How to Find the Best Place for the Shared Bikes?

By applying the exact Newton and Gauss-Newton Method

Course : TF502 Numerical Analysis Project

Name : Qianwen Zhu

Student ID : 56716616

Shanghaitech University

@SIST

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Abstract

This Numerical Analysis project will focus on the numerical methods (the exact Newton method and the Gauss-Newton method) involved in solving systems of nonlinear equations. The practical model is to find the best place for shared bikes which means to solve several distance nonlinear functions with a initial location guess. First, we will apply the exact Newton method for solving multivariable nonlinear equations, which involves using the Hessian matrix. Second, we will apply the Gauss-Newton method which is an iterative method regularly used for solving nonlinear least square problems with employing the Hessian approximation. We will also analysis the convergence rate and the results of these two methods.

Keywords: Nonlinear least square problems; the exact Newton method; the Gauss-Newton method; data fitting; shared bikes.

Acknowledgements

l would like to acknowledge and thank everyone who has helped me in this project. Although the exact Newton method and the Gauss-Newton method are classical numerical methods to solve the nonlinear least squares problems, it took me a long time to figure out an interesting practical model which is related our practical life. Thanks to Professor Boris's slides and lecture ,l have a nice frame of the model construction. What's more, without TAs' patience and advice, l can't finish my project so smoothly.

l would like to express my sincere gratitude to all the people who share the learning time in the course TF502 with me.

1.Introduction

Nowadays, with the rapid development of the shared bikes, more and more people start to use shared bikes to arrive their destinations which is much cheaper and more convenient.

The need of the users is an easy access to shared bikes when they want to use one. So, to achieve the largest social profit, the shared bikes company should find the best place to put its shared bikes so as to meet the users' need to the largest extend. Then, shared bikes company's problem to find the best place of shared bikes can turn into a problem to find where the user is, which means the locations near the people who want to use shared bikes should be the best places to place shared bikes.

The goal of this project is to examine two different numerical methods that are used to solve system of nonlinear equations. In our shared bikes location model, it means to find the location of the user which also is the best place to put the shared bikes by applying the exact Newton method and the Gauss-Newton method.

2.Problem Formulation

The unconstrained nonlinear least-squares problem is given by:

$$\min_{x} ||f(x)||_2^2$$

Now to describe our shared bikes model.Everyone who has the experience of finding a shared bike would know that when the user open the app on the mobile phone, it will show the locations of nearby shared bikes and the distances between the user and those shared bikes. As a shared bikes company, it can find where the user is by using shared bikes location data and distance data.

So, we can simulate this model in this direct way: Suppose the app shows that there are a set of 30 nearby shared bikes at known position (p_k, q_k). Each of these shared bikes is at a known distance d_k from the the person who wants to find a shared bike on his mobile phone. Based on these data, we want find the user's location (u , v).

Given these positions and distances, we expect that the distance from the k_{th} shared bike to the user is given by the following equation:

$$d(u, v, k) = \sqrt{(u - p_k)^2 + (v - q_k)^2}$$

Where u and v are the x and y coordinates of the user.

We are attempting to find the best approximation location of the user by the very nature of a least-square problem. In other word, we are merely looking for the values of u and v that minimize the sum of the squares of the residuals that is, the values of u and v that minimize the function:

$$\min_{u,v} ||r_k||_2^2$$

Where k_{th} residual r is the difference between the k_{th} known distance and the k_{th} real distance to the current iteration location (u , v) , in this model, is given by the equation:

$$r_k = d_k - \sqrt{(u - p_k)^2 + (v - q_k)^2}$$

3.Solution Method

Here I use the software MATLAB to implement the exact Newton method and the Gauss-Newton method to find user's location (u , v).

A vector of 30 random points in the range of zero to one was generated in MATLAB to simulate the 30 shared bikes with 30 randomly generated distances to the user.

