# Homework 1

## Homework 1

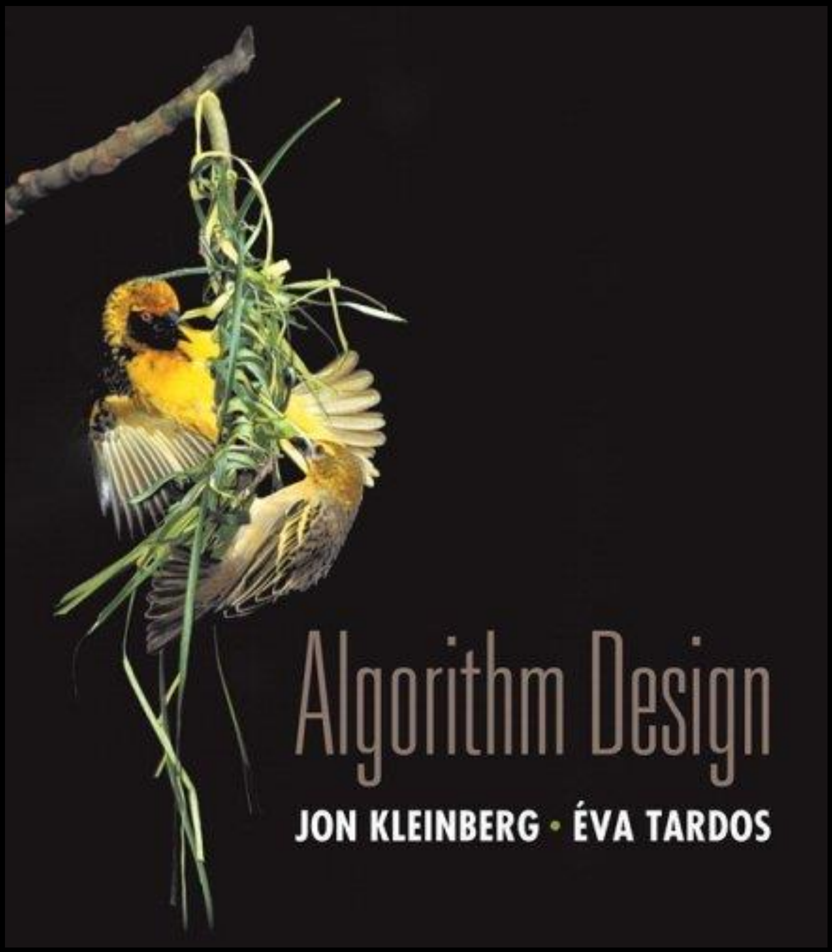
- Covers chapters 2-4
- To be released today in Blackboard
- Due: Oct. 10 before class
- No late homework will be accepted!


## Note

- Submit a paper copy of your solutions
- Write down your name and ID#
- You are encouraged to use LaTeX

## Collaboration policy

- Discussions with your fellow students are allowed, but you must write up the answers on your own.
- Copying the answers from other people or sources or sharing your answers with others will result in a grade of zero.




# Chapter 4

## Greedy Algorithms

Slides by Kevin Wayne.

# Greedy Algorithms

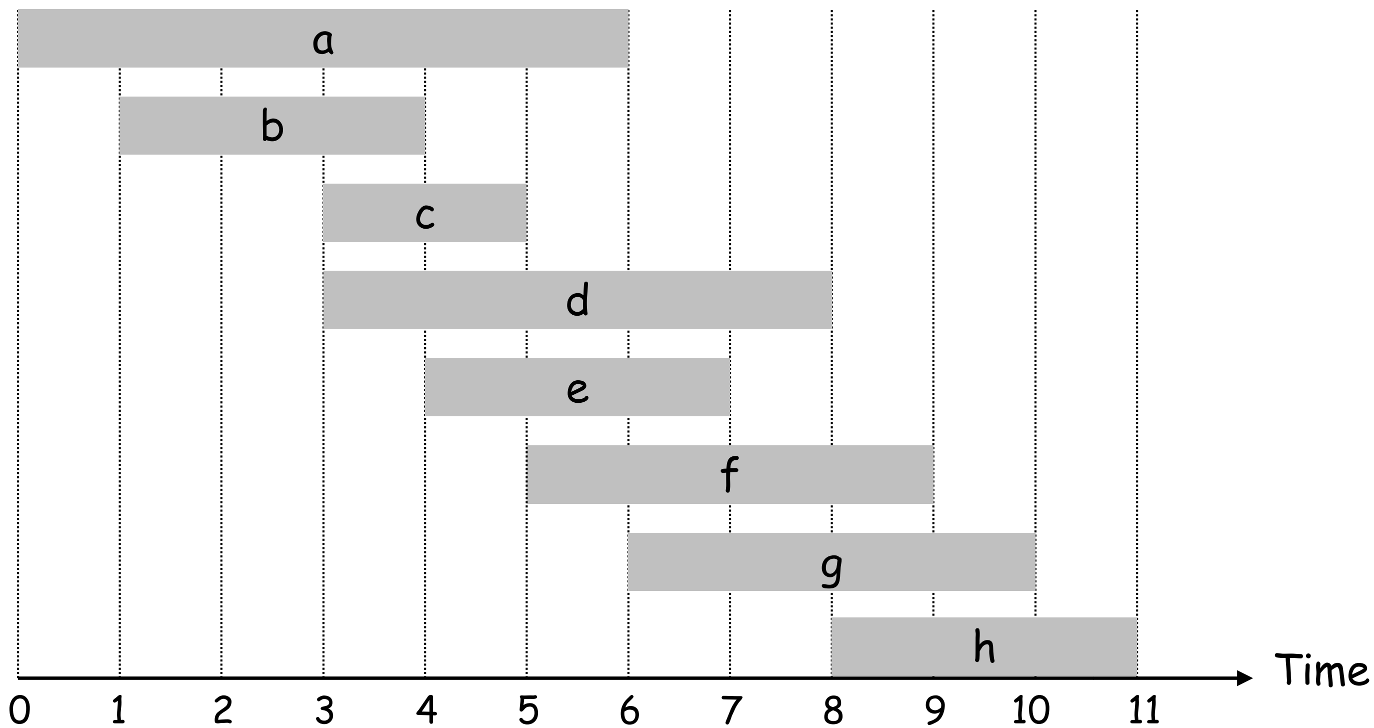
A greedy algorithm is an algorithm that makes the locally optimal choice at each step.

# 4.1 Interval Scheduling

# Interval Scheduling

Interval scheduling.

- Job j starts at $s_j$ and finishes at $f_j$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



# Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time $s_j$.
- [Earliest finish time] Consider jobs in ascending order of finish time $f_j$.
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs $c_j$. Schedule in ascending order of conflicts $c_j$.

# Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

breaks earliest start time

breaks shortest interval

breaks fewest conflicts

# Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

jobs selected

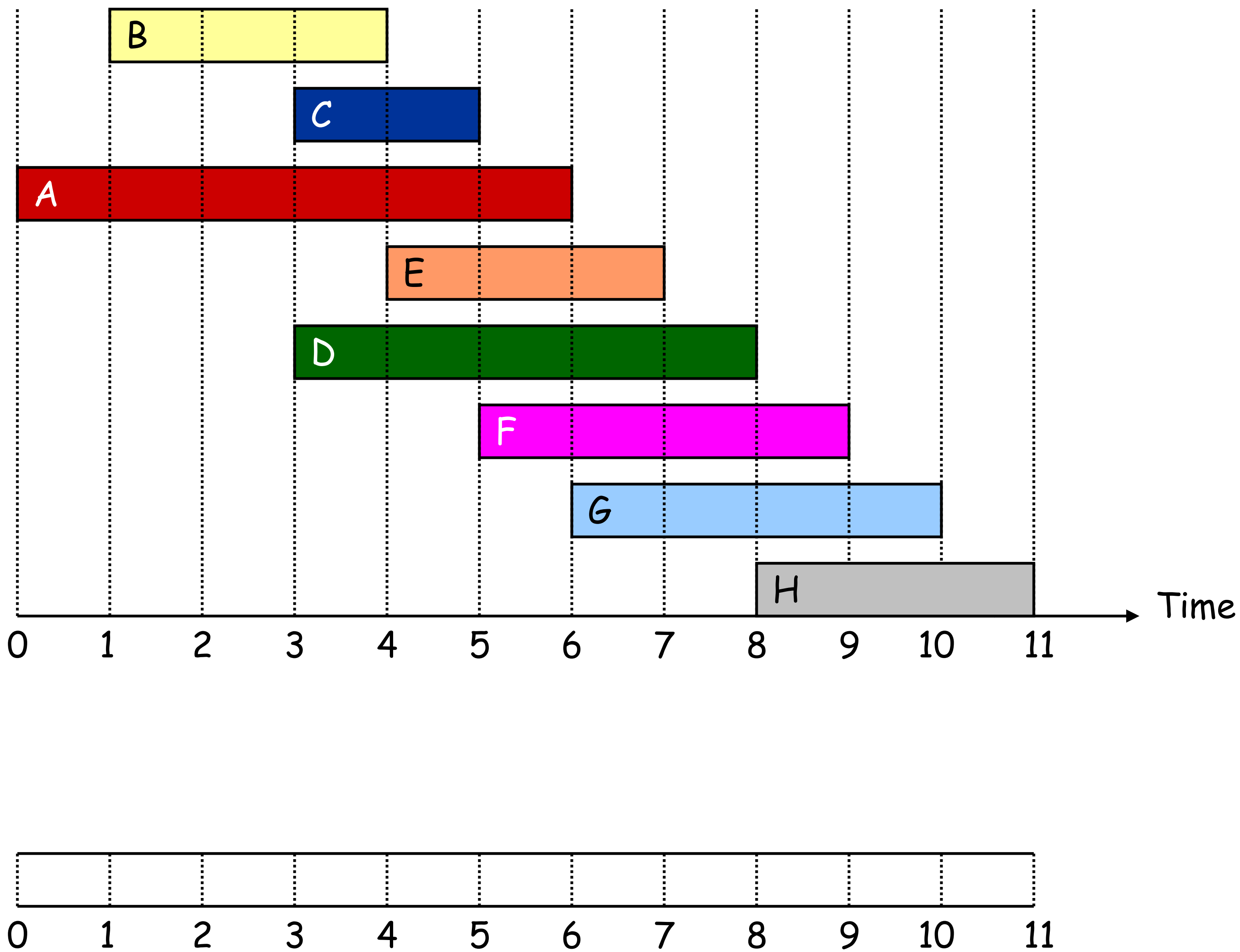
```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

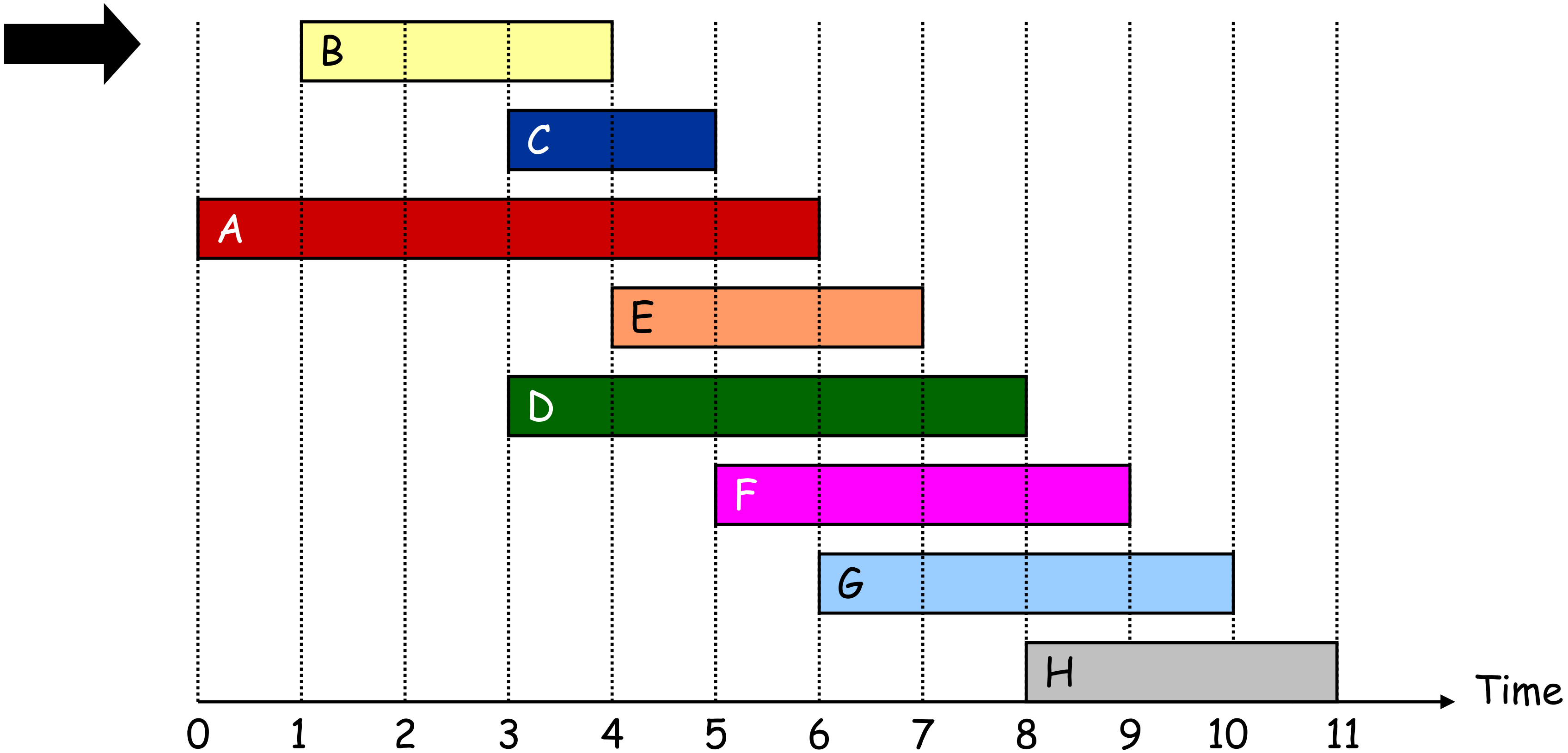
Implementation. $O(n \log n)$.

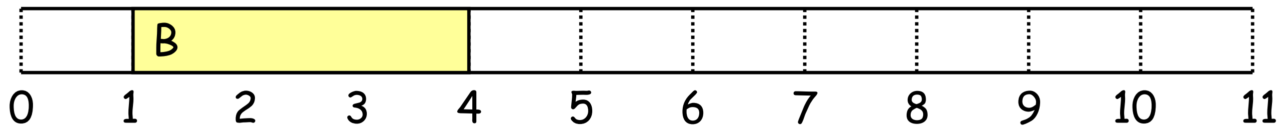
- Remember job $j^*$ that was added last to $A$.
- Job $j$ is compatible with $A$ if $s_j \geq f_{j^*}$.

# Interval Scheduling



# Interval Scheduling



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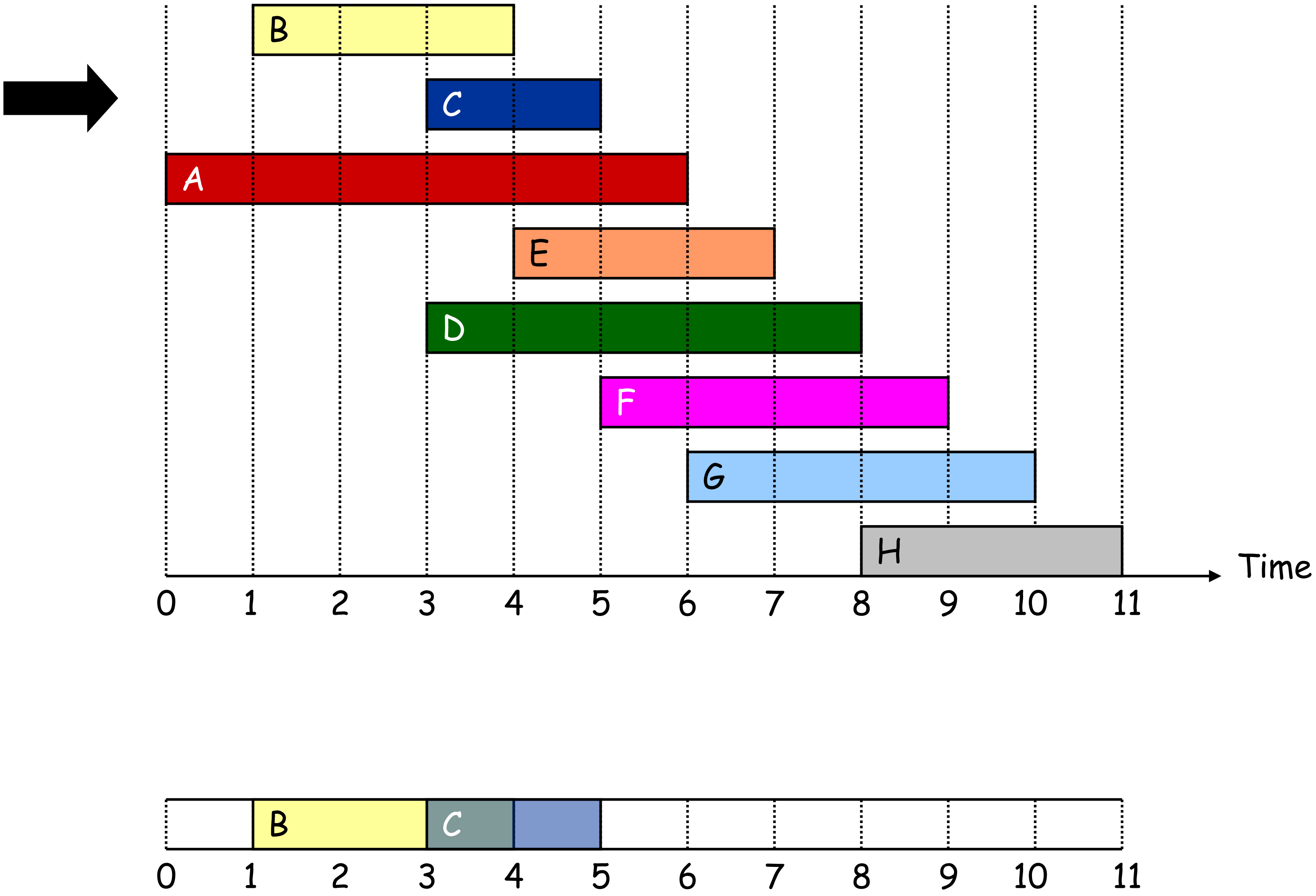

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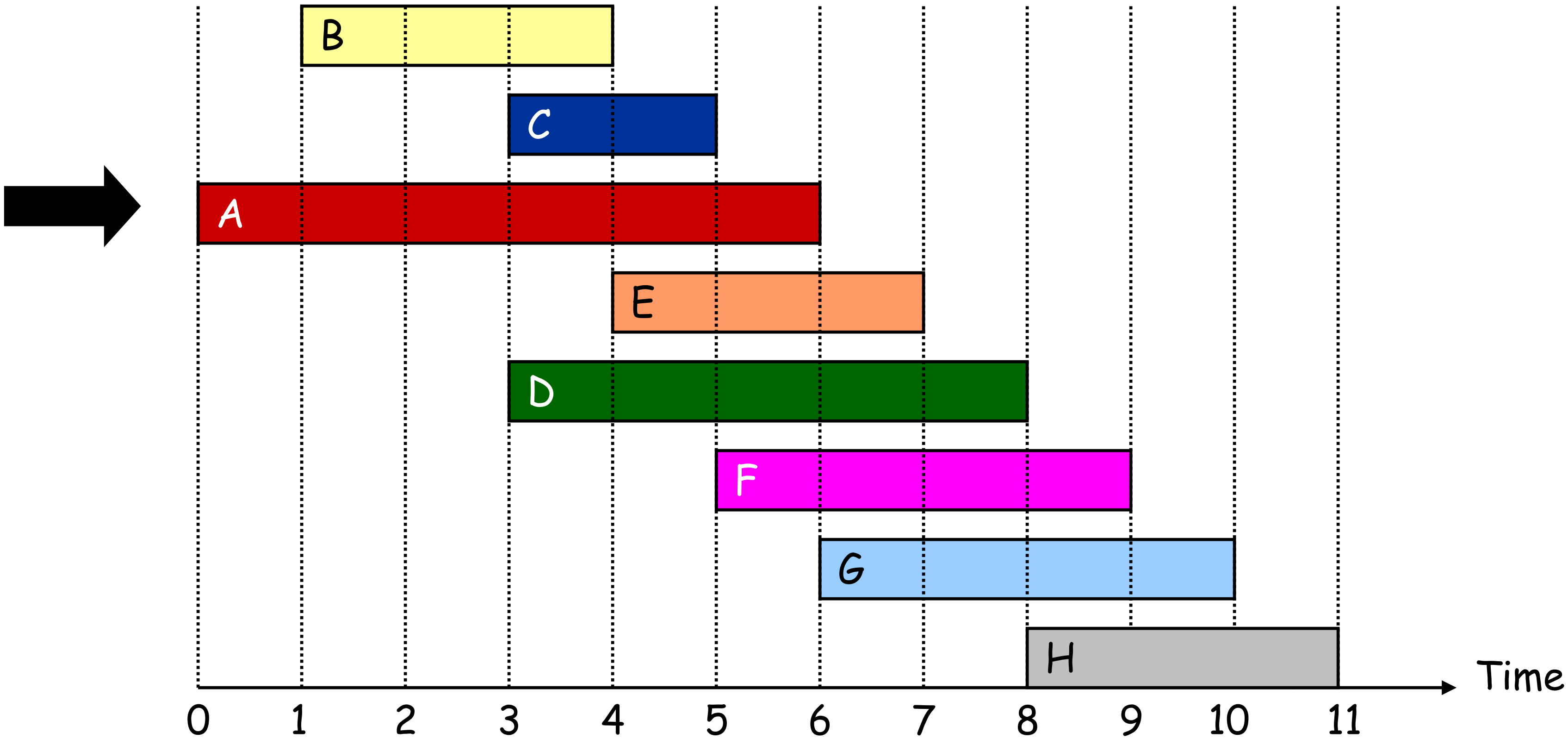


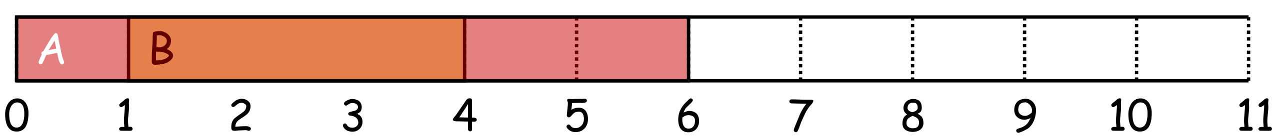

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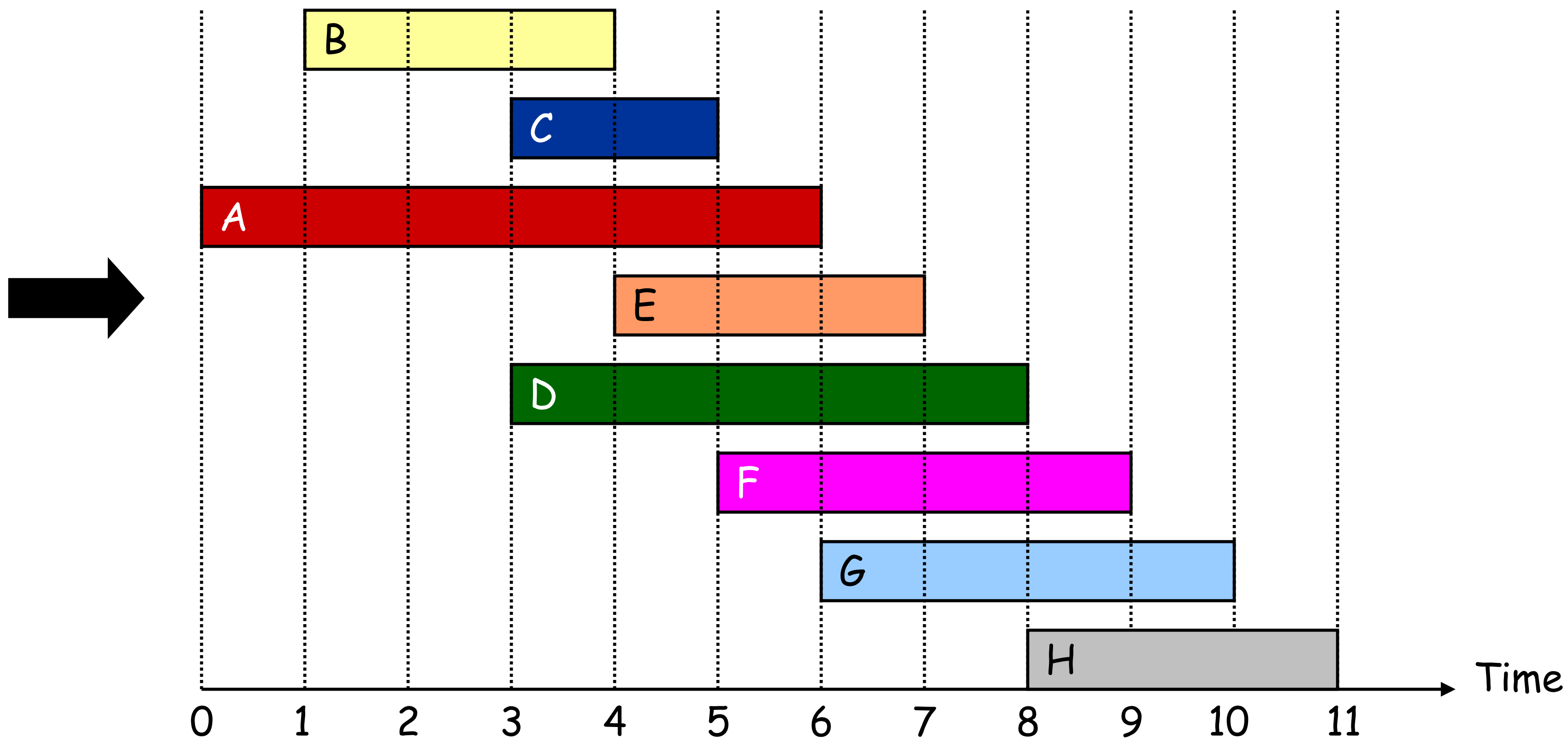


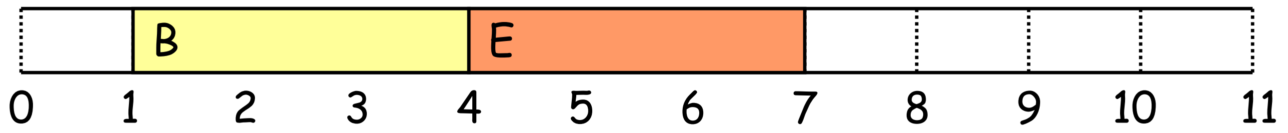

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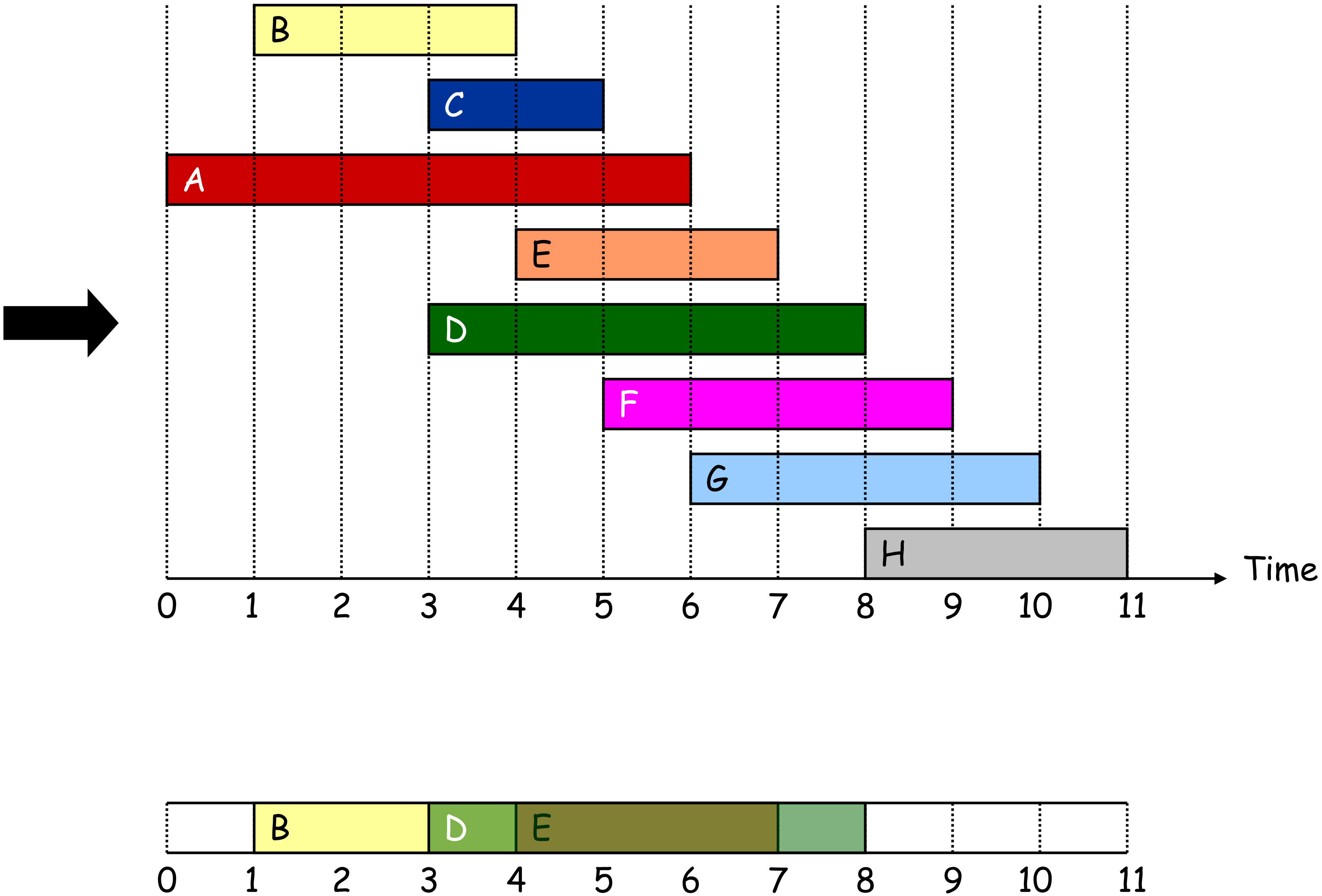

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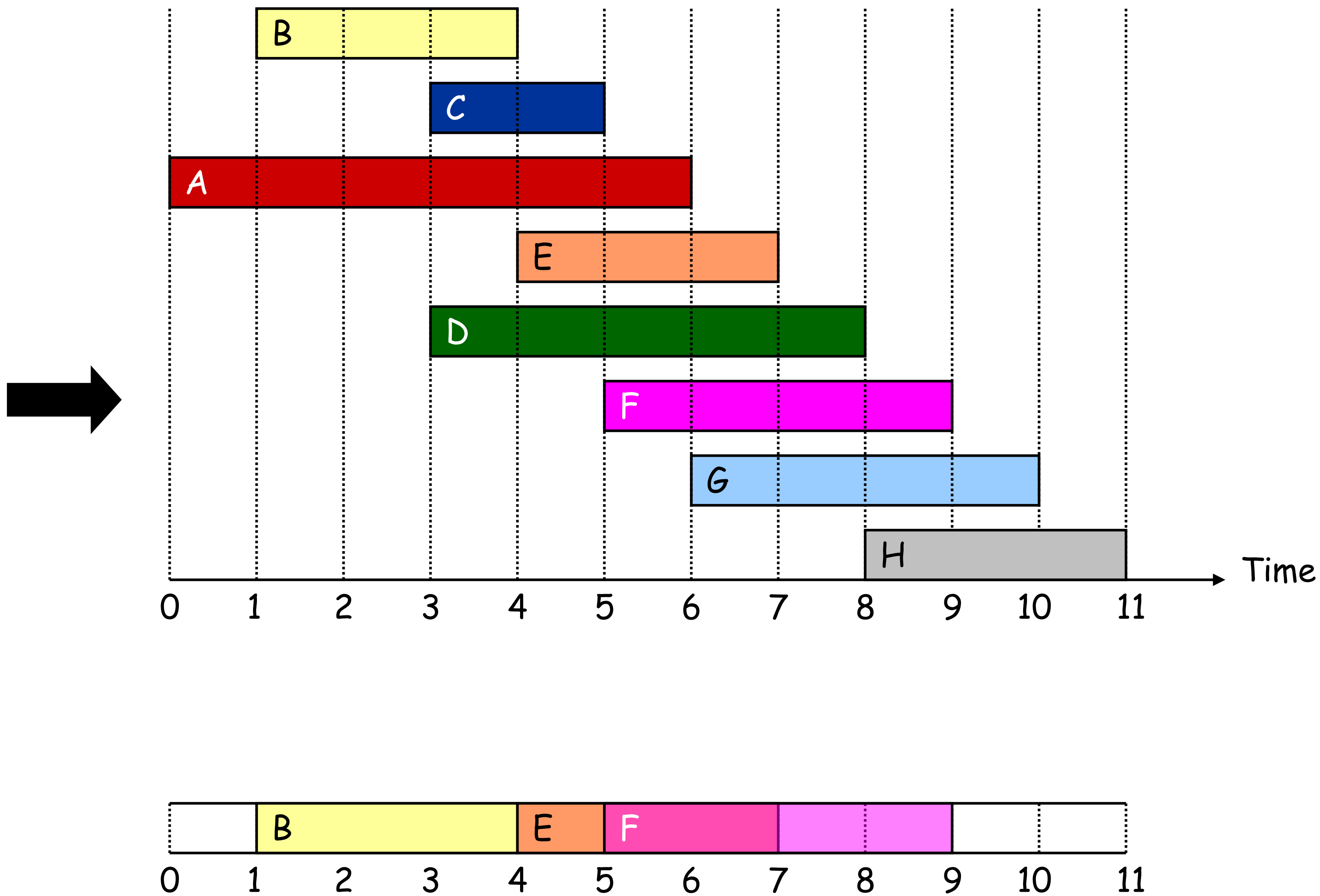

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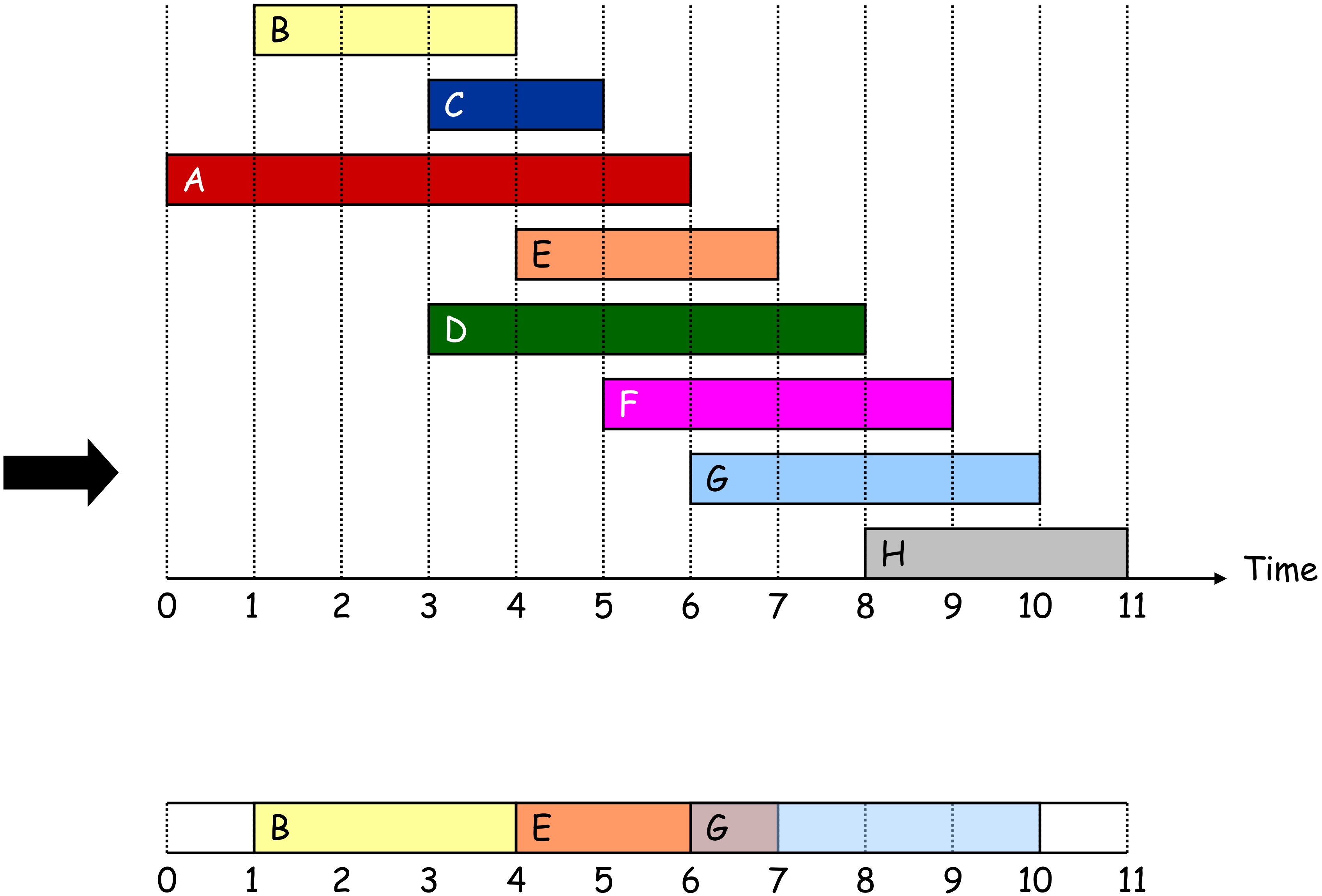
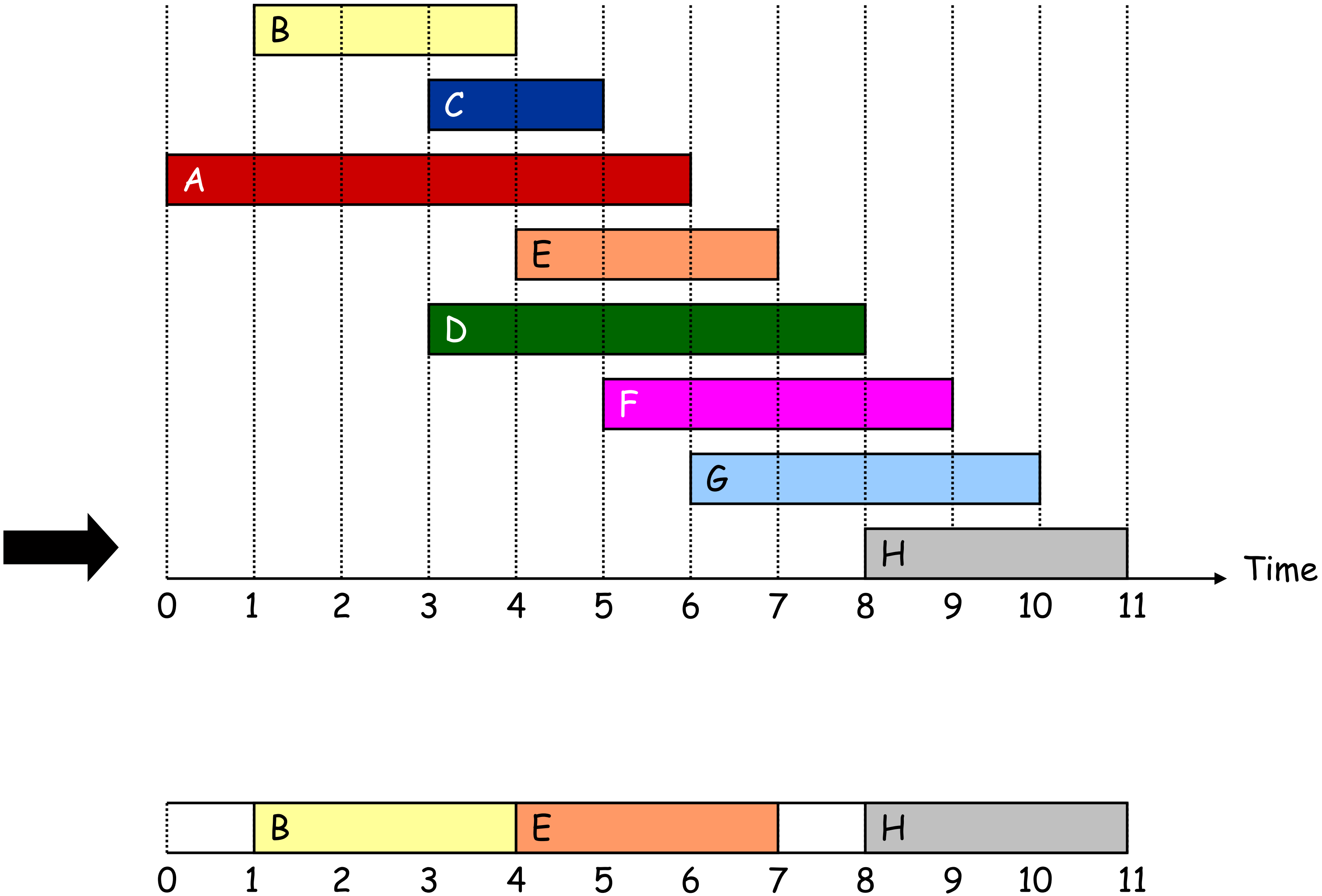


# Interval Scheduling

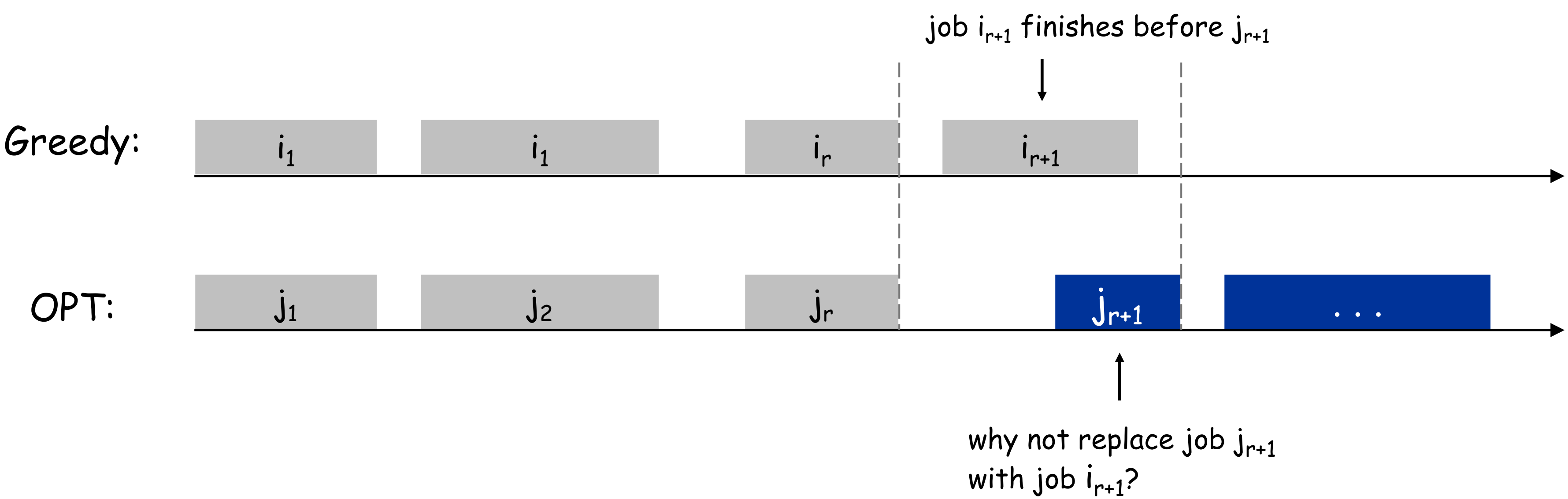


# Interval Scheduling: Analysis

**Theorem.** Greedy algorithm is optimal.

**Pf.** (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of $r$.



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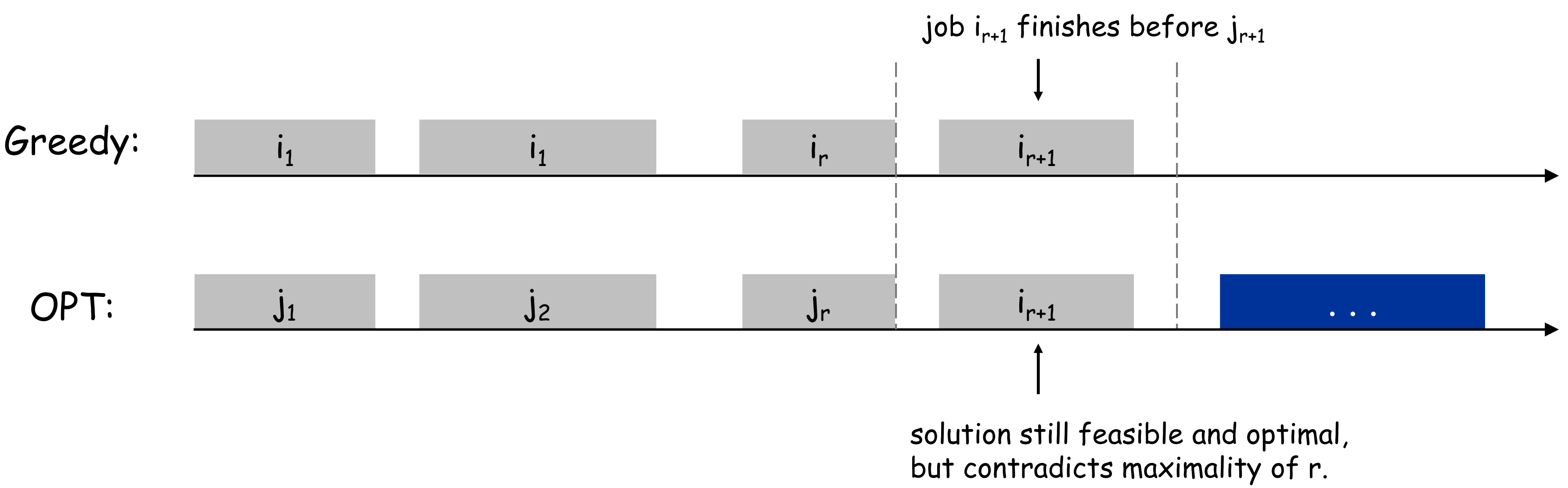
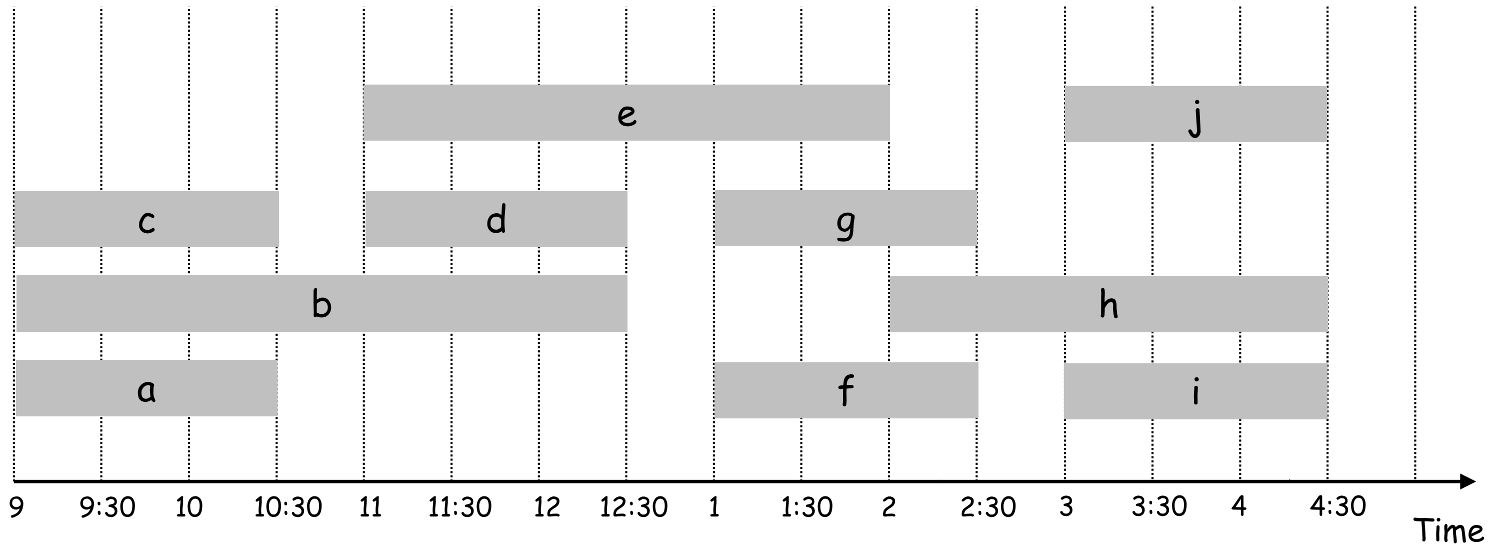


# Interval Partitioning

# Interval Partitioning

Interval partitioning.
- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

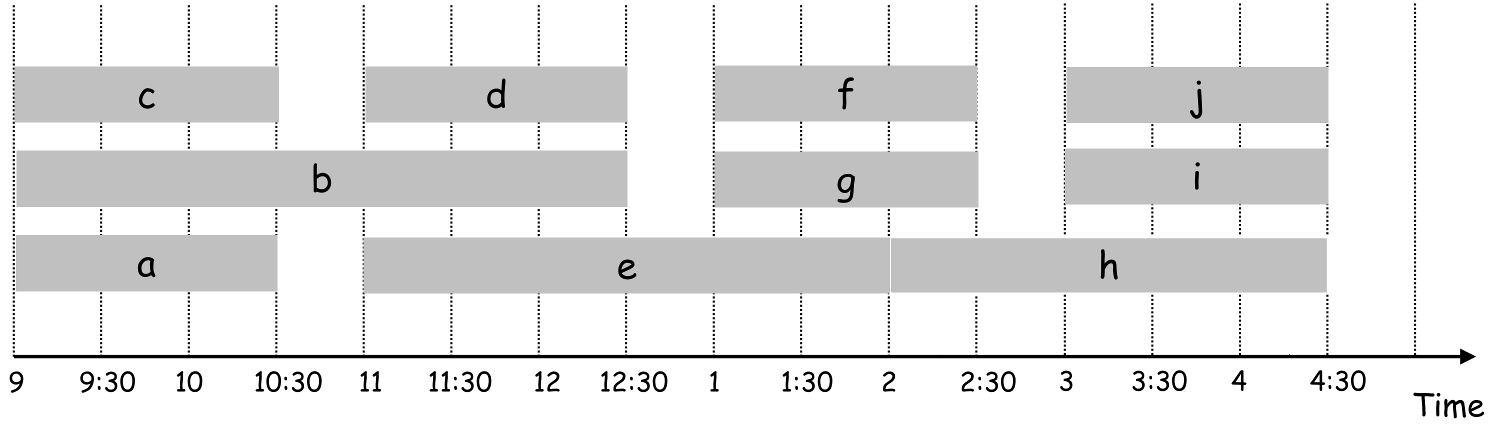


# Interval Partitioning

Interval partitioning.

- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



# Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0    ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

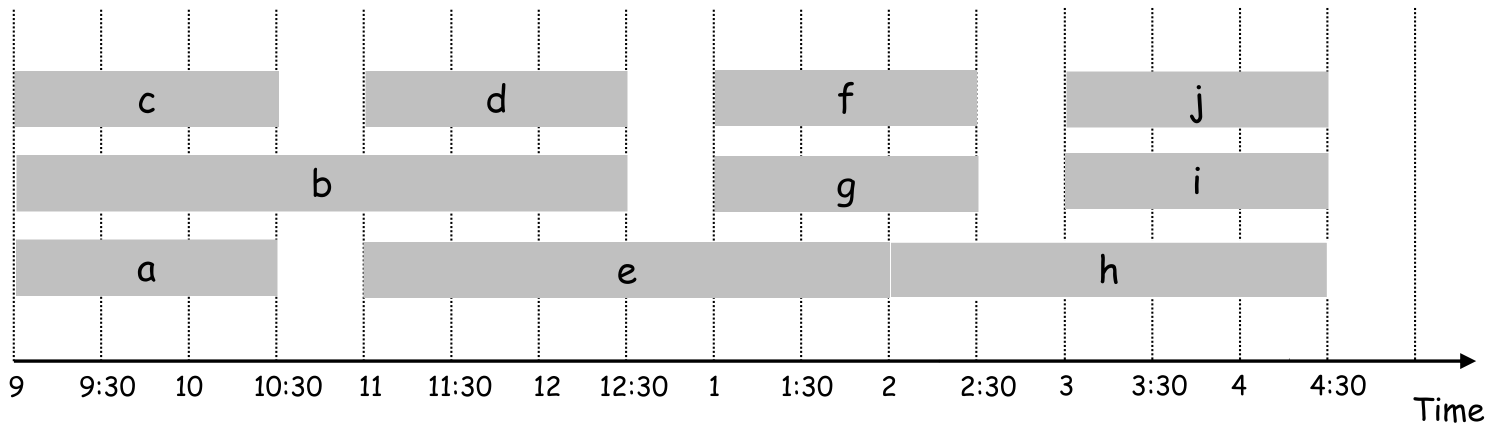
# Interval Partitioning: Lower Bound on Optimal Solution

Def. The depth of a set of open intervals is the maximum number of intervals at any time point.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

a, b, c all contain 9:30



# Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.
Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than $s_j$.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms. ▪

# 4.2 Scheduling to Minimize Lateness

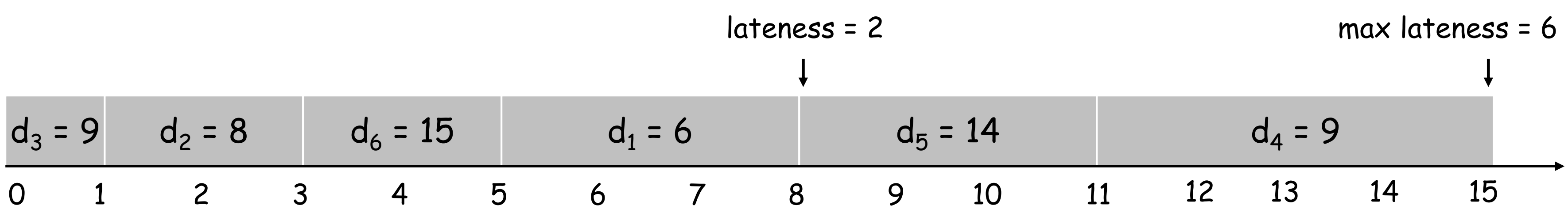
# Scheduling to Minimizing Lateness

## Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires $t_j$ units of processing time and is due at time $d_j$.
- If j starts at time $s_j$, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max\{0, f_j - d_j\}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |



# Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.

- [Earliest deadline first] Consider jobs in ascending order of deadline $d_j$.

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

# Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 100 | 10 |

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 2 | 10 |

counterexample

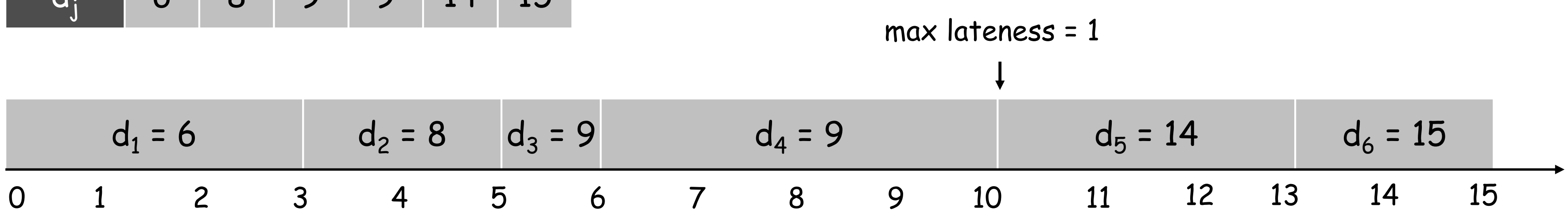
# Minimizing Lateness: Greedy Algorithm

**Greedy algorithm.** Earliest deadline first.

**Sort** n jobs by deadline so that $d_1 \le d_2 \le \ldots \le d_n$

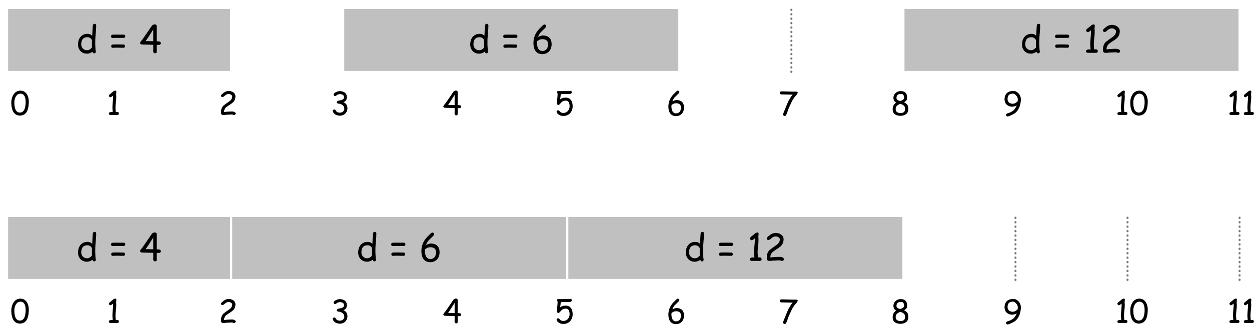
