7 Indices and Surds

Indices

Indices are the method of taking power or root of a number. The power, also known as the index, tells how many times a number has to multiply by itself.

For example: 3^4 means that you have to multiply 3 by itself four times

$= 3 \times 3 \times 3 \times 3 = 81$.

In a^b, a is called the base and b is called the power, exponent, or index.

Formulae of Indices

- $x^0 = 1$ except $x = 0$, for example $2^0 = 1$, $1^0 = 1$

  For any m, n which belongs to real numbers
- $a^m \times a^n = a^{m+n}$, for example $x^4 \times x^5 = x^{4+5} = x^9$
- $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$, for example $\dfrac{x^5}{x^4} = x^{5-4} = x^1$
- $x^{-n} = \dfrac{1}{x^n}$, for example $x^{-1} = \dfrac{1}{x^1}$
- $(x^m)^n = x^{m \times n}$ or $x^{n \times m}$, for example $(3^2)^5 = 3^{2 \times 5} = 3^{10}$
- $\dfrac{x^2}{y^2} = \left(\dfrac{x}{y}\right)^2$, $y \neq 0$, for example $\left(\dfrac{x}{y}\right)^7 = \dfrac{x^7}{y^7}$ provided $y \neq 0$
- $(x \cdot y)^m = x^m \cdot y^m$, for example $(2 \times 5)^3 = 2^3 \times 5^3$
- $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$ where n is a positive integer, for example, $125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}} = \sqrt[3]{5^3} = \sqrt[3]{5 \times 5 \times 5} = 5$
- $\left(\dfrac{a}{b}\right)^x = \left(\dfrac{b}{a}\right)^{-x}$, for example, $\left(\dfrac{2}{3}\right)^{30} = \left(\dfrac{3}{2}\right)^{-30}$

Example 1:

Simplify $32^{\frac{2}{5}} \times 5^3 - (2^2)^3 \div 3$

Solution:

$$32^{\frac{2}{5}} \times 5^3 - (2^2)^3 \div 3 = (2^5)^{\frac{2}{5}} \times 125 - \frac{2^6}{3}$$

$$= 2^{5 \times \frac{2}{5}} \times 125 - \frac{64}{3}$$

$$= 4 \times 125 - \frac{64}{3} = 500 - \frac{64}{3} = \frac{1,436}{3}$$

Example 2:

Simplify $\left(\dfrac{625^3}{441^2}\right)^{\frac{-5}{8}}$.

Solution:

$625 = 25 \times 25 = 5^2 \times 5^2 = 5^4$

$441 = 21 \times 21$

$$\left(\frac{625^3}{441^2}\right)^{\frac{-5}{8}} = \left(\frac{441^2}{625^3}\right)^{\frac{5}{8}} = \left(\frac{(21^2)^2}{(5^4)^3}\right)^{\frac{5}{8}} = \left(\frac{(21^4}{(5^{12})}\right)^{\frac{5}{8}}$$

$$\left(\text{Since} \left(\frac{A}{B}\right)^x = \left(\frac{B}{A}\right)^{-x}\right)$$

$$= \frac{(21)^{4 \times \frac{5}{8}}}{(5)^{12 \times \frac{5}{8}}} = \frac{(21)^{\frac{5}{2}}}{(5)^{\frac{15}{2}}} = \frac{(21)^{\frac{5}{2}}}{(5^3)^{\frac{5}{2}}} = \frac{(21)^{\frac{5}{2}}}{(125)^{\frac{5}{2}}}$$

$$= \left(\frac{21}{125}\right)^{\frac{5}{2}} \qquad \left(\text{Since } \frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n ;\ y \neq 0\right)$$

$$= \sqrt{\left(\frac{21}{125}\right)^5}$$ is the required answer.

Example 3:

Simplify $\left(\dfrac{(1331)^{\frac{-2}{5}}}{(3)^{\frac{-6}{5}}}\right)^{\frac{10}{3}}$.

Solution:

$$\left(\frac{(1331)^{\frac{-2}{5}}}{(3)^{\frac{-6}{5}}}\right)^{\frac{10}{3}} = \left(\frac{(3)^{\frac{-6}{5}}}{(1331)^{\frac{-2}{5}}}\right)^{-\frac{10}{3}}$$

$$\left(\text{Since} \left(\frac{A}{B}\right)^x = \left(\frac{B}{A}\right)^{-x}\right)$$

$$= \frac{3^{\frac{-6}{5}\times\frac{-10}{3}}}{1331^{\frac{-2}{5}\times\frac{-10}{3}}} = \frac{3^4}{(1331)^{\frac{4}{3}}} = \frac{3^4}{(11^3)^{\frac{4}{3}}} = \frac{(3)^4}{(11)^4}$$

$$= \left(\frac{3}{11}\right)^4 \quad \left(\text{Since } \frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n;\ y \neq 0\right)$$

Example 4:

Find the positive integral value of x which satisfies the given equation $(7^x)^{2x-3} = (7)^{x^2}$.

Solution:

According to the question, $(7^x)^{2x-3} = (7)^{x^2}$

$\Rightarrow \quad 7^{2x^2-3x} = (7)^{x^2}$

$\Rightarrow \quad 2x^2 - 3x = x^2$

Since the bases are equal and greater than 0.

$\Rightarrow \quad x^2 - 3x = 0$

$\Rightarrow \quad x(x - 3) = 0$

$\therefore \quad$ either $x = 0$ or $x = 3$

So, the required positive value of x is 3.

Example 5:

Solve for x: $10^{2x+4} = 5^{3x} \cdot 2^{x+8}$

Solution:

$10^{2x+4} = 5^{3x} \cdot 2^{x+8}$

$\Rightarrow \quad (2 \times 5)^{2x+4} = 5^{3x} \cdot 2^{x+8}$

$\Rightarrow \quad 2^{2x+4} \cdot 5^{2x+4} = 5^{3x} \cdot 2^{x+8}$

$\Rightarrow \quad \frac{2^{2x+4}}{2^{x+8}} = \frac{5^{3x}}{5^{2x+4}}$

$\Rightarrow \quad 2^{x-4} = 5^{x-4}$

$\Rightarrow \quad \frac{2^x}{2^4} = \frac{5^x}{5^4}$

$\Rightarrow \quad \frac{2^x}{5^x} = \frac{2^4}{5^4}$

$\Rightarrow \quad \left(\frac{2}{5}\right)^x = \left(\frac{2}{5}\right)^4$

$\Rightarrow \quad x = 4$ ($\because$ the bases are equal and greater than 0).

Comparing Indices

Case 1:

By taking HCF of powers and then dividing the powers by the HCF will help in comparing indices, in case powers of numbers are integers.

Example 6:

Which of the following numbers has the largest value?

(A) 3^{96} (B) 5^{48}

(C) 25^{24} (D) 2^{72}

Solution: (A)

HCF (92, 72, 48, 24) = 24

Now, $(3)^{\frac{96}{24}} = 3^4 = 81$

$(5)^{\frac{48}{24}} = 5^2 = 25$

$(25)^{\frac{24}{24}} = 25^1 = 25$

$(2)^{\frac{72}{24}} = 2^3 = 8$

Hence, 3^{96} is the largest.

Case 2:

If powers are fractions, then indices can be compared by multiplying each of the power by the LCM of denominators of fractions.

Example 7:

Which of the following is the smallest?

(A) $3^{\frac{1}{8}}$ (B) $4^{\frac{1}{12}}$

(C) $5^{\frac{1}{16}}$ (D) $9^{\frac{1}{24}}$

Solution: (D)

LCM (8, 12, 16, 24) = 48

Now, $(3)^{\frac{1}{8}\times 48} = 3^6 = 729$

$(4)^{\frac{1}{12}\times 48} = 4^4 = 256$

$(5)^{\frac{1}{16}\times 48} = 5^3 = 125$

$(9)^{\frac{1}{24}\times 48} = 9^2 = 81$

Hence, $9^{\frac{1}{24}}$ is the smallest value.

More Examples on the Comparison of Indices

Example 8:

Which is greater?

5^{3^4} or 7^{4^3}

Solution:

$5^{3^4} \neq 125^4$ and $5^{3^4} \neq 5^{3\times4}$

$5^{3^4} = 5^{81}$ (since $3^4 = 3 \times 3 \times 3 \times 3 = 81$)

$7^{4^3} = 7^{64}$ (since $4^3 = 4 \times 4 \times 4 = 64$)

5^{81} can be written as $5 \times 5^{80} = 5 \times (5^5)^{16} = 5 \times (3125)^{16}$.

7^{64} can be written as $(7^4)^{16} = (2401)^{16}$.

5^{81} is greater than 7^{64}.

Example 9:

Arrange the following numbers in ascending order.

$3^{5^4}, 4^{3^5}$, and 5^{3^4}

Solution:

$3^{5^4} = 3^{625}$ (since $5^4 = 625$)

$4^{3^5} = 4^{243}$ (since $3^5 = 243$)

$5^{3^4} = 5^{81}$ (since $3^4 = 81$)

3^{625} can be written as $(3^5)^{125} = (243)^{125}$.

4^{243} can be written as $(4^3)^{81} = (64)^{81}$.

It can be conducted that

$(243)^{125} > (64)^{81} > 5^{81}$

So, $3^{5^4} > 4^{3^5} > 5^{3^4}$.

Example 10:

Which is greater?

3^{4^5} or $(3^4)^5$

Solution:

3^{4^5} can be written as 3^{1024} (As $4^5 = 1024$).

$(3^4)^5$ can be written as $(3)^{4\times5} = (3)^{20}$.

Clearly, 3^{4^5} is greater than $(3^4)^5$.

Example 11:

Which of the following is the greatest?

$7^{\frac{1}{4}}, 3^{\frac{1}{6}}, 2^{\frac{1}{18}}, 5^{\frac{1}{12}}$

Solution:

LCM (4, 6, 8, 12) = 36

$\therefore \quad \left(7^{\frac{1}{4}}\right)^{36} = 7^9$

$\left(3^{\frac{1}{6}}\right)^{36} = 3^6$

$\left(2^{\frac{1}{18}}\right)^{36} = 2^2$

$\left(5^{\frac{1}{12}}\right)^{36} = 5^3$

$\Rightarrow$ Clearly, $7^9 > 3^6 > 5^3 > 2^2$

$\Rightarrow \quad 7^{\frac{1}{4}} > 3^{\frac{1}{6}} > 5^{\frac{1}{12}} > 2^{\frac{1}{18}}$

$\Rightarrow \quad 7^{\frac{1}{4}}$ is the greatest.

Example 12:

Arrange the following numbers in ascending order.

$\sqrt[3]{3}, \sqrt[4]{5}, \sqrt[6]{7}, \sqrt[8]{11}$

Solution:

$\sqrt[3]{3} = (3)^{\frac{1}{3}}$

$\sqrt[4]{5} = (5)^{\frac{1}{4}}$

$\sqrt[6]{7} = (7)^{\frac{1}{6}}$

$\sqrt[8]{11} = (11)^{\frac{1}{8}}$

The LCM of 3, 4, 6, and 8 is 24.

We have to make the power of each number 24.

