Unit Digit

To find the unit digit, one need not consider the complete number. The operation can be performed at unit digit only to get the results. Unit places can be directly added, subtracted or multiplied but can't be divided.

Only the unit digit of the number matters while calculating the unit digit.

For example, finding the unit digit of 76^{214} is the same as finding the unit digit of 6^{214}.

For example, finding the unit digit of $19{,}987^{21{,}567}$ is the same as finding the unit digit of $7^{21{,}567}$.

For example:

Unit digit of (5,266 × 7,869 – 24,372) = 6 × 9 – 2 = 4 – 2 = 2.

As you can see in the above example, we have taken only the unit digit of 5,266, 7,869, and 24,372 to solve the question.

Cyclicity

Unit digit of all the digits from 0 to 9, when raised to some power, repeats itself in a certain cycle.

Based on cyclicity, all ten digits {0, 1, 2, 3, ... 9} can be divided into three categories.

0		2
1	4	3
5	9	7
6		8
Cyclicity = 1	Cyclicity = 2	Cyclicity = 4

Digits with Cyclicity 1

Any number ending with 0 or 1 or 5 or 6 raised to any power (except 0) will always end in 0 or 1 or 5 or 6, respectively. That is why these digits have cyclicity as 1.

$5^1 = 5$ $6^1 = 6$

$5^2 = 25$ $6^2 = 36$

$5^3 = 125$ $6^3 = 216$

$5^4 = 625$ $6^4 = 1{,}296$

⋮ ⋮ ⋮ ⋮

Example 1:

Find the unit digit of the following:

a) $(375)^{37} \times (436)^{289} + (3741)^{2{,}895}$

b) $31^{3{,}637} + 76^{214}$

Solution: a) 1; b) 1

a) $(375)^{37} \times (436)^{289} + (3741)^{2{,}895}$

Unit digits are 5, 6, and 1, with cyclicity = 1.

$(375)^{37} \rightarrow$ Unit digit = 5

$(436)^{289} \rightarrow$ Unit digit = 6

$(3741)^{2{,}895} \rightarrow$ Unit digit = 1

Unit digit of $(375)^{37} \times (436)^{289} + (3{,}741)^{2{,}895}$ = 5 × 6 + 1 = 0 + 1 = 1

b) $31^{3{,}637} + 76^{214}$

Unit digit of $31^{3{,}637}$ = 1

Unit digit of 76^{214} = 6

Unit digit of $31^{3{,}637} + 76^{214}$ = 1 + 6 = 7

Digits with Cyclicity 2

Unit digits of numbers ending with 4 or 9 repeat themselves after every second power. It can also be said that numbers ending with 4 or 9 raised to any power (except 0) will have only two different answers of a unit digit.

Therefore, the cyclicity of 4 and 9 is 2.

$4^1 = 4$ $9^1 = 9$

$4^2 = 16$ $9^2 = 81$

$4^3 = 64$ $9^3 = 729$

$4^4 = 256$ $9^4 = 6{,}561$

$4^5 = 1{,}024$ $9^5 = 59{,}049$

⋮ ⋮ ⋮ ⋮

It can be observed that the unit digit of

$$(------4)^{Odd} = 4$$
$$(------4)^{Even} = 6$$
$$(------9)^{Odd} = 9$$
$$(------9)^{Even} = 1$$

Example 2:

Find unit digit of $(324)^{397} + (689)^{322} - (79)^{731}$

Solution: 6

Unit digit of $(324)^{397}$ = unit digit of $(4)^{397}$
= unit digit of $4^{Odd} = 4$.

Unit digit of $(689)^{322}$ = unit digit of $(9)^{322}$
= unit digit of $9^{even} = 1$.

Unit digit of $(79)^{731}$ = unit digit of $(9)^{731}$
= unit digit of $9^{odd} = 9$.

Unit digit of $(324)^{397} + (689)^{322} - (79)^{731}$
$= 4 + 1 - 9 = 5 - 9$.

{Now, the unit digit can't be negative, so take 10 carry and add}

Hence, unit digit of $(324)^{397} + (689)^{322} - (79)^{731}$
$= 5 + 10 - 9 = 6$.

(This is similar when we subtract 19 from 45, the unit digit will be 6).

Example 3:

Find the unit digit of

$$\left(14^{15^{16^{17^{\cdots\cdots\infty}}}} \times 29^{30^{31^{32^{\cdots\cdots\infty}}}}\right)$$

Solution: 4

$$(14)^{15^{16^{17^{\cdots\cdots\infty}}}}$$

Figure out whether $15^{16^{17^{\cdots\cdots\infty}}}$ would be even or odd.

15 = Odd number and $(\text{odd})^{\text{Any power}}$ = odd number

Hence, $15^{16^{17^{\cdots\cdots\infty}}}$ = odd number

Unit digit of $(14)^{odd}$ = unit digit of $4^{Odd} = 4$

Similarly, $(29)^{30^{31^{32^{\cdots\cdots\infty}}}}$

30 = Even number and $(\text{even})^{\text{Any power}}$ = even number

Hence, $30^{31^{32^{\cdots\cdots\infty}}}$ = even number

Unit digit of $(29)^{even} \equiv$ unit digit of $9^{Even} = 1$.

Unit digit of

$$\left(14^{15^{16^{17^{\cdots\cdots\infty}}}} \times 29^{30^{31^{32^{\cdots\cdots\infty}}}}\right)$$

$= 4 \times 1 = 4$

Digits with Cyclicity 4

Unit digits of numbers ending with 2 or 3 or 7 or 8 repeat themselves after every fourth power. It can also be said that numbers ending with 2 or 3 or 7 or 8 raised to any power (except 0) will have four different unit-digit answers.

Therefore, the cyclicity of 2, 3, 7, or 8 is 4.

$2^1 = 2$	$3^1 = 3$	$7^1 = 7$	$8^1 = 8$
$2^2 = 4$	$3^2 = 9$	$7^2 = 49$	$8^2 = 64$
$2^3 = 8$	$3^3 = 27$	$7^3 = 343$	$8^3 = 512$
$2^4 = 16$	$3^4 = 81$	$7^4 = 2401$	$8^4 = 4096$
$2^5 = 32$	$3^5 = 243$	$7^5 = 16807$	$8^5 = 32768$
⋮	⋮	⋮	⋮

It can be observed that each of 2, 3, 7, and 8 repeats after the fourth power.

Method to find unit digit of numbers having cyclicity 4

- Divide the power (index) by 4 and find the remainder.
- If the remainder is 1, 2, or 3, raise the unit digit of the number to the power 1, 2, and 3, respectively.
- If the remainder is 0, raise the unit digit of the number to the power 4.

For example:

Find unit digit of 23^{746}.

Remainder $\left[\frac{746}{4}\right] = 2$

Unit digit of 23^{746} = unit digit of 3^{746}

As 3 has a cyclicity of 4

So, unit digit of 3^{746} = unit digit of $3^{(\text{remainder of } 746/4)}$ = unit digit of $3^2 = 9$.

Note: Remainder can be obtained by dividing the last two digits of the number in case of divisibility by 4.

Example 4:

Find the unit digit of $342^{56} \times 767^{38}$.

Solution: 4

Rem $\left[\frac{56}{4}\right] = 0$

Unit digit of 342^{56} = unit digit of 2^{56} = unit digit of $2^4 = 6$.

Rem $\left[\frac{38}{4}\right] = 2$

Unit digit of 767^{38} = unit digit of 7^{38} = unit digit of $7^2 = 9$.

Unit digit of $342^{56} \times 767^{38} = 6 \times 9 = 4$.

Example 5:

Find the unit digit of $53^{54^{55}}$.

Solution: 1

Because cyclicity of 3 is 4, divide the index, i.e., 54^{55} by 4 and check remainder

$54^{55} = 54 \times 54 \times 54^{53} = (2 \times 27)(2 \times 27) \times 54^{53}$

$= 4 \times (27 \times 27 \times 54^{53}) = 4N$ type

Hence, Rem $\left[\frac{54^{55}}{4}\right]$ = Rem $\left[\frac{4N}{4}\right] = 0$

Unit digit of $53^{54^{55}} = 3^4 = 1$

Example 6:

If the unit digit of 48^a is 2, then which of the following can be the value of a?

(A) 20 (B) 21

(C) 22 (D) 23

Solution: (D)

Cyclicity of 8 is 4

Unit digit of 8^1 or $8^{4k+1} = 8$

Unit digit of 8^2 or $8^{4k+2} = 4$

Unit digit of 8^3 or $8^{4k+3} = 2$

Unit digit of 8^4 or $8^{4k} = 6$

a should leave remainder as 3, when divided by 4, i.e., it should be of $4K + 3$ type.

$20 = 4 \times 5 + 0$, $21 = 4 \times 5 + 1$, $22 = 4 \times 5 + 2$, and $23 = 4 \times 5 + 3$

As, only $23 = 4K + 3$ type, option (D) is correct answer.

Example 7:

Find the unit digit of $(5!)^{5!} + (4!)^{4!} + (3!)^{3!} + (2!)^{2!} + (1!)^{1!}$.

Solution: 7

It can be noticed that any factorial greater than 4 will have at least one zero at the end.

$(5!)^{5!} = (120)^{120}$ = Unit digit = 0

$(4!)^{4!} = (24)^{24} = 4^{24} = 4^{\text{Even}}$ $\Rightarrow$ Unit digit = 6

$(3!)^{3!} = 6^6$ $\Rightarrow$ Unit digit = 6

$(2!)^{2!} = 2^2$ $\Rightarrow$ Unit digit = 4

$(1!)^{1!} = 1^1$ $\Rightarrow$ Unit digit = 1

Unit digit of $(5!)^{5!} + (4!)^{4!} + (3!)^{3!} + (2!)^{2!} + (1!)^{1!}$ $= 0 + 6 + 6 + 4 + 1 = 7$

Example 8:

If n is a natural number, how many distinct values of unit digit of $(2^n + 4^n + 6^n + 8^n)$ are possible?

Solution: 2

The cyclicity of all the digits is either 1 or 2, or 4. But, for simplicity, one can also take the cyclicity of all the digits as 4. It would not affect the unit digit's results.

So, here n can be of four types:

$4k$ type $\rightarrow$ {4, 8, 12, 16,............}

$(4k + 1)$ type $\rightarrow$ {1, 5, 9, 13,............}

$(4k + 2)$ type $\rightarrow$ {2, 6, 10, 14,............}

$(4k + 3)$ type $\rightarrow$ {3, 7, 11, 15,............}

when, $n = 4k$, put $n = 4$

Unit digit of $2^4 + 4^4 + 6^4 + 8^4 = 6 + 6 + 6 + 6 = 4$

when, $n = 4k + 1$, put $n = 5$

Unit digit of $2^5 + 4^5 + 6^5 + 8^5 = 2 + 4 + 6 + 8 = 0$

when, $n = 4k + 2$, put $n = 6$

Unit digit of $= 2^6 + 4^6 + 6^6 + 8^6$
$= 4 + 6 + 6 + 4 = 0$

when, $n = 4k + 3$, put $n = 7$

Unit digit of $= 2^7 + 4^7 + 6^7 + 8^7$
$= 8 + 4 + 6 + 2 = 0$

So, it can be witnessed that for any value of n, the unit digit of $2^n + 4^n + 6^n + 8^n$ has only two distinct values, i.e., 0 and 4.

Hence, answer = 2.

Last Two Digits

This section generally asks for the last two digits of a number to be raised to some power. Only the last two digits of the number matter in the last two-digit questions.

