Assignment 1

Estimation of correlation and PSD functions for LTI systems inputs and outputs

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Question 1

[A] Theoretical expressions for ϕ

The theoretical Cross-correlation of $\phi_{yd}(l)$:

$\phi_{yd} = \phi_{xd} * h(l)$

But;

$\phi_{xd} = \phi_{dx}(-l)$

$\phi_{dx} = \phi_{xx}(l) * g(l)$; $\phi_{xx} = \sigma x^2\, \delta(l) + |\, mx\, |^2$

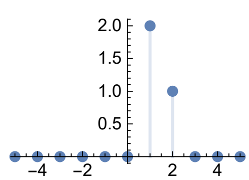
$\sigma x^2 = 1$; $mx = 5$

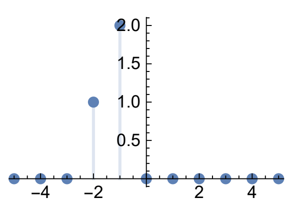
$= (\delta(l)+25)^* g(l)$

$= (\, g(l)^*\, \delta(l)\,) + (25^* g(l))$

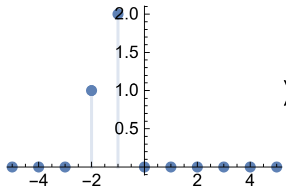
$= g(l) + 25\, \sum(g(k))$

$= g(l) + 25\, \sum(2+1)$

$=$  $+ 75$

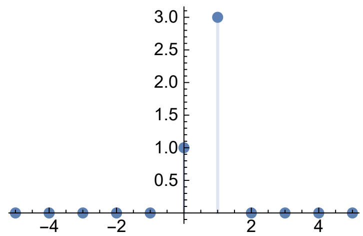
$\phi_{xd} =$  $+ 75$

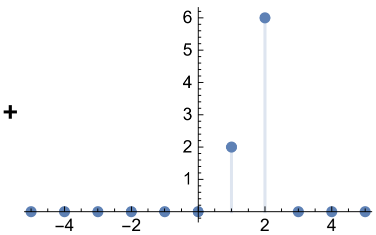
$\phi_{yd} = \phi_{xd} * h(l)$

= (75 + ) * h(l)

= (75(1+3)) + (($\phi_{xd}(-2)$ h(l+2) + ($\phi_{xd}(-1)$ h(l+1))

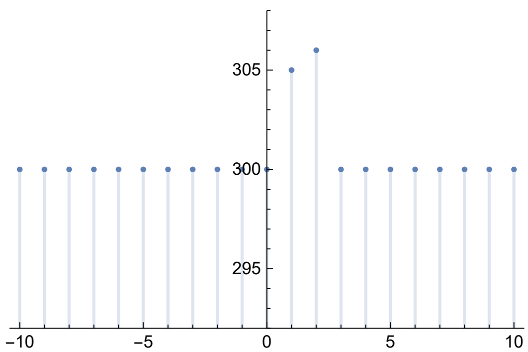
= 300 + (h(l+2) + 2h(l+1)

= 300 + 

\+ 

So, $\phi_{yd}(l)$ is

Out[18]=



The theoretical Cross-correlation of $\phi_{yx}(l)$:

ϕ_{yx} = $\phi_{xx} \star h(l)$

But;

$\phi_{xx} = \sigma x^2\, \delta(l) + |\ mx\ |^2$

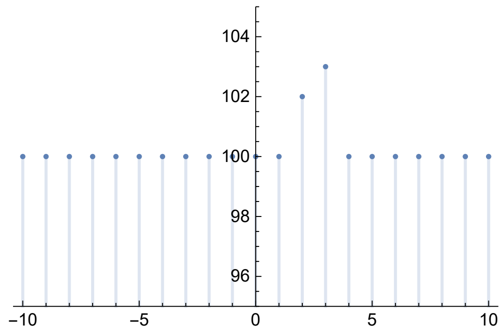
h(l) = δ(n-2) + 3δ(n-3)

ϕ_{yx} = {$\delta(n-2) + \delta(n-3)$} * {$\delta$(l) +25}

= {δ(n-2)+δ(n-3)} + {25(1+3)}

So, $\phi_{yx}(l)$ is

Out[20]=

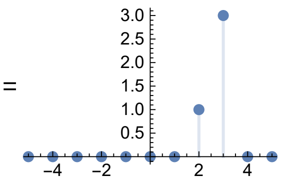


The theoretical Auto-correlation of $\phi_{yy}(l)$:

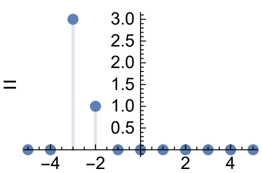
$\phi_{yy} = \phi_{xx} * h(l)*h(-l)$

But;

$h(l) = \delta(l-2) + 3\,\delta(l-3)$ =



$h(-l) = \delta(l+2) + 3\,\delta(l+3)$ =



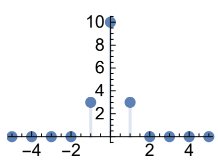
Now;

$v(l) = h(l)*h(-l)$

$= \sum_{k=-\infty}^{\infty} h(-k)\, h(l+k)$

$= \sum_{k=-3}^{-2} h(-k)\, h(l+k)$

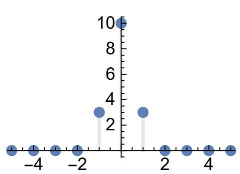
$= 3h(l+3) + 2h(l+2)$ =



Substitute;

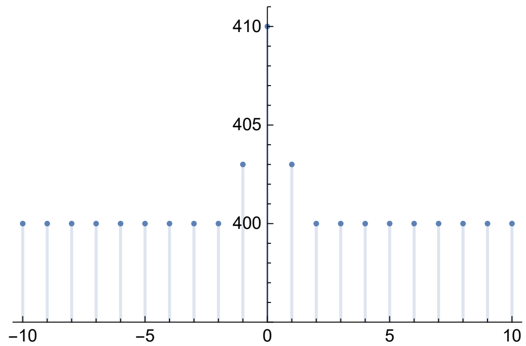
$\phi_{yy} = \phi_{xx} * v(l)$

$= (\delta(l) + 25)$ *



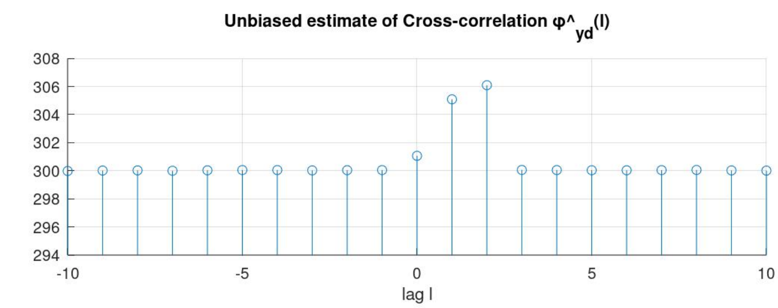
So, $\phi_{yy}(l)$ is

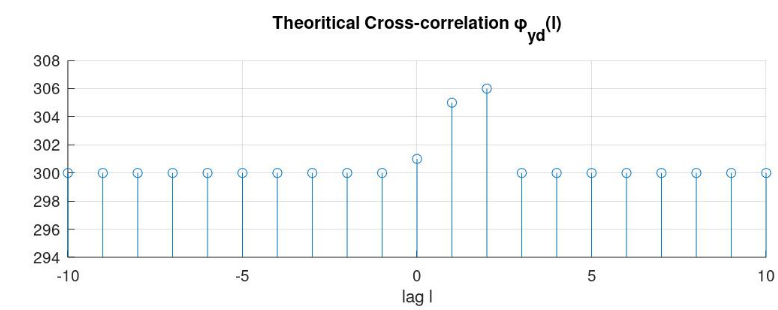
Out[21]=



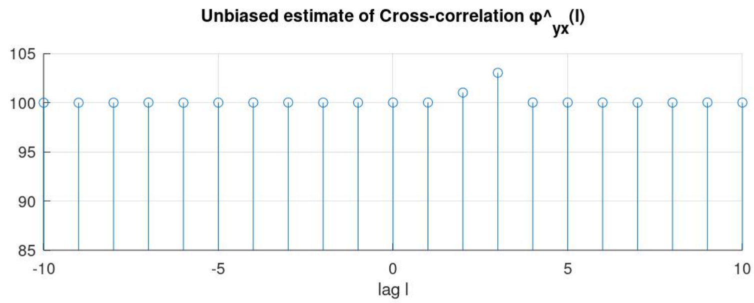
[B] Theoretical vs Estimate of ϕ

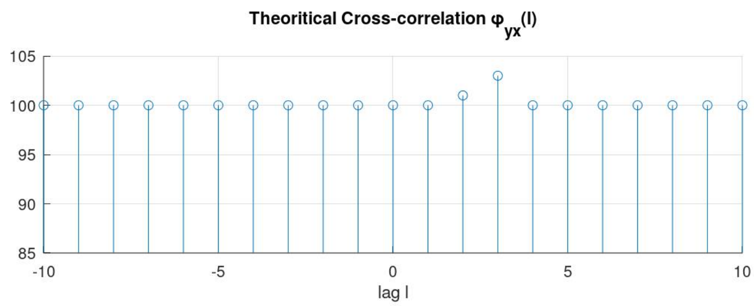
- Theoretical Cross-correlation and Unbiased estimate of Cross-correlation ϕ_{yd}



