1 Algorithms

First, we write out the Alnordi procedure with and without reorthogonization. The reorthogonalization is an usual technique that being used to insure that the vectors v_j are orthogonal.

Algorithm 1 Alnordi procedure without reorthogonization

START input m and a random v_0; calculate $v_1 = v_0/norm(v_0)$
$-\boldsymbol{For}$ j = 1, . . . , m
◇ compute $w = \boldsymbol{A}v_j$
◇ ***for*** i = 0, 1, . . . , j
$h_{i,j} = v_i^T w$
$w = w - h_{i,j}v_i$
end for
◇ $h_{j+1,j} = \|\boldsymbol{w}\|_2$
◇ If $h_{j+1,j} = 0$, ***break***
◇ $v_{j+1} = w/h_{j+1,j}$
−end For

Algorithm 2 Alnordi procedure with reorthogonization

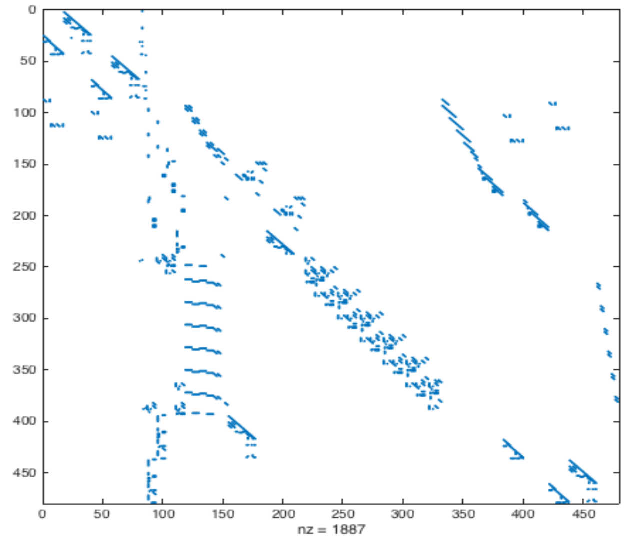
START input m and a random v_0; calculate $v_1 = v_0/norm(v_0)$
$-\boldsymbol{For}$ j = 1, . . . , m
◇ compute $w = \boldsymbol{A}v_j$
◇ ***for*** i = 1, . . . , j
$h_{i,j} = v_i^T w$
$w = w - h_{i,j}v_i$
end for
◇ ***for*** i = 1, . . . , j
$q = w^T v_i$
$w = w - q * v_i$
$h_{i,j} = h_{i,j} + q$
end for
◇ $h_{j+1,j} = \|\boldsymbol{w}\|_2$
◇ If $h_{j+1,j} = 0$, ***break***
◇ $v_{j+1} = w/h_{j+1,j}$
−end For

The first for loop in Algorithm 2 is for orthogonization and the second for loop is for re-orthogonization.

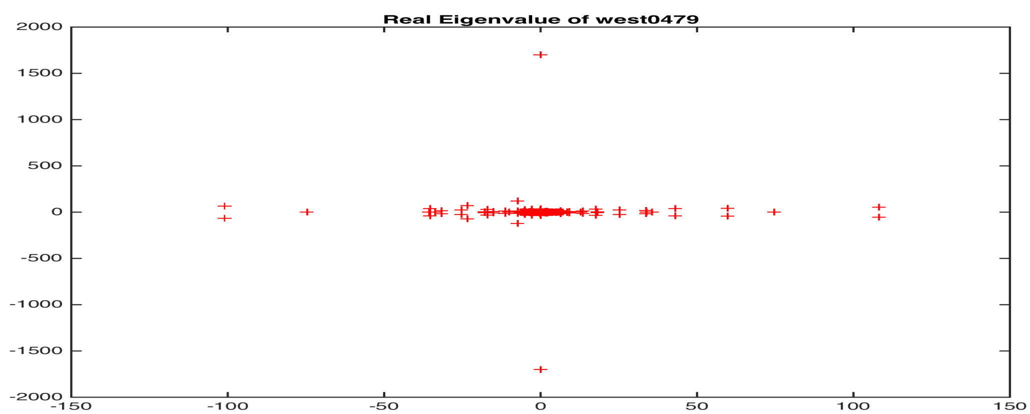
2 Results and Plots

2.1 Question 1

First, we take a glimpse on the matrix ***west*0479**.*This* 479 ×479 sparse matrix have 1887 non-zero entries.



