Teste 2 - EDO

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1. $\begin{bmatrix} I' \\ V' \end{bmatrix} = \begin{bmatrix} -1/2 & -1/8 \\ 2 & -1/2 \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}$

a) $A = \begin{bmatrix} -1/2 & -1/8 \\ 2 & -1/2 \end{bmatrix}$

$p_\lambda = \lambda^2 + \lambda + 1/2 = 0$

$\lambda_1 = -\frac{1}{2} - \frac{1}{2}i, \quad \lambda_2 = -\frac{1}{2} + \frac{1}{2}i$

$A - \lambda_1 I = \begin{bmatrix} 1/2\, i & -1/8 \\ 2 & 1/2\, i \end{bmatrix} \Rightarrow v_1 = (1, 4i), \quad v_2 = (1, -4i),$

Logo: $\underbrace{\boxed{a = -\frac{1}{2}, \quad b = \frac{1}{2}}}_{\lambda_2}, \quad \underbrace{\boxed{u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ -4 \end{pmatrix}}}_{v_2}$

$$X(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} e^{-1/2 t} \begin{bmatrix} \cos(1/2 t) & -\operatorname{sen}(1/2 t) \\ \operatorname{sen}(1/2 t) & \cos(1/2 t) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \frac{1}{4} \begin{bmatrix} I_0 \\ V_0 \end{bmatrix}$$

$$= \frac{e^{-1/2 t}}{4} \begin{bmatrix} \operatorname{sen}(1/2 t) & \cos(1/2(t)) \\ -4\cos(1/2 t) & 4\operatorname{sen}(1/2 t) \end{bmatrix} \begin{bmatrix} -V_0 \\ 4 I_0 \end{bmatrix}$$

$$= \frac{e^{-1/2 t}}{4} \begin{bmatrix} -V_0 \operatorname{sen}(1/2 t) + 4 I_0 \cos(1/2 t) \\ 4 V_0 \cos(1/2 t) + 16 I_0 \operatorname{sen}(1/2 t) \end{bmatrix} //$$

b) Para $\begin{bmatrix} I_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $X(t) = \frac{e^{-1/2 t}}{4} \begin{bmatrix} -3\operatorname{sen}(1/2 t) & 8\cos(1/2 t) \\ 12\cos(1/2 t) & 32\operatorname{sen}(1/2 t) \end{bmatrix} \Big\} (*)$

c) $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} V(t) = 0$, pois $e^{-1/2 t} \xrightarrow[t\to\infty]{} 0$, independente de I_0 e V_0, dado que $(*)$ é limitada em cada componente.

2) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$ (1) $\quad (a+d < 0$ e $ad - bc > 0)$

($\Rightarrow$) Suponha que as soluções de (1) tendem a 0. Então A tem autovalores (não necessariamente distintos) com parte real negativa. Nesse caso:

$$p_\lambda = \lambda^2 - (a+d)\lambda + (ad - bc)$$
$$\lambda_1 + \lambda_2 = (a+d) \qquad (2)$$
$$\lambda_1 \cdot \lambda_2 = ad - bc$$

Suponha que $\lambda_1 \cdot \lambda_2 \leq 0$. Se $\lambda_2 = \bar{\lambda}_1$, $\lambda_1 \cdot \bar{\lambda}_1 \geq 0 \Rightarrow \lambda_1 \cdot \bar{\lambda}_1 = 0 \Rightarrow \lambda_1 = 0$
Absurdo, pois com os dois autovalores nulos, $[x, y] \not\to 0$.
Se $\lambda_2 = \lambda_1 \Rightarrow \lambda_1 \cdot \lambda_2 = \lambda_1^2 \geq 0 \Rightarrow \lambda_1 = 0$, absurdo novamente.
Se $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \cdot \lambda_2 > 0$, outro absurdo.
Concluo que $(ad - bc) > 0$. Por fim $\lambda_1 + \lambda_2 < 0 \Rightarrow a + d < 0$.

($\Leftarrow$) Se $a + d < 0$ e $ad - bc > 0 \Rightarrow \lambda_1 + \lambda_2 < 0$ e $\lambda_1 \cdot \lambda_2 > 0$.
Se $Re(\lambda_1) > 0 \Rightarrow Re(\lambda_2) > 0 \Rightarrow Re(\lambda_1) + Re(\lambda_2) > 0$, absurdo.
Logo $Re(\lambda_1) < 0$ e $Re(\lambda_2) < 0$. Nesse caso $e^{At} \to 0$ pois é proporcional a $e^{\max\{Re(\lambda_1), Re(\lambda_2)\}} \longrightarrow 0$, quando $t \to \infty$.

