SIMATS

School of Engineering

Numerical Methods

Science & Humanities

Saveetha Institute of Medical And Technical Sciences,Chennai.

UBA10 NUMERICAL METHODS

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UNIT-I- SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

REGULAR FALSI METHOD OR FALSE POSITION METHOD

↓

Find the range of roots lies between the interval

↓

Let $f(a)<0$ + $f(b)>0$ then the roots lies between (a,b)

↓

Find the first approximate root

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

↓

If $f(x_1)<0$ + $f(b)>0$ the roots lies between (x_1, b)

↓

Calculate

$$x_2 = \frac{bf(x_1)-x_1f(b)}{f(x_1)-f(b)}$$

If $f(x_1)>0$, $f(a)<0$ the roots lies between (a, x_1)

↓

Calculate

$$x_2 = \frac{x_1f(a)-af(x_1)}{f(a)-f(x_1)}$$

↓

Same way calculate $x_3, x_4 \dots$

↓

The sequence converges to the required root.

EXAMPLE-1

Find the positive root of $x^3-2x-5=0$ by the Regula Falsi method.

Let $f(x) = x^3-2x-5=0$

$f(0) = -5 = -ve$

$f(1) = 1-2-5=-6 = -ve$

$f(2) = 8-4-5 = -1 = -ve$

$f(3) = 27-6-5 = 16 = +ve$

Roots lies between 2 and 3

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2f(3)-3f(2)}{f(3)-f(2)} = \frac{2(16)-3(-1)}{16-(-1)} = \frac{35}{17} = 2.0588$$

$f(x_1) = f(2.0588) = (2.0588)^3-2(2.0588)-5 = -0.3911$

The roots lies between 2.0588 and 3

$f(a)<0$ $f(x_1)<0$ $f(x_2)<0$ $f(b)>0$

ITERATION (r)	a	b	x_r	Sign of $f(x_r)$
1	2	3	2.0588	-0.3911
2	2.0588	3	2.0813	-0.1468
3	2.0813	3	2.0897	-0.054
4	2.0897	3	2.0928	-0.0195
5	2.0928	3	2.0939	-0.0073
6	2.0939	3	2.0943	-0.0028
7	2.0943	3	2.0945	-0.006
8	2.0945	3	2.0945	

We observe that $x_7 = x_8 = 2.0945$

Hence the root is 2.0945

(2) Find an approximate root $x\log_{10}x - 1.2 = 0$ by Regula Falsi method

$f(x) = x\log_{10}x - 1.2$

$f(1) = -1.2 = -ve$

$f(2) = -0.5979 = -ve$

$f(3) = 0.2314 = +ve$

Therefore root lies between 2 and 3.

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

r	a	b	x_r	$f(x_r)$
1	2	3	2.7210	-0.0171
2	2.7210	3	2.7402	-0.0004
3	2.7402	3	2.7407	+0.0001
4	2.7402	2.7407	2.7406	-0.00004
5	2.7406	2.7407	2.7406	–

Hence the required root is 2.7406

PROBLEMS FOR PRACTICE

(3) Solve the positive root of $x - x\cos x = 0$ by False position method

(4) Solve the equation $xe^x = 2$ by Regula Falsi method

(5) Solve the equation $3x + \sin x - e^x = 0$ by False position method

(6) Solve the equation $xe^x = \cos x$ by Regla Falsi method

Answer: (3) 0.7391 (4) 0.8526 (5) 0.3604 (6) 0.51776.

NEWTON'S METHOD [or Newton - Raphson method]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \phi(x_n)$$

Newton's formula converges if

$$|f(x) f''(x)| < |f'(x)|^2$$

Example : 1 Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to 6 decimal places.

Solution: Let $f(x) = 3x - \cos x - 1$

$f(0) = -2$; $f(1) = 1.459$

$f(0) = -ve$ $\quad f(1) = +ve$

Root lies between 0 and 1

$|f(0)| > |f(1)|$

Hence root nearer to 1

Let $x_0 = 0.6$

Formula : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\quad f'(x) = 3 + \sin x$

Iteration	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.6		-0.007101	0.607108 $(= x_1)$
1	0.607108		0.000006	$x_2 = 0.607102$
2	0.607102		0.0000004	$x_3 = 0.607102$

Here $x_3 = x_4$: The root is 0.607102.

Example - 2 Write down Newton Raphson formula for finding $\sqrt{N}$, where N is a positive real number and hence find $\sqrt{5}$

Solution : Let $x = \sqrt{N}$

$x^2 = N \Rightarrow x^2 - N = 0$

Let $f(x) = x^2 - N$; $f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow$$

$$= x_n - \frac{x_n^2 - N}{2x_n}$$

$$= x_n - \frac{x_n}{2} - \frac{N}{2x_n} = \frac{x_n}{2} - \frac{N}{2x_n}$$

$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$ is the iterative formula to find $\sqrt{N}$

To Find $\sqrt{5}$

Put

Also $x = \sqrt{5}$ lies between 2+3

Let $x_0 = 2$

$$x_1 = \frac{1}{2}\left[x_0 + \frac{5}{x_0}\right] = \frac{1}{2}\left[2 + \frac{5}{2}\right] = 2.25$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{5}{x_1}\right] = \frac{1}{2}\left[2.25 + \frac{5}{2.25}\right] = 2.2361$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{5}{x_2}\right] = \frac{1}{2}\left[2.2361 + \frac{5}{2.2361}\right]$$

$$= 2.2361$$

Here $x_2 = x_3 = 2.2361$

Hence $\sqrt{5} = 2.2361$ (Approximately)

FIXED POINT ITERATION METHOD

NOTE: The sufficient condition for convergence is $|g'(x)| < 1$ for all x in I

① Find the real root of the equation $\cos x = 3x - 1$ correct to 5 decimal places by fixed point iteration method.

Solution: Let $f(x) = \cos x - 3x + 1 = 0$

$f(0) = 2 = +ve$; $f(1) = -1.4597 = -ve$

A root lies between 0 and 1

Re arrang the equation $x = \frac{1}{3}(1 + \cos x) = g(x)$

$$|g'(1)| = \frac{1}{3}\sin 1 = 0.2804 < 1$$

Hence the iteration method may be applied.

Let $x_0 = 0.6$

$x_1 = \frac{1}{3}[1 + \cos x_0] = \frac{1}{3}[1 + \cos(0.6)] = 0.60845$

$x_2 = \frac{1}{3}[1 + \cos x_1] = 0.06884$

$x_3 = \frac{1}{3}[1 + \cos x_2] = 0.06715$

$x_4 = \frac{1}{3}[1 + \cos x_3] = 0.060709$

$x_5 = \frac{1}{3}[1 + \cos x_4] = 0.60710$

$x_6 = \frac{1}{3}[1 + \cos x_5] = 0.60710$;

$x_5 = x_6$

Hence root is 0.60710

Problems for Practice :

1) Solve the following equations by fixed point iteration method

(a) $e^x - 3x = 0$

$x_{11} = x_{12} = 0.6190$

(b) $x^3 + x^2 - 100 = 0$

$x_{12} = x_{13} = 4.33105$

Solve by Newton-Raphson method

(c) $x^3 = 6x - 4$

$x_2 = x_3 = 0.73$

(d) $x^2 + 4\sin x = 0$

$x_2 = x_3 = -1.9338$

(e) $x \log_{10} x = 12.34$ with $x_0 = 10$

$x_2 = x_3 = 11.5949$

SOLUTION OF LINEAR SYSTEM BY GAUSS JORDON METHOD

Consider n Linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \quad (1)$$

The system (1) is equivalent to

$$AX = B$$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Our aim is reduce the augmented matrix [AB] to unit matrix

$$[AB] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

All the elements in the first column of [AB] except a_{11} are made to zero, then all the element of second column except a_{22} are made to zero Similarly continue to n^{th} column. we get

$$[AB] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & x_1 \\ 0 & 1 & 0 & \cdots & 0 & x_2 \\ 0 & 0 & 1 & \cdots & 0 & x_3 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_n \end{array}\right]$$

In this method values are get immediately.

(1) Solve
$$\begin{aligned} x + 3y + 3z &= 16 \\ x + 4y + 3z &= 18 \\ x + 3y + 4z &= 19 \end{aligned}$$
by Gauss JORDON METHOD

Solution:

$$[AB] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_1 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \leftrightarrow R_1 - 3R_3$$

Hence $\boxed{x = 1, y = 2, z = 3}$

(2) Solve by using Gauss Jordon Method

$10x + y + z = 12, 2x + 10y + z = 13, x + y + 5z = 7$

Solution

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & 9 & -49 & -58 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 10R_1 \\ R_2 \leftrightarrow R_2 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad R_2 \to \frac{R_2}{8}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 9R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 \to \frac{R_3}{-59.125}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 6.125R_3 \\ R_2 \to R_2 + 1.125R_3 \end{array}$$

$\therefore x = 1; y = 1, z = 1.$

ITERATIVE METHOD : GAUSS SIEDEL METHOD

Consider the system of eqn's

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad (1)$$

Let us assume

$$\begin{aligned} |a_1| &> |b_1| + |c_1| \\ |b_2| &> |a_2| + |c_2| \\ |c_3| &> |a_3| + |b_3| \end{aligned}$$

Solve (1) for x, y, z interms of the other variables and substitute the recent values of x, y, z

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y + c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned} \quad (2)$$

Let $y^{(0)} = z^{(0)} = 0$ get $x^{(1)}$ from the first equation

$$\begin{aligned} x^{(1)} &= \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\ y^{(1)} &= \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)}) \\ z^{(1)} &= \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)}) \end{aligned}$$

To find the unknowns, we use the latest available values on the R.H.S.

