Experiment 4

Aim

Implementation of Bell circuit and Bell measurement in Qiskit.

Theory

The Bell circuit is a quantum circuit that generates an entangled state between two qubits. The circuit consists of a Hadamard gate applied to the first qubit, followed by a controlled-NOT (CNOT) gate with the first qubit as the control and the second qubit as the target. The circuit can be represented mathematically as follows:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{1}$$

where $|\psi\rangle$ represents the state of the two qubits, $|00\rangle$ and $|11\rangle$ are the basis states of the two qubits, and $\frac{1}{\sqrt{2}}$ is a normalization constant. The Hadamard gate applied to the first qubit can be represented as follows:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{2}$$

where H represents the Hadamard gate and $|0\rangle$ and $|1\rangle$ are the basis states of a single qubit. Thus, the action of the Hadamard gate on the first qubit can be written as:

$$H|0\rangle_1|0\rangle_2 = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)|0\rangle_2 \tag{3}$$

The CNOT gate applied to the two qubits can be represented as follows:

$$\text{CNOT}|x\rangle_1|y\rangle_2 = |x\rangle_1|(x \oplus y)\rangle_2 \tag{4}$$

where x and y are the basis states of the two qubits and $\oplus$ represents the bitwise XOR operation. Thus, the action of the CNOT gate on the two qubits can be written as:

$$\text{CNOT}\Big(\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)|0\rangle_2\Big) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{5}$$

which is the entangled state generated by the Bell circuit.
To perform a Bell measurement, the two entangled qubits are first subjected to a CNOT gate with the first qubit as the control and the second qubit as the target, followed by a Hadamard gate applied to the first qubit. The circuit can be represented mathematically as follows:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{6}$$

$$\text{CNOT}|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \tag{7}$$

$$(H \otimes I)\text{CNOT}|\psi\rangle = |00\rangle \tag{8}$$

identity operator applied to the second qubit. The action of the CNOT gate on the entangled state generates a superposition of the basis states $|00\rangle$ and $|10\rangle$, which are then subjected to a Hadamard gate on the first qubit. This results in the following Bell state:
The Bell measurement is performed by measuring the two qubits in the Bell basis, which consists of the following four orthonormal states:

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{9}$$

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{10}$$

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{11}$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{12}$$

The Bell basis states are all maximally entangled and are orthogonal to each other. The Bell measurement corresponds to a measurement in the Bell basis, which projects the two qubits onto one of the four Bell basis states with equal probability.

If the measurement outcome is $|\phi_+\rangle$, then the two qubits are in the state $|00\rangle$ or $|11\rangle$. If the measurement outcome is $|\phi_-\rangle$, then the two qubits are in the state $|00\rangle$ or $|11\rangle$ with a relative phase of $\pi/2$. If the measurement outcome is $|\psi_+\rangle$, then the two qubits are in the state $|01\rangle$ or $|10\rangle$. If the measurement outcome is $|\psi_-\rangle$, then the two qubits are in the state $|01\rangle$ or $|10\rangle$ with a relative phase of $\pi/2$.

In summary, the Bell circuit and Bell measurement demonstrate the generation and measurement of entangled states between two qubits, and the ability to perform measurements in the Bell basis to extract information about the entangled state. This has important implications for quantum information processing and communication, where entanglement is a crucial resource for tasks such as quantum teleportation, quantum error correction, and quantum key distribution.