$\left(3^{\frac{1}{3}}\right)^{24} = (3)^8 = (3^4)^2 = (81)^2$

$\left(5^{\frac{1}{4}}\right)^{24} = (5)^6 = (5^3)^2 = (125)^2$

$\left(7^{\frac{1}{6}}\right)^{24} = (7)^4 = (7^2)^2 = (49)^2$

$\left(11^{\frac{1}{8}}\right)^{24} = (11)^3 = \left(11\sqrt{11}\right)^2$

Clearly, $\sqrt{9} < \sqrt{11} < \sqrt{16}$.

$\Rightarrow \quad 3 < \sqrt{11} < 4$

$\therefore \quad \left(11\sqrt{11}\right)^2 = \left(11(3 \text{ to } 4)\right)^2 = \left(33 \text{ to } 44\right)^2$

So, the correct order is $(33 \text{ to } 44)^2 < (49)^2 < (81)^2 < (125)^2$

So, $\sqrt[8]{11} < \sqrt[6]{7} < \sqrt[3]{3} < \sqrt[4]{5}$.

Surds

Surds are such roots that cannot be reduced to give a whole number. They are irrational numbers.

Example of surds: $\sqrt{5}, 2+2\sqrt{5}, 2\sqrt{56}$

- If two surds are equal, then their rational parts are equal as well as their irrational parts are equal.

  For example, If $a+\sqrt{b}=3+8\sqrt{6}$

  To compare, we can write it as

  $$a+\sqrt{b}=3+\sqrt{8\times 8\times 6}$$

  $$a+\sqrt{b}=3+\sqrt{384}$$

Therefore, a = 3 and b = 384.

Conjugate of Surds

Conjugate is introduced to simplify surds.

$(a-\sqrt{b})$ is the conjugate of $(a+\sqrt{b})$ and $(a+\sqrt{b})$ is the conjugate of $(a-\sqrt{b})$.

For example, $(7-5\sqrt{6})$ is the conjugate of $(7+5\sqrt{6})$.

Rationalising a Surd

To handle expression of the form $\frac{k}{(\sqrt{a}\pm\sqrt{b})}$, where k represents any given expression. One of the most common procedures involves rationalising the denominator, where the supreme target is to convert the denominator into a rational number.

For example, consider the expression $\frac{1}{(\sqrt{5}-1)}$.

To rationalise the denominator, one ought to multiply both the numerator and denominator by the conjugate pair of the given denominator, which is $(\sqrt{5}+1)$ in this case.

So, $$\frac{1}{(\sqrt{5}-1)}=\frac{1(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}$$

$$=\frac{(\sqrt{5}+1)}{(\sqrt{5})^2-(1)^2}=\frac{(\sqrt{5}+1)}{(\sqrt{5})-(1)^2}=\frac{(\sqrt{5}+1)}{4}$$

Examples of rationalisation of a surd

Example 13:

Rationalise $\frac{1}{5+2\sqrt{2}}$

Solution:

Given term: $\frac{1}{5+2\sqrt{2}}$

Conjugate of $(5+2\sqrt{2})$ is $(5-2\sqrt{2})$.

Therefore, to simplify, we need to multiply and divide the given expression $(5-2\sqrt{2})$.

Hence, $\frac{1}{(5+2\sqrt{2})}\times\frac{(5-2\sqrt{2})}{(5+2\sqrt{2})}$

$$=\frac{(5-2\sqrt{2})}{(5)^2-(2\sqrt{2})^2}$$

[Since, $(a + b)(a - b) = a^2 - b^2$]

$$=\frac{5-2\sqrt{2}}{25-8}=\frac{5-2\sqrt{2}}{17}$$

Example 14:

If a and b are rational numbers and $a+\sqrt{b}=\frac{5+\sqrt{3}}{5-\sqrt{3}}$, find $a + 2b$.

Solution:

$$a+\sqrt{b}=\frac{5+\sqrt{3}}{5-\sqrt{3}}$$

On rationalisation,

$$a+\sqrt{b}=\frac{(5+\sqrt{3})(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})}$$

$$a+\sqrt{b}=\frac{(5)^2+(\sqrt{3})^2+2\times 5\times\sqrt{3}}{(5)^2-(\sqrt{3})^2}$$

[Using $(a + b)^2 = a^2 + b^2 + 2ab$ and $(a + b)(a - b) = a^2 - b^2$]

$$a+\sqrt{b}=\frac{25+3+10\sqrt{3}}{25-3}$$

$$a+\sqrt{b}=\frac{28+10\sqrt{3}}{22}$$

$$a+\sqrt{b}=\frac{28}{22}+\sqrt{\frac{300}{484}}$$

Now, $a=\frac{28}{22}$; $b=\frac{300}{484}$

Therefore, $a+2b=\frac{28}{22}+\frac{2\times 300}{484}$

$$a+2b=\frac{308+300}{242}$$

$$a+2b=\frac{608}{242}=\frac{304}{121}$$

Example 15:

$\dfrac{(\sqrt{32}+\sqrt{2})^2}{(\sqrt{8}-2)}$ can be written in the form of $a(b+\sqrt{2})$. If a and b are positive integers, the arithmetic mean of a and b yields ________.

(A) A prime number

(B) A composite number

(C) Neither prime nor composite

(D) An irrational number

Solution: (A)

$$(\sqrt{32}+\sqrt{2})^2=(\sqrt{32})^2+(\sqrt{2})^2+2(\sqrt{32})(\sqrt{2})$$

$$=34+2(\sqrt{64})=50$$

Therefore, $\dfrac{(\sqrt{32}+\sqrt{2})^2}{(\sqrt{8}-2)}=\dfrac{50}{(\sqrt{8}-2)}$

$$=\frac{50\left[2\left(\sqrt{2}+1\right)\right]}{4}$$

$$=\frac{100\left(\sqrt{2}+1\right)}{4}$$

$$=25(\sqrt{2}+1)$$

$$=25(1+\sqrt{2})$$

$$=a(b+\sqrt{2})$$

$\therefore$ a = 25 and b = 1.

So, the arithmetic mean of a and b is $\left(\dfrac{25+1}{2}\right)=\left(\dfrac{26}{2}\right)=13$.

13 is a prime number.

Hence, option (A) is the correct answer.

Comparison of Surds

To compare two surds, we can adopt the following approaches:

- Subtracting and adding same integer, so that we can compare easily.
- By squaring.
- We can also compare by finding their integral ranges.

Example 16:

If $a=\sqrt{7}+\sqrt{6}$ and $b=\sqrt{2}+\sqrt{11}$, which of a and b is greater?

Solution:

We have, $a=\sqrt{7}+\sqrt{6}$ and $b=\sqrt{2}+\sqrt{11}$.

On squaring: $a^2=13+2\sqrt{42}$ and $b^2=13+2\sqrt{22}$.

On comparing: clearly $\sqrt{42}>\sqrt{22}$.

Therefore, $a > b$.

Example 17:

If $a=\sqrt{4}+\sqrt{11}$ and $b=\sqrt{3}+\sqrt{13}$, find which of a and b is greater?

Solution:

We have, $a=\sqrt{4}+\sqrt{11}$ and $b=\sqrt{3}+\sqrt{13}$.

Now, $a^2=15+2\sqrt{44}$ and $b^2=16+2\sqrt{39}$.

Also, we can write the above expressions as

$a^2-15=2\sqrt{44}$ ...(i)

$b^2-15=1+2\sqrt{39}$ ...(ii)

Squaring equation (i) and (ii).

$(a^2 - 15)^2 = 176$ ...(iii)

$(b^2 - 15)^2 = 1 + 156 + 4\sqrt{39}=157+4\sqrt{39}$

Also, $6<\sqrt{39}<7$

So, $(b^2 - 15)^2$ = 157 + 4 (6 to 7) = 157 + (24 to 28) = 181 to 185 ...(iv)

From equations (iii) and (iv)

We can conclude: $a < b$.

Square Root of Surds

To determine the square root of a given surd, one of most commonly used techniques is completing the squares.

The example given below explains this technique in details.

Example 18:

Find the positive square root of $(21 + 8\sqrt{5})$.

Solution:

Method: Consider the expansion $(a^2 + 2ab + b^2)$ where the middle term of $2ab$ is represented by $8\sqrt{5}$.

Thus, $2ab = 2.4.\sqrt{5}$.

After comparing $2ab$ with the irrational part, now compare $(a^2 + b^2)$ with the rational part.

$\therefore$ $a^2 + b^2 = 21$ where $a = 4$ and $b = \sqrt{5}$.

Now, we can write, $21+8\sqrt{5}=\left(4+\sqrt{5}\right)^2$.

So, the square root of $21+8\sqrt{5}$

i.e., $\sqrt{21+8\sqrt{5}}=\sqrt{\left(4+\sqrt{5}\right)^2}=\left(4+\sqrt{5}\right)$

Example 19:

Determine the value of

$\sqrt{\left(\sqrt{21+8\sqrt{5}}\right)-\left(\sqrt{14+6\sqrt{5}}\right)}$.

Solution:

Using the method discussed above, let us first calculate the square roots of $14+6\sqrt{5}$ and $\left(21+8\sqrt{5}\right)$ individually.

$\therefore$ Look at the innermost root i.e., $\left(14+6\sqrt{5}\right)$.

Compare this with the expansion $(a^2 + 2ab + b^2)$ where the middle term of $2ab$ equals to $6\sqrt{5}$, i.e., $2ab = 6\sqrt{5} \Rightarrow ab = 3\sqrt{5}$.

After comparing $2ab$ with irrational part now compare $(a^2 + b^2)$ with the rational part.

$\therefore$ $a^2 + b^2 = 14$ where $a = 3$ and $b = \sqrt{5}$.

Now we can write,

$\sqrt{14+6\sqrt{5}}=\sqrt{9+5+2\times 3\times\sqrt{5}}$

$=\sqrt{(3)^2+\left(\sqrt{5}\right)^2+2(3)\left(\sqrt{5}\right)}$

$=\sqrt{\left(3+\sqrt{5}\right)^2}$

$=\left(3+\sqrt{5}\right)$

Similarly, we can also obtain $\left(21+8\sqrt{5}\right)=\left(4+\sqrt{5}\right)$ (as shown in the previous example).

Thus, $\sqrt{\left(\sqrt{21+8\sqrt{5}}\right)-\left(\sqrt{14+6\sqrt{5}}\right)}$

$=\sqrt{\left(4+\sqrt{5}\right)-\left(3+\sqrt{5}\right)}$

$=\sqrt{4+\sqrt{5}-3-\sqrt{5}}$

$=\sqrt{1}=1$

Therefore, the given expression $\sqrt{\left(\sqrt{21+8\sqrt{5}}\right)-\left(\sqrt{14+6\sqrt{5}}\right)}$ simplifies to 1.

