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Real Analysis I

The Hyperreals and Non-Standard Analysis

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CHAPTER 1

Introduction, History and Motivation

"That which, being added to another does not make it greater, and being taken away from another does not make it less, is nothing." - Zeno of Elea

We would like to say to Zeno, that he is wrong. In fact, and it is actually something; 'tis just infinitesimal, and it shall be the objective of this paper to convince you so.

1. Dipping our minds in the World of Infinitesimals

When you first get introduced to the world of mathematics as a young kid, from learning basic arithmetic to calculus, you believe that you gain a strong grip on the subject and its different laws to discover you know nothing at all once you try to define the things you believed you knew so well. While at the time of the Babylonians, accountants having an understanding of the concepts you are using was enough, after Greek philosophers entered the scene, they created the structure of mathematics that we follow today, which relies on having strong, comprehensive *definitions*, clear *proofs,* and concrete *theorems*.

We have discussed in class how the real numbers came to be, from the origins of their 'concept' to their much later formalization. Similarly, the concept of infinitesimals isn't as recent as people tend to believe. As with most mathematical theories, its origins begin with the Greeks; namely Pythagoras, and his concept of the *monad*. (Tropp, 3)

2. Pythagorus and Zeno

Returning to our beautiful journey of Hyperreals through the centuries, starting with Pythagoras and Zeno. After Pythagoras' students uncovered that not every number could be expressed as a rational number (using that the diagonal of a unit square is $\sqrt{2}$), (Tropp, 2) Pythagoras created the *monad*, regarded as the first infinitesimal, in order to handle the reality of the existence of the irrational numbers. The *monad* was aimed to be a small enough unit such that both the sides of the square and the diagonal could be measured in terms of it. It was an attempt to unify the rationals and irrationals under a certain unit. It is unclear whether this unit was designed and meant to be 'infinitely small.' (Tropp, 3) Zeno presented many objections, two of the most interesting ones were the Achilles and the Arrow, as they contradicted each other. The Achilles stated that Achilles is behind a tortoise and wishes to overtake it. In order to do so, he will need to reach the same point the tortoise is at, but once he does, the tortoise will no longer be at the part but will have moved forward some distance in the time it took him to reach the point the tortoise was at. Zeno, therefore, concludes that Achilles can never reach the tortoise. (Nick, 3.2) The Arrow paradox revolves around the concept that if time is made of indivisible ‘moments’ or ‘instants,’ then since the instance is indivisible, the position of the arrow must be the same at the ‘start’ and the ‘end’ of it. Therefore the arrow must be stationary during an ‘instant,’ and since the time is made of instants, the arrow cannot move at all as it is stationary in all of the ‘instants.’ (Nick, 3.3) Both paradoxes contradict each other, as the first shows that a distance cannot be infinitely divisible, and the other shows that it cannot be made of finite divisions. (Tropp, 4) This concluded that era’s trials and tribulations with infinitely small objects.

3. Exodus and Archimedes

Around half a century later, Archimedes, along with Exodus of Cnidus, wanted to be able to measure the areas and volumes of irregular geometric shapes. Exodus developed the *method of exhaustion*, which encloses the shape with straight lines and polygons that increase the number of sides until they completely enclose or ‘approximate’ the shape (the first rudimentary version of the concept of limit to infinity). (Tropp, 6) Nearly 200 years later, Archimedes introduced the concept of indivisibles, which was dividing the shape into smaller parts, *indivisibles*, to approximate the areas. However, it was not that simple, it relied on the use of parallel lines and comparing the irregular shapes with shapes that have a known area (as later developed by Caverli). (Jullien, 34) Archimedes also developed another concept similar to infinitesimals called *Laminae*, which he used in his theorem of determining area or volume by comparing to shapes that their areas or volumes are known, by placing them both on a level and determining where the fulcrum must be placed. (Tropp, 7) The method required that he imagine the various shapes as being composed of *laminae*, very small or thin strips, a similar concept to infinitesimals. (Tropp, 7)

4. Newton and Leibniz

Fast forward to the 17th century, during the time of Newton and Leibniz, their work with calculus and differentials gave another form to the elusive infinitesimal. In their discovery of calculus, Leibniz came to create the *differential*, while Newton came to create the *fluent*. Newton viewed the fluent as an element that changes continuously and has a rate of change (which is now known as the derivative). (Tropp, 8) While Newton wanted to use an infinitely small quantity, he became aware of the different problems it would pose and instead relied on using a ratio of the ‘infinitely small quantities’ as it was, at times, finite. Using this solution, he believed he did not need to use infinitesimals anymore, “I have sought to demonstrate that in the method of fluxions, it is not necessary to introduce into geometry infinitely small figures.” (Tropp, 8) On the other hand, Leibniz built his calculus with the differential, meant to be infinitely small and, similarly to Newton’s fluent, change continuously. He would say the infinitesimal was just an object that could be "as small as one wishes." (Tropp, 10) The Bernoulli brothers, who had worked so closely with Leibniz before, contributed to Leibniz's notion of the differential. Johann set a postulate that stated, "A quantity which is increased or decreased by an infinitely smaller quantity is neither increased nor decreased," further highlighting the concept of infinitely small and its intersection with the numbers we know. (Splat, 60) This era ended without any formal or rigorous conclusion of the existence of the concept of infinitesimals. Nonetheless, they proved their worthiness of being studied because of their possible applications in the essential field of calculus.

5. A Dark Era

One of the issues with the infinitesimals is that while there was a lack of rigor behind their existence, functionality, and properties, their results were often correct and eased calculations in a lot of cases. In the 19th century, a strong trend of formalization passed through mathematics that pushed mathematicians to prove theorems with rigor. At the time, no one was able to rigorously formalize infinitesimals, even though a lot of ground was set up with the formalization of limits. (Tropp, 10) Bolzano formally defined the derivative using the known famous equation that we all know from calculus:

$$\lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} \right)$$

and showed that it is a quantity, not a ratio, as theorized by Newton. (Tropp, 11) While he used limits in his work, he did not define them formally nor did he use them in a rigorous manner. Cauchy, on the other hand, did define the limit without relying on geometry and then tried to define the infinitesimal as any sequence with the limit zero. This definition, however, was not accepted. Additionally, Cauchy's definition of integrals using limits was later edited by Riemann to exclude the use of infinitesimals. (Tropp, 11) Moreover, near the end of the 19th century, Weierstrass had completely formalized the concept of limits, erasing any trace of infinitesimals within them. (Tropp, 11) They became forgotten as Dedekind and Cantor formally constructed the real numbers, which had no space for the infinitesimals anymore, and they were left like a ghost of an old urban legend. (Tropp, 12)

6. Light Prevails

While physicists and other scientists did not mind the lack of formalization behind the infinitesimals as they provided correct results, mathematicians like Hilbert valued formalization and believed that no theorem should be accepted without a sound, rigorous proof behind it. (Tropp, 12) So they stayed in the dark, waiting, until one fateful day when Robinson lit up the room and came to an astounding realization. It was that with the new mathematical discoveries of the last couple of decades and the new model of mathematical analysis, he could extend the real numbers to finally include the infinitesimals and their inverses, the unlimiteds. (Tropp, 12) He published the landmark book in 1966 named 'Non-Standard Analysis,' putting infinitesimals in the spotlight. A few years later, another model of the infinitesimals and hyperreals was developed, that used category theory to define this new realm. (Tropp, 13) This method also defined a new object called the *nil-square infinitesimal* ϵ where $\epsilon \neq 0$, but $\epsilon^2 = 0$. The use of category theory had some undeniable disadvantages. (Tropp, 13)

Overall, the addition of infinitesimals allowed proofs to seem more intuitive, along with many other advantages related to continuity and discrete models, infinite and finite combinatorics, along with certain mathematical problems. (Tropp, 14) There are three main ways with constructing the hyperreals, the first is the one that Robinson used which consists of creating a nonstandard extension of the real numbers using subsets. (Tropp, 15) Another approach is the axiomatic one.(Tropp, 16) The last one is using the ultrapower, which doesn't have the same generality as the first method, but makes that up in many other advantages. (Tropp, 17)

Now, we can end this beautiful journey through time with a quote from Gödel: "There are good reasons to believe that nonstandard analysis, in some version or other, will be the analysis of the future."

CHAPTER 2

Preliminaries

The method we are going to use for constructing the Hyperreals requires some hefty tools from both Model Theory and our previous knowledge in Set Theory, so we will be introducing these notions as we go throughout the paper. However, it is worth noting that we will not spend much time on proving all theorems' tools that we introduce in this section of the paper, so they will be properly cited when relevant as reference for the reader.

1. Ultraproducts of Sets

We start by utilizing some basic notions from (naïve) set theory, to introduce the forthcoming definitions.

Definition 1.1. (Ultra)Filter

We define a **filter $\mathcal{U}$** over a non-empty set I to be a subset of the power set of I, i.e. $\mathcal{U} \subseteq \mathcal{P}(I)$, that satisfies the following three conditions:

($\mathcal{U}$1) For any $X \in \mathcal{U}$, if there exists some $Y \subseteq I$ such that $X \subseteq Y$, then $Y \in \mathcal{U}$.
($\mathcal{U}$2) For any $X, Y \in \mathcal{U}$, we have that $X \cap Y \in \mathcal{U}$.
($\mathcal{U}$3) $I \in \mathcal{U}$, while $\emptyset \notin \mathcal{U}$.

We then say that $\mathcal{U}$ is an **ultrafilter** over I if we have that for every $X \subseteq I$, either $X \in \mathcal{U}$ or $X \setminus I \in \mathcal{U}$.

Remark. Note that in any filter over a set I we cannot have both a subset $X \subseteq I$ and its complement in the filter, as that would violate ($\mathcal{U}$2) as their intersection would be the empty set which cannot be in the filter by ($\mathcal{U}$3).

Examples.

