is not satisfied and so $\Gamma, \neg\varphi \models \Delta$ because Γ is never satisfied. If (b) holds, then for some $\chi \in \Delta$ we have that $\delta[\chi] = 1$, meaning that the conclusion of our logical implication is satisfied, so $\Gamma, \neg\varphi \models \Delta$ because if our conclusion is

satisfied, then our logical implication is always satisfied. Finally, if $\delta[\varphi] = 1$, then $\delta[\neg\varphi] = 0$, meaning that $\neg\varphi$ is not satisfied and so $\Gamma, \neg\varphi \models \Delta$ because $\neg\varphi$ is never satisfied. In all cases, $\Gamma, \neg\varphi \models \Delta$ is satisfied, which shows that the inference rule $\neg$l ensures soundness.

Similarly, we prove the same for when the last sequent is inferred by $(\neg \mathrm{r})$.

Another case is when the last sequent is inferred from $(\rightarrow \mathrm{r})$: then the premise of the last sequent is in the form $\varphi, \Gamma \vdash_{\mathrm{SQ}} \psi, \Delta$ and the conclusion is $\Gamma \vdash_{\mathrm{SQ}} (\varphi \rightarrow \psi), \Delta$ i.e., the derivation ends in $\Gamma \vdash_{\mathrm{SQ}} (\varphi \rightarrow \psi), \Delta$.
The induction hypothesis says that $\varphi, \Gamma \models \psi, \Delta$ as the derivation of $\varphi, \Gamma \vdash_{\mathrm{SQ}} \psi, \Delta$ is of length at most n. Since this is the case, at least one of the following cases occurs for any truth assignment δ: (a) $\delta[\varphi] = 0$, (b) $\delta[\psi] = 1$, (c) $\delta[\chi] = 0$ for some $\chi \in \Gamma$, or (d) $\delta[\sigma] = 1$ for some $\sigma \in \Delta$.
In cases (a) and (b), $\delta[(\varphi \rightarrow \psi)] = 1$ and so $(\varphi \rightarrow \psi)$ is satisfied, satisfying our logical implication $\Gamma \models (\varphi \rightarrow \psi), \Delta$. In case (c), for some $\chi \in \Gamma$, we have that $\delta[\chi] = 0$, so Γ is not satisfied, which satisfies our logical implication $\Gamma \models (\varphi \rightarrow \psi), \Delta$. In case (d), for some $\sigma \in \Delta$, we have that $\delta[\sigma] = 1$, so Δ is satisfied, which satisfies our logical implication $\Gamma \models (\varphi \rightarrow \psi), \Delta$.
In all cases, $\Gamma \models (\varphi \rightarrow \psi), \Delta$ is satisfied, which shows that the inference rule $(\rightarrow \mathrm{r})$ ensures soundness.

The last case is when the last sequent is inferred from $(\rightarrow \mathrm{l})$: then the premises of the last sequent are in the forms $\Gamma \vdash_{\mathrm{SQ}} \varphi, \Delta$ and $\Gamma, \psi \vdash_{\mathrm{SQ}} \Delta$ and the conclusion is $\Gamma, (\varphi \rightarrow \psi) \vdash_{\mathrm{SQ}} \Delta$ i.e., the derivation ends in $\Gamma, (\varphi \rightarrow \psi) \vdash_{\mathrm{SQ}} \Delta$.
The induction hypothesis says that $\Gamma \models \varphi, \Delta$ and $\Gamma, \psi \models \Delta$ as the derivations of $\Gamma \vdash_{\mathrm{SQ}} \varphi, \Delta$ and $\Gamma, \psi \vdash_{\mathrm{SQ}} \Delta$ are of lengths at most n.
Since this is the case, at least one of the following cases occurs for any truth assignment δ: (a) $\delta[\chi] = 0$ for some $\chi \in \Gamma$, (b) $\delta[\varphi] = 1$, (c) $\delta[\psi] = 0$, (d) $\delta[\sigma] = 1$ for some $\sigma \in \Delta$.
In case (a), $\delta[\chi] = 0$ for some $\chi \in \Gamma$, meaning that Γ is not satisfied, which satisfies the logical implication $\Gamma, (\varphi \rightarrow \psi) \models \Delta$.
In cases (b) and (c), $\delta[(\varphi \rightarrow \psi)] = 0$, so $(\varphi \rightarrow \psi)$ is not satisfied, which satisfies the logical implication $\Gamma, (\varphi \rightarrow \psi) \models \Delta$.
In case (d), $\delta[\sigma] = 1$ for some $\sigma \in \Delta$, meaning that Δ is satisfied, which satisfies the logical implication $\Gamma, (\varphi \rightarrow \psi) \models \Delta$.