Example 20:

Find the value of

$$\frac{51}{\sqrt{75+12\sqrt{21}}}-\frac{4}{\sqrt{28+16\sqrt{3}}}-\frac{38}{\sqrt{88+30\sqrt{7}}}.$$

Solution:

$\left(75+12\sqrt{21}\right)=75+2(6)\sqrt{7}\cdot\sqrt{3}$

$=63+12+2\left(3\sqrt{7}\right)\left(2\sqrt{3}\right)$

$=\left(3\sqrt{7}\right)^2+\left(2\sqrt{3}\right)^2+2\left(3\sqrt{7}\right)\left(2\sqrt{3}\right)$

$=\left(3\sqrt{7}+2\sqrt{3}\right)^2$

$\Rightarrow\ \sqrt{75+12\sqrt{21}}=\sqrt{\left(3\sqrt{7}+2\sqrt{3}\right)^2}=\left(3\sqrt{7}+2\sqrt{3}\right)$

$\left(28+16\sqrt{3}\right)=28+2\times 8\sqrt{3}$

$=16+12+2\times 4\times 2\sqrt{3}$

$=(4)^2+\left(2\sqrt{3}\right)^2+2(4)\left(2\sqrt{3}\right)$

$=\left(4+2\sqrt{3}\right)^2$

$\Rightarrow\ \sqrt{28+16\sqrt{3}}=\sqrt{\left(4+2\sqrt{3}\right)^2}=\left(4+2\sqrt{3}\right)^2$

$88+30\sqrt{7}=88+2\times 15\sqrt{7}$

$=63+25+2\left(3\sqrt{7}\right)(5)$

$=\left(3\sqrt{7}\right)^2+(5)^2+2\left(3\sqrt{7}\right)(5)$

$=\left(3\sqrt{7}+5\right)^2$

$\Rightarrow\ \sqrt{88+30\sqrt{7}}=\sqrt{\left(3\sqrt{7}+5\right)^2}=\left(3\sqrt{7}+5\right)$

So, the value of the expression is

$$\frac{51}{\left(3\sqrt{7}+2\sqrt{3}\right)}-\frac{4}{\left(4+2\sqrt{3}\right)}-\frac{38}{\left(3\sqrt{7}+5\right)}$$

$$=\frac{51\left(3\sqrt{7}-2\sqrt{3}\right)}{\left(3\sqrt{7}+2\sqrt{3}\right)\left(3\sqrt{7}-2\sqrt{3}\right)}-\frac{4\left(4-2\sqrt{3}\right)}{\left(4+2\sqrt{3}\right)\left(4-2\sqrt{3}\right)}-\frac{38\left(3\sqrt{7}-5\right)}{\left(3\sqrt{7}+5\right)\left(3\sqrt{7}-5\right)}$$

$$=\frac{51\left(3\sqrt{7}-2\sqrt{3}\right)}{(63-12)}-\frac{4\left(4-2\sqrt{3}\right)}{(16-12)}-\frac{38\left(3\sqrt{7}-5\right)}{(63-25)}$$

$$=\frac{51\left(3\sqrt{7}-2\sqrt{3}\right)}{51}-\frac{4\left(4-2\sqrt{3}\right)}{4}-\frac{38\left(3\sqrt{7}-5\right)}{38}$$

$=3\sqrt{7}-2\sqrt{3}-4+2\sqrt{3}-3\sqrt{7}+5 = 1$

Practice Exercise – 1

Level of Difficulty – 1

1. If $2^{2^{3^{2^{2}}}} = 8^{8x}$, find the value of x.

(A) $\dfrac{2^{70}}{3}$

(B) $\dfrac{2^{74}}{3}$

(C) $\dfrac{2^{78}}{3}$

(D) $\dfrac{2^{84}}{3}$

2. If $A = 2^{101}$ and $B = 2^{100} + 2^{99} + 2^{98} + 2^{97} + \ldots + 2^2 + 2^1 + 2^0$, then determine the relationship between A and B.

(A) $A > B$
(B) $A < B$
(C) $A = B$
(D) Cannot be determined

3. Find the value of $x^3 - 3x^2 + 5x - 288$, when $x = 16^{\frac{1}{4}} + 16^{\frac{2}{4}} + 16^{\frac{3}{4}}$.

(A) 1,868
(B) 1,938
(C) 2,226
(D) 2,814

4. If $u + \sqrt{v} = \left(5\sqrt{625}\right)^{1/3}$ then how many positive integral ordered pairs of u and v, respectively, are possible?

(A) 4
(B) 2
(C) 3
(D) 6

5. Find the positive square root of x, if $\dfrac{\sqrt{x}}{49} = \dfrac{9317}{x}$.

(A) 13
(B) $\sqrt{21}$
(C) 77
(D) 9

Level of Difficulty – 2

6. If $a = (63a^3 + 64)^{1/6}$ then how many positive integral values of a exist?

(A) 1
(B) 2
(C) 3
(D) More than 3

7. If $x = 9^{90} - 1$, $y = 9^{135} - 1$, $z = 9^{180} - 1$, then find the HCF (x, y, z).

(A) 9^{45}
(B) $2(9^{45}) - 1$
(C) 2
(D) $(9^{45}) - 1$

8. If $a = \sqrt{10} + \sqrt{13}$ and $b = \sqrt{3} + \sqrt{22}$, then find the relationship between a and b.

(A) $a > b$
(B) $a < b$
(C) $a = b$
(D) Cannot be determined

9. $\dfrac{\left(\sqrt{8}+1\right)}{\left(3-\sqrt{2}\right)^2}$ can be written in the form of $\left(a + b\sqrt{2}\right)$. Find the value of $\left(\dfrac{a}{b}\right)$.

(A) $\dfrac{4}{5}$

(B) $\dfrac{5}{4}$

(C) 1

(D) None of these

Level of Difficulty – 3

10. If a, b, and c are distinct positive real numbers such that $\left(a + \dfrac{1}{b}\right) = \left(b + \dfrac{1}{c}\right) = \left(c + \dfrac{1}{a}\right)$, find the positive square root of $(a \times b \times c)^2$.

(A) 1
(B) $\sqrt{2}$
(C) $\sqrt{3}$
(D) None of these

11. If $x = \dfrac{\sqrt{n+2} - \sqrt{n}}{\sqrt{n+2} + \sqrt{n}}$ and $y = \dfrac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} - \sqrt{n}}$,

where, n is a positive integer and $18x^2 + 34xy + 18y^2 = 3{,}526$, the value of n is:

12. Real numbers p and q satisfy the equation $3^p = 81^{q-2}$ and $125^q = 5^{p+3}$. What is the area of a triangle (in square units) having its sides as p units, q units, and 13 units?

(A) 30

(B) 40

(C) 25

(D) 60

13. If $\sqrt[3]{5^a \times 20^{b+1} \times 30^{c+2}}$ is a whole number, then which one of the given values of a, b, and c are consistent with it?

(A) $a = 1$, $b = 2$, $c = 2$

(B) $a = 1$, $b = 2$, $c = 3$

(C) $a = 3$, $b = 2$, $c = 4$

(D) $a = 1$, $b = 2$, $c = 5$

14. If $m = -\dfrac{1}{9}$, then which of the following option is correct?

(A) $3^{\frac{1}{m}} > 3^m$

(B) $-\dfrac{1}{m} < \dfrac{1}{\sqrt{-m}}$

(C) $\dfrac{3}{\sqrt[3]{-3m}} < 3^{-2m}$

(D) $\dfrac{-1}{3m} > 3^{-3m}$

15. The number of real values of x that satisfy the equation $\sqrt[3]{30x + \sqrt[3]{30x + 23}} = 23$ are:

(A) 1

(B) 2

(C) 3

(D) More than 3

Solutions

1. **(C)**

$$2^{2^{3^{2^2}}} = 8^{8x}$$

The order of evaluating the power of exponents is top to bottom.

$2^{2^{3^4}} = 8^{8x}$ $\Rightarrow$ $2^{2^{81}} = (2^3)^{8x}$

Equating the indices on the two sides, we get

$2^{81} = 24x$

$\Rightarrow \quad x = \dfrac{2^{81}}{24} = \dfrac{2^{78}}{3}$

Option (C) is the answer.

2. **(A)**

$B = 2^{100} + 2^{99} + 2^{98} + 2^{97} + \ldots\ldots\ldots + 2^2 + 2^1 + 2^0$

$= 2^0 + 2^1 + 2^2 + 2^3 + \ldots\ldots\ldots + 2^{100}$

B is in GP, where the common ratio (r) is 2, and the number of terms (n) is 101.

$\therefore$ The sum to n terms is $S_n = \dfrac{a(r^n - 1)}{(r-1)}$

where a is the first term ($= 2^0$).

$\Rightarrow \quad S_n = \dfrac{2^0(2^{101} - 1)}{(2-1)}$

$\Rightarrow \quad S_n = 1(2^{101} - 1)$

$\Rightarrow \quad S_n = 1\,(A - 1)$ (Since $A = 2^{101}$)

$\therefore \quad B = A - 1$

This implies that A is greater than B by 1.

$\Rightarrow \quad A > B$.

Therefore, option (A) is the correct answer.

3. **(B)**

$x = 16^{\frac{1}{4}} + 16^{\frac{2}{4}} + 16^{\frac{3}{4}} = (2^4)^{1/4} + (2^4)^{2/4} + (2^4)^{3/4}$

$= 2 + 4 + 8 = 14$

Let $f(x) = x^3 + 3x^2 + 5x - 288$.

Putting $x = 14$ in $f(x)$ we get,

$\Rightarrow f(x) = (14)^3 - 3(14)^2 + 5(14) - 288$

$= 2{,}744 - 588 + 70 - 288 = 1{,}938$.

Therefore, option (B) is the correct answer.

4. **(D)**

$u + \sqrt{v} = \left(5\sqrt{625}\right)^{1/3} = (5 \times 25)^{1/3} = 5$

$\Rightarrow u + \sqrt{v} = 5$

Possible values of u and v:

I. $u = 5$, $v = 0$

II. $u = 4$, $v = 1$

III. $u = 3$, $v = 4$

IV. $u = 2$, $v = 9$

V. $u = 1, v = 16$

VI. $u = 0, v = 25$

Thus, a total of six ordered pairs of u and v are possible.

Therefore, option (D) is the correct answer.

5. (C)

$\sqrt{x} \times x = 9317 \times 49$

$x^{3/2} = 7 \times 1331 \times 49$

$x^{3/2} = 1331 \times 343$

$x^{3/2} = (11)^3 \times (7)^3$

$x^{3/2} = (11 \times 7)^3$

$x^{1/2} = 11 \times 7$

$\sqrt{x} = 77$

Therefore, opion (C) is the correct answer.

6. (A)

Given: $a = (63a^3 + 64)^{1/6}$

$\Rightarrow a^6 = 63a^3 + 64$

$\Rightarrow a^6 - 63a^3 - 64 = 0$

$\Rightarrow a^6 - 64a^3 + a^3 - 64 = 0$

$\Rightarrow a^3(a^3 - 64) + 1(a^3 - 64) = 0$

$\Rightarrow (a^3 + 1)(a^3 - 64) = 0$

Either $(a^3 + 1) = 0$ or, $(a^3 - 64) = 0$

$\Rightarrow a^3 = -1$ $\quad\Rightarrow a^3 = 64$

$\Rightarrow a^3 = 4^3$

$\Rightarrow a = 4$

According to the question, a can take only positive integral values.

Thus, $a = 4$ is only admissible.

Hence, only one value of a exists.

Therefore, option (A) is the correct answer.

7. (D)

Given, $x = 9^{90} - 1$

$y = 9^{135} - 1$

$z = 9^{180} - 1$

HCF (90, 135, 180) = 45

So, we can rewrite x, y, and z as

$x = (9^{45})^2 - 1$

$y = (9^{45})^3 - 1$

$z = (9^{45})^4 - 1$

Let, $k = 9^{45}$

Then, $x = (k^2 - 1)$, $y = (k^3 - 1)$ and $z = (k^4 - 1)$

All of them are divisible by $(k - 1)$ as we know that $(a^n - b^n)$ is always divisible by $(a - b)$.