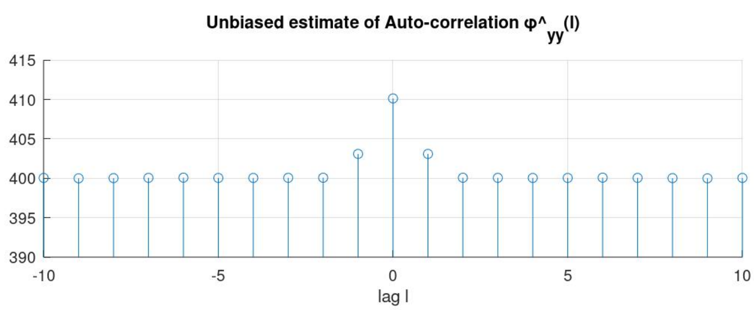


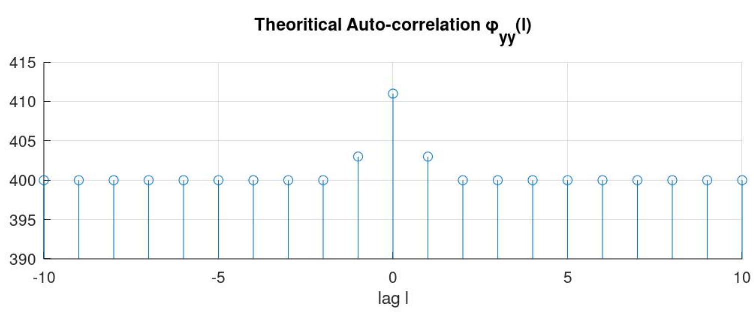
- Theoretical Cross-correlation and Unbiased estimate of Cross-correlation ϕ_{yx}





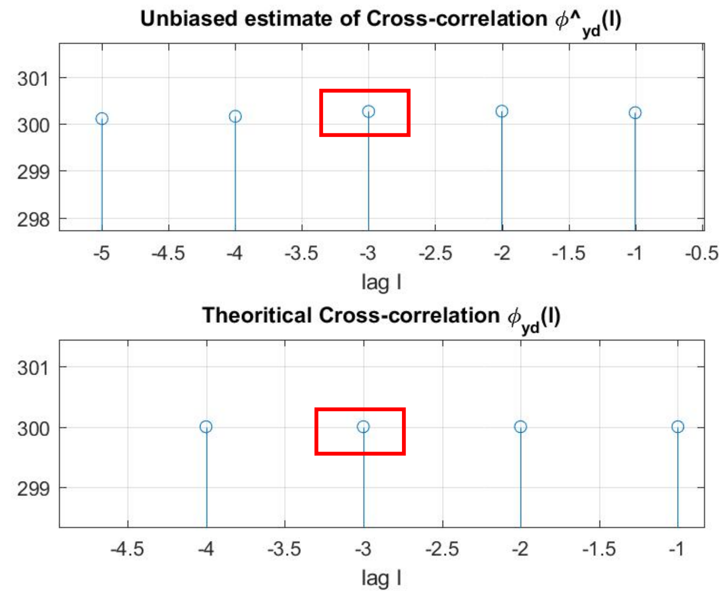
- Theoretical Auto-correlation and Unbiased estimate of Cross-correlation ϕ_{yy}





Notes

By looking at the three results above, we can see that the theoretical and measured in each plot are nearly the same -because we have a lot of samples-. However, if we zoom in a little bit (e.g. as shown below), w will see that that they are not exactly the same. The measured signal is exactly on the line, but the theoretical one is not.



