Capítulo_8.5

Thursday, October 8, 2020 12:05 AM

1. $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, μ desconhecida e σ^2 conhecida.
Defina $A = \bar{X}_n - \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\frac{\sigma}{n^{1/2}}$
$B = \bar{X}_n + \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\frac{\sigma}{n^{1/2}}$,
onde Φ é a cdf de uma normal padrão.

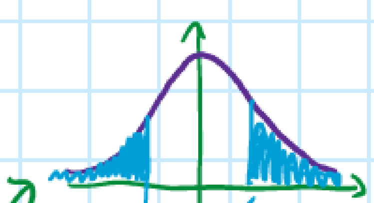
Queremos mostrar $\mathbb{P}(A < \mu < B) \geq \gamma$.

$$\mathbb{P}(A \geq \mu) = \mathbb{P}\left(\bar{X}_n - \mu \geq \Phi^{-1}\left(\tfrac{1+\gamma}{2}\right)\tfrac{\sigma}{n^{1/2}}\right)$$
$$= \mathbb{P}\left(\frac{\bar{X}_n - \mu}{\sigma/n^{1/2}} \geq \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\right)$$
$$= \mathbb{P}\left(Z \geq \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\right)$$
$$= 1 - \left(\frac{1+\gamma}{2}\right) = \frac{1-\gamma}{2},$$

Lembre que $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

dado que $\frac{\bar{X}_n - \mu}{\sigma/n^{1/2}} = Z \sim N(0,1)$

$$\mathbb{P}(B > \mu) = \mathbb{P}\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} > -\Phi^{-1}\left(\frac{1+\gamma}{2}\right)\right)$$
$$= 1 - \Phi\left(-\Phi^{-1}\left(\tfrac{1+\gamma}{2}\right)\right)$$
$$= 1 - \left(1 - \Phi\left(\Phi^{-1}\left(\tfrac{1+\gamma}{2}\right)\right)\right)$$
$$= \frac{1+\gamma}{2}$$



$\mathbb{P}(Z \leq -x) = \mathbb{P}(Z \geq x) = 1 - \mathbb{P}(Z \leq x)$

Por fim $\mathbb{P}(A < \mu < B) = \mathbb{P}(B > \mu) - \mathbb{P}(A \geq \mu)$
$$= \frac{1+\gamma}{2} - \frac{1-\gamma}{2}$$
$$= \gamma$$

Isso prova que (A, B) é int. confiança com coeficiente γ (exato!)

4. $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, μ desconhecida e σ^2 conhecida.
Queremos encontrar n tal que $\mathbb{P}(A < \mu < B) \geq 0.95$ e $(B-A) < 0.01\sigma$
Pelo exercício (1), temos A e B que satisfazem essa relação.
$A = \bar{X}_n - \Phi^{-1}(0.975)\,\sigma/\sqrt{n}$
$B = \bar{X}_n + \Phi^{-1}(0.975)\,\sigma/\sqrt{n}$
Queremos que $B - A = 2\Phi^{-1}(0.975)\cancel{\sigma}/\sqrt{n} < 0.01\cancel{\sigma}$, portanto,
$[200\,\Phi^{-1}(0.975)]^2 < n$
$\approx (200 \cdot 1{,}96)^2 < n \to n > 153658$

5. $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, desconhecidas.
Queremos um intervalo de confiança para σ^2.
Sabemos que $\sum_{i=1}^n (X_i - \bar{X}_n)^2/\sigma^2 \sim \chi^2(n-1)$. Seja G sua cdf.
Seja $c_i = G^{-1}(\gamma_i)$, $i = 1, 2$ e $\gamma = \gamma_2 - \gamma_1$. Assim temos:
$$\mathbb{P}\left(c_1 < \textstyle\sum_{i=1}^n (x_i - \bar{x}_n)^2/\sigma^2 < c_2\right) = \mathbb{P}\left(\textstyle\sum_{i=1}^n (x_i - \bar{x}_n)^2/\sigma^2 < c_2\right)$$
$$- \mathbb{P}\left(\textstyle\sum_{i=1}^n (x_i - \bar{x}_n)^2/\sigma^2 \leq c_1\right)$$
$$= \gamma_2 - \gamma_1 = \gamma.$$
Portanto: $\mathbb{P}\left(c_1 < \sum_{i=1}^n (x_i - \bar{x}_n)^2/\sigma^2 < c_2\right) = \mathbb{P}\left(\sigma^2 c_1 < \sum_{i=1}^n (x_i - \bar{x}_n)^2 < c_2\sigma^2\right)$
$$= \mathbb{P}\left(\tfrac{1}{c_2}\textstyle\sum_{i=1}^n (x_i - \bar{x}_n)^2 < \sigma^2 < \tfrac{1}{c_1}\sum_{i=1}^n (x_i - \bar{x}_n)^2\right)$$

Concluo que (A, B), tal que
$$A = \frac{1}{c_2}\sum_{i=1}^n (x_i - \bar{x}_n)^2$$
$$B = \frac{1}{c_1}\sum_{i=1}^n (x_i - \bar{x}_n)^2$$
é intervalo de confiança exato para σ^2 com coeficiente γ.

6. $X_1, \dots, X_n \sim \exp(1/\mu)$, $E[X_1] = 1/1/\mu = \mu$.
Sabemos que $\sum_{i=1}^n X_i \sim \text{Gamma}(n, 1/\mu)$ (Resumo capítulo 8)
Além disso $1/\mu \sum_{i=1}^n X_i \sim \text{Gamma}(n, 1)$ (Scaling, wikipedia, Gamma distribution).
Seja G a cdf da Gamma.
Defina $c_i := G^{-1}(\gamma_i)$, $\gamma_2 - \gamma_1 = \gamma$.
Logo:
$\mathbb{P}(c_j \geq 1/\mu \sum_{i=1}^n x_i) = G(G^{-1}(\gamma_j)) = \gamma_j$
$\mathbb{P}(c_1 < 1/\mu \sum_{i=1}^n x_i < c_2) = \gamma_2 - \gamma_1 = \gamma$.

Portanto $\mathbb{P}(\mu c_1 < \sum_{i=1}^n x_i < \mu c_2) = \mathbb{P}\left(1/c_2 \sum_{i=1}^n x_i < \mu < 1/c_1 \sum_{i=1}^n x_i\right)$
Defina
$$A = \frac{1}{G^{-1}(\gamma_2)}\sum_{i=1}^n x_i$$
$$B = \frac{1}{G^{-1}(\gamma_1)}\sum_{i=1}^n x_i$$
(A, B) é int. confiança exato com parâmetro γ.

Obs.: $2\sum_{i=1}^n x_i \sim \chi^2(2n) = \text{Gamma}(n, 1/2)$
μ
Escala