2.2 Question 2



The outlying eigenvalues are apparent like described in the instruction.

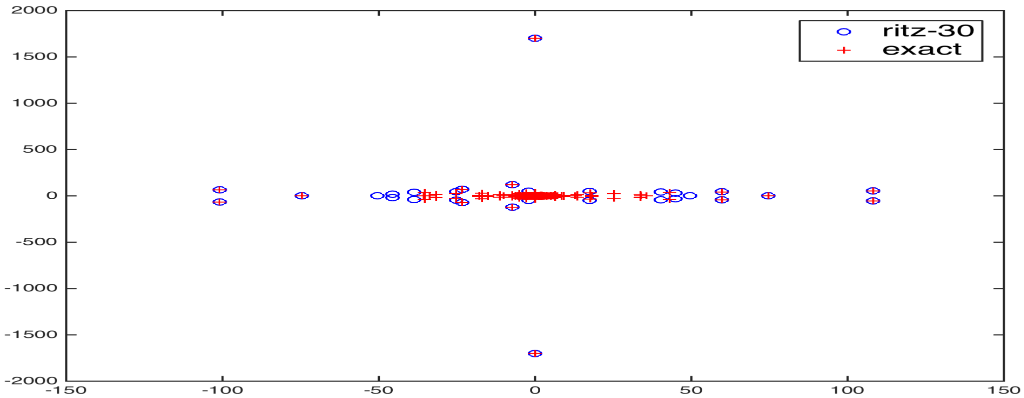
2.3 Question 3

$$\|AV_j - V_{j+1}\widehat{H_j}\| = 1.3876e-12 \qquad \|I - V_{j+1}^H V_{j+1}\| = 1.4558e-15$$

The residual we calculated is e-12 level. And the second (orthogonal check) is e-15 level. We can accept our code. The reason for the residual be at e-12 level might due to the rounding issues. Since we have two close number do addition and subtraction. But there is no doubt that our algorithm passes the orthogonal check.

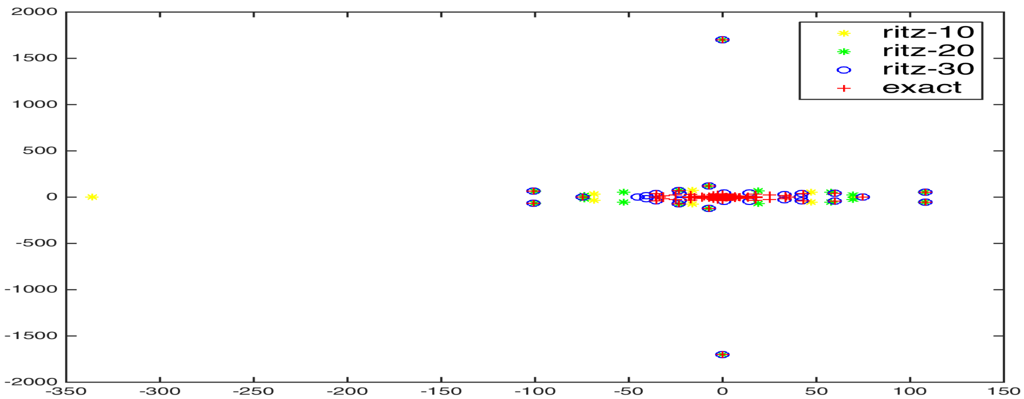
Cont.

2.4 Question 4



As showed by the plot above. The outlying eigenvalues are approximated well by Ritz values. Which can be considered that the reorthogonization Arnoldi algorithm works well on our problem.