[C.1] Theoretical Expressions for γ

The theoretical Cross-correlation of γ_{yd}

$\gamma_{yd}(l) = \gamma_{xd}(l) * h(l)$

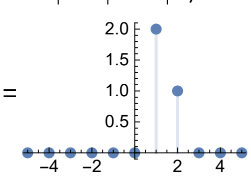
But;

$\gamma_{xd} = \gamma_{dx}(-l)$

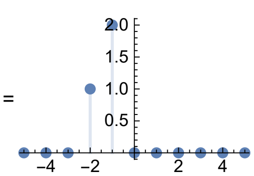
$\gamma_{dx} = \gamma_{xx}(l) * g(l)$

$\gamma_{xx}(l) = (\phi_{xx}(l) - |\,mx\,|^2)$

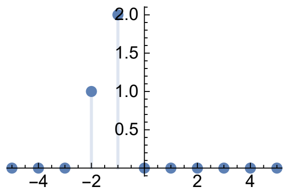
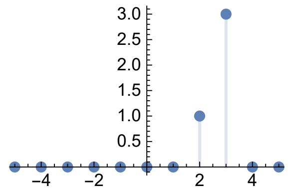
$= (\,\delta(l)\,) + |\,mx\,|^2 - |\,mx\,|^2)$

$\gamma_{dx} = \delta(l) * g(l) =$ 

So,

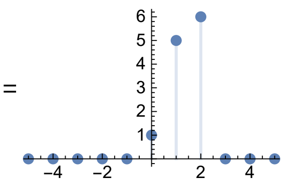
$\gamma_{xd} =$ 

Substitute;

$\gamma_{yd} = \gamma_{xd} * h(l) =$  $*$ 

$= \sum_{k=-\infty}^{\infty} \gamma_{xd}(k)\, h(1-k)$

$= \sum_{k=-2}^{-1} \gamma_{xd}(k)\, h(1-k)$

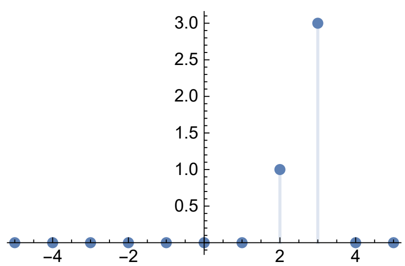
$= 1\, h(l+2) + 2h(l+1) =$ 

The theoretical Cross-correlation of γ_{yx}

$\gamma_{yx}(l) = \gamma_{xx}(l) * h(l)$

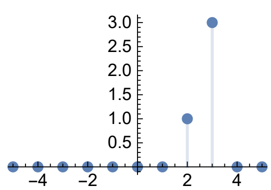
$= (\phi_{xx}(l) - |\, mx\,|^2) * h(l)$

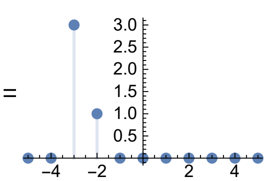
$= (\delta(l)) + |\, mx\,|^2 - |\, mx\,|^2) * h(l)$

$\gamma_{yx} = \delta(l) * h(l) =$ 

The theoretical Auto-covariance of γ_{yy}

$\gamma_{yy} = \gamma_{xx} * h(l) * h(-l)$

$h(l) = \delta(l-2) + 3\,\delta(l-3) =$ 

$h(-l) = \delta(l+2) + 3\delta(l+3) =$ 

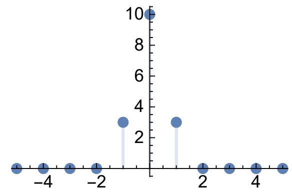
where;

$v(l) = h(l) * h(-l)$

$= \sum_{k=-\infty}^{\infty} h(-1)\, h(1+k)$

$= \sum_{k=-3}^{-2} h\,(-k)\, h(l+k)$

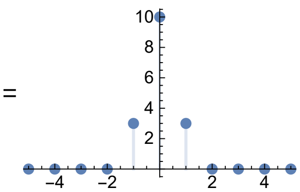
$= 3h(l+3) + 2h(l+2) \quad =$



Substitute;

