University of Ottawa

Assignment 2 - FIR causal Wiener solution

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Setup A

Question 1

$$X(z) = U(z)\,T(z)$$

$$D(z) = X(z)\,H(z) + V(z),$$

$$Y(z) = W(z)\,X(z)$$

$$E(z) = D(z) - Y(z)$$

In ideal case, the system can predict what's can be predicted, and whatever is left, is unpredictable of x.(i.e v(n) is a part of d(n) which can't be predicted.)

$$E(z) = D(z) - Y(z) = V(z)$$

$$W_{opt}(z) = H(z) = \frac{1}{1 - 0.95\, z^{-1}}$$

However, H(z) is an IIR filter, while W(z) has to be FIR filter. So, the best we can do is to approximate H(z) using very long FIR filter. I.e.

$$w_{opt}[n] = h[n]\,(u[n] - u[n-N]) = 0.95^{n}\,(u[n] - u[n-N])$$

Question 2

ϕ_{xx}

$$\phi_{xx}[l] = \phi_{uu}[l] * t[l] * t[-l]$$

- But

$$t[l] * t[-l] = \sum_{k=-\infty}^{\infty} t[l-k] \times t[-k]$$

$$= \sum_{k=-\infty}^{\infty} 0.98^{l-k}\, u[l-k]\, 0.98^{-k}\, u[-k]$$

- Interval 1: $l < 0$

$$t[l] * t[-l] = \sum_{k=-\infty}^{l} 0.98^{l-k} \times 0.98^{-k}$$

$$= 0.98^{l} \times \sum_{k=-\infty}^{l} 0.98^{-2k} \qquad ;\ \text{let } v = -k + l$$

$$= 0.98^{l} \times \sum_{v=0}^{\infty} 0.98^{-2(l-v)}$$

$$= 0.98^{l} \times 0.98^{-2l} \times \sum_{v=0}^{\infty} 0.98^{2v}$$

$$= 0.98^{-l} \left(\frac{1}{1 - 0.98^{2}} \right) = 25.25 \times 0.98^{-l}\, u[-l-1]$$

$$= 24.75 \times 0.98^{-l-1}\, u[-l-1]$$

- Interval 1: $l \geq 0$

$$t[l] * t[-l] = \sum_{k=-\infty}^{0} 0.98^{l-k} \times 0.98^{-k}$$

$$= 0.98^{l} \times \sum_{k=-\infty}^{0} 0.98^{-2k} \qquad ;\ \text{let } v = -k$$

$$= 0.98^{l} \times \sum_{v=0}^{\infty} 0.98^{-2(-v)}$$

$$= 0.98^{l} \times \sum_{v=0}^{\infty} 0.98^{2v}$$

$$= 0.98^{l} \left(\frac{1}{1 - 0.98^{2}} \right) = 25.25 \times 0.98^{l}\, u[l]$$

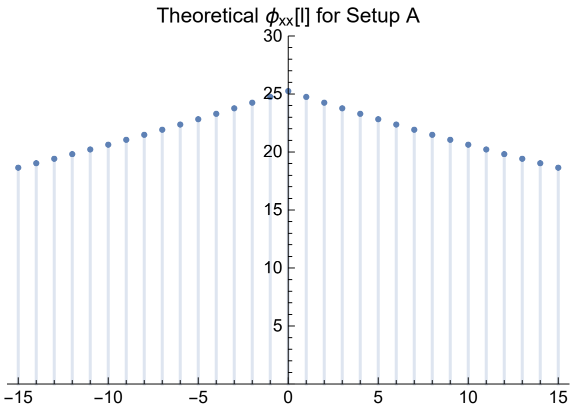
- Substitute

$$\phi_{xx}[l] = \phi_{uu}[l] * \left(25.25 \times 0.98^{l}\, u[l] + 25.25 \times 0.98^{-l}\, u[-l-1] \right)$$

