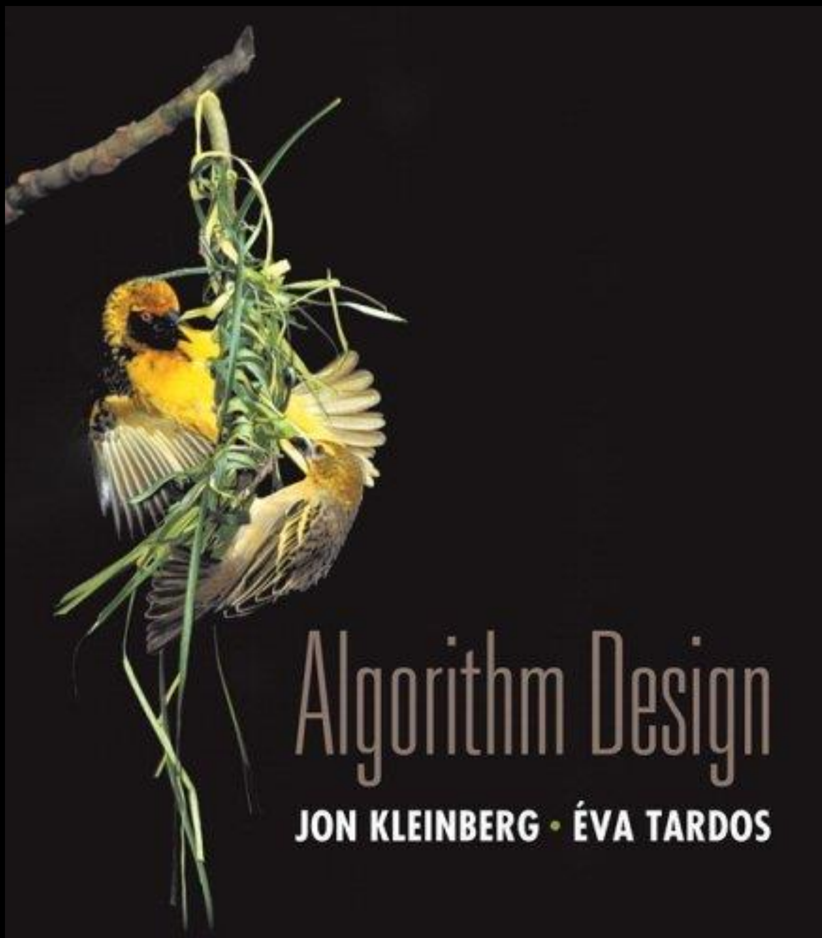


Chapter 13

Randomized Algorithms

Slides by Kevin Wayne.

Randomization

Algorithmic design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Read section 13.12 for a short review of probability theory

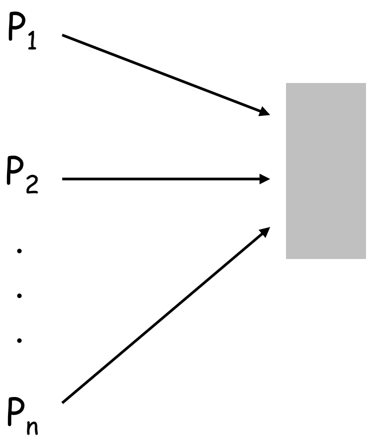
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes $P_1, \ldots, P_n$, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Contention Resolution: Randomized Protocol

Protocol. Each process requests access to the database at time t with probability $p = 1/n$.

Claim. Let $S[i, t]$ = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le \Pr[S(i, t)] \le 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p\,(1-p)^{n-1}$.

process i requests access

none of remaining n-1 processes request access

- Setting $p = 1/n$, we have $\Pr[S(i, t)] = 1/n\ \underbrace{(1 - 1/n)^{n-1}}_{\text{between } 1/e \text{ and } 1/2}$. ▪

value that maximizes Pr[S(i, t)]

Useful facts from calculus. As n increases from 2, the function:

- $(1 - 1/n)^n$ converges monotonically from 1/4 up to 1/e
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to 1/e.