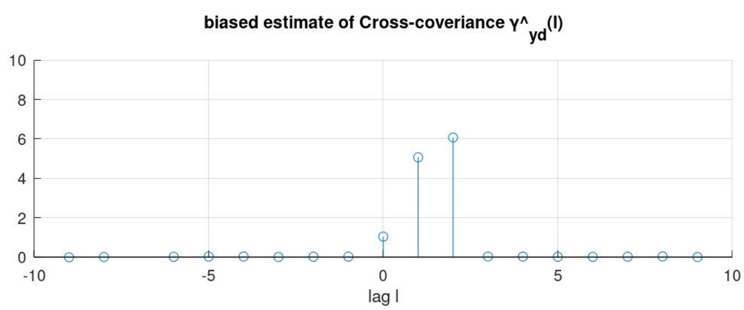
$\gamma_{yy} \;=\; \gamma_{xx} * v(l)$

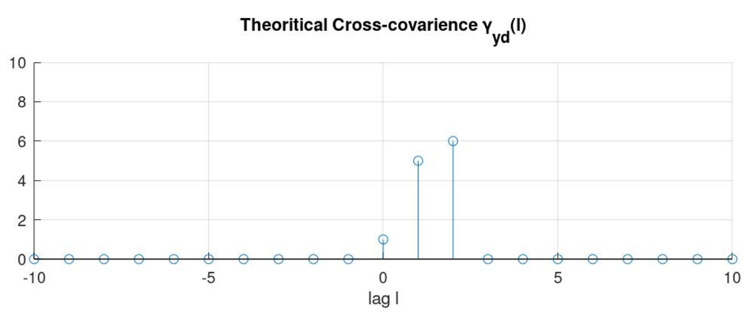
$= \delta(l) * v(l) =$



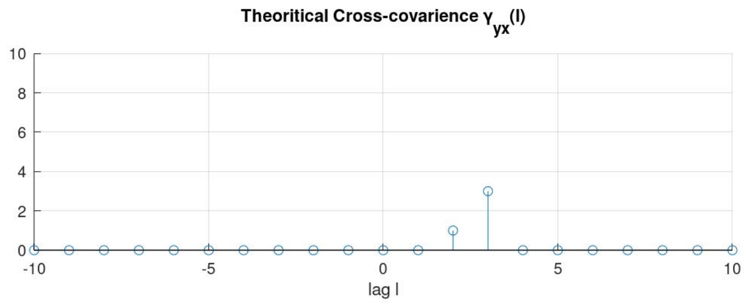
[C.2] Theoretical vs Estimate of γ

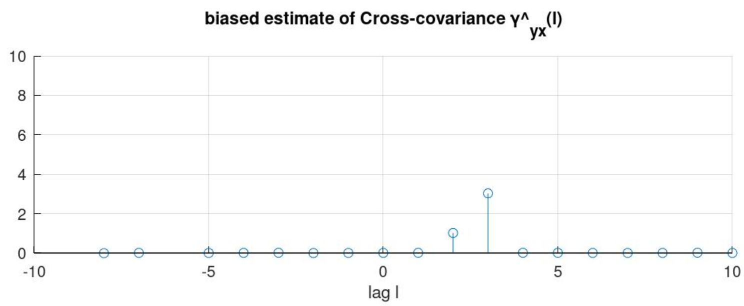
- Theoretical Cross-covariance and biased estimate of Cross-covariance γ_{yd}