- Take the set $I = \{1, 2, 3\}$. The following set forms an Ultrafilter over I:

$$\mathcal{U} = \{\{1, 2, 3\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

- Using the same set $I = \{1, 2, 3\}$, take $\mathcal{U} = \{\{1, 2, 3\}, \{1, 2\}\}$.
  We can see that $\mathcal{U}$ is a filter over I as it doesn't contain $\emptyset$, and is closed under supersets and finite intersections. However, it is not an ultrafilter since there exists $X = \{1, 3\} \subseteq I$ such that $X \notin \mathcal{U}$ and $X \backslash I = \{2\} \notin \mathcal{U}$.

- Take the set of real numbers $\mathbb{R}$, and fix some $x \in \mathbb{R}$.
  We define a filter over $\mathbb{R}$ to be $\mathcal{U} = \{A \subseteq \mathbb{R} : x \in A\}$.
  To verify it's a filter, we can see that for every $X \in \mathcal{U}$ and $Y \subseteq \mathbb{R}$, if $X \subseteq Y$, then $x \in Y$, and so $Y \in \mathcal{U}$. Also, since the intersection of any two sets in $\mathcal{U}$ must contain x, then the intersection is in $\mathcal{U}$, as well. Lastly, since $x \notin \emptyset$, then $\emptyset \notin \mathcal{U}$.
  To show it's an ultrafilter, we get that for every $X \subseteq \mathbb{R}$, if $x \in X$, then $X \in \mathcal{U}$ and if $x \notin X$, then $x \in X \backslash \mathbb{R}$, and so $X \backslash \mathbb{R} \in \mathcal{U}$. This type of ultrafilter is called a principle ultrafilter.
- Take the set of positive integers $\mathbb{Z}^+$, and define $\mathcal{U} = \mathcal{P}(2\mathbb{Z}^+)$, where $2\mathbb{Z}^+$ is the set of all positive even integers.
  Now, we claim that $\mathcal{U}$ is not a filter. To see this, take $X = \{2, 4\} \in \mathcal{U}$, and take $Y = \{1, 2, 4\} \subseteq \mathbb{Z}^+$. We know that $X \subseteq Y$, however, $Y \notin \mathcal{U}$, and so $\mathcal{U}$ violates ($\mathcal{U}1$), which means that $\mathcal{U}$ is not a filter.

Fact 1.2.

Given any non-empty set I, we have that any filter over I can be extended to an ultrafilter over I.

This fact can be easily proven by Zorn's Lemma, however, we will not be proving it. Nonetheless, we need to prove the some upcoming very important lemmata, but first, consider the following notions that we will frequently be using throughout this paper...

Definition. Proper Indexing Set

Whenever we say that I is a **Proper Indexing Set**, this means I is countable subset of $\mathbb{Z}^+$, where it is either of the form $\{1, 2, ..., n\}$ or the entire set of positive integers.

Definition. Semi-sequence

We define a **semi-sequence** α in a set A to be a mapping from a proper indexing set I into A. Similar to the notation of sequences, we will denote semi-sequences by $(\alpha_i)_{i \in I}$

We are aware that these definitions and notations might be a bit redundant for the reader, however, it will make things way easier for our discourse if our indexing sets were numeric and started from the number 1 as defined above. This is to avoid the idea of indexing tuples/sequences by other non-numeric values, e.g. $\alpha_{\text{elephant}}, \alpha_{\text{junior manager}}, \alpha_{\text{Bonzu Pippinpaddleopsicopolis (the Third)}}$ and so on...

We are also aware that one could just map elements from the set of all nonsensical things one can think of (modulo ZFC of course) to a 'proper indexing set' that makes sense to normal people, but we are not big fans of putting ourselves in situations where we have to micro-perform mappings when we can just fix the convention right from the beginning ;)

Definition 1.3. $\mathcal{U}$-Related

Given any ultrafilter $\mathcal{U}$ over an indexing set I, and a family of non-empty sets $\{A_i : i \in I\}$ we define two elements $\bar{a}_1, \bar{a}_2$ in the Cartesian product of all the sets in our family $\prod_{i \in I} A_i$ to be **$\mathcal{U}$-related** when $\{i \in I : \bar{a}_1(i) = \bar{a}_2(i)\} \in \mathcal{U}$. We then write $\mathbf{\bar{a}_1 \sim_{\mathcal{U}} \bar{a}_2}$.

Lemma 1.4.

Given an ultrafilter $\mathcal{U}$ over a set I, and an arbitrary family of non-empty sets indexed by I as such: $\{A_i : i \in I\}$, we have that the $\mathcal{U}$-relation on $\prod_{i \in I} A_i$ is an equivalence relation.

Proof. Take any $\bar{a}_1, \bar{a}_2, \bar{a}_3 \in \prod_{i \in I} A_i$ and consider the following arguments for showing that $\sim_{\mathcal{U}}$ is an equivalence relation.

Reflexivity: We want to prove that $\bar{a}_1 \sim_{\mathcal{U}} \bar{a}_1$. Since it is the same element, we have that all of its entries match, so that makes up the entirety of the set I which belongs in $\mathcal{U}$ by $(\mathcal{U}3)$, thus $\bar{a}_1 \sim_{\mathcal{U}} \bar{a}_1$ and we are done.

Symmetry: Assume that $\bar{a}_1 \sim_{\mathcal{U}} \bar{a}_2$. This means that the set of indices i, call it K, where $\bar{a}_1(i)$ and $\bar{a}_2(i)$ are equal belongs in $\mathcal{U}$, i.e. $K \in \mathcal{U}$. We can then clearly see that K is also the set of indices where $\bar{a}_2(i)$ and $\bar{a}_1(i)$ are equal, by symmetry of the equality relation. Thus $\bar{a}_2 \sim_{\mathcal{U}} \bar{a}_1$ and the relation is symmetric.

Transitivity: Assume that $\bar{a}_1 \sim_{\mathcal{U}} \bar{a}_2$ and $\bar{a}_2 \sim_{\mathcal{U}} \bar{a}_3$ that means that the set of indices i, call it K, where $\bar{a}_1$ and $\bar{a}_2$ match belongs in $\mathcal{U}$, i.e. $K \in \mathcal{U}$. Also, this means that the set of indices i, call it H, where $\bar{a}_2$ and $\bar{a}_3$ match belongs in $\mathcal{U}$, i.e. $H \in \mathcal{U}$. Since an ultrafilter is a filter first and foremost, then the intersection of any two sets in the filter is still in the filter. Thus $H \cap K \in U$, we will denote the intersection of the sets as G, such that $G = H \cap K$. The set G now contains the indices where $\bar{a}_1$ and $\bar{a}_2$ and $\bar{a}_3$ match. Additionally, since this ultrafilter is a filter, and that G is a subset of the set, let's denote it J, then $J \in U$. Now we must show which set J is such that G is its subset. We can define J as the set of indices where $\bar{a}_1$ and $\bar{a}_3$ match, since the set of indices where $\bar{a}_1$ and $\bar{a}_2$ and $\bar{a}_3$ match, G, would be considered a subset. Now since the set of indices where $\bar{a}_1$ and $\bar{a}_3$ match, J, belongs in U, then $\bar{a}_1 \sim_{\mathcal{U}} \bar{a}_3$. Thus the relation is transitive.

This concludes our proof. ∎

Definition 1.5. Ultraproduct

Given an ultrafilter $\mathcal{U}$ over a proper indexing set I and a family of non-empty sets $\{A_i : i \in I\}$, we define the **ultraproduct** $\prod_{\mathcal{U}} A_i$ to be the set of all equivalence classes of the $\mathcal{U}$-relation on $\prod_{i \in I} A_i$, or in other words; $\prod_{\mathcal{U}} A_i = \left\{[\bar{\alpha}]_{\mathcal{U}} : \bar{\alpha} \in \prod_{i \in I} A_i\right\}$.

Examples of Ultraproducts on sets include the following:

- Let $I = \{1, 2, 3\}$ and $\mathcal{U} = \{\{1, 2, 3\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.
  Now, take $A_1 = \{5, 6\}, A_2 = \{2, 1\}, A_3 = \{4, 6\}$, so we get
  that $\prod_{i \in I} A_i = \{(5, 2, 4), (5, 2, 6), (5, 1, 4), (5, 1, 6), (6, 2, 4), (6, 2, 6), (6, 1, 4), (6, 1, 6)\}$.
  We then compute the ultraproduct to be $\prod_{\mathcal{U}} A_i = \{[(5, 2, 4)]\}$.
- Let $I = \{1, 2, 3\}$ and $\mathcal{U} = \{\{1, 2, 3\}, \{2\}, \{1, 2\}, \{2, 3\}\}$.
  We can see that since the ultrafilter only contains subsets of I containing 2, then we know that any 2 triples will be $\mathcal{U}$ related if they agree on their second components.
  Now, take $A_1 = \{a, b\}, A_2 = \{x, y\}, A_3 = \{p, q\}$, so we get
  that $\prod_{i \in I} A_i = \{(a, x, p), (a, x, q), (a, y, p), (a, y, q), (b, x, p), (b, x, q), (b, y, p), (b, y, q)\}$.
  By the comment above, we can compute the ultraproduct to be
  $\prod_{\mathcal{U}} A_i = \{[(a, x, p)], [a, y, p]\}$.

2. Ultraproducts of First-Order Structures

We now move onto introducing the tools we will be borrowing from the logicians in the room, with all due consent, of course.

Definition 2.1. First-Order Language

A **First-Order Language** is a set of symbols of three kinds; constant symbols, function symbols, and relation symbols, where each function symbol and relation symbol have a positive integer associated with them called their *arity*.

Convention. *We typically denote constant symbols by c, function symbols by f, g or h and relation symbols by capital Latin letters such as R, H or E in our first-order languages.*

Definition 2.2. $\mathcal{L}$-Structure

Given a First-Order Language $\mathcal{L}$, we define an **$\mathcal{L}$-structure $\mathcal{M}$** to consist of the following:

- A **non-empty underlying set M**, that we call the **domain** of the structure.
- An element $c^{\mathcal{M}} \in M$ for each constant symbol $c \in \mathcal{L}$, that we call the **interpretation** of c in $\mathcal{M}$.
- A **closed n-ary operation** $f^{\mathcal{M}}$ on M, for every function symbol $f \in \mathcal{L}$ of arity n. Similar to constant symbols, we will refer to the such an operation as the interpretation of the function symbol f in $\mathcal{M}$.
- An **n-ary relation** $R^{\mathcal{M}} \subseteq M^n$, for every relation symbol $R \in \mathcal{L}$ of arity n. And once again, we will refer to the such an operation as the interpretation of the relation symbol R in $\mathcal{M}$.