$$= \sigma_u^2\, \delta[l] * \left(25.25 \times 0.98^{l}\, u[l] + 25.25 \times 0.98^{-l}\, u[-l-1] \right) \qquad ;\ \sigma_u^2 = 1$$

$$= 25.25 \left(0.98^{l}\, u[l] + 0.98^{-l}\, u[-l-1] \right)$$

Out[ ]=



Note

The convolution can also be calculated using the formulas given in the assignment (using z-transform)

$$t[l] * t[-l] = 0.98^l\, u[l] * 0.98^{-l}\, u[-l]$$
$$= A\, 0.98^l\, u[l] - B\, 0.98^{-l}\, u[-l-1]$$

- where

$$A = -B = \frac{1}{1-0.98^2} = 25.25$$

- So

$$t[l] * t[-l] = 25.25\left(0.98^l\, u[l] + 0.98^{-l}\, u[-l-1]\right)$$

ϕ_{dx}

$$\phi_{dx}[l] = \phi_{xx}[l] * h[l]$$
$$= 25.25\left(0.98^l\, u[l] + 0.98^{-l}\, u[-l-1]\right) * 0.95^l\, u[l]$$
$$= 25.25\left(0.98^l\, u[l] * 0.95^l\, u[l]\right) + 25.25\left(0.98^{-l}\, u[-l-1] * 0.95^l\, u[l]\right)$$

- where

$$0.98^l\, u[l] * 0.95^l\, u[l] = \mathrm{IZT}\left[\frac{1}{1-0.98\, z^{-1}}\, \frac{1}{1-0.95\, z^{-1}}\right]$$
$$= \mathrm{IZT}\left[\frac{A}{1-0.98\, z^{-1}} + \frac{B}{1-0.95\, z^{-1}}\right]$$

  - where

$$A = \left(\frac{1}{1-0.95\, z^{-1}}\right)_{z=0.98} = 32.6667$$

$$B = \left(\frac{1}{1-0.98\, z^{-1}}\right)_{z=0.95} = -31.6667$$

  - substitute

$$0.98^l\, u[l] * 0.95^l\, u[l] = 32.6667\left(0.98^l\, u[l]\right) - 31.6667\left(0.95^l\, u[l]\right)$$

- and

$$0.98^{-l}\ \mathrm{u}[-l-1] \ast 0.95^{l}\,\mathrm{u}[l] = \left(0.98^{-1}\right)^{-l}\ \mathrm{u}[-l-1] \ast 0.95^{l}\,\mathrm{u}[l]$$
$$= \mathrm{IZT}\left[\frac{-1}{1-0.98^{-1}\,z^{-1}}\ \frac{1}{1-0.95\,z^{-1}}\right]$$
$$= \mathrm{IZT}\left[\frac{B}{1-0.98^{-1}\,z^{-1}} + \frac{A}{1-0.95\,z^{-1}}\right]$$

