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Innovative Applications of O.R.

Bayesian failure-rate modeling and preventive maintenance optimization

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ABSTRACT

New results are derived for the optimal preventive maintenance schedule of a single item over a finite horizon, based on Bayesian models of a failure rate function. Two types of failure rate functions—increasing and bathtub shapes—are considered. For both cases, optimality conditions and efficient algorithms to find an optimal maintenance schedule are given. A Bayesian parametric model for bathtub-shaped failure rate functions is used, while the class of increasing failure rate functions are tackled by an extended gamma process. We illustrate both approaches using real failure time data from the South Texas Project Nuclear Operating Company in Bay City, Texas.

1. Introduction

Proper and safe operation is a goal for any electric power generation plant, especially one powered by nuclear fuel. An inefficiently functioning plant costs more to operate and produces less energy than it potentially could. Replacing or repairing items too often may substantially increase the cost of operating the plant. A decision-making rule that defines when and how an item is maintained is known as a maintenance policy. In the context of maintenance optimization, this paper's advances lie at the intersection of Bayesian (parametric and nonparametric) inference and stochastic optimization. Specifically, the use of the nonparametric extended gamma process in maintenance optimization is new. Second, characterization of the optimal policy for a parametric bathtub failure rate function is the other novel contribution of the paper; importantly, we note that the latter is a general result for any failure model with a bathtub-shaped rate.

The optimization aspects of the paper critically hinge on the behavior of the hazard rate function underlying the data-generating process. In applications, most often, these hazard rates either increase, or exhibit a U-shaped behavior. The gamma process (Noortwick, 2009) has been used in the reliability context; however, this process cannot be used as a nonparametric prior distribution on the space of increasing hazard rates. Hence, we work with the extended gamma process (Dykstra & Laud, 1981) that overcomes this constraint. A noteworthy feature of this process is that distributional assumptions about the true hazard rate are dropped. Popova, Morton, Damien, and Hanson (2010) provide a comprehensive comparative analysis that shows the advantages of a Bayesian nonparametric monotone failure rate function over a parametric one. In general, the merits of using a nonparametric model in reliability can be found in Merrick, Soyer, and Mazzuchi (2003, 2005). These authors use a mixture of Dirichlet processes to better model the true distribution function, and develop full Bayesian inference for the Cox proportional hazard model and its variants. Our focus is on directly modeling the hazard rate function rather than the distribution function (or the density). Unlike monotone failure rates, nonparametric approaches to bathtub hazard rates are notoriously difficult to implement (Dykstra & Laud, 1981). Hence, we use a parametric model that is rich enough to tackle such hazard rate functions, namely the exponentiated Weibull distribution. The key point is that the resulting characterization of maintenance policy decisions holds for any failure model with bathtub failure rates. Thus, the problem of finding an optimal maintenance schedule for a single item is resolved using a Bayesian nonparametric model for increasing failure rate systems, and a Bayesian parametric model for bathtub failure rates.

The literature on optimal maintenance and replacement strategies is vast. Wang (2002), McCall (1965), Pierskalla and Voelker (1979), and Valdez-Flores and Feldman (1989) give excellent overviews. In this paper we assume a finite time horizon, *L*. This assumption arises naturally in our setting since, in the U.S., nuclear plants are licensed to operate for a finite horizon. Additionally,

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they undergo scheduled shutdowns at set intervals (e.g., every 18 months) for fuel replacement and maintenance of large systems, and preventive maintenance planning is often done for the duration of these intervals. In contrast, the main justification given in the literature for using a finite time horizon typically seems to involve limiting attention to scheduling maintenance during a warranty period. Barlow and Proschan (1962) analyze two policies for optimal maintenance planning. The first is known as the age replacement policy and replaces the item to "as-good-as-new" state at failure or age a, whichever comes first. The second is a sequential replacement policy: the interval is selected after each replacement such that the cost over the remaining time is minimized. Jardine and Buzacott (1985) derive optimality conditions for items with increasing failure rate under various assumptions on the structure of the maintenance schedule, including assuming constant inter-maintenance intervals or age-based maintenance intervals, without making any claims of the optimality of either approach. These conditions do not have closed-form solutions in general; in their examples where they use the Weibull failure distribution, they reference a graphical procedure for finding intervals that satisfy their conditions: in this regard, related work includes Elsayed (2012), Galenko et al. (2005) and Pham and Wang (1996).

We consider a sequential policy setting and derive necessary conditions for optimality under general assumptions for the failure rate distribution. An important result from Barlow and Proschan (1962) for an optimal maintenance schedule with a strictly increasing failure rate is then seen as a special case. In this case, we present a new efficient algorithm to solve the optimization problem using the nonparametric extended gamma process—a first in the literature. Next, implications of our general optimality conditions when the failure rate has a parametric exponentiated Weibull bathtub shape function are studied. Here, a new characterization of the structure of the optimal replacement intervals is provided, and an efficient algorithm to solve the maintenance optimization model is derived; this algorithm uses as a subroutine our algorithm for an increasing failure rate function.

Background and theoretical/algorithmic optimization results are presented in Section 2. After stating the basic model and its optimality conditions, more specific results and an algorithm for the case in which the failure rate function of an item is increasing appear in Section 2.3, and those for a bathtub failure rate function are given in Section 2.4. In the next two sections, respectively, the Bayesian nonparametric and parametric models for increasing and bathtub hazard rates are detailed: within each section, via appropriate subsections, both approaches are illustrated by applying them to a set of real failure time data provided by the South Texas Project (STP) nuclear power plant in Bay City, Texas; some of the corresponding mathematical development is relegated to Appendix A. The paper concludes in Section 5.

2. Preventive maintenance optimization

2.1. Background

We consider developing an optimal maintenance policy for a single item. We say that the item has failure time distribution f, if the time to failure from when the item is "new" has probability density function $f(t)$. We consider two types of maintenance:

- *Corrective maintenance* (CM) is performed at cost a C_{cm} whenever an item fails. CM instantly restores the item to its operating state just before failure ("good-as-old" state).
- *Preventive maintenance* (PM) is scheduled to be performed at a cost C_{pm} in advance. PM instantly renews the item by replacement or overhaul ("new" state).

While we focus on single component repair, these components may be part of a much larger, complex system in which unplanned outages and overhauls may be lengthy and expensive. In such cases, it is likely that when the individual complex component fails, minimal repairs will be performed, so that the larger system's downtime is minimized; see Crow (1990). An item's failure may result in additional costs beyond simple repair of the item; C_{cm}, can represent the expected total cost of such a failure, calculated as a weighted average of various failure modes.