For the sake of simplifying the task at hand we will reduce the complexities and pedantics of defining what a first-order formula is to what we know from before as simply "predicates," as we know them from more foundational backgrounds.

Examples of First-Order Formulas (of different languages) include the following:

- $\forall x \exists y (x < y)$
- $(x = z)$
- $\exists \varepsilon \forall x (c < x \longleftrightarrow \varepsilon < x)$

Note however, that we do not specify the sets from which we quantify on, as we have that in First-Order Logic that we only talk about **one** domain of discourse.

Furthermore, we use our intuition from before to be able to tell what it means for an $\mathcal{L}$-Structure $\mathcal{M}$ to satisfy, or model, an $\mathcal{L}$-Formula. Here we define a notion that specifies a certain kind of formula, known as a sentence, or closed formula, depending on your background.

Definition 2.3. “$\mathcal{L}$-Sentence”

An **$\mathcal{L}$-sentence** —in hand-wavy terms, is just a first-order formula with no free variables, i.e. all variables are quantified over.

Examples.

- Let $\mathcal{L}_{arith} = \{0, 1, +, \cdot\}$. An example of a structure in this language is the integers. Then $\forall x \exists y(x + y = 0)$ is an $\mathcal{L}_{arith}$-sentence, capturing the existence of additive inverses.
- Let $\mathcal{L}_{ord} = \{<\}$. Then $\forall x \forall y(x < y \vee x = y \vee y < x))$ is an $\mathcal{L}_{\wr\nabla\lceil}$ sentence, capturing a property of a linear order.
- Let $\mathcal{L}_{group} = \{e, \cdot\}$. Then $\forall x\big((e \cdot x = x) \wedge (x \cdot e = x)\big)$ is an $\mathcal{L}_{\}\nabla\wr\sqcap_\surd}$ sentence, showing that e is the identity element under the binary operation $\cdot$.

Definition 2.4. $\mathcal{L}$-Theory

An **$\mathcal{L}$-Theory** is a *satisfiable* set of $\mathcal{L}$-sentences. Furthermore, we say an $\mathcal{L}$-structure $\mathcal{M}$ **satisfies** an $\mathcal{L}$-theory Γ, when $\mathcal{M} \models \varphi$ for every $\varphi \in \Gamma$; we then write $\mathcal{M} \models \Gamma$.

CONVENTION. *Given any n-tuple $\bar{a}$, we refer to its ith entry by $\bar{a}(i)$, where $1 \leq i \leq n$.*

Now that we played a bit with some Model Theoretic notions, we are now ready to define the most important notion for our current discourse.

Definition 2.5. Ultraproduct of $\mathcal{L}$-Structures

Let $\mathcal{U}$ an ultrafilter over a proper indexing set I. Given a family of $\mathcal{L}$-structures $\{\mathcal{M}_i : i \in I\}$ where M_i is the (non-empty) domain of its corresponding $\mathcal{L}$-structure $\mathcal{M}_i$, we define the **ultraproduct** $\prod_{\mathcal{U}} \mathcal{M}_i$ to be the $\mathcal{L}$-structure $\mathcal{M}^*$ with domain $\prod_{\mathcal{U}} M_i$, which satisfies the following properties:

- For any constant symbol $c \in \mathcal{L}$, we have that the interpretation of the constant symbol in $\mathcal{M}^*$ is given by the equivalence class of the semi-sequence of interpretations of c in each $\mathcal{M}_i$, or in other words:

$$\mathbf{c}^{\mathcal{M}^*} = \left[(\mathbf{c}^{\mathcal{M}_\mathbf{i}})_{\mathbf{i}\in\mathbf{I}}\right]_{\mathcal{U}}$$

- For any function symbol $f \in \mathcal{L}$ of arity k and any semi-sequences in the Cartesian product of the underlying set; $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k \in \prod_{i\in I} M_i$, we have that:

$$\mathbf{f}^{\mathcal{M}^*}\Big([\bar{\alpha}_\mathbf{1}]_{\mathcal{U}}, [\bar{\alpha}_\mathbf{2}]_{\mathcal{U}}, \ldots, [\bar{\alpha}_\mathbf{k}]_{\mathcal{U}}\Big) = \left[\Big(\mathbf{f}^{\mathcal{M}_\mathbf{i}}\big(\bar{\alpha}_\mathbf{1}(\mathbf{i}), \bar{\alpha}_\mathbf{2}(\mathbf{i}), \ldots, \bar{\alpha}_\mathbf{k}(\mathbf{i})\big)\Big)_{\mathbf{i}\in\mathbf{I}}\right]_{\mathcal{U}}$$

- For any relation symbol $R \in \mathcal{L}$ of arity k and any semi-sequences in the Cartesian product of the underlying set; $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k \in \prod_{i\in I} M_i$, we have that:

$$\big([\bar{\alpha}_\mathbf{1}]_{\mathcal{U}}, [\bar{\alpha}_\mathbf{2}]_{\mathcal{U}}, \ldots, [\bar{\alpha}_\mathbf{k}]_{\mathcal{U}}\big) \in \mathbf{R}^{\mathcal{M}^*} \text{ iff } \Big\{\mathbf{i} \in \mathbf{I} : \big(\bar{\alpha}_\mathbf{1}(\mathbf{i}), \bar{\alpha}_\mathbf{2}(\mathbf{i}), \ldots, \bar{\alpha}_\mathbf{k}(\mathbf{i})\big) \in \mathbf{R}^{\mathcal{M}_\mathbf{i}}\Big\} \in \mathcal{U}$$

Before moving forward we need to prove that the ultraproduct is well-defined. First, let us make sure that whenever we pick any representatives for the same $\mathcal{U}$-equivalence class, we have that their images under the interpretations of any of the function symbols remain the same.

Take a function symbol f and a relation symbol R of arbitrary arity n, and any semi-sequences $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ that are $\mathcal{U}$-equivalent to corresponding semi-sequences $\bar{\alpha}'_1, \bar{\alpha}'_2, \ldots, \bar{\alpha}'_n$, respectively.

We need to show that $f^{\mathcal{M}^*}(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n) = f^{\mathcal{M}^*}(\bar{\alpha}'_1, \bar{\alpha}'_2, \ldots, \bar{\alpha}'_n)$, which reduces to showing that $\big[f^{\mathcal{M}_i}(\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i))\big]_{\mathcal{U}} = \big[f^{\mathcal{M}_i}(\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i))\big]_{\mathcal{U}}$. Now consider that since $\bar{\alpha}_k \sim_{\mathcal{U}} \bar{\alpha}'_k$, for any $k \in \{1, 2, \ldots, n\}$, we have that $\{i \in I : \bar{\alpha}_k(i) = \bar{\alpha}'_k(i)\} \in \mathcal{U}$ by definition. We then get that property ($\mathcal{U}2$) of ultrafilters yields the following results:

$$\bigcap_{k=1}^{n} \{i \in I : \bar{\alpha}_k(i) = \bar{\alpha}'_k(i)\} \in \mathcal{U}$$

$$\textit{iff } \Big\{i \in I : \bigwedge_{k=1}^{n} \big(\bar{\alpha}_k(i) = \bar{\alpha}'_k(i)\big)\Big\} \in \mathcal{U}$$

$$\textit{iff } \big\{i \in I : \big(\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)\big) = \big(\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)\big)\big\} \in \mathcal{U}$$

We now want to use this information to show that;

$$\big[f^{\mathcal{M}_i}\big(\bar{\alpha}_1(i),\, \bar{\alpha}_2(i),\, \ldots,\, \bar{\alpha}_n(i)\big)\big]_{\mathcal{U}} = \big[f^{\mathcal{M}_i}\big(\bar{\alpha}'_1(i),\, \bar{\alpha}'_2(i),\, \ldots,\, \bar{\alpha}'_n(i)\big)\big]_{\mathcal{U}}.$$

For this to happen we have to have that;

$$f^{\mathcal{M}_i}\big(\bar{\alpha}_1(i),\, \bar{\alpha}_2(i),\, \ldots,\, \bar{\alpha}_n(i)\big) \sim_{\mathcal{U}} f^{\mathcal{M}_i}\big(\bar{\alpha}'_1(i),\, \bar{\alpha}'_2(i),\, \ldots,\, \bar{\alpha}'_n(i)\big),$$

and for **this** to happen we have to have that:

$$S := \big\{i \in I : f^{\mathcal{M}_i}\big(\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)\big) = f^{\mathcal{M}_i}\big(\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)\big)\big\} \in \mathcal{U}.$$

However, one can see that the aforementioned set

$$S_0 := \big\{i \in I : \big(\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)\big) = \big(\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)\big)\big\} \subseteq S \subseteq I.$$

Thus, by property $(\mathcal{U}1)$ of ultrafilters we get that $S \in \mathcal{U}$, as desired. This concludes our proof that the ultraproduct is well-defined for function symbols.

It only remains to look at the elements of the interpretations of the relation symbols and investigate whether or not they remain the same under picking different representatives for the equivalence classes, similar to what we did above. Thus, we aim to show that $\big([\bar{\alpha}_1]_{\mathcal{U}}, [\bar{\alpha}_2]_{\mathcal{U}}, \ldots, [\bar{\alpha}_n]_{\mathcal{U}}\big) \in R^{\mathcal{M}^*}$ if and only if $\big([\bar{\alpha}'_1]_{\mathcal{U}}, [\bar{\alpha}'_2]_{\mathcal{U}}, \ldots, [\bar{\alpha}'_n]_{\mathcal{U}}\big) \in R^{\mathcal{M}^*}$, which reduces to showing that $\big\{i \in I : (\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)) \in R^{\mathcal{M}_i}\big\} \in \mathcal{U}$ if and only if $\big\{i \in I : (\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)) \in R^{\mathcal{M}_i}\big\} \in \mathcal{U}$.