Contention Resolution: Randomized Protocol

Claim. The probability that process i fails to access the database in e·n rounds is at most 1/e. After e·n(c ln n) rounds, the probability is at most n^{-c}.

Pf. Let F[i, t] = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have $\Pr[F(i, t)] \leq (1 - 1/(en))^{t}$.

- Choose $t = \lceil e \cdot n \rceil$: $$\Pr[F(i,t)] \leq \left(1-\tfrac{1}{en}\right)^{\lceil en \rceil} \leq \left(1-\tfrac{1}{en}\right)^{en} \leq \tfrac{1}{e}$$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $$\Pr[F(i,t)] \leq \left(\tfrac{1}{e}\right)^{c \ln n} = n^{-c}$$

Contention Resolution: Randomized Protocol

Claim. The probability that all processes succeed within $2e \cdot n \ln n$ rounds is at least $1 - 1/n$.

Pf. Let $F[t]$ = event that at least one of the n processes fails to access database in rounds 1 through t.

$$\Pr[F[t]] = \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \le \underbrace{\sum_{i=1}^{n} \Pr[F[i,t]]}_{\text{union bound}} \le \underbrace{n\left(1 - \tfrac{1}{en}\right)^t}_{\text{previous slide}}$$

- Choosing $t = 2\lceil en \rceil \lceil \ln n \rceil$ yields $\Pr[F[t]] \le n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

13.2 Global Minimum Cut

Global Minimum Cut

Global min cut. Given a connected, undirected graph $G = (V, E)$ find a cut (A, B) of minimum cardinality.

Applications. Partitioning items in a database, identify clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

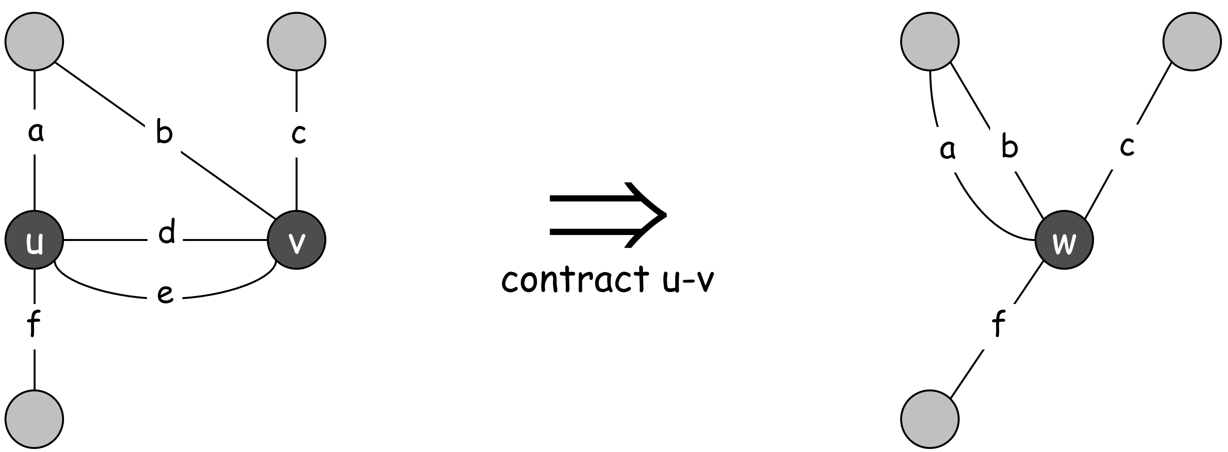
- Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u).
- Pick some vertex s and compute min s-v cut separating s from each other vertex $v \in V$.

False intuition. Global min-cut is harder than min s-t cut.

Contraction Algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge e = (u, v) uniformly at random.
- Contract edge e.
  - replace u and v by single new super-node w
  - preserve edges, updating endpoints of u and v to w
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes v_1 and v_2.
- Return the cut (all nodes that were contracted to form v_1).



