

Homework 4

Homework 4

- Covers chapters 8
- To be released today in Blackboard
- Due: **Nov. 28** before class
 - 5 extra days!
 - Start early!
- No late homework will be accepted!
- Submit your solutions with gradescope.
 - We may choose a subset of the problems to grade

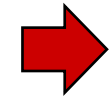
▼ 算法设计与分析



主页

公告

Slides



作业

讨论

邮件

帮助

Project: Calabash Brothers

Content

- An optimization problem that does not have an efficient exact solver
- Your task is to find a good solution given an input
 - You can use any algorithmic technique
- To be released in Blackboard by this weekend

gradebot.org

- Register & submit your solutions there

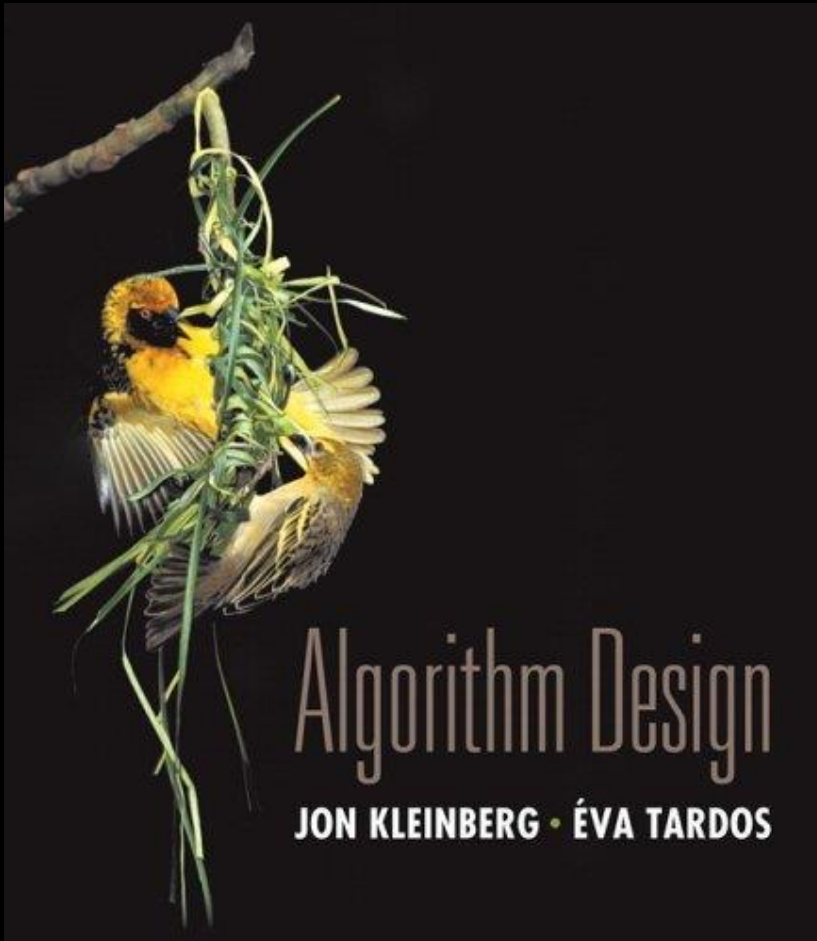
Grading

- There will be ~10 different inputs
- For each input, we will compare your solution against solutions from other students
- Your grade will depend on your rank
- 10% of the total grade



Chapter 8

NP and Computational Intractability



Slides by Kevin Wayne.
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Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Max-flow min-cut.
- Reductions.

Ex.

$O(n \log n)$ interval scheduling.

$O(n \log n)$ FFT.

$O(n^2)$ edit distance.

$O(n^3)$ bipartite matching.

Algorithm design anti-patterns.

- NP-completeness.
- PSPACE-completeness.
- Undecidability.

$O(n^k)$ algorithm unlikely.

$O(n^k)$ certification algorithm unlikely.

No algorithm possible.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966]

Those with polynomial-time algorithms. (In practice, poly-time algorithms scale to huge problems.)

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover
Primality testing (2002)	Factoring

Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Provably requires exponential-time.

- Given a program of size $\log k$, does it halt in at most k steps?
- Given a board position in an n -by- n generalization of checker, can black guarantee a win?

Frustrating news. Huge number of fundamental problems have defied classification for decades.

This chapter. Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one **really hard** problem.

Polynomial-Time Reduction

Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

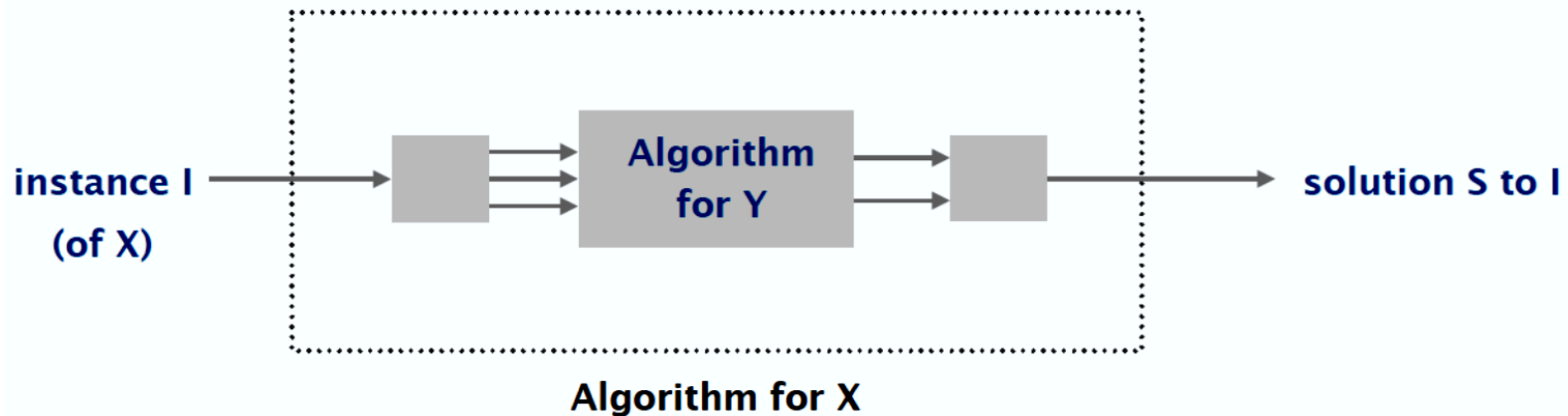
don't confuse with reduces from

Reduction. Problem X **polynomial-time reduces to** problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.



A black box that solves instances of Y in a single step



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A black box that solves instances of Y in a single step

Notation. $X \leq_p Y$.

Remarks.

- We pay for time to write down instances sent to black box \Rightarrow instances of Y must be of polynomial size.

Polynomial-Time Reduction

Purpose. Classify problems according to **relative** difficulty.

Design algorithms. If $X \leq_p Y$ and Y can be solved in polynomial-time, then X **can** also be solved in polynomial time.

Establish intractability. If $X \leq_p Y$ and X cannot be solved in polynomial-time, then Y **cannot** be solved in polynomial time.

Establish equivalence. If $X \leq_p Y$ and $Y \leq_p X$, we use notation $X \equiv_p Y$. In this case, X can be solved in polynomial time iff Y can be.

Reduction By Simple Equivalence

Basic reduction strategies.

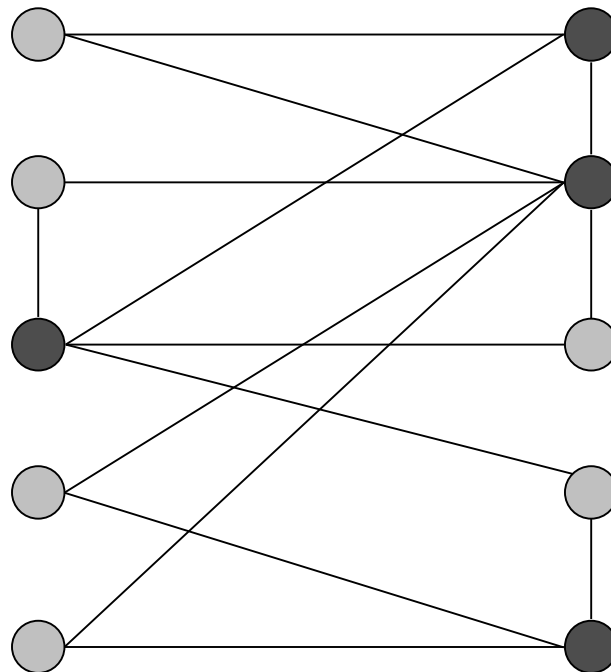
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k , is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S ?

Ex. Is there an independent set of size ≥ 6 ? Yes.

Ex. Is there an independent set of size ≥ 7 ? No.



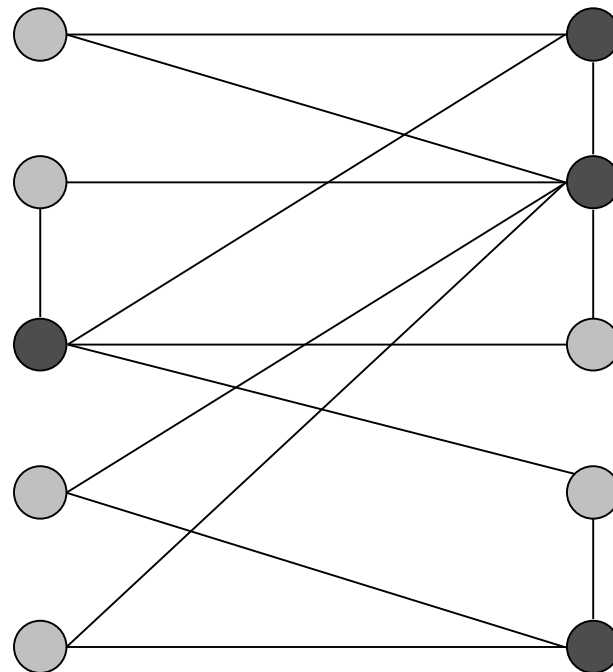
● independent set

Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k , is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S ?

Ex. Is there a vertex cover of size ≤ 4 ? Yes.

Ex. Is there a vertex cover of size ≤ 3 ? No.

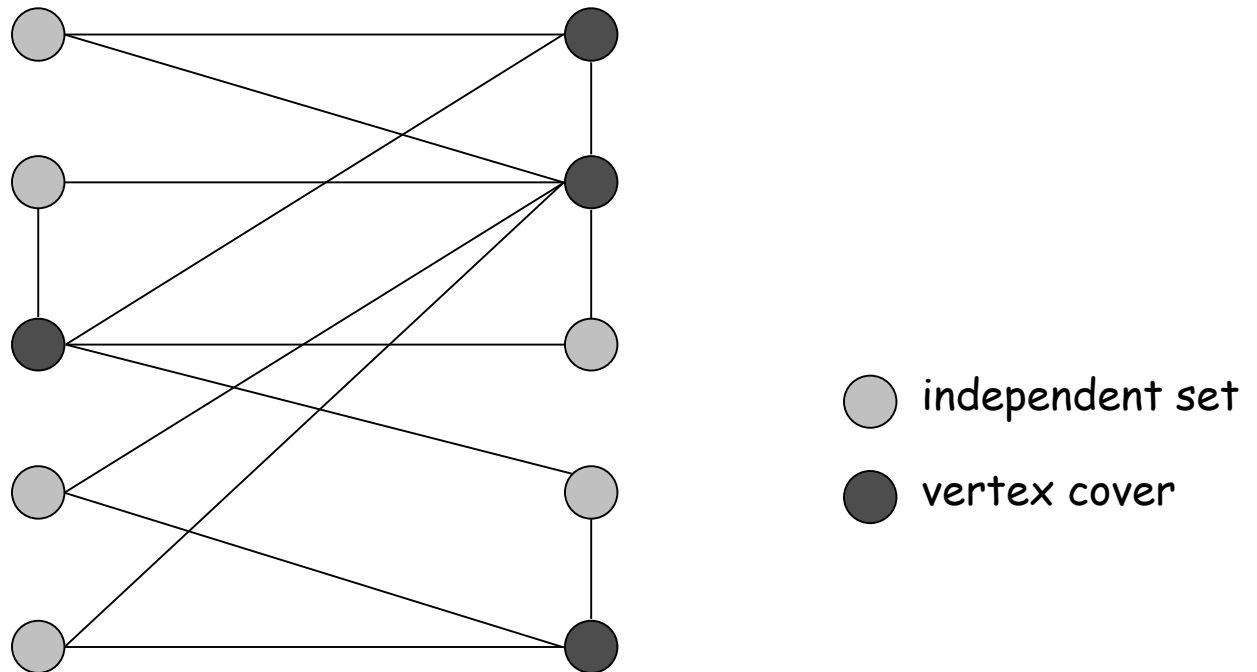


● vertex cover

Vertex Cover and Independent Set

Claim. VERTEX-COVER \equiv_p INDEPENDENT-SET.

Pf. We show S is an independent set iff $V - S$ is a vertex cover.



Vertex Cover and Independent Set

Claim. VERTEX-COVER \equiv_p INDEPENDENT-SET.

Pf. We show S is an independent set iff $V - S$ is a vertex cover.

\Rightarrow

- Let S be any independent set.
- Consider an arbitrary edge (u, v) .
- S independent $\Rightarrow u \notin S$ or $v \notin S \Rightarrow u \in V - S$ or $v \in V - S$.
- Thus, $V - S$ covers (u, v) .

\Leftarrow

- Let $V - S$ be any vertex cover.
- Consider two nodes $u \in S$ and $v \in S$.
- Observe that $(u, v) \notin E$ since $V - S$ is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Rightarrow S$ independent set. ▪

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection S_1, S_2, \dots, S_m of subsets of U , and an integer k , does there exist a collection of $\leq k$ of these sets whose union is equal to U ?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The i -th piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

$$U = \{1, 2, 3, 4, 5, 6, 7\}$$

$$k = 2$$

$$S_1 = \{3, 7\}$$

$$S_4 = \{2, 4\}$$

$$S_2 = \{3, 4, 5, 6\}$$

$$S_5 = \{5\}$$

$$S_3 = \{1\}$$

$$S_6 = \{1, 2, 6, 7\}$$

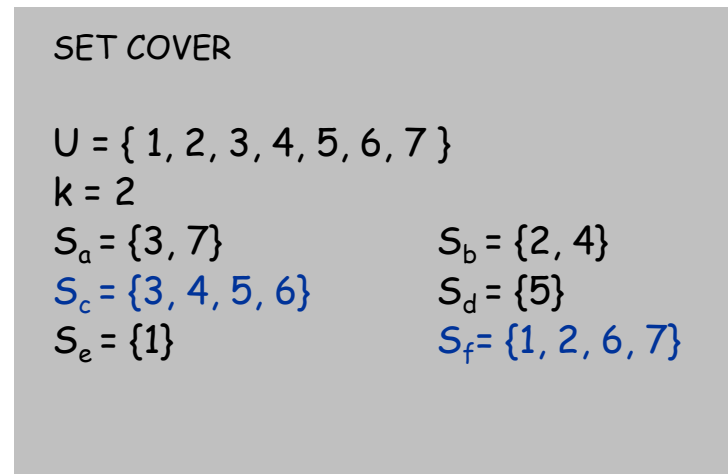
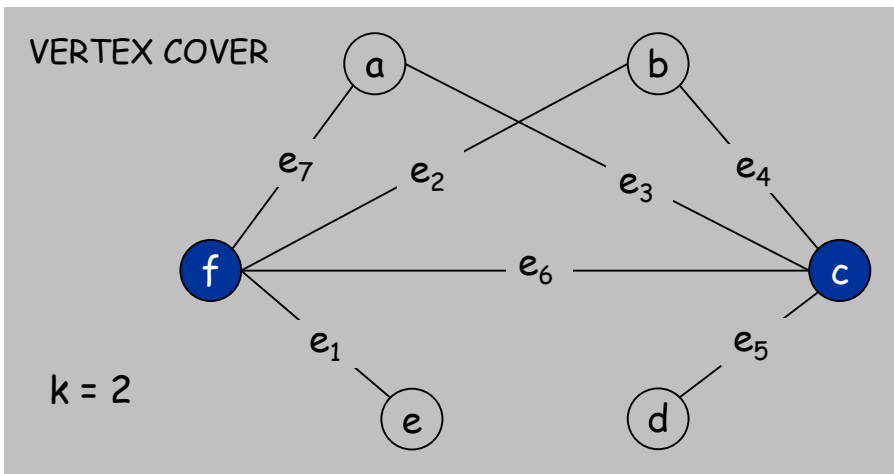
Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER \leq_p SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$, k , we construct a set cover instance (U, S) whose size equals the size of the vertex cover instance.

Construction.

- Create SET-COVER instance:
 - $k = k$, $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size $\leq k$ iff vertex cover of size $\leq k$. ▪



8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

Literal: A Boolean variable or its negation.

$$x_i \text{ or } \overline{x_i}$$

Clause: A disjunction of literals.

$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

SAT: Given CNF formula Φ , does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

↑
each corresponds to a different variable

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

Yes: $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}, x_4 = \text{false}.$

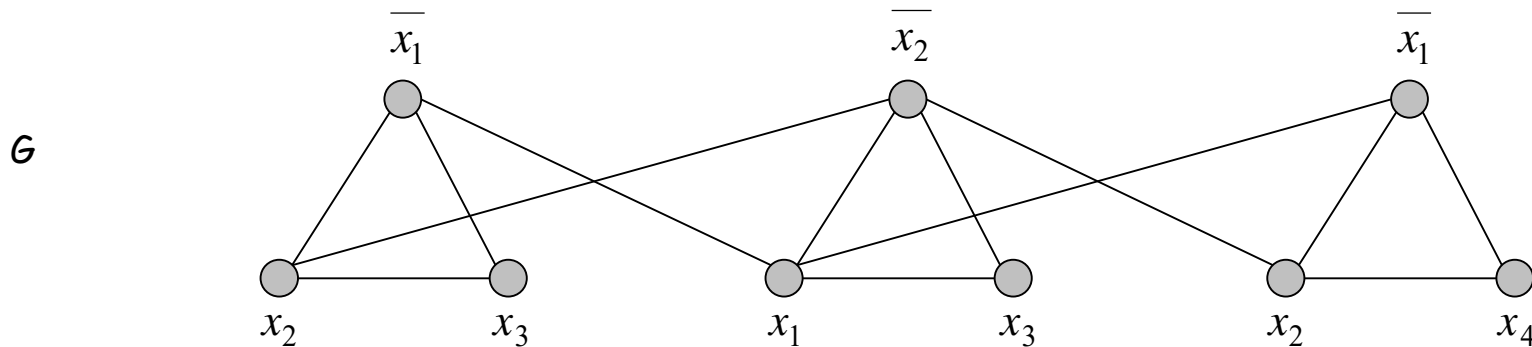
3 Satisfiability Reduces to Independent Set

Claim. $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



$k = 3$

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

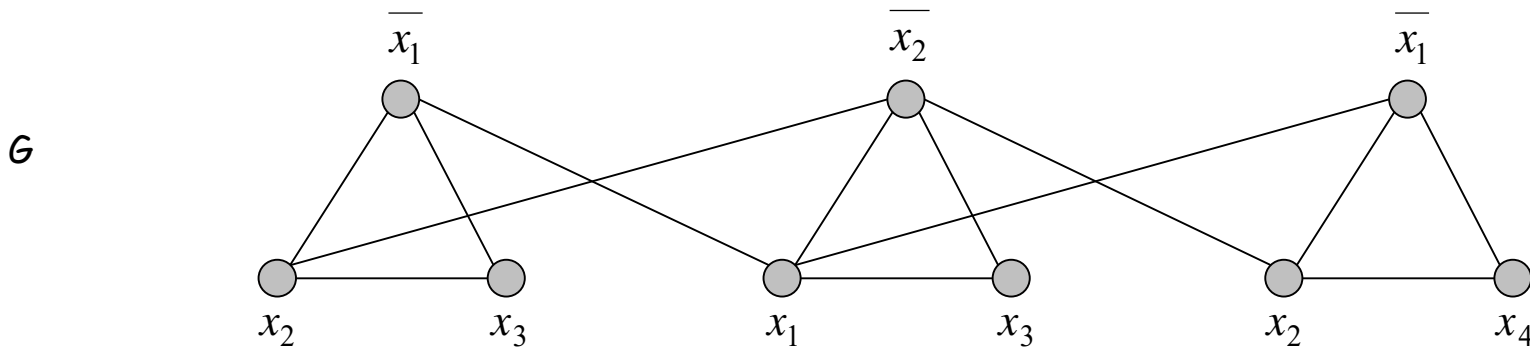
3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Pf. \Rightarrow Let S be independent set of size k .