Let us start by recalling that we proved that $S_0 \in \mathcal{U}$. Now, without loss of generality, let it be the case that $S_1 := \big\{i \in I : (\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)) \in R^{\mathcal{M}_i}\big\} \in \mathcal{U}$. We then get that $S_0 \cap S_1 \in \mathcal{U}$ by using property $(\mathcal{U}2)$ of ultrafilters, like we did before. Moreover, notice the following fact:

$$\begin{aligned} S_0 \cap S_1 = \Big\{i \in I :\ &\Big(\big(\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)\big) = \big(\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)\big)\Big) \\ &\wedge \Big((\bar{\alpha}_1(i), \bar{\alpha}_2(i), \ldots, \bar{\alpha}_n(i)) \in R^{\mathcal{M}_i}\Big)\Big\} \end{aligned}$$

This yields the following:

$$S_0 \cap S_1 \subseteq \big\{i \in I : (\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)) \in R^{\mathcal{M}_i}\big\} \subseteq I$$

From here we then do as we did before and use $(\mathcal{U}1)$ to find that $\{i \in I : (\bar{\alpha}'_1(i), \bar{\alpha}'_2(i), \ldots, \bar{\alpha}'_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}$, as desired.

This concludes our ——————— sighhhh ——————— rather lengthy proof. ■

Definition 2.6. Ultrapower

Given an ultrafilter $\mathcal{U}$ over a proper indexing set I, and an $\mathcal{L}$-structure $\mathcal{M}$, we define its **ultrapower** $\Big((\mathcal{M})\Big)^{\mathcal{U}}$ to be the following ultraproduct:

$$\Big((\mathcal{M})\Big)^{\mathcal{U}} := \prod_{\mathcal{U}} \mathcal{M}_i, \text{ where } \mathcal{M}_i = \mathcal{M}, \text{ for every } i \in I.$$

We need now to introduce some more model-theoretic notions that we will be needing later in the paper.

Definition 2.7. Substructure

We say that $\mathcal{M}_0$ is a **substructure** of $\mathcal{M}$ when $M_0 \subseteq M$ and the following three conditions hold:

- For every constant symbol $c \in \mathcal{L}$, we have that $c^{\mathcal{M}_0} = c^{\mathcal{M}}$.
- For every n-ary function symbol $f \in \mathcal{L}$, we have that for every n-tuple $(\bar{a}) \in \big(M_0\big)^n$ that $f^{\mathcal{M}_0}(\bar{a}) = f^{\mathcal{M}}(\bar{a})$.
- For every n-ary relation symbol $R \in \mathcal{L}$, we have that $R^{\mathcal{M}_0} = R^{\mathcal{M}} \cap \big(M_0\big)^n$. In other words, we have that for any n-tuple $\bar{a} \in \big(M_0\big)^n$ we have that:

$$\bar{a} \in R^{\mathcal{M}_0} \textbf{ if and only if } \bar{a} \in R^{\mathcal{M}_0}.$$

Definition 2.8. Elementary Substructure

We say that an $\mathcal{L}$-structure $\mathcal{M}$ is an **Elementary Substructure** of $\mathcal{N}$ when $\mathcal{M}$ is a substructure of $\mathcal{N}$, and whenever we have a formula with n-many free variables $\varphi(x_1, x_2, \ldots, x_n)$, for $n \in \mathbb{N}$, and any n-tuple $(\bar{a}) \in M^n$, we have that:

$$\mathcal{M} \models \varphi(\bar{a}) \textbf{ if and only if } \mathcal{N} \models \varphi(\bar{a}).$$

3. Łoś' Theorem

This entire section is devoted to discuss one very important theorem that we borrow from Model Theory. Łoś' Theorem is critical to understanding the relationship between the formulas that Ultraproducts satisfy versus the ones that its underlying 'factoring' structures satisfy.

Fact 3.1. Łoś' Theorem

Given an ultrafilter $\mathcal{U}$ over an indexing set I, and a family of non-empty $\mathcal{L}$-Structures $\{\mathcal{M}_i : i \in I\}$, we have that for any $\mathcal{L}$-formula $\varphi(x_1, x_2, \dots, x_n)$ and any semi-sequences $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n \in \prod_{i \in I} M_i$, we have that:

$$\prod_{\mathcal{U}} \mathcal{M}_i \models \varphi\big([\bar{a}_1]_{\mathcal{U}}, [\bar{a}_2]_{\mathcal{U}}, \dots, [\bar{a}_n]_{\mathcal{U}}\big) \text{ if and only if}$$
$$\{i \in I : \mathcal{M}_i \models \varphi\big(\bar{a}_1(i), \bar{a}_2(i), \dots, \bar{a}_n(i)\big)\} \in \mathcal{U}$$

The proof of Łoś' Theorem will not be done in this paper, since it is outside the scope of the topic, however it can be found in our cited works. It can be proved by induction on first-order formulas.

The following is a corollary to it, and we will be using it as well.

Corollary 3.2.

An $\mathcal{L}$-Theory T is satisfied by an ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_i$ if $\mathcal{M}_i \models T$ for every $i \in I$.

CHAPTER 3

Construction and Characterization of the Hyperreals

1. Construction

We shall now shift our focus to a case study using one specific language, that of the ordered fields, which we will denote and define as follows:

$$\mathcal{L}_{\mathbb{F}} = \{0,\ 1,\ +,\ \cdot,\ -,\ ^{-1},\ <\}$$

As a convention we will have that $\cdot$ and $+$ are binary function symbols, while $-$ and $^{-1}$ are unary function symbols, with their standard interpretations within the scope of our discourse. We aim to use this language to build an $\mathcal{L}_{\mathbb{F}}$-Structure that contains the Reals as an ordered field, however, it contains "more" elements that does not allow it to be "like" the reals in a more metaphysical sense.

Convention. *Let us denote the ordered field of real numbers to be the $\mathcal{L}_{\mathbb{F}}$-structure denoted by $\mathcal{R}$, which everyone knows from their first course in Real Analysis.*

We want this "new" structure to satisfy two main criteria, namely:

- $\mathcal{R}$ is an elementary substructure of this "new" structure.
- It has some element $\gamma > 0$ in it, where for any $r \in \mathbb{R}^+$ we have that $\gamma < r$.

We call such an element an **infinitesimal**.

Such structure is going to be our Hyperreals. ;)

In order to construct such a structure from the reals, we need a special type of filter on the natural numbers, called the Fréchet Filter, and we define it as follows...

Definition 1.1. Fréchet Filter

Given any set I, we define the **Fréchet Filter** on I to be the set of all complements of finite subsets of I.

Definition 1.2. The Hyperreals

Given an ultrafilter $\mathcal{U}_F$ extending the Fréchet Filter on $\mathbb{Z}^+$, we define the $\mathcal{L}_{\mathbb{F}}$-structure $\mathcal{HR}$ to be the following ultrapower:

$$\mathcal{HR} := \Big((\mathcal{R}) \Big)^{\mathcal{U}_F}$$

We then call this new structure the $\mathcal{L}_{\mathbb{F}}$-structure of the **Hyperreals** and its underlying set is denoted by $\mathbb{HR}$.

Theorem 1.3.

$\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$.

Proof. Let $h : \mathbb{R} \to \mathbb{HR}$ be the mapping which sends every real number to the equivalence class of its corresponding constant sequence under the $\mathcal{U}$-relation as follows:

$$r \mapsto [(\bar{r})]_{\mathcal{U}}$$

We claim that this mapping embeds $\mathcal{R}$ into $\mathcal{HR}$ in such a way which makes $\mathcal{R}$ an elementary substructure of $\mathcal{HR}$. Recall that for $\mathcal{R}$ to be en elementary substructure of $\mathcal{HR}$ we have to have that for every formula with n-many variables $\varphi(x_1, x_2, \ldots, x_n)$ that $\mathcal{R} \models \varphi(x_1, x_2, \ldots, x_n)$ if and only if $\mathcal{HR} \models \varphi(x_1, x_2, \ldots, x_n)$.

Consider the n-tuple of elements $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we want to show that $\mathcal{R} \models \varphi(a_1, a_2, \ldots, a_n)$ if and only if $\mathcal{HR} \models \varphi\big(h(a_1), h(a_2), \ldots, h(a_n)\big)$. Let us then suppose that we have $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that $\mathcal{R} \models \varphi(a_1, a_2, \ldots, a_n)$. Consider then by Łoś' Theorem we have the following:

$$\mathcal{HR} \models \varphi\big(h(a_1), h(a_2), \ldots, h(a_n)\big)$$
$$\textit{if and only if}$$
$$\big\{i \in \mathbb{Z}^+ : \mathcal{R} \models \varphi(a_1, a_2, \ldots, a_n)\big\} \in \mathcal{U}_F,$$

however by our assumption we have that this yields:

$$\mathcal{HR} \models \varphi\big(h(a_1), h(a_2), \ldots, h(a_n)\big) \textit{ if and only if } \mathbb{Z}^+ \in \mathcal{U}_F,$$

which holds by property $\mathcal{U}3$ of (ultra)filters; since F is the Fréchet filter on $\mathbb{Z}^+$. Therefore, we get that $\mathcal{HR} \models \varphi\big(h(a_1), h(a_2), \ldots, h(a_n)\big)$, whence $\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$, as desired. ∎

Corollary 1.4.

$\mathcal{HR}$ is an ordered field.

PROOF. We know that order and axioms can be written as first-order sentences. Therefore, by the theorem we just proved above, we know that since $\mathcal{R}$ satisfies these sentences, then $\mathcal{HR}$ satisfies them as well, proving that $\mathcal{HR}$ is an ordered field, as desired. ∎

Consider the following 'extended' definition of the absolute value function that we know on the reals, to the hyperreals:

Definition 1.5. Absolute Value in the Hyperreals

Given $x \in \mathbb{HR}$, we define its **absolute value** $|x| := x$, when $x > 0$, and $|x| := -x$, when $x < 0$.

We now use the absolute value we just defined, in order to define the following essential notion:

Definition 1.6. Infinitesimal

Given any element $\gamma \in \mathbb{HR}$, we call γ an **infinitesimal** if and only if $|\gamma| < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$.