$\therefore$ HCF $(x, y, z) = (9^{45}) - 1$

Therefore, option (D) is the correct answer.

8. (A)

$a = \sqrt{10} + \sqrt{13} \qquad b = \sqrt{3} + \sqrt{22}$

$\Rightarrow a^2 = 10 + 13 + 2\sqrt{130}$

$\Rightarrow b^2 = 3 + 22 + 2\sqrt{66}$

$\Rightarrow a^2 = 23 + 2\sqrt{130}$

$\Rightarrow b^2 = 25 + 2\sqrt{66}$

$\Rightarrow (a^2 - 23) = 2\sqrt{130}$

$\Rightarrow b^2 - 23 = 2 + 2\sqrt{66}$

$\Rightarrow (a^2 - 23)^2 = 4(130)$

$\Rightarrow (b^2 - 23)^2 = 4 + 264 + 8\sqrt{66}$

$\Rightarrow (a^2 - 23)^2 = 520$

$\Rightarrow (b^2 - 23)^2 = 268 + 8\sqrt{66}$

Clearly, $\sqrt{64} < \sqrt{66} < \sqrt{81}$

$\Rightarrow 8 < \sqrt{66} < 9$

So, $(b^2 - 23)^2 = 268 + 8\sqrt{66}$

$= 268 + 8(8 \text{ to } 9) = 268 + (64 \text{ to } 72)$

$= 332 \text{ to } 340$

It can be determined easily that $(a^2 - 23)^2 > (b^2 - 23)^2$

$\Rightarrow a > b$

Hence, option (A) is the correct answer.

9. (B)

$\left(3 - \sqrt{2}\right)^2 = (3)^2 + (\sqrt{2})^2 - 2(3)(\sqrt{2}) = (11 - 6\sqrt{2})$

Therefore,

$$\frac{\left(\sqrt{8}+1\right)}{\left(3-\sqrt{2}\right)^2} = \frac{\left(\sqrt{8}+1\right)}{\left(11-6\sqrt{2}\right)} = \frac{\left(\sqrt{8}+1\right)\left(11+6\sqrt{2}\right)}{\left(11-6\sqrt{2}\right)\left(11+6\sqrt{2}\right)}$$

$$= \frac{28\sqrt{2}+35}{49} = \frac{35}{49} + \frac{28\sqrt{2}}{49} = \frac{5}{7} + \frac{4}{7}\sqrt{2}$$

$\therefore \quad a = \frac{5}{7}$ and $b = \frac{4}{7}$

Therefore, $\frac{a}{b} = \frac{\left(\frac{5}{7}\right)}{\left(\frac{4}{7}\right)} = \frac{5}{4}$.

Therefore, option (B) is the correct answer.

10. (A)

a, b, and c are distinct real numbers.

$\therefore \quad a \neq b \neq c$

So, $a \neq b$ or $(a - b) \neq 0$

$b \neq c$ or $(b - c) \neq 0$

$a \neq c$ or $(a - c) \neq 0$

$$a+\frac{1}{b}=b+\frac{1}{c}$$

$$\Rightarrow \quad (a-b)=\left(\frac{1}{c}-\frac{1}{b}\right)$$

$$\Rightarrow \quad (a-b)=\left(\frac{b-c}{bc}\right)$$

$$\Rightarrow \quad bc=\left(\frac{b-c}{a-b}\right) \quad ...(i)$$

Also, $b+\frac{1}{c}=c+\frac{1}{a}$

$$\Rightarrow \quad (b-c)=\frac{1}{a}-\frac{1}{c}$$

$$\Rightarrow \quad (b-c)=\left(\frac{c-a}{ac}\right)$$

$$\Rightarrow \quad ac=\left(\frac{c-a}{b-c}\right) \quad ...(ii)$$

Similarly, $\left(a+\frac{1}{b}\right)=\left(c+\frac{1}{a}\right)$

$$\Rightarrow \quad (a-c)=\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\Rightarrow \quad (a-c)=\left(\frac{b-a}{ab}\right)$$

$$\Rightarrow \quad ab=\left(\frac{b-a}{a-c}\right) \quad ...(iii)$$

Multiplying equations (i)–(iii), we get

$$bc\times ac\times ab=\left(\frac{b-c}{a-b}\right)\times\left(\frac{c-a}{b-c}\right)\times\left(\frac{b-a}{a-c}\right)$$

$$\Rightarrow \quad (abc)^2=\left(\frac{b-c}{a-b}\right)\times\left(\frac{c-a}{b-c}\right)\times\left\{\frac{-(a-b)}{-(c-a)}\right\}$$

$\Rightarrow \quad (abc)^2 = 1$

The required positive square root of $(a \times b \times c)^2$ is $\sqrt{(abc)^2}=\sqrt{1} = 1$.

Thus, option (A) is the correct answer.

11. 6

$$x=\frac{\sqrt{n+2}-\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}=\frac{\left(\sqrt{n+2}-\sqrt{n}\right)\left(\sqrt{n+2}-\sqrt{n}\right)}{\left(\sqrt{n+2}+\sqrt{n}\right)\left(\sqrt{n+2}-\sqrt{n}\right)}$$

$$=n+1-\sqrt{n^2+2n}$$

Similarly, $y = n + 1 + \sqrt{2n+n^2}$

$\therefore \quad x + y = 2(n + 1)$ and $xy = 1$

$18x^2 + 34xy + 18y^2 = 3,526$

$18[x + y]^2 - 2xy = 3,526$

$18\ (2(n + 1))^2 - 2\ (1) = 3,526$

$(n + 1)^2 = 49$

$\Rightarrow n = 6\ (n > 0)$

12. (A)

$3^p=81^{q-2}$

$\Rightarrow\ 3^p=\left(3^4\right)^{q-2}$

$\Rightarrow\ p=4q-8 \qquad ...(i)$

Similarly

$125^q=5^{p+3}$

$\Rightarrow\ 5^{3q}=5^{p+3}$

$\Rightarrow\ 3q=p+3 \qquad ...(ii)$

Putting the value of p from equation (i) in equation (ii), we get

$3q=4q-8+3$

$\Rightarrow q = 5$

So, $p=4\ q-8=4\times5-8=12.$

Since the sides of a triangle are 12, 5, and 13 so, it is a right-angle triangle.

Thus, its area is $1/2\times12\times5=30$ square units.

Therefore, option (A) is the correct answer.

13. (C)

First of all, we have to simplify the expression $\sqrt[3]{5^a\times20^{b+1}\times30^{c+2}}$

$=\sqrt[3]{5^a\times(2^2\times5)^{b+1}\times(2\times3\times5)^{c+2}}$

$=\sqrt[3]{5^{a+b+1+c+2}\times2^{2b+2+c+2}\times3^{c+2}}$

$=\sqrt[3]{2^{2b+c+4}\times3^{c+2}\times5^{a+b+c+3}}$

$=2^{\frac{2b+c+4}{3}}\times3^{\frac{c+2}{3}}\times5^{\frac{a+b+c+3}{3}}$

We have to check the power or exponent of all the numbers, whether they are whole numbers or not, by using options.

$\therefore$ Option (A): where $a = 1$, $b = 2$, and $c = 2$

$\therefore\ 2^{\frac{2+2+2+4}{3}}\times3^{\frac{4}{3}}\times5^{\frac{8}{3}}$

$=\ 2^{\frac{10}{3}}\times3^{\frac{4}{3}}\times5^{\frac{8}{3}}$, which is not a whole number.

From option (B): where $a = 1$, $b = 2$, and $c = 3$ is given

$\therefore\ 2^{\frac{2b+c+4}{3}}\times3^{\frac{c+2}{3}}\times5^{\frac{a+b+c+3}{3}}$

$= 2^{\frac{11}{3}} \times 3^{\frac{5}{3}} \times 5^{\frac{9}{3}}$

$= 2^{11/3} \times 3^{5/3} \times 5^3$ does not yield a whole number.

Thus, by option (B) also, we are not getting a whole number.

Now we have to check for option (C): where $a = 3$, $b = 2$, and $c = 4$.

Then, $2^{\frac{2b+c+4}{3}} \times 3^{\frac{c+2}{3}} \times 5^{\frac{a+b+c+3}{3}}$

$= 2^{\frac{12}{3}} \times 3^{\frac{6}{3}} \times 5^{\frac{12}{3}}$

$= 2^4 \times 3^2 \times 5^4$ gives a whole number.

For option (D): where $a = 1$, $b = 2$, and $c = 5$.

From option (D), we will not get a whole number.

Therefore, option (C) is the correct answer.

14. (D)

Let's solve it by options.

Option 1:

$3^{\frac{1}{m}} > 3^m$ and $m = -\frac{1}{9}$ (is given) plug the value in the above condition.

$\Rightarrow \frac{1}{m} = -9$

$3^{\frac{1}{m}} > 3^m$

$3^{-9} > 3^{-1/9}$

$\frac{1}{3^9} > \frac{1}{3^{\frac{1}{9}}}$

Now, compare the denominators.

$3^9 > 3^{\frac{1}{9}}$

Hence, $\frac{1}{3^{\frac{1}{9}}} > \frac{1}{3^9}$

Hence, option (A) is wrong.

Option 2:

$-\frac{1}{m} < \frac{1}{\sqrt{-m}}$

$\text{LHS} = -\frac{1}{m}\left[m = -\frac{1}{9}\text{is given in question}\right]$

$= 9$

$\text{RHM} = \frac{1}{\sqrt{-m}} = \frac{1}{\sqrt{-\left(-\frac{1}{9}\right)}} = \frac{1}{\sqrt{\frac{1}{9}}} = 3$

Option (B) is also not correct because *LHS > RHS*.

Option 3:

$\frac{3}{\sqrt[3]{-3m}} < 3^{-2m}$ $\left[\text{Put } m = -\frac{1}{9}\right]$

$\frac{3}{\sqrt[3]{-3 \times \left(-\frac{1}{9}\right)}} < 3^{-2\times\left(-\frac{1}{9}\right)}$

$3^{(4/3)} < 3^{(2/9)}$

This option is also not correct because $3^4 > 3^{\frac{2}{9}}$.

Option 4:

$-\frac{1}{3m} > 3^{-3m}$

$-\frac{1}{m} \times \frac{1}{3} > 3^{-3\times m}\left[\text{Plug the value } m = -\frac{1}{9}\right]$

$9 \times \frac{1}{3} > 3^{-3\times\left(-\frac{1}{9}\right)}$

$3 > 3^{\frac{1}{3}}$ LHS > RHS

Hence, option (D) is the correct answer.

15. (A)

Let $f(a) = \sqrt[3]{30x + a}$.

We can think of x as a constant and a as a variable.

$f(a)$ is an increasing function of a.

The given equation can be written as

$f(f(23)) = 23$ ...(i)

If $f(23) > 23$, then $f(f(23)) > f(23) > 23$.

Similarly, if $f(23) < 23$, then $f(f(23)) < f(23) < 23$.

Only if $f(23) = 23$, equation (i) is satisfied $\sqrt[3]{30x + 23} = 23$.

Thus, x has only one possible real value.

Therefore, option (A) is the correct answer.

