STATISTICS DEPARTMENT
M.S. EXAMINATION

PART I
CLOSED BOOK

Friday, November 9, 2001

9:00 a.m. - 1:00 p.m.

School of Science Conference Room, ScN137

Instructions: **Complete *all five* problems. Each problem counts 20 points. Unless otherwise noted, points are allocated approximately equally to lettered parts of a problem. Spend your time accordingly.**

Begin each problem on a new page. Write the problem number and the page number in the specified locations at the top of each page. Also write your chosen ID code number on every page. Please write only within the black borderlines, leaving at least 1" margins on both sides, top and bottom of each page. Write on one side of the page only.

At the end of this part of the exam you will turn in your answers sheets, but you will keep the question sheets and your scratch paper.

Tables of some distributions are provided. Use them as appropriate.

1. An individual claims that she can identify the Coca-Cola drink when given an unmarked pair of drinks, one being a Coca-Cola and one being a Pepsi-Cola. The individual states that she is not perfect but can make a correct identification of the Cola drink with probability greater than 0.75. We present this individual with 60 pairs of drinks and note the number of correct identifications (out of 60).

(a) Using the binomial table output given below, at least how many correct identifications would this individual need to have, for us to conclude with at least 95% confidence that her claim is true?
(b) Obtain an approximate answer to (a) using only standard normal tables.
(c) Obtain an approximate 90% confidence interval for the P (correct identification) for this individual based on 53 correct identifications out of the 60 pairs.

Binomial Probabilities: n = 60, P(Success) = .75; missing entries have 'exact probability' = '.000000'.

k	P(Y = k)	P(Y<=k)
27	0.000001	0.00000
28	0.000002	0.00000
29	0.000006	0.00001
30	0.000018	0.00003
31	0.000053	0.00008
32	0.000145	0.00022
33	0.000368	0.00059
34	0.000877	0.00147
35	0.001954	0.00342
36	0.004071	0.00749
37	0.007922	0.01542
38	0.014385	0.02980
39	0.024343	0.05414
40	0.038340	0.09248
41	0.056108	0.14859
42	0.076146	0.22474
43	0.095626	0.32036
44	0.110839	0.43120
45	0.118228	0.54943
46	0.115658	0.66509
47	0.103354	0.76844
48	0.083975	0.85242
49	0.061696	0.91411
50	0.040719	0.95483
51	0.023953	0.97879
52	0.012437	0.99122
53	0.005632	0.99685
54	0.002190	0.99904
55	0.000717	0.99976
56	0.000192	0.99995
57	0.000040	0.99999
58	0.000006	1.00000
59	0.000001	1.00000

2. Suppose that X has the probability density function

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let Y have the probability density function

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & \text{if } y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Assume that X and Y are independent.

(a) Find the probability that $2X$ exceeds Y.

(b) Find $E\left[(XY)^{\frac{1}{2}}\right]$.

(c) Let $Z = -\log(X)$. Find the probability density function of Z.

(d) Let $W = \frac{X}{Y}$. Find the probability density of W.

4. Suppose that X has the probability density function $p(x) = P[X = x] = \theta^{x-1}(1-\theta)$ for $0 < \theta < 1$ and $x = 1, 2, \cdots$. Assume that $Y_1, Y_2, \cdots$ are random variables with the probability density function

$$f(y) = \begin{cases} \exp(-y) & \text{for } y > 0 \\ 0 & \text{elsewhere.} \end{cases} \sim \exp(1)$$

Suppose that the random variables $X,\ Y_1, Y_2, \cdots$ are independent.

(a) Find the probability density function of $Y_1 + Y_2$.

(b) Find $E\left[\sum_{j=1}^{X} Y_j\right]$.

(c) Find $P\left[\sum_{j=1}^{X} Y_j > 10 \mid X = 4\right]$.

(d) Find the moment generating function of $\sum_{j=1}^{X} Y_j$.

(e) Is Y_1 independent of $Y_1 + Y_2$? Prove your answer.

5. It can be shown that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \textit{ where } \Gamma \textit{ is the gamma function.}$$

Using the above result,

(a) write down the density function of a beta random variable X with parameters a and b.

(b) Obtain $\mu = E(X)$, the expected value or population mean of X.