We shall denote the set of all infinitesimals by $\mathbb{I}$, which one should note that it includes $[(\mathbf{0})]_{\mathcal{U}_F} \in \mathbb{R}$. This gives us that $\mathbb{I} \cap \mathbb{R} \neq \emptyset$, maybe against what one might think. Nonetheless, we want to prove the existence of the most important type of infinitesimal for the sake of our discourse; the *positive* infinitesimal. Proving its existence would make it one of the key differences between $\mathbb{R}$ and $\mathbb{HR}$ as ordered fields.

Theorem 1.7. Infinitesimals are a thing!

$\mathcal{HR}$ has a positive infinitesimal.

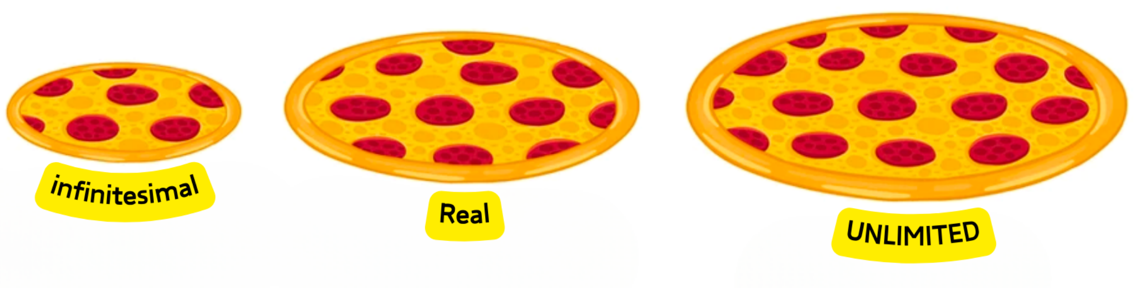
PROOF. We will start by considering the sequence $\alpha = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. We will show that $[\alpha]_{\mathcal{U}_F}$ is an infinitesimal. We can see that $[\alpha]_{\mathcal{U}_F}$ is positive, since the set of indices on which the sequence members of α are strictly greater than zero is simply $\mathbb{Z}^+$, which is in $\mathcal{U}_F$ by definition. Now, let r be an arbitrary positive real number, then by the Archimedean property, there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$. Now, we can consider a set $S = \left\{m \in \mathbb{Z}^+ \middle| m > n\right\}$. We can see that the complement of S is finite, thus the set S is a part of the Fréchet Filter over $\mathbb{Z}^+$, and since $\mathcal{U}_F$ extends the Fréchet Filter over $\mathbb{Z}^+$, then $S \in \mathcal{U}_F$. This is due to the fact that the indices i for which $\alpha(i) < r$ are in $\mathcal{U}_F$, thus $[\alpha]_{\mathcal{U}_F} < r$ for any $r \in \mathbb{R}$. ∎

Theorem 1.8.

There exists some element $\Omega \in \mathbb{HR}$ such that $r < \Omega$, for all $r \in \mathbb{R}$.

PROOF. We shall prove this statement by providing a witness, namely the equivalence class of the real-valued sequence $\omega = (n)_{n\in\mathbb{N}} = (1, 2, 3, 4, 5, \dots)$. Take any constant real sequence $(\mathbf{r}) \in \mathbb{R}^\infty$ and consider that we want to show that $[(\mathbf{r})]_{\mathcal{U}_F} < [\omega]_{\mathcal{U}_F}$ regardless of our choice of $r \in \mathbb{R}$. Note that by our choice of ω we have that the set of indices at which $r < w_i$ is the set $S = \{i \in \mathbb{Z}^+ : r < \omega_i\}$, which is the complement of the set $\{i \in \mathbb{Z}^+ : r \geq \omega_i\}$, which is finite since the sequence ω by construction is eventually greater than r. This means that S is in F by definition, whence it is in $\mathcal{U}_F$, which makes $[(\mathbf{r})]_{\mathcal{U}_F} < [\omega]_{\mathcal{U}_F}$, as desired. ■

Consider that we proved existence of two non-real elements, but let us acknowledge that by the fact that $\mathcal{HR}$ is an ordered field, we have that our positive infinitesimal, ε, and Ω from above must have additive inverses. By our previous knowledge in algebra, they are the elements obtained by multiplying each one by the additive inverse of $[(\mathbf{1})]_{\mathcal{U}_F}$. Moreover, we have other such elements as well; namely $[(\mathbf{x})]_{\mathcal{U}_F} \cdot \varepsilon$ and $[(\mathbf{x})]_{\mathcal{U}_F} \cdot \Omega$, where x $\in \mathbb{R}$. This gives us an uncountable number of both infinitesimals and what are called '**unlimiteds**' throughout some of the literature on hyperreals and non-standard analysis.



Not drawn to scale.

Before moving forward, we should be careful with our notations, before things start to get messy.

Note on Notation.

Consider the following notations for the respective sets that we will be using throughout this paper:

- A, where $A \subseteq \mathbb{R}$, will (interchangeably) denote the set of equivalence classes of constant sequences in A.
- $\mathbb{HR} \setminus \mathbb{R}$ will denote the set of all non-real hyperreals, which we like to refer to as the set of **non-real** hyperreals. :)
- $\mathbb{L}$ shall denote the set of all hyperreals bounded above by a real number, we call this set the set of all **limited** hyperreals.
- $\mathbb{U}$ will denote the set of all hyperreals which are not limited, which we will call the set of **unlimited** hyperreals.

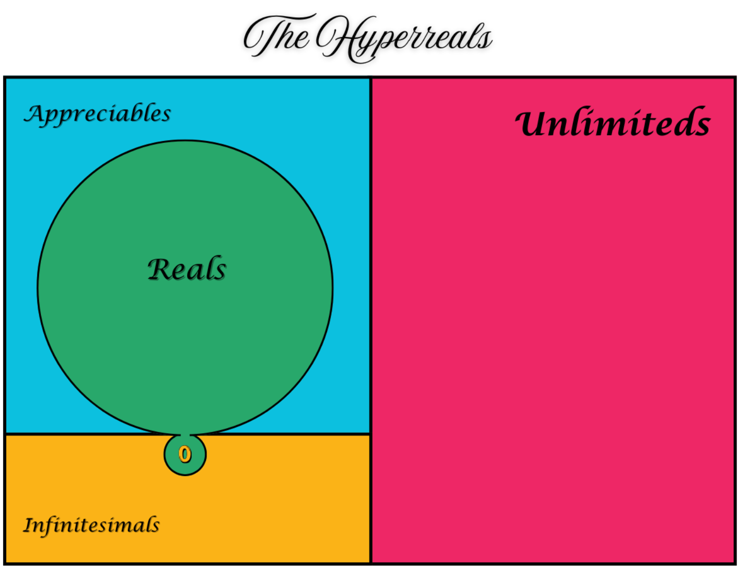
Here rises a question, are there limited hyperreals that are not infinitesimal nor real? In fact, yes, there is. This needs no formal lemma-proof section of the paper. Just consider that since we are talking about an ordered field, then we can take any positive infinitesimal γ and $1 \in \mathbb{R}$ and consider that $1 < 1+\gamma$, by the axioms that we know of ordered fields. This gives us that $1+\gamma$ is not an infinitesimal, since $\frac{1}{2} < 1+\gamma$, for instance. It remains to argue why $1+\gamma$ is **non-real**. Assume $1+\gamma$ is real, this implies that $1+\gamma-1=\gamma$ is real, since $\mathcal{R}$ is a substructure of $\mathcal{HR}$ and $+,-$ are closed operations on $\mathbb{R}$. This clearly contradicts our prior assumption that γ is infinitesimal, in particular, non-real. Thus, $1+\gamma$ is our desired non-infinitesimal non-real. Let us now consider the following definition, that encapsulates the description of all limited hyperreals that are not infinitesimals.

Definition 1.9. Appreciable

We say $x \in \mathbb{HR}$ is **appreciable** when it is a limited hyperreal that is not infinitesimal. We denote the set of all appreciables by $\mathbb{A} := (\mathbb{L} \setminus \mathbb{I})$.

Remark. *It is worth noting that zero is the only real number that is not appreciable in $\mathcal{HR}$.*

We acknowledge that the different types of numbers in the set of hyperreals has grown to be quite overwhelming to keep track of, so consider the following Venn diagram for the universe of the hyperreals.



2. Hyperreal Interactions

The concept of multiplying reals with non-reals to obtain other non-reals in our new ordered field should be quite intuitive to understand, keeping in mind that binary operations on the reals are already well-defined from our previous background in analysis, abstract algebra and even high-school mathematics. All we do is simply extend these operations to be defined on all hyperreals, rather than only on the reals. We will however elaborate on the nature of these bizarre new elements with regards to their behavior with our beloved real numbers by the following interesting factoids:

- For any two infinitesimals ϵ, $\delta \in \mathbb{I}$ we have that $\epsilon + \delta$, $\epsilon \cdot \delta \in \mathbb{I}$.
- For any two unlimiteds ω, $\upsilon \in \mathbb{U}$ we have that $\omega \cdot \upsilon \in \mathbb{U}$.
- For any $\omega \in \mathbb{U}$ and $x \in \mathbb{L}$ we have that $\omega + x \in \mathbb{U}$, i.e. an unlimited hyperreal plus a limited one gives you an unlimited hyperreal.
- The multiplicative inverse of any infinitesimal is unlimited, and the multiplicative inverse of any unlimited is an infinitesimal.

Perhaps in order to see this more clearly, one must really take a look at the hyperreals at a deeper level. Let us recall what they really are at the end of the day; $\mathcal{U}$-equivalence classes of real-valued sequences. One may wonder then, which classes constitute infinitesimals, which constitute our appreciables and which constitute unlimited hyperreals...

As we discussed before in the proof of **Theorem 1.3.** the reals are typically viewed in the hyperreals as equivalence classes of constant real valued sequences.

By observation of various examples throughout the literature, we notice that infinitesimals tend to be represented as equivalence classes of sequences **converging** to zero, while unlimiteds tend to be equivalence classes of sequences **diverging** to ∞ or $-\infty$.