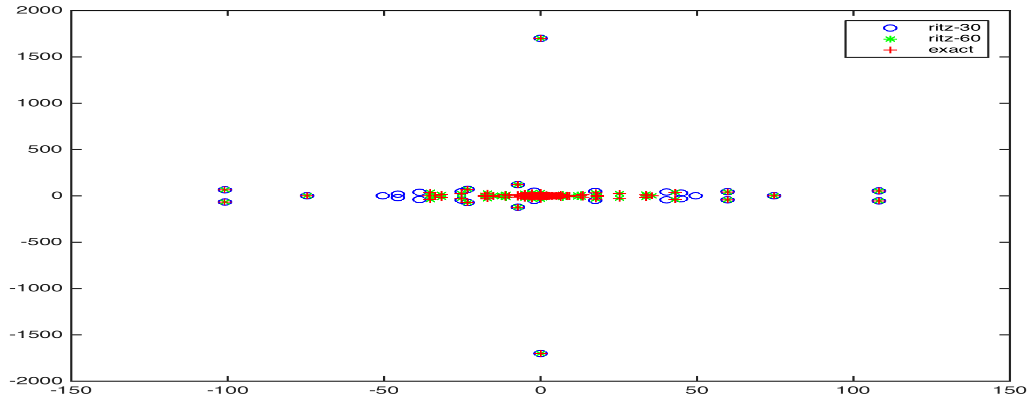
2.5 Question 5



As showed by the plot above. When we choose to take 20 and 10 steps instead of 30 steps. The Ritz value is not as good as 30-step's.

Cont.

2.6 Question 6



And we have:
For 30 steps:

$$\|I - V_{j+1}^H V_{j+1}\| = 2.1194e - 12$$

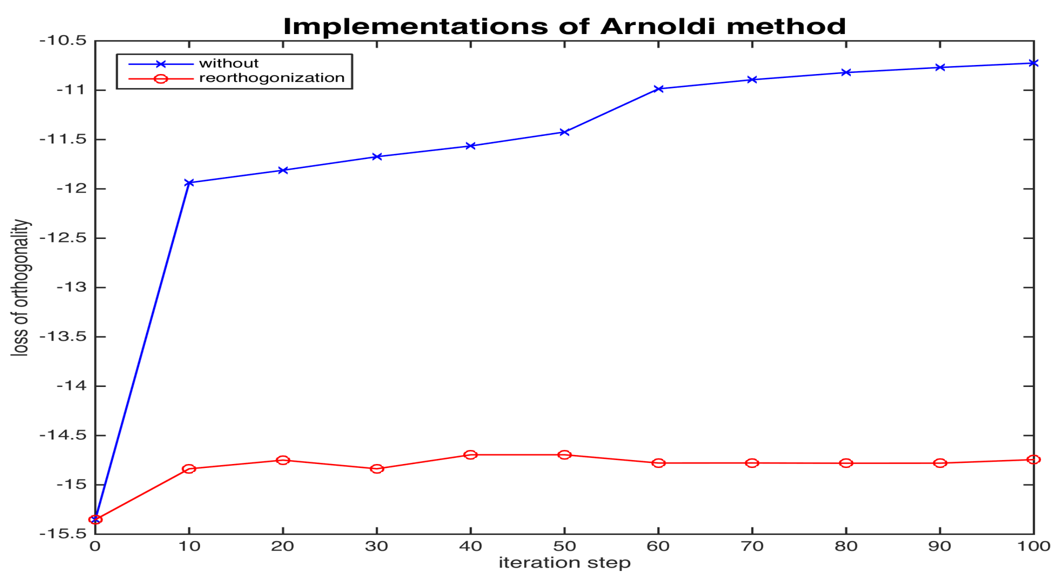
For 60 steps:

$$\|I - V_{j+1}^H V_{j+1}\| = 1.0326e - 11$$

After removing the reorthogonization in Arnoldi process. The orthogonality degraded to e-12/e-11 compared e-15/e-16 calculated through reorthogonized Arnoldi process.

Which can be considered as the reorthogonization can insure the vectors v_j remain orthogonal to working on the accuracy in the presence of rounding error.