$t \leftarrow 0$
**for** $j = 1$ to $n$
  Assign job $j$ to interval $[t, t + t_j]$
  $s_j \leftarrow t$, $f_j \leftarrow t + t_j$
  $t \leftarrow t + t_j$
**output** intervals $[s_j, f_j]$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |



# Minimizing Lateness: No Idle Time

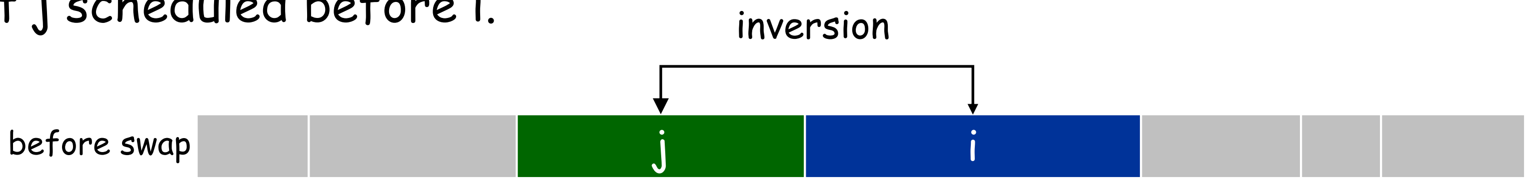
Observation. There exists an optimal schedule with no idle time.




Observation. The greedy schedule has no idle time.

# Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: i < j but j scheduled before i.

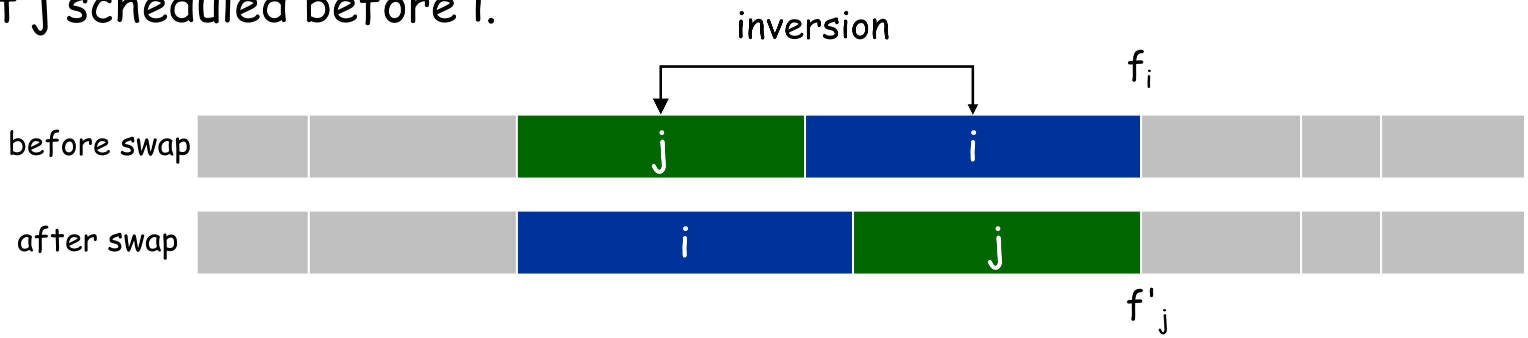



Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

# Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: i < j but j scheduled before i.




Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let $\ell$ be the lateness before the swap, and let $\ell'$ be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is late:

$$
\begin{aligned}
\ell'_j &= f'_j - d_j && \text{(definition)} \\
&= f_i - d_j && (j \text{ finishes at time } f_i) \\
&\leq f_i - d_i && (i < j) \\
&\leq \ell_i && \text{(definition)}
\end{aligned}
$$

# Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*.
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪

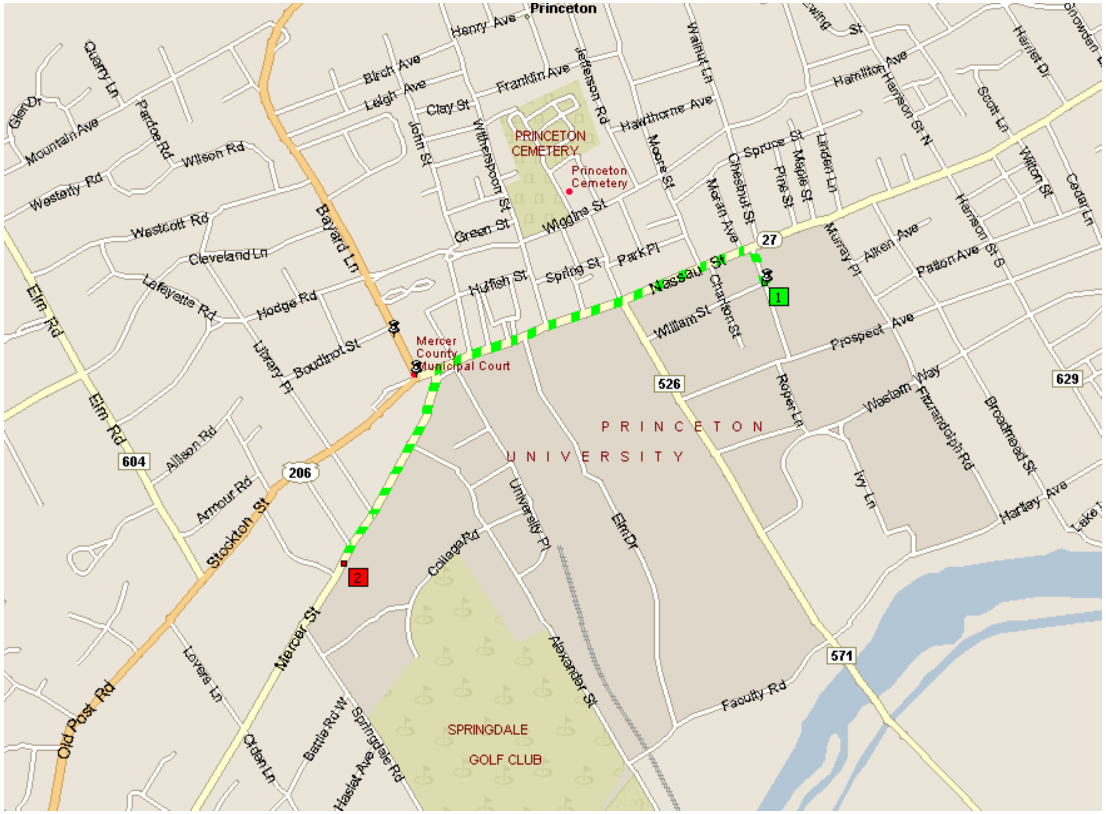
# Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

# 4.4 Shortest Paths in a Graph



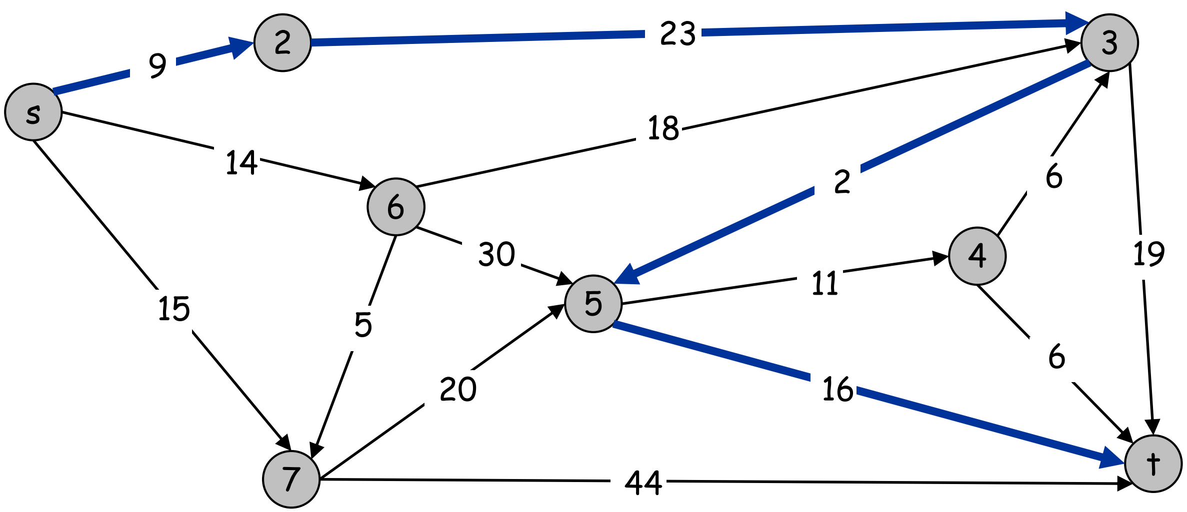

shortest path from Princeton CS department to Einstein's house

# Shortest Path Problem

Shortest path network.
- Directed graph $G = (V, E)$.
- Source $s$, destination $t$.
- Length $\ell_e$ = length of edge $e$. ← Non-negative

Shortest path problem: find shortest directed path from $s$ to $t$.

cost of path = sum of edge costs in path




Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 48.

# Dijkstra's Algorithm

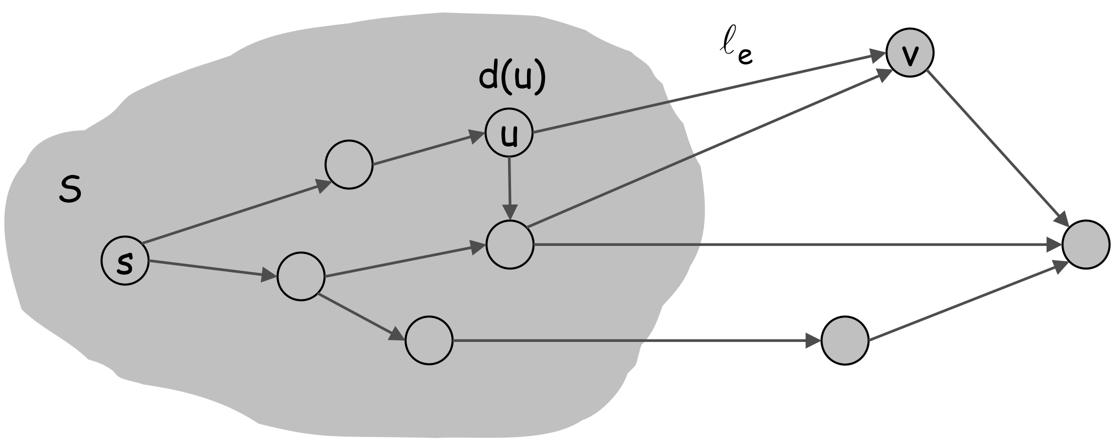
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = $\pi$(v).



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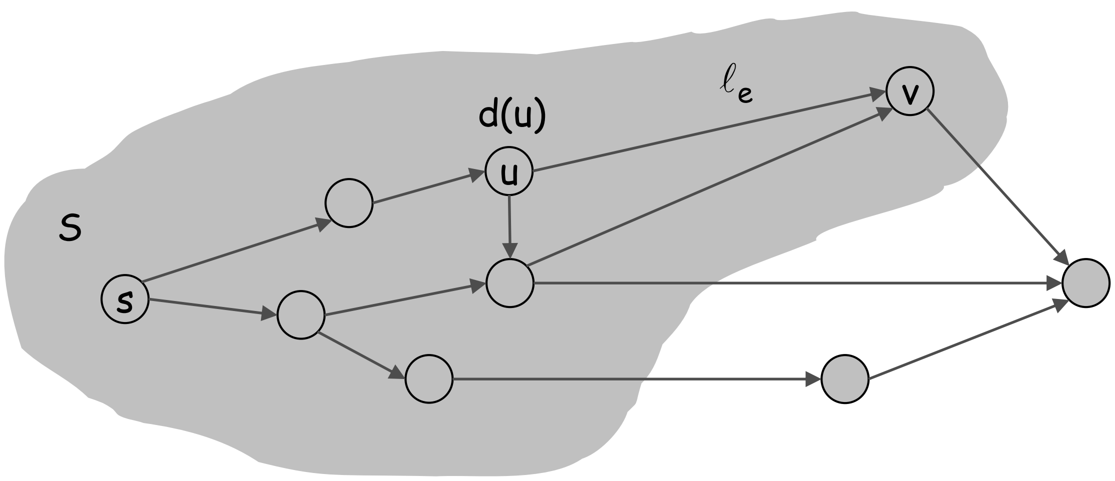
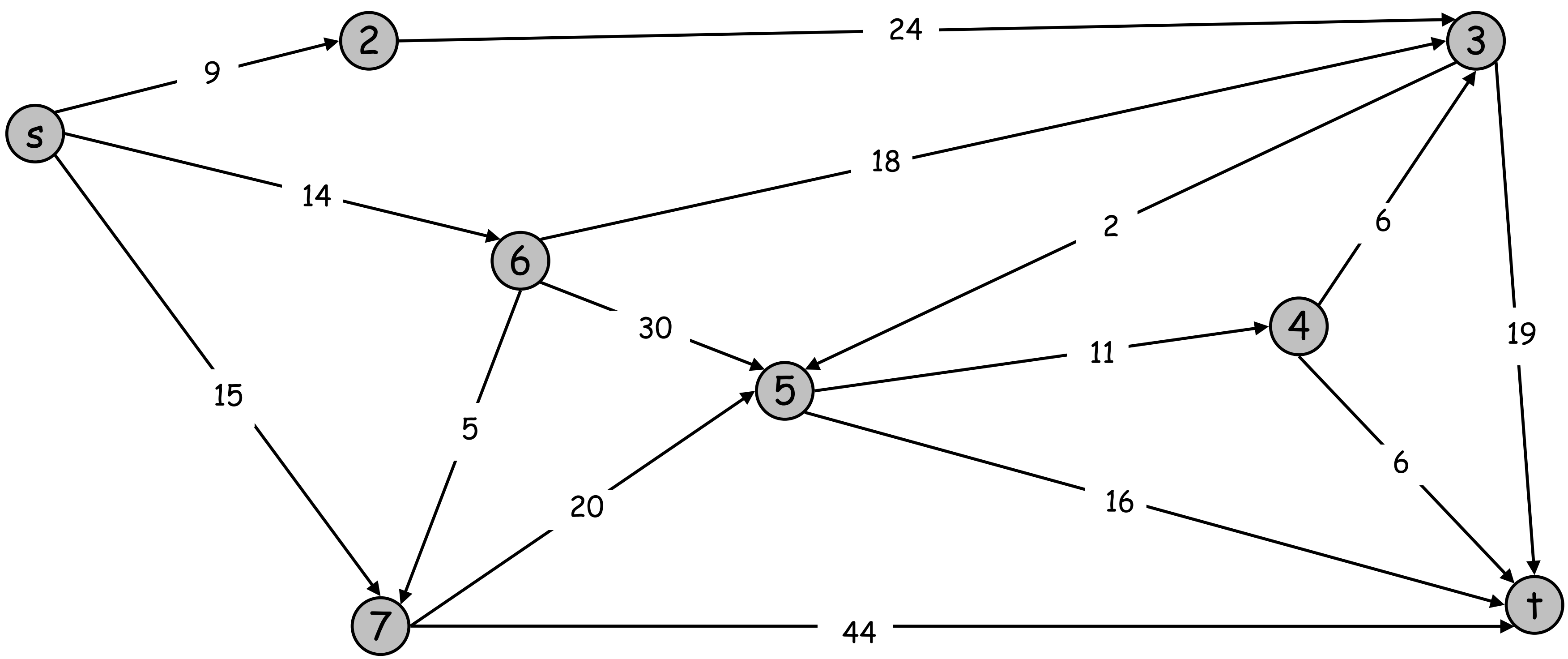


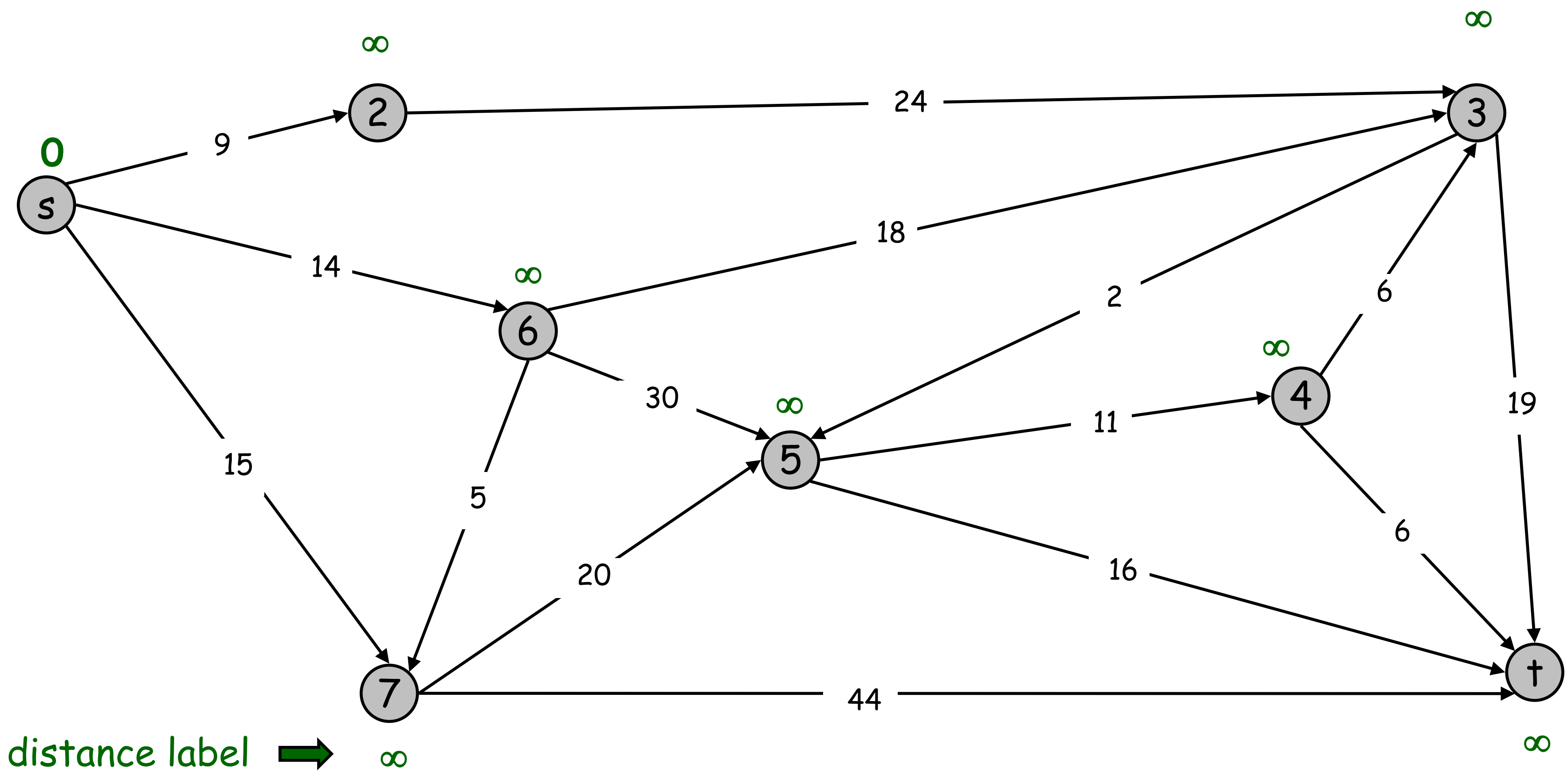
# Dijkstra's Shortest Path Algorithm

Find shortest path from s to t.



# Dijkstra's Shortest Path Algorithm

S = { }
PQ = { s, 2, 3, 4, 5, 6, 7, t }



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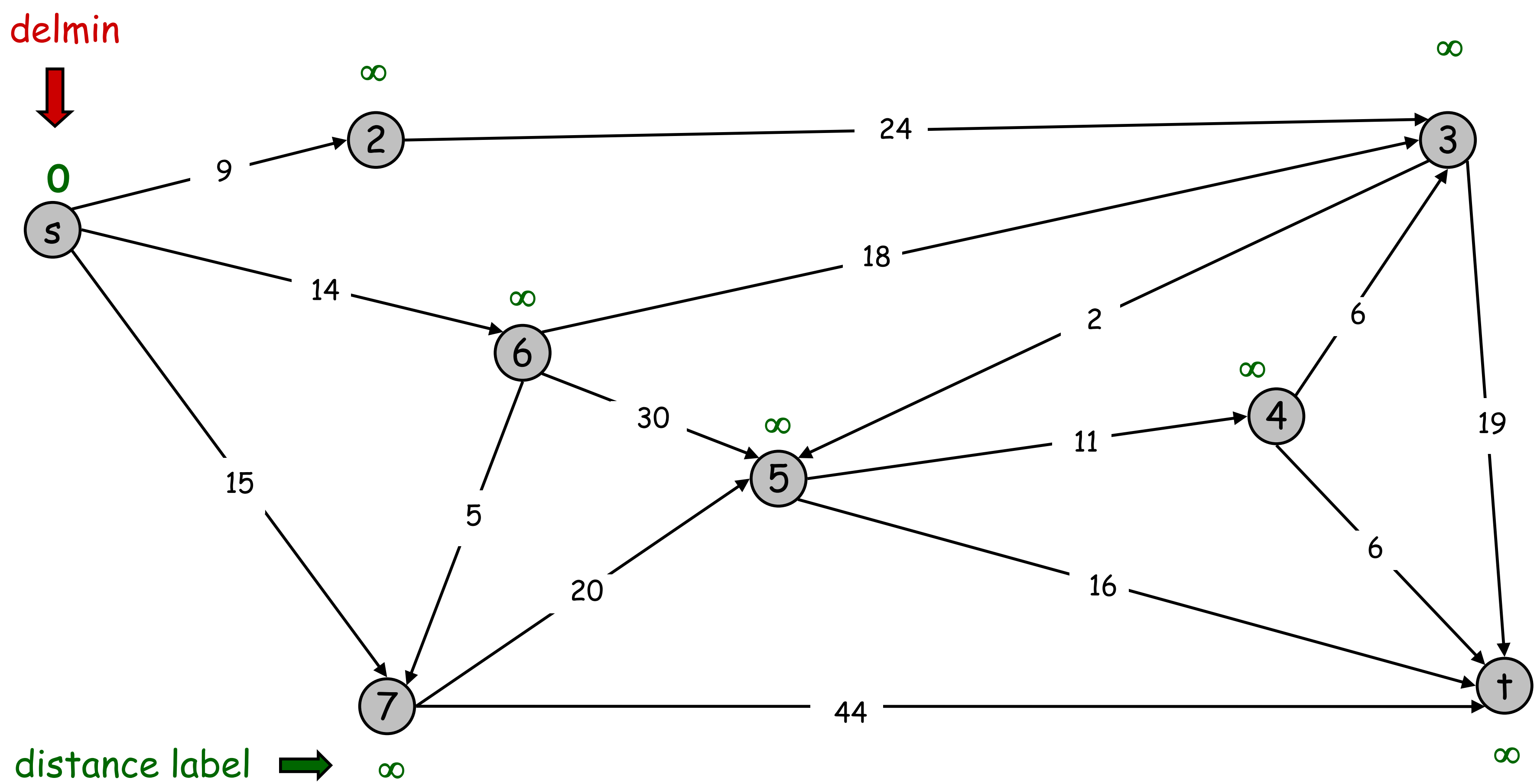

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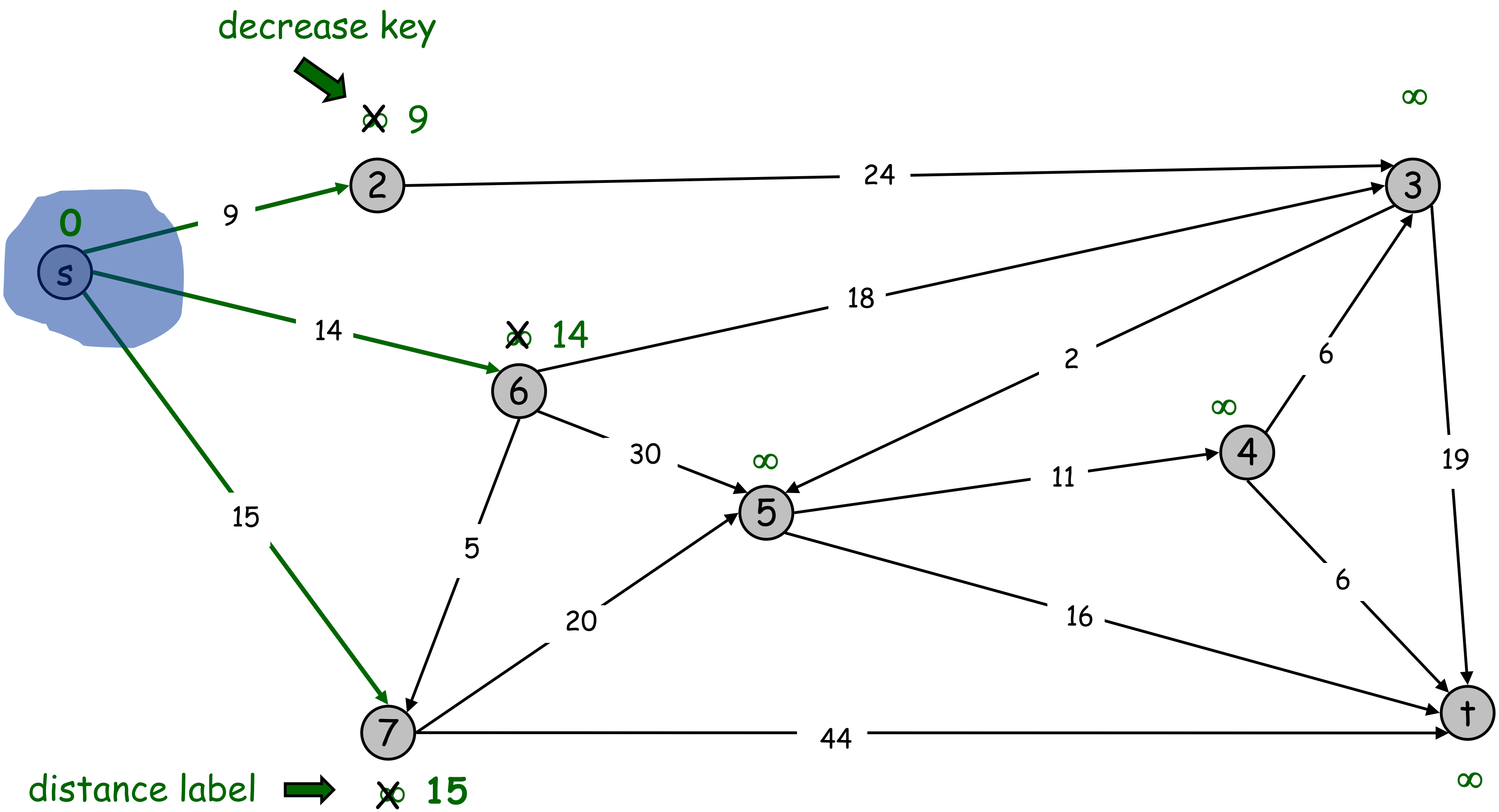

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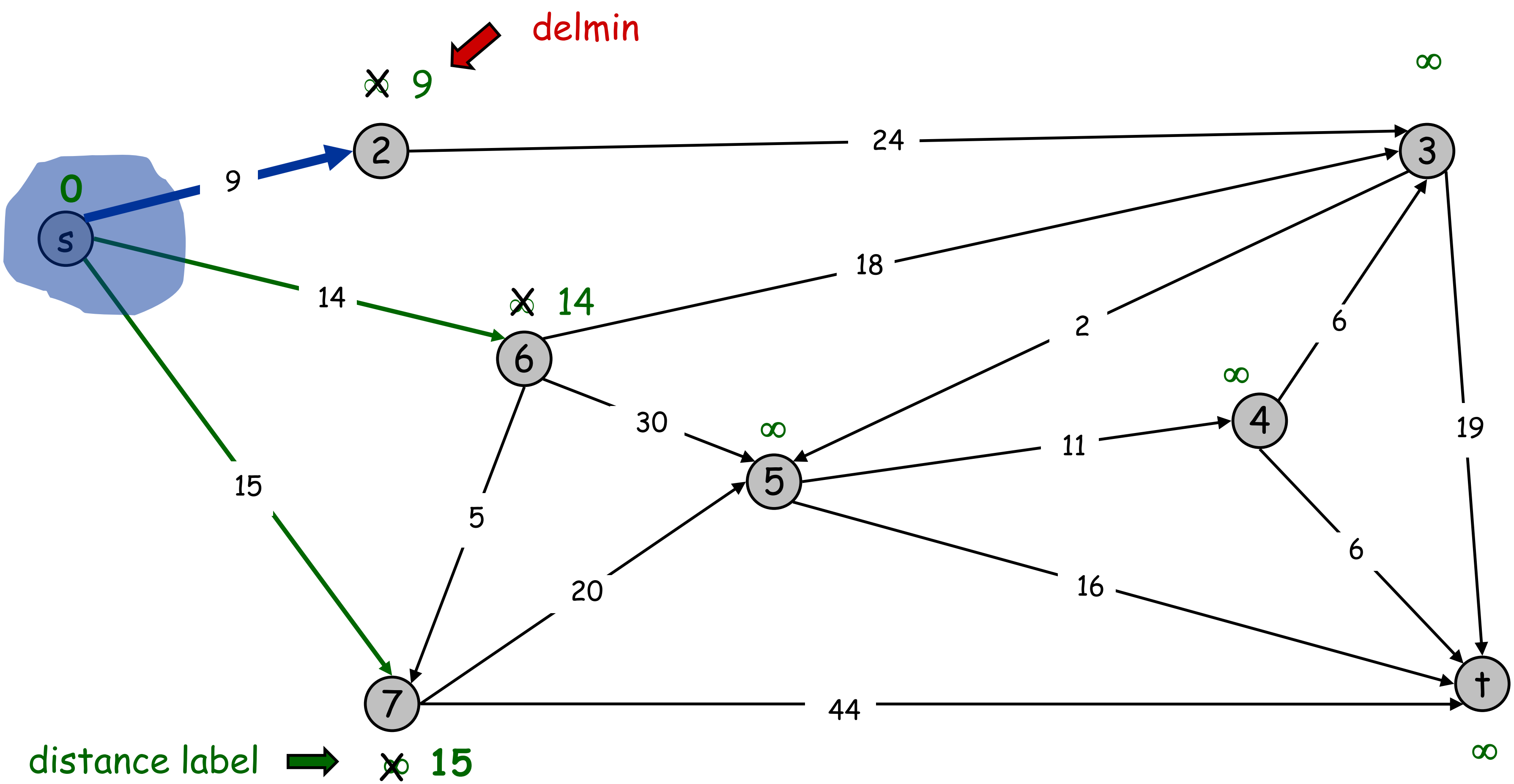
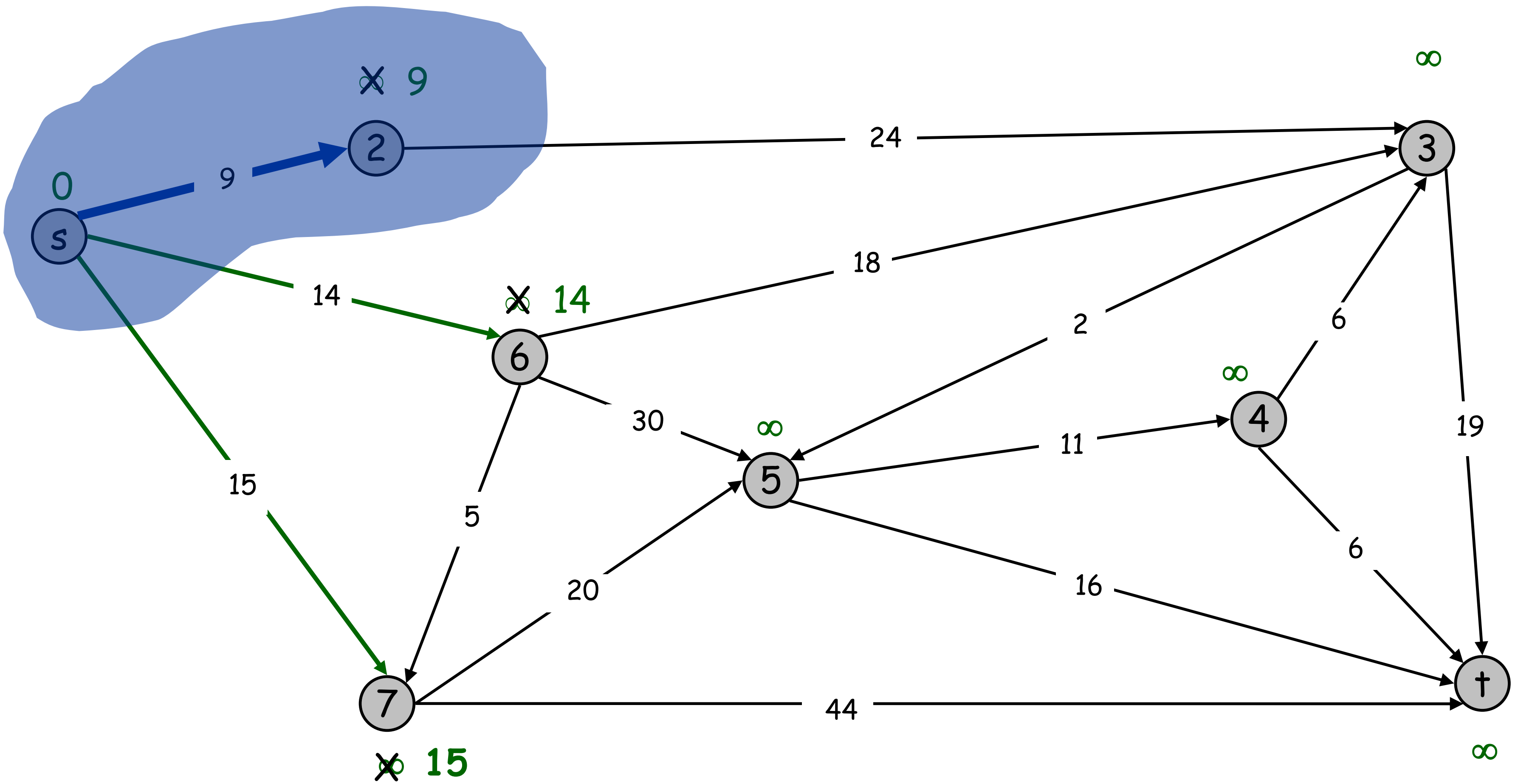


# Dijkstra's Shortest Path Algorithm

S = { s, 2 }
PQ = { 3, 4, 5, 6, 7, t }



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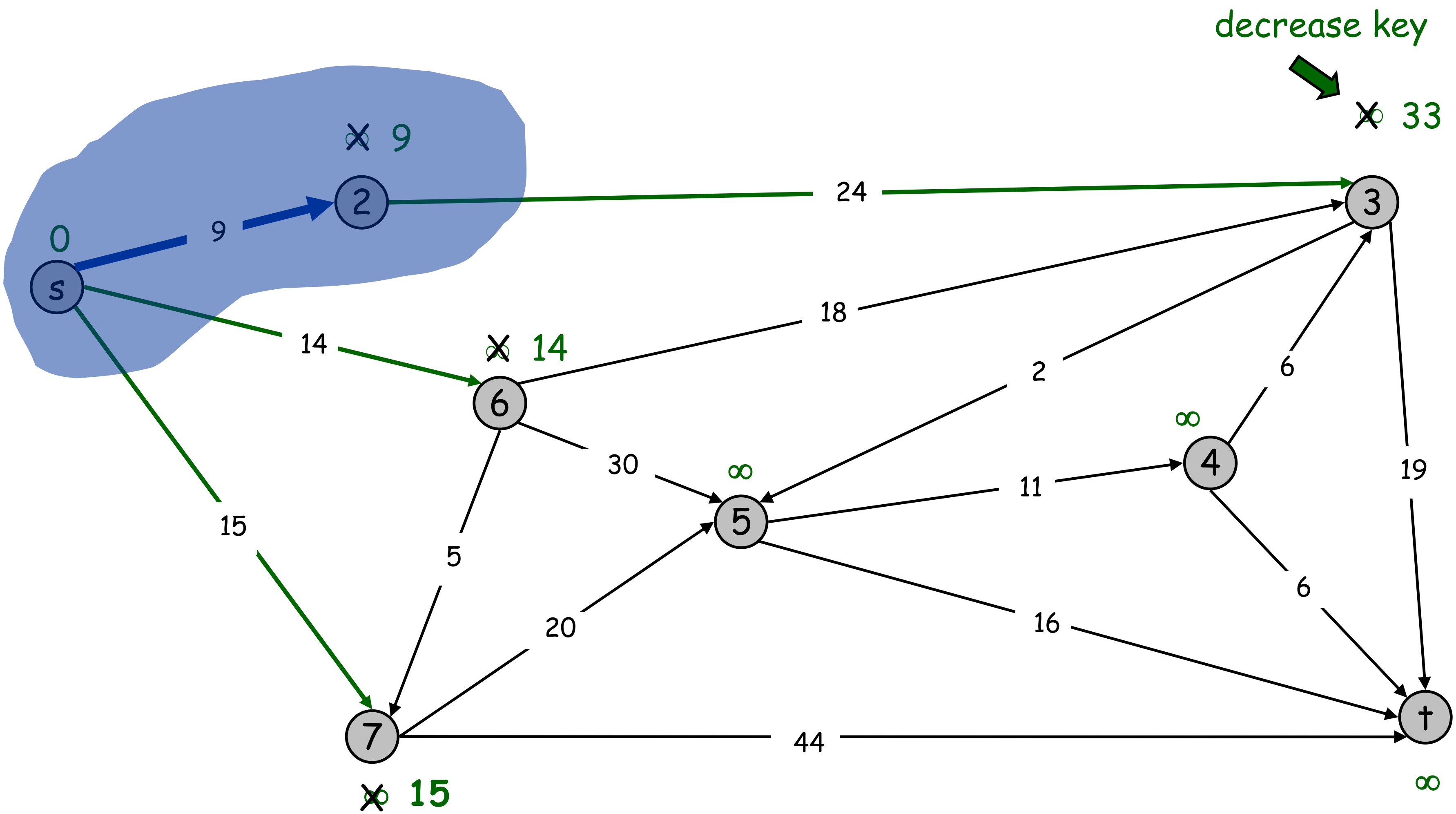

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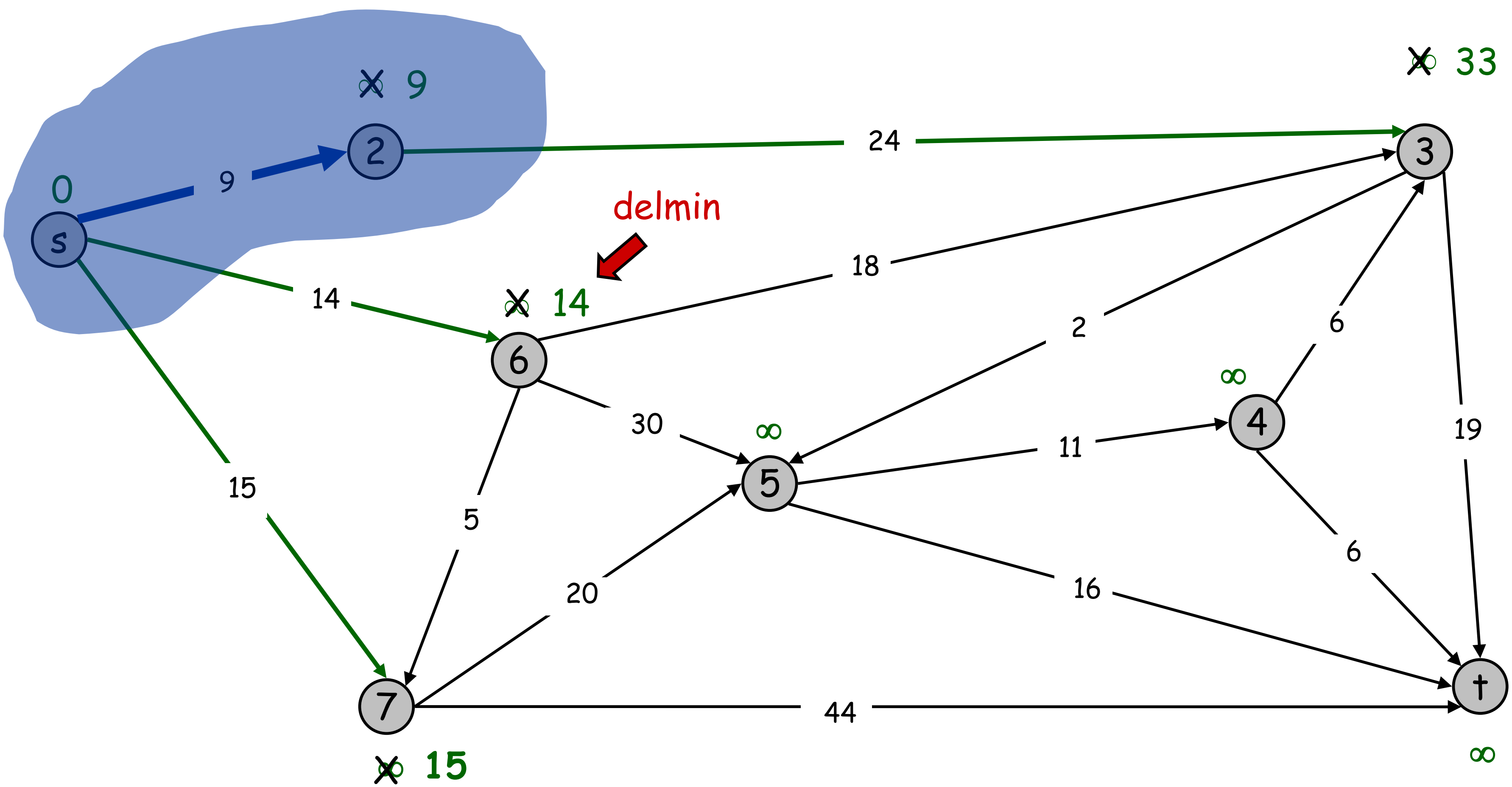
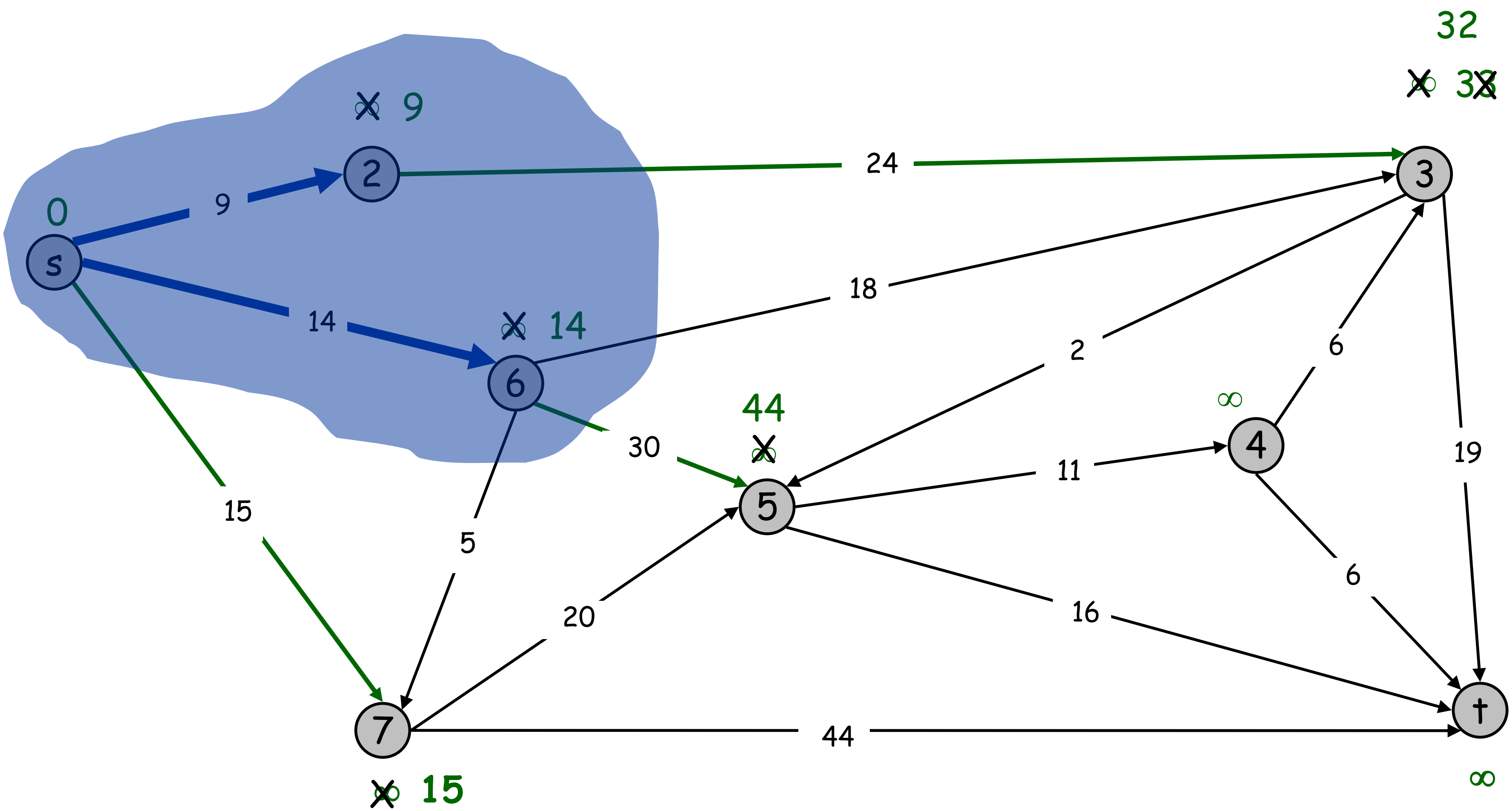


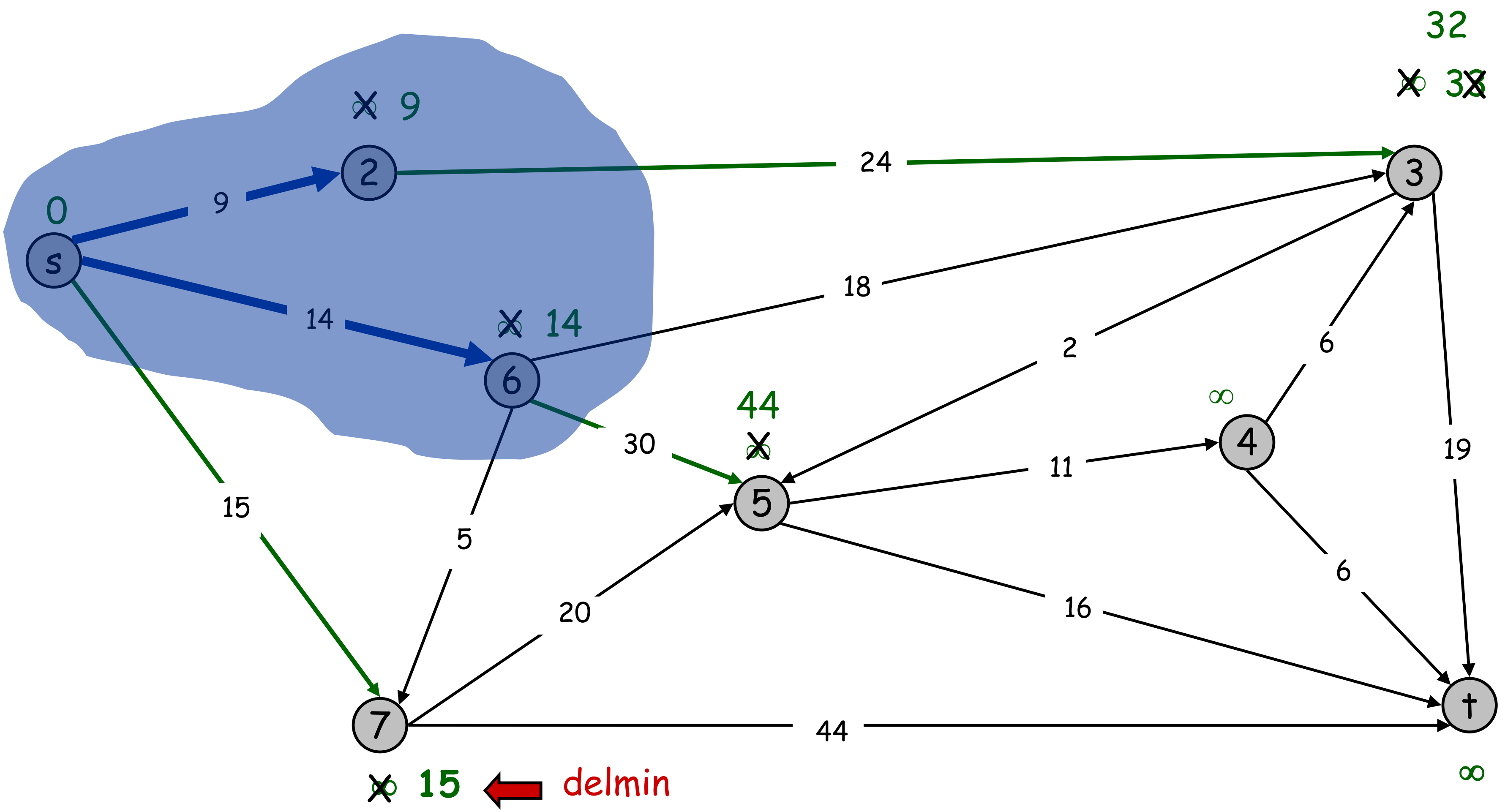
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 6 }
PQ = { 3, 4, 5, 7, t }



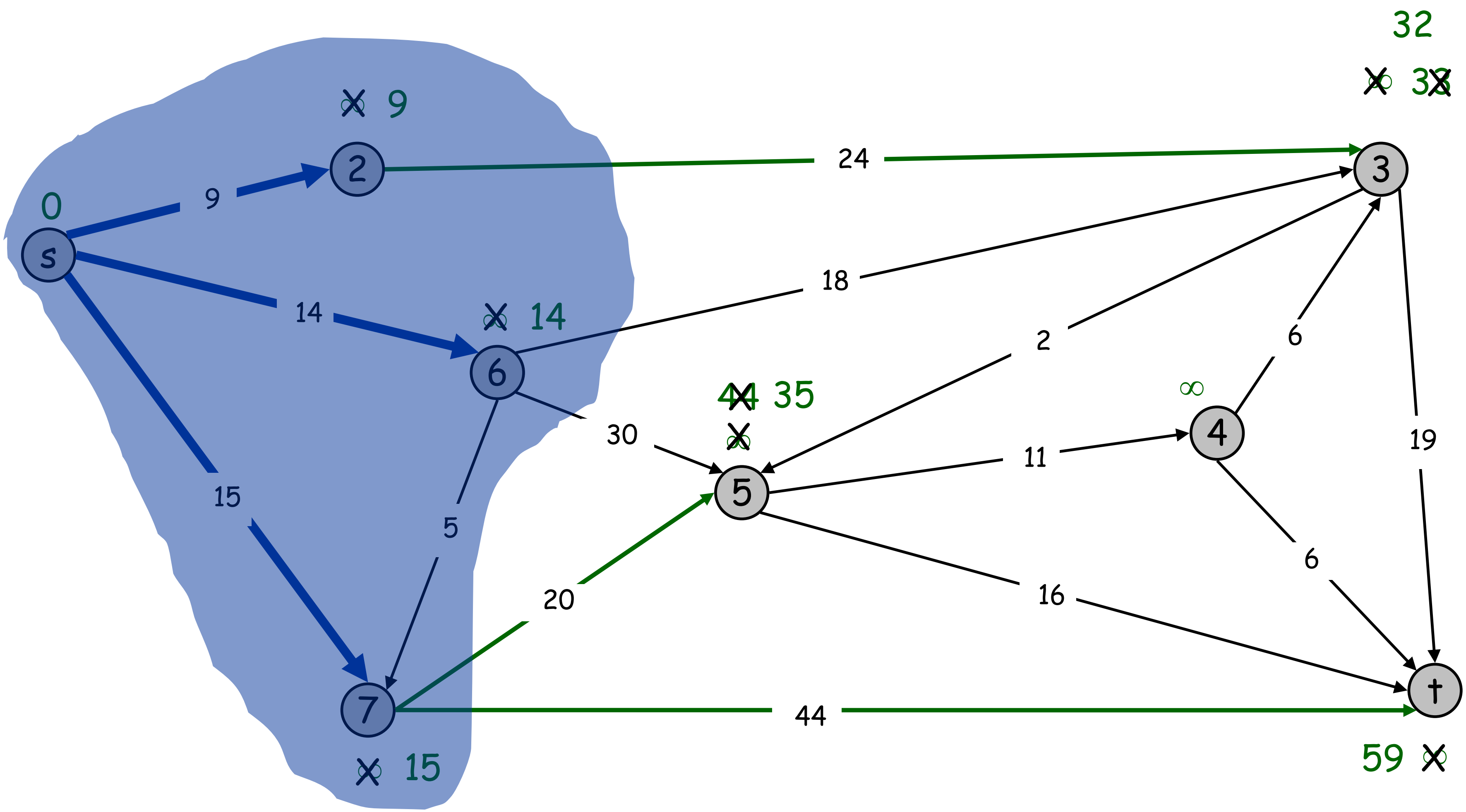
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 6 }
PQ = { 3, 4, 5, 7, t }



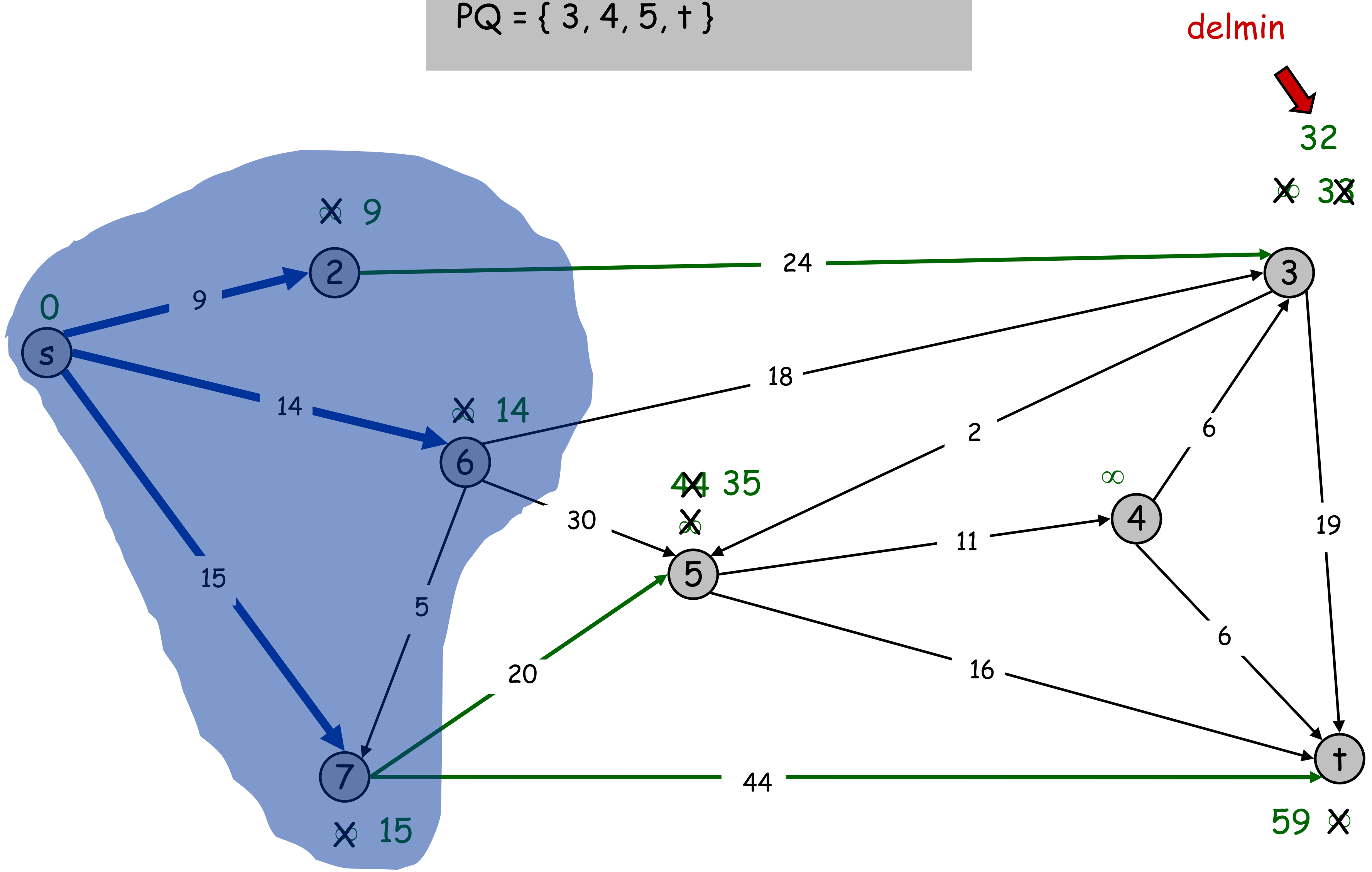
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 6, 7 }
PQ = { 3, 4, 5, t }



# Dijkstra's Shortest Path Algorithm

S = { s, 2, 6, 7 }
PQ = { 3, 4, 5, t }



