Minsk, 2020

Chapter 1

Description of the goals of the study and the structure of research

1.1 Stationary distribution of the system states at an arbitrary time

Having computed the stationary distribution $\boldsymbol{\pi}_i,\ i \geq 0$, of the embedded MC, we can compute the vectors $\mathbf{p}_i,\ i \geq 0$, defining the stationary distribution of the states of the $MAP/D/1$ system at an arbitrary time: $\mathbf{p}_i$ is the row vector composed from the stationary probabilities

$$p(i,\nu) = \lim_{t\to\infty} P\{i_t = i, \nu_t = \nu\},\ i \geq 0,\ \nu = \overline{0,W},$$

of the process $\{i_t, \nu_t\},\ t \geq 0$.

Theorem 2.1 The vectors $\mathbf{p}_i,\ i \geq 0$, of the stationary probabilities of the states of the $MAP/D/1$ system at an arbitrary time are computed as follows:

$$\mathbf{p}_0 = \lambda \boldsymbol{\pi}_0 (-D_0)^{-1}, \tag{1.1}$$

$$\mathbf{p}_i = \lambda \Big(\boldsymbol{\pi}_0 (-D_0)^{-1} \sum_{k=1}^{i} D_k \hat{\Omega}_{i-k} + \sum_{k=1}^{i} \boldsymbol{\pi}_k \hat{\Omega}_{i-k} \Big),\ i > 0, \tag{1.2}$$

where

$$\hat{\Omega}_m = \int_0^\infty P(m,u)(1 - B(u))du = \int_0^{b_1} P(m,u)du,\ m \geq 0. \tag{1.3}$$

The matrices $\hat{\Omega}_m$ can be easily calculated recursively through the matrices Ω_m computed as

$$\Omega_m = \int_0^\infty P(m,t)dB(t) = P(m,b_1),\ m \geq 0. \tag{1.4}$$

by formulas:

$$\hat{\Omega}_0 = (\Omega_0 - I)D_0^{-1}, \tag{1.5}$$

$$\hat{\Omega}_m = (\Omega_m - \hat{\Omega}_{m-1} D_1) D_0^{-1},\ m \geq 1. \tag{1.6}$$

In the case of the stationary Poisson arrival process, the vectors $\boldsymbol{\pi}_i$ and $\mathbf{p}_i$ become scalars and $\boldsymbol{\pi}_i = \mathbf{p}_i,\ i \geq 0$.

1.2 Stationary distribution of the waiting times in the system

Having computed the stationary distribution $\mathbf{p}_i,\ i \geq 0$, of the states of the $MAP/D/1$ system at an arbitrary time, we can compute the stationary distribution of the waiting times in the system.

Let w_t be the virtual waiting time at the moment $t,\ t \geq 0$. This is the time, which a customer has to wait for beginning the service, if it would arrive to the system at the moment t.

Let $\mathbf{W}(x)$ be the row vector whose νth entry denotes the stationary probability that the MAP underlying process is in state $\nu,\ \nu = \overline{0, W}$ at an arbitrary time and the virtual waiting time does not exceed x:

$$\mathbf{W}(x) = \lim_{t \to \infty} (P\{w_t < x, \nu_t = 0\}, \dots, P\{w_t < x, \nu_t = W\}).$$

Denote by $\mathbf{w}(s) = \int\limits_0^\infty e^{-sx} d\mathbf{W}(x)$ the row vector consisting of the Laplace-Stieltjes transforms (LSTs) of the entries of the vector $\mathbf{W}(x)$.

Theorem 2.2 The vector LST $\mathbf{w}(s)$ is defined as follows:

$$\mathbf{w}(s)\,(sI + D(\beta(s))) = s\mathbf{p}(0),\ \ Re\ s > 0, \tag{1.7}$$

where

$$D(\beta(s)) = D_0 + D_1 \beta(s), \tag{1.8}$$

$\beta(s) = \int\limits_0^\infty e^{-st} dB(t) = e^{-sb_1}$.