- S must contain exactly one vertex in each triangle.
- Set these literals to true. \leftarrow and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf \Leftarrow Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k . ▪



$k = 3$

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

- Simple equivalence: $\text{INDEPENDENT-SET} \equiv_p \text{VERTEX-COVER}$.
- Special case to general case: $\text{VERTEX-COVER} \leq_p \text{SET-COVER}$.
- Encoding with gadgets: $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$.

Transitivity. If $X \leq_p Y$ and $Y \leq_p Z$, then $X \leq_p Z$.

Pf idea. Compose the two algorithms.

Ex: $3\text{-SAT} \leq_p \text{INDEPENDENT-SET} \leq_p \text{VERTEX-COVER} \leq_p \text{SET-COVER}$.

Self-Reducibility

Decision problem. Does there **exist** a vertex cover of size $\leq k$?

Optimization problem. **Find** vertex cover of minimum cardinality.

Self-reducibility. Optimization problem \leq_p decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.

- (Binary) search for cardinality k^* of min vertex cover.
- Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\leq k^* - 1$.
 - v must be in a vertex cover of size $\leq k^*$
 - Prove by construction: a vertex cover of $G - \{v\}$ plus v must be a vertex cover of G
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{v\}$.

↑
delete v and all incident edges

8.3 Definition of NP

Decision Problems

Decision problem.

- X is a set of strings.
- Instance: string s .
- Algorithm A solves problem X : $A(s) = \text{yes}$ iff $s \in X$.

Polynomial time. Algorithm A runs in polynomial time if for every string s , $A(s)$ terminates in at most $p(|s|)$ "steps", where $p(\cdot)$ is some polynomial.

↑
length of s

PRIMES: $X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, \dots \}$

Instance: $s = 592335744548702854681$

Algorithm: [Agrawal-Kayal-Saxena, 2002] $p(|s|) = |s|^8$.

Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y ?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss-Edmonds elimination	$\left[\begin{array}{ccc c} 0 & 1 & 1 & 4 \\ 2 & 4 & -2 & 2 \\ 0 & 3 & 15 & 36 \end{array} \right]$	$\left[\begin{array}{ccc c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right]$

NP

Certification algorithm intuition.

- Certifier views things from "managerial" viewpoint.
- Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.

Def. Algorithm $C(s, t)$ is a **certifier** for problem X if for every string s , $s \in X$ iff there exists a string t such that $C(s, t) = \text{yes}$.

↖
"certificate" or "witness"

NP. Decision problems for which there exists a **poly-time** certifier.

- $C(s, t)$ is a poly-time algorithm
- $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

Remark. NP stands for **nondeterministic** polynomial-time.

Certifiers and Certificates: Composite

COMPOSITES. Given an integer s , is s composite?

Certificate. A nontrivial factor t of s . Note that such a certificate exists iff s is composite. Moreover $|t| \leq |s|$.

Certifier.

```
boolean C(s, t) {  
    if (t ≤ 1 or t ≥ s)  
        return false  
    else if (s is a multiple of t)  
        return true  
    else  
        return false  
}
```

Instance. $s = 437,669$.

Certificate. $t = 541$ or 809 . $\longleftarrow 437,669 = 541 \times 809$

Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

SAT. Given a CNF formula Φ , is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

Ex.

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$$

instance s

$$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$$

certificate t

Conclusion. SAT is in NP.

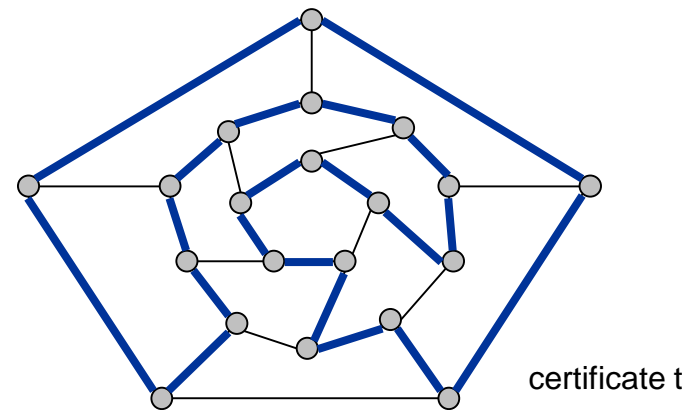
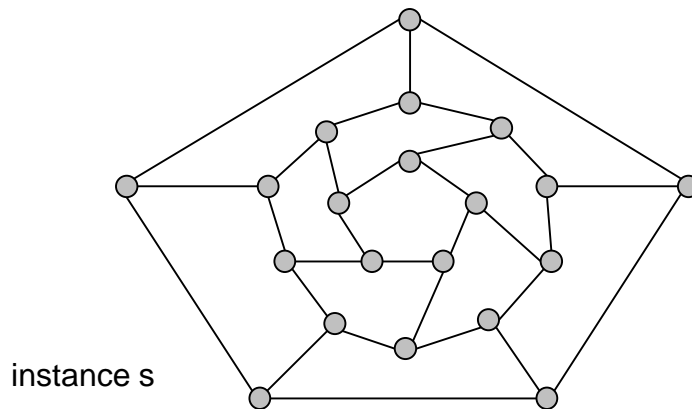
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node?

Certificate. A permutation of the n nodes.



Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.



Definition of NP

NP. Decision problems for which there exists a poly-time certifier.

Problem	Description	Algorithm	Yes	No
LSOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss-Edmonds elimination	$\left[\begin{array}{ccc c} 0 & 1 & 1 & 4 \\ 2 & 4 & -2 & 2 \\ 0 & 3 & 15 & 36 \end{array} \right]$	$\left[\begin{array}{ccc c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right]$
PRIMES	Is x prime?	AKS (2002)	53	51
FACTOR	Does x have a nontrivial factor less than y ?	?	(56159, 50)	(55687, 50)
SAT	Is there a truth assignment that satisfies the formula?	?	$\neg x_1 \vee x_2$ $x_1 \vee x_2$	$\neg x_2$ $\neg x_1 \vee x_2$ $x_1 \vee x_2$
HAM-CYCLE	Does there exist a simple cycle C that visits every node?	?		

P, NP, EXP

P. Decision problems for which there is a **poly-time algorithm**.

EXP. Decision problems for which there is an **exponential-time algorithm**.

NP. Decision problems for which there is a **poly-time certifier**.

Claim. $P \subseteq NP$.

Pf. Consider any problem X in P .

- By definition, there exists a poly-time algorithm $A(s)$ that solves X .
- Certificate: $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Claim. $NP \subseteq EXP$.

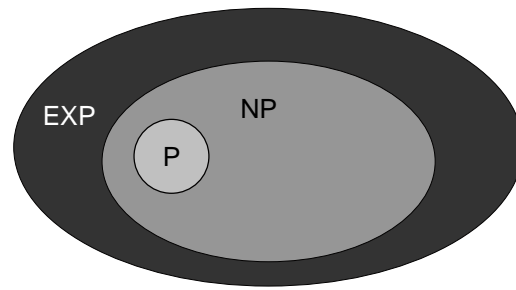
Pf. Consider any problem X in NP .

- By definition, there exists a poly-time certifier $C(s, t)$ for X .
- To solve input s , run $C(s, t)$ on all strings t with $|t| \leq p(|s|)$.
- Return **yes**, if $C(s, t)$ returns **yes** for any of these. ▪

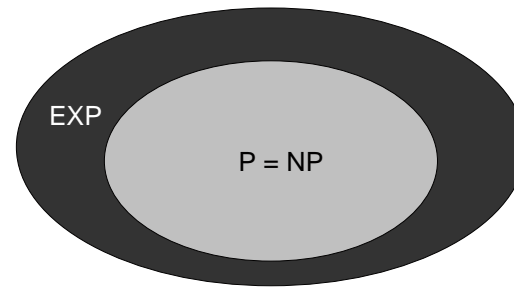
The Main Question: P Versus NP

Does $P = NP$? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

- Is the decision problem as easy as the certification problem?



If $P \neq NP$



If $P = NP$

would break RSA cryptography
(and potentially collapse economy)



If yes: Efficient algorithms for 3-COLOR, TSP, SAT, FACTOR, ...

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, ...

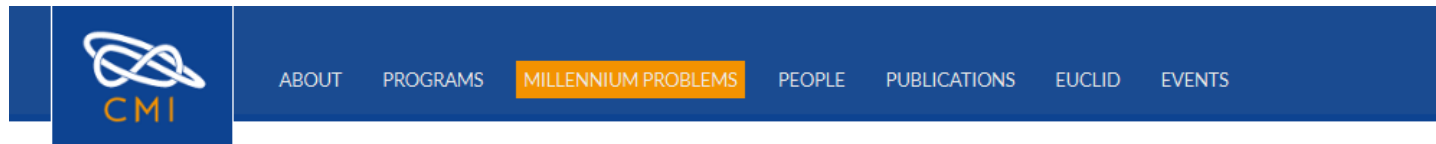
Consensus

Consensus opinion on $P = NP$? Probably no.

- 2002 poll of 100 researchers
 - 61: no
 - 9: yes
 - 22: unsure
 - 4: the question may be independent of the currently accepted axioms and therefore is impossible to prove or disprove
- 2012 poll of 151 researchers
 - 126: no
 - 12: yes
 - 8: don't know / don't care
 - 5: the question may be independent of the currently accepted axioms and therefore is impossible to prove or disprove

Millennium prize

One of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a **US\$1,000,000** prize for the first correct solution



Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part $1/2$.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

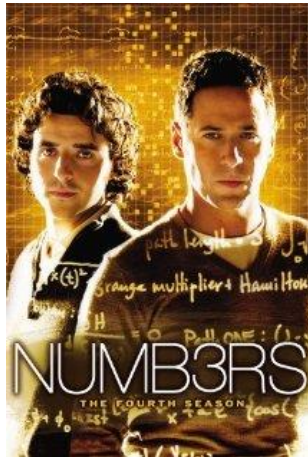
In popular culture



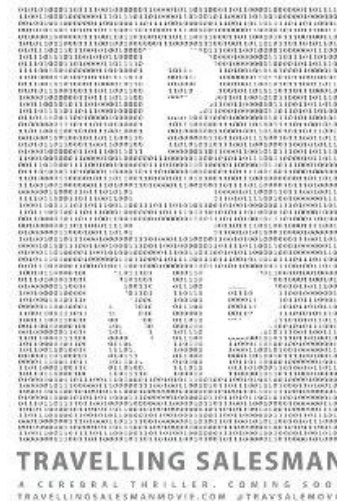
The Simpsons (season 7 episode 6)



Futurama (season 2 episode 10)



Numb3rs
(season 1
episode 2)



Travelling Salesman
(2012 film)

Princeton CS Building, West Wall



Princeton CS Building, West Wall





Character	ASCII	Bits
P	80	1010000
=	61	0111101
N	78	1001110
P	80	1010000
?	63	0111111

8.4 NP-Completeness

Polynomial Transformation

Def. Problem X **polynomial reduces** (Cook) to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Def. Problem X **polynomial transforms** (Karp) to problem Y if given any input x to X, we can construct an input y in poly-time such that x is a y_{yes} instance of X iff y is a y_{yes} instance of Y.

↑
we require $|y|$ to be of size polynomial in $|x|$

Note. Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

Open question. Are these two concepts the same?

↑
we abuse notation \leq_p and blur distinction

NP-Complete

NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff $P = NP$.

Pf. \Leftarrow If $P = NP$ then Y can be solved in poly-time since Y is in NP.

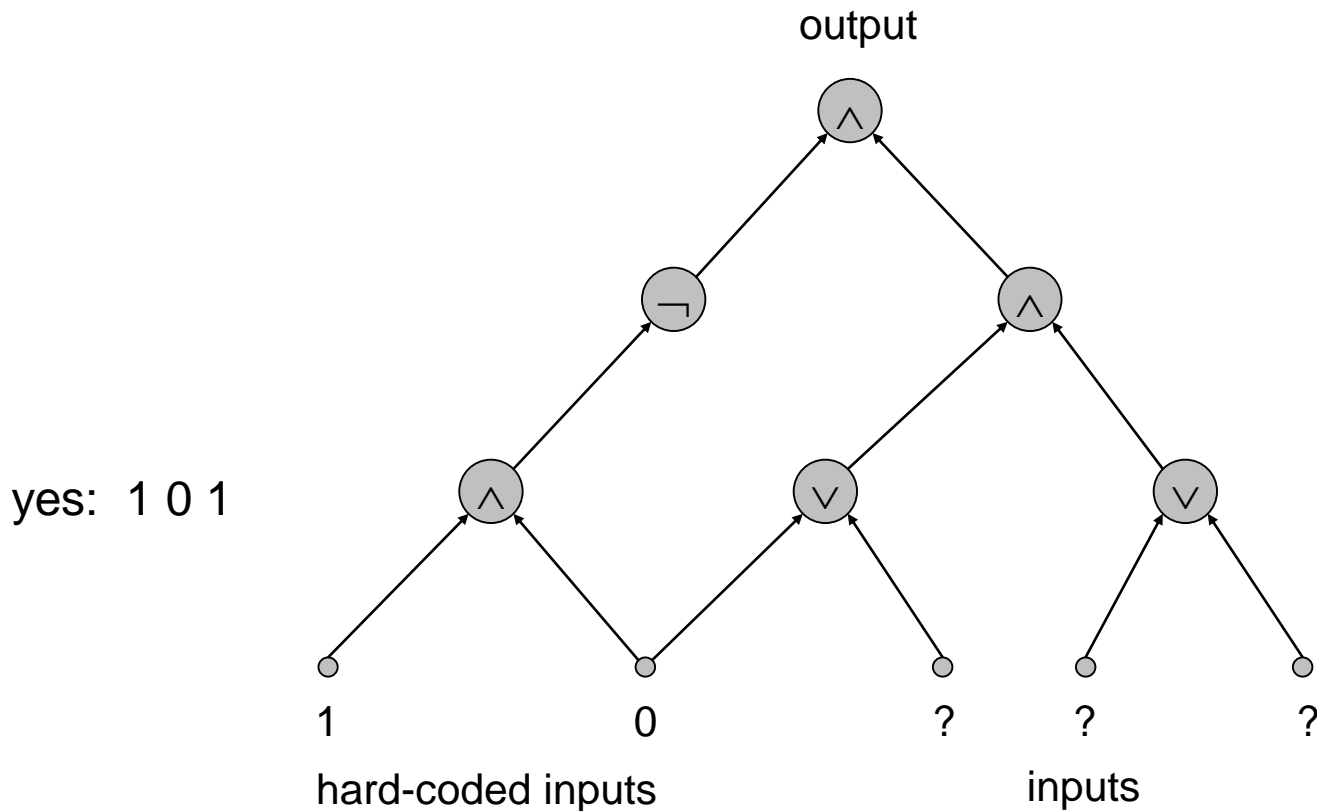
Pf. \Rightarrow Suppose Y can be solved in poly-time.

- Let X be any problem in NP. Since $X \leq_p Y$, we can solve X in poly-time. This implies $NP \subseteq P$.
- We already know $P \subseteq NP$. Thus $P = NP$. ▪

Fundamental question. Do there exist "natural" NP-complete problems?

Circuit Satisfiability

CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?



The "First" NP-Complete Problem

Theorem. CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

Pf. (sketch)

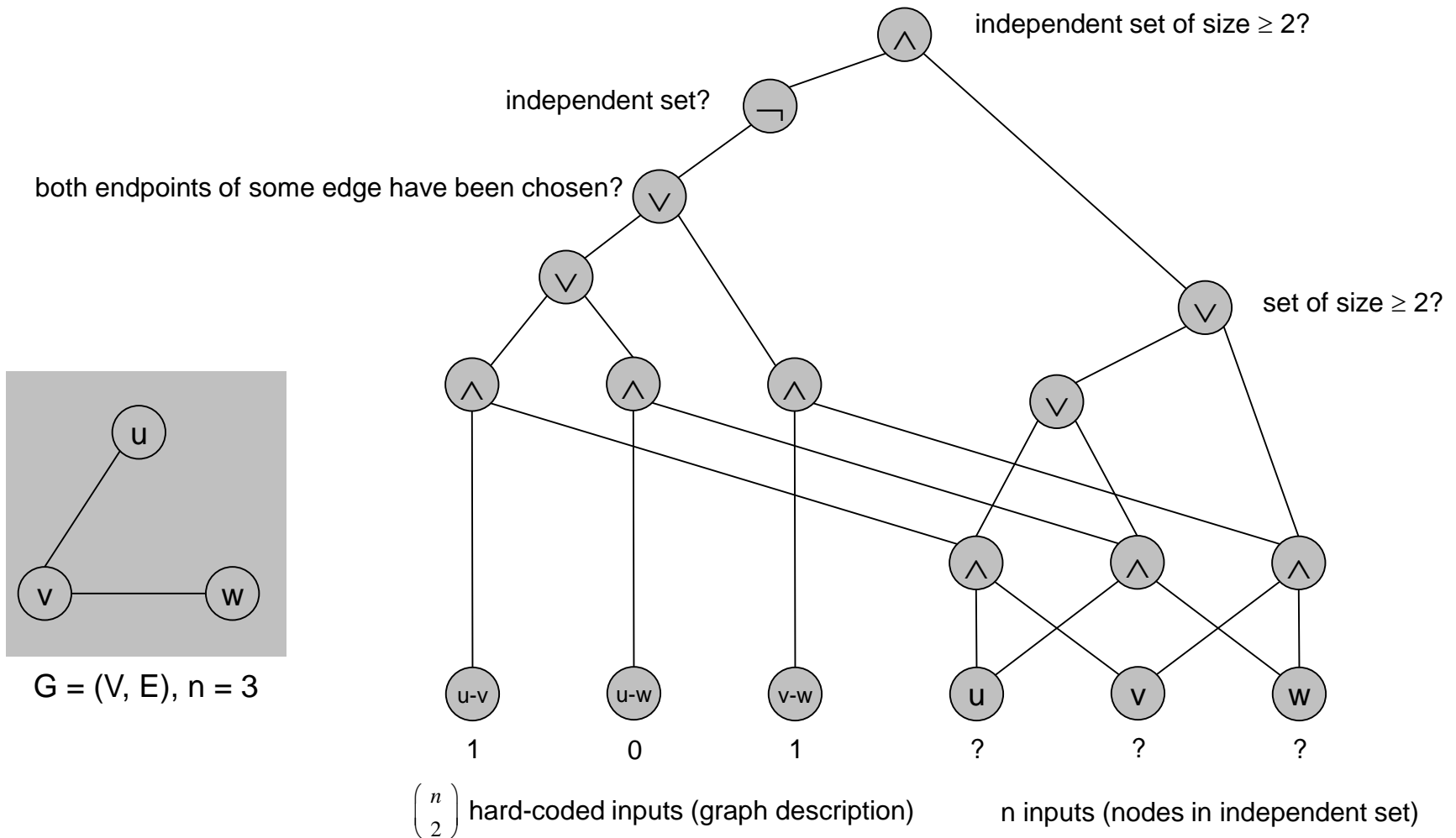
- Clearly, CIRCUIT-SAT is in NP
- Any algorithm that takes a fixed number of bits n as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

- Consider some problem X in NP. It has a poly-time certifier $C(s, t)$. To determine whether s is in X , need to know if there exists a certificate t of length $p(|s|)$ such that $C(s, t) = \text{yes}$.
- View $C(s, t)$ as an algorithm on $|s| + p(|s|)$ bits (input s , certificate t) and convert it into a poly-size circuit K .
 - first $|s|$ bits are hard-coded with s
 - remaining $p(|s|)$ bits represent (unknown) bits of t
- Circuit K is satisfiable iff $\exists t, C(s, t) = \text{yes}$.