  - where

$$A = \left(\frac{1}{1-0.98^{-1}\,z^{-1}}\right)_{z=0.95} = 13.5$$

$$B = \left(\frac{-1}{1-0.95\,z^{-1}}\right)_{z=0.98^{-1}} = -14.5$$

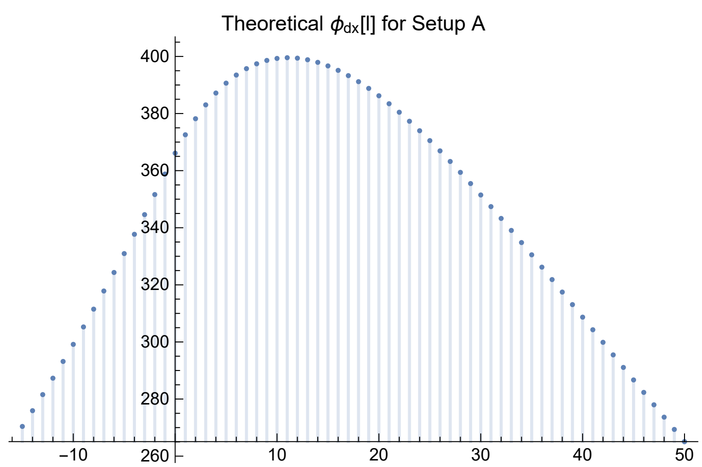
  - substitute

$$0.98^{-l}\ \mathrm{u}[-l-1] \ast 0.95^{l}\,\mathrm{u}[l] = 14.5\left(0.98^{-l}\,\mathrm{u}[-l-1]\right) + 13.5\left(0.95^{l}\,\mathrm{u}[l]\right)$$

- Substitute

$$\phi_{\mathrm{dx}}[l] = \phi_{\mathrm{xx}}[l] \ast \mathrm{h}[l]$$
$$= 25.25\left(0.98^{l}\,\mathrm{u}[l] \ast 0.95^{l}\,\mathrm{u}[l]\right) + 25.25\left(0.98^{-l}\ \mathrm{u}[-l-1] \ast 0.95^{l}\,\mathrm{u}[l]\right)$$
$$= 25.25\left(32.6667\left(0.98^{l}\,\mathrm{u}[l]\right) - 31.6667\left(0.95^{l}\,\mathrm{u}[l]\right)\right) +$$
$$25.25\left(14.5\left(0.98^{-l}\,\mathrm{u}[-l-1]\right) + 13.5\left(0.95^{l}\,\mathrm{u}[l]\right)\right)$$
$$= 824.834 \times 0.98^{l}\,\mathrm{u}[l] - 799.584 \times 0.95^{l}\,\mathrm{u}[l] +$$
$$366.125 \times 0.98^{-l}\,\mathrm{u}[-l-1] + 340.875 \times 0.95^{l}\,\mathrm{u}[l]$$

Out[∘]=



ϕ_{dd}

$$\phi_{\mathrm{dd}}[l] = \phi_{\mathrm{xx}}[l] \ast \mathrm{h}[l] \ast \mathrm{h}[-l] + \phi_{\mathrm{vv}}[l]$$
$$= \phi_{\mathrm{uu}}[l] \ast \left(\mathrm{t}[l] \ast \mathrm{t}[-l] \ast \mathrm{h}[l] \ast \mathrm{h}[-l]\right) + \phi_{\mathrm{vv}}[l]$$
$$= \sigma_{\mathrm{u}}^{2}\,\delta[l] \ast \left(0.98^{l}\,\mathrm{u}[l] \ast 0.98^{-l}\,\mathrm{u}[-l] \ast 0.95^{l}\,\mathrm{u}[l] \ast 0.95^{-l}\,\mathrm{u}[-l]\right) + \sigma_{\mathrm{v}}^{2}\,\delta[l]$$
$$= \mathrm{IZT}\left[\frac{A}{1-0.98\,z^{-1}}\ \frac{B}{1-0.98^{-1}\,z^{-1}}\ \frac{C}{1-0.95\,z^{-1}}\ \frac{D}{1-0.95^{-1}\,z^{-1}}\right] + \delta[l]$$