Last two digits of $(123,456)^{487}$ = Last two digits of $(56)^{487}$

Last two digits of $(56,789)^{369}$ = Last two digits of $(89)^{369}$

When the unit digit of base is 1

$$(61)^{43} = (1 + 60)^{43}$$

$$(x + a)^n = {}^nC_0 x^n a^0 + {}^nc_1 x^{n-1} a^1 + \cdots + {}^nC_n x^0 a^n$$

$$(1 + 60)^{43} = {}^{43}C_0 \times 1^{43} \times 60^0 + {}^{43}C_1 \times 1^{42} \times 60^1 + {}^{43}C_2 \times 1^{41} \times 60^2 + \cdots$$

In the expansion after second term, all the terms will have at least two zeros at the end. So, it will not affect the last two-digit calculation, and these terms can be neglected.

$$(1 + 60)^{43} = 1 \times 1 \times 1 + 43 \times 1 \times 60$$
$$= 1 + 2{,}580 = 2{,}581$$

Last two digit is 81.

With proper observation, it can be concluded

$(- - - b\,1)^{XYZ} \Rightarrow$ unit digit = 1, tens digit = unit digit of $b \times z$

Example 9:

Find the last two digits of $(531)^{437}$

Solution: 11

$(531)^{437} = (31)^{437}$

Unit digit = 1 and tens digit = unit digit of $3 \times 7 = 1$

Last 2 digits of $= (31)^{437} = 11$

Important observation

The last two digits of the square of any number of $(50N \pm K)$ type are the same as the last two digits of the square of K.

Last two digits of $(50N \pm K)^2$
= Last two digits of K^2

For example:

$78^2 = (50 \times 2 - 22)^2 = 6{,}084$

$22^2 = 484$

So, the last two digits of $78^2 = (50 \times 2 - 22)^2$ = last two digits of 22^2

Similarly, last two digits of $169^2 = (50 \times 3 + 19)^2$ = last two digits of $19^2 = 61$

When unit digit of bases 3, 7, or 9

$3^4 = 81, \quad 7^4 = 2{,}401, \quad 9^2 = 81$

To find out the last two digits of numbers ending with 3, 7, or 9, convert the base to get 1 at the unit place and then apply the method of finding the last two digits of the base ending with 1.

Example 10:

Find the last two digits of the following:

a) 3^{275} **b)** 217^{438} **c)** 59^{54}

Solution: a) 07; b) 09; c) 61

a) Last two digits of 3^{275}
= Last two digits of $(3^4)^{68} \times 3^3$
= Last two digits of $(81)^{68} \times 27 = 41 \times 27$
= 07

b) $217^{438} = 17^{438} = (17^4)^{109} \times 17^2$
$= (17^2 \times 17^2)^{109} \times 89$
$= (89 \times 89)^{109} \times 89$
$= (89^2)^{109} \times 89$
$= (21)^{109} \times 89$ {∴ Last two digits of $89^2 = 11^2 = 21$}

$= 81 \times 89 = 09$

c) $59^{54} = (59^2)^{27}$ $\{\therefore$ Last two digits of $59^2 = 9^2 = 81\}$

$= (9^2)^{27} = (81)^{27} = 61$

When the unit digit of the base is 5

Let's assume the base ending in 5 has its tenth digit as *b*, and the exponent is *N*.

$(\_\_ \; \_\_ \; \underline{b\ 5})^N$

Only when both *b* and *N* are odd the last two digits would be 75; otherwise, it would be 25.

Only when both the tens digit of the number and the unit digit of power are odd the last two digits will be 75; otherwise, it would be 25.

Example 11:

Find last two digits of:

a) 435^{289}

b) $5{,}835^{242} + 395^{871}$

Solution: a) 75; b) 00

a) In 435^{289} both tens digit of the number as well as unit digit of power are odd; the last two digits of $435^{289} = 75$.

b) In $5{,}835^{242}$ tens digit of the number is odd, but the unit digit of power is even; the last two digits of $5{,}535^{242} = 25$.

In 395^{871} both tens digit of the number as well as unit digit of power are odd; the last two digits of $395^{871} = 75$.

So, last 2 digits of $5{,}835^{242} + 395^{871} = 25 + 75$
$= 00$

When the unit digit of the base is even

We know that $2^{10} = 1{,}024$

Last two digits of $(1{,}024)^N$ = the last two digits of $(24)^N$

It is the speciality of the number 24 that

$(24)^{Odd} \Rightarrow$ last two digits are 24

$(24)^{Even} \Rightarrow$ the last two digits are 76

The above results can be re-written as

$(2^{10})^{Odd} \Rightarrow$ last two digits are 24

$(2^{10})^{Even} \Rightarrow$ the last two digits are 76

Example 12:

Find last two digits of:

a) 8^{34}

b) 62^{43}

c) $5{,}736^{218}$

Solution: a) 04; b) 28; c) 56

a) $8^{34} = (2^3)^{34} = 2^{102} = (2^{10})^{10} \times 2^2 = (2^{10})^{Even} \times 2^2$

$= 76 \times 04 = 04$

b) $62^{43} = (2 \times 31)^{43} = 2^{43} \times 31^{43} = (2^{10})^4 \times 2^3 \times 31^{43}$

$= 76 \times 08 \times 91 = 28$

c) $5{,}736^{218} = 36^{218} = (2^2 \times 3^2)^{218}$

$= 2^{436} \times 3^{436}$

$= (2^{10})^{43} \times 2^6 \times (3^4)^{109}$

$= 24 \times 64 \times (81)^{109}$

$= 24 \times 64 \times 21 = 56$

Shortcut to Calculate the Last Two Digits

1. Last two digits of $N^{20K} = 01$ (when *N* is a number ending in 1, 3, 7, or 9 and *K* is a natural number).
2. Last two digits of $N^{20K} = 76$ (when *N* is a number ending in 2, 4, 6, or 8 and *K* is a natural number).

   For example, last 2 digits of $789^{240} = 789^{20 \times 12} = 01$

   For example, last 2 digits of $584^{840} = 584^{20 \times 42} = 76$

Remainders

To find the remainders, replace numbers with remainders till you get the answer, keeping +, –, ×, and power as it is.

For example, when 100 is divided by 7, the remainder = 2.

$$\left(\frac{100}{7}\right)_r = 2$$

Now, we will represent 100 in different ways and divide it by 7, but still, the remainder will be the same.

As $\left(\frac{100}{7}\right)_r = \left(\frac{50+30+20}{7}\right)_r = \left(\frac{1+2+6}{7}\right)_r = \left(\frac{9}{7}\right)_r = 2$

When 50, 30, and 20 are divided by 7, the remainders are 1, 2, and 6, respectively. So, we have replaced numbers 50, 30, and 20 with remainders 1, 2, and 6, respectively, keeping + sign as it is.

Also, $\left(\frac{100}{7}\right)_r = \left(\frac{125-25}{7}\right)_r = \left(\frac{6-4}{7}\right)_r = \left(\frac{2}{7}\right)_r = 2$

When 125 and 25 are divided by 7, the remainders are 6 and 4, respectively. So, we have replaced numbers 125 and 25 with remainders 6 and 4, respectively, keeping – sign as it is.

Similarly, $\left(\frac{100}{7}\right)_r = \left(\frac{10\times 10}{7}\right)_r = \left(\frac{3\times 3}{7}\right)_r = \left(\frac{9}{7}\right)_r = 2$

Example 13:

Find the remainder when (239 × 457 – 218) is divided by 5.

Solution: 0

$$\left(\frac{239\times 457-218}{5}\right)_r$$

$$\left(\frac{239}{5}\right)_r = 4,\ \left(\frac{457}{5}\right)_r = 2,\ \left(\frac{218}{5}\right)_r = 3$$

Now, replace numbers with remainders keeping signs of × and – as it is to get the final answer.

$$\left(\frac{239\times 457-218}{5}\right)_r = \left(\frac{4\times 2-3}{5}\right)_r = \left(\frac{5}{5}\right)_r = 0$$

Hence, the final answer is 0.

Negative Remainder

The concept of the negative remainder is only introduced to reduce the complexity of finding remainders. But the negative remainder is never accepted as the final answer. So, if in any problem, one gets the negative remainder at the end of the solution; add it to the divisor, and it will give a positive remainder, which would be the final acceptable answer.

Negative remainder + divisor = Positive remainder

$$\text{Rem}\left[\frac{32}{7}\right] \begin{matrix} \to +4\ \text{(Positive remainder)} \\ \to -3\ \text{(Negative remainder)} \end{matrix}$$

32 = (28 + 4). 28/7 remainder is 0 and 4/7 remainder is 4. Now replace numbers with remainders. So, final remainder is 0 + 4 = 4.

Also, 32 = (35 – 3). 35/7 remainder is 0, and 3/7 remainder is 3. Now replace numbers with remainders. So, final remainder is 0 – 3 = –3.

Now negative remainder = –3.

So, positive remainder = Negative remainder + divisor = – 3 + 7 = 4.

Similarly, $\text{Rem}\left[\frac{60}{9}\right] \begin{matrix} \to +6 \\ \to -3 \end{matrix}$

Note: While solving the remainder problem, one can select any positive or negative remainder, but it is always advised to select the smaller out of positive and negative remainder to simplify the calculation.

Example 14:

Find remainder when (252 × 398 × 496) is divided by 25.

Solution: 16

$$\frac{\overset{+2}{252}\times \overset{-2}{398}\times \overset{-4}{496}}{25} = (2)\times(-2)\times(-4) = 16$$

Common Factor

Sometimes, it is easy to simplify the given question by cancelling the common factors from the numerator and denominator and then performing the division. But the remainder obtained must be multiplied by the common factor to get the remainder of the original question.

Example 15:

Find the remainder when 308 is divided by 12.

Solution: 8

308 = 4 × 77

12 = 4 × 3

$$\frac{308}{12} = \frac{4 \times 77}{4 \times 3} = \frac{77}{3}$$

$$\text{Rem}\left[\frac{77}{3}\right] = 2$$

$$\text{Hence, Rem}\left[\frac{308}{12}\right] = 4 \times 2 = 8$$

Simple Remainder Problems

$$\left[\frac{(a \pm 1)^m}{a}\right]_r = (\pm 1)^m$$

$$\left[\frac{(na \pm 1)^m}{a}\right]_r = (\pm 1)^m$$

$$\left[\frac{(na \pm K)^m}{a}\right]_r = (\pm K)^m$$

In this type of question, one's motive should be to make a dividend in such form that it leaves the remainder as +1 or –1. Sometimes it would not be possible to get such a form of a dividend. In that case, proceed with $+K$ or $-K$ as the remainder.

Example 16:

Find the remainder of the following:

a) $\frac{126^{37}}{5}$

b) $\frac{3^{30}}{16}$

c) $\frac{32^{19}}{24}$

Solution: a) 1; b) 9; c) 8

a) $\frac{126^{37}}{5}$

$\text{Rem}\left[\frac{126}{5}\right] = +1$. Now replace the number with the remainder, keeping power as it is.

$$\text{Rem}\left[\frac{126^{37}}{5}\right] = \text{Rem}\left[\frac{(+1)^{37}}{5}\right] = 1$$

b) $\frac{3^{30}}{16} = \frac{(3^4)^7 \times 3^2}{16} = \frac{81^7 \times 9}{16}$

$$\text{Rem}\left[\frac{81^7}{16}\right] = 1^7 = 1$$

$$\text{Rem}\left[\frac{9}{16}\right] = 9$$

Now, replace numbers with remainders keeping the sign of × as it is.