Let us try to support these claims by the following illustrative examples...

- $[(\mathbf{r})]_{\mathcal{U}_F}$ is real in $\mathcal{HR}$ for any $r \in \mathbb{R}$.
- $\left[\left(\frac{1}{n}\right)_{n\in\mathbb{Z}^+}\right]_{\mathcal{U}_F}$ is a positive infinitesimal, while $\left[\left(\frac{-1}{n}\right)_{n\in\mathbb{Z}^+}\right]_{\mathcal{U}_F}$ is a negative infinitesimal.
- $\left[(n)_{n\in\mathbb{Z}^+}\right]_{\mathcal{U}_F}$ is a positive unlimited.
- $\left[\left(-n\sin^2(n)\right)_{n\in\mathbb{Z}^+}\right]_{\mathcal{U}_F}$ is a negative unlimited.
- $\left[\left(1 \pm \frac{1}{n}\right)_{n\in\mathbb{Z}^+}\right]_{\mathcal{U}_F}$ is a positive appreciable that is non-real.
- $\left[(-\pi \pm e^{-n})_{n\in\mathbb{Z}^+}\right]_{\mathcal{U}_F}$ is a negative appreciable that is non-real.

We shall now further elaborate on our task at hand by considering a how different types of hyperreals interact with each other using the field operations via the following intuitive examples. This will become quite useful later when we come to discuss how we use hyperreals to do some 'non-standard analysis' on the reals themselves...

Convention. *We shall fix the notion (a_n) to refer to the sequence indexed by the positive integers $(a_n)_{n\in\mathbb{Z}^+}$. It shall be clear from the context moving forward.*

Examples. We will be using the definition of the ultraproduct with respect to the hyperreals in order to play with the arithmetic operations of the field as follows:

- Here we want to showcase that the addition of infinitesimals gives us an infinitesimal, so consider this computation:

$$\left[\left(\frac{1}{n}\right)\right]_{\mathcal{U}_F} + \left[\left(\frac{1}{n}\right)\right]_{\mathcal{U}_F} = \left[\left(\frac{1}{n} + \frac{1}{n}\right)\right]_{\mathcal{U}_F} = \left[\left(\frac{2}{n}\right)\right]_{\mathcal{U}_F}$$

- This is an instance of multiplication of two unlimiteds: $[(n^n)]_{\mathcal{U}_F} \cdot [(n!)]_{\mathcal{U}_F} = [(n^n \cdot n!)]_{\mathcal{U}_F}$ showcases how when we multiply two unlimiteds together gives us another unlimited.
- We want to explore how the multiplication of infinitesimals gives us an infinitesimal:

$$\left[\left(\frac{1}{n}\right)\right]_{\mathcal{U}_F} \cdot \left[\left(\frac{1}{n}\right)\right]_{\mathcal{U}_F} = \left[\left(\frac{1}{n} \cdot \frac{1}{n}\right)\right]_{\mathcal{U}_F} = \left[\left(\frac{1}{n^2}\right)\right]_{\mathcal{U}_F}$$

- We will showcase that the multiplicative inverse of an unlimited is an infinitesimal by this:

$$\left[\left(\frac{1}{n}\right)\right]_{\mathcal{U}_F} \cdot [(n)]_{\mathcal{U}_F} = \left[\left(\frac{1}{n} \cdot n\right)\right]_{\mathcal{U}_F} = [(1)]_{\mathcal{U}_F}$$

The above examples have been handpicked to be easy for the reader to see how some hyperreals interact with each other. Nonetheless, it is worth mentioning that even though the examples above might showcase the outcomes of certain types of hyperreals with each other, these examples do not *prove* this anything about their generalized patterns. Unfortunately, we will not be proving it in this paper either, however, the following is a fact that we are going to take a bit for granted regarding how hyperreals interact with the real numbers in $\mathcal{HR}$.

Fact 2.1. Arithmetic of Reals with Hyperreals

Take any binary operation $\diamond$ interpreted from the ones in $\mathcal{L}_{\mathbb{F}}$, and consider the following for any **non-zero** real number $r \in \mathbb{R}$ and any hyperreal $\beta \in \mathbb{HR}$, then consider the following case scenarios:

$$r \diamond \beta \in \begin{cases} \mathbb{L} & \text{if and only if } \beta \text{ is limited} \\ \mathbb{U} & \text{if and only if } \beta \text{ is unlimited} \\ \mathbb{R} & \text{if and only if } \beta \text{ is real} \\ \mathbb{I} & \text{if and only if } \beta \text{ is infinitesimal } \textbf{AND } \diamond = \cdot \\ \mathbb{A} & \text{if } \beta \text{ is infinitesimal } \textbf{AND } \diamond = + \end{cases}$$

It is not within the focus of this paper to prove this proposition, however, the reader is recommended to think about it and verify this hypothesis with multiple examples.

Remark. *Note that we had to specify that our real number is zero, as the above fact would fail when we talk about multiplying zero with arbitrary unlimited hyperreals.*

Fun Fact! Infinitesimal Subring

The set of all infinitesimals $\mathbb{I}$ forms a subring of $\mathbb{HR}$ that is commutative.

One is encouraged to think about this fact as a nice exercise. This especially goes out for the younger students still getting exposed to abstract algebra, who are running out of examples for rings that they would not know of during their studies ;)

Keep in mind that the reason why it is not a field is that it does not inherit the unity of $\mathcal{HR}$. One might also add that the multiplicative inverses of the non-zero elements are frankly unlimiteds, which do not belong in $\mathbb{I}$!

3. Archimedeanity

Recall that an ordered field $\mathbb{F}$ is called Archimedean if for any $x \in \mathbb{F}$, there exists an $n \in \mathbb{N}$ such that $x < n$. Moreover, we know that if $\mathbb{F}$ is Archimedean, then for all $x \in \mathbb{F}$, there exists an $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < x$. To show that $\mathbb{HR}$ is not Archimedean, we will show the contrapositive of this claim as follows:

By our work above, we know that there exists a positive infinitesimal $\delta \in \mathbb{HR}$, and by the definition of an infinitesimal, we know that $\delta < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$.

Therefore, this shows that $\mathbb{HR}$ is not Archimedean.

4. Completeness

We know that Archimedeanity is a necessary condition for completeness, and since $\mathbb{HR}$ is not Archimedean as proved above, then $\mathbb{HR}$ is not complete.

To illustrate this idea further, we know that in complete ordered fields, any subset bounded from above must have a supremum. However take the set $\mathbb{R} \subseteq \mathbb{HR}$. We know that $\mathbb{R}$ is bounded above since we proved the existence of unlimiteds in the hyperreals. However, take any unlimited Ω, then we know that $\frac{\Omega}{2}$ is an unlimited as well, and so there is no $\chi \in \mathbb{U}$ such that $\chi \leq \Omega$ for all $\Omega \in \mathbb{U}$. Therefore, there is no "least" unlimited, and so $\sup(\mathbb{R})$ does not exist, again proving that $\mathbb{HR}$ is not complete.

Food for Thought!

Since $\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$, we know that if $\mathcal{R}$ satisfies a first order formula, then $\mathcal{HR}$ must satisfy it as well. However since $\mathcal{R}$ is complete as an ordered field and $\mathcal{HR}$ is not, then we can conclude that completeness cannot be expressed as a first order formula. In fact, it cannot even be expressed in a first-order theory! This is follows from **Corollary 3.3** that we discussed earlier in the paper ;)

5. Density

CONVENTION. *We will denote the density relation between sets by the symbol $\prec$.*

Consider the following facts that we know about density of certain sets from before:

$$\mathbb{Q} \prec \mathbb{R}$$
$$\mathbb{Q} \prec \mathbb{IR} \prec \mathbb{Q}$$
$$\mathbb{IR} \prec \mathbb{R}$$

Here, we will investigate density of the reals in the hyperreals. In other words, we aim to see if $\mathbb{R} \prec \mathbb{HR}$, and whether or not between any two real numbers we can find a hyperreal, which we will denote by $\mathbb{HR} \prec \mathbb{R}$, as well, even though $\mathbb{HR} \not\subseteq \mathbb{R}$.

Actually, since $\mathbb{R}$ is a bounded subset of $\mathbb{HR}$, and there are at least two different elements $\omega, \upsilon \in \mathbb{HR} \setminus \mathbb{R}$, where ω and υ bound the reals from above, we get that there are no real numbers between ω and υ, yielding us $\mathbb{R} \not\prec \mathbb{HR}$, which answers our first question. ;)

For the question as to whether or not the Hyperreals are dense in the Reals, that actually trivially holds since between any two distinct reals, we have a real between them, by their own density within themselves. In other words, since the reals are Hyperreals after all, we get that of course the hyperreals are dense in $\mathbb{R}$. Supposedly, the better question to ask would be whether or not the set of limited non-reals are dense in $\mathbb{R}$, or in other words, whether or not $\left(\mathbb{L} \setminus \mathbb{R}\right) \prec \mathbb{R}$.

Lemma 5.1.

$\mathbb{L} \setminus \mathbb{R}$ is dense in $\mathbb{R}$.

PROOF. Take any $x, y \in \mathbb{R}$ and without loss of generality, assume that $x < y$. Since $x < y$, then $y - x = \varepsilon$, where $\varepsilon > 0$ and $\varepsilon \in \mathbb{R}$. Now, take any positive infinitesimal δ and compute $x + \delta$. We know that $y - (x + \delta) = (y - x) - \delta = \varepsilon - \delta < \varepsilon$, moreover, since $\varepsilon > 0$, then $\varepsilon - \delta > 0$, as well since $0 < \delta < \varepsilon$. Therefore, we get that $x < x + \delta < y$.
Therefore, we showed that there exists a limited non-real between any two reals, and so $\mathbb{L} \setminus \mathbb{R}$ is dense in $\mathbb{R}$, as desired. ∎

CHAPTER 4

A Hyper-Lens on Real Analysis

In this section we shall use this ordered field to investigate if it can help ease the process of doing 'standard' analysis on the reals themselves. This is one of the main reasons why one might find interest in the Hyperreals to begin with. Let us start by the core notions that we will be needing to be able to do any analysis using the hyperreals.