Practice Exercise – 2

Level of Difficulty – 1

1. a and b are positive numbers such that $a^b = b^a$ and $b = 28a$. The value of a is:

(A) $\sqrt[8]{28}$

(B) $\sqrt[2]{28}$

(C) $\sqrt[14]{28}$

(D) $\sqrt[27]{28}$

2. If $\left[\dfrac{(729)^{\frac{2}{3}}}{(343)^{\frac{2}{3}}} \times \dfrac{(2401)^{\frac{5}{2}}}{(81)^{\frac{5}{2}}} \times \dfrac{(49)^{\frac{3}{2}}}{(729)^{\frac{5}{6}}}\right] = \left(\dfrac{7}{3}\right)^k$ then find the value of k.

3. If K is a positive integer such that $\{\left(\sqrt[5]{9}\right)^1\left(\sqrt[5]{9}\right)^2\left(\sqrt[5]{9}\right)^3 \ldots\ldots\ldots\ldots \left(\sqrt[5]{9}\right)^K\} > 728$, then find the smallest value of K.

(A) 7

(B) 3

(C) 5

(D) 9

4. If $x = 3 + 7^{2/3} + 7^{1/3}$, then the value of $(x^3 - 9x^2 + 6x)$ is:

5. If $(0.777)^x = (77.7)^y = \dfrac{1}{10{,}000}$, then the value of $\left(\dfrac{1}{x} - \dfrac{1}{y}\right)$ is:

(A) $\dfrac{1}{2}$

(B) $-\dfrac{1}{2}$

(C) $\dfrac{1}{4}$

(D) $-\dfrac{1}{4}$

6. If $x + \dfrac{1}{x} = \sqrt{11}$, then find the value of $x^{11} - 79x^7 + x^3$.

(A) 1

(B) 9

(C) 79

(D) None of these

7. If $x = \sqrt{14 + 6\sqrt{5}}$, then $x + \dfrac{1}{x}$ is:

(A) $\dfrac{5\sqrt{5}}{4}(\sqrt{5} + 1)$

(B) $\dfrac{3\sqrt{5}}{4}(2\sqrt{5} + 1)$

(C) $\dfrac{3\sqrt{5}}{4}(\sqrt{5} - 1)$

(D) $\dfrac{3\sqrt{5}}{4}(\sqrt{5} + 1)$

8. $3^{6+12+18+24+\ldots+6y} = \left(\dfrac{1}{729}\right)^{-66}$, what is the value of y?

(A) 11

(B) 10

(C) 12

(D) 14

9. If a, b, and c are non-zero and $10^a = 16^b = 40^c$ then $5b\left(\dfrac{1}{c} - \dfrac{1}{a}\right)$ is equal to:

(A) 10

(B) 7.5

(C) 5

(D) 2.5

10. If $7^{x+1} = 49^y$ and $8^x = 128^y \times 16$, then find the value of 2^{x+y}.

(A) 2^{-22}

(B) 2^{22}

(C) 2^{-12}

(D) 2^{-9}

Level of Difficulty – 2

11. If $\sqrt{320\sqrt{6} + 384\sqrt{10} + 240\sqrt{15} + 1532} = p\sqrt{2} + q\sqrt{3} + r\sqrt{5},$ where, p, q, and r are positive integers. Find the value of p.

12. Arrange the following numbers in ascending order.

I. $2^{3^{2^3}}$

II. 23^{2^3}

III. 23^{23}

IV. 2^{323}

V. $2^{3^{23}}$

(A) III, II, IV, V, I
(B) II, III, IV, I, V
(C) II, III, IV, V, I
(D) II, III, I, IV, V

13. If $A=\sqrt{6}+\sqrt{26}$, $B=\sqrt{8}+\sqrt{22}$, $C=\sqrt{13}+\sqrt{19}$, and $D=\sqrt{11}+\sqrt{16}$ then which among the following is true?
(A) $A > C > B > D$
(B) $C > A > D > B$
(C) $C > B > A > D$
(D) $C > A > B > D$

14. There are 2 sets, A and B. A consists of all integers n such that $-100 \le \sqrt[3]{n}-3 \le 100$. B consists of the squares of A's elements. How many elements are common in A and B?

15. If $K = \sqrt{3} - 2$, then the value of $K^4 + 8K^3 + 24K^2 + 32K + 32$ is:
(A) 9
(B) 81
(C) 16
(D) 25

16. If x, y, and z are rational numbers are $\frac{1}{\left(\sqrt[3]{4}-\sqrt[3]{2}+1\right)}=x\sqrt[3]{7}+y\sqrt[3]{2}+z$, then find $\frac{1}{2}(x\times y\times z)$.
(A) 1
(B) 2
(C) 0
(D) -1

17. If $x=\sqrt[6]{5+2\sqrt{6}}$ and $y=\sqrt[6]{5-2\sqrt{6}}$, then the value of $(x+y)\left(x^2+y^2+2xy-3\right)$ is:
(A) $2\sqrt{2}$
(B) $2\sqrt{3}$
(C) $2\sqrt{6}$
(D) 10

18. If $(11^A \times 55^{(B+1)} \times 45^{(C+2)})^{1/3}$ is a whole number, then which of the following statements below is consistent with it?
(A) $a = 4$, $b = 5$, $c = 1$
(B) $a = 3$, $b = 5$, $c = 4$
(C) $a = 3$, $b = 5$, $c = 3$
(D) $a = 7$, $b = 6$, $c = 5$

19. If $K=\sqrt{35-19\sqrt{3}+\sqrt{7-4\sqrt{3}}}$, then which of the following is true about the value of K? (Take $\sqrt{3}=1.73$).
(A) $0 < K < 1$
(B) $1 < K < 2$
(C) $2 < K < 3$
(D) $3 < K < 4$

20. If $p^x = q^y = r^z$ and p, q, r are in GP than the value of $\left(\frac{xy+zy}{16xz}\right)$ is:
(A) 2^{-1}
(B) 2
(C) 2^{-2}
(D) None of the above

Level of Difficulty – 3

21. If $x^3 - 6x^2 + 3x - 2 = 0$, then which of the following could be a value of x?
(A) $3^{2/3} + 3^{1/3}$
(B) $3^{2/3} + 3^{1/3} + 1$
(C) $3^{2/3} + 3^{1/3} + 2$
(D) $3^{2/3} + 3^{1/3} - 1$

22. If $A = 2^{2^{2^2}}$, $B = 2^{22^2}$, $C = 2^{2^{22}}$, and $D = 2^{222}$, then their correct descending order is:
(A) $D > B > C > A$
(B) $D > C > B > A$
(C) $C > B > D > A$
(D) $C > A > D > B$

23. If $9^K + 15^K = 25^K$, then positive value of $3\left[2\left(\frac{5}{3}\right)^K-\sqrt{5}\right]$ yields:
(A) A prime number
(B) A composite number
(C) Neither prime nor composite
(D) None of these

24. If $x=\left(5\sqrt{5}+11\right)$, then what is the value of $\frac{\left(5\sqrt{5}-11\right)}{4(123+55\sqrt{5})}$ in terms of x?

(A) $\frac{1}{x^2}$

(B) $\frac{x^3}{4}$

(C) $\frac{4}{x^3}$

(D) $\frac{2}{x^3}$

25. In the given equation $(3+2\sqrt{2})^{x^3-7}+(3-2\sqrt{2})^{x^3-7}=6$, how many positive integral values of x are possible? (Given that x is a positive real number).

(A) 1
(B) 2
(C) 3
(D) None of these

26. If $x=\frac{8}{(\sqrt{5}+\sqrt{3})}$, $y=\frac{12}{(\sqrt{5}+\sqrt{2})}$, and $z=\frac{4}{(\sqrt{3}+\sqrt{3})}$, then find the value of $144\left(\frac{x}{yz}+\frac{y}{xz}+\frac{z}{xy}\right)$.

(A) $(\sqrt{5}+4\sqrt{2}+9\sqrt{5})$

(B) $(\sqrt{5}+4\sqrt{2}+9\sqrt{3})$

(C) $12(\sqrt{5}+4\sqrt{2}+9\sqrt{3})$

(D) $12(\sqrt{2}+4\sqrt{5}+9\sqrt{6})$

27. If $X=\sqrt{(70-16\sqrt{6})^{\frac{1}{2}}+(21+6\sqrt{10})^{\frac{1}{2}}}$, the value of X is necessarily:

(A) $2^{\frac{1}{2}}(\sqrt{15}-1)$

(B) $2^{\frac{1}{2}}(\sqrt{15}+1)$

(C) $2^{\frac{-1}{2}}(\sqrt{15}-1)$

(D) $2^{\frac{-1}{2}}(\sqrt{15}+1)$

28. If $x^2-\sqrt{3}x-1=0$, then the value of the expression $x^{10}-22x^6-23x^2$ is:

(A) −10
(B) −5
(C) −3
(D) −7

29. If $a=\sqrt{18}+\sqrt{12}$, $b=\frac{10}{\sqrt{20}-\sqrt{10}}$, $c=\sqrt{19}+\sqrt{11}$, and $d=\frac{4}{\sqrt{17}-\sqrt{13}}$, then which of the following is true?

(A) $d > c > a > b$
(B) $c > a > d > b$
(C) $d > a > c > b$
(D) $b > d > c > a$

30. The number of real roots of the equation $5^{\frac{8^3}{88x}+\frac{5^3}{11x}+\frac{x^3}{11x}}=25$ is:

Solutions

1. (D)

Since $a^b = b^a$ (is given).

Putting the value of $b = 28a$ in the equation $a^b = b^a$, we get

$a^{28a} = (28a)^a$

$(a^a)^{28} = 28^a \cdot a^a$

$$\frac{(a^a)^{28}}{a^a} = 28^a$$

$(a^a)^{(28-1)} = 28^a$

$(a^a)^{27} = 28^a$

$(a^{27})^a = 28^a$

$a^{27} = 28$

$a = \sqrt[27]{28}$

Hence, option (D) is the correct answer.

2. 11

$$\left[\frac{(3^6)^{\frac{2}{3}}}{(7^3)^{\frac{2}{3}}} \times \frac{(7^4)^{\frac{5}{2}}}{(3^4)^{\frac{5}{2}}} \times \frac{(7^2)^{\frac{3}{2}}}{(3^6)^{\frac{5}{6}}}\right] = \left(\frac{7}{3}\right)^k$$

$$\left[\frac{3^4}{7^2} \times \frac{7^{10}}{3^{10}} \times \frac{7^3}{3^5}\right] = \left(\frac{7}{3}\right)^k$$

$$\frac{7^{11}}{3^{11}} = \left(\frac{7}{3}\right)^{11} = \left(\frac{7}{3}\right)^k$$

$\Rightarrow \quad k = 11$

3. (C)

Since the given inequality is:

$$\{(\sqrt[5]{9})^1 (\sqrt[5]{9})^2 (\sqrt[5]{9})^3 \ldots\ldots\ldots\ldots (\sqrt[5]{9})^K\} > 728$$

$$\therefore \quad (\sqrt[5]{9})^{1+2+3+\ldots+k} > 728$$

$$9^{\left(\frac{1+2+3+\ldots+K}{5}\right)} > 728$$

Since we have to find the minimum value of K.

$$\therefore \quad \frac{1+2+3+\ldots+K}{5} = 3$$

Thus, $1 + 2 + 3 + \ldots + K = 15$.

If we consider $K = 5$.