Hence, final remainder = 1 × 9 = 9

c) $\frac{32^{19}}{24} = \frac{(2^5)^{19}}{2^3 \times 3} = \frac{2^{95}}{2^3 \times 3}$

Cancelling out the common factor of 2^3, we are left with $\frac{2^{92}}{3}$

$$\text{Rem}\left[\frac{2^{92}}{3}\right] = (-1)^{92} = 1$$

But final remainder = $\text{Rem}\left[\frac{2^{92}}{3}\right]$ × common factor = 1 × 8 = 8

Euler Theorem

If a and b are the co-prime numbers and $\phi(b)$ is the Euler totient function of b then $\text{Rem}\left[\frac{a^{\phi(b)}}{b}\right] = 1$

To solve problems, the Euler theorem can also be generalised as $\text{Rem}\left[\frac{a^m}{b}\right] = \left(\frac{a^{\left(\frac{m}{\phi(b)}\right)_{\text{Rem}}}}{b}\right)_{\text{Rem}}$

Euler Totient Function

Euler totient function of any number 'N' represents the number of numbers less than N that are coprime to N.

For example:

Let's calculate $\phi(12)$.

Write all the numbers less than 12 that are coprime to it = 1, 5, 7, 11.

So, there are total four numbers.

$\phi(12) = 4$

Method to calculate Euler totient function

Prime factorise the number N

$N = p^a \times q^b \times r^c \times$ ....

$$\phi(N) = N \times \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)................$$

Example 17:

$\phi(40)$

Solution: 16

$40 = 8 \times 5 = 2^3 \times 5$

$$\phi(40) = 40\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 40 \times \frac{1}{2} \times \frac{4}{5} = 16$$

Keynote

If P is prime number, $\phi(P) = P - 1$

Example 18:

Find the remainder when 28^{109} is divided by 37.

Solution: 28

$\phi(37) = 37 - 1 = 36$

Applying Euler theorem,

$$\text{Rem}\left[\frac{28^{109}}{37}\right] = \left(\frac{28^{\left(\frac{109}{36}\right)_{\text{Rem}}}}{37}\right)_{\text{Rem}}$$

$$= \left(\frac{28}{37}\right)^1_{\text{Rem}} = 28^1 = 28$$

Example 19:

Find the remainder when 13^{194} is divided by 48.

Solution: 25

$48 = 2^4 \times 3$

Apply Euler theorem,

$$\phi(48) = 48 \times \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 48 \times \frac{1}{2} \times \frac{2}{3} = 16$$

$$\text{Rem}\left[\frac{13^{194}}{48}\right] = \left(\frac{13^{\left(\frac{194}{16}\right)_{\text{Rem}}}}{48}\right)_{\text{Rem}} = 13^2 = 169$$

Further divide 169 by 48 = $\left(\frac{169}{48}\right)_{\text{Rem}} = 25$

Fermat's Theorem

If P is a prime number and a, P are coprime to each other.

$$\text{Rem}\left[\frac{a^P}{P}\right] = a \quad \text{or} \quad \text{Rem}\left[\frac{a^{P-1}}{P}\right] = 1$$

It can be noticed that Fermat's theorem is a special case of the Euler theorem only.

Example 20:

Find the remainder when 6^{25} is divided by 13.

Solution: 6

13 is a prime number, and 6 and 13 are co-prime to each other. Hence, apply Fermat theorem.

$$\text{Rem}\left[\frac{6^{12}}{13}\right] = 1$$

$$\text{Rem}\left[\frac{6^{25}}{13}\right] = \text{Rem}\left[\frac{6^{12} \times 6^{12} \times 6}{13}\right] = 1 \times 1 \times 6 = 6$$

Wilson Theorem

If P is a prime number.

$$\text{Rem}\left[\frac{(P-1)!}{P}\right] = (-1) \quad \text{or} \quad (P - 1) \text{ and}$$

$$\text{Rem}\left[\frac{(P-2)!}{P}\right] = 1$$

Example 21:

Find the remainder when 27! is divided by 29.

Solution: 1

$$\text{Rem}\left[\frac{27!}{29}\right] = 1$$

Example 22:

Find the remainder when 25! is divided by 23^2.

Solution: 483

$$\text{Rem}\left[\frac{25!}{23^2}\right] = \frac{25 \times 24 \times 23 \times 22!}{23^2}$$

Cancelling out the common factor of 23

$$\text{Rem}\left[\frac{25\times24\times22!}{23}\right] = 2\times1\times(-1) = -2 \text{ or } = -2 + 23 = 21$$

Final remainder = Remainder × common factor = 21 × 23 = 483.

Chinese Remainder Theorem (CRT)

CRT is applied in problems where divisor is a composite number that can be split into two co-prime numbers.

Let's understand the application of CRT by an example.

Example 23:

Find the remainder when 7^{24} is divided by 88.

Solution: 25

$$\text{Rem}\left[\frac{7^{24}}{88}\right]$$

$\Rightarrow$ 88 = 8 × 11 (split into two coprime numbers)

$$\Rightarrow \text{Rem}\left[\frac{7^{24}}{8}\right] = (-1)^{24} = 1$$

$$\Rightarrow \text{Rem}\left[\frac{7^{24}}{11}\right] = \left(\frac{7^{\left(\frac{24}{10}\right)_R}}{11}\right)_R \text{ (using Euler theorem)}$$

$$= \frac{7^4}{11} = \frac{7^2\times7^2}{11} = \frac{49\times49}{11} = \frac{5\times5}{11} = \frac{25}{11} = 3$$

It can be observed that 7^{24} is either of the forms of $8x + 1$ or $11y + 3$.

According to CRT, the final remainder would be the smallest number satisfying.

$8x + 1 = 11y + 3$ ...(i)

$8x = 11y + 2$ (x and y are non-negative integers)

At $y = 0$, $x \to$ Not integer

At $y = 1$, $x \to$ Not integer

At $y = 2$, $x = 3$

Smallest solution of equation (i) is obtained at $x = 3$ and $y = 2$.

Hence, the remainder = 8 × 3 + 1 = 25 or 11 × 2 + 3 = 25.

Example 24:

Find remainder when 21^{40} is divided by 95.

Solution: 16

95 → 5 × 19 (Co-prime pairs)

$$\text{Rem}\left[\frac{21^{40}}{95}\right] = ?$$

$$\Rightarrow \text{Rem}\left[\frac{21^{40}}{5}\right] = 1^{40} = 1$$

$$\Rightarrow \text{Rem}\left[\frac{21^{40}}{19}\right] = \text{Rem}\left[\frac{2^{40}}{19}\right]$$

$$= \frac{2^{36}\times2^4}{19} = \frac{1\times16}{19} = 16$$

Applying CRT, $5x + 1 = 19y + 16$

$5x = 19y + 15$

At, $y = 0$, $x = 3$

Hence, remainder is 5 × 3 + 1 = 16 or 19 × 0 + 16 = 16.

Base System

In mathematics, there are different ways of representing the number depending upon the number of digits used. And the number of digits used denotes the base of the system. Like in the decimal system (which is very common), digits from 0 to 9, i.e., total 10 digits, are used.

Base can take any value from 2 onwards but in general practice it is used up to base 16, i.e., hexadecimal system.

Number of digits used in any base *m* range from 0 to *m* – 1

For example:

Base		**Digits used**
Base 10 (Decimal system)	→	0 to 9
Base 8 (Octal system)	→	0 to 7

Base 7 $\rightarrow$ 0 to 6

(Septenary system)

Base 2 $\rightarrow$ 0 and 1

(Binary system)

Base 16 $\rightarrow$ 0 to 9, A, B, C, D, E, F

(Hexadecimal system)

To avoid confusion, numbers from 10 to 15 are represented by alphabets as $10 \rightarrow A$, $11 \rightarrow B$, $12 \rightarrow C$, $13 \rightarrow D$, $14 \rightarrow E$, $15 \rightarrow F$.

Hence, digits used in base 16 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.

- One should be familiar with counting numbers used in a particular system. For example, counting numbers used in base 6 would be
  1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21,...
  Because in base, six digits that can be used are 0, 1, 2, 3, 4, and 5.

Conversion from Decimal System (Base 10 to Base *m*)

To convert any number from base 10 to base m, successively divide the number by m and note down the remainders.

Let's understand by example:

Convert $(358)_{10}$ to base 6

$(358)_{10} \rightarrow (\quad)_6$

$$\begin{array}{c|l} 6 & 358 \\ \hline 6 & 59 - 4 \\ \hline 6 & 9 - 5 \\ \hline & 1 - 3 \end{array}$$

$(358)_{10} \leftrightarrow (1{,}354)_6$

Example 25:

Convert $(34{,}572)_{10}$ to base 16.

Solution: 870 C_{16}

$$\begin{array}{c|l} 16 & 34572 \\ \hline 16 & 2160 \quad - 12 \\ \hline 16 & 135 - 0 \\ \hline & 8 - 7 \end{array}$$

$(34{,}572)$ to $\leftrightarrow$ $(870C)_{16}$

Conversion from base m to decimal system (base 10)

$(abcd)_m \rightarrow (\quad)_{10}$

$= a \times m^3 + b \times m^2 + C \times m^1 + d \times m^0$

$(abcd)_m \rightarrow (am^3 + bm^2 + cm + d)_{10}$

Example 26:

Convert $(2{,}135)_7$ to the decimal system.

Solution: 761_{10}

$(2{,}135)_7 = 2 \times 7^3 + 1 \times 7^2 + 3 \times 7 + 5 \times 7^o = 686 + 49 + 21 + 5 = 761$

$(2{,}135)_7 \leftrightarrow (761)_{10}$

Example 27:

Find the value of base m in $(426)_8 = (338)_m$

Solution: 9

Convert both the bases to decimal system and equate

$\Rightarrow$ $4 \times 8^2 + 2 \times 8^1 + 6 \times 8^o = 3 \times m^2 + 3 \times m^1 + 8 \times m^o$

$\Rightarrow$ $256 + 16 + 6 = 3m^2 + 3m + 8$

$\Rightarrow$ $3m^2 + 3m + 8 = 278$

$\Rightarrow$ $m^2 + m - 90 = 0$

$\Rightarrow$ $(m - 9)(m + 10) = 0$

$m = 9, -10$

Hence, $m = 9$

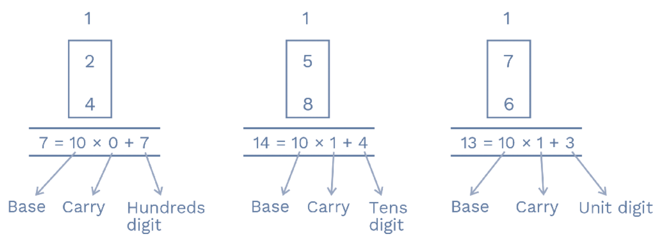
Addition, Subtraction, and Multiplication in Base System

The logic of addition, subtraction, and multiplication in any base is same as of decimal system.

Addition in base 10

Let us understand the logic of addition.

$$\begin{array}{cccc} & 2 & 5 & 7 \\ + & 4 & 8 & 6 \\ \hline & 7 & 4 & 3 \end{array}$$



Logic

Add the digits. If it is greater than the highest digit used in that base, divide it by base, keep the remainder, and send the quotient as carrying. Continue the same process till the end.

Example 28:

$(354)_6 + (432)_6$

Solution: $1{,}230_6$

$$\begin{array}{cccc} & 1 & 1 & \\ & 3 & 5 & 4 \\ + & 4 & 3 & 2 \\ \hline & 8 & 9 & 6 \\ & 6\times1+2 & 6\times1+3 & 6\times1+0 \\ \hline & 12 & 3 & 0 \end{array}$$

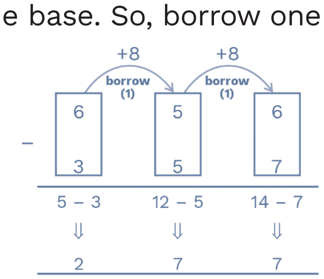
The logic used in addition can be extended for subtraction and multiplication.

Example 29:

Calculate $(656)_8 - (357)_8$

Solution: 277_8

Whenever we borrow one, we borrow a value equal to the base. So, borrow one means + 8.