(c) Obtain the population variance of X and show that it is equal to $[\mu(1-\mu)]/[K+1]$, where $K = a + b$.

(d) Suppose that X refers to the proportion of a store's inventory unsold during a particular week. Over a fifteen week period, the results are

0.1066
0.0000
0.5828
0.4562
0.4301
0.3912
0.5557
0.4625
0.5904
0.3975
0.3097
0.5893
0.5332
0.2652
0.6582 .

Note: Sum of observations = 6.3286, Sum of squares of observations = 3.1575.

Using the naive approach (referred to as 'method of moments') of setting the population mean and variance equal to the sample mean and variance, fit the above X to a beta model and determine values for a and b.

#1 CB Solution

(a) Let X = # of correct identification

51, since $P(X \geq 51) = 1 - P(X \leq 50) = 1 - .95483 \leq .05$.

(b) need $Z = \frac{X - n\pi}{\sqrt{n\pi(1-\pi)}} = \frac{X - 60(.75)}{\sqrt{60(.75)(.25)}} = \frac{X - 45}{3.354} \geq 1.65$

or $X \geq 50.53 \simeq 51$. (same answer as above)

(c) $\underbrace{\hat{\pi} - 1.65\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{60}}}_{\simeq .81} \leq \pi \leq \underbrace{\frac{53}{60} + 1.65\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{60}}}_{.95}$

$\frac{53}{60} = \hat{\pi}$

$\hat{\pi} \simeq .883$

CB Solution

4. Suppose that X has the probability density function

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let Y have the probability density function

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & \text{if } y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Assume that X and Y are independent.

(a) Find the probability that $2X$ exceeds Y. $P(2X > Y)$

(b) Find $E\left[(XY)^{\frac{1}{2}}\right]$.

(c) Let $Z = -\log(X)$. Find the probability density function of Z.

(d) Let $W = \frac{X}{Y}$. Find the probability density of W.

DF

1. Suppose that X has the probability density function

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let Y have the probability density function.

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & \text{if } y > 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Assume that X & Y are independent

(a) Find the probability that $2X$ exceeds Y.

(b) Find $E[(XY)^{\frac{1}{2}}]$.

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(d) Let $W = X/Y$. Find the probability density function of W.

#3
C B
FS

3 a) $L(\theta) = f(x_1, \ldots, x_n | \theta) = \prod^n f(x_i | \theta)$

$= \prod^n (\theta+1) x_i^\theta = (\theta+1)^n \left[\prod^n x_i \right]^\theta$

$l(\theta) = n \log(\theta+1) + \theta \sum^n \log(x_i)$

$\frac{d}{d\theta} l(\theta) = \frac{n}{\theta+1} + \sum^n \log(x_i) = 0$

$\frac{n}{\theta+1} = -\sum^n \log(x_i)$

$\hat{\theta} = -1 - \frac{n}{\sum^n \log(x_i)}$

b) $f(x|\theta) = (\theta+1) x^\theta$

$\log f(x|\theta) = \log(\theta+1) + \theta \log(x)$

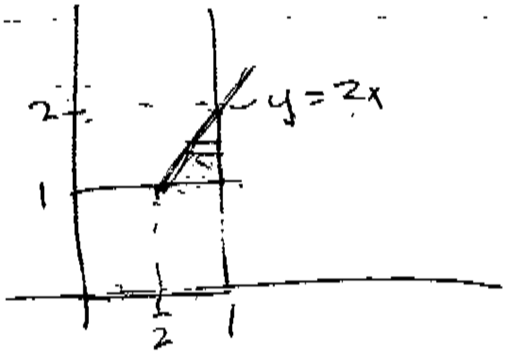
$\frac{d}{d\theta} \log f(x|\theta) = \frac{1}{\theta+1} + \log(x)$

$\frac{d^2}{d\theta^2} \log f(x|\theta) = -(\theta+1)^{-2} = -\frac{1}{(\theta+1)^2}$

$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log f(x|\theta) \right] = \frac{1}{(\theta+1)^2}$