The failure rate function, $z(t)$, is defined as

$$z(t) = \frac{f(t)}{\int_t^\infty f(u)du}. \tag{1}$$

We assume that the item's failure rate function is continuous and that the failure rate function depends on the time of the last renewal. Then, because PM restores the item to a new state and CM restores the item to a good-as-old state, the item's failure rate depends only on the last time the item underwent a PM. Next, assume a finite time horizon, L; i.e., we develop the optimal maintenance policy for the time interval $[0, L]$. Finally, we assume that the failure distribution, f, does not change during this time horizon. Let $N(t_1, t_2)$ represent the number of times the item fails in the interval (t_1, t_2), and let $E[N(t_1, t_2)]$ be the expected number of such failures. Barlow and Hunter (1960) show that

$$\int_{t_1}^{t_2} z(u)du = E[N(t_1, t_2)]. \tag{2}$$

The assumptions we outline above could be somewhat restrictive depending on the nature of the larger system which contains the item. Nonetheless, the assumptions are reasonable in many applications including our work with the nuclear power company. Importantly, we derive the structure of the optimal sequential maintenance policy under fairly general conditions regarding the shape of the failure rate function.

2.2. Optimization model

Let the item be in the new state at time 0, and assume it must be restored to the new state by PM at time L. Should an item fail, we immediately restore it to the good-as-old state by CM. Our maintenance schedule minimizes expected cost using decision variables:

- n: the number of PMs over the entire time horizon, and
- T_i: the length of the ith PM interval, $i = 1, \dots, n$.

With this notation, the number of PMs in the interval $(0, L)$ is $n-1$ because an additional nth PM must take place at time L. Thus, performing no PMs on $(0, L)$ corresponds to $n = 1$.

We can express our maintenance optimization model in nested form:

$$\min_{n\geq 1} \left[(n-1)C_{pm} + \min_{\substack{T_1,\dots,T_n \geq 0 \\ \sum_{i=1}^n T_i = L}} \sum_{i=1}^n C_{cm} E[N(0, T_i)] \right], \tag{3}$$

where we omit the cost of the mandatory nth PM at time L. Fixing n and denoting the objective function as $X(T_1, \dots, T_n)$, we obtain:

$$\min_{\substack{T_1,\dots,T_n \geq 0 \\ \sum_{i=1}^n T_i = L}} \left[X(T_1, \dots, T_n) \equiv (n-1)C_{pm} + \sum_{i=1}^n C_{cm} E[N(0, T_i)] \right]. \tag{4}$$

With λ denoting the Lagrange multiplier on the equality constraint in (4), and neglecting nonnegativity constraints, the first-order necessary conditions for an optimal solution, $(T_1^*, \ldots, T_n^*)$, yield

$$z(T_i^*) = \frac{-\lambda}{C_{cm}}, \;\; i = 1, \ldots, n,$$

which means that

$$z(T_1^*) = z(T_2^*) = \cdots = z(T_n^*). \tag{5}$$

Thus, the value of the failure rate function, at the length of each PM interval, is the same. Eq. (5) holds under fairly general conditions on the time to failure, i.e., it has a density. Condition (5) does not specify an optimal solution in general, as it may also be satisfied by nonoptimal solutions. However, if the failure rate function satisfies certain properties (e.g., being bathtub-shaped), condition (5) suffices to ensure optimality and can be used to efficiently solve model (3). And herein lies one of the significant theoretical contributions of the paper, for we will show that under nonparametric monotone failure rates, as well as *any* parametric bathtub hazard model, condition (5) is sufficient to ensure an optimal solution, and we can efficiently compute such a solution.

2.3. Increasing failure rate

We briefly consider an increasing failure rate function, because it provides a subroutine for solving model (3) in the bathtub case described next. If the failure rate function is increasing and hence invertible, we immediately obtain from Eq. (5) that optimal PM intervals must have equal length, as first established in Boland and Proschan (1982). We now discuss efficiently finding such a solution.

Under equal PM intervals, problem (3) simplifies to:

$$\min_{L/T \in \mathbb{Z}_+} \left[h(T) \equiv (LT^{-1} - 1)C_{pm} + LT^{-1} C_{cm} E[N(0,T)] \right], \tag{6}$$

where $\mathbb{Z}_+$ denotes the set of positive integers. Relaxing the integer constraint in (6), differentiating $h(T)$, and setting the result to zero yields:

$$\frac{C_{pm}}{C_{cm}} = Tz(T) - E[N(0,T)], \tag{7}$$

which has a single root, since the right-hand side of (7) is increasing because the failure rate function is increasing.

We let T_c^* solve (7), where subscript "c" indicates the continuous relaxation. Given an analytic expression for the failure rate function, we may be able to solve (7) explicitly for T_c^*. Otherwise we employ a numerical bisection search. The T_c^* that solves (7) will not typically solve the integer-constrained problem (6). However, Galenko et al. (2005) show that if $\hat{n}$ is the positive integer satisfying $L/(\hat{n}+1) \le T_c^* \le L/\hat{n}$ then the integer-constrained optimal solution, T^*, is either $L/(\hat{n}+1)$ or $L/\hat{n}$, whichever has lower cost, $h(\cdot)$. We refer to this as finding T^* by *rounding* T_c^*, and we summarize the procedure in Algorithm 1.

2.4. Bathtub failure rate

The next two propositions characterize optimal PM intervals for, and facilitate efficient solution of, model (3) in the case of a bathtub failure rate function. In these results we define a bathtub failure rate function to allow for a "flat portion" at the minimum failure rate. Specifically, we assume that $z(t)$ decreases strictly on $[0, I_1]$, is constant on $[I_1, I_2]$, and increases on $[I_2, L]$. We assume $0 < I_1 \le I_2 < L$, and if $I_1 = I_2$, we simply denote this point by I.

Proposition 1. *Consider model (3). Assume the failure rate function is continuous and satisfies the bathtub hypothesis with the minimum failure rate on* $[I_1, I_2] \subset (0, L)$.

Algorithm 1 Algorithm for increasing failure rate.

Input: instance of model (3) with C_{pm}, C_{cm}, L, and increasing $z(t)$
Output: n^* and $(T^*, \ldots, T^*)$, optimal solution of model (3)
$\underline{T} \leftarrow 0$, $\overline{T} \leftarrow L$
repeat
  $T \leftarrow \frac{1}{2}(\underline{T} + \overline{T})$
  if $\frac{C_{pm}}{C_{cm}} < Tz(T) - E[N(0,T)]$ **then**
    $\overline{T} \leftarrow T$
  else
    $\underline{T} \leftarrow T$
  end if
until $(\lceil L/\underline{T} \rceil - \lfloor L/\overline{T} \rfloor \le 1)$
$\hat{n} \leftarrow \lfloor L/\overline{T} \rfloor$
$T^* \leftarrow \arg\min_{T \in \{\frac{L}{\hat{n}+1}, \frac{L}{\hat{n}}\}} h(T)$
$n^* \leftarrow L/T^*$
return n^* and $(T^*, \ldots, T^*)$

- *If $I_1 = I_2 \equiv I$ and $(T_1^*, T_2^*, \ldots, T_n^*)$ denotes an optimal solution to model (3) then either*
  1. *all inter-PM intervals are equal, i.e., there exists T^* with $T_1^* = T_2^* = \cdots = T_n^* = T^*$; or,*
  2. *there are exactly two lengths of inter-PM intervals, $\check{T} < \hat{T}$, and $T_i^* \in \{\check{T}, \hat{T}\}$, $i = 1, \ldots, n$.*

