TF502: Numerical Analysis
Homework 1

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This homework is about basic analysis exercises recalling material that you might have learned already in previous calculus, linear algebra or other analysis lectures.

1. Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be a norm in $\mathbb{R}^n$. Prove that
$$\forall x, y \in \mathbb{R}^n, \qquad |\|x\| - \|y\|| \leq \|x - y\|$$
by using the three basic axioms of a norm. **(10 points)**

2. Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be a norm in $\mathbb{R}^n$. Find all vectors $x, y \in \mathbb{R}^n$, for which the equation
$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$
is satisfied for
   (a) the case that $\|\cdot\|$ denotes the 2-norm, **(3 points)**
   (b) the case that $\|\cdot\|$ denotes the ∞-norm, and **(3 points)**
   (c) the case that $\|\cdot\|$ denotes the 1-norm. **(3 points)**

3. Let us consider a sequence $x_1, x_2, \ldots \in \mathbb{R}$, which is defined by
$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2} \ .$$
Show that $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence. Also compute the limit
$$\lim_{k \to \infty} x_k \ .$$
(5 points)

4. Provide formal proofs of the following statements:
   (a) Any norm, $f(x) = \|x\|$, $x \in \mathbb{R}^n$, is a continuous function. **(3 points)**
   (b) Every polynomial function is a continuous functions. **(3 points)**

5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function and let $D \subseteq \mathbb{R}^n$ be a bounded set. Prove that there exists for any given norm $\|\cdot\|$ in $\mathbb{R}^m$ and $\|\cdot\|^*$ in $\mathbb{R}^n$ a constant $L < \infty$ such that
$$\forall x, y \in D, \qquad \|f(x) - f(y)\| \leq L\|x - y\|^* \ .$$
(10 points)

6. Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant $L \neq 1$. Proof that the iteration
$$x_{k+1} = \frac{2g(x_k)}{1 + L^2}$$
converges, $\lim_{k \to \infty} x_k = x^*$, where x^* is independent of the choice of $x_0 \in \mathbb{R}$. **(5 points)**

7. We denote with

$$\mathrm{mat}(x) = \begin{pmatrix} x_1 & & x_n \\ & \ddots & \\ x_{n^2-n+1} & & x_{n^2} \end{pmatrix}$$

the matrix that is obtained by associating the components of $x \in \mathbb{R}^{n^2}$ with the components of an $(n \times n)$-matrix. Now, fet $f(x) = \sigma_{\max}(\mathrm{mat}(x))$ denote the maximum singular value of $\mathrm{mat}(x)$ as a function of x. Are the following statement true or false?

(a) The function f is continuous.

(b) The function f is Lipschitz continuous.

(c) The function f is differentiable.

Please also derive / explain your conclusion (a wild guess gives 0 points!). **(10 points)**

8. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if

$$\forall x, y \in \mathbb{R}^n, s \in [0,1], \qquad f(sx + (1-s)y) \leq sf(x) + (1-s)f(y) \; .$$

Prove that a twice continuously differentiable function is convex if and only if its Hessian matrix is positive semi-definite on $\mathbb{R}^n$. **(10 points)**

9. Consider the following vectors in $\mathbb{R}^4$

$$a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} , \quad a_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Implement the Gram-Schmidt algorithm using a computer language of your choice in order to find orthogonal basis vectors $q_1, q_2, q_3 \in \mathbb{R}^4$ such that $\mathrm{span}(q_1, q_2, q_3) = \mathrm{span}(a_1, a_2, a_3)$. **(15 points)**

10. The L_2-scalar product on the interval $[0, 2\pi]$ is given by

$$\int_0^{2\pi} f(x)g(x) \,\mathrm{d}x \; .$$

Work out the scalar product of the function $f(x) = \sin(mx)$ and $g(x) = \sin(nx)$ for general $m, n \in \mathbb{N}$. Under which conditions on n and m are the functions f and g orthogonal? **(10 points)**