Continue the process until the convergence is confirmed

That is the values of n^{th} iteration will be equal to $(n+1)^{th}$ interation.

Example. Solve by Gauss Seidel method

$27x + 6y - z = 85, x + y + 54z = 110$
$6x + 15y + 2z = 72$

Solution:

Matrix is not diagonally dominant

Re write the equations

$$\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned}$$

Write x, y, z as follows

$$\begin{aligned} x &= \frac{1}{27}[85 - 6y + z] \\ y &= \frac{1}{15}[72 - 6x - 2z] \\ z &= \frac{1}{54}[110 - x - y] \end{aligned}$$

Let y = 0, z = 0 (First iteration)

$$\begin{aligned} x^{(1)} &= \frac{1}{27}[85 - 6(0) + 0] = 3.148 \\ y^{(1)} &= \frac{1}{15}[72 - 6(3.148) - 0] = 3.541 \\ z^{(1)} &= \frac{1}{54}[110 - 3.148 - 3.541] = 1.913 \end{aligned}$$

Second iteration

$$\begin{aligned} x^{(2)} &= \frac{1}{27}[85 - 6(3.541) + 1.913] = 2.432 \\ y^{(2)} &= \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.572 \\ z^{(2)} &= \frac{1}{54}[110 - 2.432 - 3.572] = 1.926 \end{aligned}$$

Third iteration

$$\begin{aligned} x^{(3)} &= \frac{1}{27}[85 - 6(3.572) + 1.926] = 2.426 \\ y^{(3)} &= \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573 \\ z^{(3)} &= \frac{1}{54}[110 - 2.426 - 3.573] = 1.926 \end{aligned}$$

Fourth iteration

$$\begin{aligned} x^{(4)} &= \frac{1}{27}[85 - 6(3.573) + 1.926] = 2.426 \\ y^{(4)} &= \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573 \\ y^{(5)} &= \frac{1}{54}[110 - 2.426 - 3.573] = 1.926 \end{aligned}$$

Hence x = 2.426, y = 3.573, z = 1.926

PROBLEMS

Solve by Gauss Jordon method

1) $10x - 2y + 3z = 23$, $2x + 10y - 5z = -33$, $3x - 4y + 10z = 41$

2) $x + 3y + 3z = 16$, $x + 4y + 3z = 18$, $x + 3y + 4z = 19$

Solve by Gauss Seidel Method

1) $4x + 2y + z = 14$, $x + 5y - z = 10$, $x + y + 8z = 20$

2) $2x + y + \quad = 4$, $x + 2y + z = 4$, $x + y + 2z = 4$

INVERSE OF A MATRIX BY GAUSS JORDON METHOD

The coefficient matrix is reduced to a diagonal matrix.

$$\begin{bmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\0&0&0&\cdots&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\\vdots\\d_n\end{bmatrix}$$

The Gauss-Jordan method gives $Ax = B \Rightarrow X = A^{-1}B$

We start with the augmented matrix of A with the identity matrix I of the same order. When the Gauss Jordan procedure is completed, we obtain $[A/I] \sim [I/A^{-1}]$

1) Using Gauss-Jordan method, find the inverse of the matrix $\begin{pmatrix}2&2&3\\2&1&1\\1&3&5\end{pmatrix}$

Solution:

$$[A - I] = \left[\begin{array}{ccc|ccc}2&2&3&1&0&0\\2&1&1&0&1&0\\1&3&5&0&0&1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&1&3/2&1/2&0&0\\2&1&1&0&1&0\\1&3&5&0&0&1\end{array}\right] \quad R_1 \leftrightarrow \frac{R_1}{2}$$

$$\sim \left[\begin{array}{ccc|ccc}1&1&3/2&1/2&0&0\\0&-1&-2&-1&1&0\\0&2&7/2&-1/2&0&1\end{array}\right] \quad \begin{array}{l}R_2 \leftrightarrow R_2-2R_1\\ R_3 \leftrightarrow R_3-R_1\end{array} \sim \left[\begin{array}{ccc|ccc}1&1&3/2&1/2&0&0\\0&1&2&1&-1&0\\0&2&7/2&-1/2&0&1\end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{-1}$$

$$\sim \left[\begin{array}{ccc|ccc}1&0&-1/2&-1/2&1&0\\0&1&2&1&-1&0\\0&0&-1/2&-5/2&2&1\end{array}\right] \quad \begin{array}{l}R_1 \leftrightarrow R_1-R_2\\ R_3 \leftrightarrow R_3-2R_2\end{array} \sim \left[\begin{array}{ccc|ccc}1&0&-1/2&-1/2&1&0\\0&1&2&1&-1&0\\0&0&1&5&-4&-2\end{array}\right] \quad R_3 \leftrightarrow R_3(-2)$$

$$\sim \left[\begin{array}{ccc|ccc}1&0&0&2&-1&-1\\0&1&0&-9&7&4\\0&0&1&5&-4&-2\end{array}\right] \quad \begin{array}{l}R_1 \leftrightarrow R_1+\frac{1}{2}R_3\\ R_2 \leftrightarrow R_2-2R_3\end{array}$$

Hence $A^{-1} = \begin{bmatrix}2&-1&-1\\-9&7&4\\5&-4&-2\end{bmatrix}$

PROBLEMS FOR PRACTICE

Using Gauss-Jordan method, find the inverse of following matrices.

(i) $A = \begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}$ (ii) $\begin{bmatrix}2&2&6\\2&6&-6\\4&-8&-8\end{bmatrix}$ (iii) $\begin{bmatrix}2&1&-1\\0&2&1\\5&2&-3\end{bmatrix}$

EIGENVALUE OF A MATRIX BY POWER METHOD

1) Find the numerically largest eigenvalue of $A = \begin{bmatrix}1&-3&2\\4&4&-1\\6&3&5\end{bmatrix}$ by power method.

Solution: Let $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ be an arbitrary initial eigenvector.

$$AX_1 = \begin{bmatrix}1&-3&2\\4&4&-1\\6&3&5\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\4\\6\end{bmatrix} = 6\begin{bmatrix}0.167\\0.667\\1\end{bmatrix} = 6X_2$$

$$AX_2 = \begin{bmatrix}1&-3&2\\4&4&-1\\6&3&5\end{bmatrix}\begin{bmatrix}0.167\\0.667\\1\end{bmatrix} = \begin{bmatrix}0.166\\2.336\\8.003\end{bmatrix} = 8.003\begin{bmatrix}0.021\\0.292\\1\end{bmatrix} = 8.003\,X_3$$

$$AX_3 = \begin{bmatrix}1&-3&2\\4&4&-1\\6&3&5\end{bmatrix}\begin{bmatrix}0.021\\0.292\\1\end{bmatrix} = \begin{bmatrix}1.145\\0.252\\6.002\end{bmatrix} = 6.002\begin{bmatrix}0.191\\0.042\\1\end{bmatrix} = 6.002\,X_4$$

$$AX_4 = \begin{bmatrix}1&-3&2\\4&4&-1\\6&3&5\end{bmatrix}\begin{bmatrix}0.191\\0.042\\1\end{bmatrix} = \begin{bmatrix}2.065\\-0.068\\6.272\end{bmatrix} = 6.272\begin{bmatrix}0.329\\-0.011\\1\end{bmatrix} = 6.272\,X_5$$

$$\vdots$$

$$AX_{12} = \begin{bmatrix}1&-3&2\\4&4&-1\\6&3&5\end{bmatrix}\begin{bmatrix}0.3\\0.066\\1\end{bmatrix} = \begin{bmatrix}2.102\\0.464\\6.998\end{bmatrix} = 6.998\begin{bmatrix}0.3\\0.066\\1\end{bmatrix}$$

The largest eigen value = 7

PROBLEMS FOR PRACTICE

Using power method, find all the eigenvalues of the following matrices

(i) $\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}$ (ii) $\begin{bmatrix}25&1&2\\1&3&0\\2&0&-4\end{bmatrix}$ (iii) $\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}$

NEWTON'S FORWARD INTERPOLATION FORMULA FOR EQUAL INTERVALS

$$y(x) = P_n(x) = P_n(x_0+uh) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \dots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \dots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

where,

$$u^{(r)} = u(u-1)(u-2)\dots(u-\overline{r-1})$$

$$u = \frac{x-x_0}{h}$$

1.) Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence, evaluate y at $x=5$.

x	4	6	8	10
y	1	3	8	10

SOLUTION: We form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
(x_0) 4	(y_0) 1			
		$3-1=2$ (Δy_0)		
(x_1) 6	(y_1) 3		$5-2=3$ $(\Delta^2 y_0)$	
		$8-3=5$ (Δy_1)		$-3-3=-6$ $(\Delta^3 y_0)$
(x_2) 8	(y_2) 8		$2-5=-3$ $(\Delta^2 y_1)$	
		$10-8=2$ $\Delta(y_2)$		
(x_3) 10	(y_3) 10			

There are only 4 data given.
Hence, the polynomial will be degree 3.

$$y(x) = P_3(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x-x_0}{h}$

Here $x_0 = 4$, $h = 6-4 = 2$ [difference]

$$y(x) = P_3(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{3!}(-6)$$

$$= x-3+\frac{3}{8}(x-4)(x-6) - \frac{1}{8}(x-4)(x-6)(x-8)$$

$$= x-3+\frac{3}{8}[x^2-10x+24] - \frac{1}{8}[x^3-10x^2+24x-8x^2+80x-192]$$

$$= \frac{1}{8}[-x^3+21x^2-126x+240]$$

$$y(5) = \frac{1}{8}[-(5)^3+21(5)^2-126(5)+240]$$

$$= \frac{1}{8}[-125+21(25)-630+240]$$

$$= \frac{1}{8}[-125+525-630+240] = \frac{1}{8}[10] = 1.25$$

PROBLEMS FOR PRACTICE

Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence find $f(2)$.

x:	0	5	10	15
y:	14	379	1444	3584

NEWTON'S BACKWARD INTERROLATION FORMULA

$$P_n(x) = P_n(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n$$

$$+ \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots + \frac{v(v+1)(v+2)\dots(v+n-1)}{n!}\nabla^n y_n$$

where $v = \frac{x - x_n}{h}$

From the following data, find θ at $x = 43$ and $x = 84$

x :	40	50	60	70	80	90
θ :	184	204	226	250	276	304

Also express θ in terms of x.