Example

Ex. Construction below creates a circuit K whose inputs can be set so that K outputs true iff graph G has an independent set of size 2.



Establishing NP-Completeness

Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

Recipe to establish NP-completeness of problem Y .

- Step 1. Show that Y is in NP.
- Step 2. Choose an NP-complete problem X .
- Step 3. Prove that $X \leq_p Y$.

Theorem. If X is an NP-complete problem and Y is a problem in NP with the property that $X \leq_p Y$, then Y is NP-complete.

Pf. Let W be any problem in NP. Then $W \leq_p X \leq_p Y$.

- By transitivity, $W \leq_p Y$.
- Hence Y is NP-complete. ▪

\uparrow \uparrow
by definition of by assumption
NP-complete

3-SAT is NP-Complete

Review of 3-SAT:

- **Literal:** A Boolean variable or its negation. x_i or $\overline{x_i}$
- **Clause:** A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$
- **Conjunctive normal form:** A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$
- **SAT:** Given CNF formula Φ , does it have a satisfying truth assignment?
- **3-SAT:** SAT where each clause contains exactly 3 literals.

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

Yes: $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}, x_4 = \text{false}.$

3-SAT is NP-Complete

Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that $\text{CIRCUIT-SAT} \leq_p \text{3-SAT}$ since 3-SAT is in NP.

- Let K be any circuit. We can construct an instance of 3-SAT Φ :

- Step 1: Create a 3-SAT variable x_i for each circuit element i .

- Step 2: Make circuit compute correct values at each node:

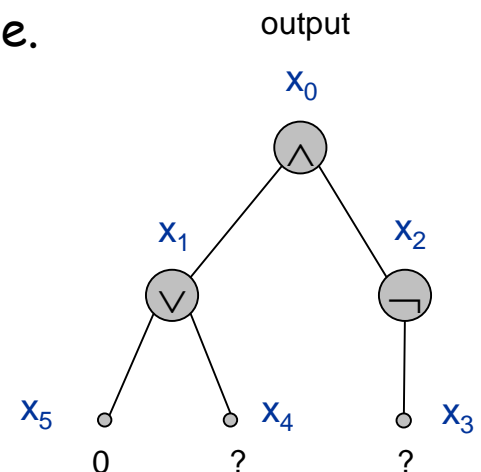
- $x_2 = \neg x_3 \Rightarrow$ add 2 clauses: $x_2 \vee x_3$, $\overline{x_2} \vee \overline{x_3}$
- $x_1 = x_4 \vee x_5 \Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}$, $x_1 \vee \overline{x_5}$, $\overline{x_1} \vee x_4 \vee x_5$
- $x_0 = x_1 \wedge x_2 \Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1$, $\overline{x_0} \vee x_2$, $x_0 \vee \overline{x_1} \vee \overline{x_2}$

- Step 3: Hard-coded input values and output value.

- $x_5 = 0 \Rightarrow$ add 1 clause: $\overline{x_5}$

- $x_0 = 1 \Rightarrow$ add 1 clause: x_0

- Final step: turn clauses of length < 3 into clauses of length exactly 3 (see book p472) .



3-SAT is NP-Complete

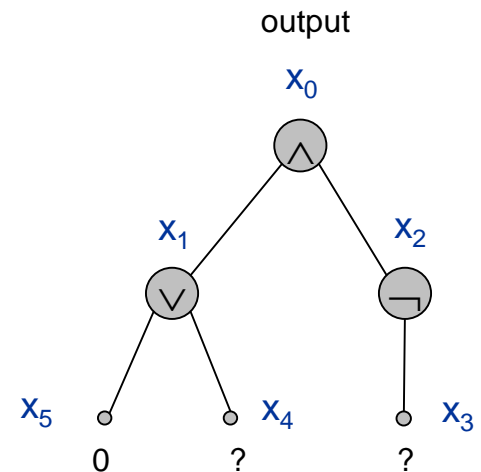
Lemma. Φ is satisfiable iff. the inputs of K can be set s.t. it outputs 1.

Pf. \Leftarrow

- Suppose there are inputs of K s.t. it outputs 1.
- Compute the values of all the nodes in K .
- According to our construction, this set of values satisfies Φ .

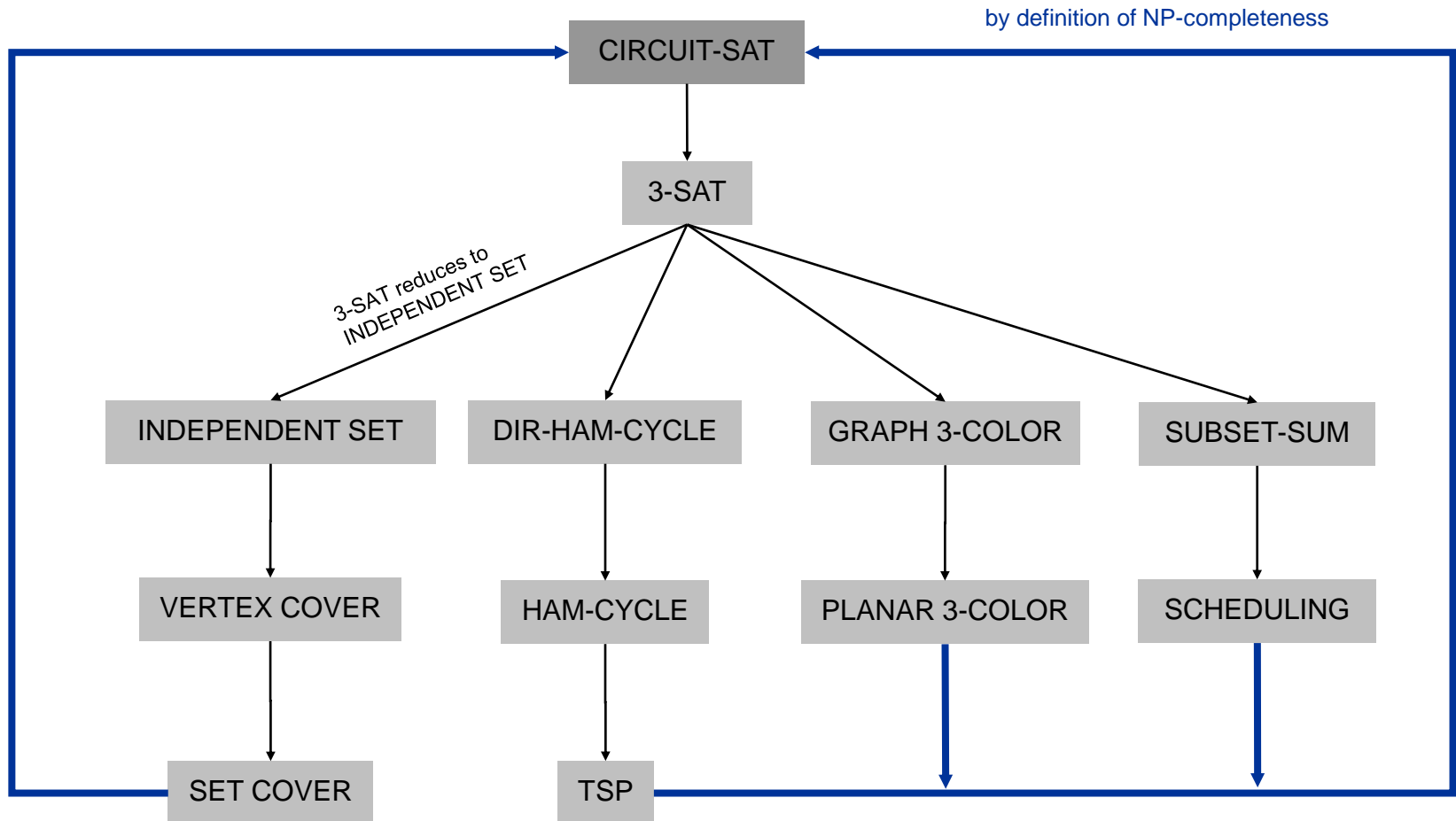
\Rightarrow

- Suppose Φ is satisfiable.
- The variable assignment of Φ specifies the values of all the nodes in K .
- Our construction ensures that all the nodes in K are correctly computed and the output is 1.



NP-Completeness

Observation. All problems below are NP-complete and polynomial reduce to one another!



Some NP-Complete Problems

Six basic genres of NP-complete problems and paradigmatic examples.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Practice. Most NP problems are known to be either in P or NP-complete.

Notable exceptions. Factoring, graph isomorphism, Nash equilibrium.

Theorem. [Ladner 1975] Unless $P = NP$, there exist problems in NP that are neither in P nor NP-complete.

Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

- 6,000 citations per year (title, abstract, keywords).
 - more than "compiler", "operating system", "database"
- Prime intellectual export of CS to other disciplines.
- Broad applicability and classification power.
- "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager finds closed-form solution to 2D-Ising.
- 19xx: Feynman and other top minds seek 3D solution.
- 2000: Istrail proves 3D-Ising is NP-complete; thus search for closed-form solution to 3D-Ising appears doomed.

More Hard Computational Problems

- Aerospace engineering:** optimal mesh partitioning for finite elements.
- Biology:** protein folding.
- Chemical engineering:** heat exchanger network synthesis.
- Civil engineering:** equilibrium of urban traffic flow.
- Economics:** computation of arbitrage in financial markets with friction.
- Electrical engineering:** VLSI layout.
- Environmental engineering:** optimal placement of contaminant sensors.
- Financial engineering:** find minimum risk portfolio of given return.
- Game theory:** find Nash equilibrium that maximizes social welfare.
- Genomics:** phylogeny reconstruction.
- Mechanical engineering:** structure of turbulence in sheared flows.
- Medicine:** reconstructing 3-D shape from biplane angiocardiogram.
- Operations research:** optimal resource allocation.
- Physics:** partition function of 3-D Ising model in statistical mechanics.
- Politics:** Shapley-Shubik voting power.
- Pop culture:** Minesweeper consistency.
- Statistics:** optimal experimental design.

Terminology

NP-complete. A problem in NP such that every problem in NP polynomial reduces to it.

NP-hard. [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]
A problem such that every problem in NP reduces to it.



Stephen Cook

Stephen Cook

- 1971 paper "The Complexity of Theorem Proving Procedures"
 - formalized the notions of polynomial-time reduction and NP-completeness
 - proved the existence of an NP-complete problem (SAT)
- Bio
 - Joined UC Berkeley in 1966 but was denied tenure in 1970
 - Joined U of Toronto in 1970
 - 1982 Turing award



Richard Karp

"It is to our everlasting shame that we were unable to persuade the math department to give him tenure." -- Richard Karp (1985 Turing award)

8.9 co-NP and the Asymmetry of NP

Asymmetry of NP

Asymmetry of NP. We only need to have short proofs of *yes* instances.

It may not be easy to show proofs of *no* instances.

Ex 1. SAT vs. TAUTOLOGY.

- Can prove a CNF formula is satisfiable by giving such an assignment.
- How could we prove that a formula is **not** satisfiable?

Ex 2. HAM-CYCLE vs. NO-HAM-CYCLE.

- Can prove a graph is Hamiltonian by giving such a Hamiltonian cycle.
- How could we prove that a graph is **not** Hamiltonian?

Q. how do we classify TAUTOLOGY?

- SAT is NP-complete and $SAT \equiv_p TAUTOLOGY$
- But TAUTOLOGY is not even known to be in NP

NP and co-NP

NP. Decision problems for which there is a poly-time certifier.

Ex. SAT, HAM-CYCLE, COMPOSITES.

Def. Given a decision problem X , its **complement** \overline{X} is the same problem with the _{yes} and _{no} answers reverse.

Ex. $\overline{X} = \{ 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, \dots \}$
 $X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, \dots \}$

co-NP. Complements of decision problems in NP.

- for _{no} instance, there is a succinct disqualifier

Ex. TAUTOLOGY, NO-HAM-CYCLE, PRIMES.

NP = co-NP ?

Fundamental question. Does $NP = co-NP$?

- Do _{yes} instances have succinct certificates iff _{no} instances do?
- Consensus opinion: no.

Theorem. If $NP \neq co-NP$, then $P \neq NP$.

Pf idea.

- P is closed under complementation.
- If $P = NP$, then NP is closed under complementation.
- In other words, $NP = co-NP$.
- This is the contrapositive of the theorem.

Good Characterizations

Good characterization. [Edmonds 1965] $NP \cap co-NP$.

- If problem X is in both NP and $co-NP$, then:
 - for yes instance, there is a succinct certificate
 - for no instance, there is a succinct disqualifier
- Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching.

- If yes, can exhibit a perfect matching.
- If no, can exhibit a set of nodes S such that $|N(S)| < |S|$.

Good Characterizations

Observation. $P \subseteq NP \cap \text{co-NP}$.

Fundamental open question. Does $P = NP \cap \text{co-NP}$?

- Mixed opinions.
- Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.
 - linear programming [Khachiyan, 1979]
 - primality testing [Agrawal-Kayal-Saxena, 2002]

Fact. Factoring is in $NP \cap \text{co-NP}$, but not known to be in P.



if poly-time algorithm for factoring,
can break RSA cryptosystem

PRIMES is in $NP \cap co-NP$

Theorem. PRIMES is in $NP \cap co-NP$.

Pf. We already know that PRIMES is in co-NP, so it suffices to prove that PRIMES is in NP.

Pratt's Theorem. An odd integer s is prime iff there exists an integer $1 < t < s$ s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$

$$t^{(s-1)/p} \not\equiv 1 \pmod{s}$$

for all prime divisors p of $s-1$

Input. $s = 437,677$

Certificate. $t = 17, 2^2 \times 3 \times 36473$



prime factorization of $s-1$
also need a recursive certificate
to assert that 3 and 36,473 are prime

Certifier.

- Check $s-1 = 2 \times 2 \times 3 \times 36473$.
- Check $17^{s-1} \equiv 1 \pmod{s}$.
- Check $17^{(s-1)/2} \equiv 437676 \pmod{s}$.
- Check $17^{(s-1)/3} \equiv 329415 \pmod{s}$.
- Check $17^{(s-1)/36473} \equiv 305452 \pmod{s}$.



use repeated squaring

FACTOR is in $NP \cap co-NP$

FACTORIZE. Given an integer x , find its prime factorization.

FACTOR. Given two integers x and y , does x have a nontrivial factor less than y ?

Theorem. $FACTOR \equiv_p FACTORIZE$.

Pf. Binary search to find a factor, divide the factor and repeat

Theorem. $FACTOR$ is in $NP \cap co-NP$.

Pf.

- **Certificate:** a factor p of x that is less than y .
- **Disqualifier:** the prime factorization of x (where each prime factor is $\geq y$), along with a Pratt's certificate that each factor is prime.

Primality Testing and Factoring

Q: Is PRIMES in P?

- Yes! (Agrawal-Kayal-Saxena, 2002)

Q: Is FACTOR in P?

- Consensus: No.

RSA cryptosystem.

- Based on dichotomy between complexity of two problems.
- To use RSA, must generate large primes efficiently.
- To break RSA, must find efficient factoring algorithm.

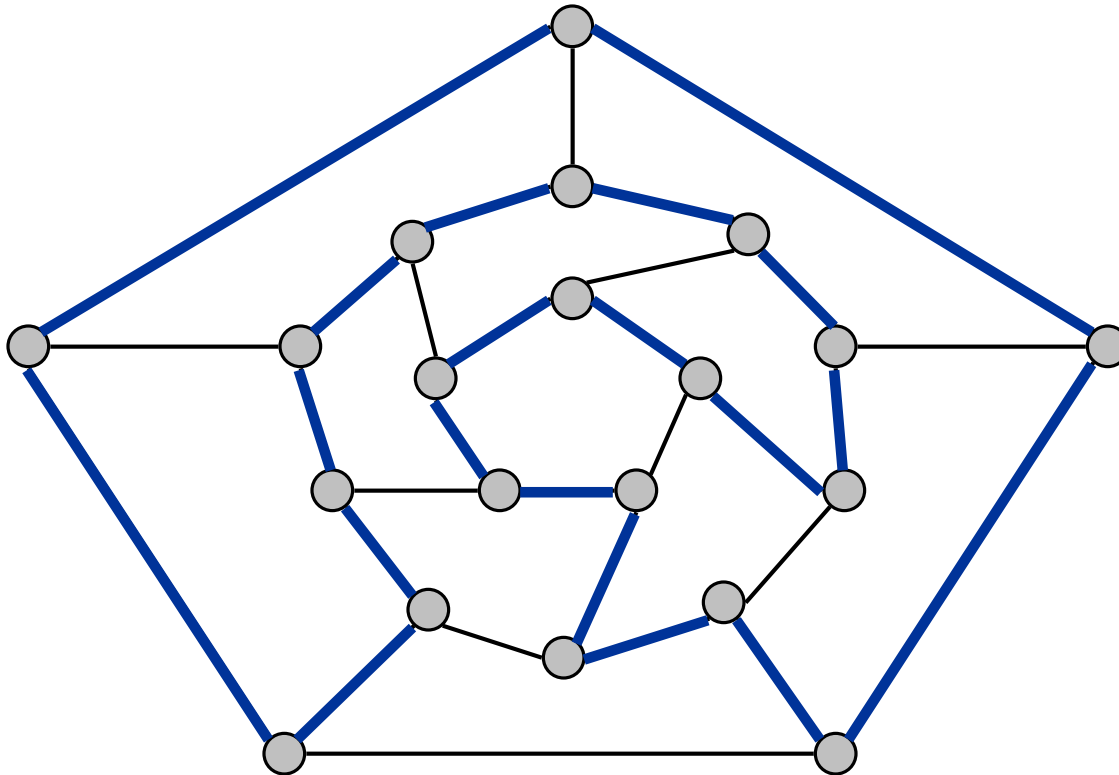
8.5 Sequencing Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- **Sequencing problems:** HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Hamiltonian Cycle

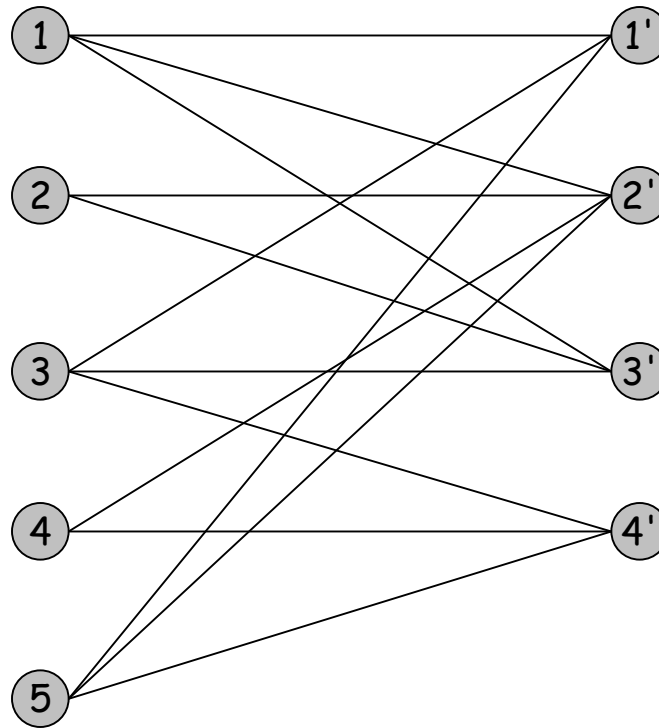
HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V .