  Under condition 1, $T^* \ge I$. Under condition 2, $(T_1^*, \ldots, T_{n-1}^*, T_n^*) = (\hat{T}, \ldots, \hat{T}, \check{T})$ is optimal, where $\check{T} < I < \hat{T}$.
- *If $I_1 < I_2$ then there exists a solution $(T_1^*, T_2^*, \ldots, T_n^*)$ to model (3) satisfying either condition 1 or condition 2. The optimal solution can be constructed so that under condition 1, $T^* \ge I_2$, and under condition 2, $(T_1^*, \ldots, T_{n-1}^*, T_n^*) = (\hat{T}, \ldots, \hat{T}, \check{T})$, where either $\check{T} < I_1 < I_2 < \hat{T}$ or $I_1 \le \check{T} < \hat{T} = I_2$.*

Proof. We know that an optimal solution must satisfy condition (5); i.e., there is a constant γ such that $z(T_1^*) = z(T_2^*) = \cdots = z(T_n^*) = \gamma$. Because the failure rate function, $z(t)$, strictly decreases on $[0, I_1]$, is constant on $[I_1, I_2]$, and strictly increases on $[I_2, L]$ the level set $\{t : z(t) = \gamma\}$ is either a singleton, a doubleton, or the entire interval $[I_1, I_2]$.

The singleton and doubleton cases lead to conditions 1 and 2, respectively, and these are the only two cases possible when $I_1 = I_2 = I$. In this situation, under the doubleton, we have that $\check{T}$ and $\hat{T}$ must straddle I, again by the bathtub assumption.

Suppose $I_1 = I_2 = I$ and we have an optimal schedule with equal inter-PM intervals and $T^* < I$. Then, we lengthen one interval while shortening another by some small amount $\varepsilon > 0$ such that $T^* + \varepsilon < I$, and we only affect the schedule's cost over those two intervals. Specifically, our objective function changes by

$$C_{cm}\left[\int_0^{T^*+\varepsilon} z(t)dt + \int_0^{T^*-\varepsilon} z(t)dt - 2\int_0^{T^*} z(t)dt\right]$$

$$= C_{cm}\left[\int_{T^*}^{T^*+\varepsilon} z(t)dt - \int_{T^*-\varepsilon}^{T^*} z(t)dt\right] < 0.$$

The change in cost is negative because $z(t)$ is decreasing over $[0, I]$. This contradicts $T^* < I$ being the length of an optimal interval.

Suppose $I_1 = I_2$ and we have an optimal schedule with two components of $(T_1^*, T_2^*, \ldots, T_n^*)$ less than I. Then, we mimic the argument just made, shortening one of those two intervals and lengthening the other by a small amount $\varepsilon > 0$ such that the longer interval is still shorter than I. This again leads to a less expensive solution, giving us the desired contradiction. The proof for $I_1 = I_2$ is completed by noting that we can exchange the order of inter-PM intervals without changing the objective function, and hence let $T_n^* = \check{T}$.

When $I_1 < I_2$ the singleton and doubleton cases are identical to that considered above. Under the singleton case we have that $T^* \geq I_2$, and under the doubleton case we have $\check{T} < I_1 < I_2 < \hat{T}$, for reasons identical to those under $I_1 = I_2$. When the level set satisfies $\{t : z(t) = \gamma\} = [I_1, I_2]$, this implies that we can have an optimal solution $(T_1^*, T_2^*, \dots, T_n^*)$ with many different lengths, provided they satisfy $I_1 \leq T_i^* \leq I_2$, $i = 1, \dots, n$, and $\sum_{i=1}^n T_i^* = L$. However, we show that we can find an optimal solution where $T_i^* < I_2$ for at most one component of $(T_1^*, T_2^*, \dots, T_n^*)$. To this end, suppose we have two intervals, without loss of generality of length T_1^* and T_2^*, satisfying $I_1 \leq T_1^* \leq T_2^* < I_2$. If $I_2 - T_2^* > T_1^* - I_1$ then the schedule cannot be optimal as we may increase T_2^* to $T_2^* + (T_1^* - I_1 + \varepsilon)$ and decrease T_1^* by the same amount to $I_1 - \varepsilon$ with the change in cost being

$$C_{cm}\left[\gamma\varepsilon - \int_{I_1-\varepsilon}^{I_1} z(t)dt\right] < 0,$$

provided $\varepsilon > 0$ is sufficiently small. Thus, we have $0 < I_2 - T_2^* \leq T_1^* - I_1$. Now, using for a fourth time the same type of change-in-cost argument we can increase T_2^* by $I_2 - T_2^*$ (up to a value of I_2) and we can decrease T_1^* by the same amount with no change in cost. □

Proposition 1 shows that we can find an optimal solution to model (3), when the failure rate function has a bathtub shape, which satisfies either condition 1 or condition 2. Of course, we do not know *a priori* which condition the optimal solution will satisfy. We close this section by developing further results that allow development of an efficient algorithm that implicitly handles both cases.

If we are in condition 1, in which our optimal solution has equal inter-PM intervals, from Proposition 1 we know that $T^* \geq I_2$. This means that if we know that our optimal solution has equal inter-PM intervals, these intervals are on the increasing portion of the failure rate. We can therefore mimic our analysis from the case of an increasing failure rate in Section 2.3. Rewrite model (3) as

$$\min_{n\in\mathbb{Z}_+}[(n-1)C_{pm} + nC_{cm}E[N(0, I_2)] + nC_{cm}E[N(I_2, L/n)]], \tag{8}$$

and rewrite this as:

$$\min_{T\geq I_2, L/T\in\mathbb{Z}_+}\left[(LT^{-1}-1)C_{pm} + LT^{-1}C_{cm}E[N(0, I_2)] + LT^{-1}C_{cm}E[N(I_2, T)]\right]. \tag{9}$$

Relaxing the integer constraint in model (9) and setting the derivative of the objective function to zero yields

$$\frac{C_{pm}}{C_{cm}} = Tz(T) - E[N(0, I_2)] - E[N(I_2, T)]. \tag{10}$$

The right-hand side of Eq. (10) is increasing in T and hence (10) has a single root. Therefore, condition (10) uniquely identifies an optimal solution, T_c^*, to the relaxed bathtub problem when we know that our optimal solution has equally-spaced PMs. And, the solution T_c^* can be rounded to obtain an optimal solution, T^*, to the integer-constrained model (9).

Viewed another way, model (9) is of the form of model (6) from Section 2.3. We can shift time ahead by I_2 because we are in condition 1 of Proposition 1. So, the failure rate function is increasing on the interval of interest, and model (9) is equivalent to model (6) with the preventive maintenance cost incremented by $C_{cm}E[N(0, I_2)]$. Thus, model (9) can be solved by Algorithm 1, which includes the rounding procedure.