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S = { s, 2, 3, 6, 7 }
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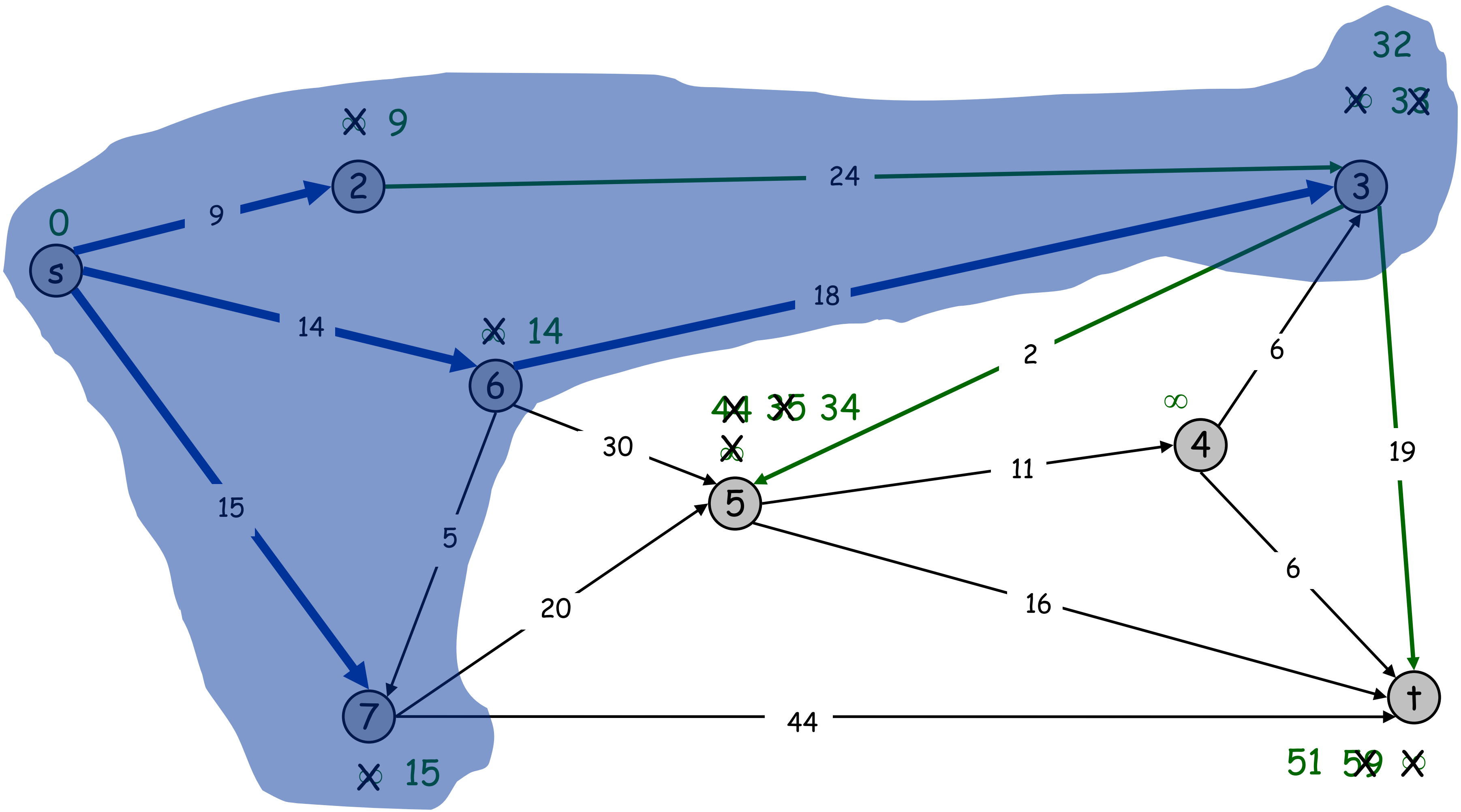

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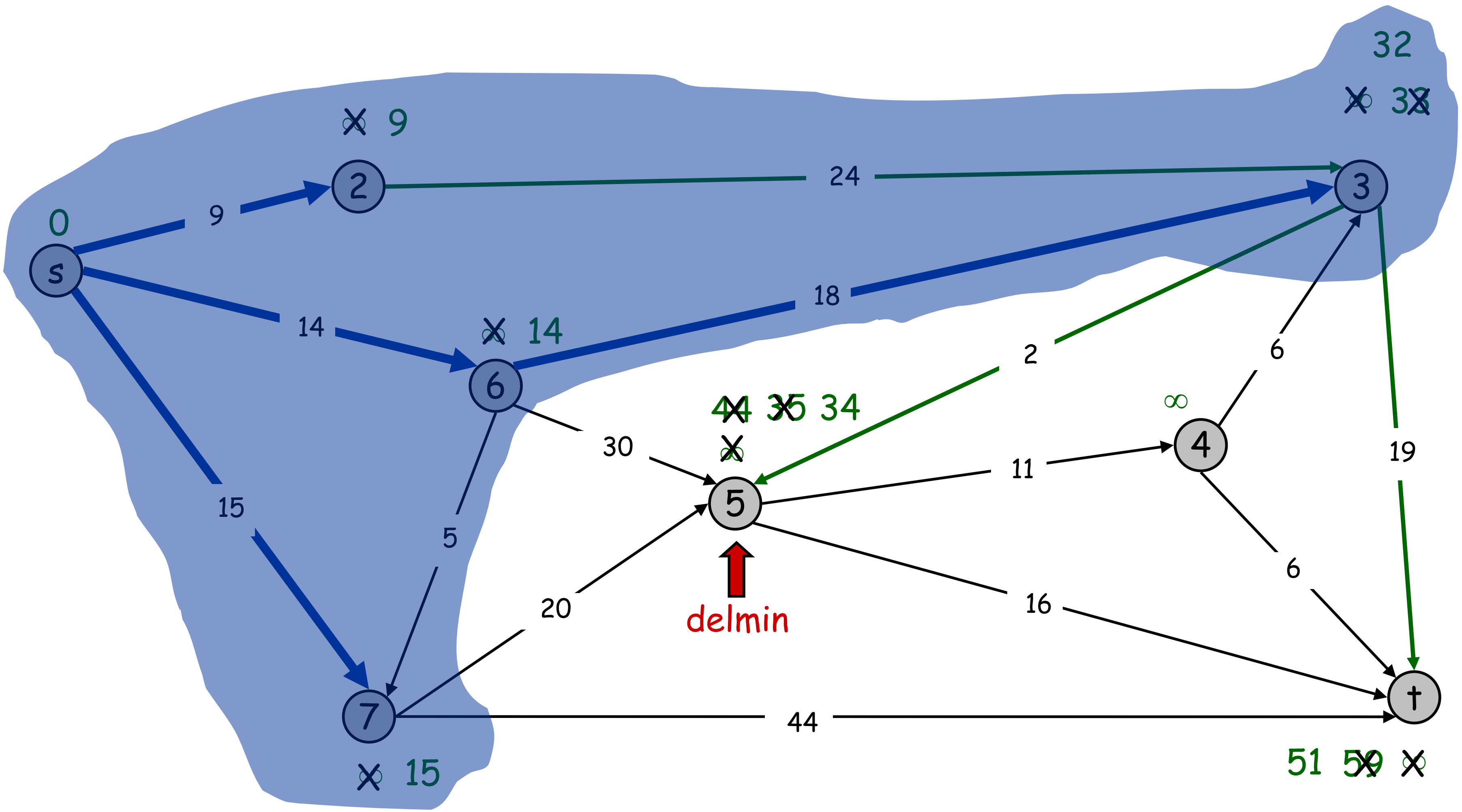
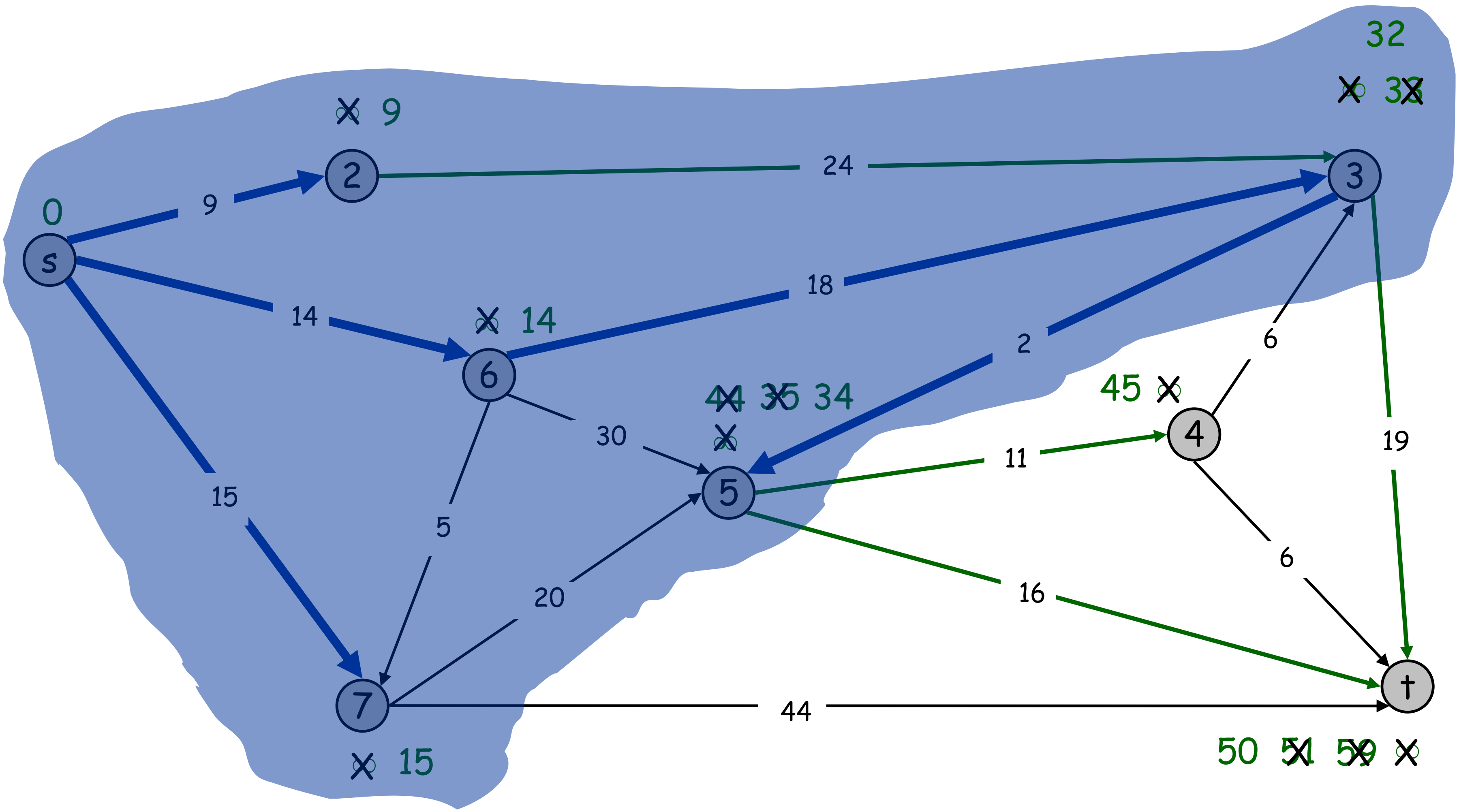
PQ = { 4, 5, t }



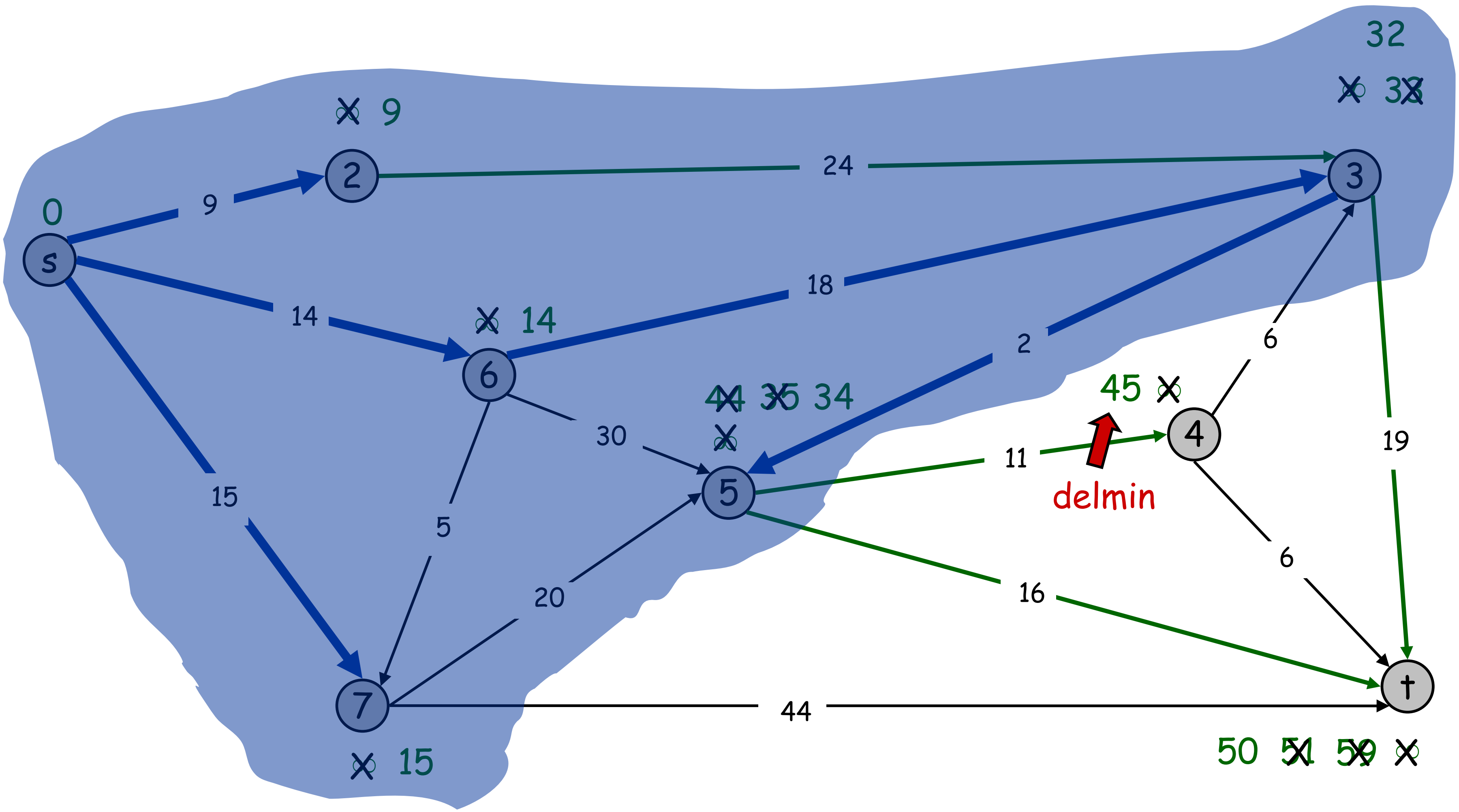
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 5, 6, 7 }
PQ = { 4, t }



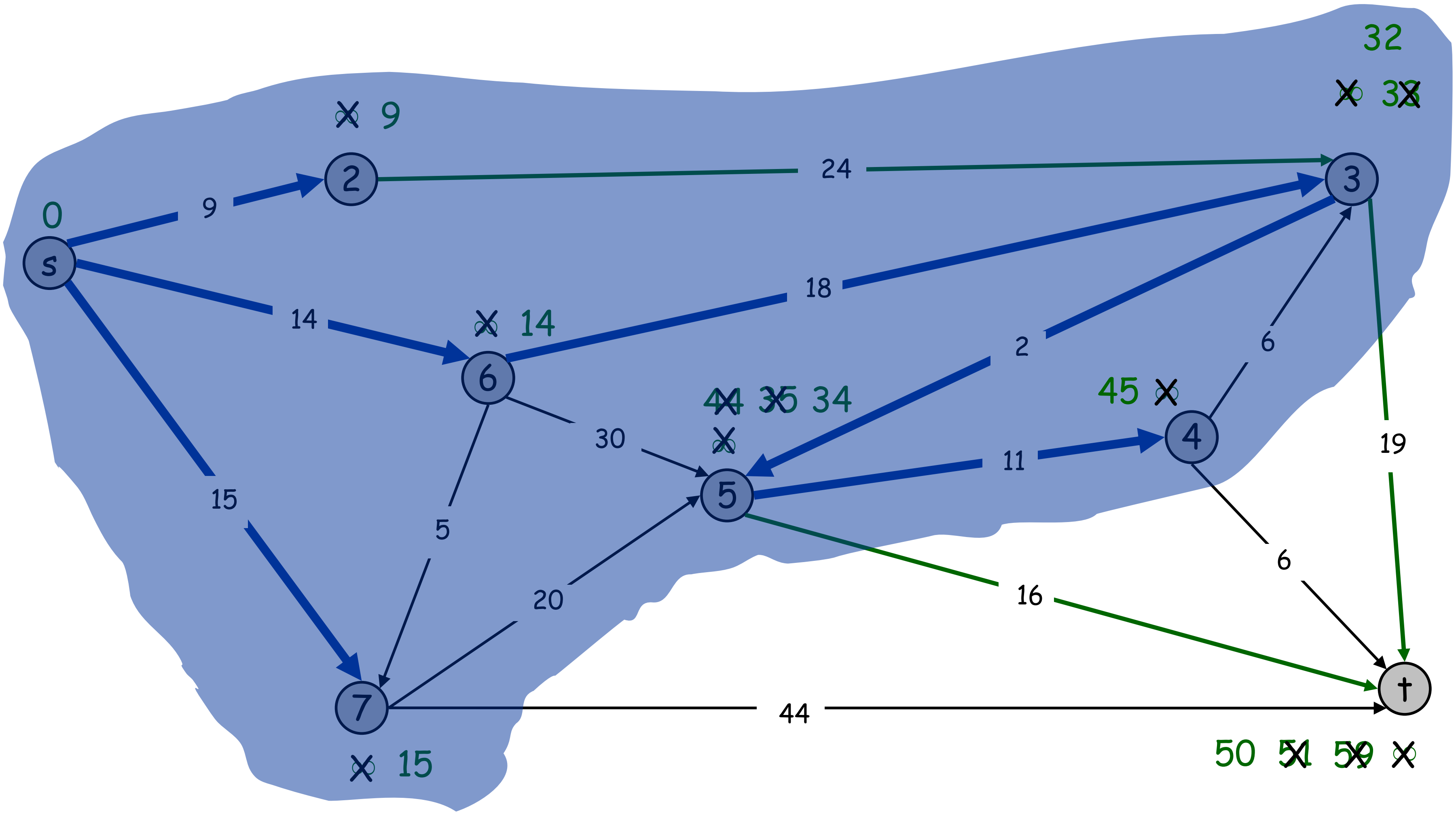
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 5, 6, 7 }
PQ = { 4, t }



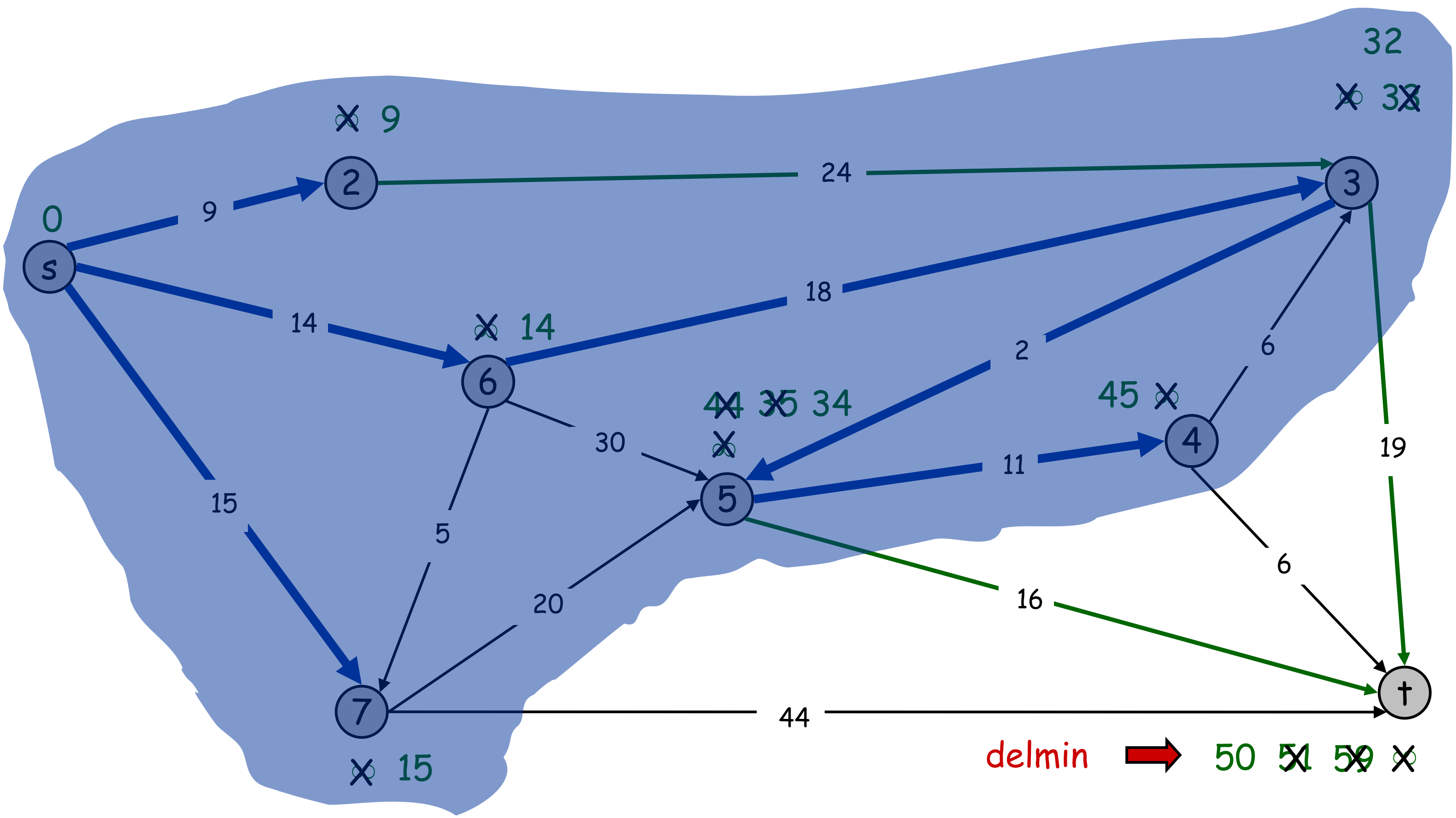
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 4, 5, 6, 7 }
PQ = { t }



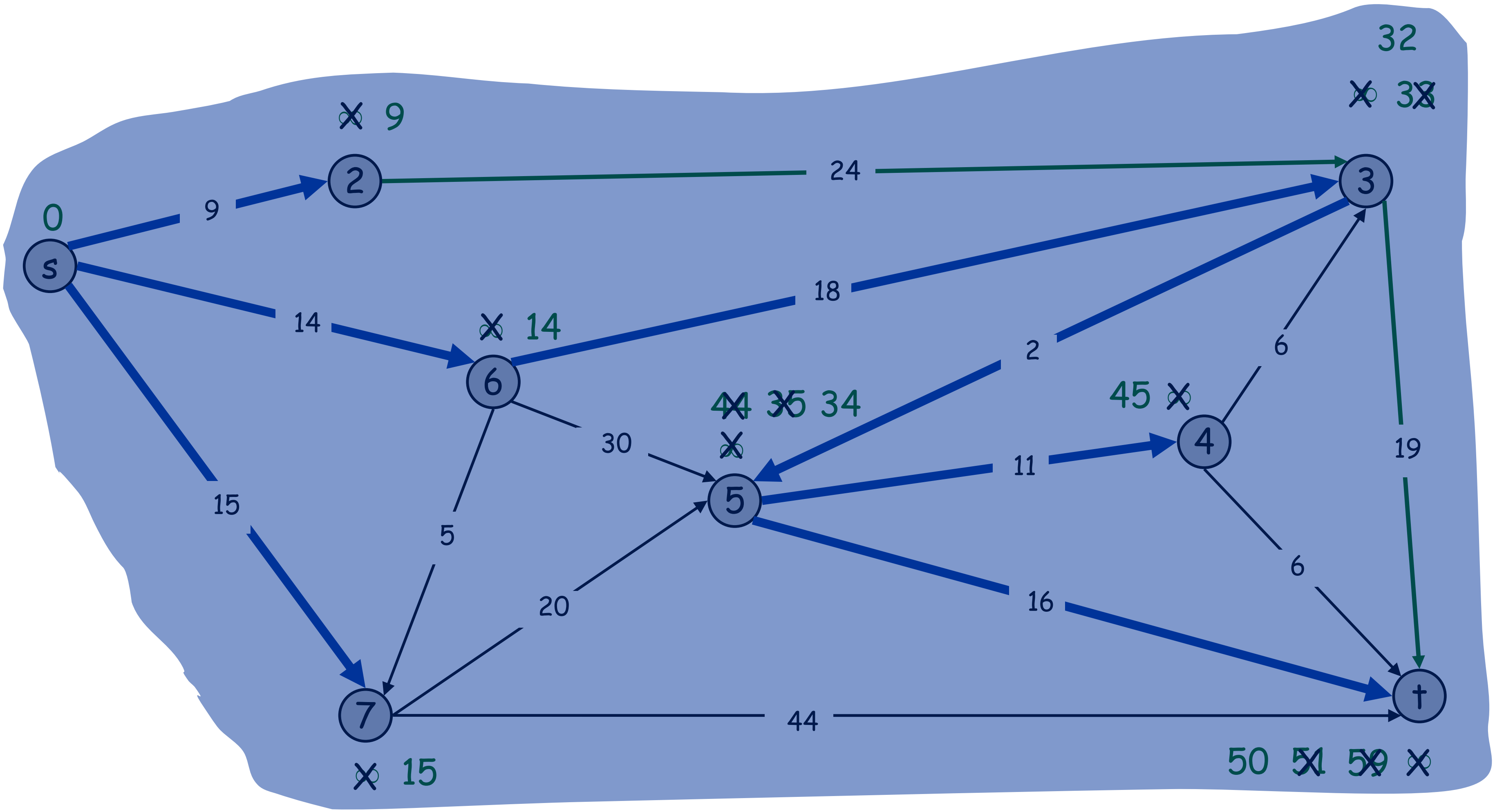
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 4, 5, 6, 7 }
PQ = { t }



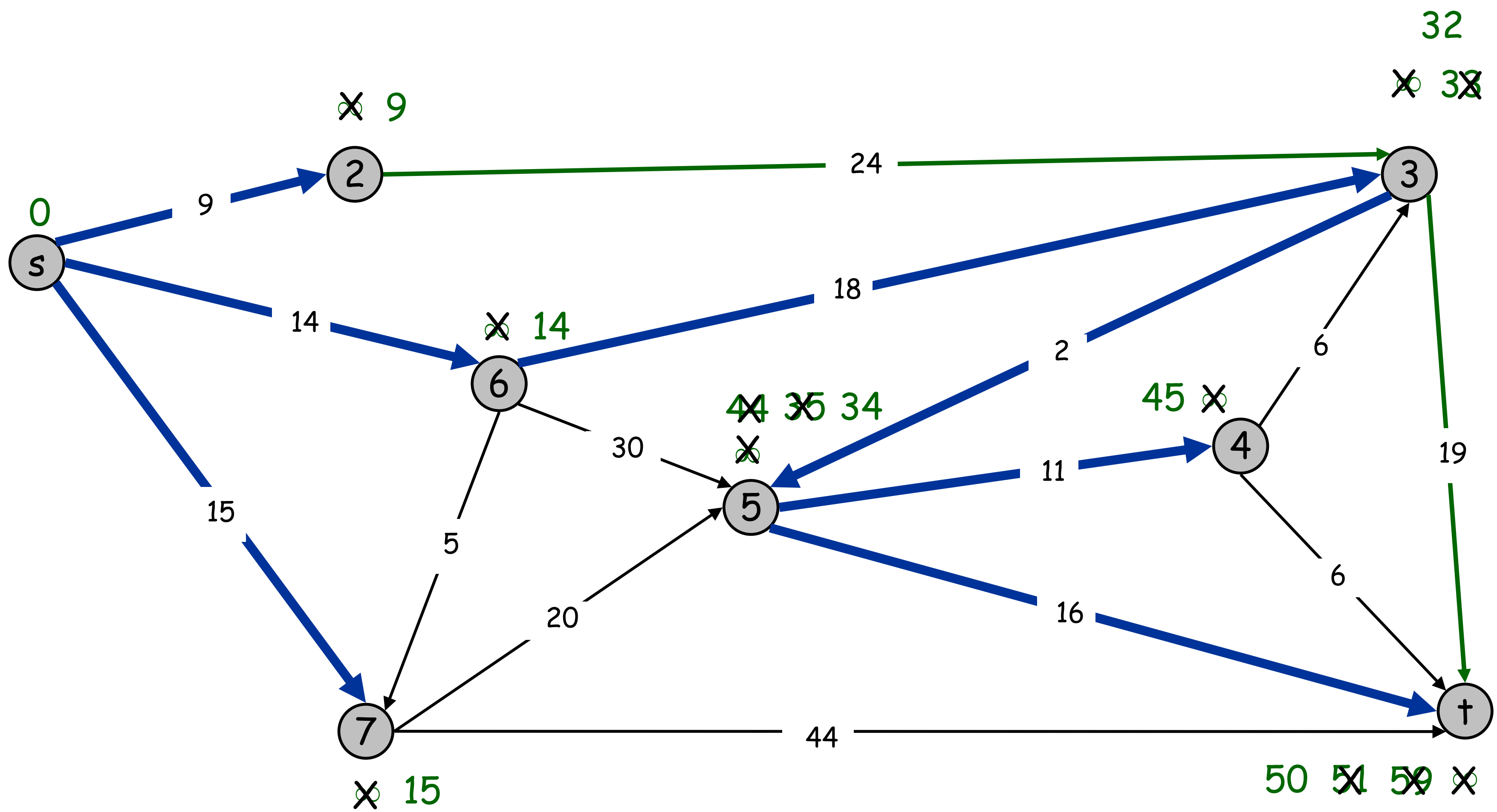
# Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 4, 5, 6, 7, t }
PQ = { }



# Dijkstra's Shortest Path Algorithm

S = { s, 2, 3, 4, 5, 6, 7, t }
PQ = { }



# Dijkstra's Algorithm: Proof of Correctness

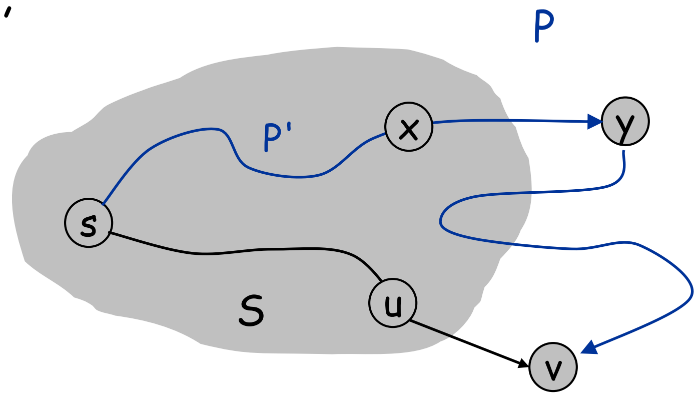
**Invariant.** For each node $u \in S$, $d(u)$ is the length of the shortest $s$-$u$ path.

**Pf.** (by induction on $|S|$)

**Base case:** $|S| = 1$ is trivial.

**Inductive hypothesis:** Assume true for $|S| = k \geq 1$.

- Let $v$ be next node added to $S$, and let $u$-$v$ be the chosen edge.
- The shortest $s$-$u$ path plus $(u, v)$ is an $s$-$v$ path of length $\pi(v)$.
- Consider any $s$-$v$ path $P$. We'll see that it's no shorter than $\pi(v)$.
- Let $x$-$y$ be the first edge in $P$ that leaves $S$, and let $P'$ be the subpath to $x$.
- $P$ is already too long as soon as it leaves $S$.




$$\underbrace{\ell(P) \geq \ell(P') + \ell(x,y)}_{\text{nonnegative weights}} \geq \underbrace{d(x) + \ell(x, y)}_{\text{inductive hypothesis}} \geq \underbrace{\pi(y)}_{\text{defn of } \pi(y)} \geq \underbrace{\pi(v)}_{\text{Dijkstra chose } v \text{ instead of } y}$$

# Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$ .

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{ \ \pi(w), \ \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

| PQ Operation | Dijkstra | Array | Binary heap | d-way Heap | Fib heap † |
|---|---|---|---|---|---|
| Insert | n | n | log n | $d \log_d n$ | 1 |
| ExtractMin | n | n | log n | $d \log_d n$ | log n |
| ChangeKey | m | 1 | log n | $\log_d n$ | 1 |
| IsEmpty | n | 1 | 1 | 1 | 1 |
| Total | | $n^2$ | m log n | $m \log_{m/n} n$ | m + n log n |

† Individual ops are amortized bounds

# 4.3 Optimal Caching

# Optimal Offline Caching

**Caching.**

- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

**Goal.** Eviction schedule that minimizes number of evictions.

**Ex:** k = 2, initial cache = ab,
requests: a, b, c, b, c, a, a, b.

**Optimal eviction schedule:** 2 evictions.

| requests | cache | |
|---|---|---|
| a | a | b |
| b | a | b |
| c | **c** | b |
| b | c | b |
| c | c | b |
| a | **a** | b |
| a | a | b |
| b | a | b |

# Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache: | a | b | c | d | e | f |

future queries: **g a b c e d a b b a c d e a f a d e f g h . . .**

↑ cache miss (g) ↑ eject this one (f)

Theorem. [Bellady, 1960s] FF is optimal eviction schedule.

Pf. Algorithm and theorem are intuitive; proof is subtle.