1. Being Infinitely Close

Definition 1.1. Infinitely Close

We say two elements $x, y \in \mathbb{HR}$ are **infinitely close** when $x - y$ is infinitesimal, and we then write $x \approx y$.

Lemma 1.2.

The infinitely close relation is an equivalence relation.

PROOF. In order to show that $\approx$ is an equivalence relation, we need to show the following three properties:

Reflexivity: Take any element $x \in \mathbb{HR}$, and consider that $-x$ is its additive inverse, so $x - x = 0$, which is infinitesimal, whence $x \approx x$, yielding us reflexivity of the relation.

Symmetry: Take any two elements $x, y \in \mathbb{HR}$ where $x \approx y$. This means that $x - y \in \mathbb{I}$. Since $\mathbb{I}$ forms a ring, we have that the additive inverse $-(x - y) = y - x \in \mathbb{I}$, thus $y \approx x$, yielding symmetry.

Transitivity: Take any three hyperreals $x, y, z \in \mathbb{HR}$, and assume $x \approx y$ and $y \approx z$. We want to show that $x \approx z$. Consider that we have that $x - y = \varepsilon_0$ and $y - z = \varepsilon_1$, for some $\varepsilon_0, \varepsilon_1 \in \mathbb{I}$. This gives us that $x - \cancel{y} + \cancel{y} - z = x - z = \varepsilon_0 + \varepsilon_1$. Since, $\varepsilon_0 + \varepsilon_1 \in \mathbb{I}$, we get that $x - z \in \mathbb{I}$, as we wanted, making $x \approx z$, yielding us transitivity.

This concludes the proof. ■

Since that we now have an equivalence relation, one would come to the conclusion that we can partition the limited hyperreals into equivalence classes of the infinitely close relation. However, instead of denoting the equivalence classes by $[x]_\approx$ or something similar, we find the following notion much cooler to describe our equivalence classes. In a way, much more magical!

Definition 1.3. Halo

For any limited hyperreal $x \in \mathbb{L}$ we define its **halo**; $\mathbf{hal(x)}$, to be the set of all hyperreals infinitely close to x, as follows:

$$\mathbf{hal(x)} = \{\, \mathbf{y} \in \mathbb{HR} : \mathbf{x} \approx \mathbf{y} \,\}$$

If you are still encountering issues imagining halos of hyperreals, fret not! Because when faced with a limited hyperreal, the "Halo" singer herself could not see it either!

Beyoncé in Renaissance World Tour Film. Source: myTalk1071.

We would like to remind the reader that the halo of a limited hyperreal is essentially just going to your favorite limited hyperreal and asking it: "OK, who are the people that you feel closest to, and I mean REALLY close..?" The response would then be its halo. All the people to whom it feels indescribably close to. Mathematically speaking, one may view the halo of a hyperreal x to be sort of an infinitesimally small 'ball' around x. We have not defined a metric on the hyperreals, so this will be the analogous notion of "really small balls" in the hyperreal line. This analogy will become more apparent when we start looking at how we can take a different lens on the standard analysis of real-valued sequences and continuous functions.

Furthermore, recall that in our previous experience with mathematical analysis, we used to have the quantification 'for all $\varepsilon > 0$, there exists some $\delta > 0$ such that P, where P is some predicate formula... We would like the reader to now consider that this quantification can be written in a more intuitive way using our constructed hyperreal line. Since we now formally have elements of infinitesimal nature, we can just say that there exists some real δ such that in $\text{hal}(\delta)$ we have that property P holds.

Theorem 1.4. Not Everyone Can be my Halo :"

Every limited hyperreal is in the halo of exactly one real number.

PROOF. Take any limited hyperreal x and take any $\varepsilon \in \mathbb{R}^+$. Now, let $A = \{r \in \mathbb{R} \mid r < x\}$. Since x is limited, we know that there exists $r, s \in \mathbb{R}$ such that $r < x < s$, and so we get that A has an upper bound s. Moreover, since $\mathbb{R}$ is complete, we know that A has a supremum $d \in \mathbb{R}$. Now, since d is the supremum of A, we know that $d + \frac{\varepsilon}{2} \notin A$, and so $x \leq d + \frac{\varepsilon}{2} < d + \varepsilon$, so $x < d + \varepsilon$. Also, we know that if $x < d - \varepsilon$, we will get that $d - \varepsilon$ is an upper bound for A, contradicting the fact that d is the least upper bound of A. Therefore, we get that $d - \varepsilon < x < d + \varepsilon$, meaning that $-\varepsilon < x - d < \varepsilon$ and so $|x - d| < \varepsilon$. Since this is true for all $\varepsilon \in \mathbb{R}$ where $\varepsilon > 0$, we get that $|x - d|$ is smaller than any positive real number, and so $x \approx d$.

For uniqueness, assume that there is a $d' \in \mathbb{R}$ such that $x \approx d'$. This means that $x \approx d$ implies $d \approx d'$, by transitivity. Consider now that since both d and d' are real, then if we have that $d \approx d'$, this would imply that $|d - d'| < r$ for all $r \in \mathbb{R}^+$, whence $d = d'$ as per our previous knowledge, so there is exactly one real number in the halo of every limited hyperreal, as desired. ■

REMARK. Now that we know more about hyperreal arithmetic and how the infinitely close relation works, we can redefine the set of infinitesimals to be $\mathbb{I} := \text{hal}(0)$. This is clear as we know that the difference between 0 and an infinitesimal is always infinitesimal since 0 is an infinitesimal in and of itself.

2. Sequences and Convergence

In order for us to discuss sequences in the hyperreals, we must add predicates to our language which will help us identify the different types of elements we have in the hyperreals, namely the naturals and the reals. To do this, let us use our existing language $\mathcal{L}$ to define the following language $\mathcal{L}'$ as follows:

$$\mathcal{L}' = \mathcal{L}_{\mathbb{F}} \cup \{R, N\} \cup \{c_r : r \in \mathbb{R}\},$$

where R and N are unary relation symbols and each c_r is a constant symbol.
We will interpret R in $\mathcal{HR}$ to be the set of all reals in the hyperreals. Likewise, we will

interpret N in $\mathcal{HR}$ to be the set of all naturals in the hyperreals. Lastly, we will interpret the constant symbols $\{c_r : r \in \mathbb{R}\}$ in $\mathcal{HR}$ such that $c_r = r$ for every r in the reals.

Definition 2.1. Non-Standard Extension (for Sets)

We define the **non-standard extension** of $\mathbf{A} \subseteq \mathbb{R}$ to be the subset A^* of $\mathbb{HR}$ such that $A^* = \{[\bar{a}]_{\mathcal{U}} : \bar{a} \in \prod_{i \in \mathbb{Z}^+} A\}$, where $\bar{a}$ is the constant sequence $(a)_{n \in \mathbb{N}} \subseteq \mathbb{R}$.

Definition 2.2. Non-Standard Extension (for Functions)

The **non-standard extension** of a function $\mathbf{f} : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, is the **function** $f^* : A^* \to \mathbb{HR}$, where $f^*\Big([(\mathbf{a})]_{\mathcal{U}}\Big) = \Big[\big(f(\mathbf{a})\big)\Big]_{\mathcal{U}}$ for any constant sequence $(\mathbf{a})_{n \in \mathbb{N}} \subseteq A$.

To obtain f^*, we start by extending the domain of f from A to all of $\mathbb{R}$, so we get a 'new' function $f' : \mathbb{R} \to \mathbb{R}$, which we can easily construct. Now, for every $x \in \mathbb{R}$, we define $f^*\big([x]_{\mathcal{U}}\big) = \big[f'(x)\big]_{\mathcal{U}}$. However, we get that f^* is a function from the hyperreals to the hyperreals, so we will restrict the function on its domain to make it $f^*\big|_{A^*}$.

Theorem 2.3.

A real-valued sequence (s_n) converges to a limit $L \in \mathbb{R}$ if and only if $s_N \in \mathrm{hal}(L)$ for every unlimited $N \in \mathbb{N}^*$.

Before we get into the proof, we will change the notation of some of the first-order logic sentences so it easier for the reader to follow along with the proof. The change is explained by the following:

- We will use $\forall n \in \mathbb{N}(\phi(n))$ to replace $\forall n(N(n) \to \phi(n))$ for a first-order formula ϕ in the variable n.

Proof. For the forward direction, take a real-valued sequence (s_n) and assume it converges to a limit $L \in \mathbb{R}$. By the definition of convergence, for all $\varepsilon \in \mathbb{R}$ where $\varepsilon > 0$, there is an $m \in \mathbb{N}$ such that

$$\mathcal{R} \models \forall n \in \mathbb{N}(n \geq m \to |s_n - L| < \varepsilon).$$

However, since $\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$, then

$$\mathcal{HR} \models \forall n \in \mathbb{N}(n \geq m \to |s_n - L| < \varepsilon).$$

We will take any unlimited $M \in \mathbb{N}$, then since $m \in \mathbb{N}$, and M is greater than all $n \in \mathbb{N}$, then we get that $M > m$. Thus, we get that $|s_M - L| < \varepsilon$, and since this holds for all $\varepsilon > 0$, then $|s_M - L|$ is smaller than any positive real number, and so $s_M \approx L$, meaning that $s_M \in$ hal (L), as desired.

For the reverse direction, we will assume that $s_m \in$ hal (L), meaning that $s_m \approx L$, so $s_m \approx L$ for all unlimited $m \in \mathbb{N}$. This means that for all $\varepsilon > 0$ where $\varepsilon \in \mathbb{R}$, we get that $|s_M - L| < \varepsilon$ for a fixed $L \in \mathbb{R}$ and some unlimited $M \in \mathbb{N}$. Then if we have any $n \in \mathbb{N}$ where $n \geq M$, we know that n is an unlimited, so we know that $|s_n - L| < \varepsilon$ by our earlier assumption. This shows that

$$\exists M \in \mathbb{N} \forall n \in \mathbb{N}(n \geq M \rightarrow |s_M - L| < \varepsilon).$$

However, since $\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$, then

$$\exists M \in \mathbb{N} \forall n \in \mathbb{N}(n \geq M \rightarrow |s_M - L| < \varepsilon).$$

Therefore, by showing both directions of the claim, we showed that a real-valued sequence (s_n) converges to a limit $L \in \mathbb{R}$ if and only if $s_M \in \text{hal}(L)$ for every unlimited $M \in \mathbb{N}$. ■

Fact 2.4.

