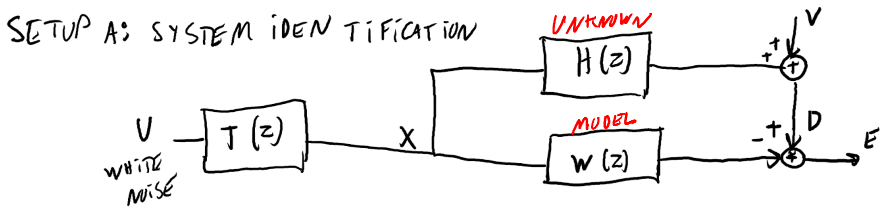
Assignment #2: FIR causal Wiener solution

Consider the two following setups:

Setup A System identification

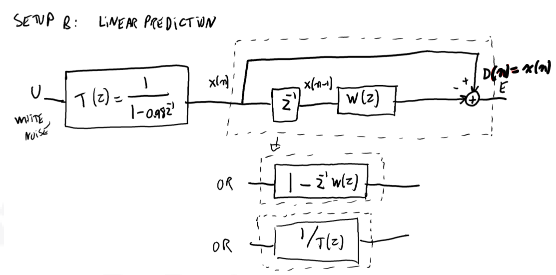


$x(n) = u(n) * t(n)$

$d(n) = x(n) * h(n) + v(n)$

$$e(n) = d(n) - y(n) = d(n) - \sum_{k=0}^{N-1} w_k x(n-k) = d(n) - \mathbf{w}^H \mathbf{x}(n) \text{ with } \mathbf{w} = \begin{bmatrix} w_0^* & w_1^* & \cdots & w_{N-1}^* \end{bmatrix}^T.$$

Setup B Linear prediction



$x(n) = u(n) * t(n)$

$d(n) = x(n)$

$$e(n) = d(n) - y(n) = d(n) - \sum_{k=0}^{N-1} w_k x(n-k-1) = d(n) - \mathbf{w}^H \mathbf{x}(n-1) \text{ with } \mathbf{w} = \begin{bmatrix} w_0^* & w_1^* & \cdots & w_{N-1}^* \end{bmatrix}^T.$$

Use $T(z) = \dfrac{1}{1-0.98z^{-1}}$ $\quad |z| > 0.98 \quad (t(n) = 0.98^n u(n))$, $\quad H(z) = \dfrac{1}{1-0.95z^{-1}}$ $\quad |z| > 0.95 \quad$ ($h(n) = 0.95^n u(n)$), and $u(n), v(n)$ from uncorrelated zero-mean Gaussian white noise stochastic processes with power (and variance) $\sigma^2 = 1$.

1) Assuming a very large value for the number of coefficients N, find an intuitive solution for $\mathbf{w}_{opt}$ in the two setups, just by simple observation (e.g. from the z-transform of the systems).

2) For both setups, using causal FIR Wiener filters $\mathbf{w}_{opt}$ with $N = 10$ coefficients:

- Compute (by coding) the Wiener solution $\mathbf{w}_{opt}$ and the normalized MMSE, based on the theoretical correlation functions $\phi_{dx}(l)$, $\phi_{xx}(l)$, $\phi_{dd}(l)$ (required for $\mathbf{R}$ and $\mathbf{p}$ in $\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{p}$ and for the normalized MMSE $\dfrac{\xi(\mathbf{w}_{opt})}{\sigma_d^2} = \dfrac{\sigma_d^2 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}{\sigma_d^2}$). Plot the resulting $\mathbf{w}_{opt}$ solution for each setup.

$\sigma_d^2 = \phi_{dd}(0)$

- Note: for Setup A, when $x(n)$ is not white noise (as here) and when the length N of $\mathbf{w}_{opt}$ is shorter than the impulse response $h(n)$ (as here), you will find that $\mathbf{w}_{opt}$ does more than just modeling $h(n)$, and the resulting shape for the coefficients in $\mathbf{w}_{opt}$ may not be fully as expected (i.e., as in part 1)). Explain why this is the case.

3) For both setups, estimate (by coding) the Wiener solution $\mathbf{w}_{opt}$ and the normalized MMSE based on $L = 10{,}000{,}000$ samples of the signals $x(n)$, $d(n)$, by estimating the $\phi_{dx}(l)$ and $\phi_{xx}(l)$ functions to compute $\mathbf{w}_{opt}$, and by direct estimation of the normalized MMSE $\dfrac{\sigma^2_{e_{opt}}}{\sigma^2_d}$ from the error signal $e_{opt}(n)$ produced with the estimated $\mathbf{w}_{opt}$. Plot the resulting $\mathbf{w}_{opt}$ solution for each setup.