In all cases, $\Gamma, (\varphi \to \psi) \models \Delta$ is satisfied, which shows that the inference rule $(\to \mathrm{l})$ ensures soundness.

Therefore, using strong induction, we have shown that whenever $\Gamma \vdash_{\mathrm{SQ}} \varphi$, then $\Gamma \models \varphi$, thus proving that SQ is a sound system. [Koe] ∎

Definition 2.4. *A set of formulas Γ is inconsistent if there exists a set Γ_0, where $\Gamma_0 \subseteq \Gamma$, such that $\Gamma_0 \vdash_{SQ}$ i.e. the succedent is empty [Pro24a].*

Definition 2.5. *A set of formulas Γ is consistent if it is not inconsistent.*

Lemma 2.6. *Let Γ be a set of propositional formulas and φ be a propositional formula. Then the set $\Gamma \cup \{\neg\varphi\}$ is inconsistent if and only if $\Gamma \vdash_{SQ} \varphi$.*

Proof. For the forward direction. Assume that the set $\Gamma \cup \{\neg\varphi\}$ is inconsistent. This means that $\Gamma, \neg\varphi \vdash_{\mathrm{SQ}}$. By the $(\neg r)$ rule, we know that $\Gamma \vdash_{\mathrm{SQ}} \neg\neg\varphi$, and by double negation, we have that $\Gamma \vdash_{\mathrm{SQ}} \varphi$. So we showed that $\Gamma \vdash_{\mathrm{SQ}} \varphi$ if $\Gamma \cup \{\neg\varphi\}$ is an inconsistent set.

For the backward direction, assume $\Gamma \vdash_{\mathrm{SQ}} \varphi$. Then by the rule $(\neg l)$, we know that $\Gamma, \neg\varphi \vdash_{\mathrm{SQ}}$. Therefore, we showed that the set $\Gamma, \neg\varphi$ is inconsistent.

Therefore, by showing both directions, we proved that the set $\Gamma \cup \{\neg\varphi\}$ is inconsistent if and only if $\Gamma \vdash_{\mathrm{SQ}} \varphi$. ∎

Corollary 2.7. *Let Γ be a set of propositional formulas and φ be a propositional formula. Then the set $\Gamma \cup \{\neg\varphi\}$ is consistent if and only if $\Gamma \nvdash_{SQ} \varphi$.*

Definition 2.8. *A set of propositional formulas Γ is complete if and only if $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for any $\varphi \in \mathcal{F}^*$ [Pro24b].*

Theorem 2.9. *Suppose that Γ is a consistent set of formulas. Then there exists a complete set of formulas Γ^* such that $\Gamma \subseteq \Gamma^*$.*

Proof. Let Γ be a consistent set. Since F^* is a countably infinite set, we can enumerate all its formulas as

$$F^* = \{\varphi_1, \varphi_2, \varphi_3, \ldots\}.$$

We proceed by constructing a chain of consistent sets of formulas

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \subseteq \Gamma_{n+1} \subseteq \cdots$$

as follows. We start by taking $\Gamma_0 = \Gamma$, and so Γ_0 is consistent. Then suppose the consistent set Γ_n has been constructed, we construct a new consistent set Γ_{n+1} as follows.:

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_{n+1}\} & \text{if } \Gamma_n \cup \{\varphi_{n+1}\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_{n+1}\} & \text{if } \Gamma_n \cup \{\varphi_{n+1}\} \text{ is inconsistent.} \end{cases}$$