Answer = 277_8

Example 30:

$(46)_7 \times (54)_7$

Solution: $3{,}603_7$

$$\begin{array}{cccc} & 46 \times 54 & & \\ \hline & 2 & 5 & 3 \\ 3 & 3 & 2 & \times \\ \hline 3 & 6 & 0 & 3 \end{array} \quad \begin{array}{l} \text{Step 1}: 6 \times 4 = 24 = 7 \times 3 + 3 \\ \text{Step 2}: 4 \times 4 + 3 = 19 = 7 \times 2 + 5 \\ \text{Step 3}: 6 \times 5 = 30 = 7 \times 4 + 2 \\ \text{Step 4}: 5 \times 4 + 4 = 24 = 7 \times 3 + 3 \end{array}$$

Answer = $(3{,}603)_7$

Divisibility in the base system

- A number in base m is divisible by $m - 1$, if the sum of digits of the numbers is divisible by $m - 1$.
- A number in base m is divisible by $m + 1$ if the difference between the sum of the digits at odd places and the sum of the digits at even places is either 0 or divisible by $m + 1$.

Example 31:

Find the remainder when $(37{,}542)_8$ is divided by 7.

Solution: 0

$$\frac{(37542)_8}{7} = \left(\frac{3+7+5+4+2}{7}\right)_{Rem}$$

$$\left(\frac{21}{7}\right)_{Rem} = 0$$

Keynote

- Maximum n digit number in base m = $(m^n - 1)$ in base 10.
- Minimum n digit number in base m = (m^{n-1}) in base 10.

Example 32:

Find the remainder, when $(23{,}795)_{12}$ divided by 13.

Solution: 2

$$\frac{(23,795)_{12}}{13} = \left[\frac{(2+7+5)-(3+9)}{13}\right]_{Rem} = \left(\frac{2}{13}\right)_{Rem} = 2$$

Trailing Zeros in the Base System

If a number in base m has K trailing zeros, the number will be multiple m^K.

Example 33:

How many trailing zeroes can be obtained when 60! in base 10 is converted to base 9?

Solution: 14

The highest power of 9 available in 60! will give the number of trailing zeroes when 60! is converted to base 9.

To find the highest power of 9 in 60! Find the highest power of 3 in it.

$$\frac{60}{3} = 20, \ \frac{20}{3} = 6, \ \frac{6}{3} = 2 = 20 + 6 + 2 = 28$$

$$60! \rightarrow 3^{28} = 9^{14}$$

There would be 14 trailing zeroes.

Binomial Theorem

The traditional form of binomial expression is $(x + a)^n$, and the expansion of $(x + a)^n$, $n \in N$ is called a binomial theorem.

Important Concepts in Binomial Theorem

If a and b are real numbers and $n \in N$, the binomial expansion of $(a + b)^n$ is given by:

$(a + b)^n = {}^nC_0 \times a^n \times b^0 + {}^nC_1 \times a^{n-1} \times b^1 + {}^nC_2 \times a^{n-2} \times b^2 + \cdots + {}^nC_r \times a^{n-r} \times b^r + \cdots + {}^nC_{n-1} \times a^1 \times b^{n-1} + {}^nC_n \times a^0 \times b^n$

i.e., $(a + b)^n = \sum_{r=0}^{n} {}^nC_r \, (a)^{n-r} \times (b)^r$

If we replace 'b' by '$-b$', we get

$$(a - b)^n = \sum_{r=0}^{n} {}^nC_r \times (a)^{n-r} \times (-b)^r$$

$$(a - b)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \times a^{n-r} \times b^r$$

Binomial Theorem Properties

1. The terms in the expansion of $(a - b)^n$ are alternatively positive and negative.

2. The last term is positive or negative depending upon the value of n, (even or odd).

3. In the expansion $(a + b)^n = \sum_{r=0}^{n} {}^nC_r \times (a)^{n-r} \times (b)^r$

r can take any value from '0' to 'n', so the total number of terms in the expansion is $(n + 1)$.

4. Since ${}^nC_r = {}^nC_{n-r}$ for $r = 0, 1, 2, 3, \ldots n$ so, the coefficient of terms equidistant from the beginning and end are equal.

5. Putting $a = 1$ and $b = x$ in the expansion of $(a + b)^n$, we get

$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r \times x^r$ (This is the expansion of ascending power of x)

$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r \times x^{n-r}$ (This is the expansion of descending power of x).

- The coefficient of $(r + 1)$th term in the expansion of $(1 + x)^n$ is ${}^nC_r$.
- The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

6. Putting $a = 1$ and $b = -x$ is the expansion of $(a + b)^n$, we get

$$(1 - x)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \times x^r$$

7. If n is odd, $[(x + a)^n + (x - a)^n]$ and $[(x + a)^n - (x - a)^n]$ both contains same number of terms which is equal to $\left(\frac{n+1}{2}\right)$ terms.

8. If n is even, $[(x + a)^n + (x - a)^n]$ contains $\left(\frac{n}{2} + 1\right)$ terms and$[(x + a)^n - (x - a)^n]$ contains $\left(\frac{n}{2}\right)$ terms.

Example 34:

Find the expansion of $(a + b)^5$.

Solution:

Here, $(a+b)^5 = {}^5C_0 \times (a)^5 \times (b)^0 + {}^5C_1 \times (a)^4 \times (b)^1 + {}^5C_2 \times (a)^3 \times (b)^2 + {}^5C_3 \times (a)^2 \times (b)^3 + {}^5C_4 \times (a)^1 \times (b)^4 + {}^5C_5 \times (a)^0 \times (b)^5$

$= 1 \times a^5 \times 1 + 5 \times a^4 \times b + 10 \times a^3 \times b^2 + 10 \times a^2 \times b^3 \times 5 + a \times b^4 + 1 \times 1 \times b^5$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Example 35:

Find the expansion of $(a-b)^5$.

Solution:

Here, $(a-b)^5 = {}^5C_0 \times (a)^5 \times (-b)^0 + {}^5C_1 \times (a)^4 \times (-b)^1 + {}^5C_2 \times (a)^3 \times (-b)^2 + {}^5C_3 \times (a)^2 \times (-b)^3 + {}^5C_4 \times (a)^1 \times (-b)^4 + {}^5C_5 \times (a)^0 \times (-b)^5$

$\Rightarrow 1 \times a^5 \times 1 + 5 \times a^4 \times (-b) + 10 \times a^3 \times b^2 + 10 \times a^2 \times (-b)^3 \times 5 + a \times b^4 + 1 \times 1 \times (-b)^5$

$\Rightarrow a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

Example 36:

Find the number of terms in the expansion $(a-b)^{99}$.

Solution:

As we know that the expansion $(a \pm b)^n$ contains $(n+1)$ terms.

Total number of terms in the expansion of $(a-b)^{99} = 99 + 1 = 100$

Hence, $(a-b)^{99}$ contains 100 terms.

Example 37:

Find the number of terms in the expansion of $[(a+b)^{55} + (a-b)^{55}]$.

Solution:

As we know that, if n is odd, the expansion of $[(a+b)^n + (a-b)^n]$ contains $\left(\frac{n+1}{2}\right)$ terms.

So, the expansion of $[a+b)^{55} + (a-b)^{55}]$ contains $\left(\frac{55+1}{2}\right) = \frac{56}{2} = 28$ terms.

Example 38:

Find the number of terms in the expansion of $[(a+b)^{54} + (a-b)^{54}]$.

Solution:

As we know that, if n is even, the expansion of $[(a+b)^n + (a-b)^n]$ contains $\left(\frac{n}{2}+1\right)$ terms.

So, the expansion of $[(a+b)^{54} + (a-b)^{54}]$ contains $\left(\frac{54}{2}+1\right) = 28$ terms.

Example 39:

Find the number of terms in the expansion of $[(a+b)^{54} - (a-b)^{54}]$.

Solution:

As we know that if n is even, the expansion of $[(a+b)^n - (a-b)^n]$ contains $\left(\frac{n}{2}\right)$ terms.

So, the expansion of $[(a+b)^{54} - (a-b)^{54}]$ contains $\left(\frac{54}{2}\right) = 27$ terms.

General Term or Indicated Term or Middle Term in a Binomial Expansion

If a and b are real numbers and all $x \in N$, the binomial expansion of $(a+b)^n$ is given by:

$(a+b)^n = {}^nC_0 \times a^n \times b^0 + {}^nC_1 \times a^{n-1} \times b^1 + {}^nC_2 \times a^{n-2} \times b^2 + \cdots + {}^nC_r \times a^{n-r} \times b^r + \ldots\ldots\ldots + {}^nC_{n-1} \times a^1 \times b^{n-1} + {}^nC_n \times a^0 \times b^n$.

From the above expansion we find that:

First term $= {}^nC_0 \times a^n \times b^0$

Second term $= {}^nC_1 \times a^{n-1} \times b^1$

Third term $= {}^nC_2 \times a^{n-2} \times b^2$ and so on.

Here we observe a pattern that the suffix of c in any term is one less than the term number. The power of b is the same as the suffix of c and the power of a is n minus the suffix of c.

Now, from the above observation, we can find the $(r+1)$th term.

$T_{(r+1)} = {}^nC_r \times a^{n-r} \times b^r$

This is known as the general term. For different values of r, one can find all the terms which exist in the expansion.

- In the binomial expansion of $(a + b)^n$, the rth term from the end is $(n + 1) - r + 1 = (n - r + 2)$th term from the beginning.
- In the binomial expansion of $(a + b)^n$, the rth term from the beginning is $(n + 1) - r + 1 = (n - r + 2)$th term from the end.
- If n is even, $\left(\frac{n}{2}+1\right)$th term is the middle term.
- If n is odd, $\left(\frac{n+1}{2}\right)$th term and $\left(\frac{n+3}{2}\right)$th term are the two middle terms.

Example 40:

Find the general term in the expansion of $(a + b)^9$.

Solution:

General term is given by

$$T_{(r+1)} = {}^nC_r \times (a)^{n-r} \times (b)^r$$
$$T_{(r+1)} = {}^9C_r \times (a)^{9-r} \times (b)^r$$

Example 41:

Find the 19th term in the expansion of $\left(\frac{x^2}{2} - 4x\right)^{21}$.

Solution:

General term is given by $T_{(r+1)} = {}^nC_r \times a^{n-r} \times b^r$

$$T_{19} = T_{(18+1)}$$

$$T_{(18+1)} = {}^{21}C_{18} \times \left(\frac{x^2}{2}\right)^{21-18} \times (-4x)^{18}$$

$$= 1,330 \times \left(\frac{x^2}{2}\right)^3 \times (-1)^{18} \times (4x)^{18}$$

$$= 1,330 \times \frac{x^6}{8} \times (4x)^{18}$$

Example 42:

Find the fifth term from the end in the expansion of $\left(x - \frac{1}{x}\right)^8$.

Solution:

Fifth term from the end = (8 − 5 + 2) = fifth term from the beginning.

$$\therefore\ T_5 = T_{(4+1)} = {}^8C_4 \times (x)^{8-4} \times \left(-\frac{1}{x}\right)^4$$

$$= 70 \times x^4 \times \frac{1}{x^4} = 70$$

Hence, the fifth term from the end of the expansion is 70.

Example 43:

Find the middle term in the expansion of $(a^2 + b)^8$.

Solution:

Here, $n = 8$, which is an even number.

So, $\left(\frac{8}{2}+1\right)^{th}$ term, i.e., fifth term, is the middle term.

$$T_5 = T_{(4+1)} = {}^8C_4 \times (a^2)^{8-4} \times (b)^4$$
$$= 70 \times a^8 \times b^4 = 70a^8b^4$$

Hence, the middle term is $70a^8b^4$.