Then, $1 + 2 + 3 + 4 + 5 = 15$

Therefore, the minimum value of K is 5.

Hence, option (C) is the correct answer.

4. 20

$x = 3 + 7^{2/3} + 7^{1/3}$

$(x - 3) = 7^{2/3} + 7^{1/3}$

Taking cube on both sides of the equation, we get

$\Rightarrow \quad x^3 - 27 - 9x(x - 3) = (7^{2/3})^3 + (7^{1/3})^3 + 3 \times 7^{2/3} \times 7^{1/3} (7^{2/3} + 7^{1/3})$

$\Rightarrow \quad x^3 - 27 - 9x^2 + 27x = 49 + 7 + 21(x - 3)$

$\Rightarrow \quad x^3 - 9x^2 + 6x = 20$

5. (A)

$(0.777)^x = (77.7)^y = 10^{-4}$

$\Rightarrow 0.777 = 10^{-4/x}$

$77.7 = 10^{-4/y}$

$\Rightarrow 0.777 \times 100 = 77.7$

$\Rightarrow 10^{-4/x} \times 10^2 = 10^{-4/y}$

$\Rightarrow 10^{-4/x + 2} = 10^{-4/y}$

$$\Rightarrow \quad -\frac{4}{x} + 2 = -\frac{4}{y}$$

$$\Rightarrow \quad \frac{4}{x} - \frac{4}{y} = 2$$

$$\Rightarrow \quad \frac{1}{x} - \frac{1}{y} = \frac{1}{2}$$

Hence, option (A) is the correct answer.

6. (D)

$$x + \frac{1}{x} = \sqrt{11}$$

Taking square on both sides of the equation, we get

$$x^2 + \frac{1}{x^2} = (\sqrt{11})^2 - 2 = 9.$$

Again taking square on both sides of the equation, we get

$$x^4 + \frac{1}{x^4} = (9)^2 - 2 = 81 - 2 = 79.$$

Now, $x^{11} - 79x^7 + x^3 = x^7\left[x^4 + \frac{1}{x^4} - 79\right]$

$= x^7[79 - 79] = 0.$

Hence, option (D) is the correct answer.

7. (D)

We have

$$x=\sqrt{14+6\sqrt{5}}\;=\left(3+\sqrt{5}\right)$$

$$\therefore\ \frac{1}{x}=\frac{1}{\sqrt{14+6\sqrt{5}}}=\frac{1}{\sqrt{14+6\sqrt{5}}}\cdot\frac{\sqrt{14-6\sqrt{5}}}{\sqrt{14-6\sqrt{5}}}$$

$$=\frac{\sqrt{\left(3-\sqrt{5}\right)^2}}{\sqrt{196-180}}=\frac{3-\sqrt{5}}{4}$$

$$\therefore\ x+\frac{1}{x}=\left(3+\sqrt{5}\right)+\frac{3-\sqrt{5}}{4}$$

$$=\frac{12+4\sqrt{5}+3-\sqrt{5}}{4}=\frac{3\sqrt{5}}{4}(\sqrt{5}+1)$$

Thus, the required option (D) is correct.

8. (A)

Given: $3^{6+12+18+24+...+6y}=(1/729)^{-66}$

$3^{6+12+18+24+...+6y}=3^{6(1+2+3+4+...+y)}$

$=3^{6(1+2+3+4+...+y)}$

$=3^{6y(y+1)/2}$ ...(i)

$(1/729)^{-66}=(1/3^6)^{-66}$

$=(3^{-6})^{-66}$

$=3^{6\times 66}$ ...(ii)

From eqations (i) and (ii)

$\Rightarrow y(y+1)/2=66$

$\Rightarrow y(y+1)=132=11\times 12$

Solving above, we get, $y=11$.

Hence, value of y satisfying the given equation is 11.

Hence, option (A) is the correct answer.

9. (D)

Let $10^a=16^b=40^c=k$

$a=\log_{10}k$, $b=\log_{16}k$, $c=\log_{40}k$

Now, we have to find:

$$5b\left(\frac{1}{c}-\frac{1}{a}\right)\;=5\times\log_{16}k\left(\frac{1}{\log_{40}k}-\frac{1}{\log_{10}k}\right)$$

$$=5\times\log_{4^2}k\left(\log_k 40-\log_k 10\right)$$

$$=\frac{5}{2}\log_4 k\left[\log_k(10\times 4)-\log_k 10\right]$$

$$=\frac{5}{2}\log_4 k\left[\log_k 10+\log_k 4-\log_k 10\right]$$

$$=\frac{5}{2}\log_4 k\times\log_k 4$$

$$=\frac{5}{2}\frac{\log k}{\log 4}\times\frac{\log 4}{\log k}$$

$$=\frac{5}{2}$$

$\therefore$ Hence, option (D) is the correct answer.

10. (A)

In each of the above equations, the bases are powers of the same number.

Therefore, $7^{x+1}=49^y$.

$\Rightarrow 7^{x+1}=7^{2y}$

$\Rightarrow x+1=2y$

$\Rightarrow x=2y-1$ ...(i)

Also, $8^x=128^y\times 16$.

$2^{3x}=2^{7y}\times 2^4$

$\Rightarrow\ 2^{3x}=2^{7y+4}$

$\Rightarrow\ 3x=7y+4$

Putting value of x from equation (i).

$3(2y-1)=7y+4$

$6y-3=7y+4$

$y=-7$

Therefore, $x=2y-1$.

$x=2x(-7)-1$

$x=-15$

Now, $2^{x+y}=2^{(-15)+(-7)}=2^{-22}$.

Thus, option (A) is the correct answer.

11. 16

$$\sqrt{320\sqrt{6}+384\sqrt{10}+240\sqrt{15}+1532}$$

$$=p\sqrt{2}+q\sqrt{3}+r\sqrt{5}$$

Squaring both sides of the equation, we get

$$320\sqrt{6}+384\sqrt{10}+240\sqrt{15}+1532$$

$$=2p^2+3q^2+5r^2+2pq\sqrt{6}+2pr\sqrt{10}+2qr\sqrt{15}.$$

As p, q, r are integers, the rational parts and the three different kinds of irrational parts on the two sides must be correspondingly equal.

$2p^2+3q^2+5r^2=1532,\ 2pq=320,\ 2pr=384,$

$2qr=240$

$pq=160$, $pr=192$, $qr=120$

$$\frac{(pq)(pr)}{qr}=p^2$$

$\therefore\ \ p^2=256$

$p=16$ ($\because\ p>0$).

12. (B)

I. $2^{3^{2^3}} = 2^{3^8} = 2^{6561}$

II. $23^{2^3} = 23^8$

Since, $2^4 < 23 < 2^5$.

So, $2^{4 \times 8} < 23^8 < 2^{5 \times 8}$ i.e., 23^{2^3} lies between 2^{32} and 2^{40}.

III. $2^4 < 23 < 2^5$, so, $2^{4 \times 23} < 23^{23} < 2^{5 \times 23}$.

Thus, 23^{23} lies between 2^{92} and 2^{115}.

IV. 2^{323} is already given as a power of 2.

V. 3^{23} is a number greater than 3^8.

Thus, V > I.

The given numbers in ascending order will be as follows:

II < III < IV < I < V

Thus, option (B) is the correct answer.

13. (D)

We have

$A = \sqrt{6} + \sqrt{26} \Rightarrow A^2 = 32 + 2\sqrt{156}$ ... (i)

$B = \sqrt{8} + \sqrt{22} \Rightarrow B^2 = 30 + 2\sqrt{176}$ ... (ii)

$C = \sqrt{13} + \sqrt{19} \Rightarrow C^2 = 32 + 2\sqrt{247}$ ... (iii)

$D = \sqrt{11} + \sqrt{16} \Rightarrow D^2 = 27 + 2\sqrt{176}$ ... (iv)

Now, from equations (i) and (iii)

We can conclude, $C > A$ ... (v)

From equations (ii) and (iv)

We can conclude, $B > D$ ... (vi)

From equations (i) and (iv)

We can conclude, $A > D$ ... (vii)

From equations (i) and (ii)

We can conclude, $A > B$ ... (viii)

Now, from equations (v), (vi), (vii), and (viii), we can conclude:

$C > A > B > D$

Hence, option (D) is correct.

14. 1,046

$-100 \le \sqrt[3]{n} - 3 \le 100$

$-97 \le \sqrt[3]{n} \le 103$

$(-97)^3 \le n \le (103)^3$

We know that all elements of B are positive. Since we are asked to find out the common elements of A and B, we can neglect the negative elements.

Now, we know that B only has perfect squares.

So, to get the common elements of A and B, we can consider only perfect squares till 103^3, i.e., 10,92,727.

Square root of 10,92,727 = 1,045.*yy*.

As zero is also included, the total number of common elements is 1,045 + 1 = 1,046 common elements.

15. (D)

$K = \sqrt{3} - 2$ or $K + 2 = \sqrt{3}$

Taking square on both sides of the equation, we get

$K^2 + 4K + 4 = 3$.

Again, taking square on both sides of the equation, we get

$(K^2 + 4K + 4)(K^2 + 4K + 4) = 9$

$K^4 + 8K^3 + 24K^2 + 32K + 16 = 9$

$K^4 + 8K^3 + 24K^2 + 32K = -7$

Now, we need to find the value of $K^4 + 8K^3 + 24K^2 + 32K + 32$.

Putting $(K^4 + 8K^3 + 24K^2 + 32K) = -7$ in the above given expression, we will get

$(K^4 + 8K^3 + 24K^2 + 32K) + 32 = -7 + 32 = 25$.

Hence, option (D) is the correct answer.

16. (C)

According to the problem

$$\frac{1}{\left(\sqrt[3]{4} - \sqrt[3]{2} + 1\right)} = \frac{1}{(2^2)^{\frac{1}{3}} - (2)^{\frac{1}{3}} + 1}$$

$$= \frac{1}{\left(2^{\frac{1}{3}}\right)^2 - \left(2^{\frac{1}{3}}\right)(1) + (1)^2}$$

$$= \frac{\left\{\left(2^{\frac{1}{3}}\right) + 1\right\}}{\left\{\left(2^{\frac{1}{3}}\right) + 1\right\}\left\{\left(2^{\frac{1}{3}}\right)^2 - \left(2^{\frac{1}{3}}\right)(1) + (1)^2\right\}}$$

We know, $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$

$$= \frac{\left(\sqrt[3]{2} + 1\right)}{\left(2^{\frac{1}{3}}\right)^3 + (1)^3}$$

$$= \frac{\left(\sqrt[3]{2} + 1\right)}{3}$$

$$= \frac{1}{3}\sqrt[3]{2} + \frac{1}{3}$$

$$= x\sqrt[3]{7} + y\sqrt[3]{2} + z$$

Clearly, $x = 0$, $y = \frac{1}{3}$, and $z = \frac{1}{3}$.

Thus, $\frac{1}{2}(x \times y \times z)$

$= 0$ (since $x = 0$).

Thus, option (C) is the correct answer.

