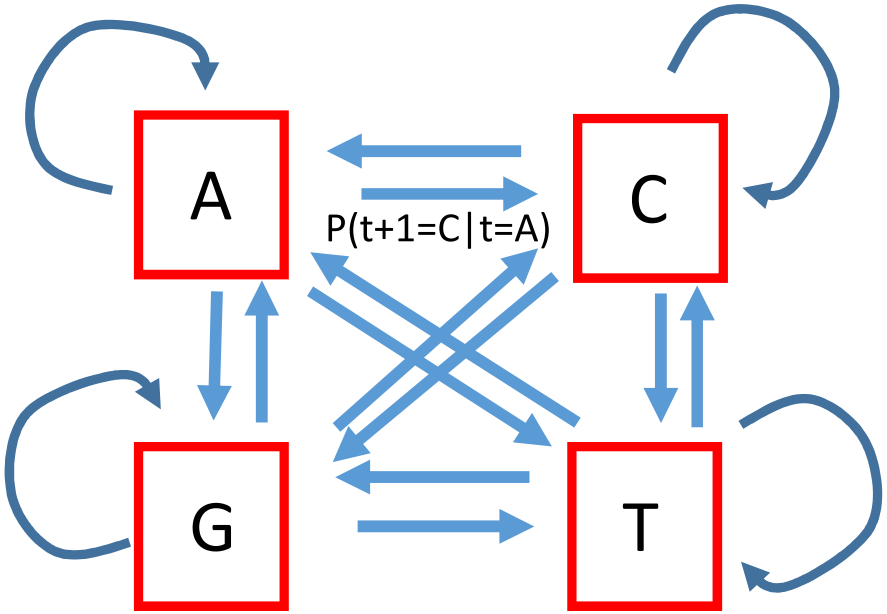
Session 2

Hidden Markov Models applied to Clustering of Biological Sequences

Markov model

	t0		t1		t2		t3		t4		t5		t6
Markov chain	A	→	A	→	C	→	A	→	G	→	G	→	T

Four possible categories



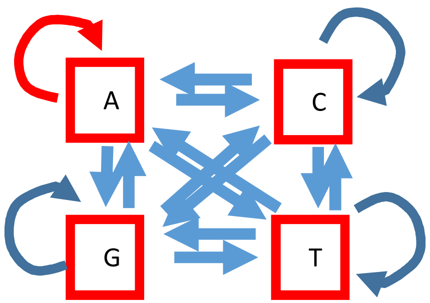
Probability transition matrix

How to identify a pattern of characters?

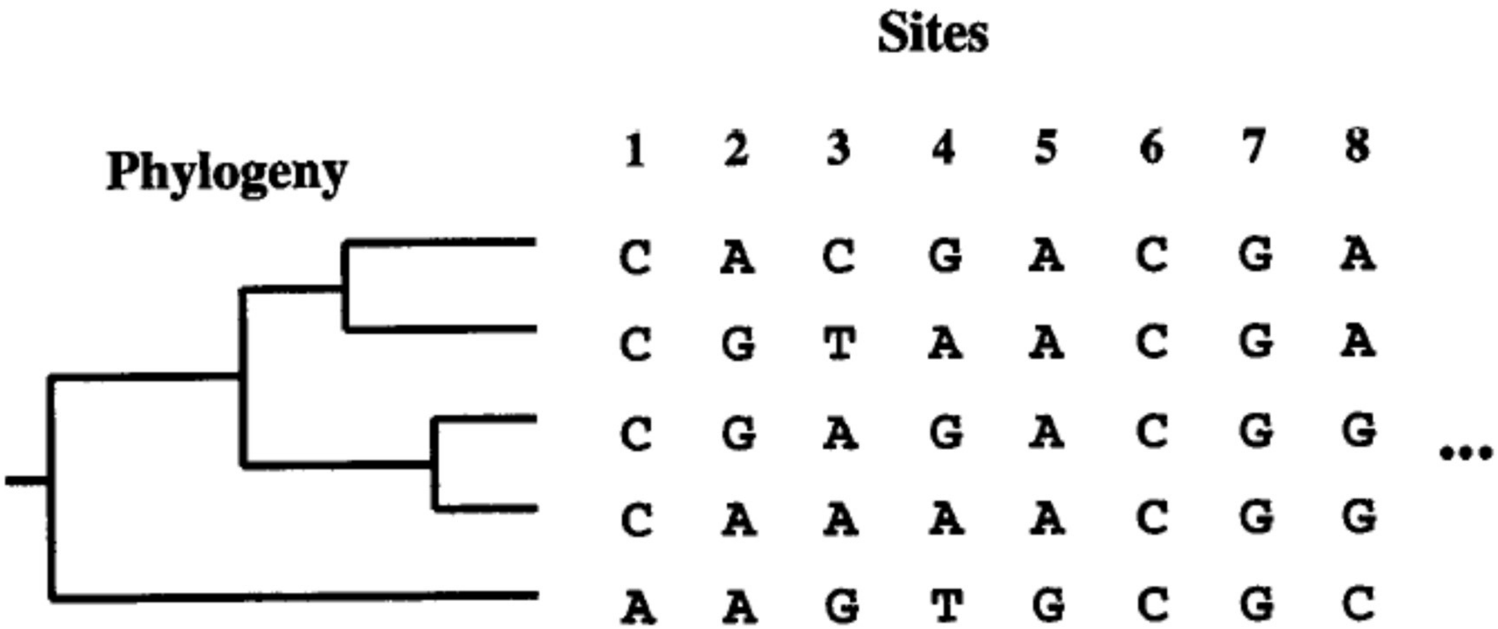
	t0		t1		t2		t3		t4		t5		t6
Markov chain	A	→	A	→	C	→	A	→	G	→	G	→	T

$$P(AACAGGT) = \boxed{P(A)}\,\boxed{P(A|A)}\,P(C|A)P(A|C)P(G|A)P(G|G)P(T|G)$$

Prior probability



How does the rates of evolution change over a sequence?



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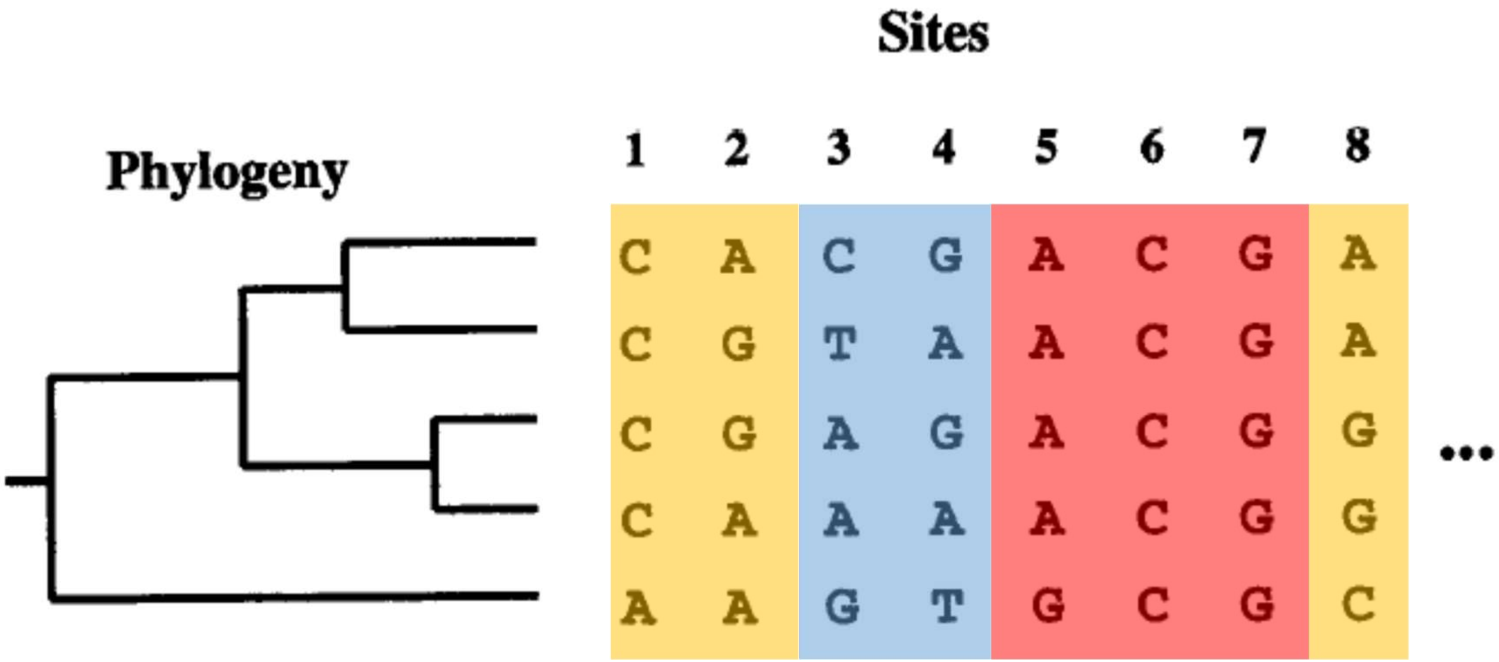
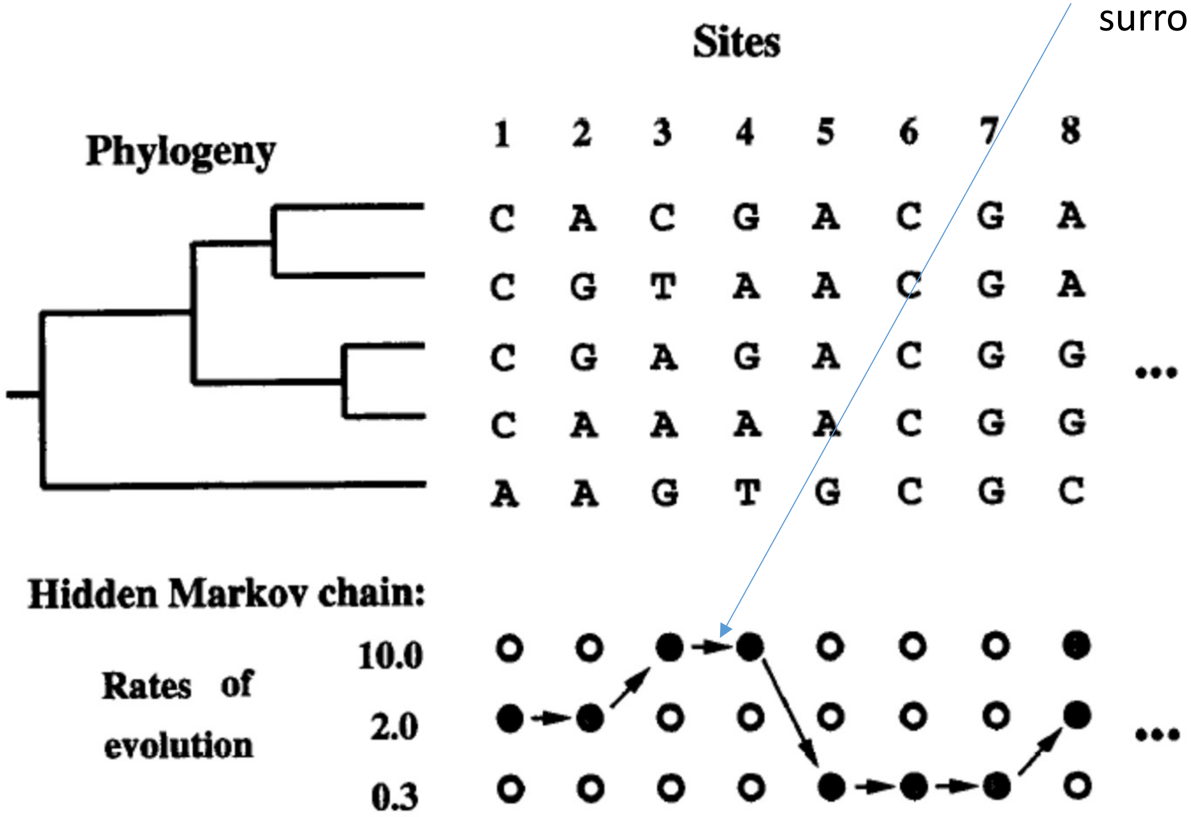