# Reduced Eviction Schedules

**Def.** A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

**Intuition.** Can transform an unreduced schedule into a reduced one with no more evictions.

| | | | |
|---|---|---|---|
| a | a | b | c |
| a | a | x | c |
| c | a | d | c |
| d | a | d | b |
| a | a | c | b |
| b | a | x | b |
| c | a | c | b |
| a | a | b | c |
| a | a | b | c |

an unreduced schedule

| | | | |
|---|---|---|---|
| a | a | b | c |
| a | a | b | c |
| c | a | b | c |
| d | a | d | c |
| a | a | d | c |
| b | a | d | b |
| c | a | c | b |
| a | a | c | b |
| a | a | c | b |

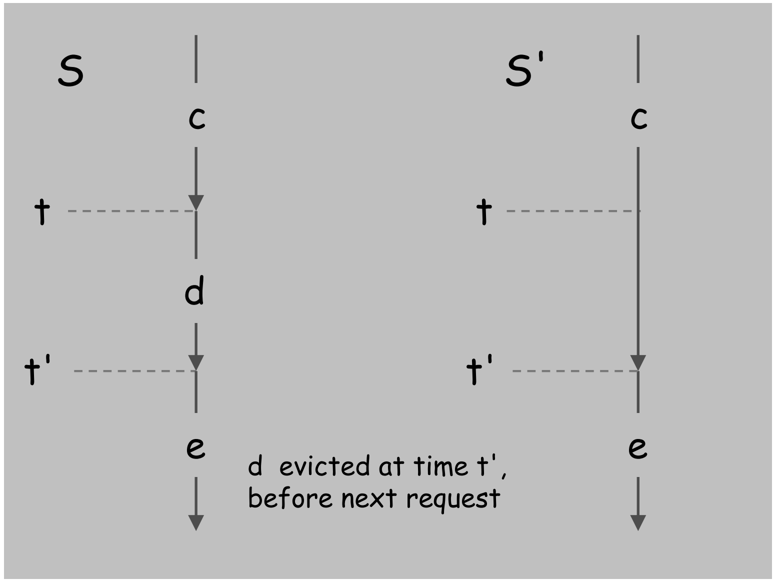
a reduced schedule

# Reduced Eviction Schedules

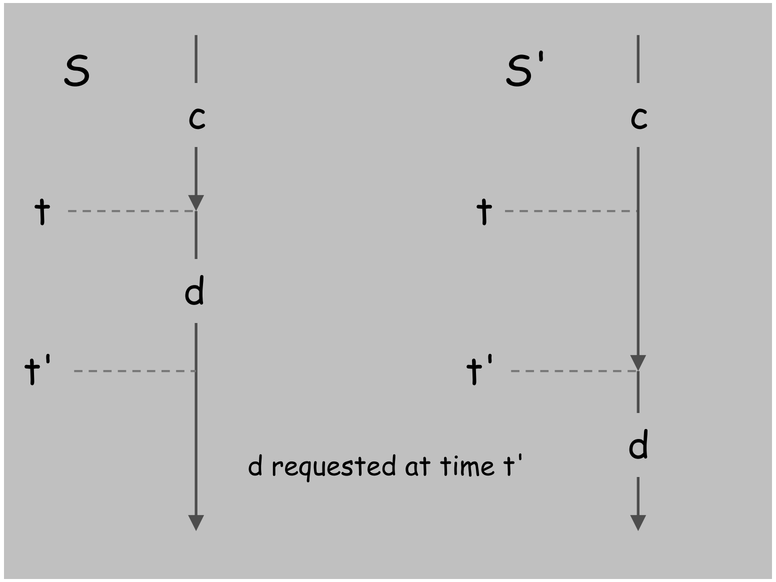
Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more evictions.

Pf. (by induction on number of unreduced items) doesn't enter cache at requested time

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- Case 1: d evicted at time t', before next request for d.
- Case 2: d requested at time t' before d is evicted. ▪




Case 1




Case 2

# Farthest-In-Future: Analysis

Lemma. Let $S$ be a reduced schedule that makes the same schedule as $S_{FF}$ through the first j requests. Then there is a reduced schedule $S'$ that makes the same schedule as $S_{FF}$ through the first j+1 requests, and incurs no more evictions than $S$ does.

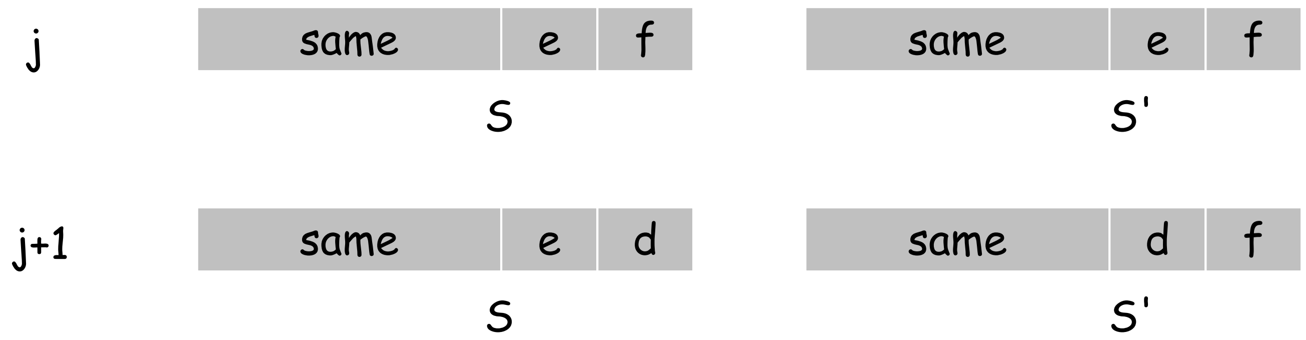
Pf.

- Consider $(j+1)^{st}$ request $d = d_{j+1}$.
- Since $S$ and $S_{FF}$ have agreed up until now, they have the same cache contents before request j+1.
- Case 1: (d is already in the cache). $S' = S$
- Case 2: (d is not in the cache and $S$ and $S_{FF}$ evict the same element). $S' = S$

# Farthest-In-Future: Analysis

Pf. (continued)

- Case 3: (d is not in the cache; $S_{FF}$ evicts e; S evicts f $\neq$ e).
  - begin construction of S' from S by evicting e instead of f




  - now S' agrees with $S_{FF}$ on first j+1 requests
  - From request j+2 onward, we make S' the same as S, but this becomes impossible when e or f is involved

# Farthest-In-Future: *Analysis*

Pf. (continued)

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)

| | | | | |
|---|---|---|---|---|
| j' | same | e | same | f |
| | S | | S' | |

- Case 3a: g = e. Can't happen with Farthest-In-Future since there must be a request for f before e.

# Farthest-In-Future: Analysis

Pf. (continued)

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

| j' | S | | S' | |
|---|---|---|---|---|
| | same | e | same | f |

otherwise S' would take the same action

- Case 3b: $g \neq e, f$. S must evict e.
  Make S' evict f; now S and S' have the same cache. ▪

| j' | S | | S' | |
|---|---|---|---|---|
| | same | g | same | g |

# Farthest-In-Future: Analysis

Pf. (continued)

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

| j' | same | e | | same | f |
|---|---|---|---|---|---|
| | S | | | S' | |

- Case 3c: g = f. Element f can't be in cache of S, so let e' be the element that S evicts.
  - if e' = e, S' accesses f from cache; now S and S' have same cache
  - if e' $\neq$ e, S' evicts e' and brings e into the cache; now S and S' have the same cache. S' is no longer reduced, but can be transformed into a reduced schedule which
    a) agrees with $S_{FF}$ through step j+1
    b) has no more evictions than S

# Farthest-In-Future: Analysis

Theorem. FF is optimal eviction algorithm.
Pf. (by induction on number of requests j)

Base case (trivial):