Solution : Since six data are given, $p(x)$ is of degree 5. To find θ at $x = 43$ use forward interpolation and to find θ at $x = 84$, use backward interpolation formula.

$$u = \frac{x - x_0}{h} = \frac{43 - 40}{10} = 0.3$$

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
40	184				
		20			
50	204		2		
		22		0	
60	226		2		0
		24		0	
70	250		2		0
		26		0	
80	276		2		
		28			
90	304				

$$\theta(x=43) = \theta[40 + (0.3)10]$$

$$= \theta_0 + u\Delta\theta_0 + \frac{u(u-1)}{2}\Delta^2\theta_0 + \dots$$

$$= 184 + (0.3)20 + \frac{(0.3)(-0.7)}{2}(2)$$

$$= 184 + 6.0 - 0.21$$

$$\theta(x=43) = 189.79$$

$$\theta(x=84) = \theta[90 + (-0.6)10] \qquad \because v = \frac{84-90}{10} = -0.6$$

$$= \theta_n + v\nabla\theta_n + \frac{v(v+1)}{2}\nabla^2\theta_n + \dots$$

$$= 304 + (-0.6)28 + \frac{(-0.6)(0.4)}{2}(2)$$

$$\theta(x=84) = 286.96$$

$$\theta = \theta_0 + u\Delta\theta_0 + \frac{u(u+1)}{2!}\Delta^2\theta_0 + \dots$$

$$= 184 + u(20) + \frac{u(u-1)}{2}(2), \qquad \text{where } u = \frac{x-40}{10}$$

$$= 184 + \frac{20(x-40)}{10} + \frac{(x-40)(x-50)}{100}$$

$$= 184 + 2x - 80 + \frac{1}{100}[x^2 - 90x + 2000]$$

$$\theta = 0.01x^2 + 1.1x + 124.$$

PROBLEMS FOR PRACTICE

From the following data find the value of tan (0.12) and tan (0.28).

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2533	0.3093

LAGRANGE'S INTERPOLATION FORMULA

Let $y_0, y_1, y_2 \cdots y_n$ be the entries corresponding to the arguments $x_0, x_1, x_2 \cdots x_n$ which are not ~~eq~~ necessarily equally spaced, then Lagrange's Interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)} y_2 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

1. Find $f(x)$ as a polynomial in x from the given data and find $f(8)$

x	3	7	9	10
$f(x)$	168	120	72	63

GIVEN: $x_0 = 3,\ x_1 = 7,\ x_2 = 9,\ x_3 = 10$

The values of x are not equally spaced, so we use Lagrange's Interpolation formula to find $y = f(x)$.

For 4 values of x:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_3)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_3)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \left[\frac{(x-7)(x-9)(x-10)}{(3-7)(3-9)(3-10)}\right](168) + \left[\frac{(x-3)(x-9)(x-10)}{(7-3)(7-9)(7-10)}\right](120) +$$

$$\left[\frac{(x-3)(x-7)(x-10)}{(9-3)(9-7)(9-10)}\right](72) + \left[\frac{(x-3)(x-7)(x-9)}{(10-3)(10-7)(10-9)}\right](63)$$

NOTE: $(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$

After simplification by using the above formula, we get

$$f(x) = -(x^3 - 26x^2 + 223x - 630) + 5(x^3 - 22x^2 + 147x - 270)$$
$$- 6(x^3 - 20x^2 + 121x - 210) + 3(x^3 - 19x^2 + 111x - 189)$$

$\therefore f(x) = x^3 - 21x^2 + 119x - 27$

$\Rightarrow f(8) = 8^3 - 21(8^2) + 119(8) - 27 = 512 - 1344 + 952 - 27$

$\therefore f(8) = 93$

PROBLEMS FOR PRACTICE

1. Find the polynomial $f(x)$ by using Lagrange's Formula and hence find $f(3)$ for the following values of x and y.

x	0	1	2	5
y	2	3	12	147

2. Use Lagrange's Interpolation formula, fit a polynomial to the data

x	-1	1	2
y	7	5	12

3. Use Lagrange's Interpolation formula to fit a polynomial to the data and hence find the value of y when $x = 2$

x	0	1	3	4
y	-12	0	6	12

DIVIDED DIFFERENCES

* Let a function $y = f(x)$ take values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, x_2 \ldots x_n$ not necessarily equally spaced.

* The first divided difference for the arguments $x_0 \& x_1$ is defined as $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

* It is denoted by $f(x_0, x_1)$ or $\underset{x_1}{\Delta} f(x_0)$ or $[x_0, x_1]$

* we shall denote $\underset{x_1}{\Delta} f(x_0) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

* Similarly $\underset{x_2}{\Delta} f(x_1) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$\underset{x_3}{\Delta} f(x_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

$$\vdots$$

$$\underset{x_n}{\Delta} f(x_{n-1}) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The second divided difference for the arguments x_0, x_1, x_2 is defined as

$$\boxed{\underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{\underset{x_2}{\Delta} f(x_1) - \underset{x_1}{\Delta} f(x_0)}{x_2 - x_0}}$$

For $x_1, x_2, x_3 \Rightarrow \underset{x_2, x_3}{\Delta^2} f(x_1) = \dfrac{\underset{x_3}{\Delta} f(x_2) - \underset{x_2}{\Delta} f(x_1)}{x_3 - x_1}$

The third divided difference for x_0, x_1, x_2, x_3 is

$$\boxed{\underset{x_1, x_2, x_3}{\Delta^3} f(x_0) = \frac{\underset{x_2 x_3}{\Delta^2} f(x_1) - \underset{x_1 x_3}{\Delta^2} f(x_0)}{x_3 - x_0}}$$ and so on

1. Construct the divided difference table for the following data and find the value of $f(2)$.

x	4	5	7	10	11	12
$f(x)$	50	102	296	800	1010	1224

GIVEN:

$x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$.

$x_4 = 11$, $x_5 = 12$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\Delta^2} f(x_0) + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) \underset{x_1 x_2 \ldots x_n}{\Delta^n} f(x_0) \quad \text{①}$$

DIVIDED DIFFERENCE TABLE:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	50					
		52				
5	102		15			
		97		-0.133		
7	296		14.2		-0.069	
		168		-0.617		-0.011
10	800		10.5		-0.115	
		210		-1.7		
11	1010		2			
		214				
12	1224					

On substituting the above values in ①, then $f(x)$ when $x = 2$ is $\boxed{f(2) = 49.19}$

EXAMPLE: By using Newton's divided difference formula find $f(8), f(6), f(9), f(15)$.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

UNIT-III: APPROXIMATION OF DERIVATIVES USING INTERPOLATION POLYNOMIALS

NEWTON'S FORWARD DIFFERENCES

FORMULA If $y[x]$ is a polynomial of degree 'n' in x and $u = \frac{x - x_0}{h}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0 \text{ (or) } u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0 \text{ (or) } u=0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0 \text{ (or) } u=0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

NEWTON'S BACKWARD DIFFERENCES

FORMULA If $y(x)$ is a polynomial of degree 'n' in x and $v = \frac{x - x_n}{h}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_n \text{ (or) } v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n,\ v=0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n,\ v=0} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Example: Compute $f'(0)$ and $f''(4)$ from the data

x	0	1	2	3	4
y	1	2.718	7.381	20.086	54.598

DIFFERENCE TABLE

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		1.718			
1	2.718		2.945		
		4.663		5.097	
2	7.381		8.042		8.668
		12.705		13.765	
3	20.086		21.807		
		34.512			
4	54.598				

Solution: Since we required $f'(0)$ & $f''(4)$, we use Newton's forward and backward formula

By Newton's forward formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[1.718 - \frac{1}{2}(2.945) + \frac{1}{3}(5.097) - \frac{1}{4}(8.668)\right]$$

$$= [1.718 - 1.4725 + 1.699 - 2.167]$$

$$= 0.2225$$

By Newton backward difference formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[34.512 + \frac{1}{2}(21.807) + \frac{1}{3}(13.765) + \frac{1}{4}(8.668)\right]$$

$$= 34.512 + 10.9035 + 4.588 + 2.167$$

$$= 52.1705$$

PROBLEMS FOR PRACTICE

1. Find the first, second and third derivatives of $f(x)$ at $x = 1.5$ if

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

2. Given the following data find $y'(6)$ and $f'(5)$

x	0	2	3	4	7	9
y	4	26	58	112	466	922

3. The following table gives the velocity v of a particle at time t. Find the acceleration at $t = 2$

t	0	2	4	6	8	10	12
v	4	6	16	34	60	94	131

NUMERICAL INTEGRATION BY TRAPEZOIDAL AND SIMPSON'S $\frac{1}{3}, \frac{3}{8}$ RULE, ROMBERG'S METHOD

TRAPEZOIDAL RULE:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$\Rightarrow$ Trapezoidal Rule $= \frac{h}{2}$[sum of first and last ordinates + 2(sum of the remaining ordinates)]