Use the following code to generate the signals $x(n)$ and $d(n)$ for Setup A and Setup B in Q3:

```
L=10000000;
u=randn(L+200,1); % white Gaussian, variance of one
x=filter(1,[1 -0.98],u); % t(n) filter
x=x(101:end);      % discard first 100 samples to remove transients
                   % in simulated signals
v=randn(L+100,1); % white Gaussian, variance of one

% setup A
dA=filter(1,[1 -0.95],x) + v;  % h(n) filter plus additive noise
dA=dA(101:end);     % discard first 100 samples to remove transients
                    % in simulated signals
xA=x(101:end);     % discard first 100 samples to remove transients
                   % in simulated signals
% setup B
dB=x(101:end);     % discard first 100 samples to remove transients
                   % in simulated signals
xB=x(100:end-1); % x(n-1), delayed by one sample
```

Appendix   Some z-transform relations for correlation functions

$$a^n u(n) \overset{z-transf.}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad |z|>|a| \qquad\qquad -a^n u(-n-1) \overset{z-transf.}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad |z|<|a|$$

$$a^{-n} u(-n) \overset{z-transf.}{\longleftrightarrow} \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|<\left|a^{-1}\right|$$

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$$a^n u(n) * b^{-n} u(-n) \overset{z-transf.}{\longleftrightarrow} \frac{-b^{-1}z^{-1}}{(1-az^{-1})(1-b^{-1}z^{-1})} = \frac{A}{1-az^{-1}} + \frac{B}{1-b^{-1}z^{-1}} \quad |a|<|z|<\left|b^{-1}\right|$$

$$A = \frac{-b^{-1}a^{-1}}{1-b^{-1}a^{-1}} = \frac{1}{1-ab}$$

$$B = \frac{-b^{-1}b}{1-ab} = \frac{-1}{1-ab}$$

$$a^n u(n) * b^{-n} u(-n) = Aa^n u(n) - Bb^{-n} u(-n-1)$$

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$$a^n u(n) * a^{-n} u(-n) * b^n u(n) \overset{z-transf.}{\longleftrightarrow} \frac{-a^{-1}z^{-1}}{(1-az^{-1})(1-a^{-1}z^{-1})(1-bz^{-1})} = \frac{A}{1-az^{-1}} + \frac{B}{1-a^{-1}z^{-1}} + \frac{C}{1-bz^{-1}} \quad \max(|a|,|b|)<|z|<\left|a^{-1}\right|$$

$$A = \frac{-a^{-1}a^{-1}}{(1-a^{-1}a^{-1})(1-ba^{-1})} = \frac{a}{(1-a^2)(a-b)}$$

$$B = \frac{-a^{-1}a}{(1-aa)(1-ba)} = \frac{-1}{(1-a^2)(1-ab)}$$

$$C = \frac{-a^{-1}b^{-1}}{(1-ab^{-1})(1-a^{-1}b^{-1})} = \frac{-b}{(a-b)(1-ab)}$$

$$a^n u(n) * a^{-n} u(-n) * b^n u(n) = Aa^n u(n) - Ba^{-n} u(-n-1) + Cb^n u(n)$$

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$$a^n u(n) * a^{-n} u(-n) * b^n u(n) * b^{-n} u(-n) \overset{z-transf.}{\longleftrightarrow} \frac{a^{-1}b^{-1}z^{-2}}{(1-az^{-1})(1-a^{-1}z^{-1})(1-bz^{-1})(1-b^{-1}z^{-1})}$$

$$= \frac{A}{1-az^{-1}} + \frac{B}{1-a^{-1}z^{-1}} + \frac{C}{1-bz^{-1}} + \frac{D}{1-b^{-1}z^{-1}} \quad \max(|a|,|b|)<|z|<\min\left(\left|a^{-1}\right|,\left|b^{-1}\right|\right)$$

$$A = \frac{a^{-1}b^{-1}a^{-2}}{(1-a^{-1}a^{-1})(1-ba^{-1})(1-b^{-1}a^{-1})} = \frac{a}{(1-a^2)(a-b)(1-ab)}$$

$$B = \frac{a^{-1}b^{-1}a^2}{(1-aa)(1-ba)(1-b^{-1}a)} = \frac{-a}{(1-a^2)(1-ab)(a-b)}$$

$$C = \frac{a^{-1}b^{-1}b^{-2}}{(1-ab^{-1})(1-a^{-1}b^{-1})(1-b^{-1}b^{-1})} = \frac{-b}{(a-b)(1-ab)(1-b^2)}$$

$$D = \frac{a^{-1}b^{-1}b^2}{(1-ab)(1-a^{-1}b)(1-bb)} = \frac{b}{(1-ab)(a-b)(1-b^2)}$$

$$a^n u(n) * a^{-n} u(-n) * b^n u(n) * b^{-n} u(-n) = Aa^n u(n) - Ba^{-n} u(-n-1) + Cb^n u(n) - Db^{-n} u(-n-1)$$