Finding the Term Independent of Variable

Here, finding the term which is independent of the variable means that the term whose power of the variable is '0'.

Let's understand this with an example.

Example 44:

Find the term independent of x in the expansion of $\left[3x^2 + \frac{2}{x^2}\right]^{10}$.

Solution:

Let $(r + 1)$th term be independent of x in the given expression.

$$T_{(r+1)} = {}^{10}C_r\ (3x^2)^{10-r} \times \left(\frac{2}{x^2}\right)^r$$

$= {}^{10}C_r \times (3)^{10-r} \times (x)^{20-2r} \times (2)^r \times (x)^{-2r}$

$= {}^{10}C_r \times (3)^{10-r} \times (x)^{20-4r} \times (2)^r$

This term is independent of x, if $20 - 4r = 0$

$\therefore\ r = 5$

$(5 + 1)$th term = sixth term is independent of x.

$T_6 = {}^{10}C_5 \times (3)^{10-5} \times (2)^5 = 252 \times (3)^5 \times (2)^5$

$= 256 \times (6)^5$

$T_C = 19,90,656$

Finding the Coefficient for a Given Power of the Variable

For example:

3a means three times a, and a is a variable and 3 is the coefficient of a.

The coefficient of an algebraic expression can be positive or negative,

Let's understand this with an example.

Example 45:

Find the coefficient of x^{57} in the expansion of $(x^4 + x^3)^{15}$.

Solution:

Let $(r + 1)$th contain x^{57} in the expansion $(x^4 + x^3)^{15}$

$T_{(r+1)} = {}^{15}C_r \times (x^4)^{15-r} \times (x^3)^r$

$= {}^{15}C_r \times (x)^{60-4r} \times (x)^{3r}$

$= {}^{15}C_r \times (x)^{60-r}$ ...(i)

For this term to contain x^{57}, $(60 - r)$ must be equal to 57.

$60 - r = 57$

$r = 3$

Then $T_{(3+1)}$, i.e., T_4 term be the term which contains x^{57}.

Put $r = 3$ in equation (i) we get

$T_4 = {}^{15}C_3 \times (x)^{60-3}$

$= \dfrac{15 \times 14 \times 13}{3 \times 2 \times 1} \times x^{57}$

$= 455x^{57}$

Hence, the coefficient of x^{57} will be 455.

Example 46:

Find the coefficient of $(x)^{-5}$ in the expansion of $\left(x^3 - \dfrac{2}{x^2}\right)^{10}$.

Solution:

Let $(r + 1)$th term contains $(x)^{-5}$ in the expansion of $\left[x^3 - \dfrac{2}{x^2}\right]^{10}$.

$T_{(r+1)} = {}^{10}C_r \times (x^3)^{10-r} \times \left(-\dfrac{2}{x^2}\right)^r$

$= {}^{10}C_r \times (x)^{30-3r} \times (-2)^r \times (x)^{-2r}$

$= {}^{10}C_r \times (x)^{30-5r} \times (-2)^r$ ...(i)

For this term to contain $(x)^{-5}$

$30 - 5r = -5$

$r = 7$

$T_{(7+1)} = T_{(8)}$ be the term which contains $(x)^{-5}$

Put $r = 7$ in equation (i) we get

$T_8 = {}^{10}C_7 \times (x)^{30-35} \times (-2)^7$

$= 120 \times (-128) \times (x)^{-5}$

Hence, the coefficient of $(x)^{-5}$ will be $120 \times (-128) = -15360$.

Consecutive Terms or Consecutive Coefficients

If one's given consecutive terms or the coefficient of consecutive terms in the expansion $(a + b)^n$, then one assumes that the consecutive terms are rth, $(r + 1)$th, and $(r + 2)$th. In other words, we can say that one can assume T_r, $T_{(r+1)}$, and $T_{(r+2)}$ terms as consecutive terms.

The ratio of $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$ and $\dfrac{{}^nC_{r+1}}{{}^nC_r} = \dfrac{n-r}{r+1}$.

Example 47:

The coefficient of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio of 21:140:780. Find the value of n.

Solution:

Let's assume the three consecutive terms be rth, $(r + 1)$th and $(r + 2)$th terms and their coefficient in the expansion of $(1 + x)^n$ be ${}^nC_{r-1}$, ${}^nC_r$, and ${}^nC_{r+1}$, respectively.

Here, it is given that, ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1}$ = (21:140:780)

Now, $\dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{21}{140}$

$\Rightarrow \dfrac{r}{n-r+1} = \dfrac{3}{20}$

$\Rightarrow 20r = 3n - 3r + 3$

$\therefore \quad n = \dfrac{23r-3}{3}$ ...(i)

Similarly, $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r} = \dfrac{140}{780}$

$\therefore \dfrac{r+1}{n-r} = \dfrac{7}{39} \quad \Rightarrow \quad 39r + 39 = 7n - 7r$

$\Rightarrow \dfrac{46r+39}{7} = n$ ...(ii)

On comparing both equations

$\dfrac{23r-3}{3} = \dfrac{46r+39}{7}$

$\therefore r = 6$

Put the value of r in equation (i) we get

$n = \dfrac{23 \times 6 - 3}{3}$

$n = 45$

Hence, the value of $n = 45$.

Finding the Number of Integral Terms Using Binomial Theorem

Let's understand this concept with the help of an example.

Example 48:

Find the number of integral terms in the expansion of $(3^{1/2} + 7^{1/4})^{2,048}$.

Solution:

The general term in the expansion of $(3^{1/2} + 7^{1/4})^{2,048}$ is given by

$T_{(r+1)} = {}^{2,048}C_r \times \left[(3)^{1/2}\right]^{2048-r} \times \left[(7)^{1/4}\right]^r$

$= {}^{2,048}C_r \times (3)^{1,024-r/2} \times (7)^{r/4}$

$= {}^{2,048}C_r \times (3)^{1,024-r} \times (3)^{r/2} \times (7)^{r/4}$

$= {}^{2,048}C_r \times (3)^{1,024-r} \times [(3)^2 \times 7]^{r/4}$

Now, $T_{(r+1)}$ is an integer if $\dfrac{r}{4}$ is an integer such that $0 \le r \le 2,048$ and r is a multiple of 4.

Hence, r can take any value from 0, 4, ... 2,048.

r = 0, 4, 8, 12, ... 2,048

r can assume 513 values.

Hence, there are 513 integral terms.

Practice Exercise – 1

Level of Difficulty – 1

1. What is the remainder when 50^{51} is divided by 17?
   (A) 17
   (B) 13
   (C) 1
   (D) 16

2. What is the difference (in base 12) between the largest five-digit number (in base 12) and the smallest five-digit number (in base 12)?
   (A) ABBBB
   (B) BABBB
   (C) BBABB
   (D) BBBAB

3. Find the unit digit of ($789^{444} + 798^{444} + 897^{444}$).
   (A) 8
   (B) 6
   (C) 4
   (D) 2

4. What is the remainder when $23^{20} + 20^{19}$ is divided by 11?
   (A) 3
   (B) 6
   (C) 5
   (D) 2

5. Find the remainder when 128^{128} is divided by 100.
   (A) 28
   (B) 84
   (C) 36
   (D) None of these

Level of Difficulty – 2

6. Find the remainder when 111222333444 .... 888999 is divided by 37.
   (A) 36
   (B) 18
   (C) 1
   (D) None of these

7. What is the remainder when 8^{567} is divided by 1,088?
   (A) 8
   (B) 64
   (C) 512
   (D) 576

8. Find the unit digit of $7^{2023} - 2^{2022}$.
   (A) 9
   (B) 7
   (C) 5
   (D) 3

9. If $x^3 + y^3 + z^3 = 177$ and $xyz = 59$, find the value of $\left(\dfrac{x^2 + y^2 + z^2}{xy + yz + zx}\right)^4$.

10. A two-digit number PQ in base 11 is one-third of the number formed by reversing its digits when considered in base 19. How many such numbers are possible?
    (A) 6
    (B) 5
    (C) 4
    (D) 3

Level of Difficulty – 3

11. If $P = 3^{8008}$, find the last four digits of P.

12. All-natural numbers that give remainders 1 and 2 when divided by 6 and 5, respectively, are written in ascending order, side by side, from left to right. What is the 99th digit from the left of the number thus formed?

13. If N = 12345678910111213 .... till 500 digits. Find the remainder when N is divided by 625.
    (A) 125
    (B) 220
    (C) 215
    (D) 203

14. Find the unit digit of the [6 × {LCM of ($3^{2021} - 1$) and ($3^{2021} + 1$)}].

(A) 4

(B) 6

(C) 2

(D) 8

15. If $(1{,}100)_x + (554)_x = (785)_{11}$, then the value of $(645)_x$ when written in decimal base system is ________.

Solutions

1. (D)

We can write 50^{51} in the form of

$$\frac{50^{51}}{17} = \frac{(51-1)^{51}}{17} = \frac{(17\times 3-1)^{51}}{17}$$

Since the number $(50)^{51}$ is of the form $17K - 1$.

Remainder when $(17K - 1)$ is divided by 17

is $\frac{(-1)^{51}}{17} = -1$ or $-1 + 17 = 16$

Hence, option (D) is the correct answer.

2. (A)

When the base of number system exceed 10, then 10, 11, 12, 13, 14, and 15 are represented by *A*, *B*, *C*, *D*, *E*, and *F*, respectively.

Largest five-digit number in base 12 = *BB,BBB*.

Smallest five-digit number in base 12 = 10,000.

Required Difference = (*BB,BBB*) – (10000) = *AB,BBB*.

Note: *B* – 1 = 11 – 1 = 10 = A

Hence, option (A) is the correct answer.

3. (A)

Unit digit of $(789^{444} + 798^{444} + 897^{444})$ = unit digit of $(9^{444} + 8^{444} + 7^{444})$

Unit digit of 7^{4K} and 9^{4K} is always 1

Unit digit of 8^{4K} is always 6

Hence, unit digit of $(9^{444} + 8^{444} + 7^{444})$ = 1 + 6 + 1 = 8.

Hence, option (A) is the correct answer.

4. (B)

$$\frac{23^{20}+20^{19}}{11}$$

$$= \frac{(22+1)^{20}+(22-2)^{19}}{11} = \frac{(+1)^{20}}{11} + \frac{(-2)^{19}}{11}$$

$$= \frac{1}{11} - \frac{(2)^{19}}{11} = \frac{1}{11} - \frac{2^4\times(2^5)^3}{11}$$

$$= \frac{1}{11} - \frac{16\times(33-1)^3}{11} = \frac{1}{11} - \frac{16\times(-1)^3}{11}$$

$$= \frac{1}{11} + \frac{16}{11} = \frac{17}{11}$$

∴ Remainder = 6

Hence, option (B) is the correct answer.

5. (C)

Remainder when *N* is divided by 100 = Last two digits of *N*

Remainder when 128^{128} is divided by 100 = Last two digits of 128^{128}

$(128)^{128} = (2^7)^{128} = 2^{896} = (2^{10})^{89} \times 2^6 = (1024)^{89} \times 2^6$

[Last two digits of $(1024)^x$ = Last two digits of $(24)^x$]

$(24)^{89} \times 64 = 24 \times 64 = 1536$

[As last two digits of $(24)^{\text{Odd power}} = 24$]

So last two digits of $128^{128} = 36$

Hence, option (C) is the correct answer.

6. (D)

Any number of the form *kkk*, where *k* is any digit from 1 to 9, is always divisible by 37.

As *kkk* = *k*(111) = *k*(37 × 3)

Now 111222333444 .... 888 999 can be written as

$111 \times 10^{24} + 222 \times 10^{21} + 333 \times 10^{18}$ .... $888 \times 10^3 + 999$

Now each of the 111, 222, 333, and so on till 999 is divisible by 37. Therefore, the remainder will be 0.