SIMPSON'S ONE THIRD RULE:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

$\Rightarrow$ Simpson's one third rule $= \frac{h}{3}$[sum of first and last ordinates +2[sum of remaining odd ordinates) +4(sum of remaining even ordinates)]

ROMBERG'S METHOD:

$$I = I_2 + \left[\frac{I_2 - I_1}{3}\right]$$

Simpson's 3/8 Rule

$$\int_{x_0}^{x_0+nh} y(x)\,dx = 3h/8\{(y_0+y_n) + 2(y_3+y_6+\cdots y_{n-3}) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})$$

EXAMPLES:

1. Evaluate $\int_0^5 \frac{dx}{4x+5}$ by Simpson's one third rule. $(n=10)$

Here, $y(x) = \frac{1}{4x+5}$

$h = \frac{5-0}{10} = 2$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0.2 y_0	0.1429 y_1	0.1111 y_2	0.0909 y_3	0.0796 y_4	0.0667 y_5	0.0588 y_6	0.0526 y_7	0.0476 y_8	0.0434 y_9	0.04 y_{10}

$$\Rightarrow \int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{1}{2(3)}\left[0.24+2(0.2944)+4(0.3964)\right] = \frac{1}{6}[2.4148]$$

$= 0.4025$

Example for practice

① Evaluate $\int_0^{\pi/2} \sin x\,dx$ using Simpson's 1/3, 3/8 rules.

2. Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method.

Let $y = \frac{1}{x^2+4}$ and Let $I = \int_0^2 \frac{dx}{x^2+4}$, Take $h=1$

The tabulated values of y are:

x	0	1	2
y	0.25	0.2	0.125

using Trapezoidal rule,

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_2)+2y_1\right]$$

$I_1 = 0.3875$

Take $h = 0.5$, The tabulated values of y are

x	0	0.5	1	1.5	2
y	0.25	0.2353	0.2	0.160	0.125

$$I_2 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$\rightarrow I_2 = 0.3914$

Take $h = 0.25$, The values of y are:

x	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2
y	0.25	0.2462	0.2353	0.2192	0.20	0.1798	0.160	0.1416	0.125

$$I_3 = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)\right]$$

$=(0.125)(3.1392)$

$\rightarrow I_3 = 0.3924$

Using Romberg's theorem for I_1 and I_2,

$$\rightarrow I = I_2 + \left[\frac{I_2 - I_1}{3}\right] = 0.3927 \quad ①$$

Using Romberg's theorem for I_2 & I_3,

$$\rightarrow I = I_3 + \left[\frac{I_3 - I_2}{3}\right] = 0.3927 \quad ②$$

① = ②

$$\rightarrow I = \int_0^2 \frac{dx}{x^2+4} = 0.3927$$

PROBLEMS FOR PRACTICE

① Evaluate $\int_1^4 f(x)$ by Simpson's 3/8 R

x	1	2	3	4
f(x)	1	8	27	64

TWO POINT GAUSSIAN QUADRATURE

Formula: $\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$

Example: Apply Gauss two point formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

Solution: Given that interval is -1 to 1 so we apply

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$$

Here $f(x) = \frac{1}{1+x^2}$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{1+1/3} = \frac{1}{4/3} = 3/4$$

$$f\left(1/\sqrt{3}\right) = \frac{1}{1+1/3} = 3/4$$

$$\therefore \int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{3}{4}+\frac{3}{4} = \frac{3}{2} = 1.5$$

THREE POINT GAUSSIAN QUADRATURE

Formula: $\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$

Example: Using Gaussian three point formula evaluate $\int_{-1}^{+1} (3x^2+5x^4)\,dx$

Solution: $f\left(-\sqrt{\frac{3}{5}}\right) = 3\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)^2 = \frac{18}{5}$

$$f\left(\sqrt{3/5}\right) = \frac{18}{5}$$

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[\frac{18}{5}+\frac{18}{5}\right]+0 = \frac{5}{9}\left(\frac{36}{5}\right) = 4$$

Example for practice

1. $\int_{-2}^{2} e^{-x/2}\,dx\,dy$ by Gauss two point formula
2. $\int_{0}^{1} (3x^2+5x^4)\,dx$ by Gauss three point formula.

DOUBLE INTEGRAL USING TRAPEZOIDAL AND SIMPSON'S RULES

TRAPEZOIDAL RULE

FORMULA:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

$= \frac{hk}{4}$ [(sum of the values of f at four corner)

+2 (sum of values of f at the remaining nodes on the boundary)

+4 (sum of the values of f at the interior node)]

SIMPSON RULE

Formula

$I = \frac{hk}{9}$ [sum of values of f at the four corners

+2 (Sum of the values of f at the odd Position on the boundary except the corners)

+4 (Sum of the values of f at the even Position on the boundary)

+{4 (sum of the values of f at odd position)
+8 (sum of the values of f at even Position) on the odd row of the matrix except boundary)}

+ {8 (sum of the values of f at odd position)
+16 (sum of the values of f at the even position) on the even row of the matrix}]

Example: Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{dx\,dy}{xy}$ using Trapezoidal and simpson's rule.

Solution: Divided the range of x and y into 4 equal parts

$h = \frac{2.4-2}{4} = 0.1$ $\qquad$ $K = \frac{1.4-1}{4} = 0.1$

TRAPEZOIDAL RULE

x \ y	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

$I = \frac{(0.1)(0.1)}{4}\{[0.5+0.4167+0.3571+0.2976]$

$+2[0.3841+0.4167+0.4545+0.4762+0.4545+0.4348+0.3788+0.3472+0.3205+0.3106+0.3247+0.3401]$

$+4[0.4329+0.4132+0.3953+0.3968+0.3788+0.3623+0.3663+0.3497+0.3344]\}$

$= \frac{0.01}{4}[1.5714+9.2864+13.7188]$

$= 0.0614$

SIMPSON RULES

$I = \left(\frac{h}{3}\right)\left(\frac{K}{3}\right)\{(0.5+0.4167+0.3571+0.2976)$

$+2(0.4167+0.4545+0.3472+0.3247)$

$+4(0.3846+0.4545+0.4762+0.4348+0.3788+0.3205+0.3106+0.3401+0.3788)$

$+8(0.3968+0.3623+0.3497+0.4132)$

$+16(0.3663+0.3344+0.4329+0.3953)\}$

$= \left(\frac{0.01}{3}\right)\left(\frac{0.01}{3}\right)[55.2116] = 0.0613$

PROBLEMS FOR PRACTICE Solve by Trapezoidal and Simpson Rule

1) Evaluate $\int_1^2\int_3^4 \frac{1}{(x+y)^2}\,dx\,dy$

2) $\int_0^2\int_1^2 \sin(9x+y)\,dx\,dy$, $h = 0.25$, $K = 0.5$

Taylor Series Method

$$y' = \frac{dy}{dx} = f(x,y)\,; \quad y(x_0) = y_0$$

Taylor's Series about $x = x_0$

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \dots \rightarrow (1)$$

Let $x_1 = x_0 + h$; $y_1 = y(x_1)$

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots \rightarrow (2)$$

$$y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \dots \rightarrow (3)$$

Procedure to find $y(x_1)$ using (2) (Method 1)

(1) Find y' from $\frac{dy}{dx} = f(x,y)$

(2) Find x_0, y_0 from the given problem

(3) Compute y_0'

(4) Compute y'' & y_0''

(5) Compute y''' & y_0'''

(6) Substitute in Taylor's series to get $y(x_1) = y_1$.

Procedure to find $y(x)$ using (1) (Method 2)

(1) Substitute values of $x_0, y_0, y_0', y_0'', \dots$ in (1)

(2) Simplify $y(x)$ as a series in 'x'

Solved Problems

1) Using Taylor's series method find y at $x = 0.1$ if $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$.

Solution:

(1) $y' = x^2y - 1$

(2) $x_0 = 0$, $y_0 = 1$, $h = 0.1$

(3) $y_0' = x_0^2 y_0 - 1 = 0^2 \times 1^2 - 1 = -1$

(4) $y'' = x^2y' + 2xy$; $y_0'' = x_0^2 y_0' + 2x_0 y_0$
$= 0^2 \times (-1) + 2 \times 0 \times 1 = 0$

(5) $y''' = x^2y'' + 2xy' + 2xy' + 2y$;
$y_0''' = x_0^2 y_0'' + 4x_0 y_0' + 2y_0$
$= 0^2 \times 0 + 4 \times 0 \times (-1) + 2 \times (1) = 2$

(6) $y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$

$= 1 + \frac{0.1}{1!}(-1) + \frac{0.1^2}{2!}(0) + \frac{0.1^3}{3!}(2)$

$= 1 - 0.1 + \frac{1}{3}(0.001)$

$= 0.90033$

$\therefore y_1 = y(0.1) = 0.90033$

Problems for Practice

① Solve $y' = x + y$; $y(0) = 1$ by Taylor's series method. find the values of y at $x = 0.1$ & $x = 0.2$

② Use Taylor's series method to find y at $x = 0.1$ correct to 4 decimal places from $\frac{d^2y}{dx^2} = x^2 - y$; $y(0) = 1$, with $h = 0.1$.