There exists an optimal reduced schedule S that makes the same schedule as $S_{FF}$ through the first 0 requests.

Inductive step (implied by the lemma):
If there exists an optimal reduced schedule S that agrees with $S_{FF}$ through the first j requests,
then there exists an optimal reduced schedule S' that agrees with $S_{FF}$ through the first j+1 requests

# Caching Perspective

Online vs. offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

LRU. Evict page whose most recent access was earliest.

FF with direction of time reversed!

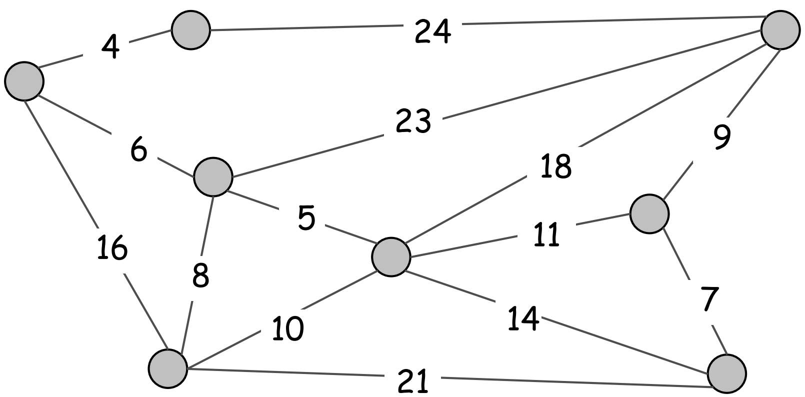
Theorem. FF is optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.

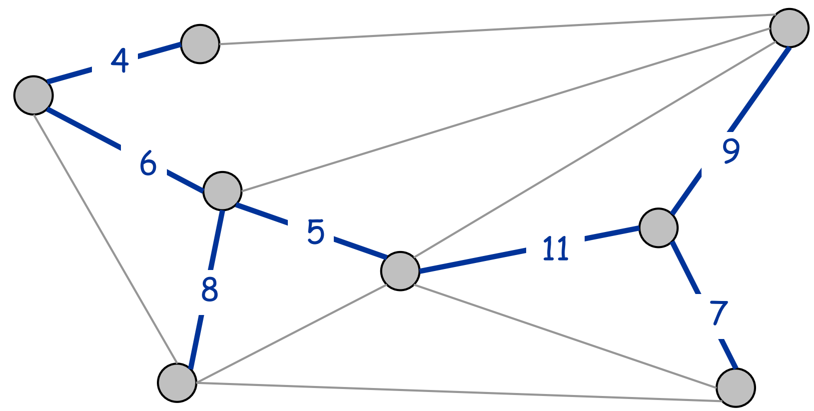
# 4.5 Minimum Spanning Tree

# Minimum Spanning Tree

**Minimum spanning tree.** Given a connected graph $G = (V, E)$ with real-valued edge weights $c_e$, an MST is a subset of the edges $T \subseteq E$ such that $T$ is a spanning tree whose sum of edge weights is minimized.




$G = (V, E)$




$T, \ \Sigma_{e \in T} c_e = 50$

**Cayley's formula.** There are $n^{n-2}$ spanning trees of $n$ vertices.

can't solve by brute force

# Applications

MST is fundamental problem with diverse applications.

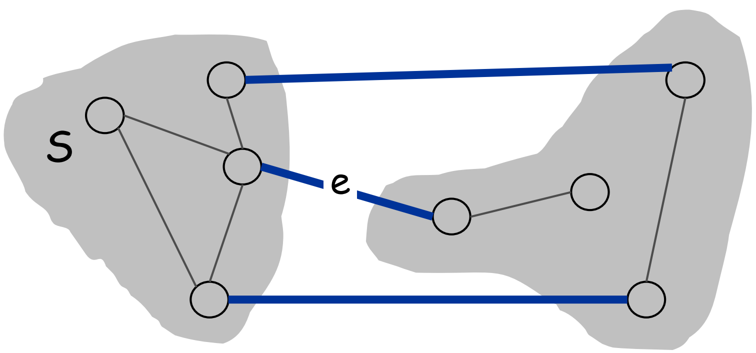
- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

# Greedy Algorithms

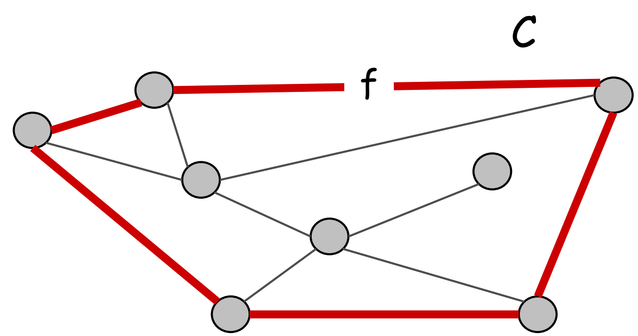
Simplifying assumption. All edge costs $c_e$ are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.




e is in the MST



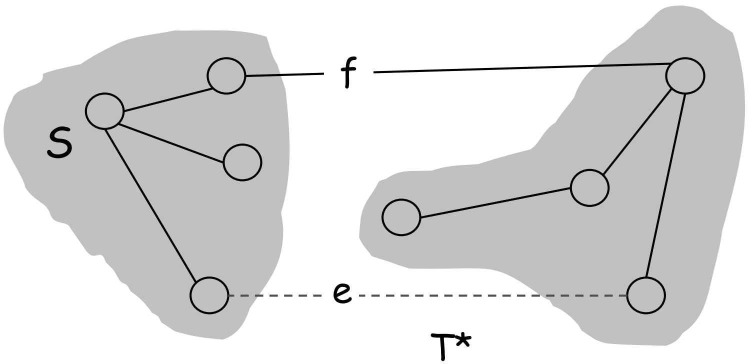

f is not in the MST

Simplifying assumption. All edge costs $c_e$ are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST $T^*$ contains e.

Pf. (exchange argument)

- Suppose e does not belong to $T^*$, and let's see what happens.
- Adding e to $T^*$ creates a cycle C in $T^*$.
- e is in a cycle C with exactly one endpoint in S $\Rightarrow$ there exists another edge f in C with exactly one endpoint in S.
- $T' = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, $\text{cost}(T') < \text{cost}(T^*)$.
- This is a contradiction. ▪

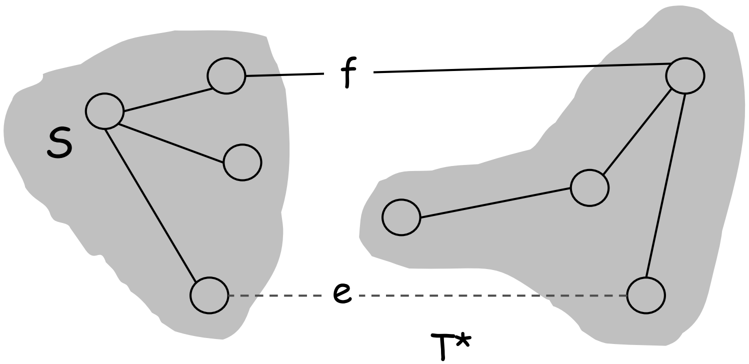


Simplifying assumption. All edge costs $c_e$ are distinct.

Cycle property. Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (exchange argument)

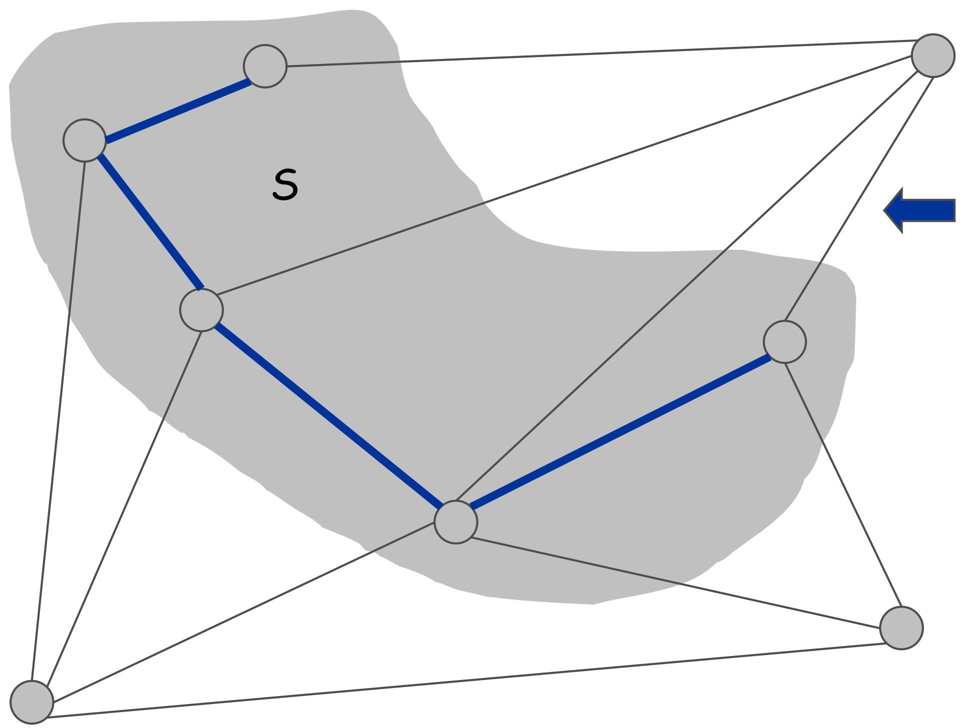
- Suppose f belongs to T*, and let's see what happens.
- Deleting f from T* creates a cut S in T*.
- f is in the cycle C with exactly one endpoint in S $\Rightarrow$ there exists another edge e in C with exactly one endpoint in S.
- $T' = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, $\text{cost}(T') < \text{cost}(T^*)$.
- This is a contradiction. ▪



# Prim's Algorithm: Proof of Correctness

Prim's algorithm. [Jarník 1930, Prim 1957, Dijkstra 1959]

- Initialize S = any node.
- Apply cut property to S.
- Add min cost edge in cutset corresponding to S to T, and add one new explored node u to S.



# Implementation: Prim's Algorithm

Implementation. Use a priority queue.

- Maintain set of explored nodes S.
- For each unexplored node v, maintain attachment cost a[v] = cost of cheapest edge from v to a node in S.
- $O(n^2)$ with an array; $O(m \log n)$ with a binary heap.

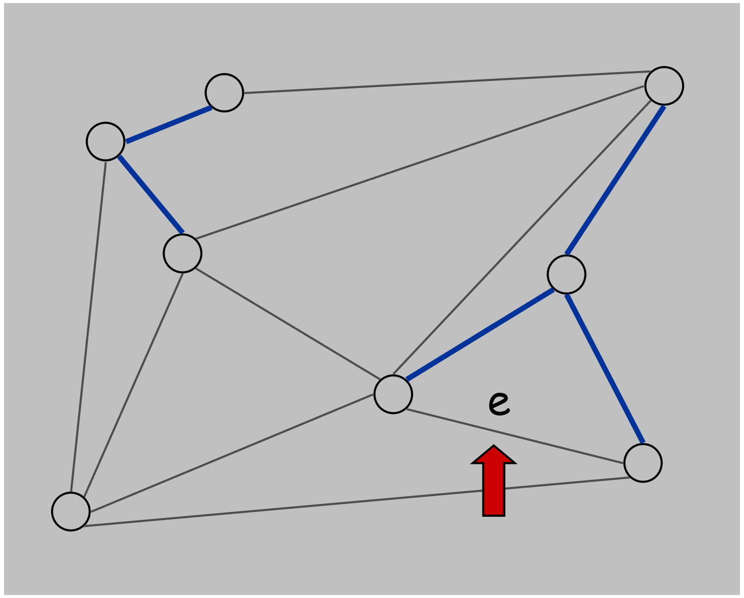
```
Prim(G, c) {
   foreach (v ∈ V) a[v] ← ∞
   Initialize an empty priority queue Q
   foreach (v ∈ V) insert v onto Q
   Initialize set of explored nodes S ← φ