Fast evolving Slow evolving Intermediate evolving

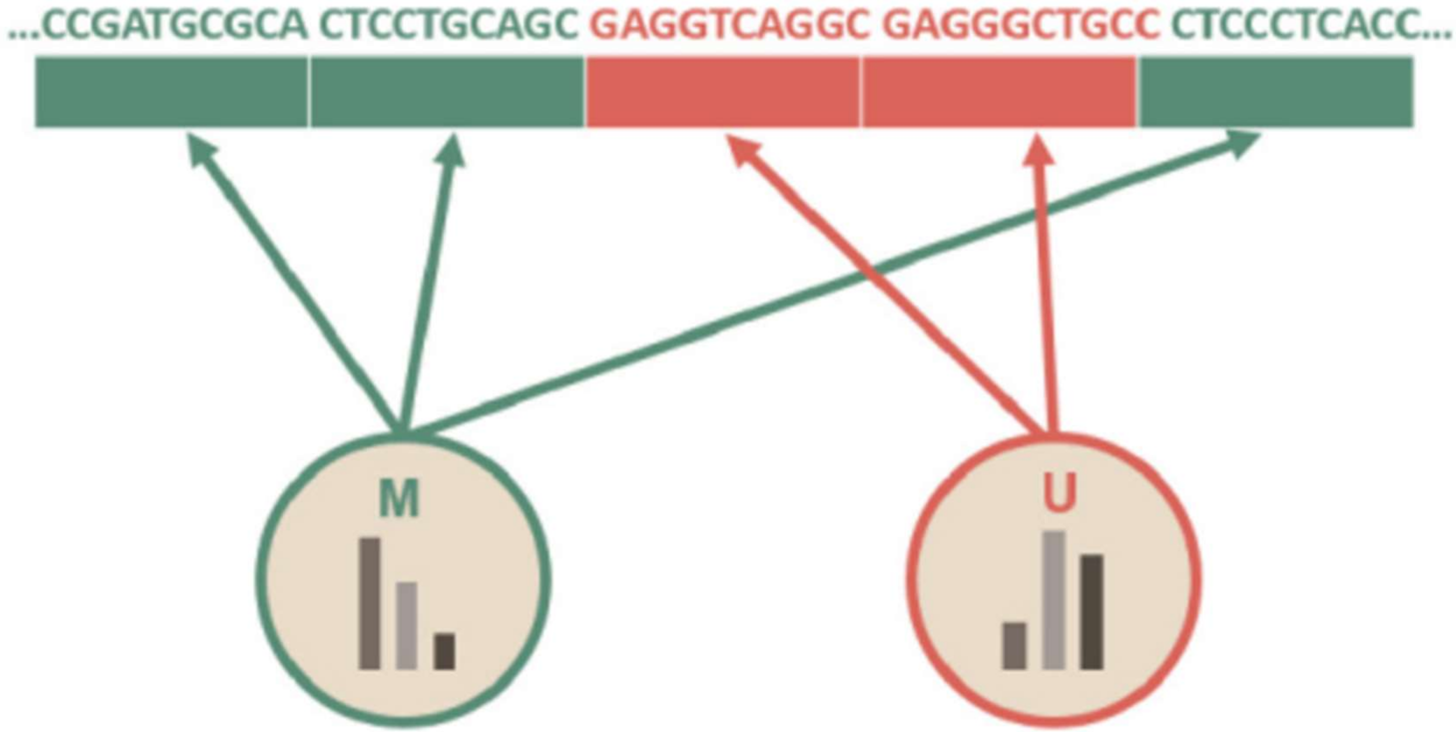
How does the rates of evolution change over a sequence?

The Probability that I will be in one state or another depends on the surrounding nucleotides



Other examples of finding a pattern in the sequence: Metilation

DNA Metilation

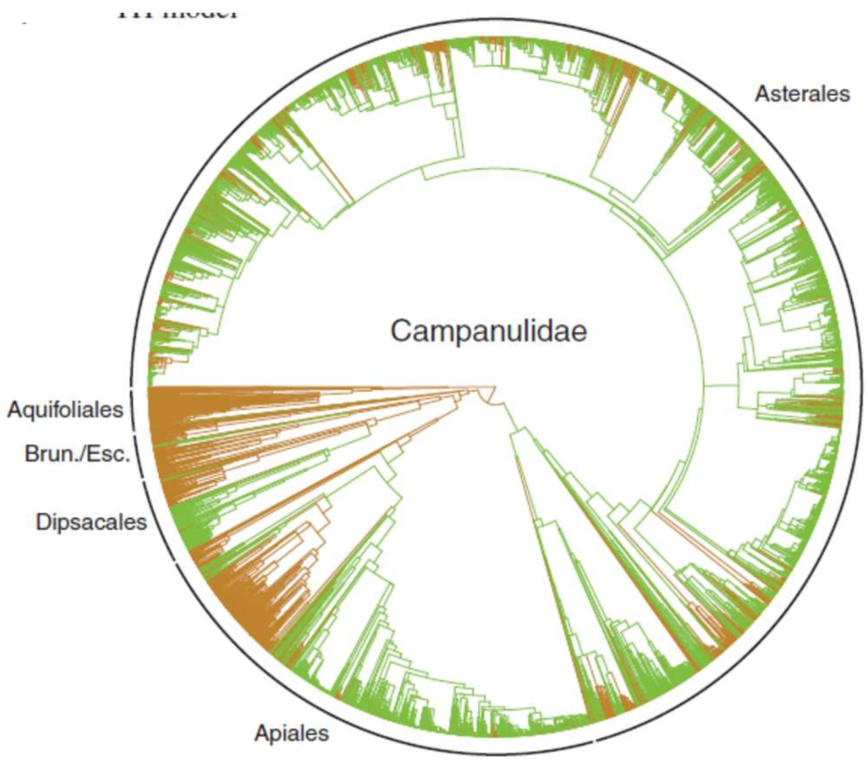


A Review of Three Different Studies on Hidden Markov Models for Epigenetic Problems: A Computational Perspective

How does a phenotype change over a phylogeny?

Identifying Hidden Rate Changes in the Evolution of a Binary Morphological Character: The Evolution of Plant Habit in Campanulid Angiosperms

JEREMY M. BEAULIEU1,∗, BRIAN C. O’MEARA2, AND MICHAEL J. DONOGHUE1



How to identify a pattern of characters?

2) To which rate of evolution each nucleotide belongs to?

Hidden because the state (Fast, Intermediate, Slow) is unknown

ATGATTTCAAAAGGACCATCATTAGGA   Observed categories

FFF I I I I I I S S S…   Hidden states

Fast (F), Intermediate (I) or Slow (S) is hidden (in fact, this is what we want to estimate!)

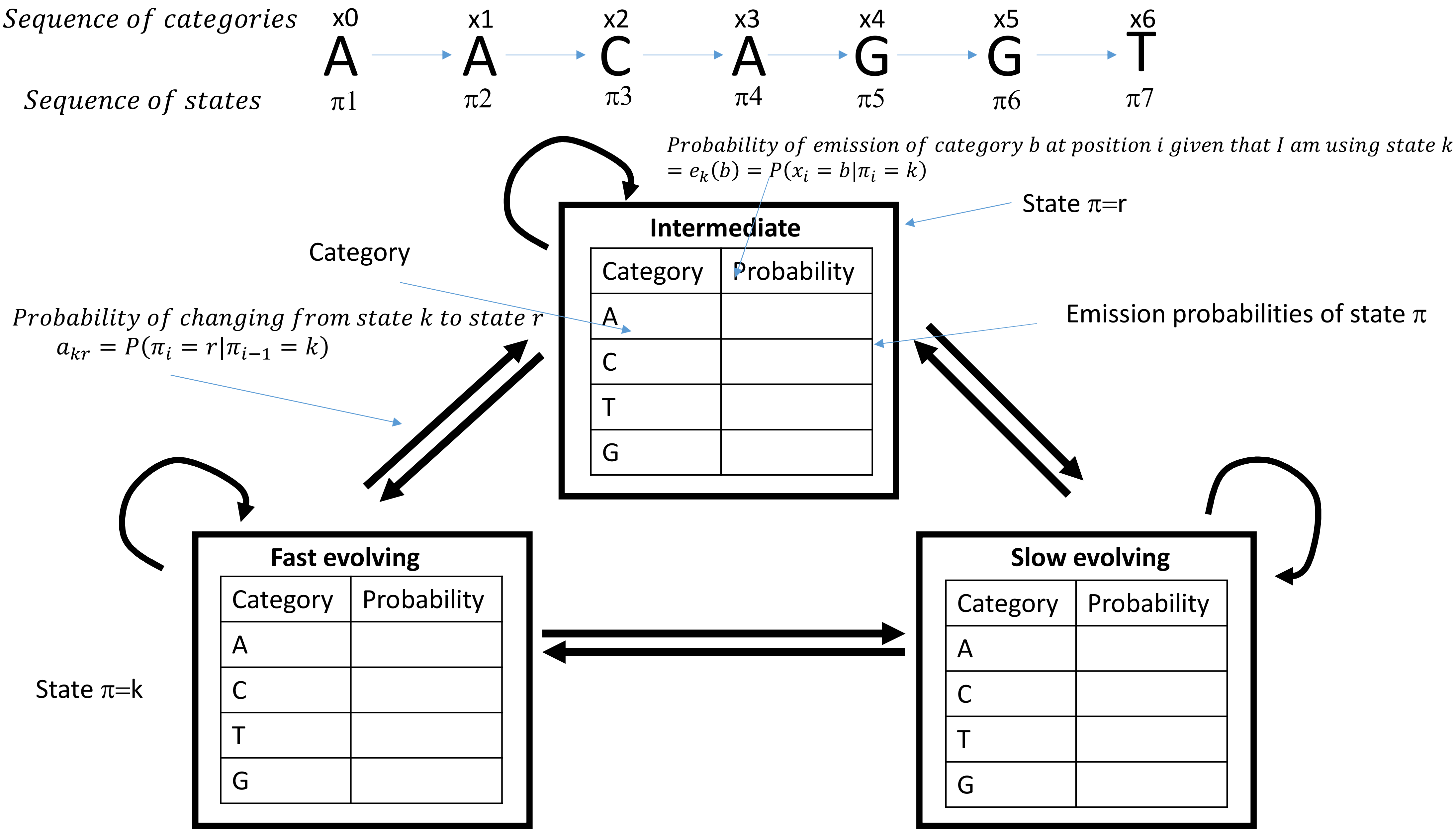
What am I saying when I use a HMM?