Contraction Algorithm

Contraction algorithm. [Karger 1995]

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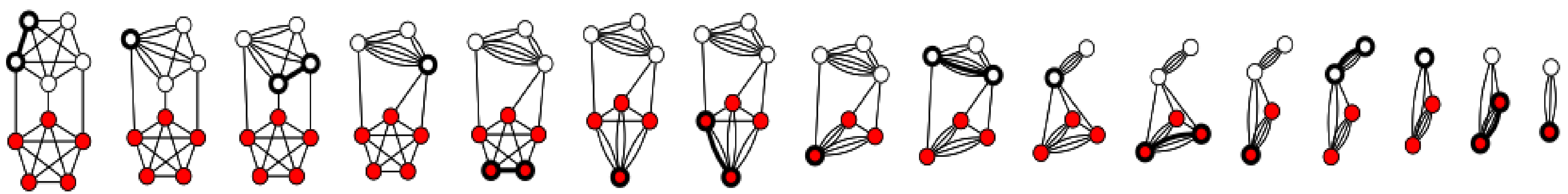
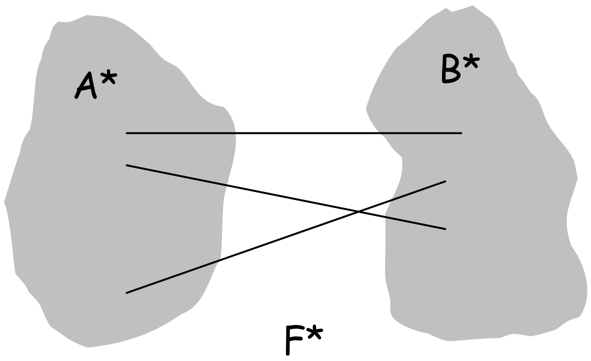


Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$.

Pf. Consider a global min-cut (A^*, B^*) of G. Let F^* be edges with one endpoint in A^* and the other in B^*. Let $k = |F^*|$ = size of min cut.

- In first step, the algorithm contracts an edge in F^* with probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be min-cut. $\Rightarrow$ $|E| \geq \frac{1}{2}kn$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n$.



Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$.

Pf. Consider a global min-cut (A*, B*) of G. Let F* be edges with one endpoint in A* and the other in B*. Let k = |F*| = size of min cut.

- Let G' be graph after j iterations. There are n' = n-j supernodes.
- Suppose no edge in F* has been contracted. The min-cut in G' is still k.
- Since value of min-cut is k, $|E'| \geq \frac{1}{2}kn'$.
- Thus, algorithm contracts an edge in F* with probability $\leq 2/n'$.

- Let E_j = event that an edge in F* is not contracted in iteration j.

$$\begin{aligned}
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}] \\
&\geq \left(1-\tfrac{2}{n}\right)\left(1-\tfrac{2}{n-1}\right)\cdots\left(1-\tfrac{2}{4}\right)\left(1-\tfrac{2}{3}\right) \\
&= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\cdots\left(\frac{2}{4}\right)\left(\frac{1}{3}\right) \\
&= \frac{2}{n(n-1)} \\
&\geq \frac{2}{n^2}
\end{aligned}$$

Contraction Algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times.

Claim. If we repeat the contraction algorithm $n^2 \ln n$ times with independent random choices, the probability of failing to find the global min-cut is at most $1/n^2$.

Pf. By independence, the probability of failure is at most

$$\left(1-\frac{2}{n^2}\right)^{n^2 \ln n} = \left[\left(1-\frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2\ln n} \underset{\substack{\uparrow \\ (1-1/x)^x \,\leq\, 1/e}}{\leq} \left(e^{-1}\right)^{2\ln n} = \frac{1}{n^2}$$

Global Min Cut: Context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph, and return best of two cuts.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm

13.3 Linearity of Expectation

Expectation

Expectation. Given a discrete random variables X, its expectation E[X] is defined by:

$$E[X] = \sum_{j=0}^{\infty} j \Pr[X = j]$$

Waiting for a first success. Coin is heads with probability p and tails with probability 1-p. How many independent flips X until first heads?