More clearly, I implemented the algorithm with and without reorthogonization and plot $log_1 0(\|I - V_{j+1}^H V_{j+1}\|)$. With the numerical results showed in the plot below, we can conclude that reorthogonization is a good technique to ensure the approximation accuracy through insuring the vectors v_j remain orthogonal.



Cont.

2.7 Question 7

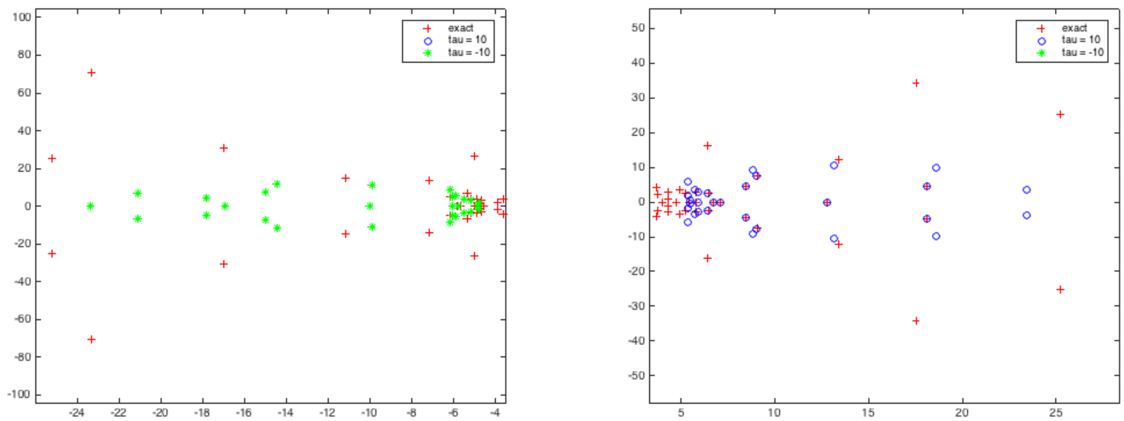
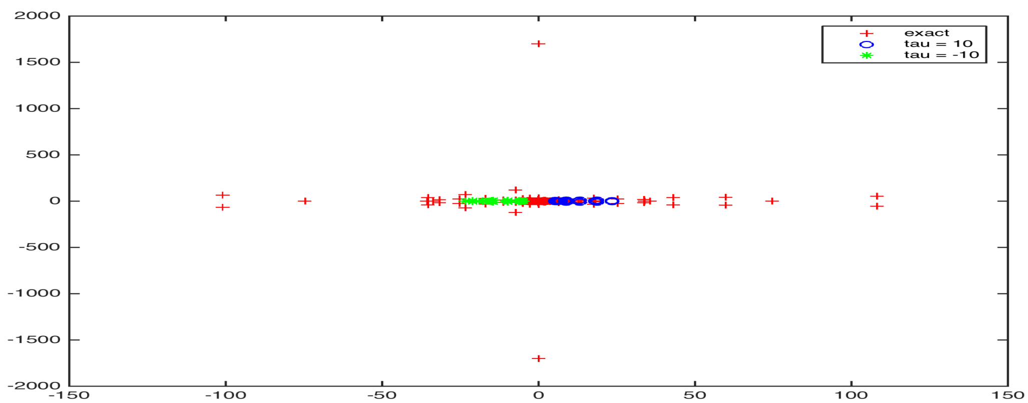


Figure 1: Figure: Shift and invert: (a) $\tau = -10(b)\tau = 10$



The shift and invert method helps us finding eigenvalues near a given number. As showed on three plots above, the calculation about eigenvalues near 10 and -10 has good performance.

3 References

1. Z.Bai. Handouts on Krylov subspace projection methods. http://web.cs.ucdavis.edu/ bai/Winter09/

2. D.S.Watkins. The matrix eigenvalue problem: GR and Krylov subspace method. Siam, 2007.

3. L.Giraud, J.Langou and M.Rozloznik. The loss of orthogonality in the gram-schmidt on orthogonization process. Computers Mathematics with Applications, 50(7):1069-1075, 2005.

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Code and original plots: https://github.com/JaneJianshi/ECS231HW4

The End.