YES: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V .



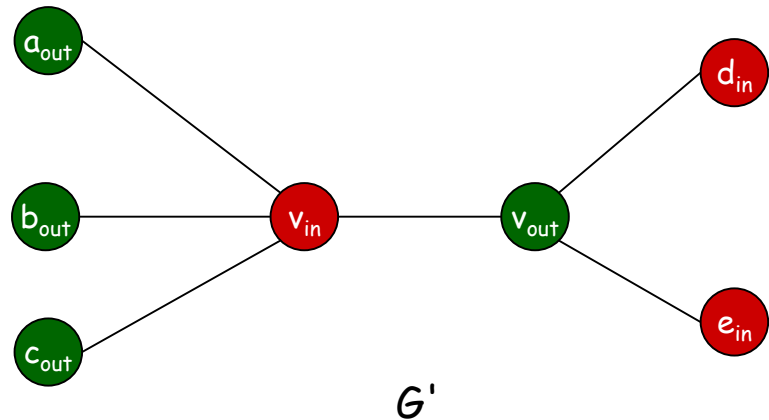
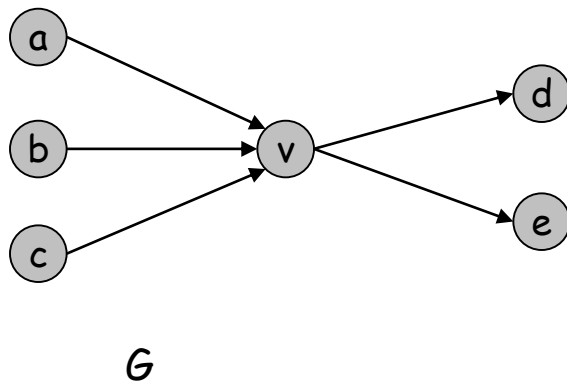
NO: bipartite graph with odd number of nodes.

Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a **digraph** $G = (V, E)$, does there exist a simple directed cycle Γ that contains every node in V ?

Claim. $\text{DIR-HAM-CYCLE} \leq_p \text{HAM-CYCLE}$.

Pf. Given a directed graph $G = (V, E)$, construct an undirected graph G'

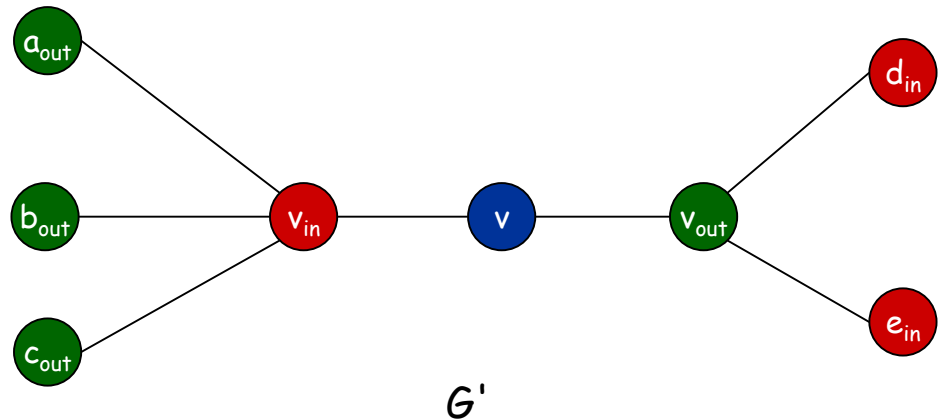
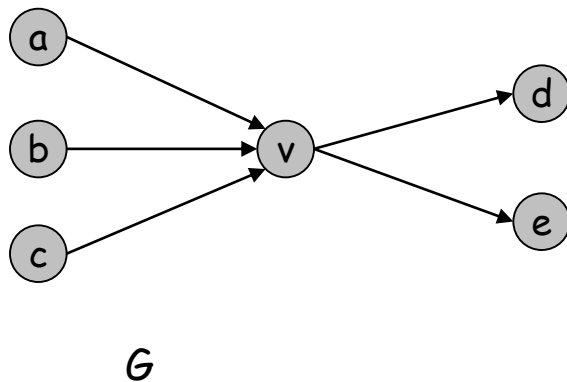


Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a **digraph** $G = (V, E)$, does there exist a simple directed cycle Γ that contains every node in V ?

Claim. $\text{DIR-HAM-CYCLE} \leq_p \text{HAM-CYCLE}$.

Pf. Given a directed graph $G = (V, E)$, construct an undirected graph G' with $3n$ nodes.



Directed Hamiltonian Cycle

Claim. G has a Hamiltonian cycle iff G' does.

Pf. \Rightarrow

- Suppose G has a directed Hamiltonian cycle Γ .
- Then G' has an undirected Hamiltonian cycle (same order).

Pf. \Leftarrow

- Suppose G' has an undirected Hamiltonian cycle Γ' .
- Γ' must visit nodes in G' using one of following two orders:
 ..., B, G, R, B, G, R, B, G, R, B, ...
 ..., B, R, G, B, R, G, B, R, G, B, ...
- Blue nodes in Γ' make up directed Hamiltonian cycle Γ in G , or reverse of one. ▪

3-SAT Reduces to Directed Hamiltonian Cycle

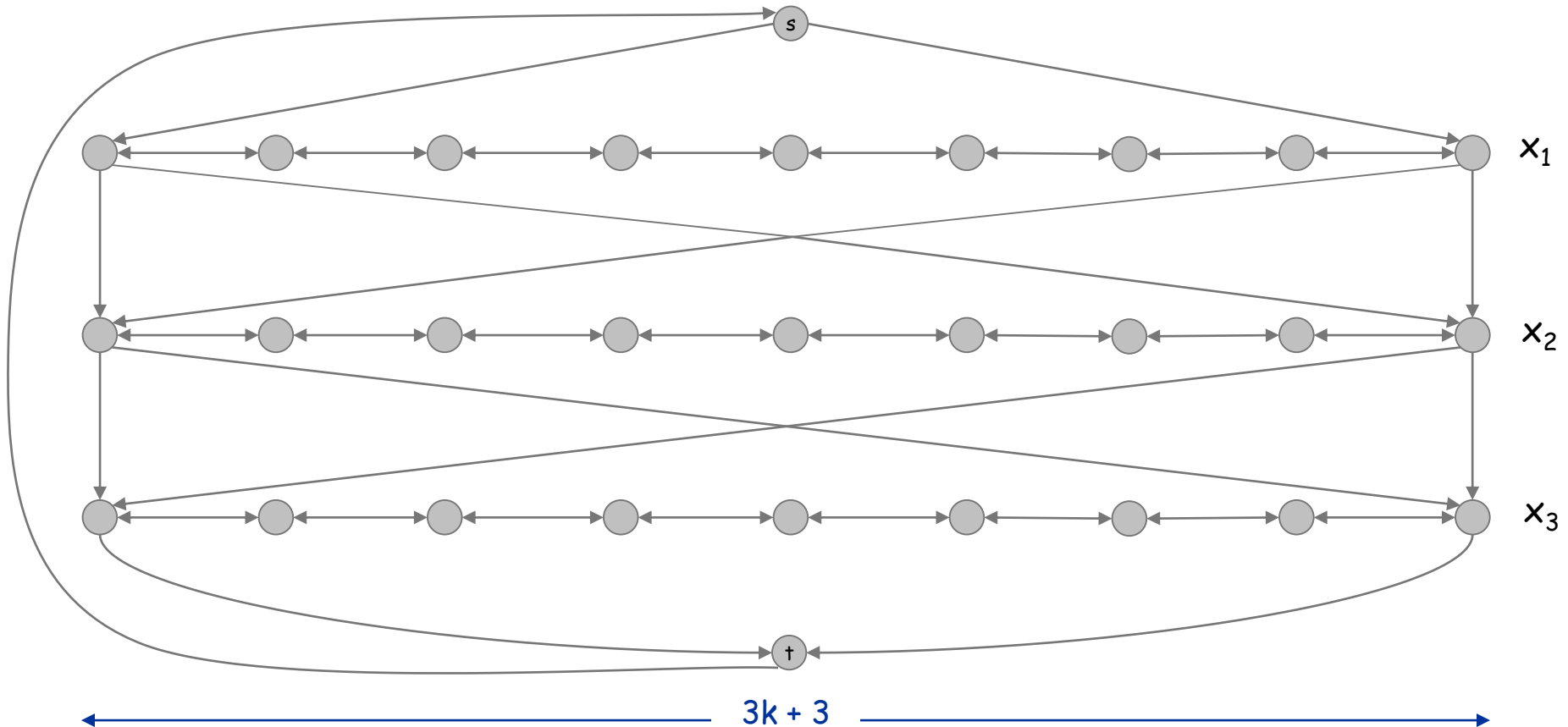
Claim. $3\text{-SAT} \leq_p \text{DIR-HAM-CYCLE}$.

Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

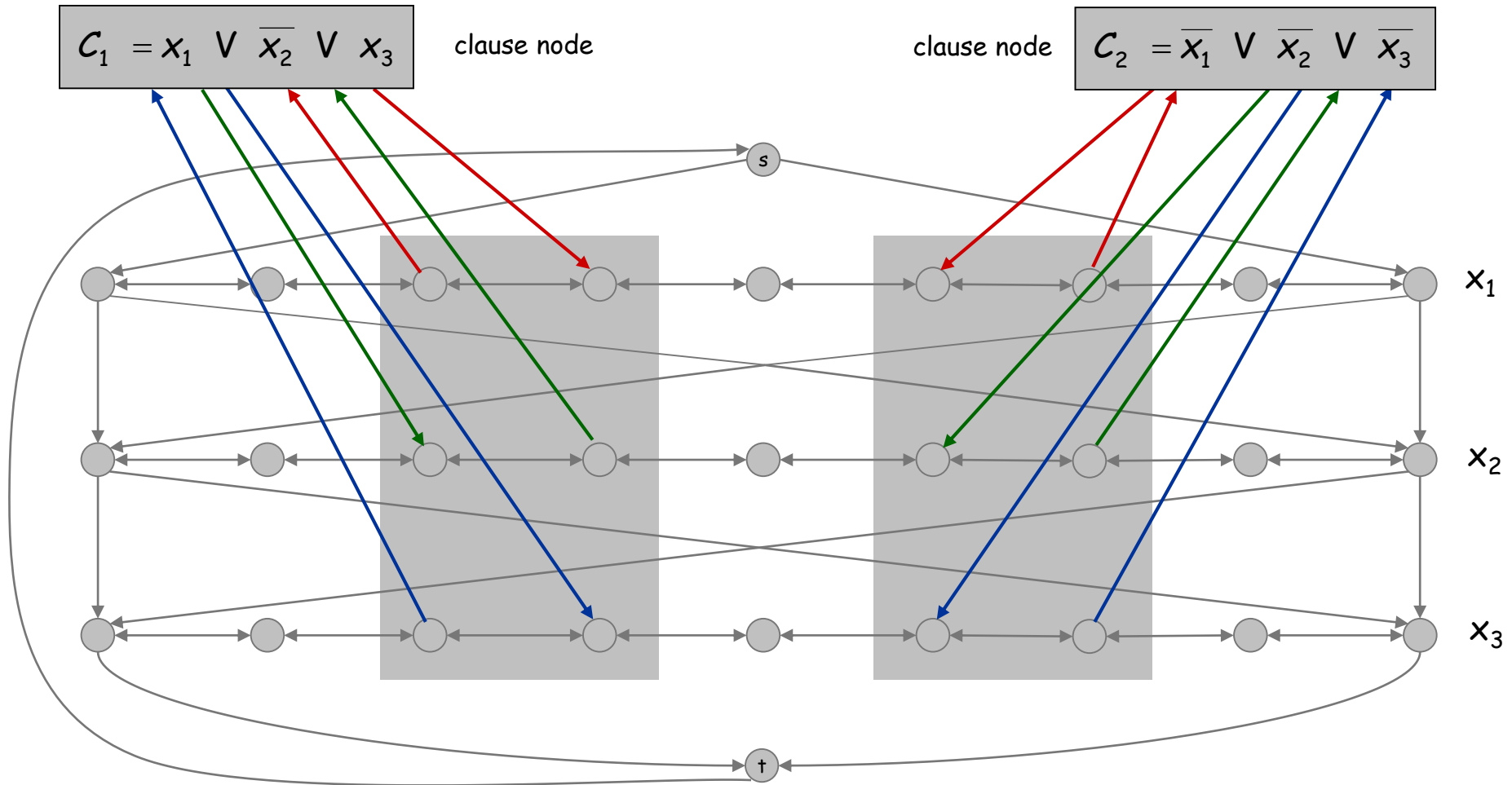
- Construct G to have 2^n Hamiltonian cycles.
- Intuition: traverse path i from left to right \Leftrightarrow set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- For each clause: add a node and 6 edges.



3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. \Rightarrow

- Suppose 3-SAT instance has satisfying assignment x^* .
- Then, define Hamiltonian cycle in G as follows:
 - if $x_i^* = 1$, traverse row i from left to right
 - if $x_i^* = 0$, traverse row i from right to left
 - for each clause C_j , there will be at least one row i in which we are going in "correct" direction to splice node C_j into tour

3-SAT Reduces to Directed Hamiltonian Cycle

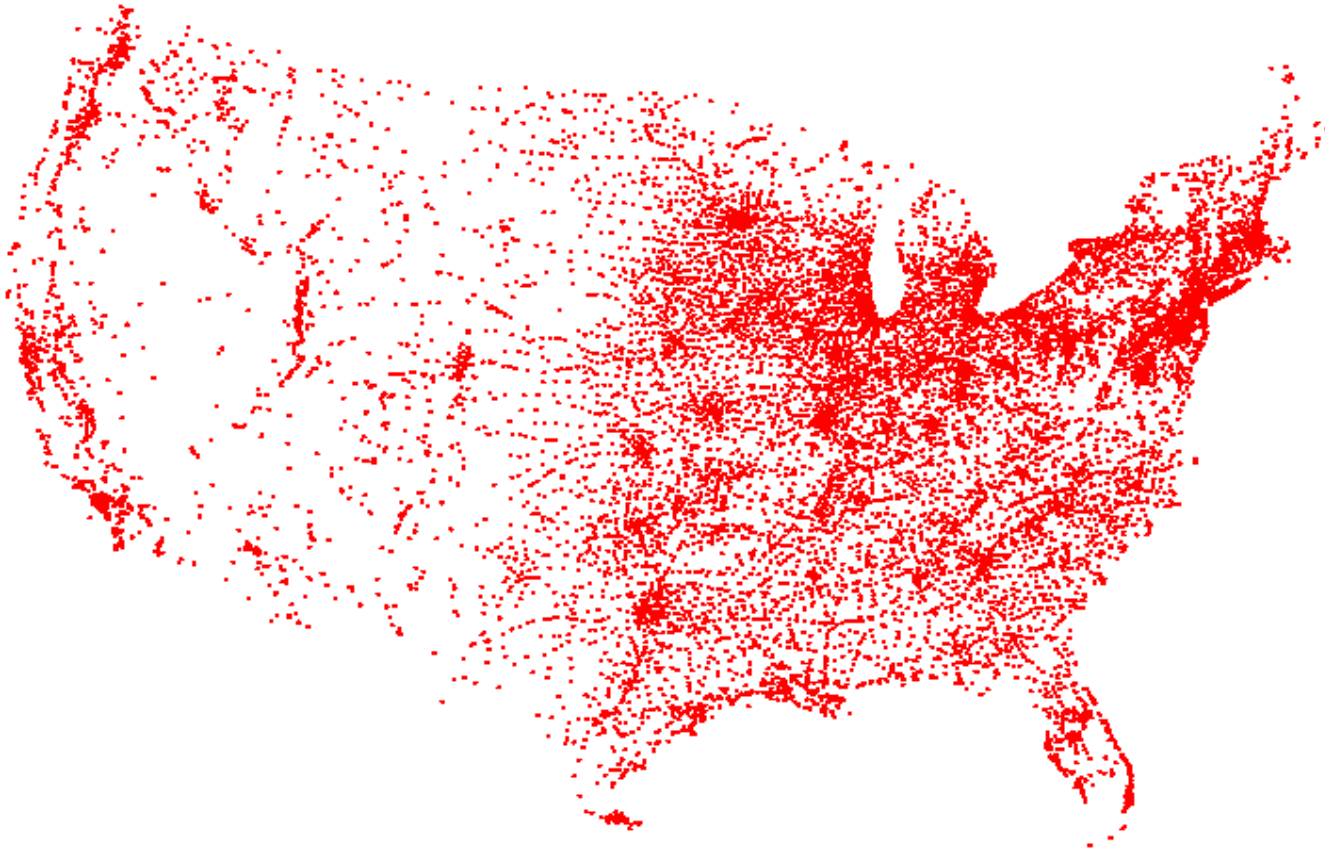
Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. \Leftarrow

- Suppose G has a Hamiltonian cycle Γ .
- If Γ enters clause node C_j , it must depart on mate edge.
 - thus, nodes immediately before and after C_j are connected by an edge e in G
 - removing C_j from cycle, and replacing it with edge e yields Hamiltonian cycle on $G - \{C_j\}$
- Continuing in this way, we are left with Hamiltonian cycle Γ' in $G - \{C_1, C_2, \dots, C_k\}$.
- Set $x_i^* = 1$ iff Γ' traverses row i left to right.
- Since Γ visits each clause node C_j , at least one of the paths is traversed in "correct" direction, and each clause is satisfied. ▪