Hence, option (D) is the correct answer.

7. (D)

$$\frac{8^{567}}{1{,}088} = \frac{8^2\times 8^{565}}{8^2\times 17}$$

Let's keep the common factor (64) aside and find the remainder when 8^{565} is divided by 17.

$$\frac{8^{565}}{17} = \frac{\left(8^4\right)^{141}\times 8^1}{17} = \frac{\left(4{,}096\right)^{141}\times 8^1}{17}$$

Now replace the number with a remainder

$$\frac{\left(4{,}096\right)^{141}\times 8}{17} = \frac{\left(-1\right)^{141}\times 8^1}{17} = \frac{-1\times 8}{17}$$

Remainder = (–8) or (–8 + 17 = 9)

But, final remainder = remainder of $\left(\frac{8^{565}}{17}\right)$ × common factor = 9 × 64 = 576

Hence, option (D) is the correct answer.

8. (A)

As per the cyclicities of 7, unit digit of 7^{4k} = 1 (where k is a natural number),

and as per the cyclicity of 2, unit digit of 2^{4k} = 6 (where k is a natural number).

Now unit digit of $7^{2,023}$ = unit digit of $(7^{2,020} \times 7^3)$

= unit digit of $(7^4)^{504} \times 7^3 = 1 \times 3 = (3)$

Unit digit of $2^{2,022}$ = unit digit of $2^{2,020} \times 2^2$

$= (2^4)^{504} \times 4 = 6 \times 4 = (4)$

Unit digit of $7^{2,023} - 2^{2,022} = 3 - 4 = (9)$

[As $7^{2,023} > 2^{2,022}$, so unit digit of $7^{2,023}$ will act like 13 not 3.

For example, 143 – 74 = 69].

Hence, option (A) is the correct answer.

9. 16

$xyz = 59$

Thus, $3xyz = 177$

Thus, $3xyz = x^3 + y^3 + z^3$

When $3xyz = x^3 + y^3 + z^3$

$\Rightarrow \ x + y + z = 0$

$\Rightarrow \ (x + y + z)^2 = 0$

$\Rightarrow \ x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 0$

Thus, $x^2 + y^2 + z^2 = -(2xy + 2xz + 2yz)$

Thus, $= \frac{x^2 + y^2 + z^2}{xy + yz + xz} = -2$

Thus, $\left(\frac{x^2 + y^2 + z^2}{xz}\right)^4 = 16$

10. (B)

$(PQ)_{11} = \frac{1}{3}(QP)_{19}$

convert both (PQ) and (QP) into the base 10

$11P + Q = \frac{1}{3}(19Q + P)$

$33P + 3Q = 19Q + P$

$32P = 16Q$

$\frac{P}{Q} = \frac{1}{2}$

Now PQ could be 12, 24, 36, 48, and 5A.

Hence, 5 values of PQ are possible.

Hence, option (B) is the correct answer.

11. 6,561

We have $P = 3^{8,008}$

We can relate the concept of ending digit with the remainder as follows:

Unit digit of a number can be expressed as the remainder when the number is divided by 10.

The last two digits of a number can be expressed as the remainder when the number is divided by 100.

Last three digits of a number can be expressed as remainder when number is divided by 1,000.

Last four digits of a number can be expressed as remainder when number is divided by 10,000.

Therefore, we need to find remainder when $3^{8,008}$ is divided by 10,000.

Using Euler theorem:

First, we will find Euler totient function of 10,000.

$\phi(10,000) = 10,000 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$

$= 10,000 \times \frac{1}{2} \times \frac{4}{5} = 4,000$

Now, we know that using Euler theorem:

$\frac{3^{\phi(10000)}}{10000}$ will give remainder 1.

Therefore, $\frac{\left(3^{4000}\right)^2 \times 3^8}{10000} = \frac{1 \times 6561}{10000}$

Remainder will be 6,561 which are also the last four digits of the number.

Hence, 6,561 is the correct answer.

12. 0

Natural number N which when divided by 5 give a remainder 2 are given by $5K + 2$ ['k' is a whole number].

Natural number N which when divided by 6 give a remainder 1 are given by $6m + 1$ ['m' is a whole number]

$N = 5k + 2 = 6m + 1$

$\Rightarrow 6m = 5k + 1$

Min value of k for which m would be integer is 1.
Min value of $N = 7$ (putting the value of $k = 1$).
Next values of $N = 7 + \{\text{LCM (5 and 6)}y\} = 7 + 30y$ (where y is a whole number)
Putting the value of y as 0, 1, 2, 3, 4, and so on we will get all such natural numbers as 7, 37, 67, 97, 127, and so on.
When numbers written one beside the other,
737679712715718721724727730733 7...
In the above sequence: The first four numbers give first seven digits. After these four numbers, three-digit numbers follow. The three-digit numbers from an AP with common difference 30.
First three-digit number = 127
The 30th three-digit number = 127 + (30 – 1) × 30 = 997.
So, the first 34 numbers give 97 digits with 34th number as 997.
The next number in this AP is 997 + 30 = 1,027.
Hence, the 99th digit is 0.

13. (B)

Range	Numbers	Digits
1–9	9	9
10–99	90	180
100–199	100	300

So, if we write 1 2 3 4 5 6 7 8 9 10 11 ... 198 199.
A total of 489 digits will be written.
For the next three numbers after 199, 200, 201, and 202, nine digits will be written.
So, 1 2 3 4 5 6 7 8 9 10 11 12 ... 199 200 201 202.
We have 498 digits being written.
Now, if we write 203, three more digits will be added, and the total digits would then be 501.
So, the unit digit 203 has to be removed.
1 2 3 4 5 6 7 8 9 10 11 ... 199 200 201 202 20 will be our number with 500 digits.
Remainder when N is divided by 625 = Remainder when the last four digits of N are divided by 625 $\rightarrow \dfrac{0220}{625} \rightarrow$
Remainder = 220.
Hence, option (B) is the correct answer.

14. (A)

Since $(3^{2021} - 1)$ and $(3^{2021} + 1)$ are consecutive even numbers.
$\therefore$ their HCF = 2
We know that
LCM of $(a, b) \times$ HCF $(a, b) = a \times b$
LCM $\times\, 2 = (3^{2021} - 1) \times (3^{2021} + 1) = (3^{2021})^2 - 1^2$

$$\text{LCM} = \frac{3^{4042} - 1}{2}$$

Now $6 \times \text{LCM} = 3 \times (3^{4042} - 1)$

Unit digit of $3^{4042} = 3^{4040} \times 3^2 = 1 \times 9 = 9$
Note: Since cyclicity of 3 is 4 and unit digit of $3^{4K} = 1$
$\therefore$ Unit digit of $= 3 \times (3^{4042} - 1)$
$= 3 \times (9 - 1) = 24 = 4$
Hence, option (A) is the correct answer.

15. 421

$(1100)_x + (554)_x = (785)_{11}$
Now, writing each of them in decimal base system:
$(1100)_x = x^3 + x^2$
$(554)_x = 5x^2 + 5x + 4$
$(785)_{11} = 7 \times 11^2 + 8 \times 11 + 5 \times 1 = 940$
So, $x^3 + x^2 + 5x^2 + 5x + 4 = 940$
$x^3 + 6x^2 + 5x + 4 = 940$
$x\,(x^2 + 6x + 5) = 936$
$x(x + 5)\,(x + 1) = 936$
Solving we get $x = 8$
Therefore, $(645)_x = (645)_8 = 6 \times 8^2 + 4 \times 8 + 5 \times 1 = 421$

Practice Exercise – 2

Level of Difficulty – 1

1. Constant term in the expansion of $\left(x-\frac{1}{x}\right)^{10}$ is:

(A) –252
(B) 252
(C) 125
(D) –125

2. What is the remainder when $(2938475)_{18}$ is divided by 19?

(A) 17
(B) 18
(C) 9
(D) 8

3. Which of the following will completely divide $118^{2022} - 61^{2022}$?

(A) 573
(B) 537
(C) 579
(D) 597

4. Find the remainder when $3^{5555} + 5^{3333}$ is divided by 23.

(A) 1
(B) 22
(C) 8
(D) None of these

5. Find the remainder when 490 × 565 × 662 × 720 is divided by 47.

(A) 22
(B) 25
(C) 30
(D) 17

6. Let $k = 2004^2 + 2^{2004}$. What is the unit digit of $k^2 + 2^k$?

7. Find the remainder when the product of first 19 consecutive positive integers is divided by 361.

(A) 0
(B) 1
(C) 18
(D) None of these

8. What is the remainder when $4^{2468} - 9^{1357}$ is divided by 10?

9. A number *N*, when expressed in base 12, has 7 in its unit's place. What is the digit in the unit's place when the number N is converted into base 5?

(A) 1
(B) 2
(C) 3
(D) Cannot be determined

10. Find the unit digit of $(1!)^{99!} + (2!)^{98!} + (3!)^{97!} + (4!)^{96!} + \ldots + (97!)^{3!} + (98!)^{2!} + (99!)^{1!}$.

(A) 9
(B) 7
(C) 5
(D) 1

Level of Difficulty – 2

11. Find the coefficient of x^{30} in the expansion of $\left[x^3 - \frac{1}{x^2}\right]^{15}$.

(A) 455
(B) –455
(C) 1365
(D) –1365

12. A four-digit number of the form AB89 is a perfect square, where A and B are distinct digits. How many such numbers are possible?

(A) 1
(B) 2
(C) 3
(D) 4

13. Find the number of terms in the expansion $(1 + 2a + a^2)^{30}$.

(A) 61
(B) 496

(C) 71
(D) 248

14. Find the unit digit of $7^{7^{8^{9^{10}}}}$.
(A) 7
(B) 9
(C) 3
(D) 1

15. A 3 digits number in base 9, when expressed in base 11, has its digits reversed in order. What is the value of the original number in the decimal system?
(A) 245
(B) 490
(C) 732
(D) Cannot be determined

16. Let *N* be a set of integers {2, 7, 12, 17, ..., 242, 247, 252} and *K* be a subset of *N* such that the sum of no two elements of *N* is 254. Find the maximum possible number of elements in *K*.
(A) 25
(B) 26
(C) 50
(D) 51

17. Find the remainder when $11^{67} + 13^{17}$ is divided by 144.
(A) 1
(B) 143
(C) 40
(D) None of these

18. How many natural numbers exist such that in base 5, a number is a four-digit number, and in base 6, the number is a three-digit number?
(A) 89
(B) 90
(C) 91
(D) 92

19. The sum of weights of 11 boxes is 49 kg, where the weight of each box is an integral multiple of kilograms. What could be the minimum number of possible pairs of boxes of the same weights?
(A) 0
(B) 1
(C) 2
(D) 3

20. A number *N*, when divided by 24, leaves a remainder of 17. What will the sum of all possible remainders be when *N* is divided by 96?

Level of Difficulty – 3

21. Find the term independent of *x* in the expansion of $\left(5x^2 - \dfrac{1}{4x^4}\right)^{12}$.
(A) $\dfrac{{}^{12}C_4 \times (5)^8}{256}$
(B) $\dfrac{{}^{12}C_6 \times (5)^{12}}{256}$
(C) $\dfrac{{}^{12}C_5 \times (2)^{12}}{256}$
(D) Cannot be determined

22. How many factors of 9! are there whose unit digit is 5?
(A) 10
(B) 16
(C) 40
(D) 80

23. Find the sum of digits of the smallest natural number, which when multiplied by 123 ends in a number whose thousands, hundreds, tens, and unit digits are 2, 0, 1, and 4, respectively.
(A) 11
(B) 13
(C) 15
(D) 17

24. A two-digit number has distinct digits. How many such two-digit positive numbers is the difference between the number itself and the number formed on reversing its digits is the perfect square of an integer?