"My data is a mixture of k probability distributions, each determined by its own parameters"

"The probability of using one probability distribution or another depends on which one I used in the previous step"

Sequence of categories	x0	x1	x2	x3	x4	xi	x6	= x
	A →	A →	C →	A →	G →	G →	T	
Sequence of states	π1	π2	π3	π4	π5	πi	π7	= π

SOME NOTATION



Sequence of categories
x0 x1 x2 x3 x4 x5 x6
A A C A G G T
Sequence of states
π1 π2 π3 π4 π5 π6 π7
Probability of emission of category b at position i given that I am using state k
$= e_k(b) = P(x_i = b|\pi_i = k)$
State π=r
Intermediate
Category
Probability
A
C
T
G
Category
Emission probabilities of state π
Probability of changing from state k to state r
$a_{kr} = P(\pi_i = r|\pi_{i-1} = k)$
Fast evolving
Category
Probability
A
C
T
G
State π=k
Slow evolving
Category
Probability
A
C
T
G

NOTATION HMM

$State\ at\ position\ i = \pi_i$

$Sequence\ of\ states = \pi$

$Category\ at\ position\ i = x_i$

$Sequence\ of\ categories\ = x$

$$P(x,\pi) = a_{0\pi} \prod_{i}^{N} e_{\pi_i}(x_i) a_{\pi_i \pi_i+1}$$

Initial state ("Probability that I started the chain at state π")

$$Probability\ of\ changing\ from\ state\ k\ to\ state\ r = a_{kr} = P(\pi_i = r|\pi_{i-1} = k)$$

Given that I was in the previous position using state *k*

I am using state *r*

Transmission matrix

	Fast Evolving	Intermediate Evolving	Slow Evolving
Fast Evolving			
Intermediate Evolving			
Slow Evolving			

Emission Probability matrix for each possible state

Fast evolving

Category	Probability
A	
C	
T	
G	

Intermediate

Category	Probability
A	
C	
T	
G	

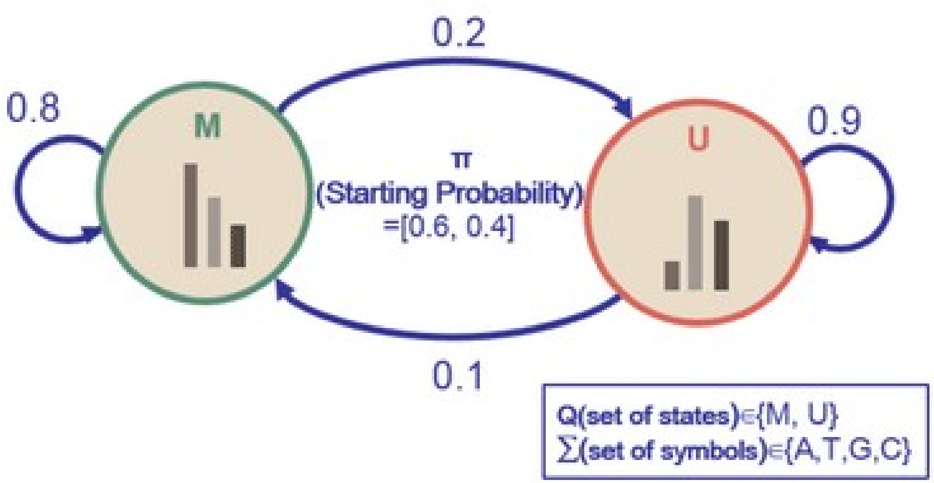
Slow evolving

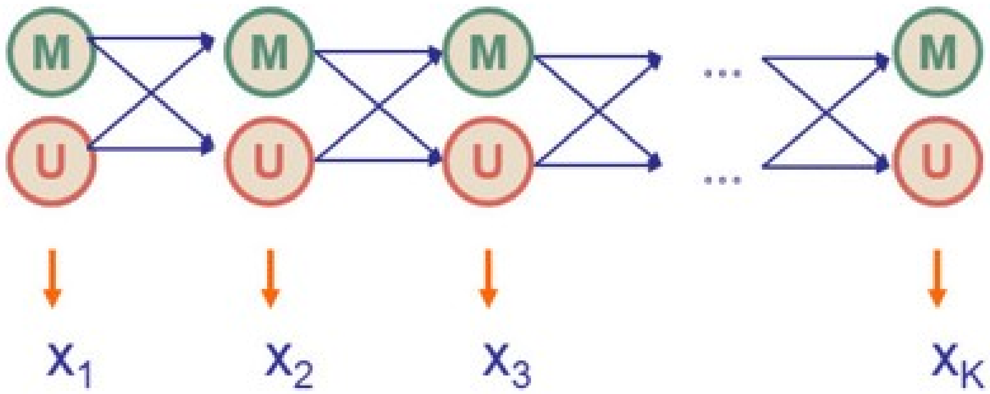
Category	Probability
A	
C	
T	
G	

$$Probability\ of\ emission\ of\ category\ b\ at\ position\ i\ given\ that\ I\ am\ using\ state\ k = e_k(b) = P(x_i = b|\pi_i = k)$$

HMM: Multiple ways of modelling

DNA Metilation

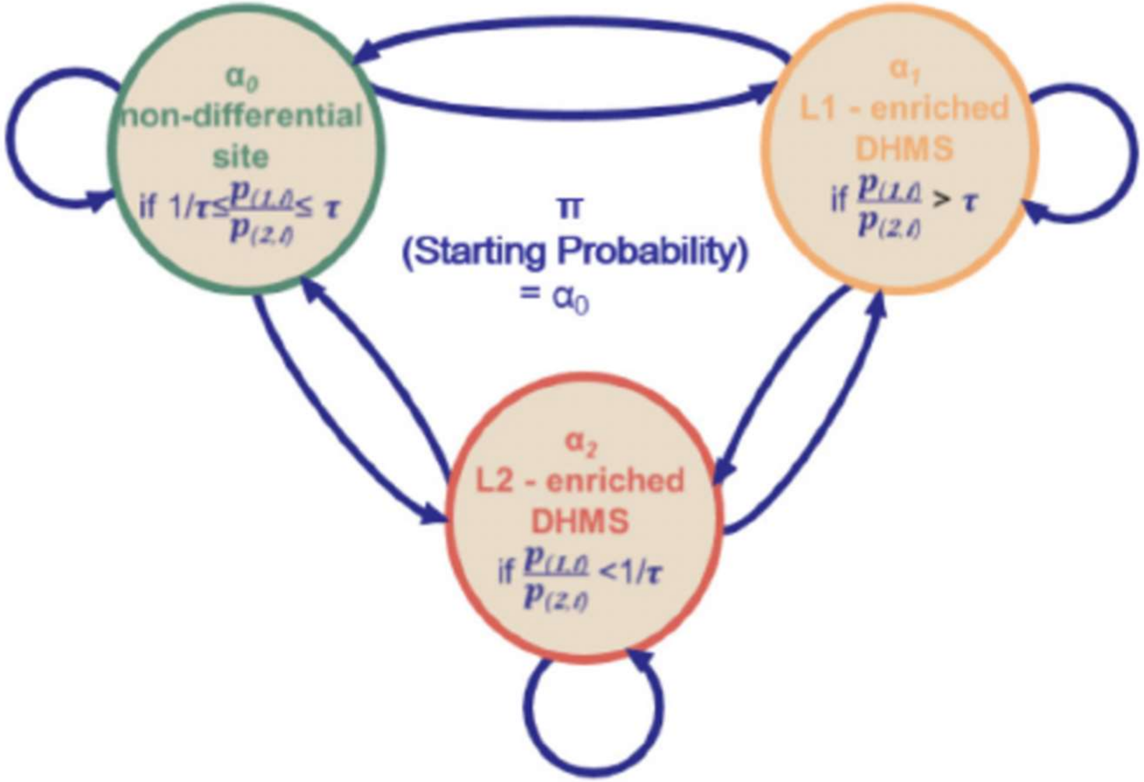




A Review of Three Different Studies on Hidden Markov Models for Epigenetic Problems: A Computational Perspective

HMM: Multiple ways of modelling

DNA Metilation

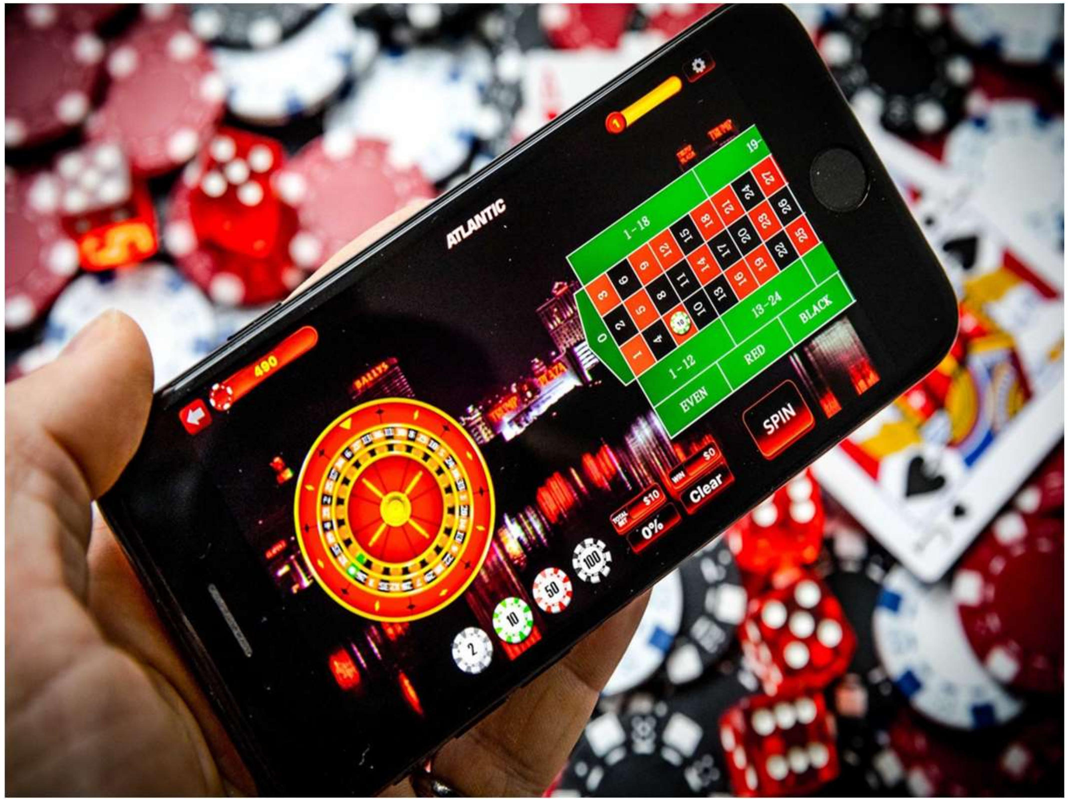


differential histone modification sites (DHMSs)

A Review of Three Different Studies on Hidden Markov Models for Epigenetic Problems: A Computational Perspective

HMM: A simple working case

The occasionally dishonest casino problem



With some probability, the casino uses a dice that is **LOADED** so the number six (the bench wins) occurs more often (P(6)=0.5 than expected at random.

HOW CAN WE KNOW WHEN IT IS LOADED OF FAIR?

HMM

The occasionally dishonest casino problem.

0.95

State Fair

Probability of making a change from one state to another

State Loaded

0.9

0.1

0.05

Emission probabilities of fair dice

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

Fair (F) dice

Emission probabilities of loaded dice

1	2	3	4	5	6
1/10	1/10	1/10	1/10	1/10	1/2

Loaded (L) dice

HMM:

1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$$\boldsymbol{P(C|F) = 0.01}$$

$$\boldsymbol{P(F|C) = 0.1}$$

x	$P(x\|C)$
1	$\frac{1}{10}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

HMM: Probability of x given model?

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

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1	$\frac{1}{10}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

It refers to the values of emission and transmission that are used for the k considered states

HMM

Probability of a sequence x given that the states of the sequence are known and we have the HMM model

Imagine we observe the sequence x

$$x = \langle x_1, x_2, x_3 \rangle = \langle 6,1,6 \rangle$$

Assume that the prior probability of starting at one state or at the other is the same (0.5)

What would be the probability $P(x,\pi)$ if the state sequence was

$$\pi = \langle \pi_1, \pi_2, \pi_3 \rangle = \langle F, F, F \rangle$$

$$P(x,\pi) = a_{0\pi} \prod_{i}^{N} e_{\pi_i}(x_i) a_{\pi_i \pi_i+1}$$

$$P(x,\pi) = 0.5 * 0.95 * \frac{1}{6} * 0.95 * \frac{1}{6} * 0.95 * \frac{1}{6} = 0.001984664$$

Prior of starting at F

Prob of staying at state F

Emission prob of 6 at state F

HMM: Probability of x given model?