The initial guess of the user's location is (0.5,0.5).

```
p = rand(30,1);
q = rand(30,1);
d = rand(30,1);
a = [0.5;0.5];
```

(1)In the exact Newton method, the twice difference is calculated at each iteration using the newest approximations of u and v:

$$(u,v)_{k+1} = (u,v)_k - M((u,v)_k)^{-1} r_k((u,v)_k)^T$$

where

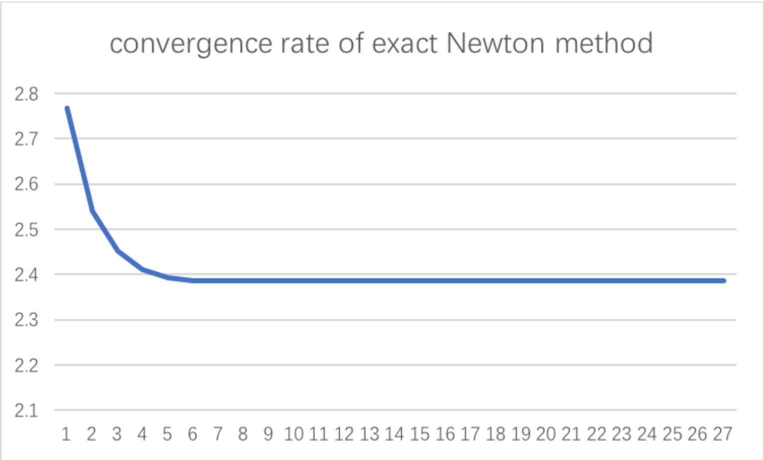
$$M((u,v)_k) \approx r_k''((u,v)_k)$$

is a suitable Hessian approximation.

In Matlab, the main code is :

```
d_X = twice_difference\F_prime_X.';
```

The convergence rate is:



(2)In the Gauss-Newton method, the Jacobian matrix is calculated at each iteration using the newest approximation of u and v:

$$(u,v)_{k+1} = (u,v)_k - M((u,v)_k)^{-1} r_k'((u,v)_k)^T r_k((u,v)_k)$$

where

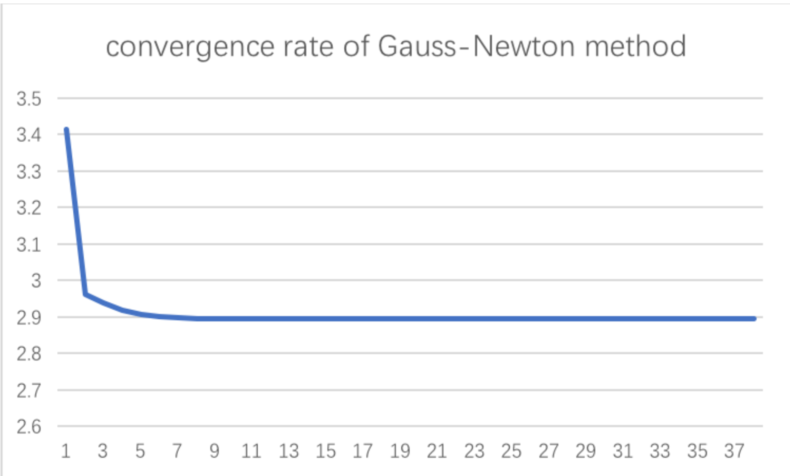
$$M((u,v)_k = r_k'((u,v)_k)^T r_k'((u,v)_k)$$

is a suitable Hessian approximation.