- where

$$A = \frac{0.98}{(1-0.98^2)(0.98-0.95)(1-0.98\times 0.95)} = 11\,955.3$$

$$B = \frac{-0.98}{(1-0.98^2)(0.98-0.95)(1-0.98\times 0.95)} = -11\,955.3$$

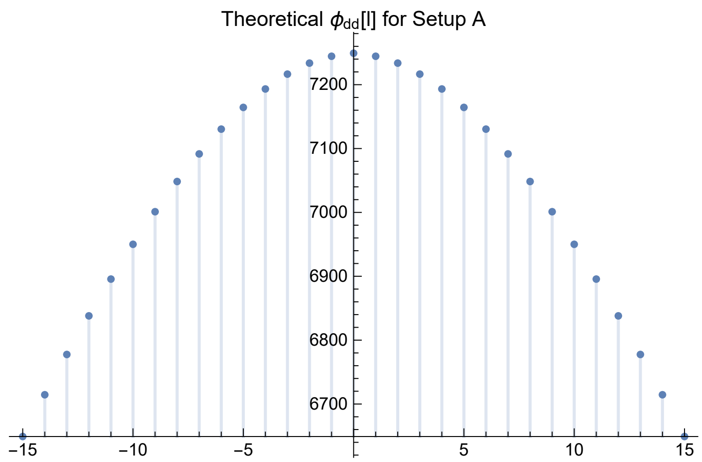
$$C = \frac{-0.95}{(1-0.95^2)(0.98-0.95)(1-0.98\times 0.95)} = -4707.05$$

$$D = \frac{0.95}{(1-0.95^2)(0.98-0.95)(1-0.98\times 0.95)} = 4707.05$$

- substitute

$$\phi_{dd}[l] = \phi_{xx}[l] * h[l] * h[-l] + \phi_{vv}[l]$$
$$= 11\,955.3\left(0.98^l\,u[l] + 0.98^{-l}\,u[-l-1]\right) -$$
$$4707.05\left(0.95^l\,u[l] + 0.95^{-l}\,u[-l-1]\right) + \delta[l]$$

Out[ ]=

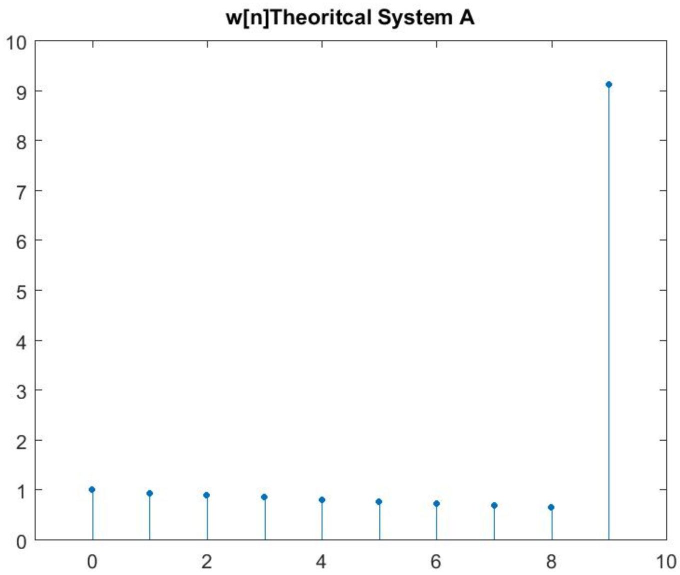


Note

$\sigma_d^2 = \phi_{dd}[0] = 7249.25$

Theoretical W_{opt}

$W_{opt} = R^{-1}\,P$; the plot is shown below



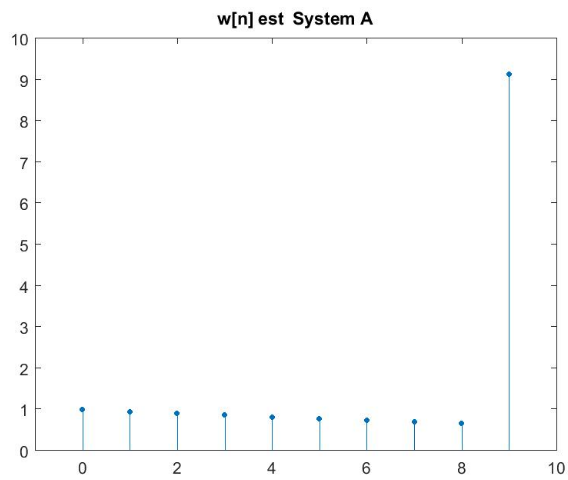
$$\text{Normalized MMSE} = \frac{\sigma_d^2 - P^H R^{-1} P}{\sigma_d^2} = 0.1062$$