Observe that each Γ_n is consistent: Γ_0 is consistent by definition. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_{n+1}\}$, this is because $\Gamma_n \cup \{\varphi_{n+1}\}$ is consistent. If it is not, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_{n+1}\}$. We have to prove that $\Gamma_n \cup \{\neg\varphi_{n+1}\}$ is consistent. For the sake of contradiction, assume it is inconsistent. This means that there exists a derivation such that $\Gamma_n, \neg\varphi_{n+1} \vdash_{\mathrm{SQ}}$. By applying $(\neg\mathrm{l})$, we get that $\Gamma_n \vdash_{\mathrm{SQ}} \varphi_{n+1}$, meaning that the set $\Gamma_n \cup \{\varphi_{n+1}\}$ is consistent. However, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_{n+1}\}$ is only the case when $\Gamma_n \cup \{\varphi_{n+1}\}$ is inconsistent, so we reach a contradiction by assuming that $\Gamma_n \cup \{\neg\varphi_{n+1}\}$ is inconsistent when $\Gamma_n \cup \{\varphi_{n+1}\}$ is inconsistent. Lastly, let

$$\Gamma^* = \bigcup_{n \in \mathbb{N}} \Gamma_n.$$

We claim that Γ^* is a complete set of formulas. For the sake of contradiction, assume that Γ^* is inconsistent. This means that there exists an inconsistent set Γ' such that $\Gamma' \subseteq \Gamma^*$. We also know that $\Gamma' \subseteq \Gamma_i$ for some $1 \leq i \leq n$, making Γ_i an inconsistent set. However, this contradicts our proof above that every Γ_n is consistent, therefore such a set Γ' cannot exist, meaning that the set Γ^* is consistent.
It remains to show that Γ^* is a complete set, meaning for any formula φ, either $\varphi \in \Gamma^*$, or $\neg\varphi \in \Gamma^*$. Since we know that all $\varphi_i \in \mathcal{F}^*$ were used in the construction of the set Γ^*, then we know any formula $\varphi \in \Gamma^*$. We also know that if $\varphi \notin \Gamma^*$, then that is because $\Gamma_n \cup \{\varphi\}$ was inconsistent. But in that case, we know that $\neg\varphi \in \Gamma^*$, so we can see that Γ^* is a complete, consistent set [Pro24b]. ■

Theorem 2.10. *If Γ is a complete set of propositional formulas, then Γ is satisfiable.*

Proof. Assume the set Γ is complete. Let the set of propositional variables be $P = \{p_0, p_1, p_2, \ldots\}$. As each propositional variable is a formula and as Γ is complete we get that either $p_i \in \Gamma$ or $\neg p_i \in \Gamma$, but not both, for every $i \in \mathbb{N}$. Let

$$\delta : F \to \{0, 1\}$$

be the truth assignment determined by setting:

$$\delta[p_i] = \begin{cases} 1 & \text{if } p_i \in \Gamma; \\ 0 & \text{if } \neg p_i \in \Gamma. \end{cases}$$

Claim. We claim that for any formula $\varphi \in F^*$ we have:

$$\delta[\varphi] = 1 \text{ if and only if } \varphi \in \Gamma.$$

We prove the claim by induction on formulas. By definition of the truth assignment δ we have that $\delta[p] = 1$ if and only if $p \in \Gamma$ for every propositional variable $p \in P$.

Now let φ, ψ be formulas in F^* for which the claim is satisfied. We have to show that the claim holds for $\neg\varphi$ and $(\varphi \to \psi)$.

For the first, we know that if $\delta[\neg\varphi] = 1$, then $\delta[\neg\varphi] = 0$, which means that $\varphi \notin \Gamma$ because of our induction hypothesis. However, since Γ is complete, we know that $\neg\varphi \in \Gamma$, successfully showing the claim holds for $\neg\varphi$.

It remains to show that the claim holds for the formula $(\varphi \to \psi)$.
If we have that $\delta[(\varphi \to \psi)] = 1$, then we know that $\delta[\varphi] = 0$ or $\delta[\psi] = 1$. Using our induction hypothesis, we know that $\varphi \notin \Gamma$ or $\psi \in \Gamma$. But since Γ is complete, then we know that $\neg\varphi \in \Gamma$ or $\psi \in \Gamma$, which implies that $(\varphi \to \psi) \in \Gamma$.