We now turn to condition 2 from Proposition 1, where our optimal solution is of the form $(\hat{T}, \dots, \hat{T}, \check{T})$, with only one interval having shorter length. If L is large compared to I_2 then this schedule obtained from applying Algorithm 1 is near-optimal because the contribution of the last interval is small. However, we can say more.

Proposition 2. *Assume that the hypotheses of Proposition 1 hold. And, assume* $(T_1^*, \dots, T_{n-1}^*, T_n^*) = (\hat{T}, \dots, \hat{T}, \check{T})$ *is an optimal solution to (3). Then,* $(T_1^*, \dots, T_{n-1}^*) = (\hat{T}, \dots, \hat{T})$ *solves model (3) redefined over interval* $[0, \bar{L}]$*, where* $\bar{L} = L - \check{T}$.

Proof. Suppose $(\hat{T}, \dots, \hat{T})$ is suboptimal for model (3) defined on the interval $[0, \bar{L}]$. Then, there exists a lower-cost schedule over $[0, \bar{L}]$. Taking that schedule and appending interval $\check{T}$ at the end yields a feasible solution to (3) defined on the interval $[0, L]$. Moreover, that schedule has lower cost than $(\hat{T}, \dots, \hat{T}, \check{T})$ over $[0, L]$, a contradiction. □

Proposition 2 implies that we can optimize the PM schedule over $[0, \bar{L}]$ by applying Algorithm 1 using condition (10) in place of condition (7) in that algorithm's "if" statement. Of course, this requires knowing $\bar{L}$. From Proposition 1, we know that $L - I_2 \leq \bar{L} \leq L$. So, we can build a grid of points on an interval of length I_2 for possible values of $\bar{L}$ and among those select the lowest cost solution, effectively solving model (3) under the bathtub assumption. A procedure for doing so is given in Algorithm 2. The cost of the PM schedule produced by Algorithm 2 is within $z(0)\varepsilon$ of the cost of an optimal PM schedule, where ε is the width of a cell in our grid for approximating the interval $[0, \bar{L}]$. Restated, Algorithm 2 yields a schedule that is only guaranteed to be near optimal, where the cost of the feasible schedule is suboptimal by at most $z(0)\varepsilon$.

Algorithm 2 Algorithm for bathtub failure rate.

Input: instance of model (3) with I_2, C_{pm}, C_{cm}, L, bathtub $z(t)$, and grid resolution $\varepsilon > 0$
Output: Near-optimal solution n^* and $(\hat{T}, \dots, \hat{T}, \check{T})$ of model (3)
Initialize $\hat{T} \leftarrow L, \check{T} \leftarrow 0, n^* \leftarrow 1$
for $i = 0$ to $\lceil I_2/\varepsilon \rceil$ **do**
  $\bar{L} \leftarrow L - I_2 + i\varepsilon$
  Solve (9) to obtain $\hat{n}$ and $\hat{T}_i$ using Algorithm 1 with $\bar{L}$ instead of L and (10) instead of (7)
  $\check{T}_i \leftarrow L - \hat{T}_i$.
  if $X(\hat{T}_i, \dots, \hat{T}_i, \check{T}_i) < X(\hat{T}, \dots, \hat{T}, \check{T})$ **then**
    $\hat{T} \leftarrow \hat{T}_i, \check{T} \leftarrow \check{T}_i, n^* \leftarrow \hat{n}$
  end if
end for
if $\check{T} > 0$ **then**
  $n^* \leftarrow n^* + 1$
end if
return $(\hat{T}, \dots, \hat{T}, \check{T})$.

If we could analytically express $z(t)$, we may be able to explicitly solve Eq. (10) for $\hat{T}$ when carrying out Algorithm 2 instead of calling Algorithm 1. Indeed, one of the key advances in this research is that $z(t)$ can be expressed analytically for all the existing parametric bathtub hazard rate models. In Section 4, we demonstrate this using the exponentiated Weibull distributions (Mudholkar & Srivastava, 1993), which is a generalization of the standard Weibull.

3. Nonparametric failure rate estimation and real data example

Recall from the previous section that the key to finding optimal maintenance decisions hinges on the hazard rate function. Here we describe two classes of such functions that guarantee the existence of optimal solutions for monotone hazards and bathtub hazards. Specifically, (i) Bayesian nonparametric models for increasing failure rates, and (ii) parametric Bayesian models for *bathtub* shape functions.

For both real data examples in this section, data from the STP nuclear plant are used. STP typically schedules maintenance based on recommendations from system engineers or other experts in the field. These experts weigh manufacturer suggestions, safety guidelines, past item failure history, the importance of the item to plant operation, as well as many other factors when designing maintenance schedules. This results in somewhat subjective maintenance schedules where items are sometimes maintained too often or not often enough. The methods herein provide data-driven recommendations that should result in optimal or near-optimal maintenance schedules.

3.1. Extended gamma process for monotone hazard rates

For an increasing hazard rate, Algorithm 1 can be applied in a straight-forward manner to solve for the optimal PM schedule. To this end, following Laud, Smith, and Damien (1996) and Dykstra and Laud (1981) we adapt a Markov chain Monte Carlo (MCMC) algorithm to simulate from the posterior distribution of a failure rate function, using exact and censored observations. Dykstra and Laud (1981) develop nonparametric theoretical results to handle monotone failure rates via the extended gamma process defined as follows.

The density of a gamma distribution, given parameters $\alpha > 0$ and $\beta > 0$ is

$$Gamma(x|\alpha, \beta) = x^{\alpha-1}\beta^{\alpha}\frac{e^{-\beta x}}{\Gamma(\alpha)}, \tag{11}$$

where $x > 0$. In developing the extended gamma (EG) process, extend the scalar parameters α and β in (15) to functions of time. This is useful since there is no *a priori* reason to suppose that the failure rate is constant throughout the time horizon. Specifically, let $\alpha(t)$, $t \geq 0$, be a non-decreasing, left-continuous function such that $\alpha(0) = 0$, which implies non-negativity. Let $\beta(t)$, $t \geq 0$, be a positive, continuous function, bounded away from 0 and ∞. Define $Z(t)$, $t \geq 0$, to be a gamma process with parameter $\alpha(\cdot)$. In other words,

- $Z(0) = 0$,
- $Z(t)$ has independent increments,
- for $t > s$, $Z(t) - Z(s) \sim Gamma(\alpha(t) - \alpha(s), 1)$ (increments may be non-stationary).

Then, the EG process is the process $h(t)$, $t \geq 0$, where

$$h(t) = \int_{[0,t]} [\beta(s)]^{-1} dZ(s). \tag{12}$$

This process is used as a nonparametric prior for increasing functions; specifically, we use it as a prior for an increasing failure rate function $z(t)$. Stated differently, the sample paths of the EG process index the class of increasing failure rate functions. (A similar formulation for decreasing hazards could be derived; see Dykstra and Laud, 1981 for details.)