Let us denote by $\mathbf{w}_r$, the rth initial moment of the vector distribution of the virtual waiting time, $r \geq 0$. In particular, the value $\mathbf{w}_1\mathbf{e}$ defines the mean value of the virtual waiting time and the value $\mathbf{w}_2\mathbf{e} - (\mathbf{w}_1\mathbf{e})^2$ defines the variance of the virtual waiting time.

1.3 Moments of the virtual waiting time in the $BMAP/G/1$ queue

We will use the following notation:

- $\mathbf{w}_r$ is the $(W+1)$-size vector of rth initial moments of virtual waiting time, $r = \overline{0, 3}$;
- $\hat{\boldsymbol{e}} = (1, 0, \dots, 0)$;
- $\tilde{I}$ is a diagonal matrix with the diagonal entries $(0, 1, \dots, 1)$;
- $A_0 = D(1)$;
- $A_1 = -D^{'}(1)b_1$;
- $A_2 = \frac{1}{2}[D^{''}(1)b_1^2 + D^{'}(1)b_2]$;
- $A_3 = -\frac{1}{6}[D^{(3)}(1)b_1^3 + 3D^{''}(1)b_1b_2 + D^{'}(1)b_3]$;
- $A_4 = \frac{1}{24}[D^{(4)}(1)b_1^4 + 6D^{(3)}(1)b_1^2b_2 + 4D^{''}(1)b_1b_3 + 3D^{''}(1)b_2^2 + D^{'}(1)b_4]$;
- $A = A_0\tilde{I} + (I + A_1)\mathbf{e}\hat{\mathbf{e}}$.

The vectors $\mathbf{w}_r, r = \overline{0,3}$, are calculated using the following recursive formulas.

$$\mathbf{w}_0 = \boldsymbol{\theta},$$

$$\mathbf{w}_1 = \Big\{[\mathbf{w}_0(I + A_1) - \mathbf{p}_0]\tilde{I} + \mathbf{w}_0 A_2 \mathbf{e}\hat{\mathbf{e}}\Big\}A^{-1},$$

$$\mathbf{w}_2 = -2\Big\{[\mathbf{w}_0 A_2 - \mathbf{w}_1(I + A_1)]\tilde{I} + (\mathbf{w}_0 A_3 + \mathbf{w}_1 A_2)\mathbf{e}\hat{\mathbf{e}}\Big\}A^{-1},$$

$$\mathbf{w}_3 = 3\Big\{[2\mathbf{w}_0 A_3 - 2\mathbf{w}_1 A_2 + \mathbf{w}_2(I + A_1)]\tilde{I} + (2\mathbf{w}_0 A_4 - 2\mathbf{w}_1 A_3 + \mathbf{w}_2 A_2)\mathbf{e}\hat{\mathbf{e}}\Big\}A^{-1}.$$

1.4 Moments of the actual waiting time in the $BMAP/G/1$ queue

We will additionally use the following notation:

- v_r is the rth initial moment of actual waiting time, $r = \overline{0,3}$;

We will calculate values v_r using the above expressions for the vectors $\mathbf{w}_r$ and the following recursive formulas.

$$v_0 = 1,$$

$$v_1 = \rho^{-1}[(\mathbf{w}_0 A_2 - \mathbf{w}_1 A_1)\mathbf{e} - \frac{\lambda b_2}{2}],$$

$$v_2 = \rho^{-1}[(-2\mathbf{w}_0 A_3 + 2\mathbf{w}_1 A_2 - \mathbf{w}_2 A_1)\mathbf{e} - v_0\frac{\lambda b_3}{3} - v_1 \lambda b_2],$$

$$v_3 = \rho^{-1}[(6\mathbf{w}_0 A_4 - 6\mathbf{w}_1 A_3 + 3\mathbf{w}_2 A_2 - \mathbf{w}_3 A_1)\mathbf{e} - \lambda(v_0\frac{b_4}{4} + v_1 b_3 + \frac{3}{2}v_2 b_2)].$$

Remark. In the case of degenerate distribution of the service time with parameter T

$$b_k = T^k, k \geq 1.$$