25. How many five-digit numbers are there such that digits at hundred's place, unit's place and ten-thousands place are the first three terms of a geometric progression in any order?
(A) 1,500
(B) 900
(C) 3,300
(D) 1,700

26. What is the sum of the last two digits of $\frac{6^{100}-1}{5}$?

27. In a certain number system to the base K, we define a sum, S, as given below.
$S = (11)_K + (22)_K + (33)_K + \ldots + \{(K-1)(K-1)\}_K$
Here $\{(K-1)(K-1)\}_K$ denotes the largest two digit number in that number system. Find the value of $4S + (20)_K$.
(A) $(1{,}000)_K$
(B) $(2{,}000)_K$
(C) $(3{,}000)_K$
(D) $(4{,}000)_K$

28. The coefficients of three consecutive terms in the expansion $(1 + a)^n$ are in the ratio 11:45:165. Find the value of n.
(A) 55
(B) 54
(C) 56
(D) 57

29. A trader wants to measure all integral weights from 1 to 320 kg, using a common balance where weights can be kept in both the pans. What is the minimum number of weights required?

30. Find the number of integral terms in the expansion of $(3^{1/2} + 7^{1/4})^{512}$.
(A) 511
(B) 128
(C) 512
(D) 129

Solutions

1. (A)

We have, $\left(x - \frac{1}{x}\right)^{10}$

The general term of the expansion

$\left(x - \frac{1}{x}\right)^{10}$ is given by

General term $= (r+1)$th term $= {}^{10}C_r \times (x)^{10-r} \times \left(-\frac{1}{x}\right)^r$

$= (-1)^r \times {}^{10}C_r \times (x)^{10-2r}$ ...(1)

The given expression is constant when $10 - 2r = 0$, i.e., when $10 = 2r$ or $r = 5$

Put $r = 5$ in equation (1), we get $(-1)^5 \times {}^{10}C_5$

$= -1 \times \frac{10!}{5! \times 5!} = (-252)$

Hence, the constant term is –252.

Hence, option (A) is the correct answer.

2. (C)

$$\frac{(2938475)_{18}}{19}$$

$$= \frac{(2 \times 18^6 + 9 \times 18^5 + 3 \times 18^4 + 8 \times 18^3 + 4 \times 18^2 + 7 \times 18^1 + 5)_{10}}{19}$$

Now, $\frac{(18)^{even}}{19}$, remainder = 1 and $\frac{(18)^{odd}}{19}$, remainder = 18 or –1

$$\frac{\{2 \times 1 + 9 \times (-1) + 3 \times 1 + 8 \times (-1) + 4 \times 1 + 7 \times (-1) + 5\}}{19}$$

$$= \frac{-10}{19}$$

Remainder = –10 or –10 + 19 = 9.

Hence, option (C) is the correct answer.

3. (B)

$A^N - B^N$ is divisible by $(A + B)$ and $(A - B)$ whenever N is even.

$\Rightarrow 118^{2022} - 61^{2022}$ is divisible by both (118 – 61) and (118 + 61) and also by (118 – 61)(118 + 61)

$\Rightarrow 118^{2022} - 61^{2022}$ is divisible by 57 (19 × 3), 179 and also by (57 × 179)

$\Rightarrow 118^{2022} - 61^{2022}$ is divisible by (179 × 3) = 537

Hence, option (B) is the correct answer.

4. (D)

Concept: $(x^N + y^N)$ is always divisible by $(x + y)$ when N is an odd number.

Now $3^{5555} + 5^{3333} = (3^5)^{1111} + (5^3)^{1111} = (243)^{1111} + (125)^{1111}$ which is divisible by (243 + 125) = 368 = 16 × 23

So, $3^{5555} + 5^{3333}$ is divisible by 16 × 23.

So, the remainder when $(3^{5555} + 5^{3333})$ is divided by 23 will be zero.

Hence, option (D) is the correct answer.

5. (B)

When 490, 565, 662, and 720 are divided by 47, we get 20, 1, 4, and 15, respectively, as the remainder. Their product = 20 × 1 × 4 × 15 = 1,200.

We get the remainder 25 by dividing 1,200 by 47.

So, the remainder when 490 × 565 × 662 × 720 is divided by 47 is 25.

Hence, option (B) is the correct answer.

6. 0

Unit digit of $\left(2004^2 + 2^{2004}\right)$

$=$ unit digit of $\left(4^2 + 2^4\right)$

$=$ unit digit of $(6 + 6) = 2$

So, the unit digit of k is 2.

Unit digit of $k^2 = 4$

Since $k = 2004^2 + 2^{2004}$ is a multiple of 4, therefore 2^{4k} will have its unit digit as 2^4, which is equal to 6.

Thus, the unit digit of $k^2 + 2^k = 4 + 6 = 0$.

7. (D)

The product of the first 19 consecutive positive integers will be = 19!.

Now we need to find out the remainder when 19! is divided by 361 (19 × 19)

$$= \frac{19!}{19 \times 19} = \frac{19 \times 18!}{19 \times 19}$$

Now taking 19 common from both numerator and denominator, first we will find the remainder when 18! is divided by 19.

Remainder when 18! Is divided by 19 = 18 (by Wilson's theorem).
Now, final remainder = remainder × common factor = 18 × 19 = 342.
Hence, option (D) is the correct answer.

8. 7

When any number is divided by 10, remainder = unit digit of that number
So, when $4^{2468} - 9^{1357}$ is divided by 10,
Remainder = unit digit of $4^{2468} - 9^{1357}$
Unit digit of any even power of 4 = 6
Unit digit of any odd power of 9 = 9
Unit digit of $4^{2468} - 9^{1357} = 6 - 9 = 7$
(As $4^{2468} > 9^{1357}$ and 6 is smaller than 9, so 6 will act as 16 and then unit digit will be 7).

9. (D)

Given that N when expressed in base 12, has 7 in its unit's place.
Hence $N = 12k + 7$.
Now, when $12k + 7$ is converted to base 5, the unit's place depends on the remainder when $12k + 7$ is divided by 5.
A unique value cannot be determined as for $k = 1$, the remainder is 4, whereas for $k = 2$, the remainder is 1.
Hence, option (D) is the correct answer.

10. (A)

For all $(N!)^{N!}$ (where $N \geq 5$), unit digit will be 0.
So, we only need to find the unit digit of $(1!)^{99!} + (2!)^{98!} + (3!)^{97!} + (4!)^{96!}$
Unit digit of $(1)^{99!} + (2)^{98!} + (6)^{97!} + (24)^{96!}$
= unit digit of $(1)^{99!} + (2)^{98!} + (6)^{97!} + (4)^{96!}$
= unit digit of 1 + 6 + 6 + 6
= unit digit of 19 = 9
Hence, option (A) is the correct answer.

11. (B)

Suppose $(r + 1)$th term involves x^{30} in the expansion of $\left(x^3 - \dfrac{1}{x^2}\right)^{15}$.

As we know that, $(x - a)^n$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \, (x)^{n-r} \, (a)^r$$

$$T_{(r+1)} = (-1)^r \times {}^{15}C_r \, (x^3)^{15-r} \times \left(\frac{1}{x^2}\right)^r$$

$$= (-1)^r \times {}^{15}C_r \, (x)^{45-5r} \qquad \text{...(i)}$$

For this term to contain $(x)^{30}$, we must have
$45 - 5r = 30 \Rightarrow r = 3$
So, (3 + 1)th i.e., 4th term contain x^{30}
Put $r = 3$ in equation (i) we get
$= (-1)^3 \times {}^{15}C_3 \times (x)^{30} = -455\,(x)^{30}$
Hence, the coefficient of x^{30} = (–455)
Hence, option (B) is the correct answer.

12. (B)

The last two digits of the perfect square repeat in such a way that
Last two digits of x^2 = last 2 digits of $(50K \pm x)^2$.
Where $1 \leq x \leq 25$ and K = natural number
So, the first perfect square with the last two digits as $89 = 17^2$.
Other squares with the last two digits as 89.

$(50K \pm 17)^2$
K = 1: $(50 - 17)^2 = 1089$; $(50 + 17)^2 = 67^2 = 4489$
K = 2: $(100 - 17)^2 = 83^2 = 6889$; $(100 + 17)^2$ = 5 digit number

Only 1,089 and 6,889 are valid.
As in 4,489 both the digits A and B are the same.
Hence, two such numbers are possible.
Hence, option (B) is the correct answer.

13. (A)

Here the given expansion is $(1 + 2a + a^2)^{30}$
The given expansion can be written as $[(1 + a)^2]^{30} = (1 + a)^{60}$
We know that the expansion $(a + b)^n$ contains ${}^{n+r-1}C_{r-1}$ terms.
hence, $(1 + a)^{60} \Rightarrow {}^{60+2-1}C_{2-1} \Rightarrow {}^{61}C_1 \Rightarrow 61$ terms
Hence, option (A) is the correct answer.

14. (A)

We know that the unit digit of $7^{4k} = 1$, where k is a natural number

Here the power of 7 is $7^{8^{9^{10}}}$. Now we will divide this power by 4.

$7^{8^{9^{10}}}$ when divided by 4, the remainder would be 1

So, we can write $7^{8^{9^{10}}} = 4k + 1$ (As number = divisor × quotient + remainder)

Unit digit of $7^{7^{8^{9^{10}}}}$ = unit digit of $7^{(4k+1)}$

= unit digit of $7^{4k} \times 7^1 = 1 \times 7 = 7$.

Hence, option (A) is the correct answer.

15. (D)

Let's assume the three-digit number = ABC

According to the question, $(ABC)_9 = (CBA)_{11}$

Converting both $(ABC)_9$ and $(CBA)_{11}$ to the decimal system, we will get

$A \times 9^2 + B \times 9 + C \times 1 = C \times 11^2 + B \times 11 + A$

$\Rightarrow 81A + 9B + C = 121C + 11B + A$

$\Rightarrow 80A = 2B + 120C$

$\Rightarrow 40A = B + 60C$

As $40A$ and $60C$ always end in 0, B should definitely be 0.

Now, $40A = 60C$

$$\frac{A}{C} = \frac{3}{2}$$

Which means A = 3 or 6 and B = 2 or 4

As $A \le 8$ because the number (ABC) is in base 9.

So, ABC = 302 or 604

Now, $(ABC)_9 = (81A + 10B + C)_{10}$

$(302)_9 = (81 \times 3 + 10 \times 0 + 2)_{10} = (245)_{10}$

$(604)_9 = (81 \times 6 + 10 \times 0 + 4)_{10} = (490)_{10}$

So, the value of ABC in the decimal number system could be either 245 or 490.

Hence, option (D) is the correct answer.

16. (B)

As you can see that the numbers in the set are in AP with first term 2 and the common difference of 5 and last term of 252.

So, Nth term (last term) of the numbers $T_N = 2 + (N - 1)5 = 252$, solving which we will get $N = 51$.

So, there are 51 terms with middle term = 26th term = $2 + 25 \times 5 = 127$.

If we take any two of the first 26 terms, their sum will be less than 254, so the first 26 terms of the set can definitely be part of K.

If we take any two of the last 26 terms, their sum will be more than 254, so the last 26 terms of the set can be part of K.

Hence, the maximum number of elements in K = 26.

Hence, option (B) is the correct answer.

17. (D)

$11^{67} + 13^{17} = (12 - 1)^{67} + (12 + 1)^{17}$

Now, if we expand $(12 - 1)^{67} + (12 + 1)^{17}$, then all the terms in the expansion are divisible by 144 except the last two terms in $(12 - 1)^{67}$ and $(12 + 1)^{17}$ each.

Last two terms in $(12-1)^{67} = {}^{67}C_{66}(12) - 1$ = 804 – 1 = 803.