$AV = \frac{(\theta+1)^2}{n}$

(a)



$$f_{X,Y}(x,y) = \frac{1}{y^2} I_{(0,1)}(x) I_{(1,\infty)}(y)$$

$$P[Y \leq 2X]$$

$$= \int_1^2 \int_{\frac{y}{2}}^1 \frac{1}{y^2}\, dx\, dy$$

$$= \int_1^2 \frac{1}{y^2}\left(1-\frac{y}{2}\right) dy$$

$$= -\frac{1}{y} - \frac{1}{2}\ln(y)\Big|_1^2$$

$$= \left(1-\frac{1}{2}\right) = \frac{1}{2}(\ln(2) - \ln(1$$

(b)

$$E\left[(XY)^{\frac{1}{2}}\right] = E(X^{\frac{1}{2}})\, E(Y^{\frac{1}{2}})$$

$$= \int_0^1 x^{\frac{1}{2}}\, dy \int_1^\infty y^{\frac{1}{2}} y^{-2}\, dy$$

$$= \left[\frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1}\Big|_0^1\right]\left[\frac{y^{-2+\frac{1}{2}+1}}{-2+\frac{1}{2}+1}\Big|_1^\infty\right]$$

$$= \left[\frac{2}{3}\right]\left[\frac{-1}{-\frac{1}{2}}\right] = \frac{4}{3}$$

(c)

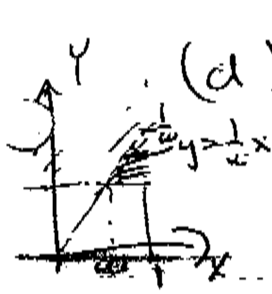
$$F_Z(z) = P[Z \leq z] = P[-\log(X) \leq z]$$

$$= P[X \geq e^{-z}] = 1 - e^{-z} \text{ so } f_Z(z) = e^{-z}$$

$0 < w < 1 \Rightarrow$

(d)

$$1 - F_W(w) = P[X/Y > w] = P\left[Y < \frac{1}{w} X\right]$$



$$= \int_1^{\frac{1}{w}} \int_{wy}^1 \frac{1}{y^2}\, dx\, dy = \int_1^{\frac{1}{w}} (1-wy)\frac{1}{y^2}\, dy$$

$$= \left(-\frac{1}{y}\right)\Big|_1^{\frac{1}{w}} - w\ln(y)\Big|_1^{\frac{1}{w}} = 1 - w - w\ln\left(\frac{1}{w}\right)$$

$$f_W(w) = 1 + \ln(w)$$
$$= -\ln(w)$$

c) $\hat{\theta} \pm z_{\alpha/2} \frac{1}{\sqrt{n I(\hat{\theta})}}$

$\hat{\theta} \pm z_{\alpha/2} \frac{\hat{\theta}+1}{\sqrt{n}}$

d) $n=12$ is a small sample.

Parametric Bootstrap:

Get $\hat{\theta}$ MLE

① $\underset{\sim}{x}_i^* \sim f(x|\hat{\theta}) = (\hat{\theta}+1)x^{\hat{\theta}}$

$i=1, \ldots, B$

② List of $\hat{\theta}_i^* = -\frac{n}{\sum \log(x_i^*)} - 1$

$i=1, \ldots, B$

③ $S_{\hat{\theta}} = \sqrt{\frac{\sum(\hat{\theta}_i^* - \bar{\hat{\theta}}^*)^2}{B-1}}$

④ $[\hat{\theta}^*_{(.025)}, \hat{\theta}^*_{(.975)}]$

DF Solution (15) VF

4 Suppose that X has the probability density function $p(x) = P[X = x] = \theta^{x-1}(1-\theta)$ for $0 < \theta < 1$ and $x = 1, 2, \cdots$. Assume that $Y_1, Y_2, \cdots$ are random variables with the probability density function

$$f(y) = \begin{cases} \exp(-y) & \text{for } y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Suppose that the random variables X, $Y_1, Y_2, \cdots$ are independent.

(a) Find the probability density function of $Y_1 + Y_2$.

(b) Find $E\left[\sum_{j=1}^{X} Y_j\right]$.

(c) Find $P\left[\sum_{j=1}^{X} Y_j > 10 \mid X = 4\right]$.

9 (d) Find the moment generating function of $\sum_{j=1}^{X} Y_j$.