Lab Exercise

Code:

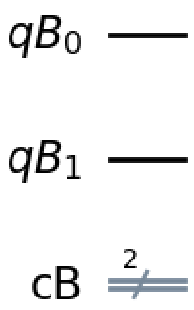
In [1]:
```
from qiskit import *
from qiskit.quantum_info import Statevector
import numpy as np

qB = QuantumRegister(2, name='qB')
cB = ClassicalRegister(2, name='cB')
circuit = QuantumCircuit(qB, cB)

%matplotlib inline
```

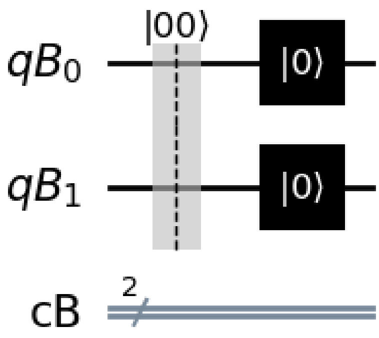
In [2]:
```
circuit.draw(output='mpl')
```

Out[2]:



In [3]:
```
circuit.barrier(qB, label='|00⟩')
circuit.reset(qB)
circuit.draw(output='mpl')
```

Out[3]:



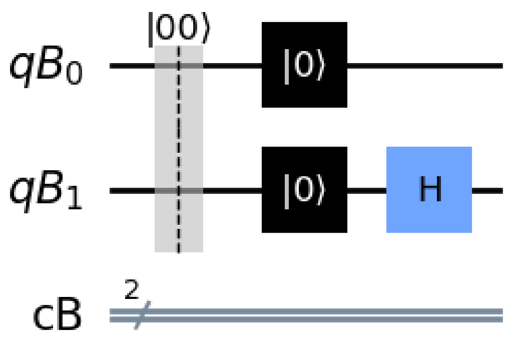
In [4]:
```
state_H_1 = Statevector.from_int(0, 2**2)
state_H_1 = state_H_1.evolve(circuit)
state_H_1.draw('latex')
```

Out[4]:

$$|00\rangle$$

In [5]:
```
circuit.h(qB[1])
circuit.draw(output='mpl')
```

Out[5]:



In [6]:
```
state_C_1 = Statevector.from_int(0, 2**2)
state_C_1 = state_C_1.evolve(circuit)
state_C_1.draw('latex')
```

Out[6]:

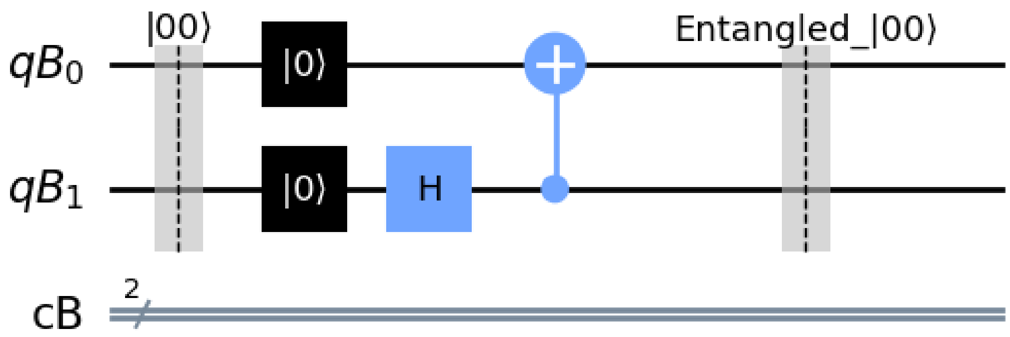
$$\frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}}{2}|10\rangle$$

In [7]:
```
circuit.cx(qB[1], qB[0])
circuit.barrier(label = 'Entangled_|00⟩')
```

Out[7]: <qiskit.circuit.instructionset.InstructionSet at 0x24321b6f880>

In [8]:
```
circuit.draw(output='mpl')
```

Out[8]:



In [9]:
```
state_E_1 = Statevector.from_int(0, 2**2)
state_E_1 = state_E_1.evolve(circuit)
state_E_1.draw('latex')
```