Last two terms in $(12 + 1)^{17} = {}^{17}C_{16}(12) + 1$ = 17 × 12 + 1 = 204 + 1 = 205.

Required remainder will be the remainder, when 803 + 205 = 1008 is divided by 144, which is 0.

Hence, option (D) is the correct answer.

18. (C)

Let us find the natural numbers (base 10) corresponding to a four-digit number in the base five systems.

The smallest such number being $(1,000)_5$ = 125 and the largest such number is $(4,444)_5$ = 624 (i.e., 625 – 1).

Similarly, the smallest and the largest three-digit numbers in the base six system are $(100)_6$ = 36 and $(555)_6$ = 215.

From the above observations, we can say that all numbers from 125 to 215 satisfy both conditions.

Hence, 215 – 125 + 1 = 91 such numbers exist.

Hence, option (C) is the correct answer.

19. (C)

To minimise the pairs of boxes with the same weights, we must maximise the number of boxes with different weights.

Let us take the weight of different boxes, starting from 1 kg as following:

1 kg, 2 kg, 3 kg, 4 kg, 5 kg, 6 kg, 7 kg, 8 kg, and 9 kg

A total of nine boxes with a total weight = 45 kg.

Now we have to accommodate two boxes such that their combined weight is 4 kg. There are two possibilities – (2 kg and 2 kg) and (1 kg and 3 kg).

If we take 2 kg and 2 kg, then there will be three boxes with each weighing 2 kg, and there will be three pairs possible.

[For example, if boxes A, B, and C are 2 kg each, there will be three pairs of boxes with the same weights (A, B), (B, C), and (A, C)].

But if we take boxes of 3 kg and 1 kg, there will be two pairs of boxes with the same weight, i.e., two boxes weighing 3 kg and two boxes weighing 1 kg.

Hence, the minimum number of pairs of boxes with the same weights will be two.

Hence, option (C) is the correct answer.

20. 212

Number = Divisor $\times$ quotient + remainder

$N = 24Q + 17.$

When $Q = 0$ and N is divided by 96, the remainder would be = 17.

When $Q = 1$ and N is divided by 96, the remainder would be = 41.

When $Q = 2$ and N is divided by 96, the remainder would be = 65.

When $Q = 3$ and N is divided by 96, the remainder would be = 89.

When $Q = 4$ and N is divided by 96, the remainder would be = 17.

When $Q = 5$ and N is divided by 96, the remainder would be = 41.

And this process goes on till infinity, but remainders will repeat, and only four possible remainders would be there, which are 17, 41, 65, and 89.

Thus, the sum of all possible remainders = 17 + 41 + 65 + 89 = 212.

21. (A)

Let $(r + 1)^{th}$ term be independent of x in the given expansion.

Now, $T_{(r+1)} = {}^{12}C_r \times (5x^2)^{12-r} \times \left(-\dfrac{1}{4x^4}\right)^r$

$\Rightarrow {}^{12}C_r \times (5)^{12-r} \times (x)^{24-2r} \times \left(-\dfrac{1}{4}\right)^r \times (x)^{-4r}$

$\Rightarrow {}^{12}C_r \times (5)^{12-r} \times \left(-\dfrac{1}{4}\right)^r \times x^{24-6r}$ ...(i)

This term is independent of x, if $24 - 6r = 0 \Rightarrow r = 4$

So, (4 + 1), i.e., fifth term is independent of x.

Putting $r = 4$ in equation (i), we get

$$T_5 = {}^{12}C_4 \times (5)^{12-4} \times \left(-\frac{1}{4}\right)^4$$

$\Rightarrow {}^{12}C_4 \times (5)^8 \times \left(-\dfrac{1}{4}\right)^4$

Hence, required term $= \dfrac{{}^{12}C_4 \times (5)^8}{256}$

Hence, option (A) is the correct answer.

22. (A)

The prime factorised form of $9! = 2^7 \times 3^4 \times 5^1 \times 7^1$.

When any factor of $3^4 \times 7^1$ is multiplied by 5^1, the unit digit of the resulting number will be 5.

The number of factors of $3^4 \times 7^1 = 5 \times 2 = 10$.

The number of factors of 9! whose unit digit is 5 = 10 × 1 = 10.

Hence, option (A) is the correct answer.

23. (D)

Since the product ends with 4, the required number should end with 8.

$123 \times 8 = 984$

So, the product of ten's place of the required number and 123 should end with 3.

Hence, ten's place of the required number has to be 1.

$123 \times 18 = 2{,}214$

So, the product of hundred's place of the required number and 123 should end with 8. Hence, the hundred's place of the required number has to be 6.

$123 \times 618 = 76{,}014$

So, the product of thousand's place of the required number and 123 should end with 6. Hence, the thousand's place of the required number has to be 2.

$123 \times 2618 = 3{,}22{,}014$

So, the required number is 2,618.

Sum of digits of the required number (2,618) = 17.

Hence, option (D) is the correct answer.

24. 29

Let the two-digit number be xy

The difference between the number and the number obtained when its digits are reversed is,

$(10y - x) - (10x - y) = 9(y - x)$

This must be a perfect square.

So, it can only take the values of 9, 36, and 81.

It cannot take the value 0 because it is given that the digits of the two-digit number are distinct.

If $9(y - x) = 9$, then $(y - x) = 1$. There will be 17 numbers such that either $y - x = 1$ or $x - y = 1$.

If $9(y - x) = 36$, then $(y - x) = 4$. There will be 11 numbers such that either $y - x = 4$ or $x - y = 4$.

If $9(y - x) = 81$, then $(y - x) = 9$.

There will be only 1 number such that $x - y = 9$, that is 90.

So, the total number of numbers possible are 17 + 11 + 1 = 29.

25. (C)

Number is of the form $a_1a_2a_3a_4a_5$ where $a_1 \neq 0$

Given that a_1, a_3, and a_5 are in G.P. in any order, so possible geometric progressions are

(1, 1, 1), (2, 2, 2), (3, 3, 3) .... (9, 9, 9) and (1, 2, 4), (1, 3, 9), (2, 4, 8), (4, 6, 9)

$\Rightarrow 9 \times (1 \times 10 \times 1 \times 10 \times 1) + 4 \times (3! \times 10 \times 10) = 3{,}300$ numbers in all.

Hence, option (C) is the correct answer.

26. 12

$$\frac{6^{100}-1}{5} = \frac{(6-1)\left(1+6+6^2+6^3+\cdots+6^{99}\right)}{5}$$

Therefore, the expression reduces to $\left(1+6+6^2+6^3+\cdots+6^{99}\right)$

The last two digits will be obtained when the expression is divided by 100. We can write 100 as the product of two co-primes in the following manner:

$100 = 4 \times 25$ [$N = a \times b$, where $a = 4$, $b = 25$ are co-primes].

Using the Chinese remainder theorem:

$$\left(1+6+6^2+6^3+\cdots+6^{99}\right) \bmod 4 = r_1 = 3$$

$$\Rightarrow \left(1+6+6^2+6^3+\cdots+6^{99}\right) = 4x + 3$$

$$\left(1+6+6^2+6^3+\cdots+6^{99}\right) \bmod 25 = r_2 = 0$$

$$\left(1+6+6^2+6^3+\cdots+6^{99}\right) = 25y$$

$$\Rightarrow \quad 4x + 3 = 25y$$

The minimum value of x for which y is a whole number is 18. If x is 18, $y = 3$.

So, minimum number = 75.

General number satisfying the above equation = 75 + k\{LCM (25, 4)\} = $100k + 75$

So, $\left(1+6+6^2+6^3+......+6^{99}\right) = 100k + 75$

Which means when $\left(1+6+6^2+6^3+\cdots+6^{99}\right)$ is divided by 100, the quotient is k and the remainder is 75.

Hence, the last 2 digits of $\left(1+6+6^2+6^3+\cdots+6^{99}\right) = 75$.

The sum of the last two digits = (7 + 5) = 12.

27. (B)

First convert each term of S into base 10, we will get

$$S = (K + 1) + (2K + 2) + (3K + 3) + \cdots + \{(K - 1)K + K - 1\}$$

$$S = K\{(1 + 2 + 3 + \cdots + (K - 1)\} + \{(1 + 2 + 3 + \cdots + (K - 1)\}$$

$$S = K\left(\frac{(K-1)K}{2}\right) + \left(\frac{(K-1)K}{2}\right)$$

$$= \frac{1}{2}\left(K^3 - K\right)$$

$$4S = 2\left(K^3 - K\right) = \left(2K^3 - 2K\right)$$

$$= (2{,}000)_K - (20)_K$$

Thus, $4S + (20)_K = (2{,}000)_K - (20)_K + (20)_K = (2{,}000)_K$

Hence, option (B) is the correct answer.

28. (A)

Let the three consecutive terms in the expansion be r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ term, their coefficient in the expansion $(1 + a)^n$ are ${}^nC_{r-1}$, ${}^nC_r$, and ${}^nC_{r+1}$, respectively.

The ratio of ${}^nC_{r-1}:{}^nC_r:{}^nC_{r+1} = 11:45:165$

Now $\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{11}{45} \Rightarrow \frac{r}{n-r+1} = \frac{11}{45}$

$$\Rightarrow n = \frac{56r - 11}{11} \quad \text{...(i)}$$

Now $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{45}{165} \Rightarrow \frac{r+1}{n-r} = \frac{9}{33}$

$$\Rightarrow n = \frac{42r + 33}{9} \quad \text{...(ii)}$$

On comparing both equations we get

$$\frac{42r + 33}{9} = \frac{56r - 11}{11} \Rightarrow r = 11$$

Put the value of r in equation (i) we get

$$\Rightarrow n = \frac{56 \times 11 - 11}{11} = 55$$

Hence, $n = 55$.

Hence, option (A) is the correct answer.

29. 6

To measure 1 kg, the trader must have a 1 kg weight. Now, as he can put weights on both sides of the pan, he can measure the sum and the difference between the weights.

So, to measure 2 kg, instead of using a 2 kg weight, he will take a new weight of 3 kg. As the trader has 1 kg and 3 kg weights, he can also measure 2 kg (3 – 1) and 4 kg. To measure a 5 kg, a new weight of 9 kg is required.

With 1 kg, 3 kg, and 9 kg weights, he can measure all weights up to 13 kg.

S. No.	Weight	Upper Limit for Measurement
1	1 kg	1 kg
2	3 kg	4 kg
3	9 kg	13 kg
4	27 kg	40 kg
5	81 kg	121 kg
6	243 kg	364 kg

$\therefore$ The minimum number of weights required is 6.

30. (D)

T_{r+1} term in the expansion $(3^{1/2} + 7^{1/4})^{512}$ is given by

$$T_{r+1} = {}^{512}C_r \times \left[(3)^{1/2}\right]^{512-r} \times \left[(7)^{1/4}\right]^r$$

$$= {}^{512}C_r \times (3)^{256-\frac{r}{2}} \times (7)^{\frac{r}{4}}$$

$$= {}^{512}C_r \times (3)^{256-r} \times (3)^{\left(\frac{r}{2}\right)} \times (7)^{\frac{r}{4}}$$

$$= {}^{512}C_r \times (3)^{256-r} \times \left(3^2 \times 7\right)^{\frac{r}{4}}$$

$T_{(r+1)}$ will be an integer, if $\frac{r}{4}$ is an integer such that $0 \le r \le 512$

r is a multiple of 4, satisfying $0 \le r \le 512$

$r = 0, 4, 8, 12, \ldots, 512$

$T_n = a + (n - 1) \times d \Rightarrow 512 = 0 + (n - 1) \times 4$

$\Rightarrow n = 129$

So, r can assume 129 values.

Hence, there are 129 integral terms in the expansion of $(3^{1/2} + 7^{1/4})^{512}$.

Hence, option (D) is the correct answer.