The computational issue confronting us is this: how does one sample from (12)? The answer is provided in Appendix A, along with a generic algorithm, which is used in the real data illustration below. The main idea is to discretize continuous time. In the limit, as the discretization becomes finer, the process converges to the failure-rate process in (12). Of course, the finer the partition, the better the approximation. However, too fine of a partition can make the simulation quite difficult to execute, since the Gibbs sampler has to iterate in a very high-dimensional space. Laud et al. (1996) describe the trade-offs.

3.2. Extended gamma process application using STP data

The algorithm in Appendix A was applied to an on-line chemistry analytical device measuring plant cooling water salinity. In this case, the item tended to fail more and more with age. This suggests the need for an increasing failure rate model. We have 25 exact observations of failure times from STP, taken from February 2001 to August 2009, shown in Table 1. We use STP maintenance records to estimate costs associated with maintenance(C_{pm}, C_{cm}). Currently, STP employs a parametric (Weibull) model for these data. STP engineers are also critically aware that some of their items have monotone failure rate distributions. Hence, our nonparametric monotone hazard approach could lead to better PM protocols, ensuring higher reliability and lower costs in the long-term. Also of interest is that STP uses an empirical nonparametric estimation approach to model increasing failure rates; this, as we will show, could be much improved via the extended gamma process model. Fig. 1 shows the median, upper quartile, and lower quartile functional estimates along the time axis using an extended gamma process model. These are also compared to the Kaplan–Meier survivor function-based empirical failure rate estimate of the failure rate that STP uses. We see that the empirical estimator of the failure rate function behaves erratically toward the end of the time horizon, showing a failure rate approaching zero at approximately 7000 days. (Relatively speaking, the Bayesian quartile estimators remain smooth.) STP engineers concede that it is unlikely that this is an accurate estimate of the true failure rate function at that time. In contrast, using the Bayesian nonparametric model, we can select and analyze various percentiles of the simulated failure rate for more robust results. We use $L = 14600$ days (or 40 years), $C_{pm} = \$2000$ and $C_{cm} = \$8000$ as parameters for Algorithm 1. We select the median model of the failure rate and use this when numerically evaluating condition (7) in Algorithm 1. This yields $T_c^* = 3709$ days as the solution to the relaxed problem. As we discussed in Section 2.3, the optimal PM interval must divide L. We complete Algorithm 1 by evaluating the objective function at the adjacent integer solutions and selecting the lower of the two; we get $T^* = 3650$ days or 10 years. The current PM schedule for this item at STP is to perform no PM, or only perform maintenance if something breaks. The model-based insight into the performance of STP's systems could prove invaluable in the long-run. For this item, the estimated cost of maintenance over a 20 year

Table 1
Failure times of a salinity-measuring device. All observation types are exact.

Failure time (days)
3413.57
3632.62
5862.41
7132.61
7494.66
7668.45
4037.61
4966.56
4997.51
5317.28
5500.31
6024.40
6310.58
7220.73
7236.44
7690.51
2357.67
2983.47
3919.65
4051.66
7354.53
4786.44
5466.57
5883.52
7339.53

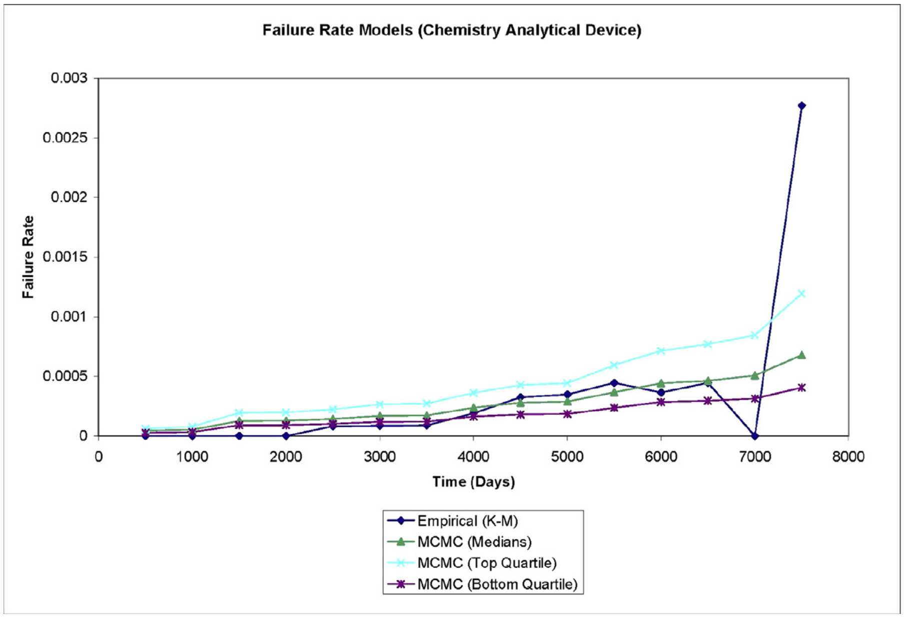

Fig. 1. MCMC simulated versus the Kaplan–Meier empirical failure rate using medians, top quartiles, and bottom quartiles. The empirical estimator behaves erratically past 6500 days where data are scarce.

interval when performing no PM is about $13,800; performing the PM yields an estimated cost of about $8000, a 42% savings.

Computational notes: in this example, the dimension of the grid (denoted Δ in Appendix A) is 15; i.e., we simulate from distributions of 15 increments of the failure rate. We drew 5000 samples from the distributions of each of the 15 δ's, and discard the first 1000 samples as "burn-in" observations. For these data, we did not find noticeable differences in the posterior estimates when the grid changed from, say 5–15. Of course, the computation time increases as the grid size increases, and was roughly 75–235 seconds. To assess convergence, we deployed the usual arsenal of MCMC diagnostics: these included Geweke statistics and mixing trace plots for the δs. The Geweke standard normal statistic tests the hypothesis that the means from the last 10% and last 50% of the samples from the posterior distributions of the parameters are the same versus the alternative hypothesis they are different. If the MCMC chain has converged, then the null is not rejected.

For the final samples used in the analyses above, the Geweke statistics for the δ's ranged from $-.477$ to $+1.392$. Once again, these values were invariant to choice of grid size.

4. Parametric failure rate estimation and real data example

4.1. Exponentiated Weibull distribution for bathtub hazard rates

We use an exponentiated Weibull distribution (Mudholkar & Srivastava, 1993) as a parametric model for an item's bathtub failure rate. With scale parameter λ and shape parameters θ and k, the EWD density function is given by:

$$f(t) = [1 - e^{-(t/\lambda)^k}]^{\theta-1} e^{-(t/\lambda)^k} \theta k \lambda^{-k} t^{k-1}, \quad 0 \le t < \infty.$$

The corresponding cumulative distribution function is given by:

$$F(t) = [1 - e^{-(t/\lambda)^k}]^{\theta}. \tag{13}$$

The hazard rate and survival (or reliability) functions are: $h(t) = f(t)/F(t)$ and $S(t) = 1 - F(t)$, respectively. This family offers flexibility in modeling different types of hazard rates:

1. If $k > 1$ and $k\theta < 1$, $h(t)$ is bathtub shaped.
2. If $k \ge 1$ and $k\theta \ge 1$, $h(t)$ is monotone increasing.
3. If $k \le 1$ and $k\theta \le 1$, $h(t)$ is monotone decreasing.
4. If $k < 1$ and $k\theta > 1$, $h(t)$ is unimodal.