- Theoretical Cross-covariance and biased estimate of Cross-covariance γ_{yx}

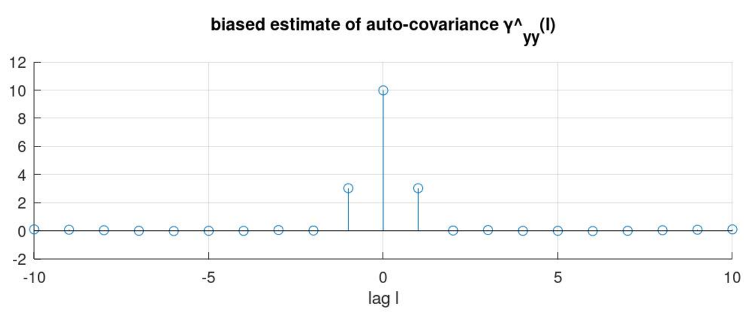


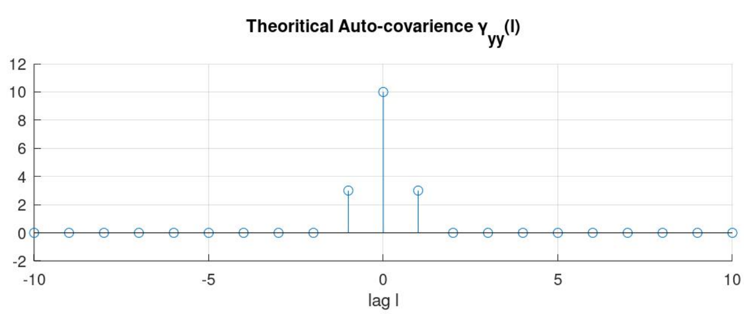


Notes

Theoretical Auto-covariance and biased estimate of Cross-covariance γ_{yy}

Theoretical Auto-covariance and Unbiased estimate of Auto-covariance γ_{yy}



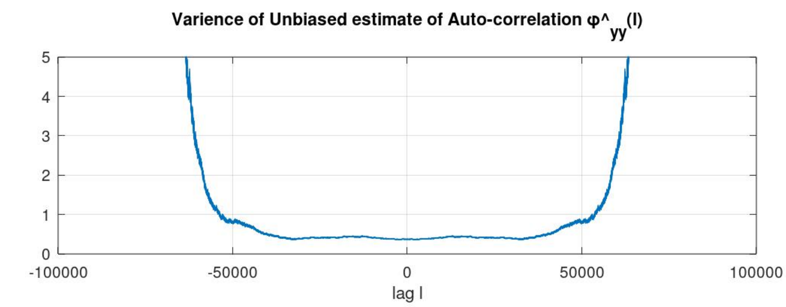


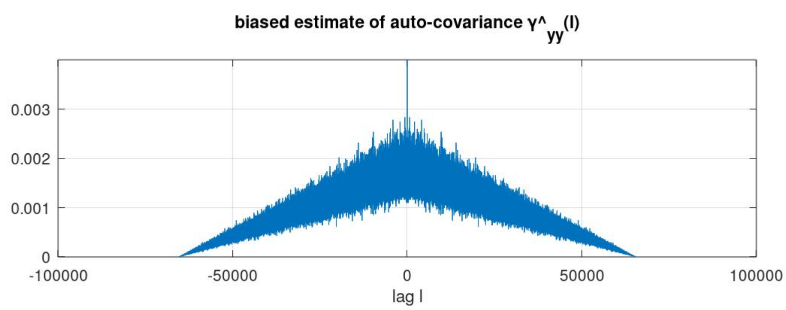
Notes

- P is defined from the signal which we removed the mean .
- The results look a little bit like the previous part, But here there is no DC component.
- The cross covariance is nonzero just for few samples.
- Also, the estimate is look very close to the ideal value [because there is a lot of samples].

[D] Variance of ϕ vs. γ Estimates

Generating N = 100 sequences of y(n), and computing the variance of the estimate





Notes

- In addition the fact that covariance (γ)has mean removed but correlation(ϕ) does not , **their variance is different**.
- There are few samples that are used in computing the ϕ function estimate, resulting in a **very noisy estimate** and estimate with a **lot of Variance** in the ϕ function estimate.
- Also, there is a **lot of fluctuations** from one estimate to the next in the ϕ function estimate.
- However, for the γ function, the importance of the samples at the edges is **reduced.**
- From the the figure above, it’s clearly shown that for ϕ, as the **lags larger**, the **bias increase** (at both edges; the positive and negative large values of lags).
- However, for γ, the **variance of the estimate is getting smaller** and smaller as the **lags increase**.

- In conclusion, the second estimate is better in terms of the variance, with less fluctuations and less noisy in it's fourier transform.

Question 2

[A] Theoretical Expressions for ϕ

The theoretical Cross-correlation of $\phi_{yd}(e^{j\omega})$:

$$\Phi_{yd}(e^{j\omega}) = \phi_{xd}(e^{j\omega}) H(e^{j\omega})$$

But;

$$\phi_{dx}(e^{j\omega}) = \phi_{xd}(e^{-j\omega})$$
$$= \phi_{xx}(e^{j\omega})\, G(e^{j\omega})$$

where

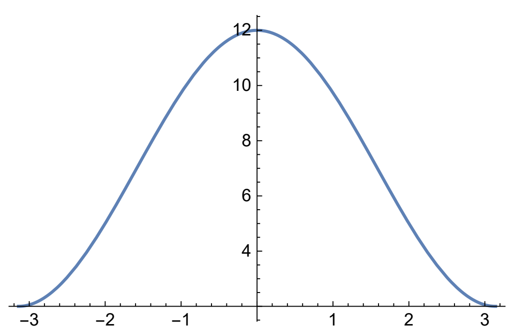
$$\phi_{xx}(e^{j\omega}) = \text{FT}\{\gamma\}$$
$$= FT\{\sigma_x^2\, \delta(l) + |\, m_x\,|^2 - |\, m_x\,|^2\}$$
$$= 1$$
$$G(e^{j\omega}) = 2\, e^{j\omega} + e^{j2\omega}$$

$$\phi_{dx}(e^{j\omega}) = 2\, e^{j\omega} + e^{j2\omega}$$
$$\phi_{xd}(e^{-j\omega}) = 2\, e^{-j\omega} + e^{-j2\omega}$$

$$H(e^{j\omega}) = e^{j2\omega} + 3\, e^{j3\omega}$$

$$\phi_{yd}(e^{j\omega}) = \phi_{xd}(e^{j\omega})\, H(e^{j\omega})$$
$$= 2\, e^{j\omega} + 1 + 6\, e^{j2\omega} + 3\, e^{j\omega}$$