③ By means of Taylor's series expansion find y at $x = 0.1, 0.2$ given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

2) Solve $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 1$. use Taylor series at $x = 0.2$ & 0.4. Also, find $y(0.1)$

Solution:

(1) $y' = x^2 + y^2$

(2) $x_0 = 0$, $y_0 = 1$, $h = 0.2$

(3) $y_0' = x_0^2 + y_0^2 = 0^2 + 1^2 = 1$

(4) $y'' = 2x + 2yy'$; $y_0'' = 2x_0 + 2y_0y_0'$
$= 2 \times 0 + 2 \times 1 \times 1 = 2$

(5) $y''' = 2 + 2y'^2 + 2yy''$;
$y_0''' = 2 + 2y_0'^2 + 2y_0y_0''$
$= 2 + 2 \times 1^2 + 2 \times 1 \times 2$
$= 8$

(6) $y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \dots$

$= 1 + (x-0)\cdot 1 + \frac{(x-0)^2}{2}(2) + \frac{(x-0)^3}{6}(8) + \dots$

$= 1 + x + x^2 + \frac{4}{3}x^3 + \dots$

$y(0.2) = 1 + 0.2 + 0.2^2 + \frac{4}{3}(0.2)^3$
$= 1.25067$

$y(0.4) = 1 + 0.4 + (0.4)^2 + \frac{4}{3}(0.4)^3$
$= 1.64533$

$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4}{3}(0.1)^3$
$= 1.11133$

4) Using Taylor series method find $y(0.1)$ given $\frac{dy}{dx} = e^x - y^2$, $y(0) = 1$.

EULER'S METHOD

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0$$

$x_1 = x_0 + h \; ; \; y_1 = y(x_1)$

$x_2 = x_1 + h \; ; \; y_2 = y(x_2)$

$x_n = x_{n-1} \; ; \; y_n = y(x_n)$

$x_{n+1} = h + h_3$

$$\boxed{y(x_{n+1}) = y_{n+1} = y_n + h + (x_n y_n)} \rightarrow ①$$

PROCEDURE

① Find $f(x,y)$, x_0 y_0

② Find y_1 using $y_1 = y_0 \cdot h + (x_0, y_0)$ etc.

SOLVED PROBLEMS

① Using Euler's method find the solution of the initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$

Soln

Given $f(x,y) = \log(x+y)$

$x_0 = 0$, $y_0 = 2$, $h = 0.2$

$x_1 = x_0 + h = 0.2$

$y_1 = y_0 + h f(x_0, y_0)$

$= 2 + 0.2 \times \log(x_0 + y_0)$

$= 2 + 0.2 \times \log(0 + 2)$

$= 2 + 0.2 \times$

$$\boxed{y(0.2) = 2.0602}$$

② Using Euler's method solve $y' = x + y + xy$, $y(0) = 1$, compute y at $x = 0.1$ by taking $h = 0.05$.

Soln

Given $f(x,y) = x + y + xy$; $x_0 = 0$, $y_0 = 1$, $h = 0.05$

$x_1 = x_0 + h = 0 + 0.05 = 0.05$

$y(x_1) = y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.05 \times (x_0 + y_0 + x_0 y_0)$

$= 1 + 0.05 \times (0 + 1 + 0)$

$$\boxed{y_1 = y(0.05) = 1.05}$$

$x_2 = x_1 + h = 0.05 = 0.1$

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.05 \times 0.05 \times (x_1 + y_1 + x_1 y_1)$

$= 1.05 \times 0.05 \times (0.05 + 1.05 + 0.05 \times 1.05)$

$= 1.05 + 0.05 \times (1.1525)$

$y_2 = 1.10762$

$\therefore y_2 = y(0.1) = 1.10762$

x	y
0.05	1.05
0.1	1.10762

Problems for practice

① Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem $\frac{dy}{dx} = \frac{1}{2}(x^2+1)y^2$, $y(0.2) = 1.1114$

② Using modified Euler's method, find $y(0.1)$ if $dy/dx = x^2 + y^2$, $y(0) = 1$.

MODIFIED EULER'S METHOD :-

$$\boxed{y_{n+1} = y_n + h\left[f\left(x_n + \tfrac{1}{2}h,\; y_n + \tfrac{1}{2}h\, f(x_n, y_n)\right)\right]}$$

SOLVED PROBLEMS

① Using modified Euler's method, compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$

$f(x,y) = y - \frac{2x}{y}$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$y_1 = y_0 + h\left[f\left(x_0 + \frac{1}{2}h,\; y_0 + \frac{1}{2}h + (x_0, y_0)\right)\right]$

$= y_0 + h\left[f\left(0 + \frac{0.1}{2},\; 1 + \frac{0.1}{2} \times \left(y_0 - \frac{2x_0}{y_0}\right)\right)\right]$

$= 1 + 0.1 \times [f(0.05), 1 + 0.05 \times (1 - 0)]$

$= 1 + 0.1 \times f(0.05, 1.05)$

$= 1 + 0.1 \times \left(1.05 - \frac{2 \times 0.05}{1.05}\right) = 1 + 0.1 \times 0.9548 = 1.09548$

② Evaluate $y(1.2)$ correct to three decimal places, by the modified Euler's method given that $\frac{dy}{dx} = (y - x^2)^3$, $y(1) = 0$ taking $h = 0.2$

Soln

$f(x,y) = (y - x^2)^3$; $x_0 = 1$, $y_0 = 0$, $h = 0.2$; $x_1 = 1.2$

$y_1 = y(1.2) = y_0 + h\left[f\left(x_0 + \frac{1}{2}h,\; y_0 + \frac{1}{2}h\right.\right.$

$= y_0 + h\left[f(x_0 + 0.1,\; y_0 + 0.1 \times f(x_0, y_0))\right]$

$= 0 + 0.2 \times f(1 + 0.1,\; 0 + 0.1 \times (y_0 - x_0^2)^3)$

$= 0.2 \times f(1.1,\; 0.1 \times ((0 - 1)^2)^3)$

$= 0.2 \times (-0.1 - 1.1^2)^3$

$= 0.2\,(-0.1 - 1.21)^3 = 0.2 \times (-2.2 + 8091)$

$y_1 = y(1.2) = -0.4496 \simeq -0.449$

FOURTH ORDER RANGE-KUTTA METHOD (R-K Method)

Algorithm

compute :- $k_1 = hf(x, y)$

compute :- $k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$

compute :- $k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right)$

compute :- $k_4 = hf\left(x + \frac{h}{2}, y + k_3\right)$

and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(x+h) = y(x) + \Delta y.$$

Working Rule :-

To solve $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

calculate $k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_1 = y_0 + \Delta y$$

when (x_1, y_1) is started, repeat the process.

Solved examples

① Using R-K Method of 4^{th} order, solve

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} \quad \text{with } y(0) = 1 \text{ at } x = 0.2$$

solution

$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0$, $y_0 = 1$, $h = 0.2$

step-1

$k_1 = hf(x_0, y_0) = 0.2 \times f(0, 1)$

$= 0.2 \times \frac{1^2 - 0^2}{1^2 + 0^2} = 0.2$

step-2

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f(0 + 0.1, 1 + 0.2)$

$= 0.2 \times f(0.1, 1.1)$

$= 0.2 \times \frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}$

$= 0.19672$

step-3

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.2 \times f(0 + 0.1, 1 + 0.09836)$

$= 0.2 \times f(0.1, 1.09836)$

$= 0.2 \times \left(\frac{1.09836^2 - 0.1^2}{1.09836^2 + 0.1^2}\right)$

$\simeq 0.1967$

step-4

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2 \times f(0 + 0.2, 1 + 0.1967)$

$= 0.2 \times f(0.2, 1.1967)$

$= 0.2 \times \left(\frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2}\right)$

$= 0.1891$

step-5

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$

$= 0.19598$

$y(0.2) = y_1 = y_0 + \Delta y = 1 + 0.19598 = 1.19598$

② Apply fourth order R-K method, to determine $y(0.2)$ with $h = 0.1$ from $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

Solution: $f(x, y) = x^2 + y^2$; $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$k_1 = hf(x_0, y_0) = 0.1 \times f(0, 1) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1105$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \times f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1105}{2}\right) = 0.1116$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 \times f(0 + 0.1, 1 + 0.1116)$

$= 0.1246$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0.1 + 2(0.1105) + 2(0.1116) + 0.1246]$

$\Delta y = 0.1147$

$y(0.1) = y_1 = y_0 + \Delta y = 1 + 0.1147 = 1.1147$

Now, $x_1 = 0.1$, $y_1 = 1.1147 \simeq 1.1115$ (correct to 4 decimals)

$k_1 = hf(x_1, y_1) = 0.1245$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.14$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1418$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1611$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1415$

$y_2 = y(0.2) = y_1 + \Delta y = 1.1115 + 0.1415 = 1.253$

Problems for practice

① Find $y(0.7)$ given that $y' = x - y^2$, $y(0.6) = 1.7379$ using R-K method of fourth order. Take $h = 0.1$

② Use 4^{th} order R-K method to compute y for $x = 0.1$ given $y' = \frac{xy}{1 + x^2}$, $y(0) = 1$, take $h = 0.1$

③ Find the Value of $y(1.1)$ using R-K method of 4^{th} order for $\frac{dy}{dx} = y^2 + xy$; $y(1) = 1$

④ Given $\frac{dy}{dx} = x^3 + \frac{y}{2}$, $y(1) = 2$, find $y(1.1)$ using R-K method of fourth order.