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{\infty} j \, \underbrace{(1-p)^{j-1}}_{\text{j-1 tails}} \underbrace{p}_{\text{1 head}} = \frac{p}{1-p} \sum_{j=0}^{\infty} j \, (1-p)^j = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

Expectation: Two Properties

Useful property. If X is a 0/1 random variable, E[X] = Pr[X = 1].

Pf.

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{1} j \cdot \Pr[X = j] = \Pr[X = 1]$$

not necessarily independent

Linearity of expectation. Given two random variables X and Y defined over the same probability space, E[X + Y] = E[X] + E[Y].

Decouples a complex calculation into simpler pieces.

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.

Pf. (surprisingly effortless using linearity of expectation)

- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1/n$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/n = 1$. ▪

linearity of expectation

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.

Pf.

- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / (n - i + 1)$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/2 + 1/1 = H(n)$. ▪

linearity of expectation

$\ln(n+1) < H(n) < 1 + \ln n$

Coupon Collector

Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have ≥ 1 coupon of each type?

Claim. The expected number of steps is $\Theta(n \log n)$.

Pf.

- Phase j = time between j and j+1 distinct coupons.
- Let X_j = number of steps you spend in phase j.
- Let X = number of steps in total = $X_0 + X_1 + \ldots + X_{n-1}$.

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = n H(n)$$

prob of success = (n-j)/n

$\Rightarrow$ expected waiting time = n/(n-j)

13.4 MAX 3-SAT

Maximum 3-Satisfiability

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$
\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}
$$

Remark. NP-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

Maximum 3-Satisfiability: Analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is 7k/8.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = number of clauses satisfied.

$$
\begin{aligned}
E[Z] &= \sum_{j=1}^{k} E[Z_j] \\
&= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\
&= \tfrac{7}{8} k
\end{aligned}
$$

linearity of expectation

The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a 7/8-approximation algorithm? In general, a random variable can almost always be below its mean.

Lemma. The probability that a random assignment satisfies $\geq 7k/8$ clauses is at least $1/(8k)$.

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k/8$ clauses are satisfied.

$$\begin{aligned}
\tfrac{7}{8}k \;=\; E[Z] \;&=\; \sum_{j \geq 0} j\, p_j \\
&=\; \sum_{j < 7k/8} j\, p_j \;+\; \sum_{j \geq 7k/8} j\, p_j \\
&\leq\; \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j \;+\; k \sum_{j \geq 7k/8} p_j \\
&\leq\; \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 \;+\; k\, p
\end{aligned}$$

Rearranging terms yields $p \geq 1 / (8k)$. ▪

Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k/8$ clauses.

Theorem. Johnson's algorithm is a 7/8-approximation algorithm with polynomial expected running time.

Pf. By previous lemma, each iteration succeeds with probability at least $1/(8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. ▪

Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX-3SAT where each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless P = NP, no ρ-approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any $\rho > 7/8$.

very unlikely to improve over simple randomized algorithm for MAX-3SAT

Two Types of Randomized Algorithms

Two Algorithms

Contraction algorithm for global min cut

- $\Omega(m)$ running time
- Returns a min cut with prob $\geq 2/n^2$

Johnson's algorithm for MAX-3SAT

- Guarantees 7/8-approximation (by repeating until success)
- Polynomial expected running time

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithm. Guaranteed running time, likely to find correct answer.
Ex: Contraction algorithm for global min cut.

Las Vegas algorithm. Guaranteed to find correct answer, likely to run in certain time.
Ex: Randomized quicksort, Johnson's MAX-3SAT algorithm.

