TF502: Numerical Analysis
Homework 4

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Deadline: November 6, 2017

1. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 1$, $f(1) = 3$, and $f(2) = 19$. Construct a polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2$ such that p interpolates f at $x \in \{0, 1, 2\}$. What are a_0, a_1, a_2? **(10 points)**

2. Assume that a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$f(0,0) = 1\,, \quad f(0,1) = 3\,, \quad f(0,2) = 19\,, \quad f(1,0) = 3\,, \quad f(2,0) = 19\,, \quad f(1,1) = 0$$

   Construct a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$p(x) = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_2 + a_4 x_2^2 + a_5 x_1 x_2$$

   such that p interpolates f at all 6 points. What are $a_0, a_1, a_2, a_3, a_4, a_5$? **(20 points)**

3. Implement a Julia code, which interpolates the function

$$f(x) = \frac{1}{1 + x^2}$$

   at the points $x_1 = -5, x_2 = -4, \ldots, x_{11} = 5$ with a polynomial p of order 10. Plot the function f as well as the polynomial p. Do you think that p approximates f well? What can you say about the approximation error? **(30 point)**

4. Implement a JULIA function named `natural_spline(f,a,b,N)` that accepts as an input a function $f \in [a, b] \to \mathbb{R}$, interval bounds $a < b$, and an integer $N > 0$. The function should return the natural spline, a piecewise polynomial function s approximating f on the interval $[a, b]$ with N polynomials of order 3. Test your implementation with the code

```
f(x) = 1/(1+x*x);
s = natural_spline(f,-5,5,10)
plot(s,f);
```

   and plot your result. **(30 points)**

5. Write a compute program, which solves the Gauss' approximation problem

$$\min_{p \in P_n} \int_{-5}^{5} \left(f(x) - p(x) \right)^2 \, \mathrm{d}x$$

   for the function $f(x) = \frac{1}{1+x^2}$ on the interval $[-5, 5]$ for $n = 10$. Plot your result. **(30 points)**

6. Implement a JULIA function named `DFT(y)` (discrete Fourier transform) that accepts as an input a vector $y \in \mathbb{R}^n$ of data points and whose output are the coefficients a_k and b_k of a function

$$p_m(x) = \frac{1}{2}a_0 + \sum_{k=1}^{m} \left(a_k \sin(kx) + b_k \cos(kx)\right)$$

satisfying $p_m(\frac{2\pi j}{n+1}) = y_j$ for all data points y_j. Use the routine `DFT(y)` to interpolate the function

$$f(x) = \operatorname{atan}\left(10^2 * \sin(x)\right) \qquad \text{i.e.} \quad y_j = f\left(\frac{2\pi j}{n+1}\right)$$

on the interval $[0, 2\pi]$ using $m = 2$, $m = 4$, $m = 8$, $m = 16$, and $m = 32$ and plot your results for p_m and f. [You may but you don't have to implement FFT.] **(30 points)**