MULTISTEP METHODS

Milne's Predictor and corrector Methods

Predictor formula: $y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$

Corrector formula: $y_{n+1,C} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$

Solved Problems

① Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$, $y(0.2) = 2.073$, $y(0.4) = 2.452$ and $y(0.6) = 3.023$. Find $y(0.8)$ by Milne's Predictor-corrector method taking $h = 0.2$

Solution:

$x_0 = 0$,	$y_0 = 2$;	$y' = f(x,y) = x^3 + y$
$x_1 = 0.2$,	$y_1 = 2.073$;	$y'_1 = x_1^3 + y_1 = 0.2^3 + 2.073 = 2.081$
$x_2 = 0.4$,	$y_2 = 2.452$;	$y'_2 = x_2^3 + y_2 = 0.4^3 + 2.452 = 2.516$
$x_3 = 0.6$,	$y_3 = 3.023$;	$y'_3 = x_3^3 + y_3 = 0.6^3 + 3.023 = 3.239$

Predictor formula:

$y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$

Put $n=3$

$y_{4,P} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) = 2 + \frac{4\times 0.2}{3}(2\times 2.081 - 2.516 + 2\times 3.239)$

$\boxed{y_{4,P} = 4.1664}$

Corrector formula:

$y_{n+1,C} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$

Put $n=3$

$y_{4,C} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$y'_4 = x_4^3 + y_4 = 0.8^3 + 4.1664 = 4.6784$

$= 2.452 + \frac{0.2}{3}(2.516 + 4\times 3.239 + 4.6784)$

$\boxed{y_{n+1,C} = 3.79536}$

corrected value of y at $x = 0.8$ is 3.79356

Problems for practice:

1) Solve $y' = x - y^2$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$ by Milne's method to find $y(0.8)$ and $y(1)$.

② Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$

soln

$y' = \frac{2 - y^2}{5x}$, $x_0 = 4$, $x_1 = 4.1$, $x_2 = 4.2$, $x_3 = 4.3$, $x_4 = 4.4$

$y_0 = 1$, $y_1 = 1.0049$, $y_2 = 1.0097$, $y_3 = 1.0143$

$y'_1 = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5\times 4.1} = 0.0493$

$y'_2 = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5\times 4.2} = 0.0467$

$y'_3 = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5\times 4.3} = 0.0452$

Predictor formula:

$y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$

Put $n=3$

$y_{4,P} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$

$= 1 + \frac{4\times 0.1}{3}(2\times 0.0493 - 0.0467 + 2\times 0.0452)$

$\boxed{y_{4,P} = 1.01897}$

$y'_4 = \frac{2 - y_4^2}{5x_4} = \frac{2 - (1.01897)^2}{5\times 4.4} = 0.0437$

Corrector formula:

$y_{n+1,C} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$

Put $n=3$

$y_{4,C} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$= 1.0097 + \frac{0.1}{3}(0.0467 + 4\times 0.0452 + 0.0437)$

$\boxed{y_{4,C} = 1.01874}$

corrected value of y at $x = 4.4$ is 1.01874

② Using Milne's method find $y(2)$, if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$, given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$

Finite difference solution of second order ordinary differential equation

Finite difference approximations to the derivatives are

$$\boxed{y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})}$$

$$\boxed{y_i'' = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1})}$$

* Replace the derivatives in the given equation and boundary conditions by their finite difference approximations
* This method reduces the problem to the solution of linear algebraic equations

Solved Problems

① Using finite difference method, find $y(0.25)$, $y(0.5)$, $y(0.75)$ satisfying the differential equation

$\frac{d^2y}{dx^2} + y = x$, subject to the boundary conditions $y(0) = 0$, $y(1) = 2$.

Solution

x	0 (x_0)	0.25 (x_1)	0.5 (x_2)	0.75 (x_3)	1 (x_4)
y	0 (y_0)	(y_1)	(y_2)	(y_3)	(y_4)

$y'' + y = x$

$y_i'' + y_i = x_i$

$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + y_i = x_i$

$16(y_{i-1} - 2y_i - y_{i+1}) + y_i = x_i$

$16y_{i-1} - 32y_i + 16y_{i+1} + y_i = x_i$

$16y_{i-1} - 31y_i + 16y_{i+1} = x_i$ — (1)

$\boxed{i=1 \text{ in } (1)}$

$16y_0 - 31y_1 + 16y_2 = x_1$

$\Rightarrow 0 - 31y_1 + 16y_2 = 0.25$ — (2)

$\boxed{i=2 \text{ in } (1)}$

$16y_1 - 31y_2 + 16y_3 = x_2$

$\Rightarrow 16y_1 - 31y_2 + 16y_3 = 0.5$ — (3)

$\boxed{i=3 \text{ in } (1)}$

$16y_2 - 31y_3 + 16y_4 = x_3$

$\Rightarrow 16y_2 - 31y_3 + 32 = 0.75$

$\Rightarrow 16y_2 - 31y_3 = -31.25$ — (4)

solving (2), (3) and (4) using calculator, we get

$y_1 = 6.5443$

$y_2 = 1.0702$

$y_3 = 1.5604$

② Using the finite difference Method, compute $y(0.5)$ given $y'' - 64y + 10 = 0$, $y(0) = y(1) = 0$, subdividing the interval into 2 equal parts.

Solution

Take $h = \frac{1-0}{2} = 0.5$

x	0 (x_0)	0.5 (x_1)	1 (x_2)
y	0 (y_0)	(y_1)	0 (y_2)

$y'' - 64y + 10 = 0 \Rightarrow y'' - 64y = -10$

$\Rightarrow \frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1}] - 64y_i = -10$

$\boxed{h^2 = \frac{1}{4} \Rightarrow \frac{1}{h^2} = 4}$

$\Rightarrow 4(y_{i-1} - 2y_i + y_{i+1}) - 64y_i = -10$

$\Rightarrow 4y_{i-1} - 72y_i + 4y_{i+1} = -10$ — (1)

$\boxed{i=1 \text{ in } (1)}$ $\Rightarrow 4y_0 - 72y_1 + 4y_2 = -10$

$\Rightarrow 4(0) - 72y_1 + 4(0) = -10$

$y_1 = 10/72 = 0.1389$

x	0	0.5	1
y	0	0.1389	0

Problems for practice

1) Solve $y'' - y = x$, given $y(0) = y(1) = 0$, using finite difference dividing the interval into 4 equal parts

2) Using finite differences, solve $y'' - 3y' + 2y = 0$ given $y(0) = 2$, $y(1) = 10.1$ dividing the interval into 2 equal parts

ONE DIMENSIONAL HEAT EQUATION BY EXPLICIT AND IMPLICIT METHODS

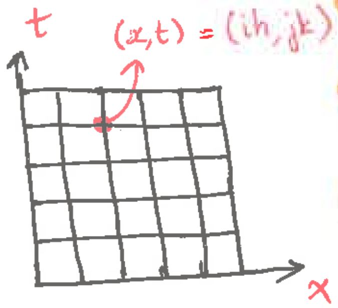
Explicit Method (or) Bendre - Schmidt method :

1-D Heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \; ; \quad c^2 = \frac{k}{\rho} \qquad \text{——— (1)}$$

(or) $\frac{\partial^2 u}{\partial x^2} - a \frac{\partial u}{\partial t} = 0$ where $c^2 = \frac{1}{a}$

In a rectangular mesh in the xt-plane with spacing h along x direction and spacing k along t direction, denote $(x,t) = (ih, jk)$



$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \; ;$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$(1) \Rightarrow u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i-1,j} \qquad \text{——— (2)}$$

where $\lambda = \frac{k}{ah^2}$

(2) is called the schmidt explicit formula which is valid only for $0 < \lambda \le \frac{1}{2}$.

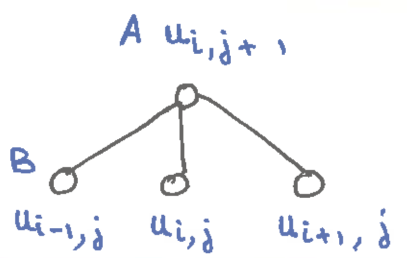
If $\lambda = \frac{1}{2}$, then (2) reduces to

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

where $k = \frac{ah^2}{2}$

Select k such that $\lambda = \frac{1}{2}$.

SCHEMATIC DIAGRAM



Value of u at A $= \frac{1}{2}$ [Value of u at B + Value of u at C].

Example :

Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = x(4-x)$. Assume $h = 1$. Find the values of u upto $t = 5$.

Solution :

Compare $\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0$ with $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0 \Rightarrow \boxed{a = 2}$

$$k = \frac{ah^2}{2} = \frac{2 \times 1^2}{2} = 1$$

$$\lambda = \frac{k}{ah^2} = \frac{1}{2 \times 1^2} = \frac{1}{2}$$

$u(0,t) = 0$ $\Downarrow$ $u(0,0) = 0$, $u(0,1) = 0$, $u(0,2) = 0$, $u(0,3) = 0$, $u(0,4) = 0$, $u(0,5) = 0$

$u(4,t) = 0$ $\Downarrow$ $u(4,0) = 0$, $u(4,1) = 0$, $u(4,2) = 0$, $u(4,3) = 0$, $u(4,4) = 0$, $u(4,5) = 0$

$u(x,0) = x(4-x)$:
- $u(1,0) = 1(4-1) = 3$
- $u(2,0) = 2(4-2) = 4$
- $u(3,0) = 3(4-3) = 3$

t \ x	0	1	2	3	4
0	0	3	4	3	0
1	0	$\frac{0+4}{2} = 2$	$\frac{3+3}{2} = 3$	$\frac{4+0}{2} = 2$	0
2	0	$\frac{0+3}{2} = 1.5$	$\frac{2+2}{2} = 2$	$\frac{3+0}{2} = 1.5$	0
3	0	$\frac{0+2}{2} = 1$	$\frac{1.5+1.5}{2} = 1.5$	$\frac{2+0}{2} = 1$	0
4	0	$\frac{0+1.5}{2} = 0.75$	$\frac{1+1}{2} = 1$	$\frac{1.5+0}{2} = 0.75$	0
5	0	$\frac{0+1}{2} = 0.5$	$\frac{0.75+0.75}{2} = 0.25$	$\frac{1+0}{2} = 0.5$	0

PROBLEMS FOR PRACTICE

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, $t \ge 0$ with $u(x,0) = x(1-x)$, $0 < x < 1$ and $u(0,t) = u(1,t) = 0$, for all $t > 0$, using explicit method with $h = 0.2$ for 3 time steps.