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

We know that in a MC it is just:

P(S|Markov Chain) = P(1)P(3|1)P(4|3)...

P(S|Hidden Markov Chain): P(1)P(3|1)P(4|3)...

In our previous HMM notation

In a HMC this chain explodes!

P(C)	P(1|C)	P(3|F)P(F|C)	$e_{\pi_i(x_i)}$	$a_{\pi_i\pi_{i+1}}$
		P(3|C)P(C|C)		
P(F)	P(1|F)	P(3|F)P(F|F)		
		P(3|C)P(C|F)		

HMM: Probability of x given model? Forward

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

	1	3	4	…
F	0.5*1/6			
L	0.5*1/10			

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$\boldsymbol{P(C|F) = 0.01}$

$\boldsymbol{P(F|C) = 0.1}$

x	$P(x\|C)$
1	$\frac{1}{10}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

HMM: Probability of x given model? Forward

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

F,F

	1	3	4	...
F	0.5*1/6	0.5*1/6*0.99*1/6		
L	0.5*1/10			

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

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4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

HMM: Probability of x given model? Forward

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

F,F

C,F

	1	3	4	…
F	0.5*1/6	0.5*1/6*0.99*1/6+ 0.5*1/10*0.1*1/6		
L	0.5*1/10			

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
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4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

HMM: Probability of x given model? Forward

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

F,F

C,F

	1	3	4	…
F	0.5*1/6	0.5*1/6*0.99*1/6+ 0.5*1/10*0.1*1/6		
L	0.5*1/10	0.5*1/10*0.9*1/10		

C,C

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$P(C|F) = 0.01$

$P(F|C) = 0.1$

x	$P(x\|C)$
1	$\frac{1}{10}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

HMM: Probability of x given model? Forward

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

F,F

C,F

	1	3	4	…
F	0.5*1/6	0.5*1/6*0.99*1/6+ 0.5*1/10*0.1*1/6		
L	0.5*1/10	0.5*1/10*0.9*1/10 + 0.5*1/6*0.01*1/6		

C,C

F,C

x	$P(x\|F)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
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1	$\frac{1}{10}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{2}$

We have searched all the combinations

HMM: Probability of x given model? Forward

P(1,3,4,5,1,2,6,6,3,6,6,2,1,2,4,5|HMM)

	1	3	4	…
F	0.5*1/6	0.5*1/6*0.99*1/6+ 0.5*1/10*0.1*1/6		
L	0.5*1/10	0.5*1/10*0.9*1/10 + 0.5*1/6*0.01*1/6		

$$\left.\begin{array}{l}P1\\[2em]P2\end{array}\right\}\ P1+P2 = P(S|HMM)$$

Some computational considerations….

- Each step implies multiplying a small number by a number between 0 and 1 (the probability of doing that step).
- We end having VERY small numbers, which ultimately will produce underflow
  - Do a log transformation (you can do this whenever you multiply probabilities)
  - Scale the probability at each step (you do this when you multiply and add probabilities)

HMM

- Scale the probability at each step (Rabiner 1989)

State	P(i)	P(i+1)
State1		
State2		
StateK		

$$\widehat{f_l}(i) = \frac{f_l(i)}{\prod_{j=1}^{i} s_i}$$

Scaling factor

HMM

- Scale the probability at each step (Rabiner 1989)

State	$\widehat{f_l}(i)$	$\widehat{f_l}(i+1)$
State1		
State2		
StateK		

$$\widehat{f_l}(i) = \frac{f_l(i)}{\prod_{j=1}^{i} s_j}$$

New scaled variable

HMM

- Scale the probability at each step (Rabiner 1989)

State	$\widehat{f}_l(i)$	$\widehat{f}_l(i+1)$
State1		
State2		
StateK		

Probability of moving from state r to state l in i+1

$$\widehat{f}_l(i+1) = \frac{1}{s_{i+1}} e_l(x_{i+1}) \sum_{r=1}^{K} \widehat{f}_r(i) a_{rl}$$

Scaled forward

Emission probability of category at xi+1 using state l

HMM

- Scale the probability at each step (Rabiner 1989)

State	$\hat{f}_l(i)$	$\hat{f}_l(i+1)$
State1		
State2		
StateK		

Total of the scaled value

If we impose that

$$\sum_l \hat{f}_l(i) = 1$$

Then si+1 is

$$s_{i+1} = \sum_l e_l(x_{i+1}) \sum_k \hat{f}_k(i) a_{kl}$$

$$P(X|HMM) = \prod_i^L s_i$$

HMM

What happens when the parameters are not known?

?

State Fair

Probability of making a change from one state to another

?

?

State Loaded

?

Emission probabilities of fair dice

1	2	3	4	5	6
?	?	?	?	?	?

Fair (F) dice

Emission probabilities of loaded dice

1	2	3	4	5	6
?	?	?	?	?	?

Loaded (L) dice

HMM

What happens when the parameters are not known?

1 We have a training dataset where we know the hidden state

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,F,L,L

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,F,L,L

?

State Fair

Probability of making a change from one state to another

State Loaded

?

?

?

Emission probabilities of fair dice

1	2	3	4	5	6
?	?	?	?	?	?

Fair (F) dice

Emission probabilities of loaded dice

1	2	3	4	5	6
?	?	?	?	?	?

Loaded (L) dice

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,F,L,L

?

State Fair

Probability of making a change from one state to another

State Loaded

?

?

?

Emission probabilities of fair dice

1	2	3	4	5	6
?	?	?	?	?	?

Emission probabilities of charged dice

1	2	3	4	5	6
?	?	?	?	?	?

Fair (F) dice

Loaded (L) dice

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,F,L,L

How would you do it?

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,L,L

Come from->Goes to	F	L
F		
L		

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π $\boxed{\text{F,F,}}$F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,F,L,L

Come from->Goes to	F	L
F	1	
L		

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,$\boxed{\text{F,F,}}$F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,F,L,L

Come from->Goes to	F	L
F	2	
L		

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,$\boxed{\text{L,L}}$

Come from->Goes to	F	L
F	α	β
L	γ	δ

a_{FF}

a_{FC}

a_{CF}

a_{CC}

?

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,L,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,L,L,L,L,L,L,L,F,F,F,F,F,F,F,F,F,F,F,F,$\boxed{\text{L,L}}$

Marginal of the row

Come from->Goes to	F	C	Total
F	α	β	α+β
C	γ	δ	γ+δ

Maximum likelihood estimates

$$a_{FF} = \frac{\alpha}{\alpha + \beta} \qquad a_{FC} = \frac{\beta}{\alpha + \beta} \qquad a_{kl} = \frac{A_{kl}}{\sum_{l'} A_{kl'}}$$

Counts in cell row k, column l

Counts in row k (Marginal)

Row k
“come from”

Column l
“go to”

HMM

Why the maximum likelihood can be computed like this?

HMM

For a given "I come from" state, we have two possible movements

"*I stay at the state*" with probability *p*

"*I move to the other state*" with probability *1-p*

We observe "*I stay at the state*" *y* observations out of *n* observations

Which is the statistical distribution that could model this process?

$$P(y|p;n) = \frac{n!}{y!\,(n-y)!}(p)^y(1-p)^{n-y}$$

HMM

$$P(y|p;n) = \frac{n!}{y!\,(n-y)!}(p)^y(1-p)^{n-y}$$

$$\log\big(L(p)\big) = \log(C) + ylog(p) + \ (n-y)\log(1-p)$$

$$\frac{dL}{dp} = \frac{h}{p} - \frac{n-y}{1-p}$$

$$\frac{dL}{dp} = 0; p = \frac{y}{n}$$

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,C,C,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F,F

a_{FF}

State Fair

Probability of making a change from one state to another

a_{CF}

a_{FC}

State Charged

a_{CC}

Emission probabilities of fair dice

1	2	3	4	5	6
?	?	?	?	?	?

Fair (F) dice

Emission probabilities of charged dice

1	2	3	4	5	6
?	?	?	?	?	?

Loaded (L) dice

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,C,C,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F,F

Emission probabilities of fair dice

1	2	3	4	5	6
?	?	?	?	?	?

Fair (F) dice

Emission probabilities of loaded dice

1	2	3	4	5	6
?	?	?	?	?	?

Loaded (L) dice

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,C,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F,F

How would you do it?

HMM

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,C,C,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F,F

x	F
1	n_{1F}
2	n_{2F}
3	n_{3F}
4	n_{4F}
5	n_{5F}
6	n_{6F}

x	$P(x\|F)$
1	n_{2F}/N_F
2	n_{2F}/N_F
3	n_{3F}/N_F
4	n_{4F}/N_F
5	n_{5F}/N_F
6	n_{6F}/N_F

$$e_k(b) = \frac{E_k(b)}{\sum_{b'} E_k(b')}$$

Can you identify the different variables?

HMM

Once the parameters have been estimated in training, they must be validated in a different dataset not used for training (replication)

Confusion matrix. In classification tasks, it allows to estimate how good is the model

Viterbi	**F is predicted given HMM**	**C is predicted given HMM**
F is true in replication		
C is true in replication		

I would like this to be as close as possible to 100%

Check also $P(S|HMM)$

HMM

The occasionally dishonest casino problem

Unknown paths

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

How to find HMM that maximizes P(X|HMM)?

HMM

The occasionally dishonest casino problem

Unknown paths

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$P(x|HMM{=}a_{rk}; e_k(b))$ Likelihood of an observed sequence given a set of transition and emission probabilities from a given HMM, computed using the forward algorithm

How to find HMM that maximizes P(X|HMM)?

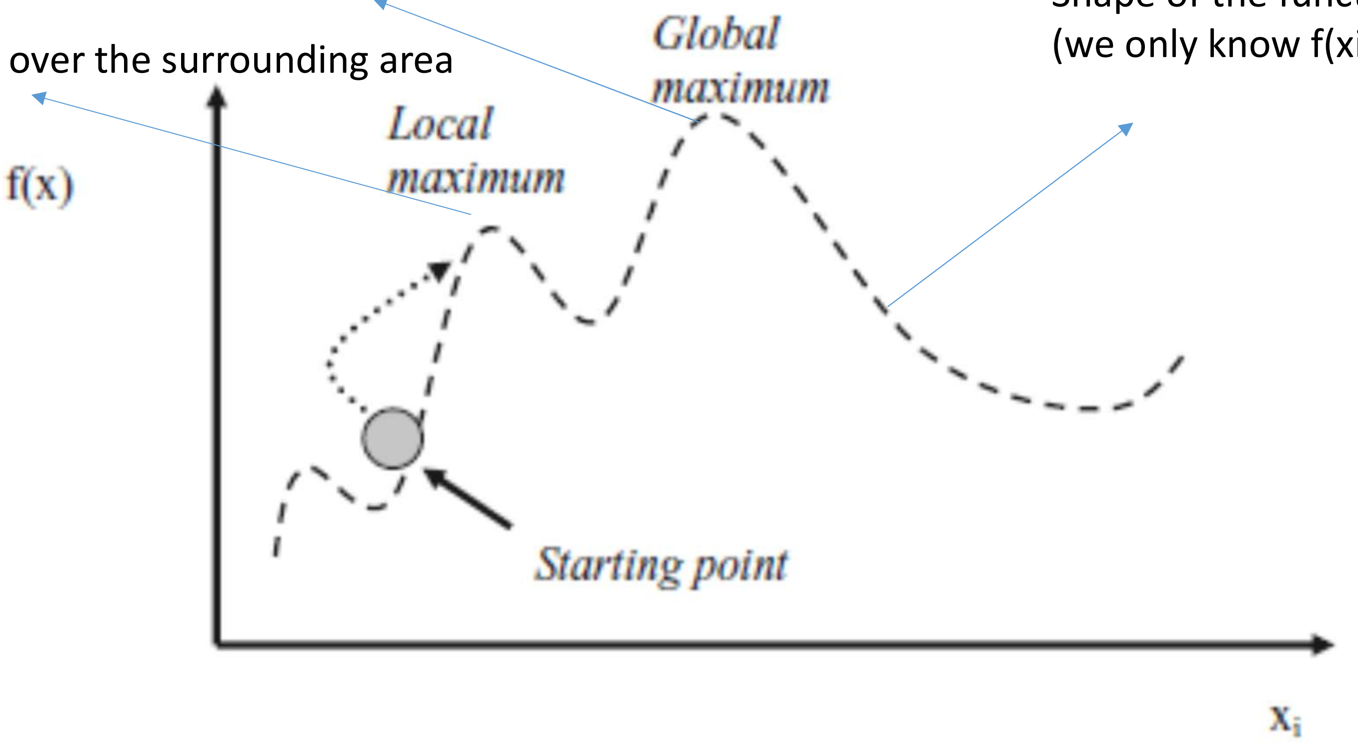
HMM

How to find HMM that maximizes P(X|HMM)?

