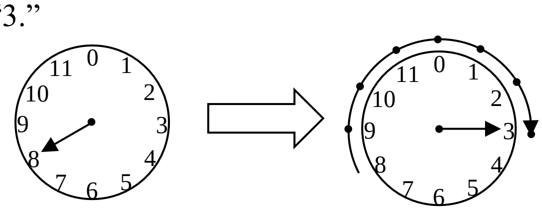
Clock arithmetic

In the digital age, many things we take for granted use codes. These include CD's, cellphones, and the internet. Often a code will have a mistake in it, but the machine will be able to interpret the code anyway. That's because these systems use *error correcting codes*. We'll study a very common code that can detect errors: the UPC code. First we need to learn about a particular kind of arithmetic – clock arithmetic.

1. Suppose you are working on an international espionage case. It's 9 am and you find out you have exactly 374 hours until a major bomb detonates. What time of day is the bomb going to explode? How did you figure it out?

2. In 2009, January 31st was on a Saturday. Without looking at a calendar, what day of the week will Christmas (December 25th) be? How did you figure it out? Are there any of the same ideas that you used to figure out the previous problem? Could you figure out what day of the week your next birthday will be?

Clock arithmetic works much like a clock, where the number reset to 0 after getting too big. For clocks, let's rename twelve as zero. For example, if we wanted to add 8 and 7 on a clock, we'd start a hand on "8" and move it 7 more spaces clockwise, as seen below. This would land us on "3."



We would write this $8 + 7 \equiv 3 \pmod{12}$. We could imagine doing other clock arithmetic, such as subtraction or multiplication, or even changing the number of slots it takes to repeat. For example, $5 \times 7 \equiv 8 \pmod{9}$ because $5 \times 7 = 35$, which goes around a clock with 9 numbers 3 times (27 spaces) and then has 8 left over.

3. Figure out the answers to the following clock arithmetic questions:
   a. $5 + 7 + 6 + 4 \equiv\ ? \pmod{8}$
   b. $6 \times 9 \equiv\ ? \pmod{11}$
   c. $11 \times 8 + 3 \equiv\ ? \pmod{13}$
   d. $26 + 15^5 \equiv\ ? \pmod{31}$

Practice with clock (modular) arithmetic

Here's a sheet where we can practice doing some clock arithmetic using "mod" notation. It is often called "modular arithmetic."

1. Figure out the following problems. Try explaining to a friend how you did it.
   a. 46 (mod 10)
   b. 57 (mod 8)
   c. 97 (mod 5)
   d. 72 (mod 12)
   e. 38,159 (mod 2)

2. Figure out the three numbers for each problem. What relationship between them did you discover?
   a. 8 (mod 6), 7 (mod 6), and 8 + 7 (mod 6)

   b. 23 (mod 7), 38 (mod 7), and 23 + 38 (mod 7)

   c. 49 (mod 5), 23 (mod 5), and 49 + 23 (mod 5)

3. Figure out the three numbers for each problem. What relationship between them did you discover?
   a. 8 (mod 6), 7 (mod 6), and 8×7 (mod 6)

   b. 9 (mod 5), 8 (mod 5), and 9×8 (mod 5)

   c. 28 (mod 12), 15 (mod 12), and 28×15 (mod 12)

4. For each problem, find four numbers for x so that:
   a. $x \equiv 6 \pmod 7$

   b. $x \equiv 1 \pmod{11}$

   c. $x \equiv 0 \pmod 9$

5. For each problem, find two pairs of different numbers (x and y but $x \neq y$) so that
   a. $x \equiv y \pmod 4$

   b. $|x - y| \equiv 2 \pmod 5$

Modular arithmetic and checking UPC codes

Accidents happen. We probably know this all too well. However, it's important that we try to avoid them as much as possible. This is especially crucial for businesses. Most checkout clerks simply slide the UPC section of a product over a laser scanner to register it in the computer to check out. But a smudge on the glass or a mark on the UPC part of the box could have the computer misread the product. This could be very expensive to the customer, since a pound of apples is about \$1.50, but a pound of filet mignon might be \$25 or a pound of saffron might be \$550.

A UPC code has 6 digits $(d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6)$ which indicate the manufacturer, 5 digits which indicate the product $(d_7,\ d_8,\ d_9,\ d_{10},\ d_{11})$ and a final check digit (d_{12}). The formula that the UPC code must meet is:

$$3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + d_{12} \equiv 0 \text{ (mod 10)}$$

1. Here's the UPC code on a box of staples. Confirm that this is a valid code.

6 54249 61039 3

2. Take something that you have that has a UPC code. If you didn't bring something, look on with a neighbor. Check to see that it works as well.

3. Suppose we change one of the numbers: 654249610593. (Which number changed?) Is this a valid code? Can you change any number and get it to work?

Modular arithmetic and checking UPC codes continued

4. Suppose we switch two of the numbers next to each other in the UPC code above. Here’s the wrong code: 654246910393. (Which numbers got switched?) Does this work? Here’s a challenge: Can you find any two adjacent numbers such that if you switched them, it would still work?

5. Here’s another challenge: If you mess up enough of the UPC code, the check might not work. Change two of the digits in my UPC code so that the check still works.

6. The following is the UPC for Hellmann's 8-oz Real Mayonnaise. Find the missing digit.

   0 48001 26_04 2

7. In the UPC, why is 3 the number every other digit is multiplied by rather than 6? (Hint: Multiply every digit from 0 to 9 by 3 and look at the answers mod 10. Do the same with 6 and compare your results.) Are there other numbers besides 3 that would function effectively? What number might you try?

8. Every book gets an International Standard Book Number (ISBN), a ten-digit number. Look on your Heart of Mathematics book or any other book for it. These are also error detecting codes. For this sequence, the sum

$$d_1 + 2d_2 + 3d_3 + 4d_4 + 5d_5 + 6d_6 + 7d_7 + 8d_8 + 9d_9 + 10d_{10}$$

has to equal 0 (mod 11). Grab a book from the swap shelf outside and check that it satisfies this requirement.

9. Here's a weird thing I discovered about Master lock combination (which changed a few years ago). Suppose your Master lock had a combination a_1, a_2, a_3, where each number is between 0 and 39 (thus 40 possibilities for each). It turned out that two things were always true:
   a. $a_1 \equiv a_3$ (mod 4)
   b. a_2 was off by 2 (mod 4) from either a_1 or a_3. For example, if $a_1 \equiv 1$ (mod 4), then $a_2 \equiv 3$ (mod 4). If $a_3 \equiv 2$ (mod 4), then $a_2 \equiv 0$ (mod 4).

   An example of a valid Master lock combination would be 34-16-10 because $34 \equiv 10$ (mod 4) and while $34 \equiv 10 \equiv 2$ (mod 4), $16 \equiv 0$ (mod 4).

   a) List 4 different possible Master lock combinations under these restrictions. For the first example, $a_1 \equiv 0$, the second one $a_1 \equiv 1$, the third $a_1 \equiv 2$, the fourth $a_1 \equiv 3$.

   b) Suppose you were a locker room thief who knew this and you noticed someone didn't spin the dial after they closed their Master combination lock. As a result, you know the last number in their combination. Suppose it was 26. What are the possibilities are there for the first number? What are the possibilities are there for the middle number?

   c) How many combinations would you have to try to open the locker? If it took 5 seconds to try a combination, how long it would take to break into this poor sap's locker?