In MATLAB, the main code is :

```
Jz = JT*J;  %Hessian Approximation
d_X = (Jz\JT) * r;
```

The convergence rate is:



To compare the convergence rate of exact Newton method and Gauss-Newton method, we can find that ：

1. The exact Newton method needs less iteration to meet the tolerance than Gauss-Newton method,

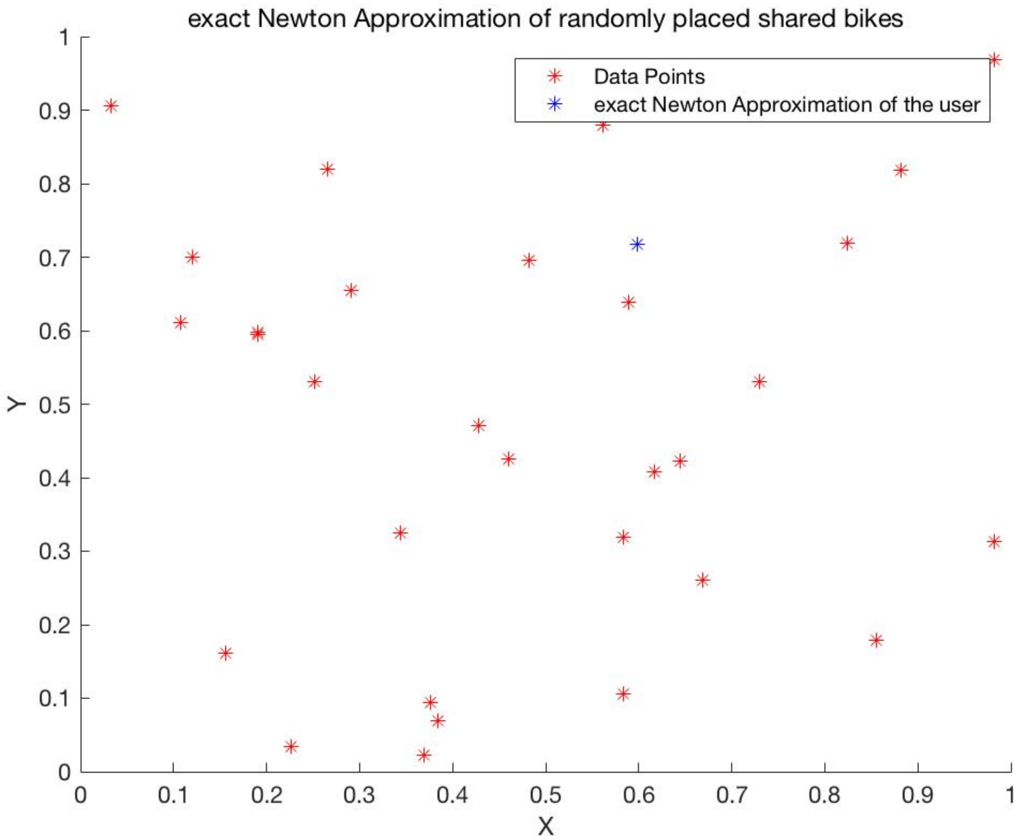
2.The exact Newton has smaller final residual than Gauss-Newton method.