17. (B)

Given

$x = \sqrt[6]{5+2\sqrt{6}}$ and $y = \sqrt[6]{5-2\sqrt{6}}$

$\Rightarrow xy = \sqrt[6]{(5+2\sqrt{6})(5-2\sqrt{6})}$ which yields

$xy = 1$

Now $(x+y)(x^2+y^2+2xy-3)$

$=(x+y)[(x+y)^2-3)]$

$=(x+y)^3-3(x+y)$

$=x^3+y^3+3xy(x+y)-3(x+y)$

$=x^3+y^3$ (since $xy = 1$)

$= \left(\sqrt[6]{5+2\sqrt{6}}\right)^3 + \left(\sqrt[6]{5-2\sqrt{6}}\right)^3$

$=\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}$

$=\sqrt{3}+\sqrt{2}+\sqrt{3}-\sqrt{2} = 2\sqrt{3}$

Hence, option (B) is the correct answer.

18. (B)

$\left[11^A \times 55^{B+1} \times 45^{(C+2)}\right]^{1/3}$

$\rightarrow \left[11^A \times 11^{B+1} \times 5^{B+1} \times 9^{C+2} \times 5^{C+2}\right]^{1/3}$

$\rightarrow \left[11^{(A+B+1)} \times 5^{(B+C+3)} \times 3^{2C+4}\right]^{1/3}$

$11^{\left(\frac{A+B+1}{3}\right)} \times 5^{\left(\frac{B+C+3}{3}\right)} \times 3^{\left(\frac{2C+4}{3}\right)} =$ whole number

Only option (B) will satisfy the given condition.

19. (B)

$\sqrt{7-4\sqrt{3}}=\sqrt{(2)^2+(\sqrt{3})^2-2(2)(\sqrt{3})}$

$= \sqrt{(2-\sqrt{3})^2} = 2-\sqrt{3}$

$K=\sqrt{35-19\sqrt{3}+\sqrt{7-4\sqrt{3}}}$

Or, $K=\sqrt{35-19\sqrt{3}+2-\sqrt{3}}$

Or, $K=\sqrt{37-20\sqrt{3}}=\sqrt{(5)^2+(2\sqrt{3})^2-2\times5\times2\sqrt{3}}$

Or, $K=\sqrt{(5-2\sqrt{3})^2}=5-2\sqrt{3}=5-2(1.73)$

$= 5 - 3.46 = 1.54$.

Hence, option (B) is the correct answer.

20. (D)

Let $p^x = q^y = r^z = k$

$\Rightarrow p^x = k \Rightarrow \quad k^{1/x} = p$

$\Rightarrow q^y = k \Rightarrow \quad k^{1/y} = q$

$\Rightarrow r^z = k \Rightarrow \quad k^{1/z} = r$

p, q, and r are in GP.

So, $\frac{q}{p}=\frac{r}{q} \quad \Rightarrow q^2 = pr$

$\Rightarrow \left(k^{\frac{1}{y}}\right)^2 = \left(k^{\frac{1}{x}}\right)\left(k^{\frac{1}{z}}\right)$

$\Rightarrow (k)^{\frac{2}{y}} = (k)^{\frac{x+z}{xz}}$

$\Rightarrow \frac{2}{y}=\frac{x+z}{xz}$

$\Rightarrow \left(\frac{xy+zy}{xz}\right)=2$

Multiplying both sides by $\left(\frac{1}{16}\right)$, we get

$\left(\frac{1}{16}\right)\left(\frac{xy+zy}{xz}\right)=\frac{1}{16}(2)=\frac{1}{8}=(8)^{-1}=(2^3)^{-1}=2^{-3}$

$\Rightarrow \left(\frac{xy+zy}{16xz}\right) = 2^{-3}$

Therefore, option (D) is the correct answer.

21. (C)

We have to check by taking each and every given answer option.

Let's check option (a).

If $x = 3^{2/3} + 3^{1/3}$.

Taking cube on both sides of the equation, we get

$x^3 = (3^{2/3} + 3^{1/3})^3 = 3^2 + 3^1 + 3 \times 3^{2/3} \times 3^{1/3}(3^{2/3} + 3^{1/3})^3$

$x^3 = 12 + 3 \times 3\,(x)$

$x^3 - 9x - 12 = 0$, which is not the equation mentioned in the question.

Similarly, option (B) will also not satisfy.

Let's check option (c)

$x = 3^{2/3} + 3^{1/3} + 2$

$x - 2 = 3^{2/3} + 3^{1/3}$

Taking cube on both sides of the equation, we get

$(x - 2)^3 = (3^{2/3} + 3^{1/3})^3$

$x^3 - 8 - 3 \times (x) \times 2(x - 2) = 3^2 + 3^1 + 3 \times 3^{2/3} \times 3^{1/3} \times (3^{2/3} + 3^{1/3})$

$x^3 - 8 - 6x(x - 2) = 12 + 9(x - 2)$ (since $(x - 2) = 3^{2/3} + 3^{1/3}$)

$x^3 - 8 - 6x^2 + 12x = 12 + 9x - 18$

$x^3 - 6x^2 + 3x - 2 = 0$ which is the equation mentioned in the question.

Hence, option (C) is the correct answer.

22. (C)

$$A = 2^{2^{2^2}} = 2^{2^4} = 2^{16}$$

$$B = 2^{22^2} = 2^{484}$$

$$C = 2^{2^{22}} = 2^{(2^{11} \times 2^{11})} = 2^{(2048 \times 2048)}$$

$$D = 2^{222}$$

So, $C > B > D > A$

Hence, option (C) is the correct answer.

23. (A)

By the problem, $9^k + 15^k = 25^k$

Dividing both sides by 9^k, we get

$$\frac{9^K}{9^K} + \frac{15^K}{9^K} = \frac{25^K}{9^K}$$

$$\Rightarrow \quad 1 + \left(\frac{5}{3}\right)^K = \left(\left(\frac{5}{3}\right)^2\right)^K \left(\text{Since } \frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n\right)$$

$$\Rightarrow \quad 1 + \left(\frac{5}{3}\right)^K = \left(\frac{5}{3}\right)^{2K} \left(\text{Since } (x^n)^m = x^{m \times n}\right)$$

Let, $\left(\frac{5}{3}\right)^K = x$

So, $1 + x = x^2$

$\Rightarrow \quad x^2 - x - 1 = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{5}}{2}$

Ignoring the imaginary value, we get $x = \dfrac{1 + \sqrt{5}}{2}$

$\Rightarrow \quad 2x = \left(1 + \sqrt{5}\right) \quad$ ...(i)

Therefore, $3\left[2\left(\frac{5}{3}\right)^K - \sqrt{5}\right]$

$= 3\left[2x - \sqrt{5}\right] \quad \left(\text{Since } \left(\frac{5}{3}\right)^K = x\right)$

$= 3\left[1 + \sqrt{5} - \sqrt{5}\right] \quad$ [From equation (i)]

$= 3(1)$

$= 3$ which is a prime number.

Thus, option (A) is the correct answer.

24. (D)

Given: $x = \left(5\sqrt{5} + 11\right)$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{\left(5\sqrt{5} + 11\right)}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{\left(5\sqrt{5} - 11\right)}{\left(5\sqrt{5} + 11\right)\left(5\sqrt{5} - 11\right)}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{\left(5\sqrt{5} - 11\right)}{(125 - 121)}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{\left(5\sqrt{5} - 11\right)}{4}$$

$$\Rightarrow \quad \left(5\sqrt{5} - 11\right) = \frac{4}{x} \quad \text{...(i)}$$

Now, $x = \left(5\sqrt{5} + 11\right)$

Squaring both sides of the equation, we get

$$x^2 = \left(5\sqrt{5} + 11\right)^2$$

or, $\quad x^2 = \left(5\sqrt{5}\right)^2 + (11)^2 + 2\left(5\sqrt{5}\right)(11)$

or, $\quad x^2 = 125 + 121 + 110\sqrt{5}$

or, $\quad x^2 = 246 + 110\sqrt{5}$

or, $\quad x^2 = 2\left(123 + 55\sqrt{5}\right)$

or, $\quad \left(\dfrac{x^2}{2}\right) = \left(123 + 55\sqrt{5}\right) \quad$ ...(ii)

From equations (i) and (ii), we get the values of $\left(5\sqrt{5} - 11\right)$ and $\left(123 + 55\sqrt{5}\right)$.

Putting the above values in the expression

$$\frac{\left(5\sqrt{5} - 11\right)}{4\left(123 + 55\sqrt{5}\right)}, \text{ we get } \frac{\left(\frac{4}{x}\right)}{4\left(\frac{x^2}{2}\right)} = \left(\frac{2}{x^3}\right).$$

Therefore, option (D) is the correct answer.

25. (A)

$(3+2\sqrt{2})^{x^3-7}+(3-2\sqrt{2})^{x^3-7}=6$

$(3+2\sqrt{2})=\dfrac{(3+2\sqrt{2})\,(3-2\sqrt{2})}{(3-2\sqrt{2})}$

(applying the concept of rationalisation of a surd)

$=\dfrac{1}{(3-2\sqrt{2})}$

$\therefore\quad (3+2\sqrt{2})^{x^3-7}\dfrac{1}{(3-2\sqrt{2})^{x^3-7}}$

$\therefore$ The equation becomes $\dfrac{1}{(3-2\sqrt{2})^{x^3-7}}$

$+\ (3-2\sqrt{2})^{x^3-7}=6.$

Let, $(3-2\sqrt{2})^{x^3-7}=k$.

$\therefore\quad \dfrac{1}{k}+k=6 \qquad \Rightarrow \quad k^2-6k+1=0$

$\therefore\quad k=\dfrac{6\pm\sqrt{(-6)^2-4(1)(1)}}{2}=\dfrac{6\pm\sqrt{32}}{2}$

$=\dfrac{6\pm4\sqrt{2}}{2}$

$\Rightarrow\quad k=(3+2\sqrt{2})$

$\therefore$ k equals to either $(3+2\sqrt{2})$ or $(3-2\sqrt{2})$.

When $k=(3-2\sqrt{2})$ then $(3-2\sqrt{2})^{x^3-7}$

$=(3-2\sqrt{2})^1$

$\Rightarrow\quad x^3-7=1 \qquad \Rightarrow\quad x^3=8$

$\Rightarrow\quad x=2$

When $k=(3+2\sqrt{2})$ then $(3-2\sqrt{2})^{x^3-7}$

$=(3+2\sqrt{2})$

Also, we know that $(3+2\sqrt{2})=\dfrac{1}{(3-2\sqrt{2})}$

$=(3-2\sqrt{2})^{-1}.$

$\therefore\quad (3-2\sqrt{2})^{x^3-7}=(3-2\sqrt{2})^{-1}.$

$\Rightarrow\quad x^3-7=-1 \qquad \Rightarrow\quad x^3=6$

$\Rightarrow\quad x=(6)^{1/3}$

The required values of x are 2 and $(6)^{1/3}$.

2 is the only possible integral value of x.

Thus, option (A) is the correct answer.