Traveling Salesperson Problem

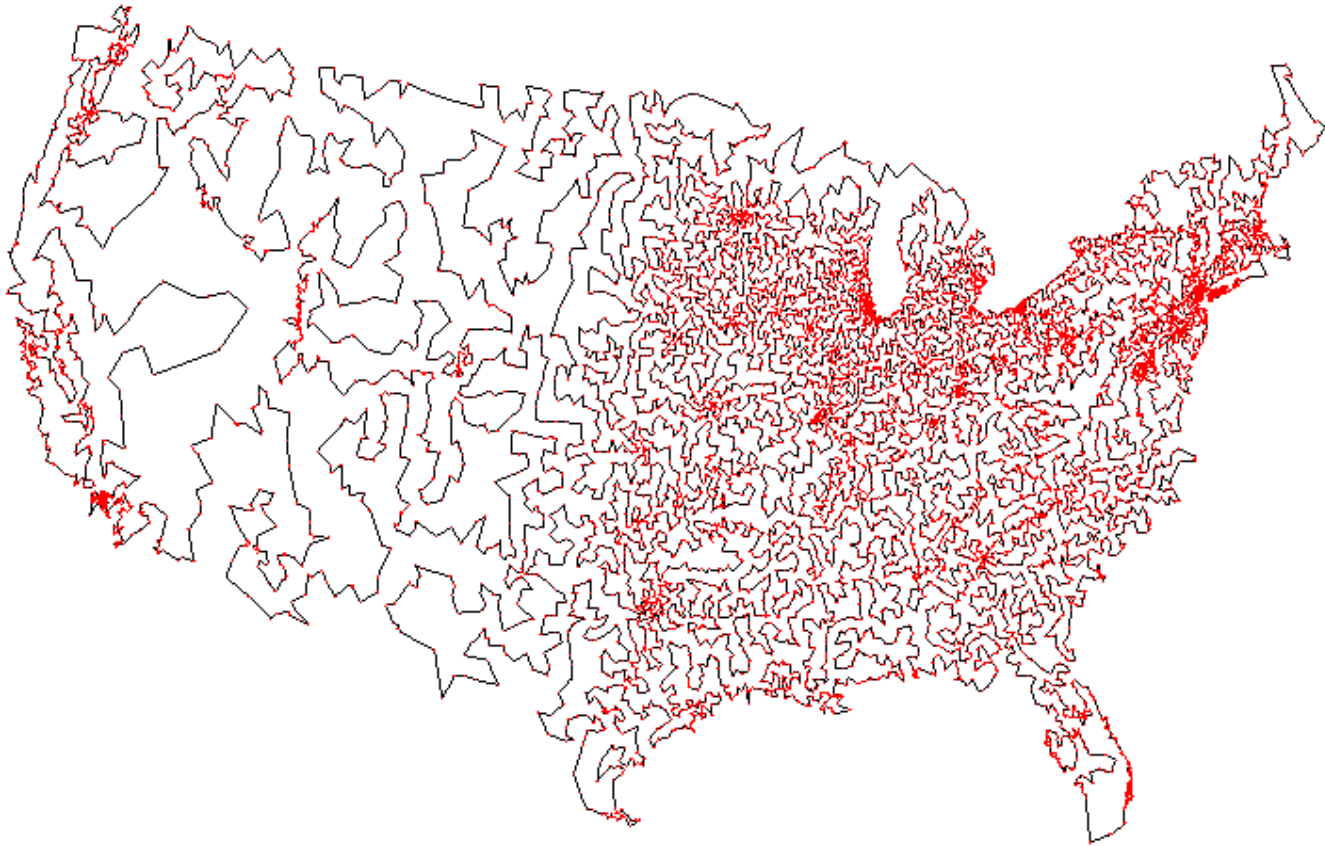
TSP. Given a set of n cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$?



All 13,509 cities in US with a population of at least 500
Reference: <http://www.tsp.gatech.edu>

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$?



Optimal TSP tour
Reference: <http://www.tsp.gatech.edu>

Traveling Salesperson Problem

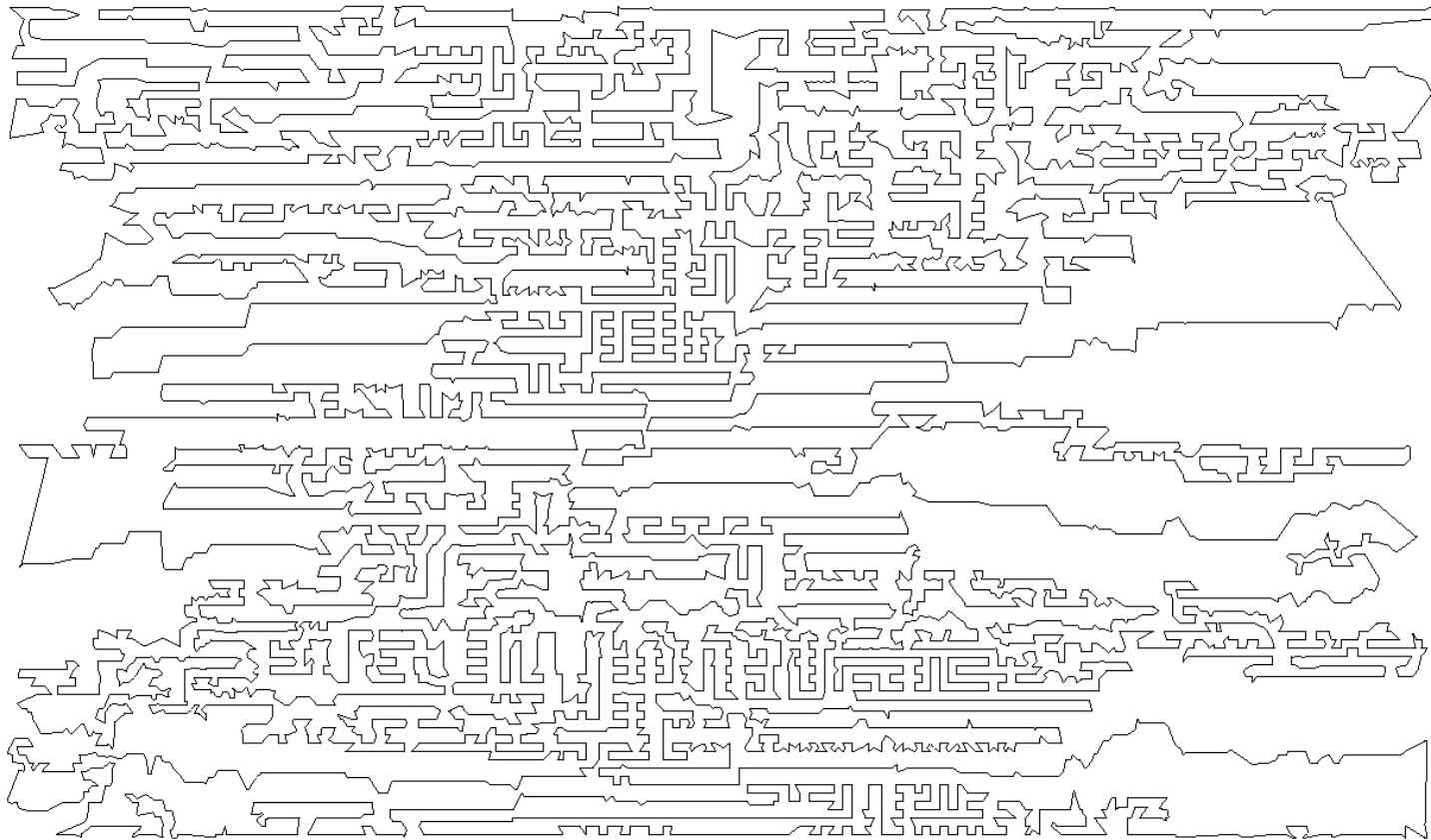
TSP. Given a set of n cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$?



11,849 holes to drill in a programmed logic array
Reference: <http://www.tsp.gatech.edu>

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$?



Optimal TSP tour
Reference: <http://www.tsp.gatech.edu>

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$?

HAM-CYCLE: given a graph $G = (V, E)$, does there exist a simple cycle that contains every node in V ?

Claim. $\text{HAM-CYCLE} \leq_p \text{TSP}$.

Pf.

- Given instance $G = (V, E)$ of HAM-CYCLE, create n cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

- TSP instance has tour of length $\leq n$ iff G is Hamiltonian. ▪

Longest Path

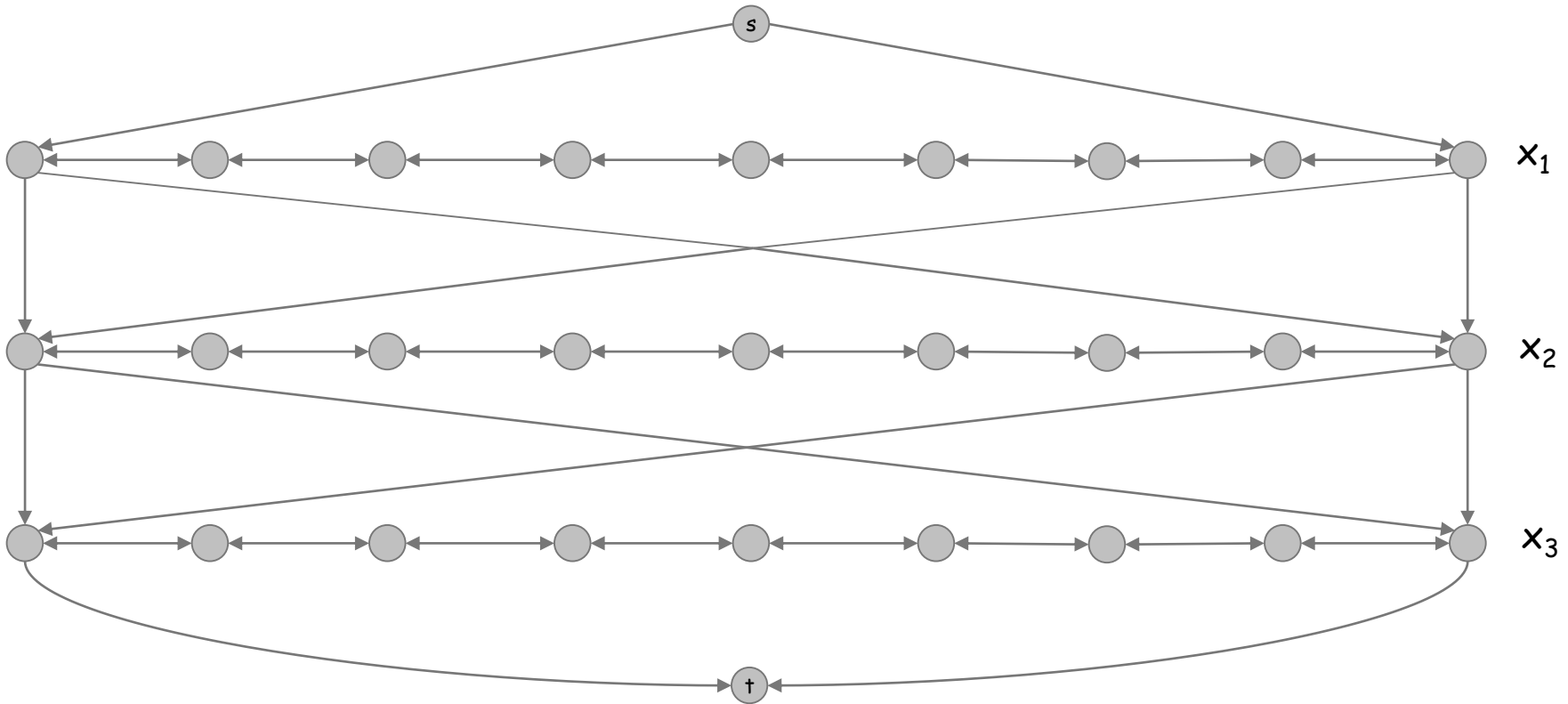
SHORTEST-PATH. Given a digraph $G = (V, E)$, does there exist a simple path of length **at most** k edges?

LONGEST-PATH. Given a digraph $G = (V, E)$, does there exist a simple path of length **at least** k edges?

Longest Path

Claim. $3\text{-SAT} \leq_p \text{LONGEST-PATH}$.

Pf 1. Redo proof for DIR-HAM-CYCLE, ignoring back-edge from t to s .

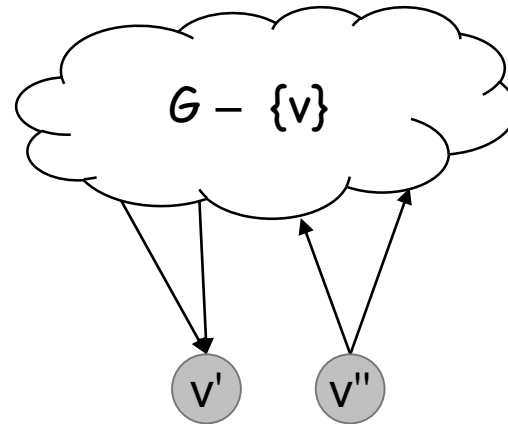
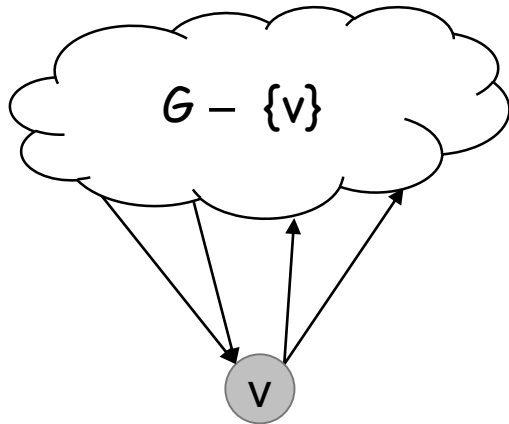


Longest Path

Claim. $3\text{-SAT} \leq_p \text{LONGEST-PATH}$.

Pf 2. Show $\text{DIR-HAM-CYCLE} \leq_p \text{LONGEST-PATH}$.

- Given an instance of DIR-HAM-CYCLE, split an arbitrary node v of the digraph into two nodes v' and v'' ; connect all the incoming edges to v' and all the outgoing edges to v'' .



The Longest Path †

Lyrics. Copyright © 1988 by Daniel J. Barrett.

Music. Sung to the tune of *The Longest Time* by Billy Joel.



Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,
There would still be papers left to write,
I have a weakness,
I'm addicted to completeness,
And I keep searching for the longest
path.

The algorithm I would like to see
Is of polynomial degree,
But it's elusive:
Nobody has found conclusive
Evidence that we can find a longest path.

I have been hard working for so long.
I swear it's right, and he marks it wrong.
Some how I'll feel sorry when it's done:
GPA 2.1
Is more than I hope for.

Garey, Johnson, Karp and other men (and
women)
Tried to make it order $N \log N$.
Am I a mad fool
If I spend my life in grad school,
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path.

† Recorded by Dan Barrett while a grad student at Johns Hopkins during a difficult algorithms final.

8.6 Partitioning Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- **Partitioning problems:** 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

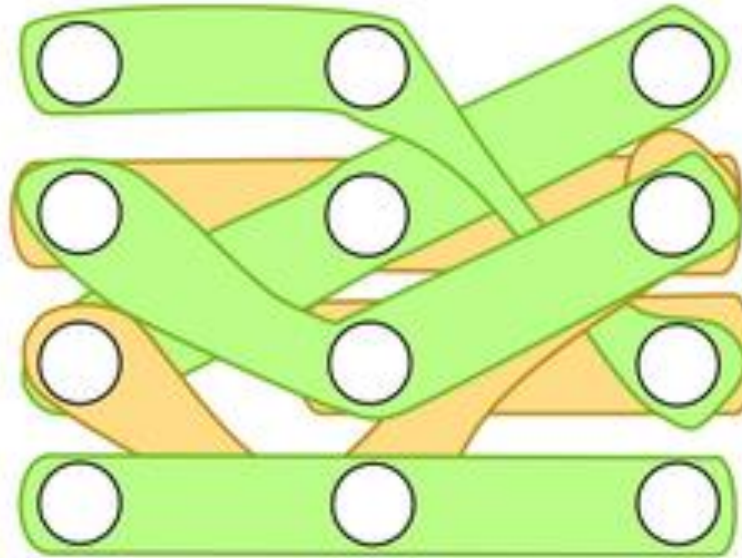
3-Dimensional Matching

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

Instructor	Course	Time
Wayne	COS 423	MW 11-12:20
Wayne	COS 423	TTh 11-12:20
Wayne	COS 226	TTh 11-12:20
Tardos	COS 523	TTh 3-4:20
Tardos	COS 423	TTh 11-12:20
Tardos	COS 423	TTh 3-4:20
Kleinberg	COS 226	TTh 3-4:20
Kleinberg	COS 226	MW 11-12:20
Kleinberg	COS 423	MW 11-12:20

3-Dimensional Matching

3D-MATCHING. Given disjoint sets X , Y , and Z , each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?



3-Dimensional Matching

3D-MATCHING. Given disjoint sets X , Y , and Z , each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

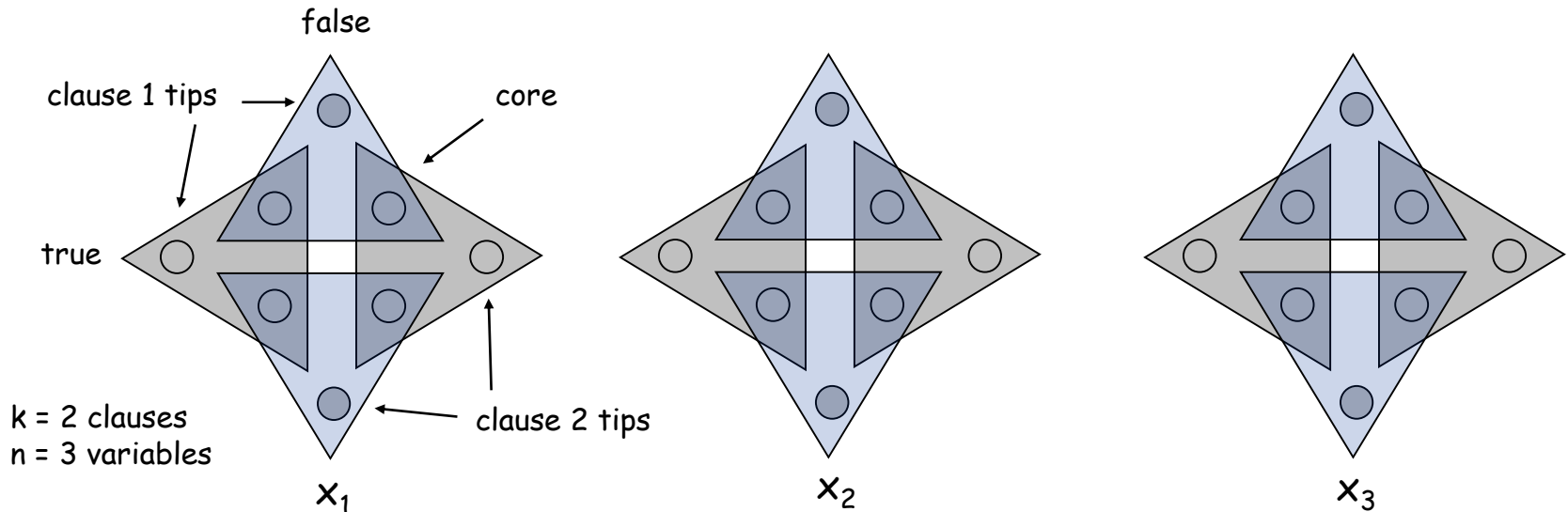
Claim. $3\text{-SAT} \leq_p 3\text{D-MATCHING}$.