Xi with maximum f(x) over all the function

Shape of the function is unknown
(we only know f(xi) when we evaluate x)

Xi with maximum f(x) over the surrounding area



HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

```
Program Genetic Algorithm ()
{
    initialize population;
    evaluate population;
    while (termination conditions not reached)
    {
        select solutions for next population;
        perform crossover and mutation
        evaluate population;
    }
}
```

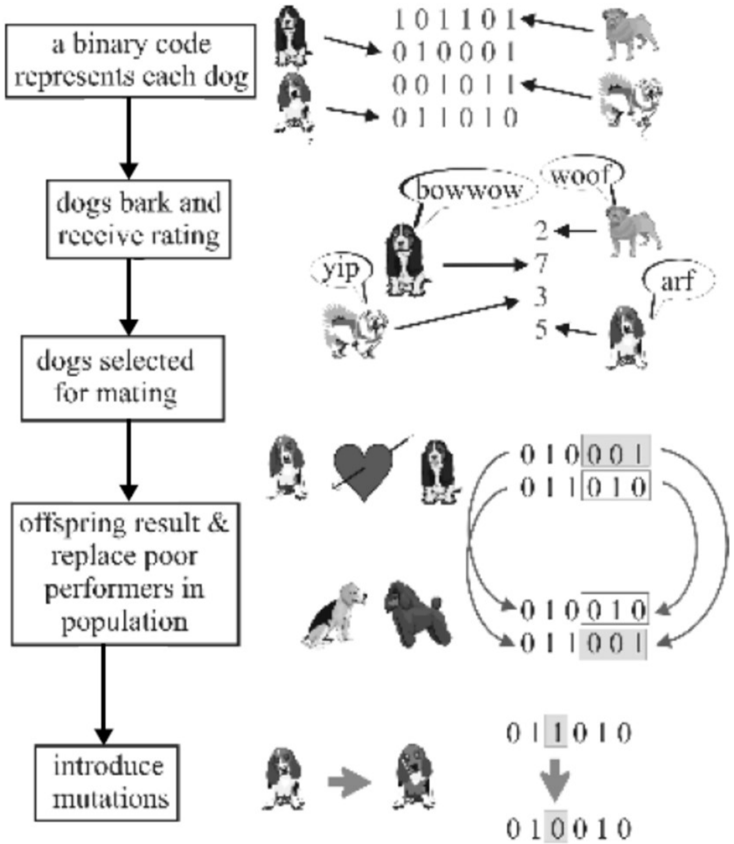
Classical Genetic algorithm

$$chromosome = [p_1, p_2, p_3, \ldots, p_{N_{var}}]$$

Algorithm 3.1: Canonical Genetic Algorithm

Determine how the solution is to be encoded as a genotype and define the fitness function;
Create an initial population of genotypes;
Decode each genotype into a solution and calculate the fitness of each of the n solution candidates in the population;

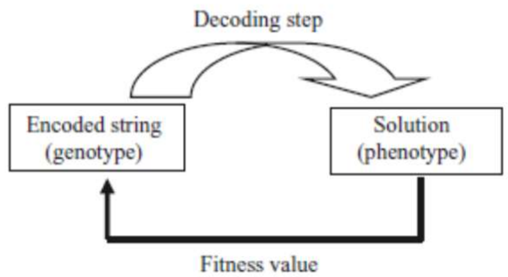
repeat
  Select n members from the current population of encodings (the *parents*) in order to create a mating pool;
  repeat
    Select two parents randomly from the mating pool;
    With probability p_{cross}, perform a crossover process on the encodings of the selected parent solutions, to produce two new (*child*) solutions;
    Otherwise, crossover is not performed and the two children are simply copies of their parents;
    With probability p_{mut}, apply a mutation process to each element of the encodings of the two child solutions;
  until *n new child solutions have been created*;
  Replace the old population with the newly created one (this constitutes a generation);
until *terminating condition*;

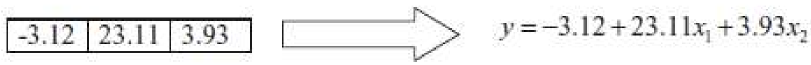


Classical Genetic algorithm

- What do we need?
    - A chromosome based representation of the solution that we are searching for:

$$chromosome = [p_1, p_2, p_3, \ldots, p_{N_{var}}]$$





Classical Genetic algorithm

- What do we need?
  - A way to generate new solutions by recombination from previous ones (Mating & Recombination):

$$parent_1 = [\, p_{m1}, p_{m2}, p_{m3}, p_{m4}, p_{m5}, p_{m6}, \ldots, p_{mN_{var}} \,]$$

$$parent_2 = [\, p_{d1}, p_{d2}, p_{d3}, p_{d4}, p_{d5}, p_{d6}, \ldots, p_{dN_{var}} \,]$$

Continuous

Binary

…

By setting few crossover points

$$offspring_1 = [\, p_{m1}, p_{m2}, \uparrow p_{d3}, p_{d4}, \uparrow p_{m5}, p_{m6}, \ldots, p_{mN_{var}} \,]$$

$$offspring_2 = [\, p_{d1}, p_{d2}, \uparrow p_{m3}, p_{m4}, \uparrow p_{d5}, p_{d6}, \ldots, p_{dN_{var}} \,]$$

By setting random uniform crossovers

$$offspring_1 = [\, p_{m1}, p_{d2}, p_{d3}, p_{d4}, p_{d5}, p_{m6}, \ldots, p_{dN_{var}} \,]$$

$$offspring_2 = [\, p_{d1}, p_{m2}, p_{m3}, p_{m4}, p_{m5}, p_{d6}, \ldots, p_{mN_{var}} \,]$$

Not allowing repeated values within each offspring

By mixing at each gene the proportions of the parents

$$p_{new} = \beta\, p_{mn} + (1-\beta) p_{dn}$$

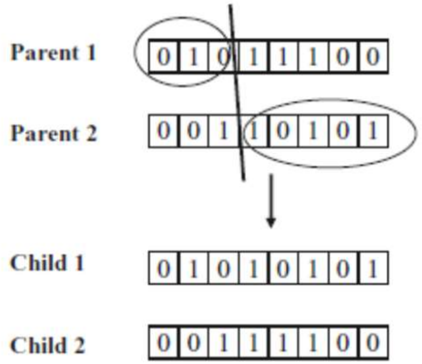
$$p_{new} = \beta (p_{mn} - p_{dn}) + p_{mn}$$

Clustering based labeling

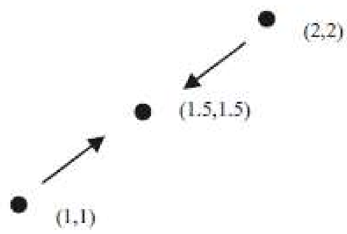
Classical Genetic algorithm

- What do we need?
  - A way to generate new solutions by recombination from previous ones (Mating & Recombination):

Single crossover



Real valued genotypes



Classical Genetic algorithm

- How to introduce new solutions (mutation)?

$$chromosome = [p_1, p_2, p_3, \ldots, p_{N_{var}}]$$

$$\Big\downarrow \quad \text{f(px) ?}$$

$$chromosome = [p_1, \mathbf{p_2}, p_3, \ldots, p_{N_{var}}]$$

Classical Genetic algorithm
Other hyperparameters to consider

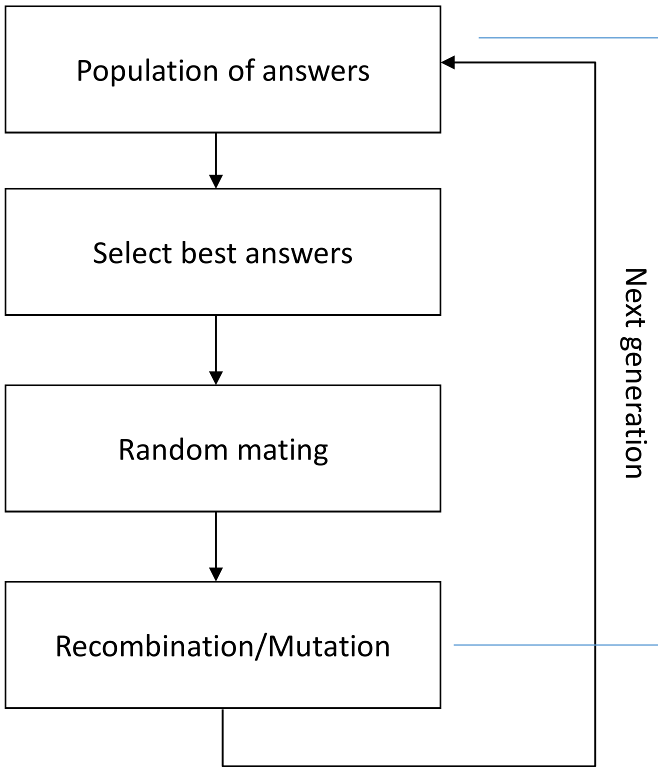
Which fitness? $F(x) = g(f(x))$

Which elitism?
(too small = a lot of bad solutions
Too high = risk of ending in a local solution)

How do I allow the best to mate?
At random?
Best have the higher frequency of mating?

$$f_{\text{rank}} = 2 - P + 2(P-1)\frac{(\text{rank}-1)}{(n-1)}.$$

Tournament selection?



How many?
Too high = time consuming. Slow convergence
Too small = noise will tend to predominate over selection

For how long?

Which mutation frequency?
Too small = result will tend to be local
Too high = random walking (no need of doing a GA!)