CRANK-NICHOLSON'S METHOD OF SOLVING 1-D HEAT EQUATION

IMPLICIT METHOD

1-D HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ; \quad c^2 = \frac{k}{s\rho}$$

put $c^2 = \frac{1}{a}$

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0$$

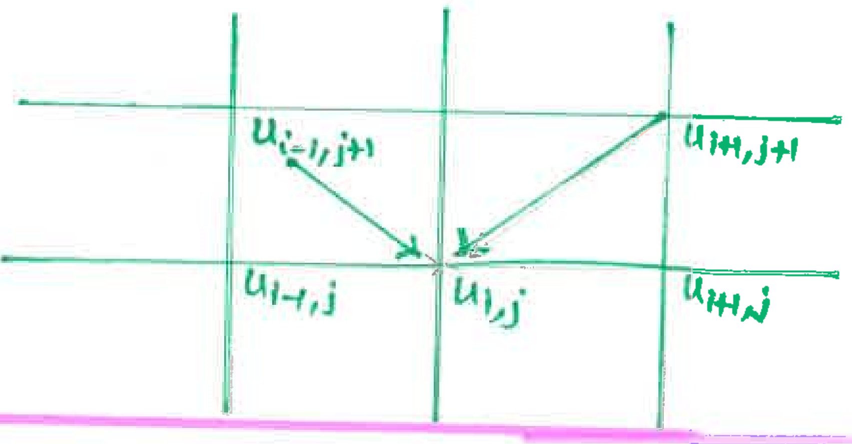
Taking $\frac{ka^2}{h^2} = \lambda$, difference equation is

$$\boxed{\begin{aligned} -\lambda u_{i-1,j+1} + (2\lambda+2)u_{i,j+1} - \lambda u_{i+1,j+1} \\ = \lambda u_{i-1,j} + (2-2\lambda)u_{i,j} + \lambda u_{i+1,j} \end{aligned}}$$

Choose k such that $\lambda = \frac{ka^2}{h^2} = 1$. Then

$$\boxed{\begin{aligned} -u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} \\ = u_{i-1,j} + u_{i+1,j} \end{aligned}}$$

(or) $u_{i,j+1} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j})$



SOLVED PROBLEM

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in $0<x<5$, $t \geq 0$ given that $u(x,0)=20$, $u(0,t)=0$, $u(5,t)=100$. Compute u for time-step with $h=1$ by Crank-Nicholson's method.

Here $\boxed{a=1}$, choose h such that $\boxed{\lambda=1} \Rightarrow \frac{ka^2}{h^2} = 1 \Rightarrow k = h^2 = 1$

$\boxed{\therefore k=1, h=1}$

$\boxed{u(0,t)=0 \Downarrow u(0,0)=0,\ u(0,1)=0}$

$\boxed{u(5,t)=100 \Downarrow u(5,0)=100,\ u(5,1)=100}$

$\boxed{u(x,0)=20,\ 0<x<5 \Rightarrow u(1,0)=20,\ u(2,0)=20,\ u(3,0)=20,\ u(4,0)=20,\ u(5,0)=20}$

$u_1 = \frac{1}{4}(0+0+20+u_2)$
$\Rightarrow 4u_1 - u_2 = 20 \rightarrow (1)$

$u_3 = \frac{1}{4}(u_2+20+20+u_4)$
$\Rightarrow -u_2 + 4u_3 - u_4 = 40 \rightarrow (3)$

$u_2 = \frac{1}{4}(u_1+20+20+u_3)$
$\Rightarrow -u_1 + 4u_2 - u_3 = 40 \rightarrow (2)$

$u_4 = \frac{1}{4}(u_3+20+100+100)$
$\Rightarrow -u_3 + 4u_4 = 220 \rightarrow (4)$

Solving (1), (2), (3) & (4) $\boxed{u_1 = 10.05}$, $\boxed{u_2 = 20.19}$, $\boxed{u_3 = 30.71}$, $\boxed{u_4 = 62.68}$

$(3)\times 4 + (4) \Rightarrow -4u_2 + 15u_3 = 380 \rightarrow (5)$

$(2)\times 15 + (5) \Rightarrow -15u_1 + 56u_2 = 980 \rightarrow (6)$

$(1)\times 15 + 4\times(6) \Rightarrow 209u_2 = 4220 \quad \therefore u_2 = 20.19$

$(1) \Rightarrow 4u_1 = 20 + u_2 \Rightarrow u_1 = 10.05$

$(2) \Rightarrow u_3 = -u_1 + 4u_2 - 40 \Rightarrow u_3 = 30.71$

$(4) \Rightarrow 4u_4 = 220 + u_3 \Rightarrow u_4 = 62.68$

By Crank-Nicholson's formula
$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

PROBLEM FOR PRACTICE

Use Crank-Nicholson scheme to solve $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t}$, $0<x<1$ and $t>0$ given $u(x,0)=0$, $u(0,t)=0$ and $u(1,t)=100t$. Compute $u(x,t)$ for one time step, taking $h=\frac{1}{4}$

One Dimensional wave equation:

1-D wave equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}}$$

choose $k = \frac{h}{a}$

Explicit formula : $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

$u_t(x,0) = f(x)$: $c = \frac{1}{2}(a+b) + \frac{1}{2} f(x)$

$$\begin{array}{cc} a & b \\ \searrow & \swarrow \\ & c \end{array}$$

Solved Problem:

1. Solve : $u_{tt} = u_{xx}$, $0<x<1$, $t>0$, $u(0,t) = u(1,t) = 0$, $t>0$, $u(x,0) = \sin(2\pi x)$, $0 \le x \le 1$ and $\frac{\partial u}{\partial t}(x,0) = 0$, $0 \le x \le 1$ with $h = 0.25$ and $k = 0.25$

Soln:

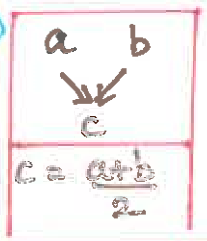
$a^2 = 1 \Rightarrow a = 1$; $k = \frac{h}{a} = \frac{0.25}{1} = 0.25$

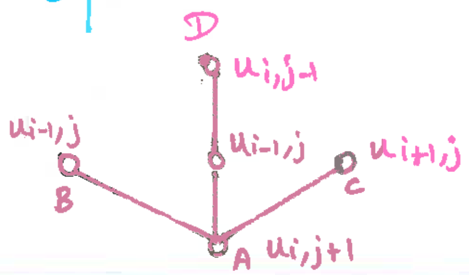
$u(0,t) = 0$
$\Downarrow$
$u(0,0) = 0$
$u(0,0.25) = 0$
$u(0,0.5) = 0$
$u(0,0.75) = 0$
$u(0,1) = 0$

$u(1,t) = 0$
$\Downarrow$
$u(1,0) = 0$
$u(1,0.25) = 0$
$u(1,0.5) = 0$
$u(1,0.75) = 0$
$u(1,1) = 0$

$u(x,0) = \sin(2\pi x)$
$\Downarrow$
$u(0.25,0) = \sin\left(\frac{\pi}{2}\right) = 1$
$u(0.5,0) = \sin(\pi) = 0$
$u(0.75,0) = \sin\left(\frac{3\pi}{2}\right) = -1$

$u_t(x,0) = 0 \Rightarrow$





j \ i	0	0.25	0.5	0.75	1
0	0	1	0	-1	0
0.25	0	$\frac{0+0}{2}=0$	$\frac{1-1}{2}=0$	$\frac{0+0}{2}=0$	0
0.5	0	-1	0	1	0
0.75	0	0	0	0	0
1	0	1	0	-1	0

Two Dimensional Laplace Equation:

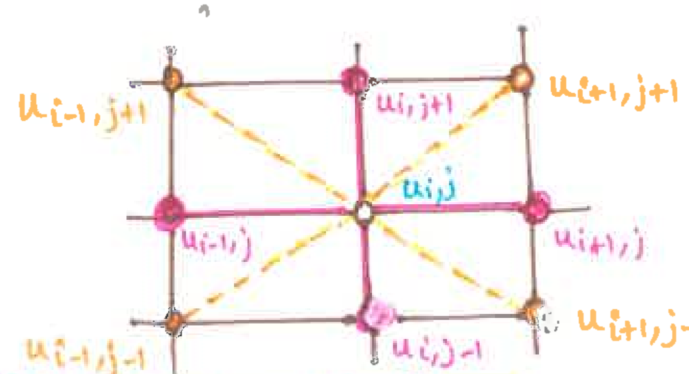
$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Standard Five Point Formula (SFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