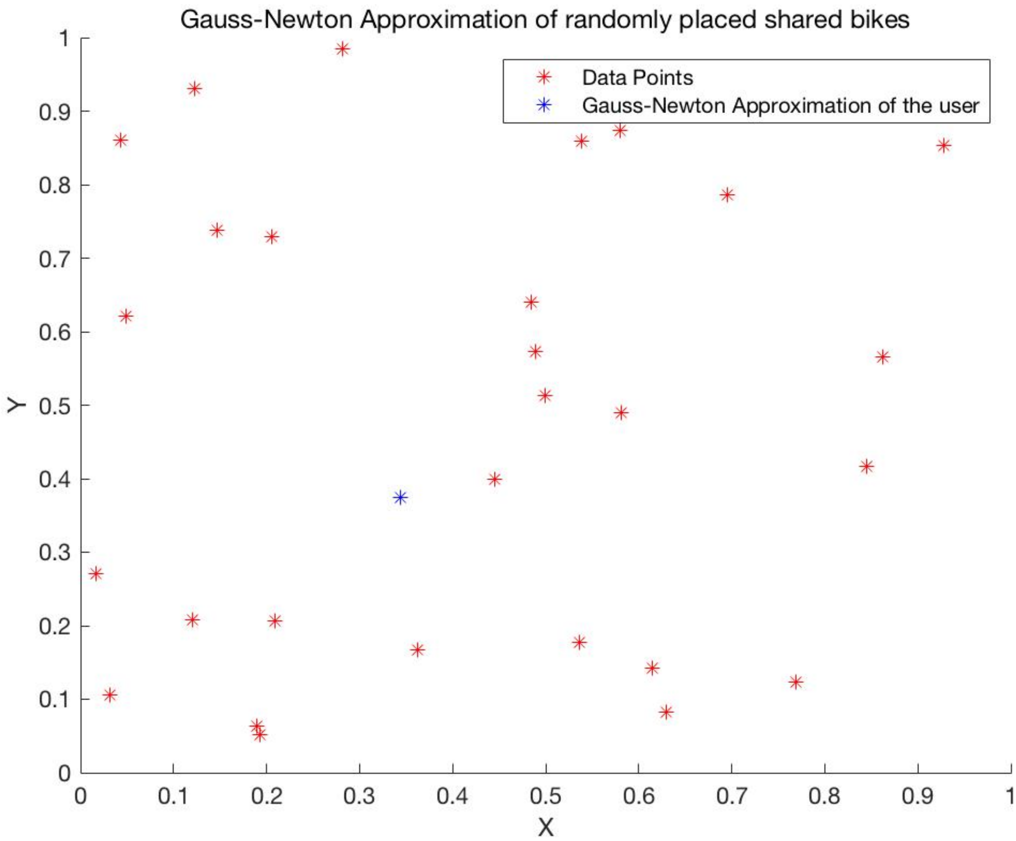
3.The exact Newton method converges faster than the Gauss-Newton method.

4.Numerical Results

I plot two result pictures generated by my MATLAB code. The red dots means the random shared bikes and the blue dots means the best approximation of the user's location by applying the exact Newton method and the Gauss-Newton method.

Since l use MATLAB to randomly generate the locations and distances of 30 shared bikes each time, so it's obvious the final user's location won' be the same.





And l I made a chart to show the comparison between the exact Newton and the Gauss-Newton method.

	Exact Newton	Gauss-Newton
Iteration time	27	38
Final (u , v)	u=0.598624679737749 v=0.717694830656237	u=0.344401046679064 v= 0.373960501797317
Running time	1867s	5s

The below two charts shows the residuals' change with iterations of these two methods. To be simpler, l just list the first two iterations and the last two iterations.

Iteration time	Exact Newton Method's residual
1	2.76671419863304
2	2.45226047842284
26	2.38470400818751
27	2.3847040081875

Iteration time	Gauss - Newton Method's residual
1	3.41398351497181
2	2.96273530066406
37	2.8948034514938
38	2.8948034514938

5.Conclusions

We have described the problem we want solve and showed how to apply the exact Newton Method and the Gauss-Newton method to get the best approximation.

By getting the best approximation of the user, we also find the best place to put shared bikes because the best place of a product is always where the need is.

From this report, it is safe to say that numerical methods are a vital strand of mathematics. They are a powerful tool in not only solving nonlinear algebraic equations with one variable, but also systems of nonlinear algebraic equations.It is hard to judge which method is better, but what is sure is that from this project, l have a better understanding of the properties of exact Newton and Gauss-Newton method.

The main results of this report can be highlighted into the following conclusions:

1.The exact Newton method need less iterations to reach the best approximation.

2.However, since the exact Newton method needs to calculate the exact Hessian matrix which is much more complicated than the Hessian approximation in the Gauss-Newton and also because of the 30 nonlinear equations, the exact Newton need much more time to get the final approximation.

3. The exact Newton method converges little bit faster than the Gauss-Newton method in this model.

4.By the way, the residual has not been decreased by very much from initial approximation when both of two methods meet the tolerance. The reason is that the data points have been distributed randomly and there is no perfect solution that will fit the theoretical model given.

5.So in practice, when the model contains large data set, it is better to use Gauss-Newton method to get the answer in a shorter time.