The monotone shapes are strict ones except when $k = \theta = 1$, in which case the EWD reduces to the negative exponential distribution.

A related class of models that could be used in practice is the generalized gamma distribution. Cox and Matheson (2014) compare this distribution and the EWD. Their overarching conclusion is that, from an inference perspective, the results obtained under the two models are indistinguishable. This is good to know since the optimality results in this paper are applicable for any parametric model with bathtub shape failure rate—a key contribution of this research. Thus, for the EWD model, we can express Eq. (10) as:

$$\frac{C_{pm}}{C_{cm}} = T\,\frac{\frac{k\theta}{\lambda}[1-\exp(-(T/\lambda)^k)]^{\theta-1}\exp(-(T/\lambda)^k)(\frac{T}{\lambda})^{k-1}}{1-[1-\exp(-(T/\lambda)^k)]^{\theta}} + \ln[1-(1-\exp(-(T/\lambda)^k))^{\theta}]. \tag{14}$$

This expression, as an inequality, is then used in the "if" statement of Algorithm 1's bisection search. Of course, the results and algorithms hold more generally for any continuous bathtub-shaped failure rate function.

Random walk Metropolis–Hastings algorithm for the EWD model

To estimate the parameters of the EWD distribution, one could use Gibbs sampling or Metropolis–Hastings. Here, we use a variant of the latter with noninformative prior distributions for the three parameters, and do not impose any *a priori* restrictions on the shape; see, also, Nassar, Eissa, and Aarset (2004). Let $\Theta = (\lambda, k, \theta)$, namely the vector comprising the EWD parameters.

1. Set initial values $\Theta^{(0)}$
2. For $t = 1, \ldots, T$ repeat the following steps:
   (a) Set $\Theta = \Theta^{(t-1)}$
   (b) Generate new candidate parameter values Θ' from a proposal distribution $q(\Theta'|\Theta) \sim N_d(\Theta, S_\Theta)$, where $N_d(\Theta, S_\Theta)$ is a multivariate normal distribution and $d = 3$ is the dimension of Θ.
   (c) Calculate

$$\alpha = \min\left(1, \frac{f(\Theta'|y)q(\Theta|\Theta')}{f(\Theta|y)q(\Theta'|\Theta)}\right)$$

Table 2
Failure times of a low pressure switch. All observation types are exact.

Failure time (days)
50.77
112.02
16.02
1164.33
24.51
1261.91
1309.64
1180.84
1237.40

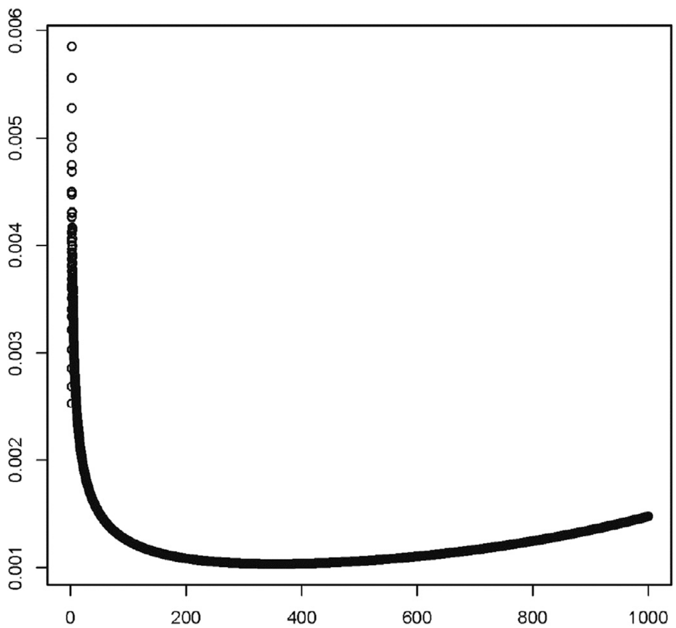

Fig. 2. Fitted bathtub failure rate for containment door.

(d) Update $\Theta^{(t)} = \Theta^{'}$ with probability α and $\Theta^{(t)} = \Theta = \Theta^{(t-1)}$ with probability $1 - \alpha$.

An important characteristic of the algorithm is that we do not need to evaluate the normalizing constant involved in the posterior distribution $f(\Theta|y)$ since it cancels out in α. Hence we can simplify α to:

$$\alpha = \min\left(1, \frac{f(y|\Theta^{'})f(\Theta^{'})q(\Theta|\Theta^{'})}{f(y|\Theta)f(\Theta)q(\Theta^{'}|\Theta)}\right)$$

4.2. Bathtub EWD optimization using STP data

For a bathtub failure rate example, we turn to low pressure switches for one of STP's containment doors comprising nine observations taken from January 1991 to August 2009, and shown in Table 2. For this illustration, as noted above, non-informative priors sans restrictions were considered; in other words, we allow the data to ascertain the true underlying shape of the hazard rate function. We ran the random walk Metropolis–Hastings algorithm for 110,000 iterations, with thinning interval of 2500 observations. Figs. 3–5 show the estimated posterior distributions of parameters λ, k, and θ. The means of these posterior distributions are: $\lambda = 1728.25, k = 5.45, \theta = 0.12$. Now, using the various shape constraints listed above, it is seen that this distribution has a bathtub failure rate. Fig. 2 shows this function. Thus, the selected policy discussed below will be optimal. Since the survival (or reliability) function, $S(t)$, is valuable to practitioners, it is easy to

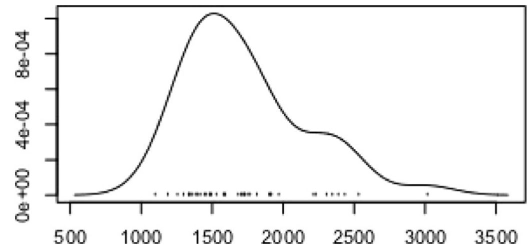

Fig. 3. Posterior distribution of λ.

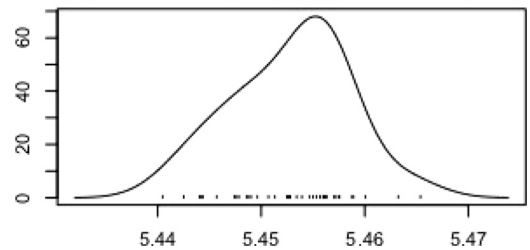

Fig. 4. Posterior distribution of k.