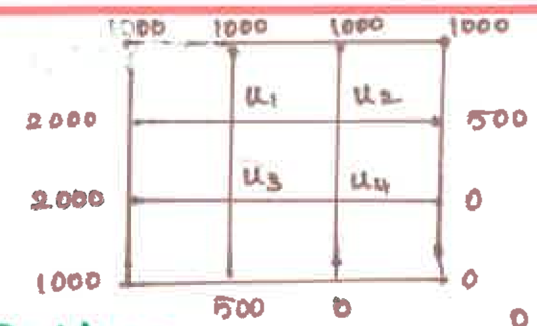
Diagonal Five Point Formula (DFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$



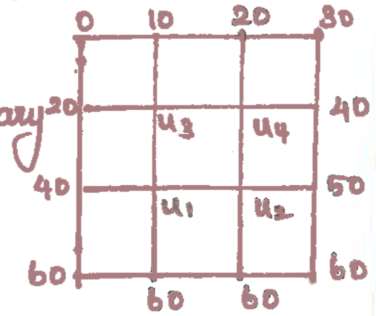
Problems for Practice:

1. Solve $16u_{xx} = u_{tt}$ given $u(0,t) = u(5,t) = 0$, $u(x,0) = x^2(5-x)$ and $u_t(x,0) = 0$
2. Solve $u_{xx} + u_{yy} = 0$, the boundary conditions are given below (give 3 iterations)



Solved Problem:

Solve $\nabla^2 u = 0$, the boundary conditions are given below (give only 3 iterations)



Assume $u_4 = 0$

$u_1 = \frac{1}{4}(20 + u_4 + 60 + 60) = 35$ (DFPF)

$u_2 = \frac{1}{4}(u_1 + u_4 + 50 + 60) = 36.3$ (SFPF)

$u_3 = \frac{1}{4}(20 + 10 + u_4 + u_1) = 16.3$ (SFPF)

$u_4 = \frac{1}{4}(u_3 + 20 + 40 + u_2) = 28.2$ (SFPF)

Hereafter use SFPF

Iteration 1 :

$u_1^{(1)} = \frac{1}{4}(40 + u_3 + u_2 + 60) = \frac{1}{4}(40 + 16.3 + 36.3 + 60)$
$= 38.2$

$u_2^{(1)} = \frac{1}{4}(u_1^{(1)} + u_4 + 50 + 60) = \frac{1}{4}(38.2 + 28.2 + 50 + 60)$
$= 44.1$

$u_3^{(1)} = \frac{1}{4}(20 + 10 + u_4 + u_1^{(1)}) = \frac{1}{4}(20 + 10 + 28.2 + 38.2)$
$= 24.1$

$u_4^{(1)} = \frac{1}{4}(u_3^{(1)} + 20 + 40 + u_2^{(1)}) = \frac{1}{4}(24.1 + 20 + 40 + 44.1)$
$= 32.1$

$|||^{ly}$

$u_1^{(2)} = 42.1$; $u_2^{(2)} = 46.1$; $u_3^{(2)} = 26.1$; $u_4^{(2)} = 33.1$

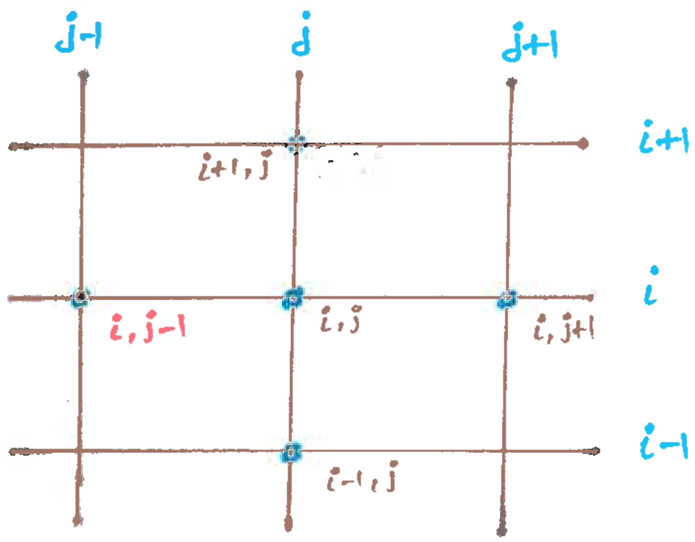
$u_1^{(3)} = 43.1$; $u_2^{(3)} = 46.6$; $u_3^{(3)} = 26.6$; $u_4^{(3)} = 33.3$

Two dimensional Poission equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

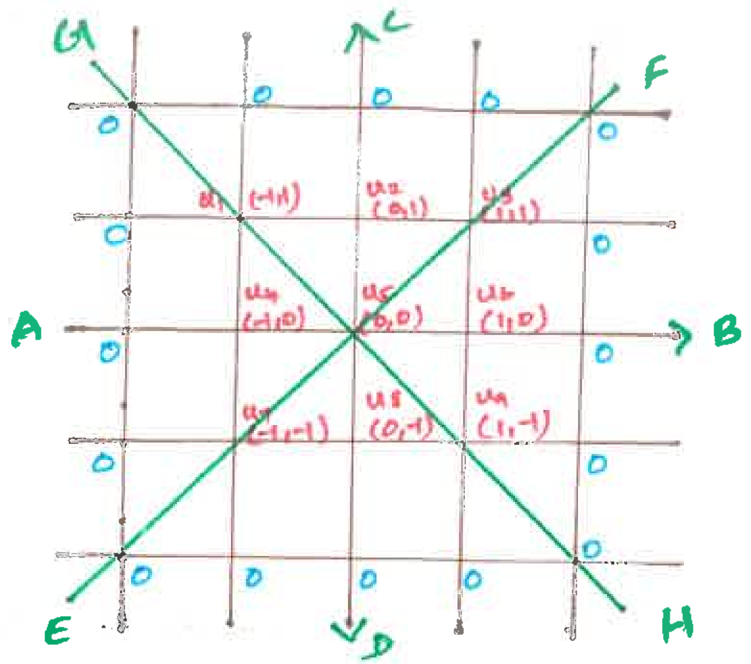
Difference equation is

$$u_{ij} = \frac{1}{4}\left[u_{i,j-1} + u_{i+1,j} + u_{i,j+1} + u_{i-1,j}\right] - \frac{h^2}{4} f(x_i, y_j)$$



Solved PROBLEMS

① Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given $u=0$ on the four boundaries dividing the square into 16 subsquares of length 1 unit.



Given mesh is Symmetrical about AB

$\Rightarrow u_1 = u_7,\ u_2 = u_8,\ u_3 = u_4 \quad \Rightarrow$ (i)

Given mesh is Symmetrical about CD

$\Rightarrow u_1 = u_3,\ u_4 = u_6,\ u_7 = u_9 \quad \Rightarrow$ (ii)

Given mesh is Symmetrical about GH

$\Rightarrow u_2 = u_4,\ u_3 = u_7,\ u_6 = u_8 \quad \Rightarrow$ (iii)

Given mesh is Symmetrical about EF

$u_1 = u_4,\ u_4 = u_8,\ u_2 = u_6 \quad \Rightarrow$ (iv)

From (i), (ii), (iii), (iv)

$$u_1 = u_3 = u_7 = u_9$$
$$u_2 = u_4 = u_6 = u_8$$

We have to find only $u_1, u_2, \& u_5$

$$u_{ij} = \frac{1}{4}\left[u_{i,j-1} + u_{i+1,j} + u_{i,j+1} + u_{i-1,j}\right] - \frac{h^2}{4} f(x_i, y_j) \Rightarrow ①$$

$$u_1 = u(-1,1) = \frac{1}{4}(0+0+u_2+u_4) - \frac{1}{4}\left[8(-1)^2(1)^2\right]$$
$$= \frac{1}{4}(2u_2) - 2 = -2 \quad [\text{put } u_2 = 0] \Rightarrow ②$$

$$u_2 = u(0,1) = \frac{1}{4}(u_1 + 0 + u_3 + u_5) - \frac{1}{4}[8.0.1]$$
$$= \frac{1}{4}[-4] = -1 \quad [\text{put } u_5 = 0] \Rightarrow ③$$

$$u_5 = u(0,0) = \frac{1}{4}(u_4 + u_2 + u_6 + u_8) - \frac{1}{4}(8.0.1)$$
$$= \frac{1}{4}(4u_2) = -1 \Rightarrow ④$$

$\therefore u_5 = u_2$

Iteration 1

$$u_1^{(1)} = \tfrac{1}{2} u_2 - 2 = \tfrac{1}{2}(-1) - 2 = -3$$
$$u_2^{(1)} = \tfrac{1}{4}(2u_1 + u_5) = \tfrac{1}{4}[-6-1] = -2$$

Iteration 2

$$u_1^{(2)} = \tfrac{1}{2} u_2^{(1)} - 2 = \tfrac{1}{2}(-2) - 2 = -3$$
$$u_2^{(2)} = \tfrac{1}{4}\left[2u_1^{(2)} + u_5^{(1)}\right] = \tfrac{1}{4}[2(-3)-2] = -2$$

First iteration values = Second iteration values

$\therefore u_1 = -3 \quad u_2 = -2,\ u_5 = u_2 = -2$

$\therefore ⑤ \Rightarrow u_1 = u_3 = u_7 = u_9 = -3 \quad | \quad u_5 = u_2 = -2.$

$\quad\quad u_2 = u_4 = u_6 = u_8 = -2$

problem for practice.

① Solve the poisson's equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x=0,\ y=0,\ x=3,\ y=3$ with $u=0$ on the boundary taking $h=1$

SIMATS

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