Note

- x[n] is colored noise (i.e. is band limited unlike the white noise). h[n] is an IIR filter and $W_{opt}[n]$ is FIR filter(*i.e.* it has *a* shorter impuse response).
- Since h[n] is a decaying function, it is possible to approximate the IIR filter using FIR Wiener solution. However, this requires length N to be large enough.
- Using N = 10 samples, as shown above, resulted in poor prediction of h[n] because
  - the number of samples is not enough.
  - W_{opt} is not only trying to model h[n] but also predict the sensor white noise (v[n]).
- This can be observed by looking at the last sample of $w_{opt}[n]$. The explanation for this behaviour is as follows:
  - since v[n] is a white noise that should not be included, it's desired that to have a narrow bandwidth FIR Wiener solution. This can be achieved by increasing the filter length (in the time domain).
  - In the case with N = 10 sample, the filter is very short leading to a wideband frequency response. This allows many frequencies from the v[n] to be included (resulting in corrupting w_{opt}).

Question 3

- The Estimated w_{opt} for Setup A (using MATLAB)



- The Estimated normalized MMSE for Setup A = 0.1067

Note

- It's noticeable that the Estimate w_{opt}, MMSE agree with the theoretical ones.

System B

Question 1

Let's define B(z) to be the equivalent system of $1 - z^{-1}$W(z). Then

$$B(z) = \frac{E(z)}{X(z)}$$

$$= \frac{D(z) - Y(z)}{X(z)} = \frac{X(z) - Y(z)}{X(z)}$$

$$= 1 - \frac{Y(z)}{X(z)}$$

$$= 1 - \frac{X(z) Z^{-1} W(z)}{X(z)}$$

$$= 1 - Z^{-1} W(z)$$

Since U (z), E (z) are white noise, then :

$$B(z) = T^{-1}(z)$$

$$1 - Z^{-1} W(z) = \frac{1}{T(z)} = 1 - 0.98 Z^{-1}$$

$$W(z) = 0.98$$

$$w(n) = 0.98\,\delta(n)$$

In ideal case, Current signal can be predicted from past sample, and the error will be white noise.

Question 2

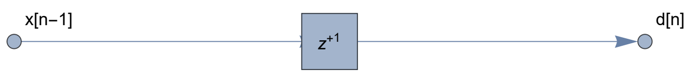
ϕ_{xx}

ϕ_{xx} should be calculated at the input of Wiener filter $w[n]$ (i.e. from x[n-1] not x[n]). However, since the delay system has an impulse response of $\delta[l\text{-}1]$.
Then $\delta[l\text{-}1] * \delta[\text{-}l\text{-}1] = \delta[l]$. i.e. calculating the correlation from x[n-1] is the same as correlation from x[n].

$$\begin{aligned}\phi_{xx}[l] &= \phi_{uu}[l] * \mathrm{t}[l] * \mathrm{t}^{*}[-l] \\ &= 25.25\left(0.98^{l}\,\mathrm{u}[l] + 0.98^{-l}\,\mathrm{u}[-l-1]\right) \quad ;\ \text{from Setup A}\end{aligned}$$

ϕ_{dx}

The relationship between $d[n]$ and $x[n-1]$ is shown below

Out[∘]=


$$\begin{aligned}\phi_{dx}[l] &= \phi_{xx}[l] * \delta[l+1] \\ &= \phi_{xx}(l+1) \\ &= 25.25\left(0.98^{l+1}\,\mathrm{u}[l+1] + 0.98^{-l-1}\,\mathrm{u}[-l-2]\right)\end{aligned}$$

ϕ_{dd}

$$\begin{aligned}\phi_{dd}[l] &= \phi_{xx}[l] * \delta[l] * \delta[-l] \\ &= \phi_{xx}[l] \\ &= 25.25\left(0.98^{l}\,\mathrm{u}[l] + 0.98^{-l}\,\mathrm{u}[-l-1]\right) \quad ;\ \text{from Setup A}\end{aligned}$$