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithm. Guaranteed running time, likely to find correct answer.
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Las Vegas algorithm. Guaranteed to find correct answer, likely to run in certain time.
Ex: Randomized quicksort, Johnson's MAX-3SAT algorithm.

stop algorithm after a certain point

Remark. Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way.

RP and ZPP

RP. [Monte Carlo] Decision problems solvable with one-sided error in poly-time.

Can decrease probability of false negative to 2^{-100} by 100 independent repetitions

One-sided error.

- If the correct answer is `no`, always return `no`.
- If the correct answer is `yes`, return `yes` with probability $\geq \frac{1}{2}$.

ZPP. [Las Vegas] Decision problems solvable in expected poly-time.

running time can be unbounded, but on average it is fast

Theorem. $P \subseteq ZPP \subseteq RP \subseteq NP$.

Fundamental open questions. To what extent does randomization help? Does P = ZPP? Does ZPP = RP? Does RP = NP?

13.5 Randomized Divide-and-Conquer

Quicksort

- Pick one element to use as pivot.
- Partition elements into two sub-arrays:
  - Elements less than or equal to pivot
  - Elements greater than pivot
- Quicksort two sub-arrays

40	20	10	80	60	50	7	30	100

40	20	10	80	60	50	7	30	100

Select pivot

20	10	7	30	**40**	80	60	50	100

Partition

<= pivot | | > pivot

7	10	20	30	**40**	50	60	80	100

Quicksort sub-arrays

Quicksort

Randomized Quicksort

- Pick one element to use as pivot uniformly at random.
- Partition elements into two sub-arrays:
  - Elements less than or equal to pivot
  - Elements greater than pivot
- Quicksort two sub-arrays

40	20	10	80	60	50	7	30	100	
40	20	10	80	60	50	7	30	100	Select pivot
20	10	7	30	**40**	80	60	50	100	Partition
		<= pivot			> pivot				
7	10	20	30	**40**	50	60	80	100	Quicksort sub-arrays

Quicksort

Running time.

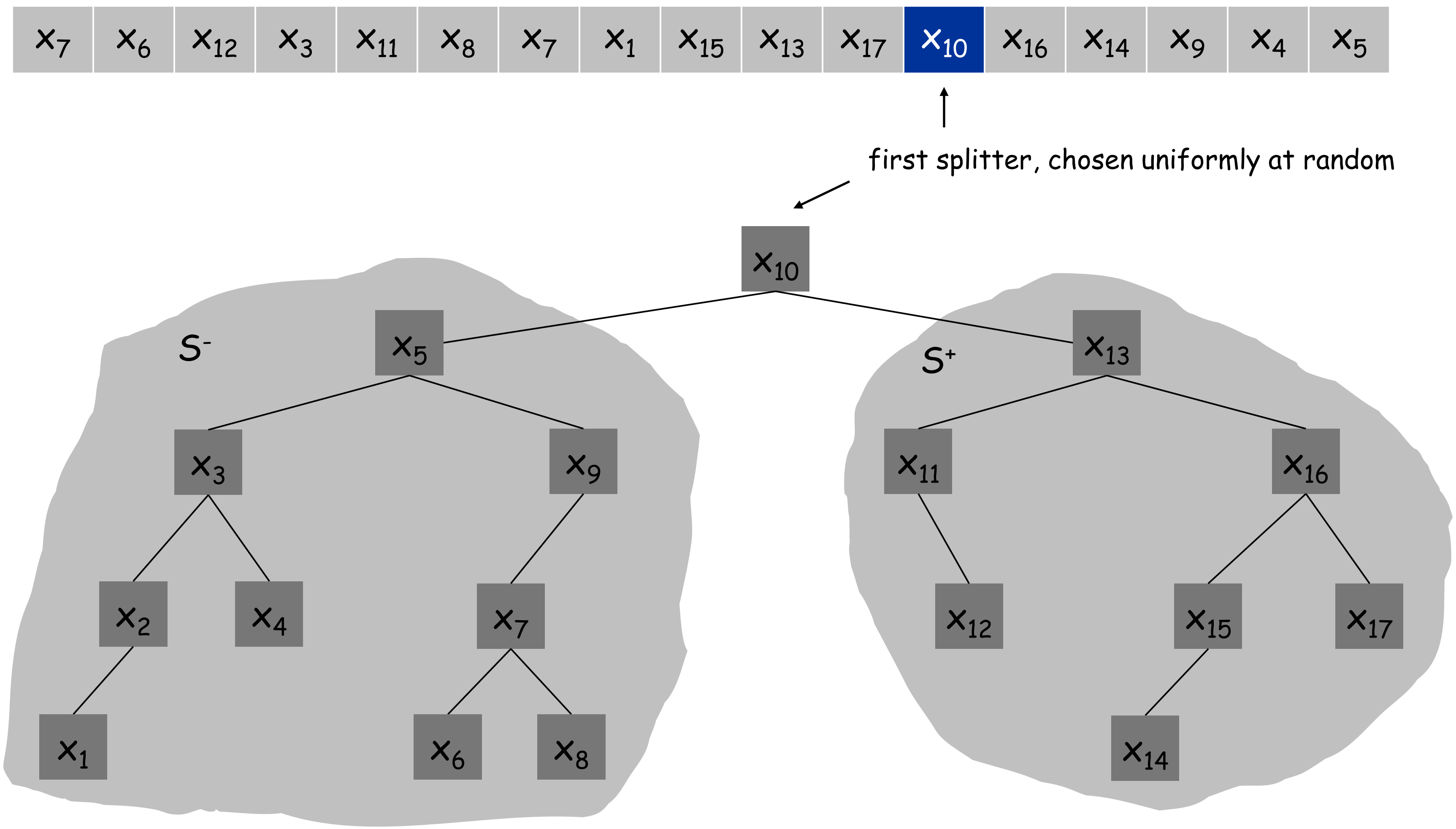
- [Best case.] Select the median element as the splitter: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest/largest element as the splitter: quicksort makes $\Theta(n^2)$ comparisons.