   while (Q is not empty) {
      u ← delete min element from Q
      S ← S ∪ { u }
      foreach (edge e = (u, v) incident to u)
          if ((v ∉ S) and (c_e < a[v]))
             decrease priority a[v] to c_e
}
```

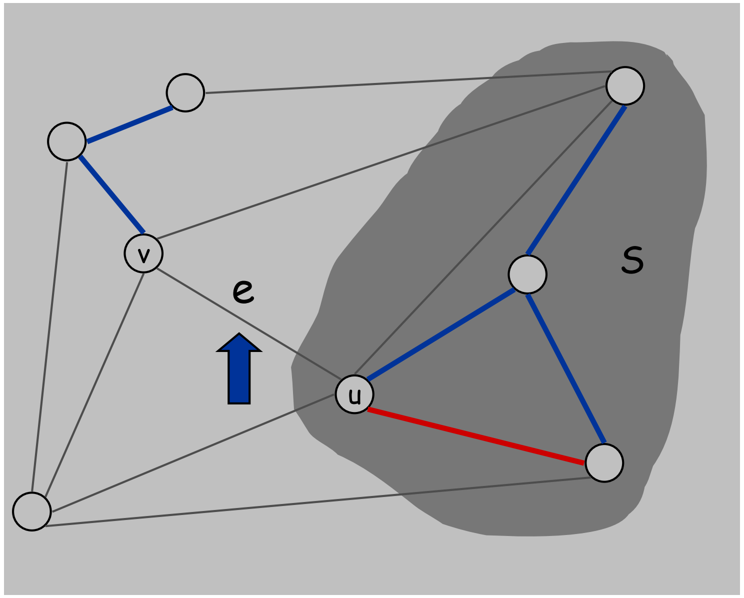
# Kruskal's Algorithm: Proof of Correctness

Kruskal's algorithm. [Kruskal, 1956]

- Consider edges in ascending order of weight.
- Case 1: If adding e to T creates a cycle, discard e according to cycle property.
- Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.




Case 1




Case 2

# Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain set for each connected component.
- $O(m \log m)$ for sorting and $O(m \underbrace{\alpha(m, n)}_{\text{essentially a constant}})$ for union-find.