Out[22]=



The theoretical expressions of $\phi_{yx}(e^{j\omega})$:

$$\phi_{yx}(e^{j\omega}) = \phi_{xx}(e^{j\omega})\, H(e^{j\omega})$$

$$\phi_{xx}(l) = \sigma x^2\, \delta(l) + |\text{ mx }|^2$$

$\phi_{xx}(e^{j\omega}) = F.T(\delta(l) + |mx|^2 - |mx|^2)$

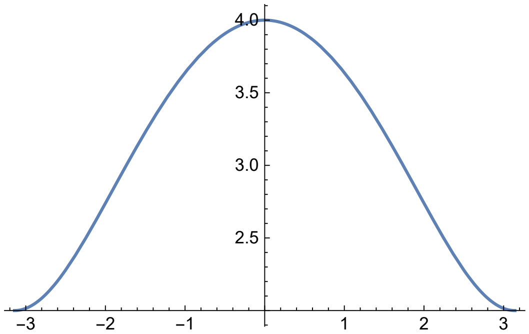
$= 1$

$H(e^{j\omega}) = e^{j2\omega} + 3\,e^{j3\omega}$

$\phi_{yx}(e^{j\omega}) = e^{j2\omega} + 3\,e^{j3\omega}$

$= e^{j2\omega} + 3\,e^{j3\omega}$

Out[23]=



The theoretical expressions of $\phi_{yy}(e^{j\omega})$:

$\phi_{yy}(e^{j\omega}) = \phi_{xx}(e^{j\omega})\,|H(e^{j\omega})|^2$

But;

$|H(e^{j\omega})|^2 = \{\cos(2\omega) + 3\cos(3\omega)\}^2 + \{\sin(2\omega) + 3\sin(3\omega)\}^2$

$= 1 + 9 + 6(\cos(2) + \cos(3) + \sin(2) + \sin(3))$

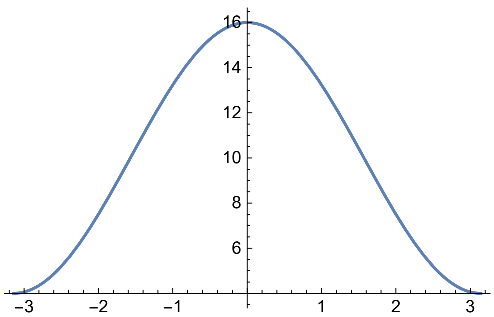
$= 10 + 6\cos(w)$

$\phi_{yy}(e^{j\omega}) = \phi_{xx}(e^{j\omega})\,|H(e^{j\omega})|^2$

$= \{10 + 6\cos(w)\}\{1\}$

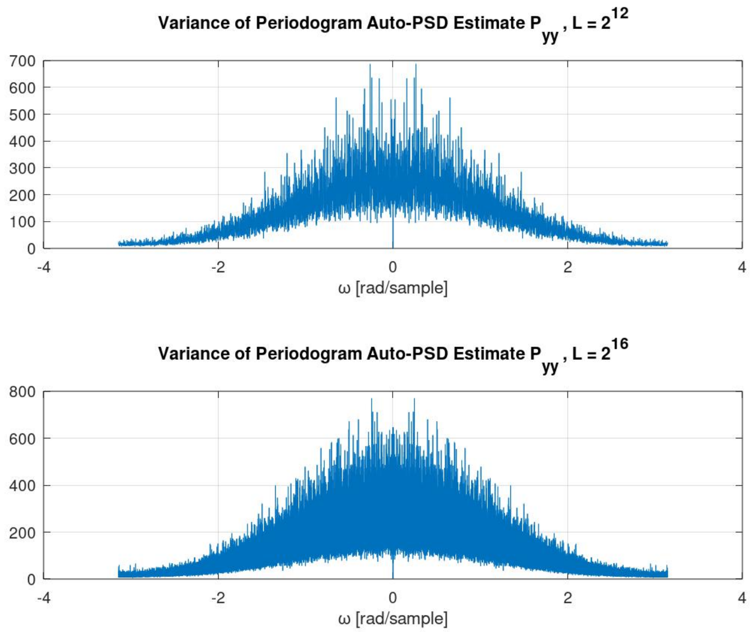
$= 10 + 6\cos(w)$

Out[27]=



[B] Variance of P under Different L

Generating N = 100 sequences of y(n), and computing the variance of the Periodogram estimate