3) $mu'' + Ku = 0$

a) $x_1 = u \Rightarrow m \cdot x_2' + K \cdot x_1 = 0 \Rightarrow x_2' = -K/m\, x_1$
$x_2 = u'$

Assim $\begin{cases} x_1' = u' = x_2 \\ x_2' = -K/m\, x_1 \end{cases} \Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} //$

b) $p_\lambda = \lambda^2 + K/m = 0 \Rightarrow \lambda_1 = \sqrt{-K/m}$ e $\lambda_2 = -\sqrt{-K/m}$.
Como $K, m > 0$, $\lambda_1 = i \cdot \sqrt{K/m}$
$\lambda_2 = -i \cdot \sqrt{K/m} //$

c) Autovetores $A - \lambda_1 I = \begin{bmatrix} -i\sqrt{K/m} & 1 \\ -K/m & -i\sqrt{K/m} \end{bmatrix} \Rightarrow \begin{array}{l} v_1 = (1, i\sqrt{K/m}) \\ v_2 = (1, -i\sqrt{K/m}) \end{array}$

$u = (1, 0)$, $\quad a = 0$
$v = (0, \sqrt{K/m})$ $\quad b = \sqrt{K/m}$

$$X(t) = \begin{bmatrix} 0 & 1 \\ \sqrt{K/m} & 0 \end{bmatrix} e^0 \begin{bmatrix} \cos(\sqrt{K/m}\, t) & -\operatorname{sen}(\sqrt{K/m}\, t) \\ \operatorname{sen}(\sqrt{K/m}\, t) & \cos(\sqrt{K/m}\, t) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -\sqrt{K/m} & 0 \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{sen}(\sqrt{K/m}\, t) & \cos(\sqrt{K/m}\, t) \\ \sqrt{K/m} \cos(\sqrt{K/m}\, t) & -\sqrt{K/m} \operatorname{sen}(\sqrt{K/m}\, t) \end{bmatrix} \begin{bmatrix} x_{2,0} \\ -\sqrt{K/m}\, x_{1,0} \end{bmatrix}$$

$$= \begin{bmatrix} x_{2,0} \operatorname{sen}(\sqrt{K/m}\, t) - \sqrt{K/m}\, x_{1,0} \cos(\sqrt{K/m}\, t) \\ \sqrt{K/m}\, x_{2,0} \cos(\sqrt{K/m}\, t) + K/m\, x_{1,0} \operatorname{sen}(\sqrt{K/m}\, t) \end{bmatrix} //$$

4) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}, \quad t > 0$

Homogênea:

$p_\lambda = \lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$
$v_1 = (1, 2)$

Para encontrar v_2: $(A - \lambda I) v_2 = v_1 \Rightarrow A v_2 = v_1$
$\begin{bmatrix} 4 & -2 & 1 \\ 8 & -4 & 2 \end{bmatrix} \Rightarrow 4x - 2y = 1$. Tomo $y = 0$ e $x = 1/4 \Rightarrow v_2 = (1/4, 0)$

$$X_h(t) = \begin{pmatrix} 1 & 1/4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} e^{0t} & t \cdot e^{0t} \\ 0 & e^{0t} \end{pmatrix} \begin{pmatrix} 0 & -1/4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t + 1/4 \\ 2 & 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} \right]$$

Particular

$P^{-1} X_p(t) = P^{-1} P e^{Bt} u(t)$

$X_p'(t) = AX + g(t) = P B \overbrace{e^{Bt} u(t)}^{P^{-1} X_p} + P e^{Bt} u'(t)$

$= \underbrace{P B P^{-1}}_{A} X_p(t) + P e^{Bt} u'(t)$

Logo $P e^{Bt} u'(t) = g(t) \Rightarrow u'(t) = e^{-Bt} P^{-1} g(t)$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1/4 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ -2 & 1 \end{bmatrix} (-2) \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 2t & -1/4 - t \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2t^{-3} \\ 2t^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} -4t^{-2} - 1/2 t^{-2} - 2t^{-1} \\ 4t^{-3} + 2t^{-2} \end{bmatrix}$$

Assim: $u(t) = \begin{bmatrix} 9/2\, t^{-1} - 2\ln t \\ -2t^{-2} - 2t^{-1} \end{bmatrix}$

$$X_p(t) = \begin{bmatrix} 1 & 1/4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9/2\, t^{-1} - 2\ln t \\ -2t^{-2} - 2t^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t + 1/4 \\ 2 & 2t \end{bmatrix} \begin{bmatrix} 9/2\, t^{-1} - 2\ln t \\ -2t^{-2} - 2t^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 9/2\, t^{-1} - 2\ln t - 2t^{-1} - 1/2\, t^{-2} - 2 - 1/2\, t^{-1} \\ 9t^{-1} - 4\ln t - 4t^{-1} - 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix} t^{-1} - \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} t^{-2} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \ln t - \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$X(t) = X_h(t) + X_p(t)$