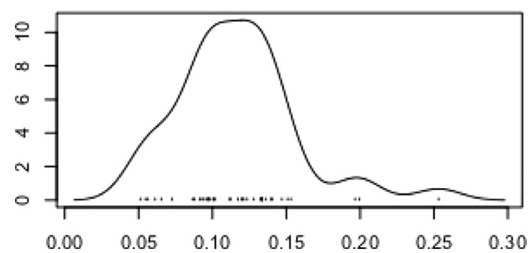

Fig. 5. Posterior distribution of θ.

provide summaries for it using samples from the Metropolis–Hastings algorithm. Hence, the mean, median, and 95% interval are 1148 days, 537 days, and [3.7 , 5239] days, respectively, for the survival function.

It is difficult to exactly solve for I in the exponentiated Weibull by setting the derivative of the failure rate to 0, so we use $I = 400$ days for our algorithm. This is not the exact I for this distribution, but it is an upper bound for the exact value, which will not affect our optimization result. Since the search space from Algorithm 2 is the time interval between $L - I$ and L, over-estimating I expands the search space, but still guarantees that the optimal solution is in the search space. We use $L = 14,600$ days, $C_{pm} = \$500$ and $C_{cm} = \$2000$ as inputs for Algorithm 2. Using $\varepsilon = 100$ days, we compare feasible solutions for $\bar{L}$ ranging from 14,200 days to 14,600 days. The feasible solution with the lowest cost has $\bar{L} = 14,600$ days, corresponding to the case when every inter-PM interval has the same length. The resulting solution is $T^* = 973$ days (or 2 years and 8 months). The current PM rate for this item is approximately once every 3.5 years. Using our model, the expected total cost over a 20 year period under the current policy is estimated to be about \$21,200, while the cost of our policy is about \$20,835, saving about 2%. These recommendations were evaluated by system engineers who found them reasonable as well as useful inputs for scheduling maintenance, which is significant because this means that the algorithms can provide practically viable answers.

5. Conclusion

The contributions of this paper lie at the useful intersection of Bayesian inference and stochastic optimization. This symbiotic theoretical relationship is of particular relevance in reliability contexts. Here, we address the problem of optimizing maintenance schedules for a single system where the relevant actions consist of corrective and preventive maintenance. The paper adapts the extended gamma process to nonparametrically model monotone hazard rates. Attendant output from a Bayesian Markov chain Monte Carlo procedure serves as inputs into a new optimality algorithm—a first in this literature. Another key contribution is the use of the exponentiated Weibull distribution to model bathtub shaped hazard rates. Here again, parameter estimates from appropriate posterior distributions serve as inputs into another algorithm to find an optimal preventive maintenance schedule. Importantly, these scheduling algorithms will work for *any* continuous bathtub shaped hazard rate model, not just the exponentiated Weibull. The models and algorithms are demonstrated on real data sets from a nuclear power plant in Bay City, Texas.

Acknowledgments

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Appendix A. Sampling the extended gamma process

For an increasing The density of a gamma distribution, given parameters $\alpha > 0$ and $\beta > 0$ is

$$Gamma(x|\alpha,\beta) = x^{\alpha-1}\beta^{\alpha}\frac{e^{-\beta x}}{\Gamma(\alpha)}, \tag{15}$$

where $x > 0$. Let $\alpha(t)$, $t \geq 0$, be a non-decreasing, left-continuous function such that $\alpha(0) = 0$, which implies non-negativity. Let $\beta(t)$, $t \geq 0$, be a positive, continuous function, bounded away from 0 and ∞. Define $Z(t)$, $t \geq 0$, to be a gamma process with parameter $\alpha(\,\cdot\,)$. We have:

- $Z(0) = 0$,
- $Z(t)$ has independent increments,
- for $t > s$, $Z(t) - Z(s) \sim Gamma(\alpha(t) - \alpha(s), 1)$ (increments may be non-stationary).

Then, the EG process is the process $h(t)$, $t \geq 0$, where

$$h(t) = \int_{[0,t]} [\beta(s)]^{-1} dZ(s). \tag{16}$$

This process is used as a nonparametric prior for increasing functions; specifically, we use it as a prior for an increasing failure rate function $z(t)$. Stated differently, the sample paths of the EG process index the class of increasing failure rate functions. (A similar formulation for decreasing hazards could be formulated; see Dykstra and Laud, 1981 for details.)

Let $0 \leq s_0 < s_1 < \cdots < s_M = L$ denote a set of times along the time interval $[0, L]$. Then, let $[0, s_1), [s_1, s_2), \ldots, [s_{M-1}, s_M)$ denote a partition of the time axis, dividing it into intervals of lengths $s_1, s_2 - s_1, s_3 - s_2, \ldots, s_M - s_{M-1}$. Let δ_i be the increment of the failure rate function over the ith interval $(s_{i-1}, s_i]$, for $i = 1, 2, \ldots, M$; i.e., $h(s_j) = \sum_{i=1}^{i=j} \delta_i$. The prior distribution for each δ_i is defined by the underlying density corresponding to the EG process. We refer to this prior density of δ_i, for $i = 1, 2, \ldots, M$, as $f^*_{\delta_i}$; in other words,

$$\delta_i \sim f^*_{\delta_i}.$$

From reliability theory, the failure rate function $z(t)$ corresponds to the cumulative failure distribution function, which we denote $F(t)$, via

$$F(t) = 1 - \exp\left[-\int_{[0,t]} z(u)du\right].$$

Using $h(t)$ to model $z(t)$, we have

$$F(t) = 1 - \exp\left[-\int_{[0,t]} h(u)du\right] \approx 1 - \exp\left[-\sum_{j:s_j\in[0,t]} \delta_j(s_j - s_{j-1})\right].$$

Let d_i be the number of observed failures in the ith interval $(s_{i-1}, s_i]$, for $i = 1, 2, \ldots, M$. Using Bayes' theorem, the posterior of $\Delta = (\delta_1, \delta_2, \ldots, \delta_M)$ can be expressed as follows:

$$[\Delta|observations] \propto \prod_{j=1}^{M} \exp\left[-d_j\sum_{i=1}^{j-1}\delta_i(s_{j-1} - s_{i-1})\right] \left(1 - \exp\left[-(s_j - s_{j-1})\sum_{i=1}^{j}\delta_i\right]\right)^{d_j} f^*_{\delta_j}. \tag{17}$$

Some items may still be operating by the time we stop observing the system. We account for these items by using them as special observations recorded at the time when we stop observing the items. These observations, known as right-censored observations (see, for example, Rausand & Høyland, 2004), provide additional information that is useful in modeling the failure rate. We account for right-censored data by updating the parameter $\beta(s_i)$, $i = 1, \ldots, M$. Consider n identical right-censored items, and let $x_1^c, x_2^c, \ldots, x_n^c$ be the censored times. Then, $\beta(s_i)$ is updated as follows:

$$\beta(s_i) \leftarrow \beta(s_i) + \sum_{j\in[1,\ldots,n], x_j^c \geq s_i} (x_j^c - s_i). \tag{18}$$

The posterior distribution described in (17) is difficult if not impossible to explicitly compute or numerically estimate, for a reasonably-sized sample and a moderately-dense grid. Instead, we use a Markov chain Monte Carlo (MCMC) algorithm that samples from that posterior distribution. To this end, first re-write the posterior distribution (17) as:

$$[\Delta|observations] \propto \prod_{j=1}^{M}\left(1 - e^{-T_j(\Delta)}\right)^{d_j} f^*_{\delta_j} e^{-a_j d_j} \tag{19}$$

where

- $a_j = \sum_{i=1}^{j-1} \delta_i(s_{i-1} - s_{j-1})$, and
- $T_j(\Delta) = (s_j - s_{j-1})\sum_{i=1}^{j}\delta_i$.