Randomize. Protect against worst case by choosing splitter at random.

Quicksort: BST Representation of Splitters

Notation. Label elements so that $x_1 < x_2 < \ldots < x_n$.

BST representation. Draw recursive BST of splitters.

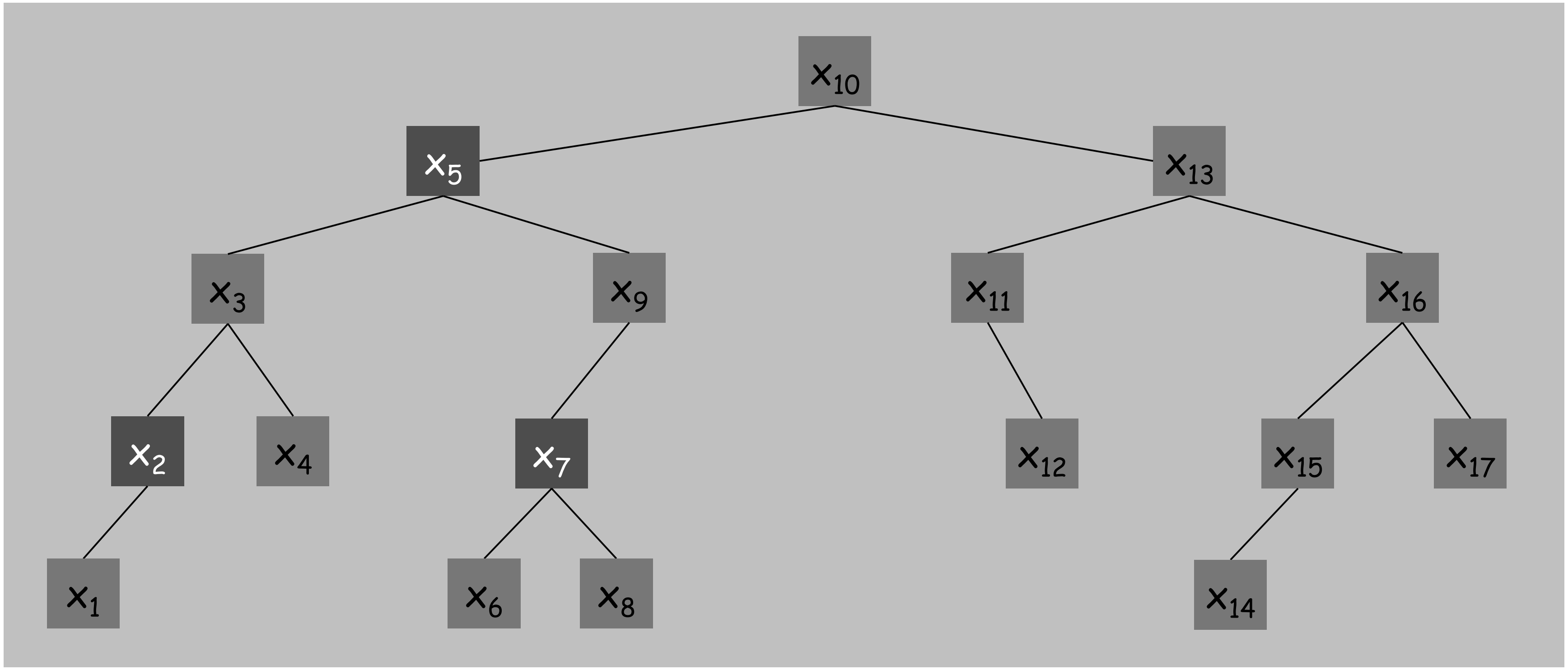


Quicksort: BST Representation of Splitters

Observation. Element only compared with its ancestors and descendants.

- x_2 and x_7 are compared if their lca = x_2 or x_7.
- x_2 and x_7 are not compared if their lca = x_3 or x_4 or x_5 or x_6.

Claim. $\Pr[x_i \text{ and } x_j \text{ are compared}, i<j] = 2 / (j - i + 1)$.



Quicksort: BST Representation of Splitters

Observation. Element only compared with its ancestors and descendants.

- x_2 and x_7 are compared if their lca = x_2 or x_7.
- x_2 and x_7 are not compared if their lca = x_3 or x_4 or x_5 or x_6.

Claim. $\Pr[x_i$ and x_j are compared, $i<j] = 2 / (j - i + 1)$.

Proof:

- The lca of x_i and x_j must be x_k where $i \leq k \leq j$
- The first pivot among $x_i, \ldots, x_j$ is the lca of x_i and x_j
- x_i and x_j are compared if and only if their lca = x_i or x_j.

Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is O(n log n).

Pf.

k = j - i + 1

$$E[C] = \sum_{1 \le i < j \le n} E[C_{ij}] = \sum_{1 \le i < j \le n} \frac{2}{j - i + 1} = 2 \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{1}{k} \le 2n \sum_{k=2}^{n} \frac{1}{k} = 2n\mathrm{H}(n)$$

probability that i and j are compared

Theorem. [Knuth 1973] Stddev of number of comparisons is ~ 0.65n.

Ex. If n = 1 million, the probability that randomized quicksort takes less than 4n ln n comparisons is at least 99.94%.

Chebyshev's inequality. $\Pr[|X - \mu| \ge k\sigma] \le 1 / k^2$.

Chapter Summary

Basics in Probability Theory

Union bound. Given events E1, …, En, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

Waiting time expectation. Expected number of trials for a first success is 1/p.

Linearity of expectation. Given two random variables X and Y defined over the same probability space, E[X + Y] = E[X] + E[Y].

Algorithms

Monte Carlo algorithm. Guaranteed to run in poly-time, likely to find correct answer.

- Contraction algorithm for global min cut

Las Vegas algorithm. Guaranteed to find correct answer, likely to run in poly-time.

- Johnson's MAX-3SAT algorithm
- Randomized quicksort