A real-valued sequence (s_n) diverges to negative/positive infinity if and only if $\mathcal{S}_N$ is a negative/positive unlimited hyperreal for every unlimited $N \in \mathbb{N}^*$.

We will not prove this fact in the paper to avoid unnecessary length and repetition, as the proof is quite straightforward.

Examples of converging and diverging sequences:

- The real-valued sequence (x_n) with $x_n := \frac{1}{n}$ converges to $L = 0$ in $\mathbb{R}$. To see this, consider $x_n = \frac{1}{n}$, however, because n is an unlimited in $\mathbb{HR}$, we get that $\frac{1}{n}$ is an infinitesimal. We know that all infinitesimals are in the halo of 0, and so $x_n \approx 0$, so 0 is the limit of (x_n).
- The real-valued sequence (s_n) with $s_n := \frac{n+2}{n}$ converges to $L = 1$ in $\mathbb{R}$. To see this, consider $s_n = \frac{n+2}{n} = 1 + \frac{2}{n}$, and so $s_n - 1 = \frac{2}{n}$. However, since n is an unlimited in $\mathbb{HR}$, we get that $\frac{2}{n}$ is an infinitesimal, and so we know that $s_n \approx 1$, and so (s_n) converges to 1.
- The real valued sequence (c_n) with $c_n := n$ diverges to ∞. Then by the construction of the sequence, we know that $c_n > x$ for any $x \in \mathbb{R}$, and by definition of unlimiteds, we get that c_n is a positive unlimited hyperreal, and so (c_n) diverges to ∞.

3. Continuity in the Reals

Theorem 3.1.

A real-valued function $f : A \rightarrow \mathbb{R}$, where $A \subseteq \mathbb{R}$ is continuous at the point $c \in A$ if and only if whenever $x \in \text{hal}(c)$, we get that $f^*(x) \in \text{hal}\big(f^*(c)\big)$ for all $x \in A^*$.

Before we prove this theorem, we will elaborate on some notation, similar to what we did above.
1. We will use $\forall x \in \mathbb{R}(\phi(x))$ to replace $\forall x(R(x) \rightarrow \phi(x))$ for a first-order formula ϕ in the variable x.
2. We will use $\forall x \in \mathbb{R}^+(\phi(x))$ to replace $\forall x((R(x) \wedge x > 0) \rightarrow \phi(x))$ for a first-order formula ϕ in the variable x.
3. We will use $\forall x \in \mathbb{HR}(\phi(x))$ instead of $\forall x \phi(x)$ when discussing variables in the hyperreals.

PROOF. For the forward direction, take a real-valued function $f : A \rightarrow \mathbb{R}$, where $A \subseteq \mathbb{R}$ and assume that f is continuous at the point $c \in A$. By the definition of continuity, this means that

$$\mathcal{R} \models \forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in A(|x - c| < \delta \rightarrow |f(x) - f(c)| < \varepsilon).$$

However, since $\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$, then

$$\mathcal{HR} \models \forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{HR} \forall x \in A^*((\delta > 0 \wedge |x - c| < \delta) \rightarrow |f^*(x) - f^*(c)| < \varepsilon).$$

We will further assume that $x \in$ hal (c), meaning that $x \approx c$. Therefore, this means that $|x - c| < \delta$ for all positive δ in the hyperreals, and so the hypothesis of the above sentence is true, and so the conclusion which states $|f^*(x) - f^*(c)| < \varepsilon$ is true as well, and since ε is a positive real number, then $f^*(x) \approx f^*(c)$, which means that $f^*(x) \in$ hal $(f^*(c))$.
Thus, we showed that if f is continuous at c and $x \in$ hal (c), then $f^*(x) \in$ hal $(f^*(c))$, as desired.

For the converse, assume that if $x \in$ hal (c), then $f^*(x) \in$ hal $(f^*(c))$, which means that

$$\mathcal{HR} \models \forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{HR} \forall x \in A^*((\delta > 0 \wedge |x - c| < \delta) \rightarrow |f^*(x) - f^*(c)| < \varepsilon).$$

However, since $\mathcal{R}$ is an elementary substructure of $\mathcal{HR}$, then

$$\mathcal{R} \models \forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in A(|x - c| < \delta \rightarrow |f(x) - f(c)| < \varepsilon).$$

Given that this is the definition of convergence, we get that whenever $x \in \text{hal}(c)$, we get that $f^*(x) \in \text{hal}\big(f^*(c)\big)$ for all $x \in A^*$, then f is continuous at a point c.

Therefore, by showing both directions of the proof, we showed that a real-valued function $f : A \rightarrow \mathbb{R}$, where $A \subseteq \mathbb{R}$ is continuous at the point $c \in A$ if and only if whenever $x \in \text{hal}(c)$, we get that $f^*(x) \in \text{hal}\big(f^*(c)\big)$ for all $x \in A^*$, as desired. ∎

Corollary 3.2.

Take $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$. Then the following are equivalent:

(i) f is continuous at c.

(ii) If $x \approx c$ for every $x \in A^*$, then $f^*(x) \approx f^*(c)$.

(iii) There is some $\delta \in \mathbb{I}^+$ such that for every $x \in A^*$, if $|x - c| < \delta$, then $f^*(x) \approx f^*(c)$.

Examples of continuous and discontinuous functions:

- The real-valued function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := k$ for some fixed $k \in \mathbb{R}$. Take any $x \in \mathbb{R}$ and some $c \in \mathbb{R}$ such that $x \approx c$ and consider the images of x and c under the extended function f^*. Then we have that $f^*(x) = k = f^*(c)$, and so $|f^*(x) - f^*(c)| = 0$, and so $f^*(x) \approx f^*(c)$, showing that f is continuous at any point c in $\mathbb{R}$, and so f is continuous on $\mathbb{R}$.
- The real-valued function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) := x^2$. Take any $x \in \mathbb{R}$ and some $c \in \mathbb{R}$ such that $x \approx c$, where $|x - c| = \delta$ for some infinitesimal δ, and consider the images of x and c under the extended function g^*. Then we have that $g^*(x) = x^2$ and $g^*(c) = c^2$, and so $|g^*(x)-g^*(c)| = |x^2-c^2| = |(x-c)(x+c)| = |x-c|\cdot|x+c| < \delta\cdot|x+c|$. However, since δ is an infinitesimal, then $\delta\cdot|x+c|$ is an infinitesimal as well, meaning that $g^*(x) \approx g^*(c)$, showing that g is continuous at any point c in $\mathbb{R}$, and so g is continuous on $\mathbb{R}$.
- Take the sine function. For any $x \in \mathbb{R}$ and some $c \in \mathbb{R}$, assume that $x \approx c$ so $|x - c| = \delta$ for some infinitesimal δ, and without loss of generality, assume that $x \geq c$. Consider the images of x and c under the extended function $\sin^*$. We get that $|\sin^*(x)-\sin^*(c)| = |\sin^*(c+\delta)-\sin^*(c)| = |\sin^*(c)\cos^*(\delta)+\cos^*(c)\sin^*(\delta)-\sin^*(c)| = |\sin^*(c)(\cos^*(\delta) - 1) + \cos^*(c)\sin^*(\delta)|$. However, since $\cos^*(\delta) \approx 1$ and $\sin^*(\delta) \approx 0$ and $\cos^*(c)$ and $\sin^*(c)$ are real, then $|\sin^*(c)(\cos^*(\delta) - 1) + \cos^*(c)\sin^*(\delta)|$ is an infinitesimal. This means that $\sin^*(x) \approx \sin^*(c)$, and so the sine function is continuous at any point c in $\mathbb{R}$, and so it is continuous on $\mathbb{R}$.
- Take $\chi : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function. Take any $x \in \mathbb{R}$ and some $c \in \mathbb{R}$ and assume that $x \approx c$. Consider the images of x and c under the extended function χ^*. We know that if $c \in \mathbb{Q}^*$, then $\chi^*(c) = 1$, otherwise $\chi^*(c) = 0$. Either way, we have 2 cases for $\chi^*(x)$:
  1. If $x \in \mathbb{Q}^*$, then $\chi^*(x) = 1$.
  2. If $x \notin \mathbb{Q}^*$, then $\chi^*(x) = 0$.

  In both cases, $\chi^*(x)$ does not stay consistently infinitely close to $\chi^*(c)$ since $\chi^*(x)$ oscillates between 0 and 1 when $x \approx c$. Therefore, it cannot be the case that χ is continuous.

CHAPTER 5

Conclusion

In this paper, we explored the rich mathematical structure and profound theory of the ordered field of the hyperreals. Beginning with foundational preliminaries, we examined its construction and unique properties, and the journey into first-order logic as part of the hyperreal construction serves as an inspiring bridge between the domains of logic and real analysis. By grounding the hyperreal system in a rigorous logical framework, we not only secure its mathematical legitimacy but also highlight the interplay between abstract logical principles and concrete analytical applications. This intersection underscores the unity of mathematical disciplines, where tools from one area enrich our understanding in another, leading to novel perspectives and approaches.

One of the most compelling aspects of this exploration is how hyperreals enable us to redefine and reinterpret fundamental notions such as convergence and continuity. These concepts, long understood in the context of real numbers, take on new dimensions when viewed through the lens of hyperreals, offering fresh insights and a deeper intuition for the behavior of functions and sequences.

The hyperreal field not only bridges abstract mathematical theory with intuitive notions of the infinite and the infinitesimal, or as we like to say "the really really big and the really really small," but it also deepens our understanding of real analysis and topics we studied thoroughly throughout the years. The hyperreals stand as a powerful example of the power of mathematical abstraction, opening new avenues for theoretical and applied exploration.

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