TF502: Numerical Analysis
Homework 5

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Deadline: November 8, 2017

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be 4-times continuously differentiable. Simpson's numerical integration rule is given by

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) .$$

The corresponding approximation error is bounded by

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq \frac{(b-a)^5}{2880} \max_{\xi \in [a,b]} |f^{(4)}(\xi)| .$$

(a) Explain why Simpson's formula cannot be used directly to approximate the integral

$$\int_1^\infty e^{-x}\mathrm{d}x .$$

(10 points)

(b) Construct a function $g : [0,1] \to \mathbb{R}$ such that

$$\int_1^\infty e^{-x}\mathrm{d}x \; = \; \int_0^1 g(y)\mathrm{d}y$$

by using the variable transformation $y = \frac{1}{x}$. Apply Simpson's formula to approximate the integral on the right hand side. How large is the approximation error (a rough bound is enough to get full points)? **(10 points)**

(c) Explain how you can construct a numerical approximation of the integral

$$\int_{-\infty}^\infty e^{-x^4}\mathrm{d}x .$$

by using a suitable variable transformation and Simpson's rule. **(10 points)**

2. Prove that the polynomials

$$q_k(x) = \frac{k!}{(2k)!}\frac{\partial^k}{\partial x^k}(x^2-1)^k$$

are orthogonal with respect to the L_2-scalar product on the interval $[-1,1]$. **(40 points)**

3. Implement the closed Newton Cotes formulas as well as Gauss Quadrature for $n = 2, 3, 4$ in JULIA and compute the integral

$$I = \int_1^2 \frac{1}{x}\,\mathrm{d}x$$

with all six methods (3$\times$ Newton Cotes and 3$\times$ Gauss Quadrature for the different values of n) and compare your results with the exact value for the integral. Rank all methods with respect to their accuracy and interpret the result. **(30 points)**