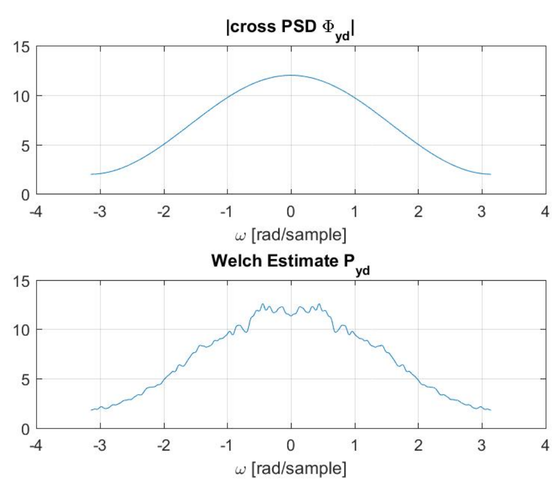


Note

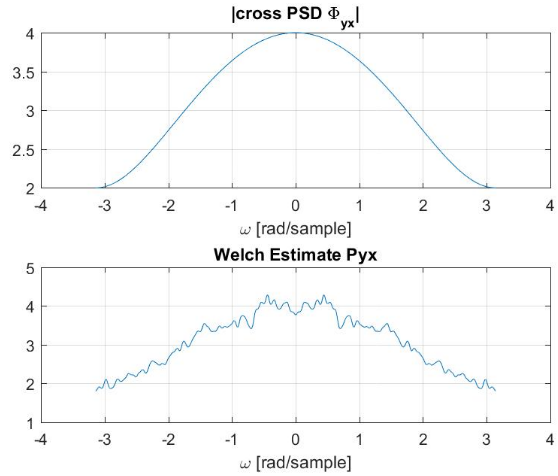
- By looking at the result, we notice that the variance for the periodogram doesn't get better for different values of L.
- Even though the peaks look worse for smaller values of L, the average values doesn't get better for larger values of L.

[C] Theoretical vs Estimate of Φ

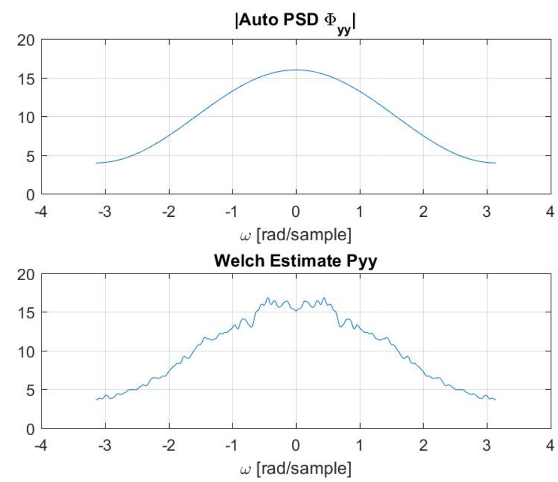
- Theoretical Cross-PSD and Welch estimate of Φ_{yd}



- Theoretical Cross-PSD and Welch estimate of Φ_{yx}



- Theoretical Auto-PSD and Welch estimate of Φ_{yx}

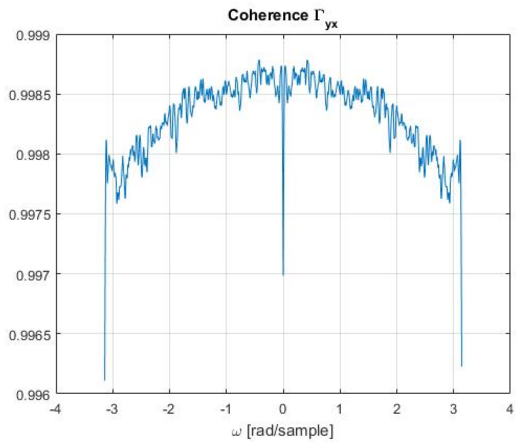


Notes

By looking at the results above, it’s noticeably that :

- Welch Estimate is a better estimate than Periodogram Estimate.
- The correct average value can be got at any frequency.

[D] Coherence



Notes

By looking at the result above, it’s noticeably that :

- The practical coherence (almost 0.9985) is very close to the theoretical one (which has the value of 1 at all frequencies).