Theoretical W_{opt}

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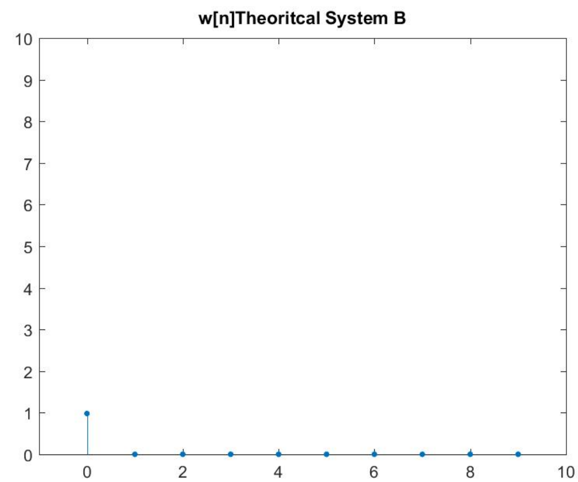


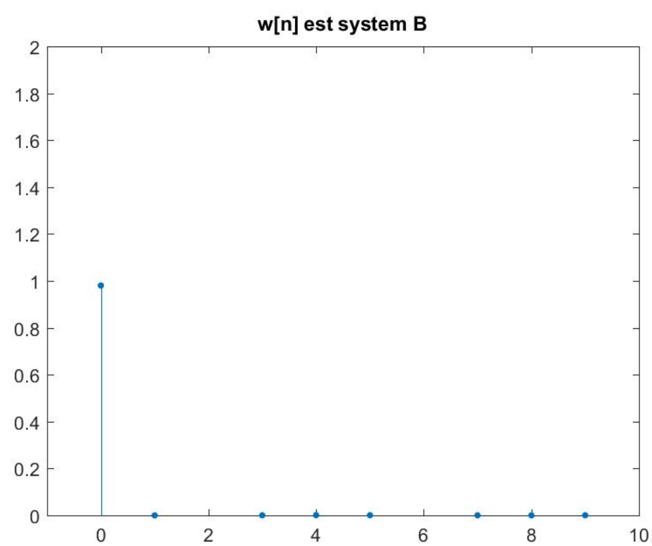
$$\text{Normalized MMSE} = \frac{\sigma_d^2 - P^H R^{-1} P}{\sigma_d^2} = 0.0396$$

Note

- As we can see, the Wiener solution agrees with the solution in Question 1. Only one sample is required to predict the next sample of x[n] from its previous value x[n-1] (i.e. $w_{opt}[n] = 0.98\ \delta[n]$)

Question 3

- The Estimated Wiener solution w_{opt} for Setup A (using MATLAB)



- The Estimated normalized MMSE for Setup B = 0.0395

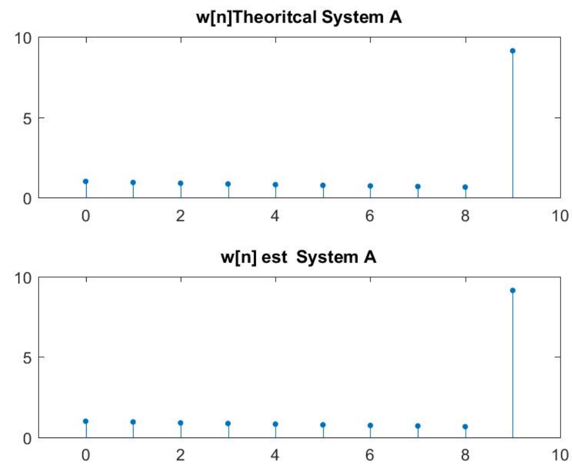
Note

- It's noticeable that the Estimate w_{opt}, MMSE agree with the theoretical ones.

Appendix

Below is a side-by-side comparison of the theoretical and estimated solution for setup A and B.

Setup A



Setup B