Pf. Given an instance Φ of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff Φ is satisfiable.

3-Dimensional Matching

Construction. (part 1)

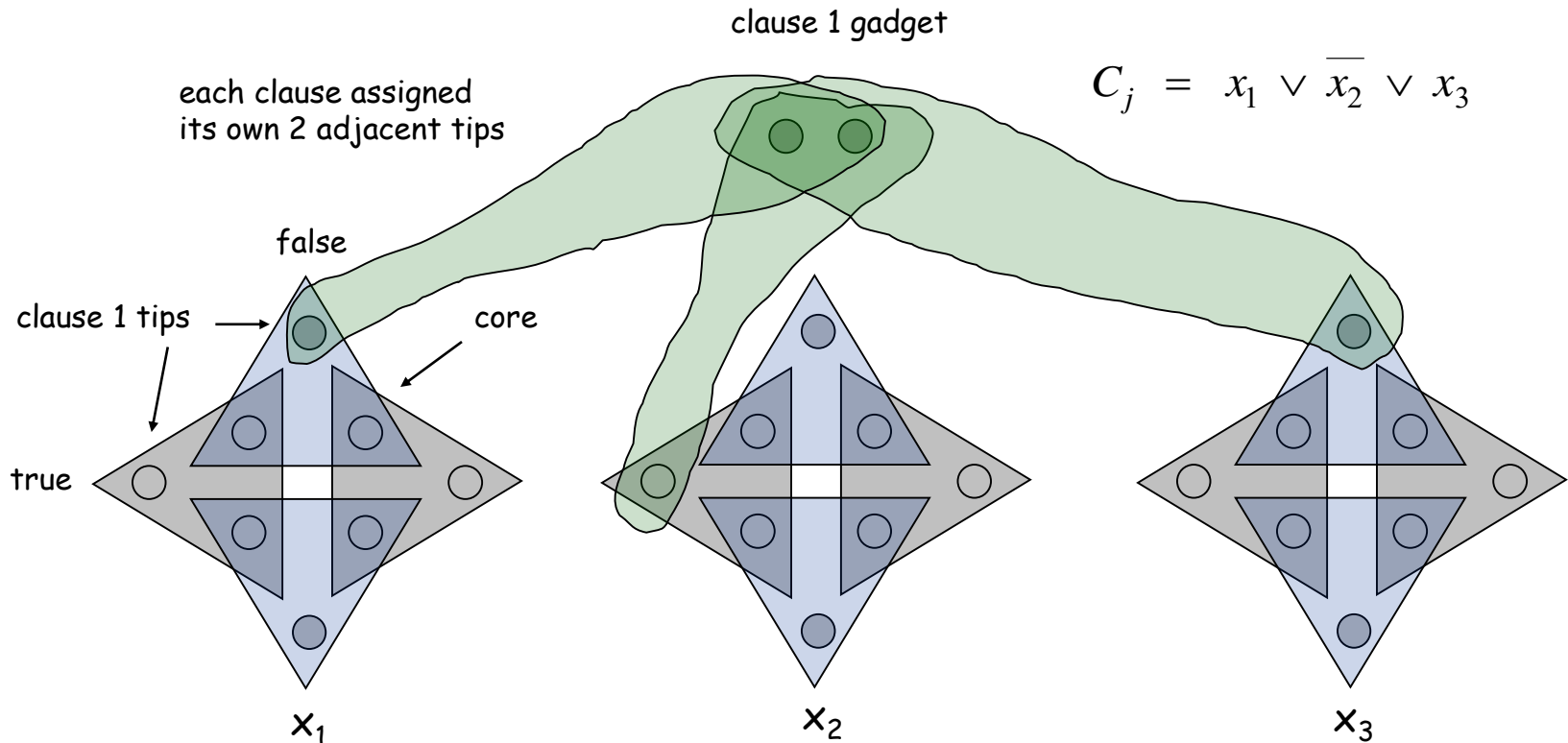
- Create gadget for each variable x_i with $2k$ core elements and $2k$ tip elements.
- No other triples will use core elements.
- In gadget i , 3D-matching must use either all the grey triples (corresponding to $x_i = \text{true}$) or all the blue ones (corresponding to $x_i = \text{false}$).



3-Dimensional Matching

Construction. (part 2)

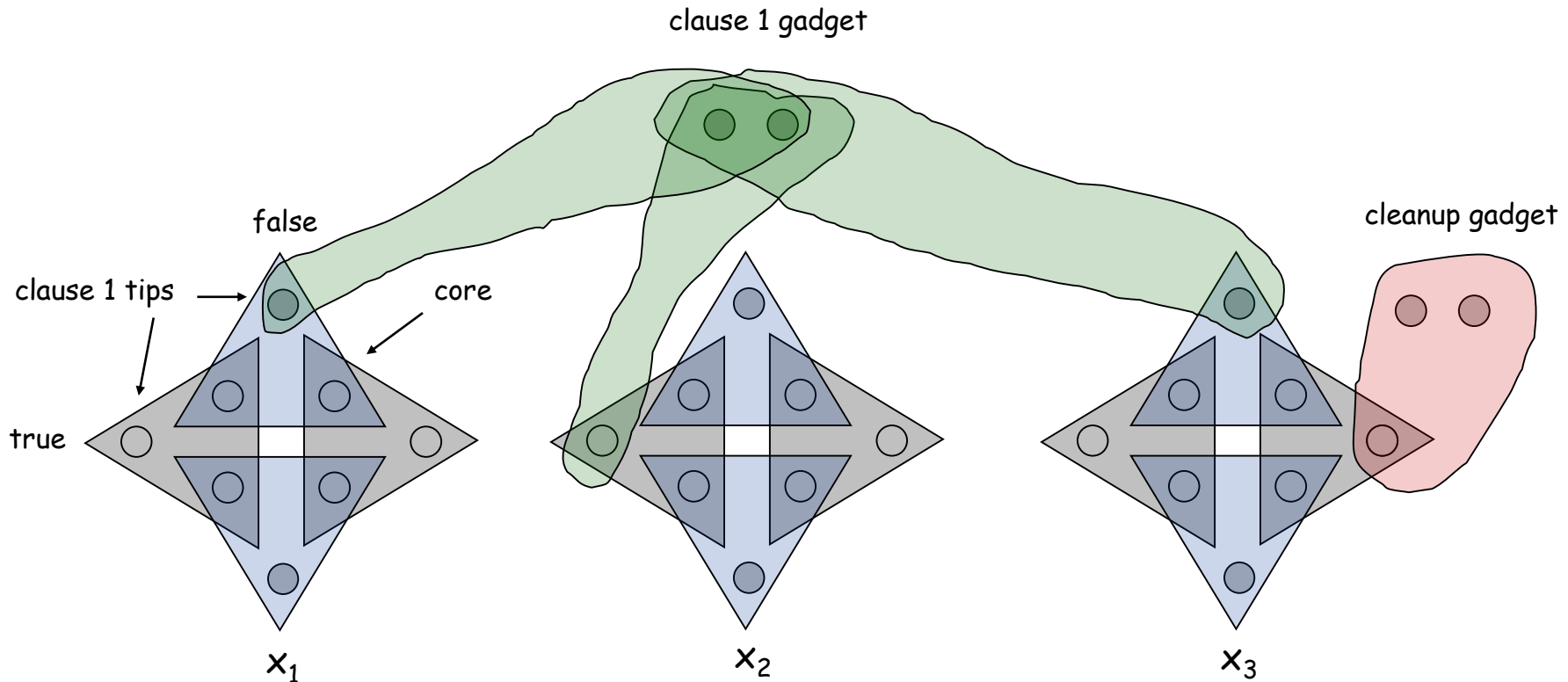
- For each clause C_j create gadget with two elements and three triples.
- Exactly one of these triples will be used in any 3D-matching.
- Ensures any 3D-matching uses either (i) grey core of x_1 or (ii) blue core of x_2 or (iii) grey core of x_3 .



3-Dimensional Matching

Construction. (part 3)

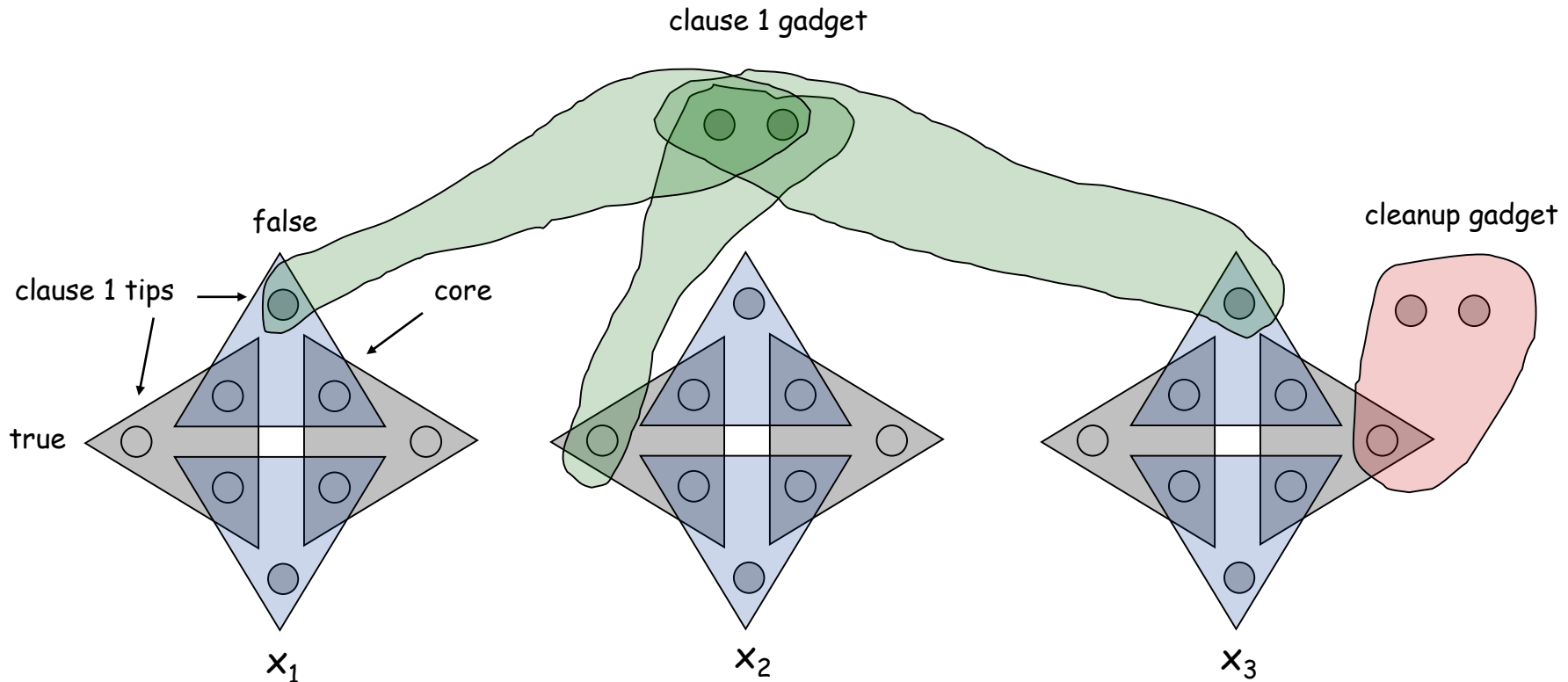
- There are $2nk$ tips. nk covered by blue/gray triples, k covered by clause triples.
- To cover the remaining $(n-1)k$ tips, create $(n-1)k$ cleanup gadgets, each connected to all the $2nk$ tips.



3-Dimensional Matching

Claim. Instance has a 3D-matching iff Φ is satisfiable.

Q. What are X , Y , and Z ? Does each triple contain one element from each of X , Y , Z ?

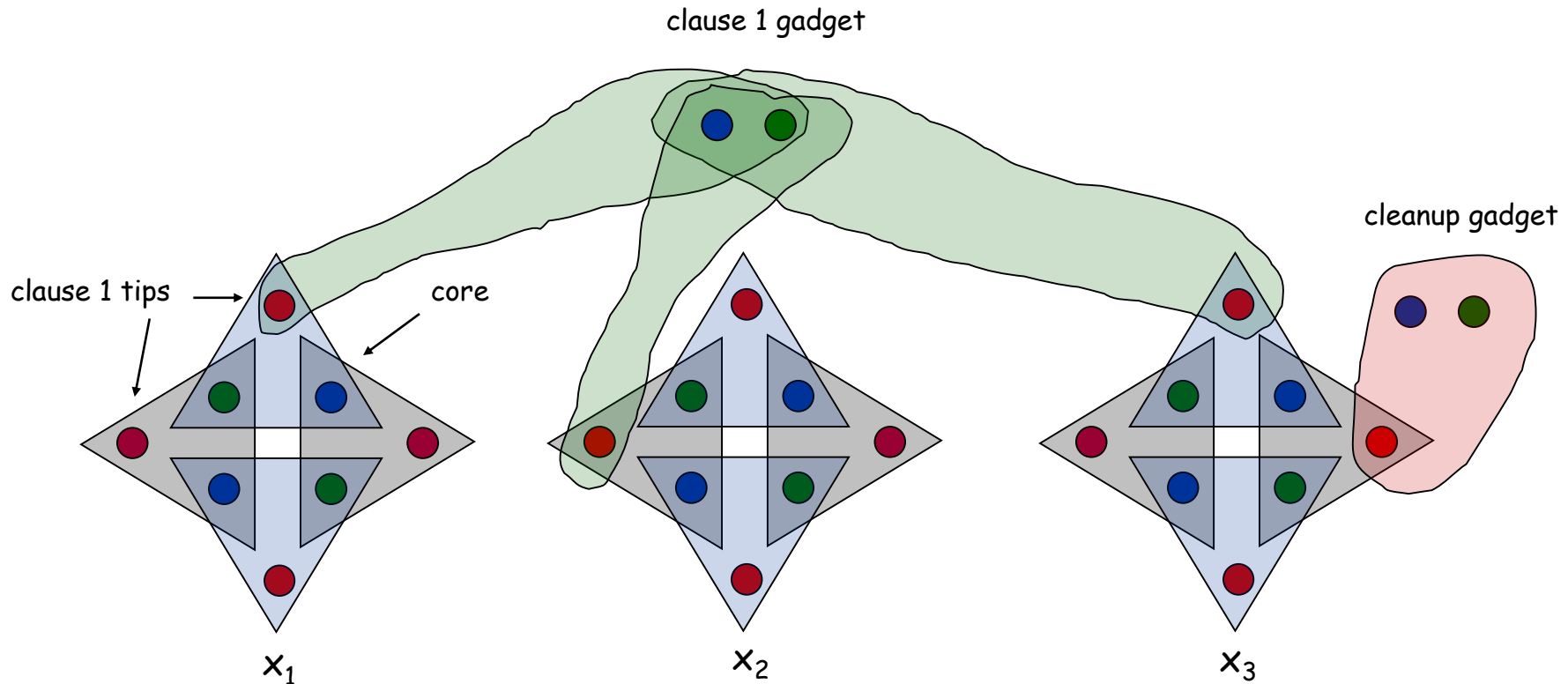


3-Dimensional Matching

Claim. Instance has a 3D-matching iff Φ is satisfiable.

Q. What are X , Y , and Z ? Does each triple contain one element from each of X , Y , Z ?

A. X = red, Y = green, Z = blue

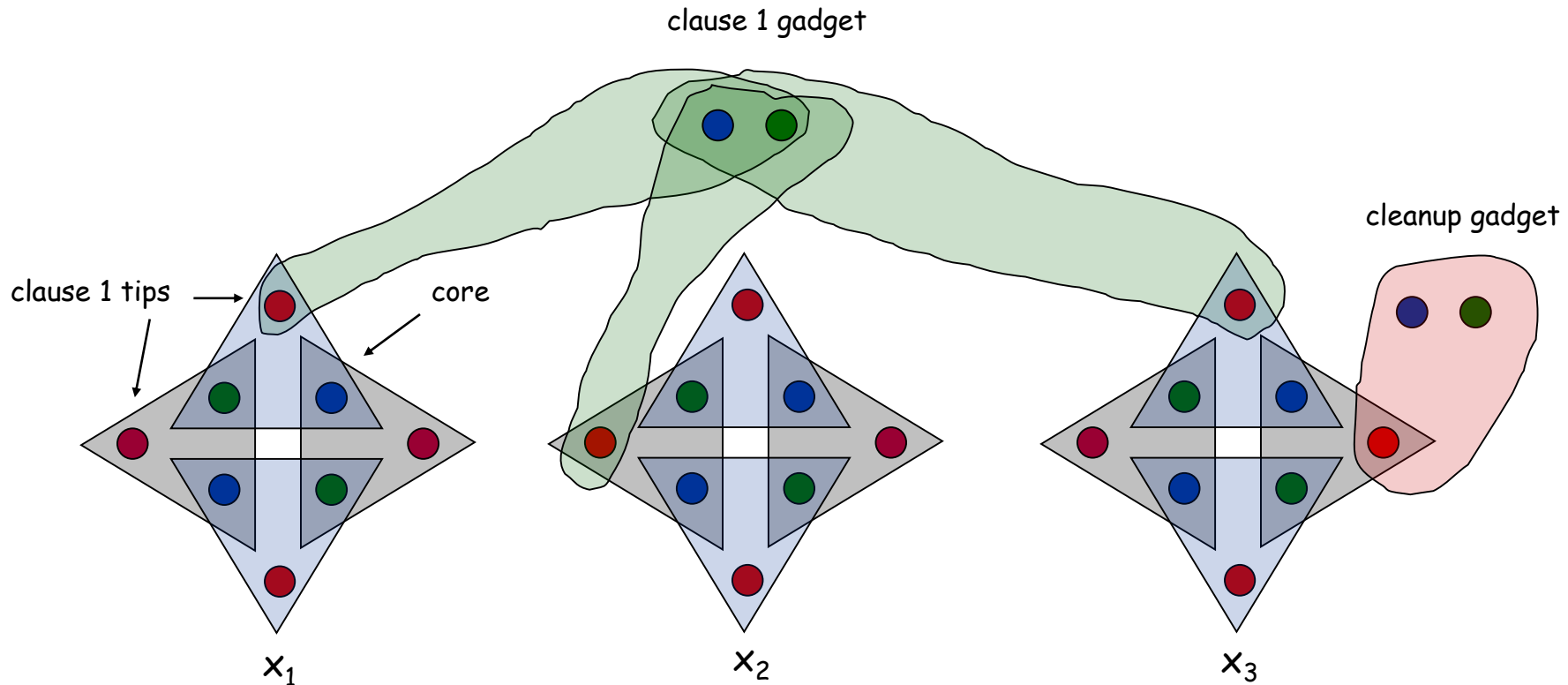


3-Dimensional Matching

Claim. Instance has a 3D-matching iff Φ is satisfiable.