(e) Is Y_1 independent of $Y_1 + Y_2$? Prove your answer.

DF

Suppose that X has the probability function $p(x) = P[X=x] = \theta^{x-1}(1-\theta)$ for $x = 1, 2, \ldots$. Assume that $Y_1, Y_2, \ldots$ are random variables with the probability density function

$$f(y) = \begin{cases} \exp(-y) & y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Suppose that the random variables $X, Y_1, Y_2, \ldots$ are independent.

(a) Find the probability density function of $Y_1 + Y_2$.

(b) Find $E\left(\sum_{j=1}^{X} Y_j\right)$

(c) Find $P\left[\sum_{j=1}^{X} Y_j > 10 \,\middle|\, X = 4\right]$

(d) Find the moment generating function of $\sum_{j=1}^{X} Y_j$.

(e) Is Y_1 independent of $Y_1 + Y_2$? Prove your answer.

8. (a) $f_{Y_1+Y_2}(z) = z\exp(-z) I_{(0,\infty)}(z)$

pf. m.g.f of Y_1+Y_2 is $\frac{1}{(1-t)^2}$

& this is m.g.f of $z\exp(-z) I_{(0,\infty)}(z)$

(b) $E(\sum_{j=1}^{X} Y_j) = E(X)\,E(Y_1)$

$= \frac{1}{1-\theta} \cdot 1 = \frac{1}{1-\theta}$

(c) $P[\sum_{j=1}^{X} Y_j > 10 \mid X=4]$

$= P[\sum_{j=1}^{4} Y_j > 10]$

m.g.f of $\sum_{j=1}^{4} Y_j = \frac{1}{(1-t)^4}$

$E(e^{tY_1})$

$= \int_0^\infty e^{ty} e^{-y} dy$

$= \frac{1}{1-t}\int_0^\infty (1-t)\, e^{-(1-t)y} dy$

$= \frac{1}{1-t}$

$\therefore$ density of $\sum_{j=1}^{4} Y_j = U$ is $\frac{u^3 e^{-u}}{3!} I_{(0,\infty)}(u)$

$P[U > 10] = \sum_{j=0}^{3} \frac{10^j e^{-10}}{j!}$ integ by parts

(d) $E(e^{t(\sum_{i}^{X} Y_i)}) = \sum_{x=1}^{\infty} E[e^{tY_1}]^x \theta^{x-1}(1-\theta)$

$= (1-\theta)E(e^{tY_1}) \sum_{x=1}^{\infty} (\theta E(e^{tY_1}))^{x-1} = \frac{(1-\theta)E(e^{tY_1})}{1-\theta E(e^{tY_1})} = \frac{(1-\theta)\frac{1}{1-t}}{1-\theta\frac{1}{1-t}}$

(e) No. $Cov(Y_1, Y_1+Y_2) = Var(Y_1) > 0$

$\therefore$ not indep.

#5 Solutions

(a)

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0<x<1$$
$$= 0, \text{ elsewhere}$$

(b)

$$\mu = E(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a}(1-x)^{b-1}dx = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}$$
$$= \frac{a}{a+b}$$

(c) $$E(x^2) = \frac{(a+1)a}{(a+b+1)(a+b)}$$

$$Var(x) = \frac{(a+1)a(a+b) - a^2(a+b+1)}{(a+b+1)(a+b)^2} = \frac{a[(a+1)(a+b) - a(a+b+1)]}{(a+b+1)(a+b)^2} = \frac{ab}{(a+b+1)(a+b)^2}$$
$$= \frac{\mu(1-\mu)}{K+1}$$

(d) 1st $\bar{x} = .4219$, $s^2 = (.1866)^2$

Set $\mu = \frac{a}{a+b} = \frac{a}{K} = .4219$ or $a = .4219K$

Set $Var(x) = \frac{\mu(1-\mu)}{K+1} = (.1866)^2$

or $K = \left[\frac{\mu(1-\mu)}{(.1866)^2} - 1\right] = \frac{(.4219)(.5781)}{(.1866)^2} - 1$
$= \underline{\underline{6}}$

Thus $a \approx .42(6) = 2.52$
$b = 6 - a \approx 3.48$