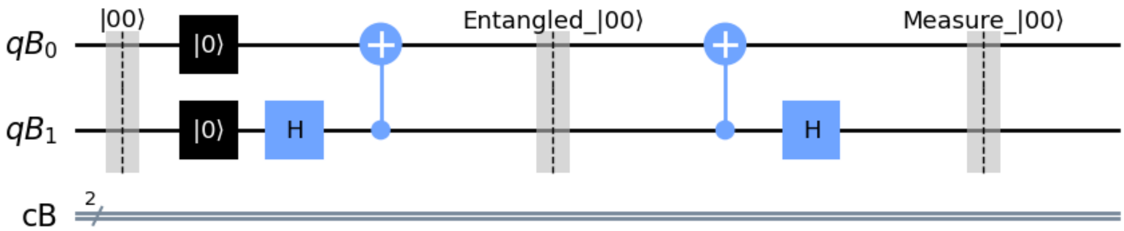
Out[9]:

$$\frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}}{2}|11\rangle$$

In [10]:
```
circuit.cx(qB[1], qB[0])
circuit.h(qB[1])
```

```
circuit.barrier(label = 'Measure_|00⟩')
circuit.draw(output='mpl')
```

Out[10]:



In [11]:
```
state_M_1 = Statevector.from_int(0, 2**2)
state_M_1 = state_M_1.evolve(circuit)
state_M_1.draw('latex')
```

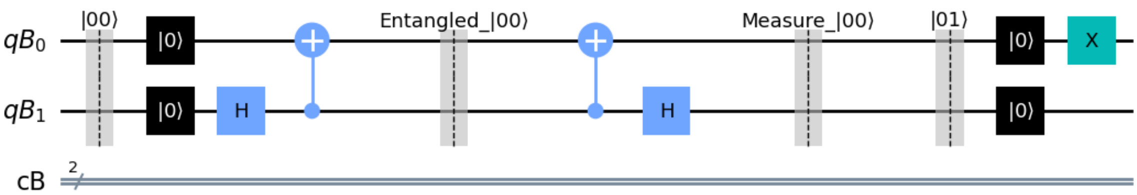
Out[11]:

$$|00\rangle$$

In [ ]:

In [12]:
```
circuit.barrier(qB, label='|01⟩')
circuit.reset(qB)
circuit.x(qB[0])
circuit.draw(output='mpl')
```

Out[12]:



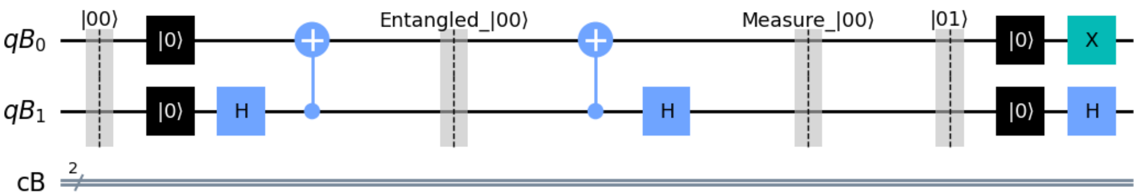
In [13]:
```
state_H_2 = Statevector.from_int(0, 2**2)
state_H_2 = state_H_2.evolve(circuit)
state_H_2.draw('latex')
```

Out[13]:

$$|01\rangle$$

In [14]:
```
circuit.h(qB[1])
circuit.draw(output='mpl')
```

Out[14]:



In [15]:
```
state_C_2 = Statevector.from_int(0, 2**2)
state_C_2 = state_C_2.evolve(circuit)
state_C_2.draw('latex')
```

Out[15]:

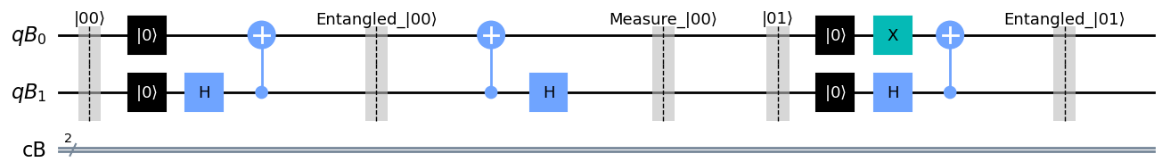
$$\frac{\sqrt{2}}{2}|01\rangle + \frac{\sqrt{2}}{2}|11\rangle$$