Pf. \Rightarrow If 3D-matching, then assign x_i according to the color of the selected triples in gadget x_i

\Leftarrow If Φ is satisfiable, then use assignment of x_i to select gadget x_i triples and use true literal in C_j to select gadget C_j triple



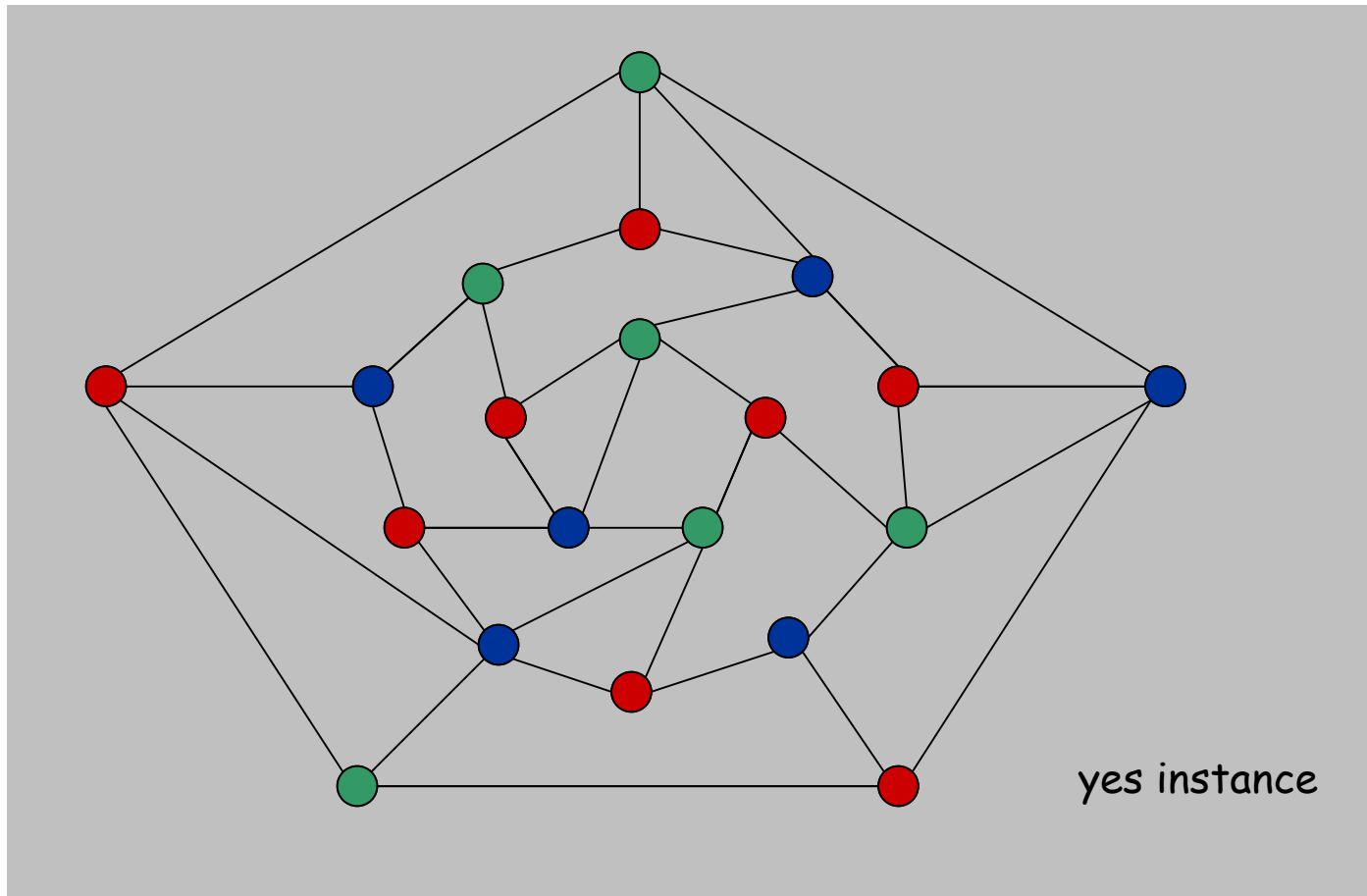
8.7 Graph Coloring

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- **Partitioning problems:** 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Colorability

3-COLOR: Given an undirected graph G does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



Application: Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k -colorable.

Fact. $3\text{-COLOR} \leq_p k\text{-REGISTER-ALLOCATION}$ for any constant $k \geq 3$.

3-Colorability

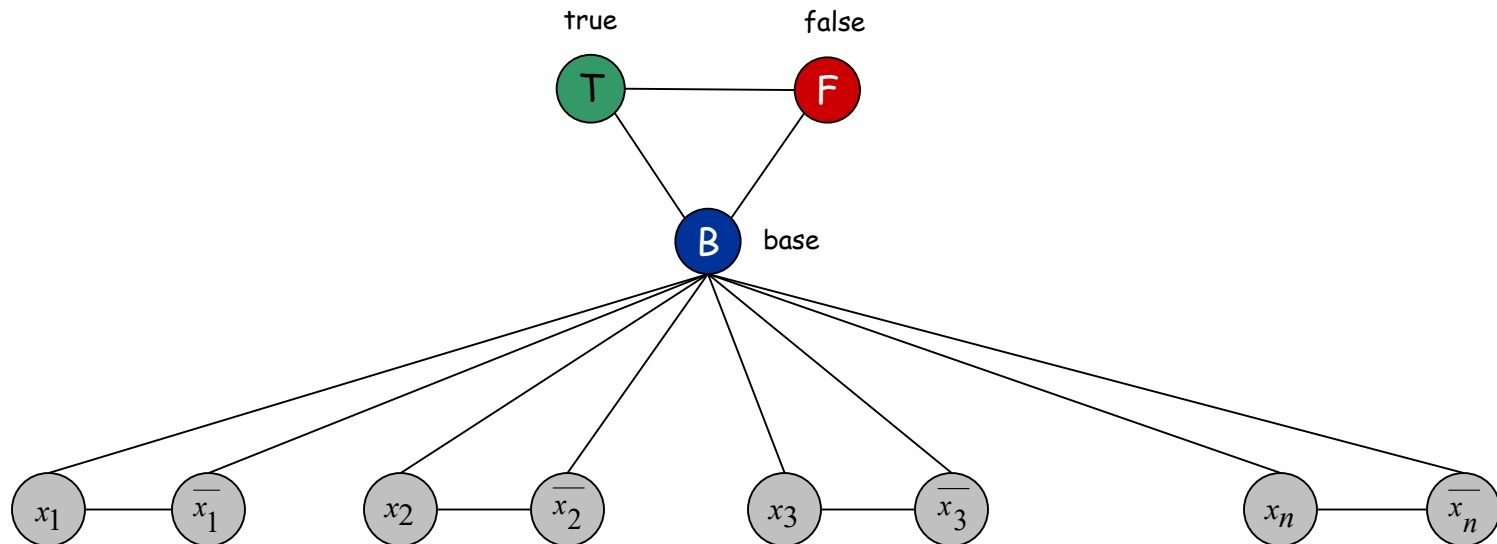
Claim. $3\text{-SAT} \leq_p 3\text{-COLOR}$.

Pf. Given 3-SAT instance Φ , we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable.

3-Colorability

Construction.

- i. For each literal, create a node.
- ii. Connect each literal to its negation.
- iii. Create 3 new nodes T, F, B; connect them in a triangle
- iv. Connect each literal to B.
- v. For each clause, add gadget of 6 nodes and 13 edges (to be described later)

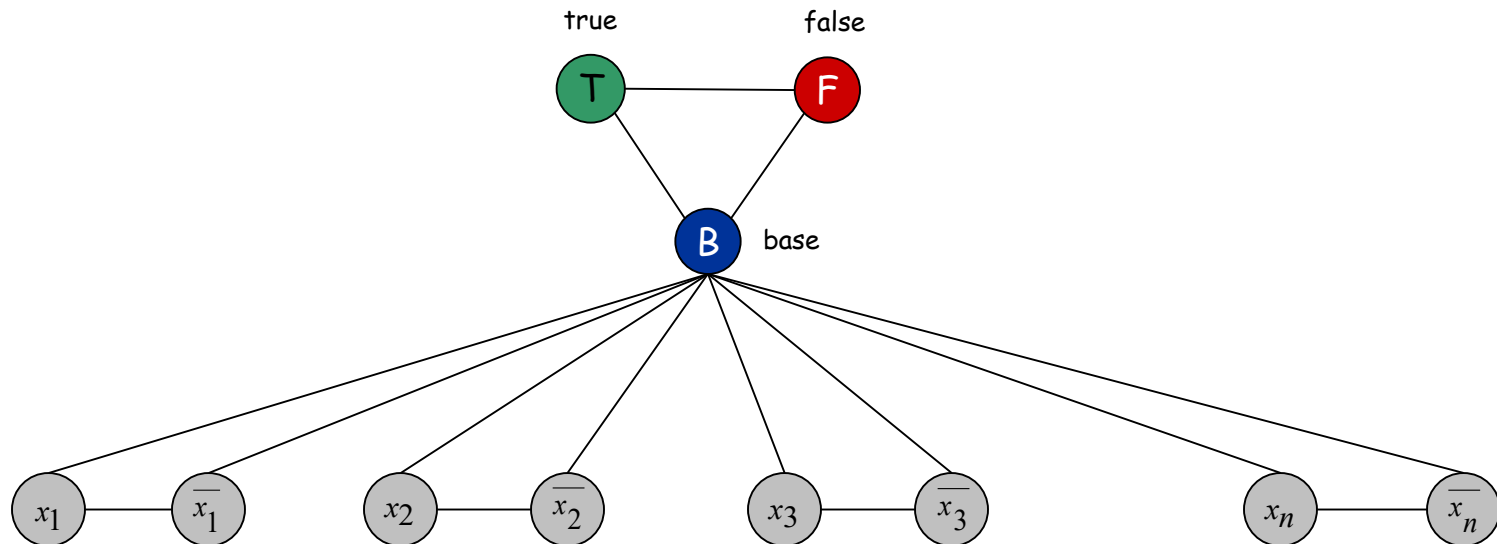


3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. \Rightarrow Suppose graph is 3-colorable.

- Set all the literals with T color to true.
- (iv) ensures each literal is T or F.
- (ii) ensures a literal and its negation are opposites.

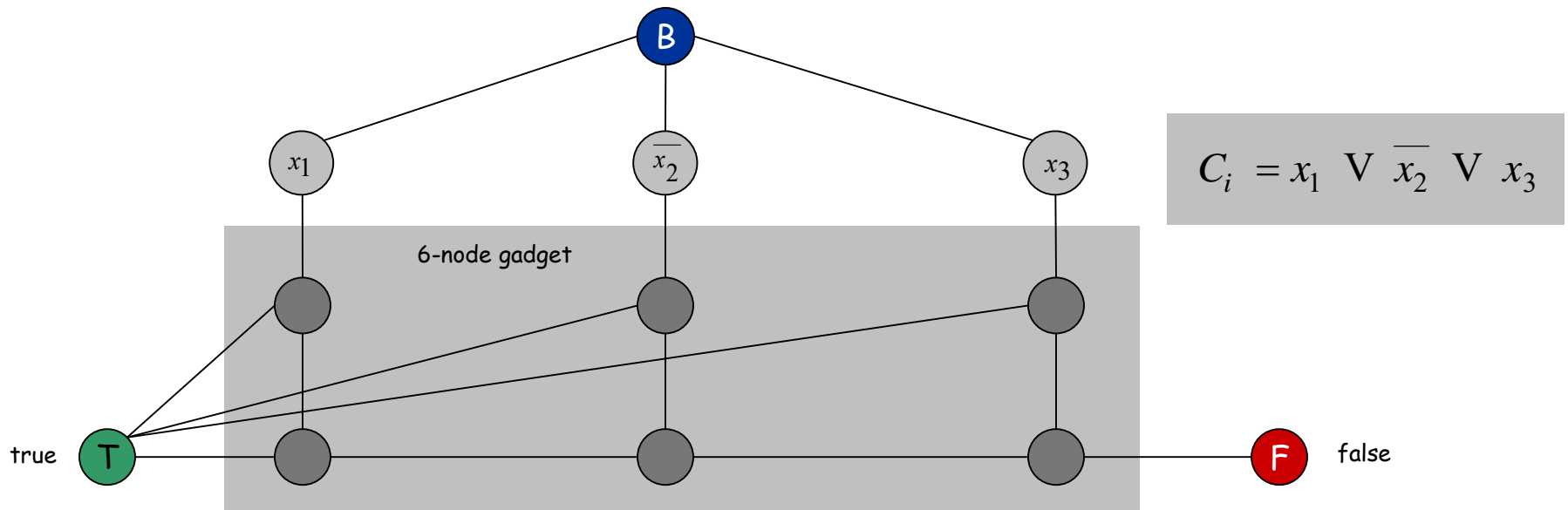


3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. \Rightarrow Suppose graph is 3-colorable.

- Set all the literals with T color to true.
- (iv) ensures each literal is T or F.
- (ii) ensures a literal and its negation are opposites.
- (v) ensures at least one literal in each clause is T.

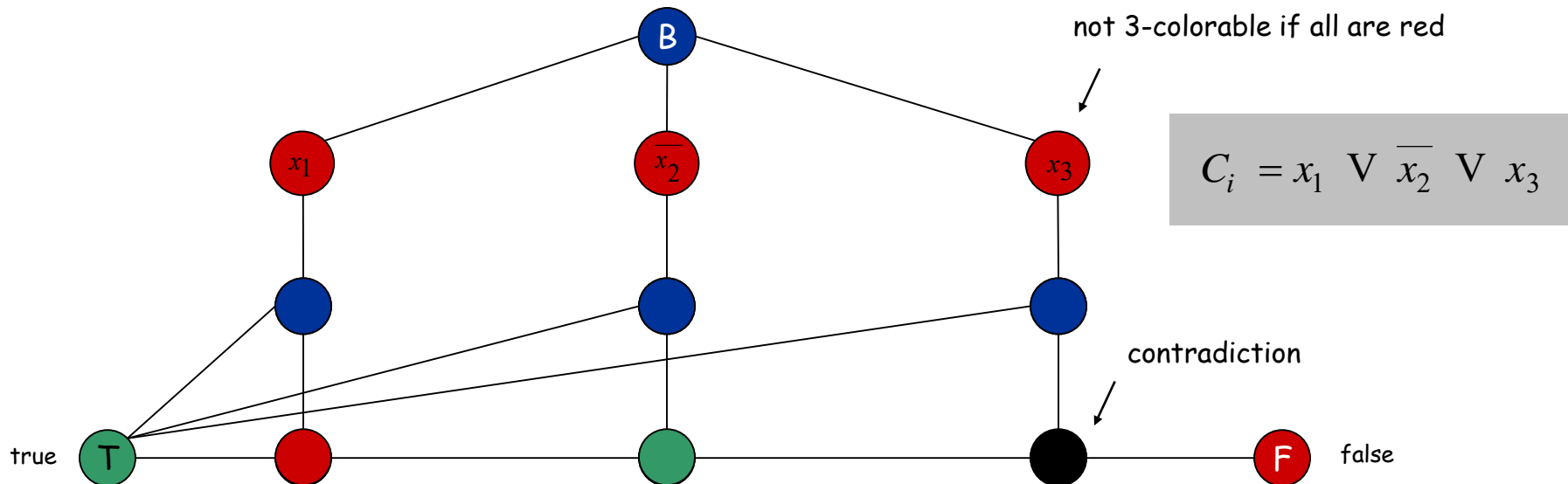


3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

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- (v) ensures at least one literal in each clause is T.



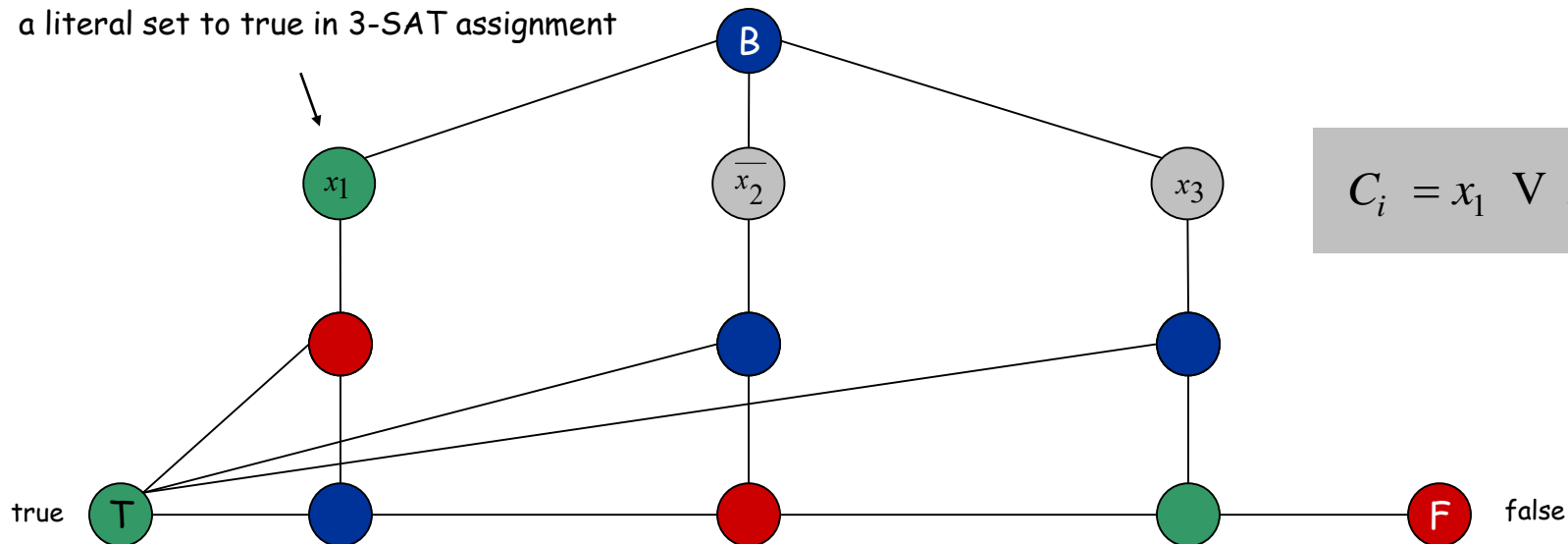
3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. \Leftarrow Suppose 3-SAT formula Φ is satisfiable.