26. (C)

$x=\dfrac{8}{(\sqrt{5}+\sqrt{3})}=\dfrac{8\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}$

$=\dfrac{8\left(\sqrt{5}-\sqrt{3}\right)}{2}=4\left(\sqrt{5}-\sqrt{3}\right)$

$y=\dfrac{12}{(\sqrt{5}+\sqrt{2})}=\dfrac{12\left(\sqrt{5}-\sqrt{2}\right)}{\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{2}\right)}$

$=\dfrac{12\left(\sqrt{5}-\sqrt{2}\right)}{3}=4\left(\sqrt{5}-\sqrt{2}\right)$

$z=\dfrac{4}{(\sqrt{3}+\sqrt{2})}=\dfrac{4\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}$

$=\dfrac{4\left(\sqrt{3}-\sqrt{2}\right)}{1}=4\left(\sqrt{3}-\sqrt{2}\right)$

Now, $(x-y+z)$

$=4\left(\sqrt{5}-\sqrt{3}\right)-4\left(\sqrt{5}-\sqrt{2}\right)+4\left(\sqrt{3}-\sqrt{2}\right)$

$=4\sqrt{5}-4\sqrt{3}-4\sqrt{5}+4\sqrt{2}+4\sqrt{3}-4\sqrt{2}=0$

$\Rightarrow\quad (x-y+z)=0$

Also, $(x-y+z)^2=(x)^2+(-y)^2+(z)^2+2[(x)(-y)+(-y)(z)+(x(z)]$

$=x^2+y^2+z^2+2(-xy-yz+xz)$

$\Rightarrow\quad (0)^2=x^2+y^2+z^2-2(xy+yz-xz)$

$\Rightarrow\quad x^2+y^2+z^2=2(xy+yz-zx)$

Dividing both sides by (xyz), we get

$\dfrac{1}{xyz}(x^2+y^2+z^2)=\dfrac{1}{xyz}\times2(xy+yz-zx)$

$\Rightarrow\quad \dfrac{x^2}{xyz}+\dfrac{y^2}{xyz}+\dfrac{z^2}{xyz}=2\left(\dfrac{xy}{xyz}+\dfrac{yz}{xyz}-\dfrac{zx}{xyz}\right)$

$\Rightarrow\quad \left(\dfrac{x}{yz}+\dfrac{y}{xz}+\dfrac{z}{xy}\right)=2\left(\dfrac{1}{z}+\dfrac{1}{x}-\dfrac{1}{y}\right)$

We know, $x=4(\sqrt{5}-\sqrt{3})$

$\Rightarrow\quad \dfrac{1}{x}=\left(\dfrac{\sqrt{5}+\sqrt{3}}{8}\right)$

$y=4(\sqrt{5}-\sqrt{2}) \qquad \Rightarrow\quad \dfrac{1}{y}=\left(\dfrac{\sqrt{5}+\sqrt{2}}{12}\right)$

$z=4(\sqrt{3}-\sqrt{2}) \qquad \Rightarrow\quad \dfrac{1}{z}=\left(\dfrac{\sqrt{3}+\sqrt{2}}{4}\right)$

$\therefore\quad 2\left(\dfrac{1}{x}+\dfrac{1}{z}-\dfrac{1}{y}\right)$

$$=2\left[\frac{\sqrt{5}+\sqrt{3}}{8}+\frac{\sqrt{3}+\sqrt{2}}{4}-\left(\frac{\sqrt{5}+\sqrt{2}}{12}\right)\right]$$

$$=2\left[\frac{3\left(\sqrt{5}+\sqrt{3}\right)}{24}+\frac{6\left(\sqrt{3}+\sqrt{2}\right)}{24}-\left(\frac{2\left(\sqrt{5}+\sqrt{2}\right)}{24}\right)\right]$$

$$=2\left[\frac{3\sqrt{5}+3\sqrt{3}+6\sqrt{3}+6\sqrt{2}-2\sqrt{5}-2\sqrt{2}}{24}\right]$$

$$=2\left[\frac{\sqrt{5}+9\sqrt{3}+4\sqrt{2}}{24}\right]=\left(\frac{\sqrt{5}+9\sqrt{3}+4\sqrt{2}}{12}\right)$$

We are required to calculate the value of

$$144\left(\frac{x}{yz}+\frac{y}{xz}+\frac{z}{xy}\right).$$

So, $\left(\frac{x}{yz}+\frac{y}{xz}+\frac{z}{xy}\right)=2\left(\frac{1}{z}+\frac{1}{x}-\frac{1}{y}\right)$

$$=\left(\frac{\sqrt{5}+4\sqrt{2}+9\sqrt{3}}{12}\right)$$

$$\Rightarrow 144\left(\frac{x}{yz}+\frac{y}{xz}+\frac{z}{xy}\right)=144\left(\frac{\sqrt{5}+4\sqrt{2}+9\sqrt{3}}{12}\right)$$

$$=12\left(\sqrt{5}+4\sqrt{2}+9\sqrt{3}\right)$$

Thus, the correct answer is option (C).

27. (D)

We know that $\left(\sqrt{a}+\sqrt{b}\right)^2=a+b+2\sqrt{ab}$

and $\left(\sqrt{a}-\sqrt{b}\right)^2=a+b-2\sqrt{ab}$.

Let $\left(\sqrt{a}-\sqrt{b}\right)^2=70-16\sqrt{6}$

$a+b-2\sqrt{ab}=70-2\sqrt{6\times 64}$.

Thus, $a + b = 70$ and $ab = 6 \times 64 = 384$. Since $64 + 6 = 70$, we conclude that $a = 64$ and $b = 6$.

Therefore, $(70-16\sqrt{6})^{\frac{1}{2}}=\left[\left(\sqrt{64}-\sqrt{6}\right)^2\right]^{\frac{1}{2}}$

$=\sqrt{64}-\sqrt{6}$

$=8-\sqrt{6}$

Again, let $\left(\sqrt{p}+\sqrt{q}\right)^2=21+6\sqrt{10}$

$p+q+2\sqrt{pq}=21+2\sqrt{15\times 6}$

Thus, $p + q = 21$ and $pq = 15 \times 6 = 90$. Since, $15 + 6 = 21$, we can conclude that $p = 15$ and $q = 6$.

Therefore,

$$(21+6\sqrt{10})^{\frac{1}{2}}=\left[\left(\sqrt{15}+\sqrt{6}\right)^2\right]^{\frac{1}{2}}=\sqrt{15}+\sqrt{6}$$

$$X=\sqrt{(70-16\sqrt{6})^{\frac{1}{2}}+(21+6\sqrt{10})^{\frac{1}{2}}}$$

$$=\sqrt{8-\sqrt{6}+\sqrt{15}+\sqrt{6}}$$

$$=\sqrt{8+\sqrt{15}}$$

Now, $8+\sqrt{15}=\frac{2\left(8+\sqrt{15}\right)}{2}=\frac{16+2\sqrt{15}\times 1}{2}$

Since, $15 + 1 = 16$.

$$\frac{16+2\sqrt{15}}{2}=\frac{\left(\sqrt{15}+1\right)^2}{2}=\left[\frac{\sqrt{15}+1}{\sqrt{2}}\right]^2$$

Therefore, $X=\sqrt{8+\sqrt{15}}$

$$=\sqrt{\left(\frac{\sqrt{15}+1}{\sqrt{2}}\right)^2}$$

$$=\frac{\sqrt{15}+1}{\sqrt{2}}$$

$$=2^{\frac{-1}{2}}(\sqrt{15}+1)$$

Therefore, option (D) is the correct answer.

28. (B)

Since the given equation is $x^2-\sqrt{3}x-1=0$ or $x^2 - 1 = \sqrt{3}x$.

Dividing both sides of the equation by x, we get

$$x-\frac{1}{x}=\sqrt{3}.$$

Now taking square on both sides of the equation, we get

$$\left(x-\frac{1}{x}\right)^2=(\sqrt{3})^2$$

Or, $x^2+\frac{1}{x^2}-2=3$

Or, $x^2+\frac{1}{x^2}=5$ ...(i)

Again, taking square on both sides of the equation, we will get

$$\left(x^2+\frac{1}{x^2}\right)^2=(5)^2$$

Or, $x^4+\frac{1}{x^4}+2.x^2\times\frac{1}{x^2}=25$

Or, $x^4+\frac{1}{x^4}=23$ ...(ii)

We have to find the value of the expression $x^{10} - 22x^6 - 23x^2$.

We can write the above given expression as:

$x^{10} - 22x^6 - 23x^2 = x^{10} - (23 - 1)\,x^6 - 23x^2$.

If we use equation (ii) in this expression, then:

$$= x^{10} - \left(x^4 + \frac{1}{x^4} - 1\right)x^6 - \left(x^4 + \frac{1}{x^4}\right)x^2$$

$$= x^{10} - \left(\frac{x^8 + 1 - x^4}{x^4}\right)x^6 - \left(x^6 + \frac{1}{x^2}\right)$$

$$= x^{10} - x^{10} - x^2 + x^6 - x^6 - \frac{1}{x^2}$$

$$= -\left(x^2 + \frac{1}{x^2}\right) = -5$$

Hence, option (B) is the correct answer.

29. (C)

We know that for two numbers x, y greater than 1, if $x > y$, then $x^2 > y^2$.

It is given that,

$$a = \sqrt{18} + \sqrt{12}$$

$$a^2 = \left(\sqrt{18} + \sqrt{12}\right)^2 = 30 + 2\sqrt{216}$$

$$b = \frac{10}{\sqrt{20} - \sqrt{10}} = \frac{10(\sqrt{20} + \sqrt{10})}{(\sqrt{20} - \sqrt{10})(\sqrt{20} + \sqrt{10})}$$

$$= \frac{10(\sqrt{20} + \sqrt{10})}{10}$$

Therefore,

$$b^2 = (\sqrt{20} + \sqrt{10})(\sqrt{20} + \sqrt{10}) = 30 + 2\sqrt{200}$$

$$c = (\sqrt{19} + \sqrt{11})$$

$$\text{or, } c^2 = (\sqrt{19} + \sqrt{11})^2 = 30 + 2\sqrt{209}$$

$$d = \frac{4}{\sqrt{12} - \sqrt{13}} = \frac{4\left(\sqrt{17} + \sqrt{13}\right)}{\left(\sqrt{17} - \sqrt{13}\right)\left(\sqrt{17} + \sqrt{13}\right)}$$

$$= \frac{4\left(\sqrt{17} + \sqrt{13}\right)}{4}$$

Therefore, $d^2 = (\sqrt{17} + \sqrt{13})^2 = 30 + 2\sqrt{221}$.

Since, $d^2 > a^2 > c^2 > b^2$

$\left[\text{As, } \sqrt{221} > \sqrt{216} > \sqrt{209} > \sqrt{200}\right]$

Therefore, $d > a > c > b$.

Thus, option (C) is the correct answer.

30. 1

$$5^{\frac{8^3}{88x} + \frac{5^3}{11x} + \frac{x^3}{11x}} = 25$$

Equating the indices on the two sides,

$$\frac{8^3}{88x} + \frac{5^3}{11x} + \frac{x^3}{11x} = 2$$

$x^3 - 22x + 189 = 0$.

Let $f(x) = x^3 - 22x + 189$.

The number of sign changes in $f(x)$ is 2.

($\therefore$ There are 2 or 0 positive roots. We can verify that there are 0 positive roots.)

The number of sign changes in $f(-x)$ $= (-x^3) + 22x + 189$ is 1.

$\therefore$ There are exactly 1 negative root.

(We can verify that -7 is a root of the equation.)

$\therefore$ $x + 7$ is a factor of the LHS.

The other factor is $x^2 - 7x + 27$.

$$x^2 - 7x + 27 = \left(x - \frac{7}{2}\right)^2 + \frac{59}{4}.$$

This is always positive.

The only real root of the given equation is -7.

Hence, the number of real roots is 1.

The answer is 1.