In [16]:
```
circuit.cx(qB[1], qB[0])
circuit.barrier(label = 'Entangled_|01⟩')
```

Out[16]: <qiskit.circuit.instructionset.InstructionSet at 0x243222f7160>

In [17]:
```
circuit.draw(output='mpl')
```

Out[17]:



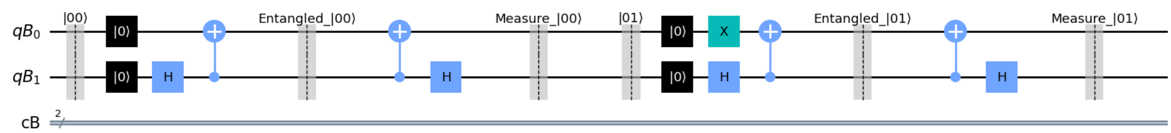
In [18]:
```
state_E_2 = Statevector.from_int(0, 2**2)
state_E_2 = state_E_2.evolve(circuit)
state_E_2.draw('latex')
```

Out[18]:

$$\frac{\sqrt{2}}{2}|01\rangle + \frac{\sqrt{2}}{2}|10\rangle$$

In [19]:
```
circuit.cx(qB[1], qB[0])
circuit.h(qB[1])
circuit.barrier(label = 'Measure_|01⟩')
circuit.draw(output='mpl')
```

Out[19]:



In [20]:
```
state_M_2 = Statevector.from_int(0, 2**2)
state_M_2 = state_M_2.evolve(circuit)
state_M_2.draw('latex')
```

Out[20]:

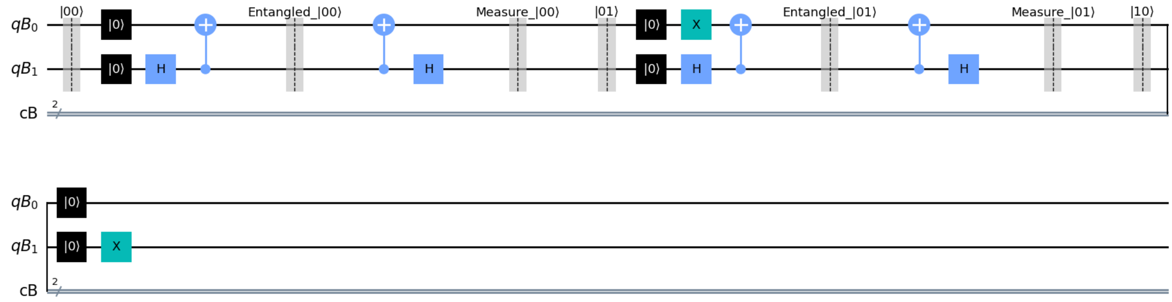
$$|01\rangle$$

In [ ]:

In [21]:
```
circuit.barrier(qB, label='|10⟩')
circuit.reset(qB)
```

```
circuit.x(qB[1])
circuit.draw(output='mpl')
```

Out[21]:



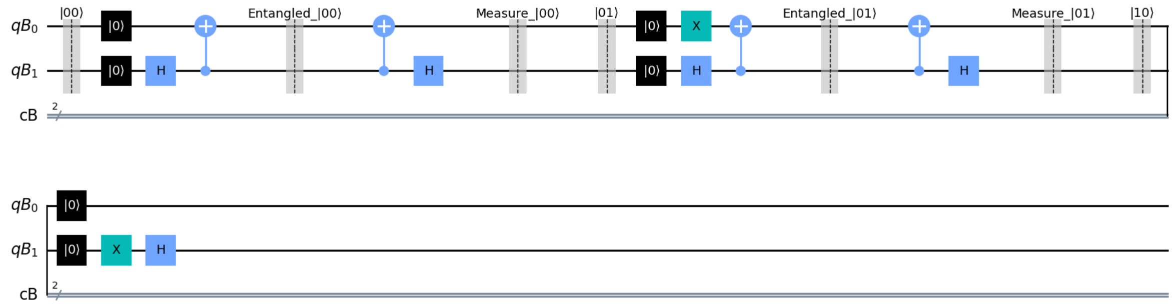
In [22]:
```
state_H_3 = Statevector.from_int(0, 2**2)
state_H_3 = state_H_3.evolve(circuit)
state_H_3.draw('latex')
```