- Color all true literals T.
- Color node below green node F, and node below that B.
- Color remaining middle row nodes B.
- Color remaining bottom nodes T or F as forced.

a literal set to true in 3-SAT assignment

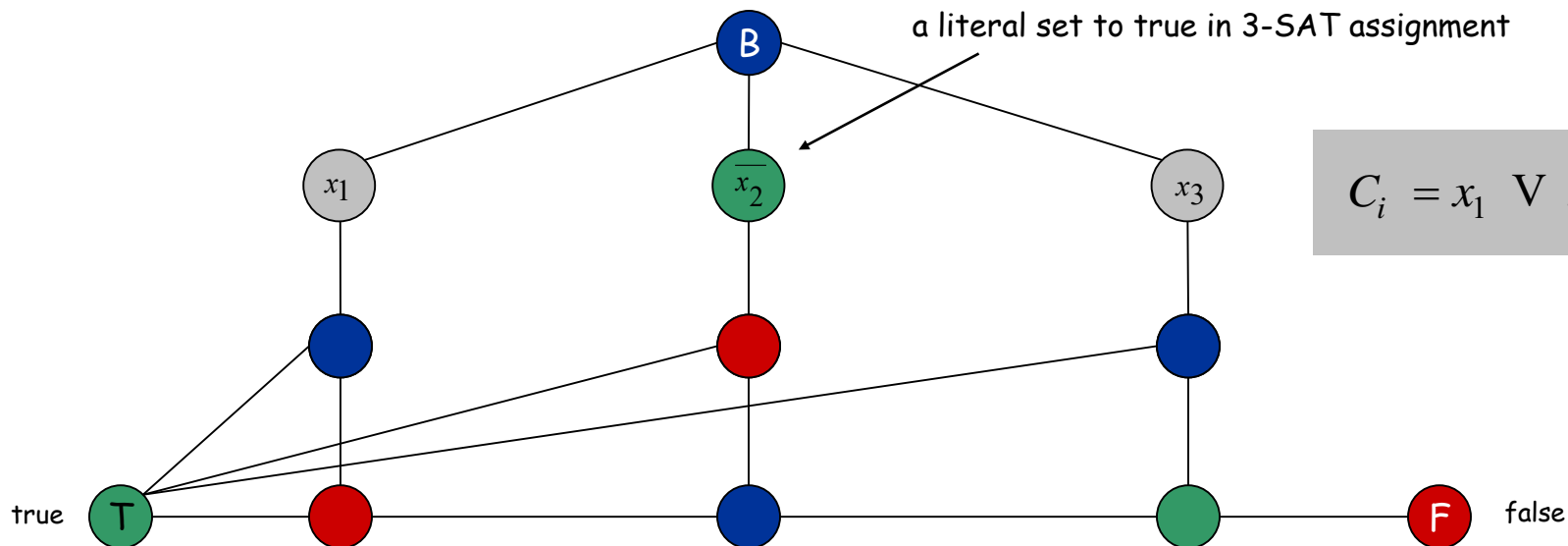


3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

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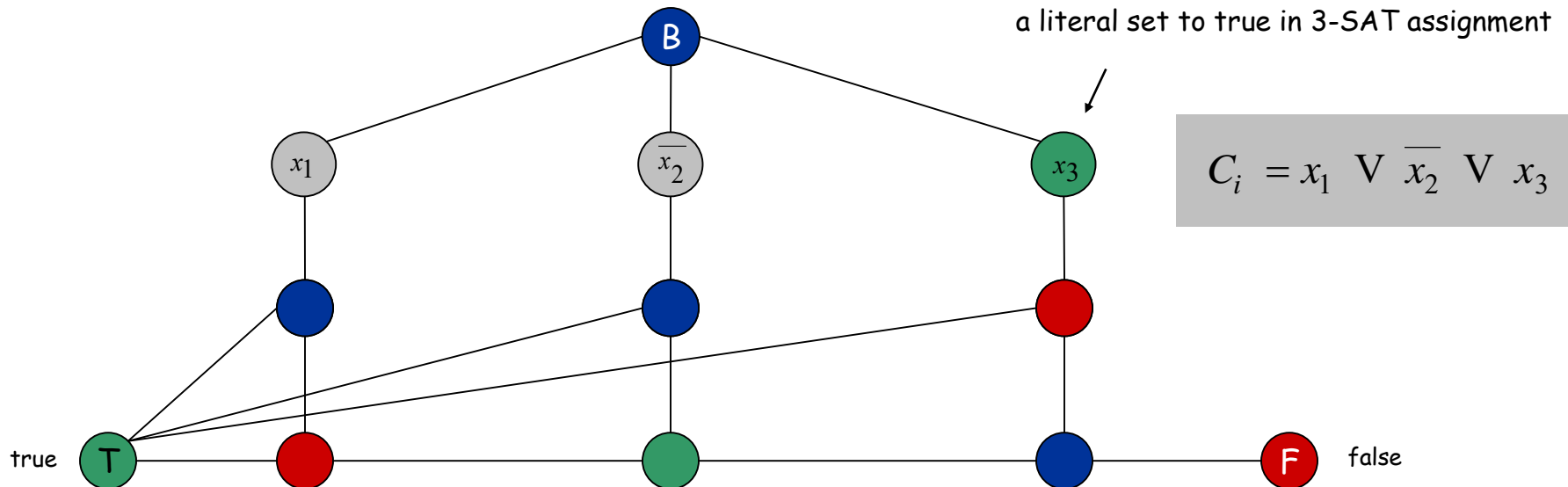


3-Colorability

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- Color remaining middle row nodes B.
- Color remaining bottom nodes T or F as forced.



8.8 Numerical Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Subset Sum

SUBSET-SUM. Given natural numbers w_1, \dots, w_n and an integer W , is there a subset that adds up to exactly W ?

Ex: $\{ 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 \}$, $W = 3754$.

Yes. $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$.

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in **binary** encoding.

3-SAT reduces to SUBSET-SUM

Claim. $3\text{-SAT} \leq_p \text{SUBSET-SUM}$.

Pf. Given an instance Φ of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff Φ is satisfiable.

3-SAT reduces to SUBSET-SUM

Construction. Given 3-SAT instance Φ with n variables and k clauses, form $2n + 2k$ decimal integers, each of $n+k$ digits, as illustrated below.

- One digit for each variable and for each clause
- Two numbers for each variable
- Two numbers for each clause
- Sum of each variable digit is 1;
sum of each clause digit is 4.

$$\begin{aligned} C_1 &= \bar{x} \vee y \vee z \\ C_2 &= x \vee \bar{y} \vee z \\ C_3 &= \bar{x} \vee \bar{y} \vee \bar{z} \end{aligned}$$

dummies to get
clause columns
to sum to 4

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,010
$\neg x$	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
$\neg y$	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
$\neg z$	0	0	1	0	0	1	1,001
}	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

3-SAT reduces to SUBSET-SUM

Claim. Φ is satisfiable iff there exists a subset that sums to W .

Pf. \Rightarrow Suppose Φ is satisfiable

- Choose numbers corresponding to each true literal
- Since Φ is satisfiable, each clause digit sums to at least 1 from the chosen numbers
- Choose dummy numbers to make each clause digit sum to 4.

$$\begin{aligned} C_1 &= \bar{x} \vee y \vee z \\ C_2 &= x \vee \bar{y} \vee z \\ C_3 &= \bar{x} \vee \bar{y} \vee \bar{z} \end{aligned}$$

dummies to get
clause columns
to sum to 4

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,010
$\neg x$	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
$\neg y$	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
$\neg z$	0	0	1	0	0	1	1,001
}	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

3-SAT reduces to SUBSET-SUM

Claim. Φ is satisfiable iff there exists a subset that sums to W .

Pf. \Leftarrow Suppose there is a subset that sums to W

- Each variable digit forces the subset to choose exactly one from x and $\neg x$
- Each clause digit forces the subset to choose at least one literal in the clause
- Assign each variable according to the chosen literal

$$C_1 = \bar{x} \vee y \vee z$$

$$C_2 = x \vee \bar{y} \vee z$$

$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

dummies to get
clause columns
to sum to 4

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,010
$\neg x$	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
$\neg y$	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
$\neg z$	0	0	1	0	0	1	1,001
}	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

From book: 3D-MATCHING reduces to SUBSET-SUM

Construction. Let $X \cup Y \cup Z$ be an instance of 3D-MATCHING with triplet set T . Let $n = |X| = |Y| = |Z|$ and $m = |T|$.

- Let $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Z = \{z_1, z_2, z_3, z_4\}$
- For each triplet $t = (x_i, y_j, z_k) \in T$, create an integer w_t in base $m+1$ with $3n$ digits that has a 1 in positions i , $n+j$, and $2n+k$.

Claim. 3D-matching iff some subset sums to $W = 111, \dots, 111$.

Triplet t_i			x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	z_1	z_2	z_3	z_4	w_i
x_1	y_2	z_3	1	0	0	0	0	1	0	0	0	0	1	0	100,001,000,010
x_2	y_4	z_2	0	1	0	0	0	0	0	1	0	1	0	0	10,000,010,100
x_1	y_1	z_1	1	0	0	0	1	0	0	0	1	0	0	0	100,010,001,000
x_2	y_2	z_4	0	1	0	0	0	1	0	0	0	0	0	1	10,001,000,001
x_4	y_3	z_4	0	0	0	1	0	0	1	0	0	0	0	1	100,100,001
x_3	y_1	z_2	0	0	1	0	1	0	0	0	0	1	0	0	1,010,000,100
x_3	y_1	z_3	0	0	1	0	1	0	0	0	0	0	1	0	1,010,000,010
x_3	y_1	z_1	0	0	1	0	1	0	0	0	1	0	0	0	1,010,001,000
x_4	y_4	z_4	0	0	0	1	0	0	0	1	0	0	0	1	100,010,001
															111,111,111,111

Partition

SUBSET-SUM. Given natural numbers w_1, \dots, w_n and an integer W , is there a subset that adds up to exactly W ?

PARTITION. Given natural numbers v_1, \dots, v_m , can they be partitioned into two subsets that add up to the same value?

$$\nwarrow \frac{1}{2} \sum_i v_i$$

Claim. $\text{SUBSET-SUM} \leq_p \text{PARTITION}$.

Pf. Let W, w_1, \dots, w_n be an instance of SUBSET-SUM.

- Create instance of PARTITION with $m = n+2$ elements.
 - $v_1 = w_1, v_2 = w_2, \dots, v_n = w_n, v_{n+1} = 2 \sum_i w_i - W, v_{n+2} = \sum_i w_i + W$
- There exists a subset that sums to W iff there exists a partition since two new elements cannot be in the same partition. •

$v_{n+1} = 2 \sum_i w_i - W$	W	subset A
$v_{n+2} = \sum_i w_i + W$	$\sum_i w_i - W$	subset B

Scheduling With Release Times

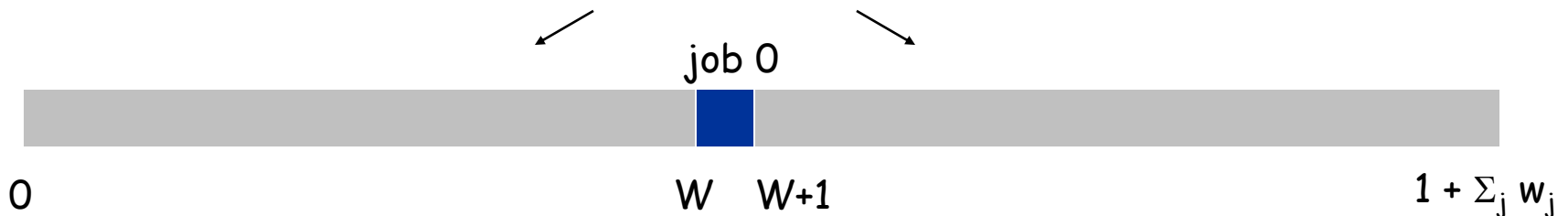
SCHEDULE-RELEASE-TIMES. Given a set of n jobs with processing time t_i , release time r_i , and deadline d_i , is it possible to schedule all jobs on a single machine such that job i is processed with a contiguous slot of t_i time units in the interval $[r_i, d_i]$?

Claim. $\text{SUBSET-SUM} \leq_p \text{SCHEDULE-RELEASE-TIMES}$.

Pf. Given an instance of SUBSET-SUM w_1, \dots, w_n , and target W ,

- Create n jobs with processing time $t_i = w_i$, release time $r_i = 0$, and deadline $d_i = 1 + \sum_j w_j$.
- Create job 0 with $t_0 = 1$, release time $r_0 = W$, and deadline $d_0 = W+1$.

Can schedule jobs 1 to n anywhere but $[W, W+1]$

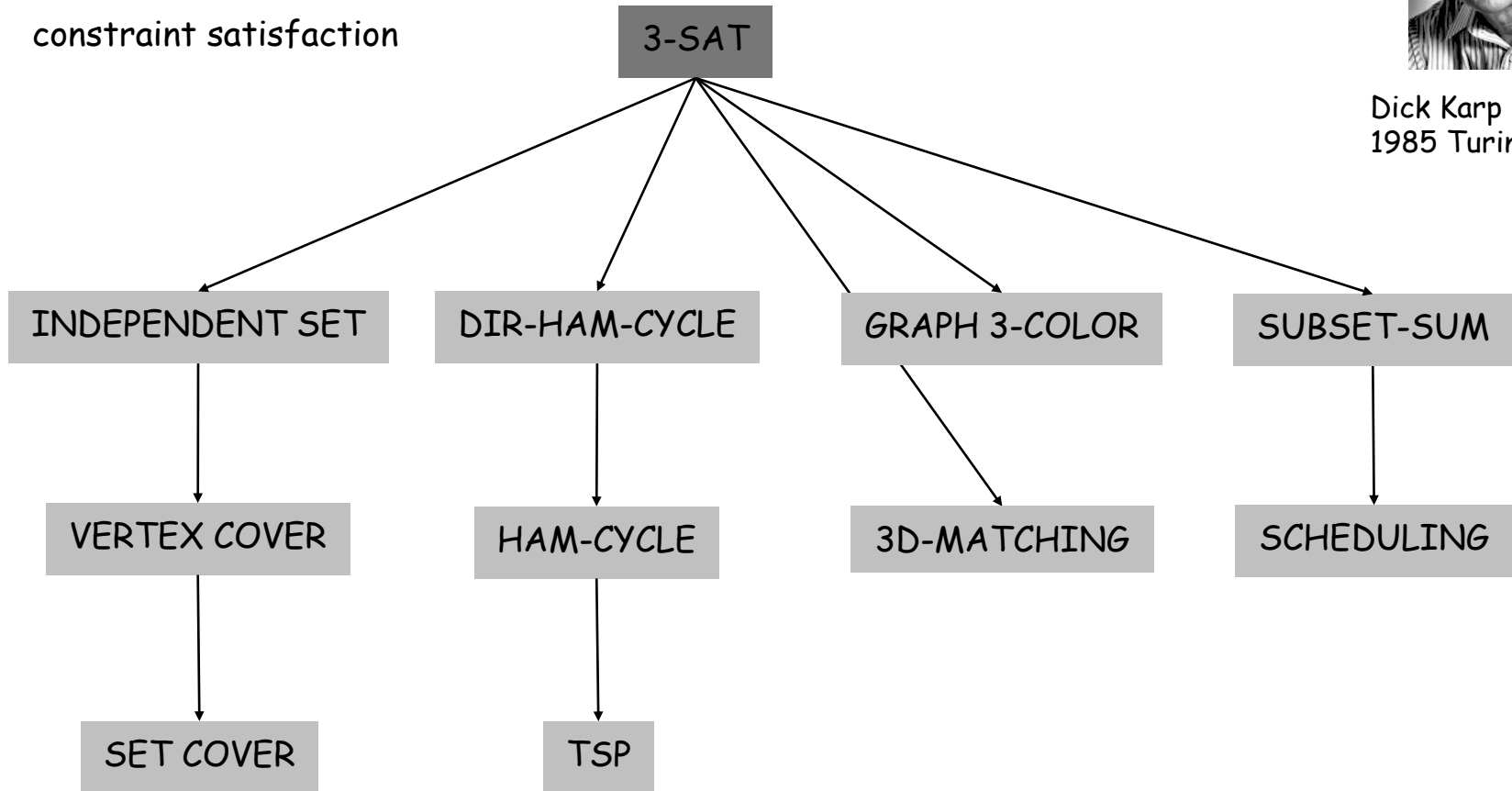


8.10 A Partial Taxonomy of Hard Problems

Polynomial-Time Reductions



Dick Karp (1972)
1985 Turing Award



packing and covering

sequencing

partitioning

numerical

Chapter Summary

Key Concepts

Decision problem

- Answer yes/no

P. Decision problems for which there is a **poly-time** algorithm.

NP. Decision problems for which there exists a **poly-time** certifier.

- Algorithm $C(s, t)$ is a **certifier** for problem X if for every string s , $s \in X$ iff there exists a string t such that $C(s, t) = \text{yes}$.

co-NP. Complements of decision problems in NP.

EXP. Decision problems for which there is an **exponential-time** algorithm.

Claim. $P \subseteq NP, \text{co-NP} \subseteq \text{EXP}$

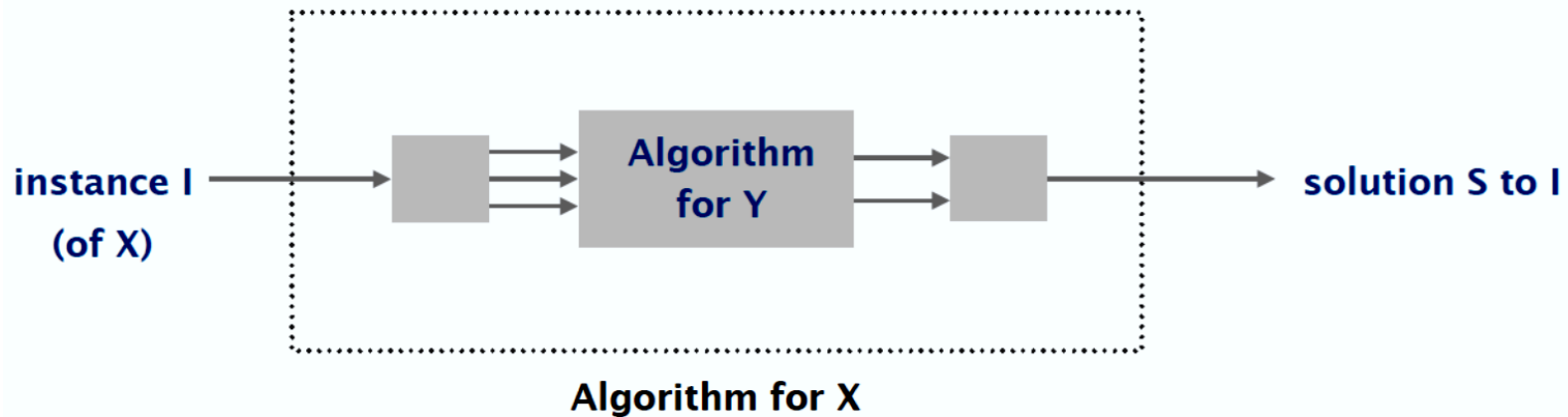
Polynomial-Time Reduction

Reduction. Problem X **polynomial-time reduces to** problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Notation. $X \leq_p Y$.

Common approach: Reduction by encoding with gadgets



NP-Completeness

NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

Recipe to establish NP-completeness of problem Y .

- Step 1. Show that Y is in NP.
- Step 2. Choose an NP-complete problem X .
- Step 3. Prove that $X \leq_p Y$.