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

"Put in a vector the parameters you want to estimate"

Which are the parameters?

x	$P(x\|F)$
1	
2	
3	
4	
5	
6	

$P(C|F)$

$P(F|C)$

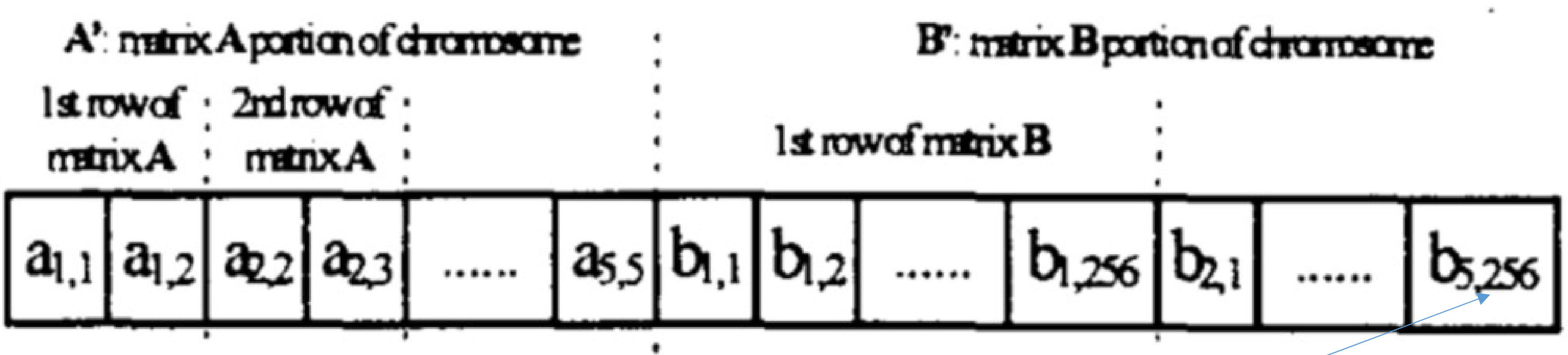
x	$P(x\|C)$
1	
2	
3	
4	
5	
6	

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Coding of the matrices in a vector (=Chromosome in GA). Example



This is because in this case they are using characters for speech recognition problems

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

"Put in a vector the parameters you want to estimate"

"taking into account possible conditions"

All these variables must add to one

Which are the parameters?

x	$P(x\|F)$
1	
2	
3	
4	
5	
6	

$P(C|F)$

$P(F|C)$

x	$P(x\|C)$
1	
2	
3	
4	
5	
6	

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

“Put in a vector the parameters you want to estimate”

“taking into account possible conditions”

Which are the parameters?

x	$P(x\|F)$
1	
2	
3	
4	
5	
6	

$P(C|F)$

$P(F|C)$

x	$P(x\|C)$
1	
2	
3	
4	
5	
6	

All these variables must add to one

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

"Put in a vector the parameters you want to estimate"

"taking into account possible conditions"

Which are the parameters?

States P(Sj|Si)	S_1	S_2	S_n
S_1			
S_2			
S_n			

All these variables must add to one

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

"Put in a vector the parameters you want to estimate"

"taking into account possible conditions"

How many independent variables do we have here?

Which are the parameters?

x	$P(x\|F)$
1	
2	
3	
4	
5	
6	

$P(C|F)$

$P(F|C)$

x	$P(x\|C)$
1	
2	
3	
4	
5	
6	

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

"Put in a vector the parameters you want to estimate"

"taking into account possible conditions"

Which are the parameters?

x	$P(x\|F)$
1	
2	
3	
4	
5	
6	

$P(C|F)$

$P(F|C)$

x	$P(x\|C)$
1	
2	
3	
4	
5	
6	

How many independent variables do we have here?
5 (because all the variables must add to 1)

In statistical jargon we would say n-1 degrees of freedom

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

"Put in a vector the parameters you want to estimate"

"taking into account possible conditions"

Some subsets of parameters must add to 1 (we can exclude one)
All parameters cannot be greater than 1

Conditions of this particular problem

Find a way how to transform the variables so the conditions are fulfilled

HMM

Discrete probability function

$$P(i) = p_i$$

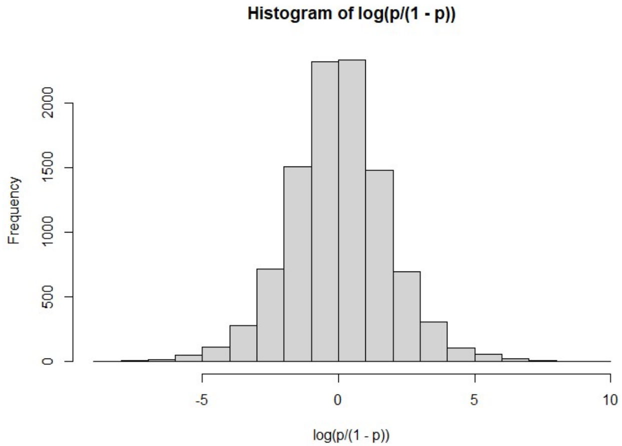
$$\sum_{i}^{K} p_i = 1$$

Logit transformation for a binary variable

$$\boxed{\theta_i} = \log\left(\boxed{\frac{p_i}{1-p_i}}\right)$$

Which is the range?

Which is the range?



$p \sim U(0,1)$

Looks like a Normal distribution, but it is not

HMM

Discrete probability function

$$P(i) = p_i$$

$$\sum_{i}^{K} p_i = 1$$

Logit transformation for a binary variable

$$\theta_i = \log\left(\frac{p_i}{1 - p_i}\right)$$

How to estimate pi?

HMM

Discrete probability function

$$P(i) = p_i$$

$$\sum_i^K p_i = 1$$

"Logit transformation of multinomial distribution"

$$p_i = \frac{e^{\theta_i}}{1 + \sum_j^{K-1} e^{\theta_j}}$$

$$p_K = 1 - \sum_j^{K-1} p_j$$

HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Chromosome structure

e_1	e_1	e_1	e_1	e_1	...			
θ_1	θ_2	θ_3	θ_4	θ_5				

We need to keep track to which table each parameter refers to (BUT IT IS NOT CODED IN THE chromosome)

Which are the parameters?

$$p_i = \frac{e^{\theta_i}}{1 + \sum_j^{K-1} e^{\theta_j}}$$

Decode the information of the chromosome

x	$P(x\|F)$
1	
2	
3	
4	
5	
6	

$P(C|F)$

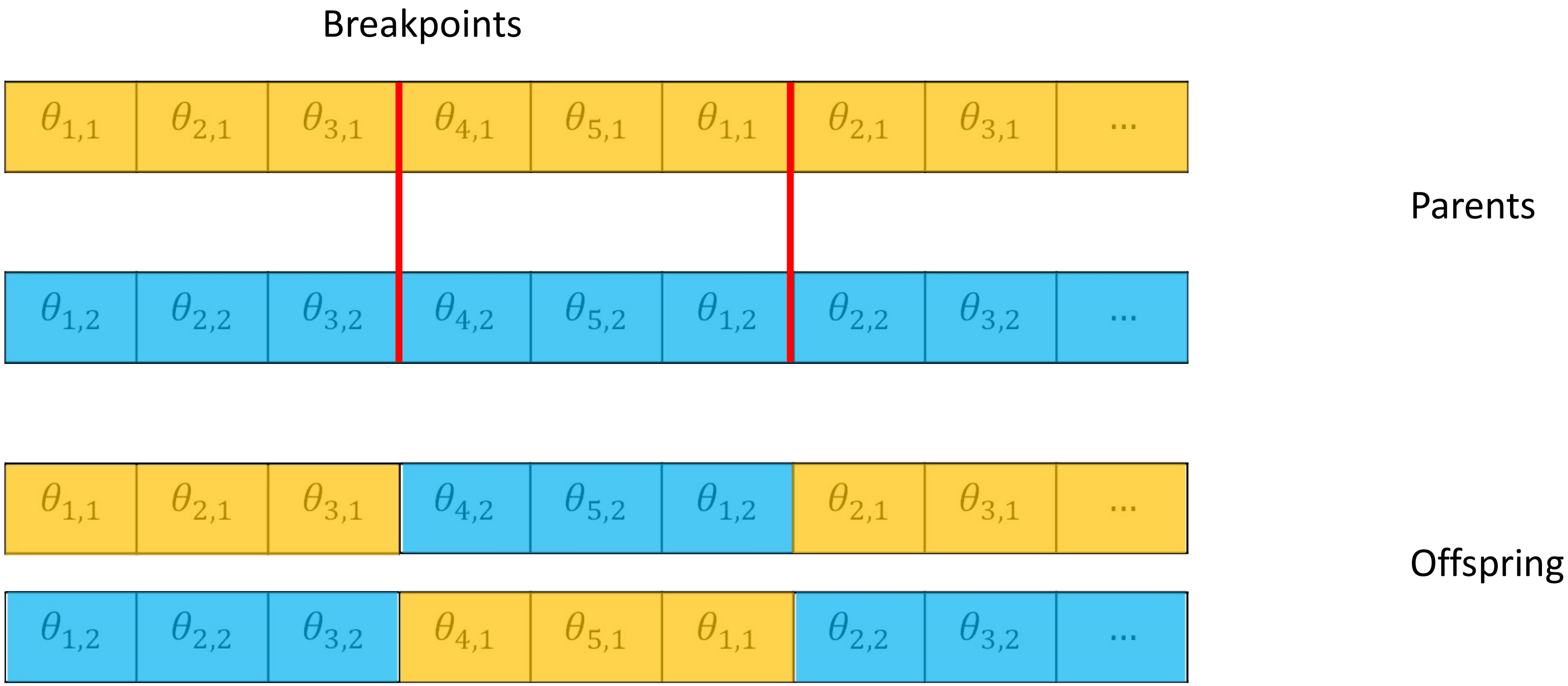
$P(F|C)$

x	$P(x\|C)$
1	
2	
3	
4	
5	
6	

HMM

The occasionally dishonest casino problem

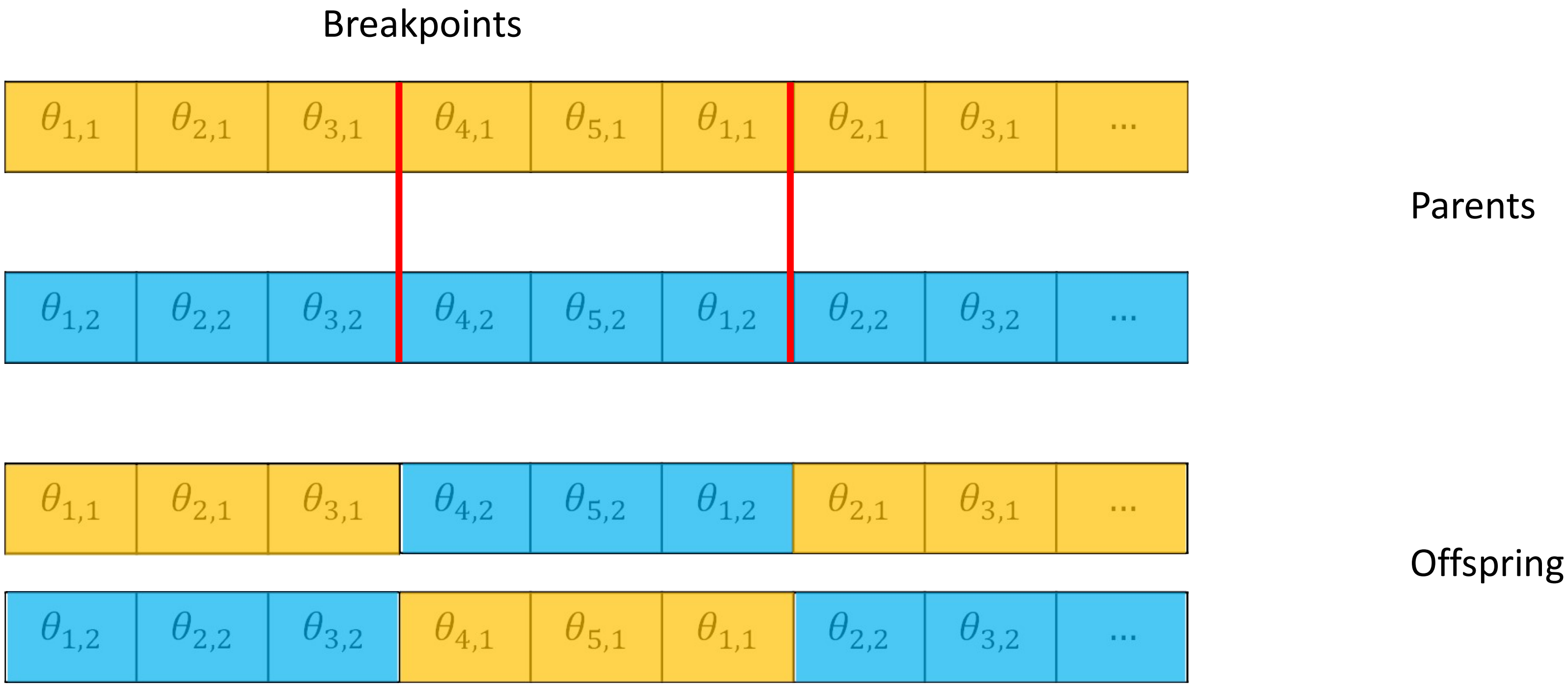
Optimization of a and e using The Genetic Algorithm



HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm



HMM

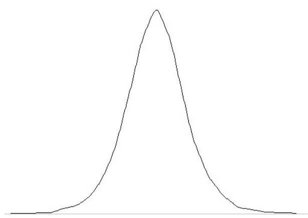
The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

$\theta_{1,2}$	$\theta_{2,2}$	$\theta_{3,2}$	$\theta_{4,1}$	$\theta_{5,1}$	$\theta_{1,1}$	$\theta_{2,2}$	$\theta_{3,2}$	…

Pick one of the variables at random

Mutation

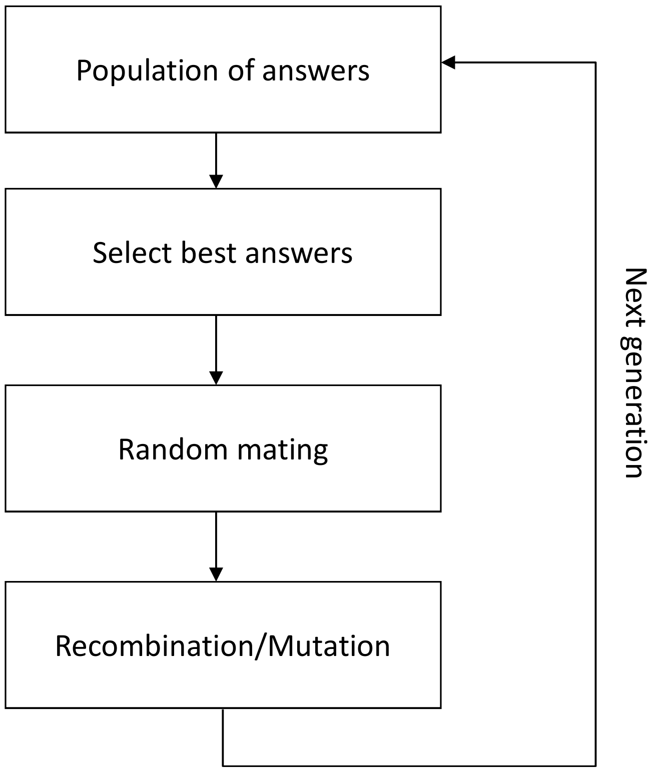


$\theta_{4,1}$

Propose a new $\theta_{4,1}^{t+1}$ using $N\left(\theta_{4,1}^{t}, sd\right)$

Classical Genetic algorithm
Other hyperparameters to consider

Which fitness?



HMM

The occasionally dishonest casino problem

Optimization of a and e using The Genetic Algorithm

Experimen	#1	#2	#3	#4	#5
Genetic Algorithm					
P_{same}	-4.9473	-3.5693	-3.2932	-3.0982	-4.2345
$P_{different}$	-7.4982	-8.9727	-8.6473	-8.5291	-9.1483
Forward-Backward Procedure					
P_{same}	-4.7359	-4.2125	-4.9843	-4.3908	-4.3876
$P_{different}$	-7.2714	-8.6137	-7.5914	-7.7634	-7.1007

Experimen	#6	#7	#8	#9	#10
Genetic Algorithm					
P_{same}	-3.3281	-4.1869	-4.2322	-4.3872	-3.1539
$P_{different}$	-7.5581	-7.6257	-8.6274	-8.7812	-8.3641
Forward-Backward Procedure					
P_{same}	-4.9811	-4.3481	-4.0567	-4.4860	-4.9251
$P_{different}$	-7.3825	-7.7351	-7.9328	-7.7514	-8.2254

HMM

The occasionally dishonest casino problem

Baum–Welch algorithm

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$\mathrm{P}(\mathrm{x}|\mathrm{HMM}{=}a_{rk}; e_k(b))$

Likelihood of an observed sequence given a set of **transition** and **emission** probabilities from a given HMM, computed using the forward algorithm

HMM

Imagine I could establish the probability of observing F,F given the sequence and the observed symbols and a proposed a_{kl}

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,$\boxed{3,6}$,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

π F,F,F,F,F,F,F,C,C,C,C,C,C,C,C,C,F,F,F,F,F,$\boxed{F,F}$,F,F,F,F,C,C,C,C,C,C,C,F,F,F,F,F,F,F,F,F,F,F,F

Which is the probability of observing this sequence given a transition a_{kl} probability and all the observed sequence?

$$P(\pi_i = k; \pi_{i+1} = l|x; HMM) = \frac{P(\pi_i = k|x_1..x_i; HMM)P(x_i|\pi_i = k; HMM)P(\pi_{i+1} = l|\pi_i = k; HMM)P(x_{i+1}|\pi_{i+1} = l; HMM)\ P(\pi_{i+1} = l|x_{i+1} \ldots x_L; HMM)}{P(x)}$$

Probability of observing state k at position I, state l at osition i+1 given the observed sequence of symbols and a proposal of parameters

Emission probability of symbol x at position i using state k

Transition probability of state k to l

Emission probability of symbol x at position i+1 using state l

$$P(\pi_i = k; \pi_{i+1} = l|x; HMM) = \frac{P(\pi_i = k|x_1..x_i; HMM)e_k(b = x_i)a_{kl}e_l(b = x_{i+1})P(\pi_{i+1} = l|x_{i+1} \ldots x_L; HMM)}{P(x)}$$

Knowing that, I would be able to get a new estimate of the expected number of times that I expect to see such combination of k and l and, given that, propose a new a_{kl} to maximize the likelihood of the data given the parameters.

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(\pi_i = k|x) = \frac{P(\pi_i = k, x)}{P(x)}$$

"I want to know the probability of state k at position i given that I have observed the whole sequence x"

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(\pi_i = k, x_1 \ldots x_i) \qquad P(x_{i+1} \ldots x_L | \pi_i = k, x_1 \ldots x_i)$$

$$P(\pi_i = k, x) = P(\pi_i = k, x_1 \ldots x_i) P(x_{i+1} \ldots x_L | \pi_i = k, x_1 \ldots x_i)$$

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(\pi_i = k, x_1 \dots x_i) \qquad P(x_{i+1} \dots x_L | \pi_i = k)$$

Because it only depends on the last element!

$$P(\pi_i = k, x) = P(\pi_i = k, x_1 \dots x_i) P(x_{i+1} \dots x_L | \pi_i = k)$$

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$P(\pi_i = k, x_1 \dots x_i)$ $\quad$ $P(x_{i+1} \dots x_L | \pi_i = k) = b_k(i)$

$$P(\pi_i = k|x) = \frac{P(\pi_i = k)P(x|\pi_i = k)}{P(x)}$$

HMM

The occasionally dishonest casino problem

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

Imagine we have observed this part of the sequence
Imagine we know the parameters of the HMM (A, B)

$$P(X_1, X_2 \dots X_i, \pi_i = l | HMM)$$

"Which is the probability that I observe the sequence of symbols and at position i the state is l given the parameters of my HMM?"

HMM

The occasionally dishonest casino problem

x $\boxed{1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4}$,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

Imagine we have observed this part of the sequence
Imagine we know the parameters of the HMM (A, B)

Transition probability matrix

Emission probability matrix

$$P(X_1, X_2 \ldots X_i, \pi_i = l | HMM)$$

"Which is the probability that I observe the sequence of symbols and at position i the state is l given the parameters of my HMM?"

HMM

Forward algorithm: how to estimate the probability of a sequence ending in state k given a HMM

$$P(X_1, X_2 \dots X_i, \pi_i = l | HMM)$$

"Which is the probability that I observe the sequence of symbols and at position i the state is k given the parameters of my HMM?"

HMM

Forward algorithm: how to estimate the probability of a sequence ending in state k given a HMM

	1	3	4	...
F	0.5*1/6	0.5*1/6*0.99*1/6+ 0.5*1/10*0.1*1/6		
C	0.5*1/10	0.5*1/10*0.9*1/10 + 0.5*1/6*0.01*1/6		

P1

P2

$$\left.\begin{matrix} P1 \\ P2 \end{matrix}\right\} P1+P2 = P(S|HMM)$$

$P(X_1, X_2, \pi_i = F|HMM)$

$P(X_1, X_2, \pi_i = C|HMM)$

HMM

Forward algorithm: how to estimate the probability of a sequence ending in state k given a HMM

$$P(X_1, X_2 \dots X_{i+1}, \pi_{i+1} = l|HMM) = f_l(X_{i+1}) = e_l(X_{i+1}) \sum_{r}^{K} f_r(X_i) a_{rl}$$

Transition probability to move from state *r* to *l*

Emission probability of X at time *i+1* given that the state is l

Probability of the sequence up to *i* , ending at *i* in state *r*

"Which is the probability that I observe the sequence of symbols and at position i the state is k given the parameters of my HMM?"

HMM

Forward algorithm: how to estimate the probability of a sequence ending in state k given a HMM

$e_l(X_{i+1})$

	1	3	4	...
F	0.5*1/6	0.5*1/6*0.99*1/6+ 0.5*1/10*0.1*1/6		
C	0.5*1/10	0.5*1/10*0.9*1/10 + 0.5*1/6*0.01*1/6		

P1

P2

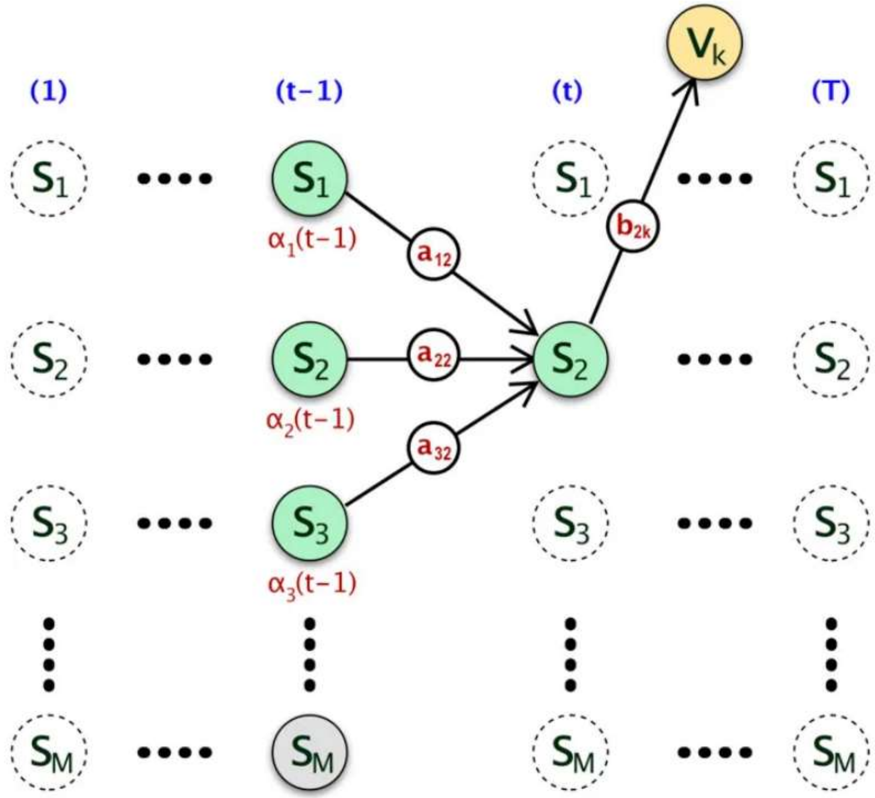
P1+P2 =P(S|HMM)

a_{rl}

$f_r(X_i)$

$$f_l(X_{i+1}) = e_l(X_{i+1}) \sum_{r}^{K} f_r(X_i) a_{rl}$$

Another way of visualizing the forward algorithm



http://www.adeveloperdiary.com/data-science/machine-learning/forward-and-backward-algorithm-in-hidden-markov-model/