```
Kruskal(G, c) {
   Sort edges weights so that c₁ ≤ c₂ ≤ ... ≤ cₘ.
   T ← φ

   foreach (u ∈ V) make a set containing singleton u

   for i = 1 to m
      (u,v) = eᵢ
      if (u and v are in different sets) {
         T ← T ∪ {eᵢ}
         merge the sets containing u and v
      }
   return T
}
```

are u and v in different connected components?

merge two components

# Minimum Spanning Tree

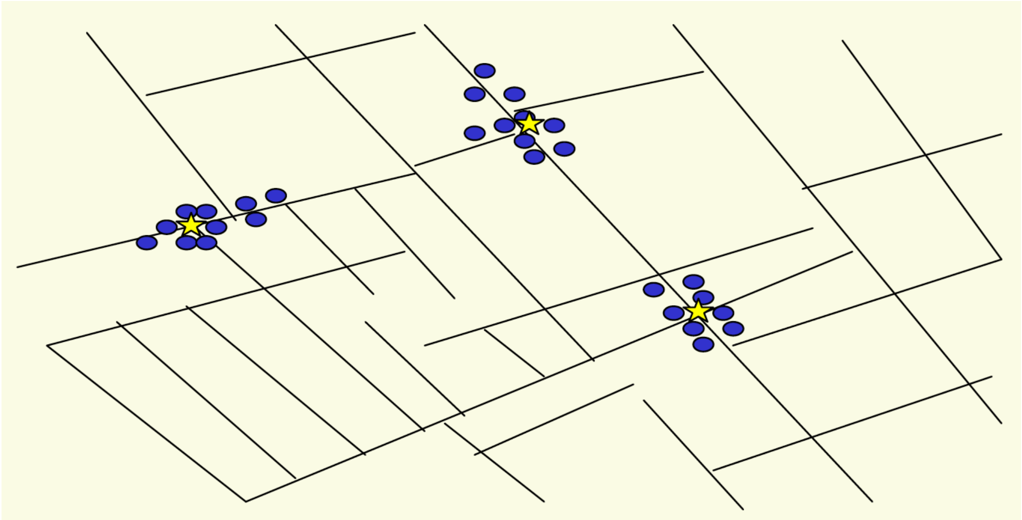
Prim's algorithm. Start with some root node s and greedily grow a tree T from s outward. At each step, add the cheapest edge e to T that has exactly one endpoint in T.

Kruskal's algorithm. Start with $T = \phi$. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle.

Reverse-Delete algorithm. Start with $T = E$. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.

Remark. All three algorithms produce an MST.

# 4.7 Clustering



Outbreak of cholera deaths in London in 1850s.
Reference: Nina Mishra, HP Labs

# Clustering

photos, documents. micro-organisms

**Clustering.** Given a set U of n objects labeled $p_1, \ldots, p_n$, partition into clusters s.t. objects in different clusters are far apart.

e.g., a large number of corresponding pixels whose intensities differ over some threshold

**Fundamental problem.** Divide into clusters so that points in different clusters are far apart.

- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.

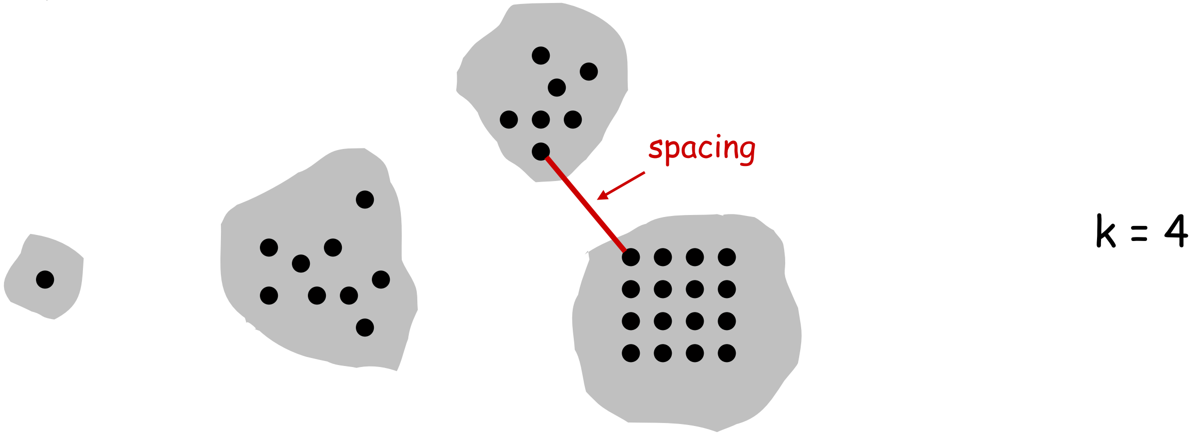
# Clustering of Maximum Spacing

k-clustering. Divide objects into k non-empty groups.

Distance function. Assume it satisfies several natural properties.
- $d(p_i, p_j) = 0$ iff $p_i = p_j$ (identity of indiscernibles)
- $d(p_i, p_j) \geq 0$ (nonnegativity)
- $d(p_i, p_j) = d(p_j, p_i)$ (symmetry)

Spacing. Min distance between any pair of points in different clusters.

Clustering of maximum spacing. Given an integer k, find a k-clustering of maximum spacing.



# Greedy Clustering Algorithm

Single-link k-clustering algorithm.

- Create n clusters, one for each object.
- Find the closest pair of objects such that each object is in a different cluster; add an edge between them and merge the two clusters.
- Repeat n-k times until there are exactly k clusters.

Key observation. This procedure is precisely Kruskal's algorithm (except we stop when there are k connected components).

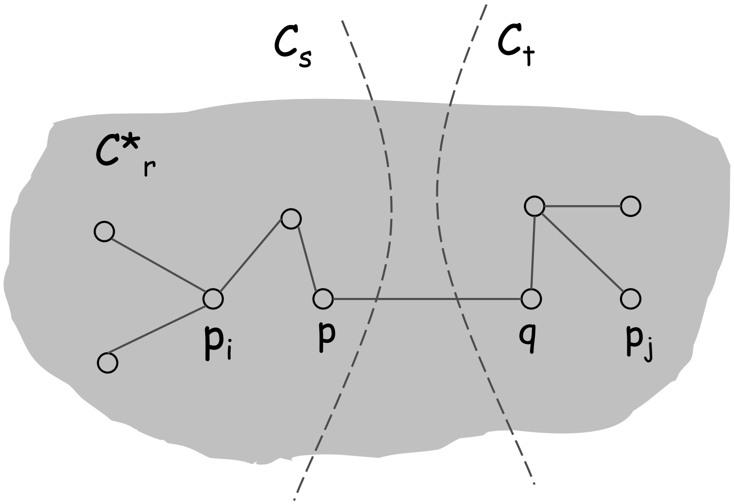
Remark. Equivalent to finding an MST and deleting the k-1 most expensive edges.

# Greedy Clustering Algorithm: Analysis

Theorem. Let $C^*$ denote the clustering $C^*_1, \ldots, C^*_k$ formed by deleting the k-1 most expensive edges of a MST. $C^*$ is a k-clustering of max spacing.

Pf. Let $C$ denote some other clustering $C_1, \ldots, C_k$.

- The spacing of $C^*$ is the length $d^*$ of the $(k-1)^{st}$ most expensive edge in MST.
- Let $p_i$, $p_j$ be in the same cluster in $C^*$, say $C^*_r$, but different clusters in $C$, say $C_s$ and $C_t$.
- Some edge $(p, q)$ on $p_i$-$p_j$ path in $C^*_r$ spans two different clusters in $C$.
- All edges on $p_i$-$p_j$ path have length $\leq d^*$ since Kruskal chose them.
- Spacing of $C$ is $\leq d^*$ since $p$ and $q$ are in different clusters. ▪



# Coin Changing

# Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: $2.89.

# Coin-Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: c1 < c2 < … < cn.

  coins selected
S ← φ
while (x ≠ 0) {
   let k be largest integer such that ck ≤ x
   if (k = 0)
      return "no solution found"
   x ← x - ck
   S ← S ∪ {k}
}
return S
```

Q. Is cashier's algorithm optimal?

# Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greed is optimal for U.S. coinage: 1, 5, 10, 25, 100.

Pf. (by induction on x)

- Consider optimal way to change $c_k \leq x < c_{k+1}$ : greedy takes coin k.
- We claim that any optimal solution must also take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm. ▪

| k | $c_k$ | All optimal solutions must satisfy | Max value of coins 1, 2, …, k-1 in any OPT |
|---|---|---|---|
| 1 | 1 | $P \leq 4$ | - |
| 2 | 5 | $N \leq 1$ | 4 |
| 3 | 10 | $N + D \leq 2$ | 4 + 5 = 9 |
| 4 | 25 | $Q \leq 3$ | 20 + 4 = 24 |
| 5 | 100 | no limit | 75 + 24 = 99 |

# Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

# Greedy Algorithms: Chapter Summary

# Greedy Algorithms

## Basic idea

- Make the locally optimal choice at each step.

## Algorithms

- Interval Scheduling
  - Choose the job with the earliest finish time
- Interval Partitioning
  - Assign lectures to classrooms in increasing order of start time
- Scheduling to Minimize Lateness
  - Choose the job with the earliest deadline
- Optimal Caching
  - Evict item that is requested farthest in future
- Shortest Paths in a Graph
  - Dijkstra: choose the node with the shortest known path
- Minimum Spanning Tree
  - Kruskal, Prim, reverse-deletion
- Clustering
  - Single-link k-clustering (=Kruskal)
- Coin Changing
  - Casher's algorithm

# Greedy Algorithms

## Proof skills

- Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.
- Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.
- Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.