Out[22]:

$$|10\rangle$$

In [23]:
```
circuit.h(qB[1])
circuit.draw(output='mpl')
```

Out[23]:



In [24]:
```
state_C_3 = Statevector.from_int(0, 2**2)
state_C_3 = state_C_3.evolve(circuit)
state_C_3.draw('latex')
```

Out[24]:

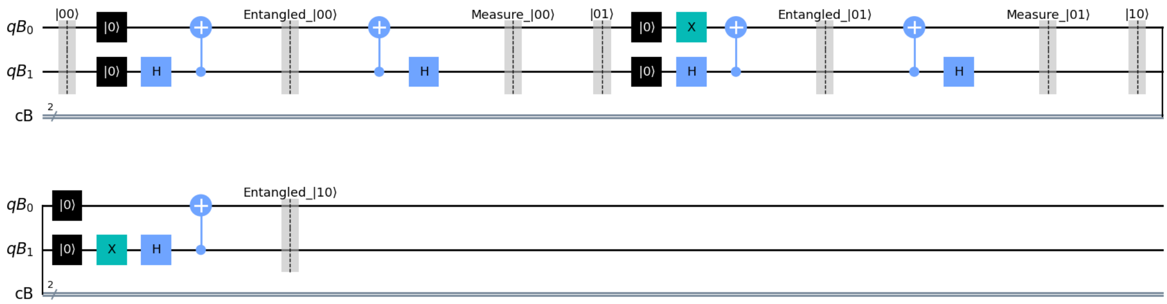
$$\frac{\sqrt{2}}{2}|00\rangle - \frac{\sqrt{2}}{2}|10\rangle$$

In [25]:
```
circuit.cx(qB[1], qB[0])
circuit.barrier(label = 'Entangled_|10⟩')
```

Out[25]:
```
<qiskit.circuit.instructionset.InstructionSet at 0x24324673670>
```

In [26]:
```
circuit.draw(output='mpl')
```

Out[26]:



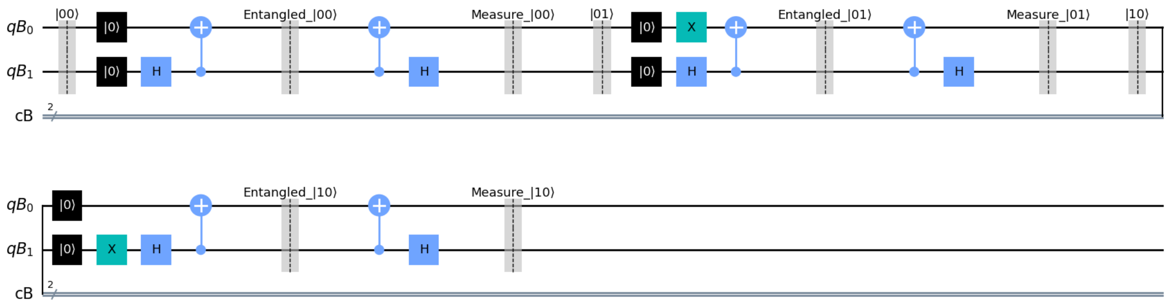
In [27]:
```
state_E_3 = Statevector.from_int(0, 2**2)
state_E_3 = state_E_3.evolve(circuit)
state_E_3.draw('latex')
```

Out[27]:

$$\frac{\sqrt{2}}{2}|00\rangle - \frac{\sqrt{2}}{2}|11\rangle$$

In [28]:
```
circuit.cx(qB[1], qB[0])
circuit.h(qB[1])
circuit.barrier(label = 'Measure_|10⟩')
circuit.draw(output='mpl')
```