Therefore, by induction on formulas, we get that there exists a truth assignment δ constructed as above which satisfies any formula $\varphi \in \Gamma$, so Γ is satisfiable [Sin]. ■

Corollary 2.11. *If Γ is a consistent set of propositional formulas, then Γ is satisfiable.*

Proof. Let Γ be a consistent set of formulas. By Theorem 2.9, there exists a complete set of formulas Γ^* such that $\Gamma \subseteq \Gamma^*$. By Theorem 2.10, we get that Γ^* is satisfiable. Thus, there exists a truth assignment δ which satisfies every formula in Γ^*. Since $\Gamma \subseteq \Gamma^*$, the same truth assignment δ satisfies every formula in Γ, therefore, Γ is satisfiable. ■

Theorem 2.12. *(Completeness Theorem for SQ). Let Γ be any set of propositional formulas and let φ be any propositional formula. Then if Γ logically implies φ, then Γ proves φ. In symbols, if $\Gamma \models \varphi$, then $\Gamma \vdash_{SQ} \varphi$.*

Proof. We will aim to prove this theorem's contrapositive, so we will assume that $\Gamma \nvdash_{\mathrm{SQ}} \varphi$ and show that $\Gamma \nvDash \varphi$.
By Corollary 2.7, we know that $\Gamma \cup \{\neg\varphi\}$ is a consistent set if and only if $\Gamma \nvdash_{\mathrm{SQ}} \varphi$, so with our assumption, $\Gamma \cup \{\neg\varphi\}$ is consistent. We also showed that since $\Gamma \cup \{\neg\varphi\}$ is a consistent set, then $\Gamma \cup \{\neg\varphi\}$ is satisfiable by Collary 2.11. Since we know that $\Gamma \cup \{\neg\varphi\}$ is satisfiable, then there exists a truth assignment δ such that $\delta[\neg\varphi] = 1$, which means that $\delta[\varphi] = 0$, so δ satisfies Γ and does not satisfy φ, meaning that $\Gamma \nvDash \varphi$.
Therefore, we showed that if $\Gamma \nvdash_{\mathrm{SQ}} \varphi$, then $\Gamma \nvDash \varphi$. Taking the contrapositive of this, we get that if $\Gamma \models \varphi$, then $\Gamma \vdash_{\mathrm{SQ}} \varphi$, proving that SQ is complete [Pro24b]. ■

Now, we aim to prove that Sequent Calculus is equivalent to the Hilbert-Style system we covered in our lectures.

Definition 2.13. *Two proof systems are equivalent if and only if they prove the same set of formulas. This means that $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if $\Gamma \vdash_M \varphi$, where $\mathcal{S}$, M represent the two systems, Γ is a set of propositional formulas, and φ is a propositional formula.*

Theorem 2.14. *Sequent Calculus and Hilbert-Style systems are equivalent.*

Proof. For notation, we will use $\mathcal{S}$ to represent the Hilbert-Style system we covered in the lectures, and SQ to represent Sequent Calculus.
Assume we have any set of propositional formulas Γ and any $\varphi \in \mathcal{F}^*$. If $\Gamma \models \varphi$, then $\Gamma \vdash_{\mathcal{S}} \varphi$ because $\mathcal{S}$ is a complete system as proven in the lecture before. Also, if $\Gamma \models \varphi$, then $\Gamma \vdash_{\mathrm{SQ}} \varphi$ because SQ is a complete system as previously proven in this paper. Moreover, if $\Gamma \nvDash \varphi$, then because $\mathcal{S}$ is a sound system as proven in the lecture before, $\Gamma \nvdash \varphi$. Also, if $\Gamma \nvDash \varphi$, then because SQ is a sound system as as previously proven in this paper, $\Gamma \nvdash \varphi$.
Therefore, we've proven that $\mathcal{S}$ and SQ prove the same set of formulas, so they are equivalent proof systems. ■

Corollary 2.15. *Any two sound and complete proof systems which describe properties about formulas of the same logical language are equivalent. This can be proved using a similar proof to the one above.*

References

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