Now, consider the following random multi-dimensional variables:

- $\mathbf{m}_j = (m_{j1}, m_{j2}, \ldots, m_{jj})$, independent multinomials, where each $\mathbf{m_j}$ is a j-cell multinomial of d_j independent trials, with the probability of the kth cell being $p_{jk} = \frac{\delta_k}{\sum_{i=1}^{j}\delta_i}$, and
- $\mathbf{g}_j = (g_{j1}, g_{j2}, \ldots, g_{jd_j})$, a collection of independent exponential random variables, with mean $\frac{1}{T_j(\Delta)}$, and truncated at 1.

We define $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_M)$ and $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_M)$, and re-express the posterior distribution (19) with $\mathbf{m}$ and $\mathbf{g}$ as:

$$[\Delta|observations, \mathbf{g}, \mathbf{m}] \propto \prod_{j=1}^{M} \delta_j^{\sum_{i=j}^{M} m_{ij}}$$
$$\times \exp\left[-\left(a_j + \sum_{i=j}^{M}(s_i - s_{i-1})\sum_{k=1}^{d_j} g_{ik}\right)\delta_j\right] f^*_{\delta_j}. \quad (20)$$

We can then sample from the posterior of Δ using the Gibbs sampling procedure given in Algorithm 3.

Algorithm 3 Gibbs sampling algorithm.

Input: the number of iterations k, failure/censored data, prior distributions $f^*_{\delta_j}, j = 1, \ldots, M$
Output: simulated observations of Δ
$i \leftarrow 0$
for $i < k$ **do**
  Simulate $[\mathbf{g}_j,\ 1 \le j \le M|\mathbf{observations}, \boldsymbol{\Delta}, \mathbf{m}] \propto$ vectors of exponential random variables with means $\frac{1}{T_j(\Delta)}$, truncated at 1, $j = 1, \ldots, M$.
  Simulate $[\mathbf{m}_j,\ 1 \le j \le M|\mathbf{observations}, \boldsymbol{\Delta}, \mathbf{g}] \propto$ vectors of j-cell multinomial random variables of d_j trials, probability of kth cell $= \frac{\delta_k}{\sum_{i=1}^{j} \delta_i}$, $j = 1, \ldots, M$.
  Simulate and output $[\delta_j,\ \ 1 \le j \le M|\mathbf{observations}, \mathbf{m}, \mathbf{g}] \propto$ Gamma$(\alpha_{s_j}, \beta_{s_j})$ random variables with a rejection algorithm using $f^*_{\delta_j}$ as the importance sampling function (described in detail in the Appendix of **?**), for $j = 1, \ldots, M$.
  $i \leftarrow i + 1$
end for

References

Barlow, R., & Hunter, L. (1960). Optimum preventive maintenance policies. *Operations Research, 8*, 90–100.
Barlow, R. E., & Proschan, F. (1962). Planned replacement. In K. Arrow, S. Karlin, & H. Scarf (Eds.), *Studies in applied probability and management science*. California: Stanford University Press.
Boland, P., & Proschan, F. (1982). Periodic replacement with increasing minimal repair costs at failure. *Operations Research, 30*, 1183–1189.
Cox, C., & Matheson, M. (2014). A comparison of the generalized gamma and exponentiated Weibull distributions. *Statistics in Medicine, 33*, 3772–3780.
Crow, L. H. (1990). Evaluating the reliability of repairable systems. In *Annual reliability and maintainability symposium* (pp. 275–279). 200 Cordoba Court, Merritt Island, FL 32953: RAMS.
Dykstra, R., & Laud, P. (1981). A Bayesian nonparametric approach to reliablity. *Annals of Statistics, 9*, 356–367.
Elsayed, E. A. (2012). *Reliability engineering*. Hoboken, NJ: John Wiley & Sons.
Galenko, A., Morton, D., Popova, E., Kee, E., Grantom, R, & Sun, A. (2005). Operational level models and methods for risk informed nuclear asset management. Proceedings of the American nuclear society international topical meeting on probabilistic safety assessment.
Jardine, A., & Buzacott, J. (1985). Equipment reliability and maintenance. *European Journal of Operational Research, 19*, 285–296.
Laud, P., Smith, A., & Damien, P. (1996). Monte Carlo methods for approximating a posterior hazard rate process. *Statistics and Computing, 6*, 77–83.
McCall, J. J. (1965). Maintenance policies for stochastically failing equipment: A survey. *Management Science, 11*, 493–524.
Merrick, J., Soyer, R., & Mazzuchi, T. (2003). A Bayesian semi-parametric analysis of the reliability and maintenance of machine tools. *Technometrics, 48*, 58–69.
Merrick, J., Soyer, R., & Mazzuchi, T. (2005). Are maintenance practices for railroad tracks effective? *Journal of the American Statistical Association, 100*, 17–25.
Mudholkar, G., & Srivastava, D. (1993). Exponentiated Weibull family for analyzing bathtub failure rate data. *IEEE Transactions on Reliability, 42*, 299–302.
Nassar, M., Eissa, F., & Aarset, M. (2004). Bayesian estimation for the exponentiated Weibull model. *Communications in Statistics, 33*, 2243–2362.
Noortwick, J. V. (2009). A survey of the application of gamma processes in maintenance. *Reliability Engineering and System Safety, 94*, 2–21.
Pham, H., & Wang, H. (1996). Imperfect maintenance. *European Journal of Operational Research, 94*(3), 425–438.
Pierskalla, W. P., & Voelker, J. A. (1979). A survey of maintenance models: The control and surveillance of deteriorating systems. *Naval Research Logistics Quarterly, 23*, 353–388.
Popova, E., Morton, D., Damien, P., & Hanson, T. (2010). Bayesian analysis and decisions in nuclear power plant maintenance. In *The oxford handbook of applied Bayesian analysis* (pp. 219–239). Oxford University Press.
Rausand, M., & Høyland, A. (2004). *System reliability theory: Models, statistical methods, and applications*. New York, NY, USA: John Wiley & Sons Inc.
Valdez-Flores, C., & Feldman, R. (1989). A survey of preventive maintenance models for stochastically deteriorating single-unit systems. *Naval Research Logistics Quarterly, 39*, 419–446.
Wang, H. (2002). A survey of maintenance policies of deteriorating systems. *European Journal of Operational Research, 139*(3), 469–489.