Out[28]:



In [29]:
```
state_M_3 = Statevector.from_int(0, 2**2)
state_M_3 = state_M_3.evolve(circuit)
state_M_3.draw('latex')
```

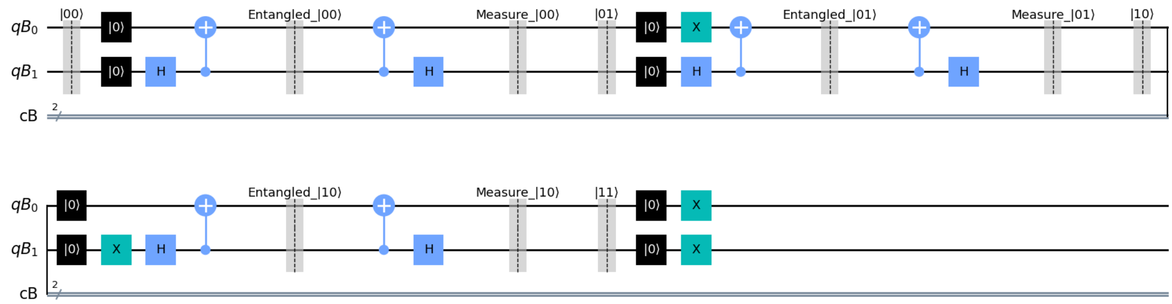
Out[29]:

$$|10\rangle$$

In [ ]:

In [30]:
```
circuit.barrier(qB, label='|11⟩')
circuit.reset(qB)
circuit.x(qB[0])
circuit.x(qB[1])
circuit.draw(output='mpl')
```

Out[30]:



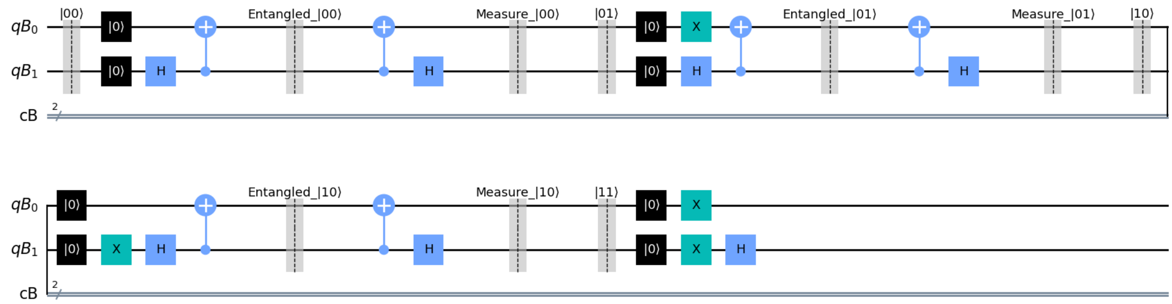
In [31]:
```
state_H_4 = Statevector.from_int(0, 2**2)
state_H_4 = state_H_4.evolve(circuit)
state_H_4.draw('latex')
```

Out[31]:

$$|11\rangle$$

In [32]:
```
circuit.h(qB[1])
circuit.draw(output='mpl')
```

Out[32]:



In [33]:
```
state_C_4 = Statevector.from_int(0, 2**2)
state_C_4 = state_C_4.evolve(circuit)
state_C_4.draw('latex')
```

Out[33]:

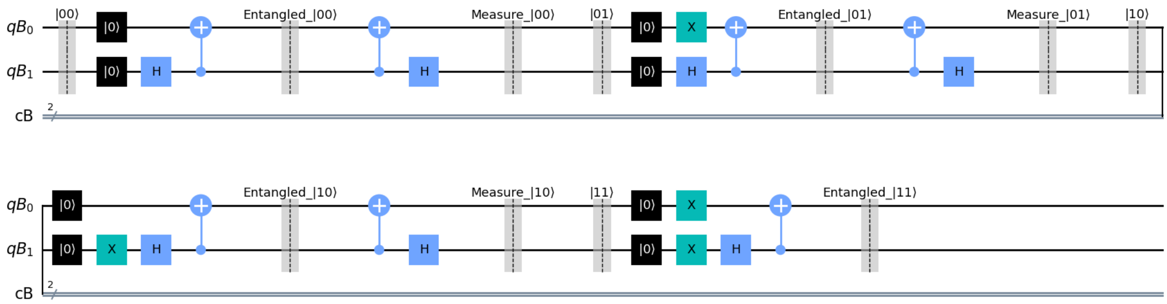
$$\frac{\sqrt{2}}{2}|01\rangle - \frac{\sqrt{2}}{2}|11\rangle$$

In [34]:
```
circuit.cx(qB[1], qB[0])
circuit.barrier(label = 'Entangled_|11⟩')
```

Out[34]:
```
<qiskit.circuit.instructionset.InstructionSet at 0x243222af0d0>
```

In [35]:
```
circuit.draw(output='mpl')
```

Out[35]:



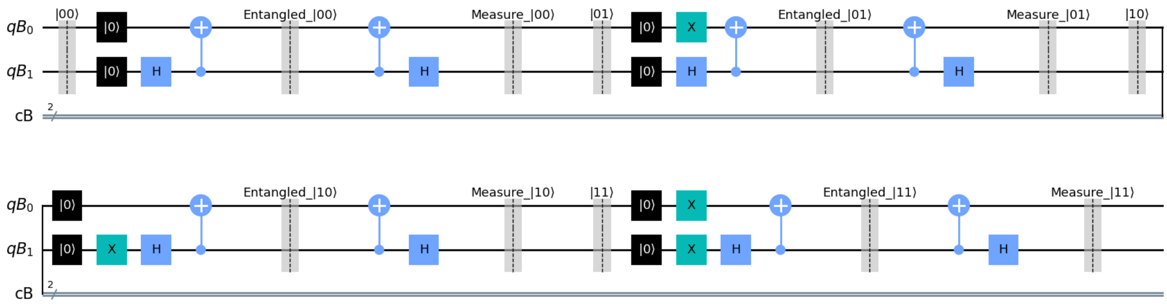
In [36]:
```
state_E_4 = Statevector.from_int(0, 2**2)
state_E_4 = state_E_4.evolve(circuit)
state_E_4.draw('latex')
```

Out[36]:

$$\frac{\sqrt{2}}{2}|01\rangle - \frac{\sqrt{2}}{2}|10\rangle$$

In [37]:
```
circuit.cx(qB[1], qB[0])
circuit.h(qB[1])
circuit.barrier(label = 'Measure_|11⟩')
circuit.draw(output='mpl')
```

Out[37]:



In [38]:
```
state_M_3 = Statevector.from_int(0, 2**2)
state_M_3 = state_M_3.evolve(circuit)
state_M_3.draw('latex')
```

Out[38]:

$$|11\rangle$$

In [ ]:

Conclusion

Fundamental ideas in quantum information science like the Bell circuit and Bell measurement show the strength and potential of quantum entanglement. The Bell circuit demonstrates how two qubits can become maximally entangled by using a controlled-NOT gate and a Hadamard gate. The Bell measurement, on the other hand, shows how to perform a measurement in the Bell basis, which enables one to learn about the correlations between the two qubits, to extract information from an entangled state.

The significance of the Bell circuit and Bell measurement lies in their ability to create and manipulate quantum entanglement, which is a crucial resource for various quantum information processing tasks. The ability to create and measure entangled states is central to quantum teleportation, quantum error correction, and quantum key distribution, all of which are promising applications in the field of quantum information science.

In conclusion, the Bell circuit and Bell measurement are important building blocks for the development of quantum technologies and will continue to be crucial in advancing our understanding and utilization of quantum entanglement.