HMM

Forward algorithm: how to estimate the probability of a sequence ending in state k given a HMM. Pseudocode (From the book of Durbin)

Algorithm: Forward algorithm

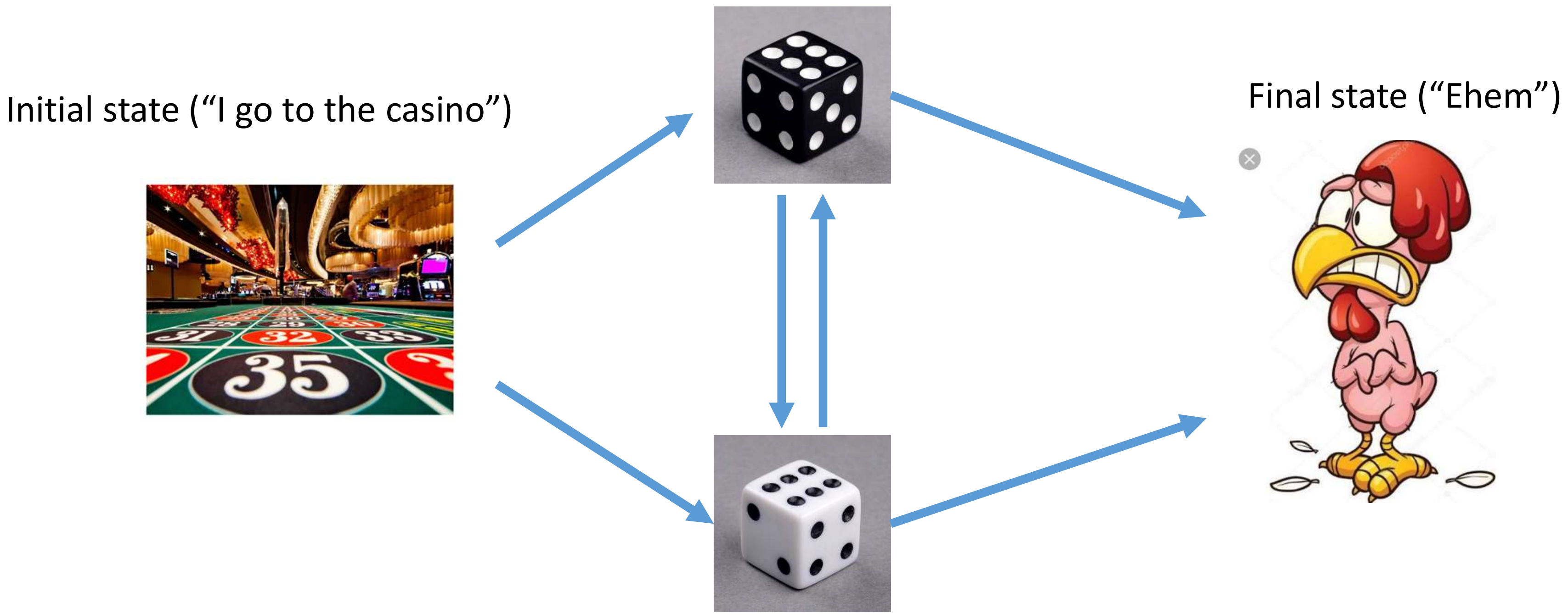
Initialisation ($i = 0$): $\boxed{f_0(0) = 1,\ f_k(0) = 0 \text{ for } k > 0.}$

Recursion ($i = 1 \dots L$): $f_l(i) = e_l(x_i) \sum_k f_k(i-1) a_{kl}.$

Termination: $P(x) = \sum_k f_k(L) a_{k0}.$

HMM

In fact, there is an initial state and a final state



HMM

Forward algorithm: There are two other states!

“After we start the sequence, the probability of going to initial state is 0”

“Before we start the sequence, we are at the initial state”

	NA	1	3	4	…	L+1
Initial State	1	0	0	0	0	0
F	0					0
C	0					0
Final State	0	0	0	0	0	P(S|HMM)

HMM

Forward algorithm: There are two other states!

"After we finish the sequence, the probability akl of going to the final state is 1"

	NA	1	3	4	…	L+1
Initial State	1	0	0	0	0	0
F	0					0
C	0					0
Final State	0	0	0	0	0	P(S|HMM)

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,$\boxed{\text{2,1,3,5,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4}}$

$$P(x_{i+1} \dots x_L | \pi_i = k) = b_k(i)$$

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(x_{i+1} \ldots x_L | \pi_i = k) = b_k(i)$$

	...	5	5	4
F				
C				

Backward algorithm: how to estimate the probability of a sequence given an initial k state

Backward algorithm

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(x_{i+1} \ldots x_L | \pi_i = k) = b_k(i)$$

	…	5	5	4
F				1
C				1

Starting at the last element, the entries in this column denote the probability moving to the end state from each state after generating the entire sequence.

Backward algorithm

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(x_{i+1} \ldots x_L | \pi_i = k) = b_k(i)$$

	...	5	5	4
F				1
C				1

$a_{kr}e_r(X_i)b_k(i)$

$0.95 * \frac{1}{6} * 1$

Backward algorithm

HMM

i

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(x_{i+1} \dots x_L | \pi_i = k) = b_k(i)$$

	…	5	5	4
F			$0.95 * \frac{1}{6} * 1$	1
C				1

$$a_{kr} e_r(X_i) b_k(i)$$

$$0.05 * \frac{1}{6} * 1$$

Backward algorithm

HMM

$$x \quad 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,\overset{i}{}\boxed{2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4}$$

$$P(x_{i+1} \ldots x_L | \pi_i = k) = b_k(i)$$

	…	5	5	4
F			$0.95 * \frac{1}{6} * 1 + 0.05 * \frac{1}{6} * 1$	1
C				1

$$a_{kr} e_r(X_i) b_k(i)$$

$$0.05 * \frac{1}{6} * 1$$

Backward algorithm

HMM

i

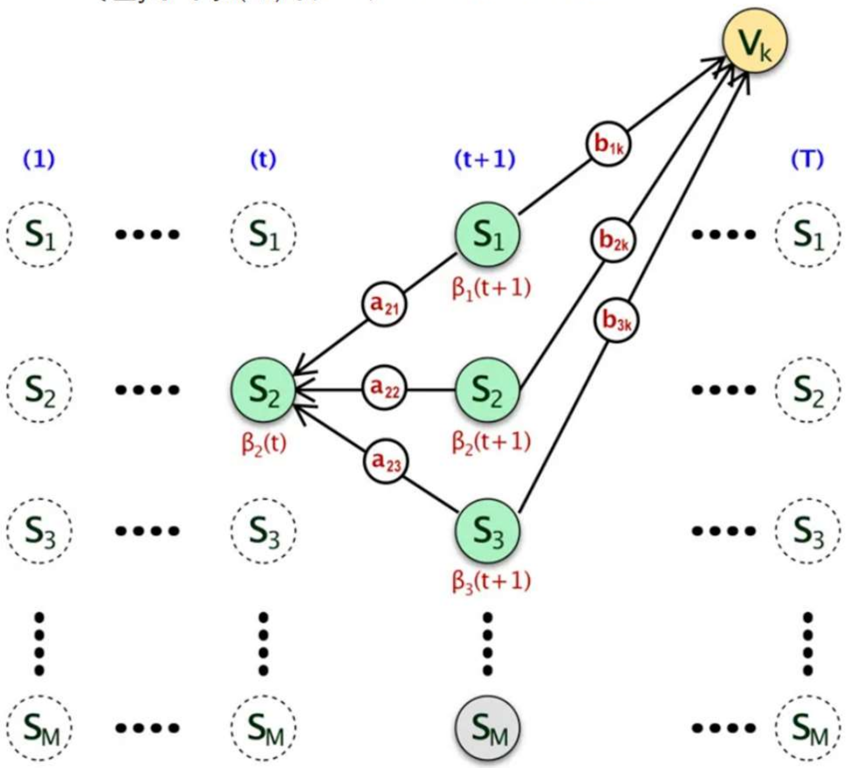
x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

$$P(x_{i+1} \dots x_L | \pi_i = k) = b_k(i)$$

	...	5	5	4
F			$0.95 * \frac{1}{6} * 1 + 0.05 * \frac{1}{6} * 1$	1
C			$0.9 * \frac{1}{10} * 1 + 0.1 * \frac{1}{10} * 1$	1

Another way of visualizing the backward algorithm

$$\beta_i(t) = \begin{cases} 1 & \text{when } t = T \\ \sum_{j=0}^{M} a_{ij} b_{jkv(t+1)} \beta_j(t+1) & \text{when } t \text{ less than } T \end{cases}$$



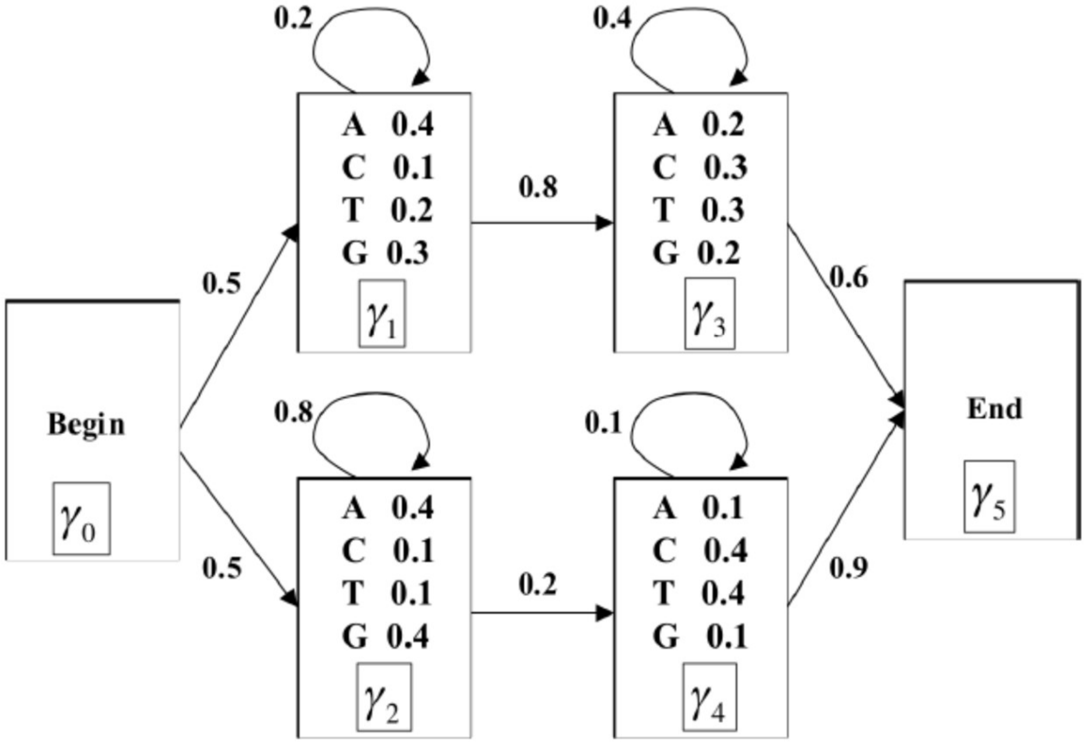
http://www.adeveloperdiary.com/data-science/machine-learning/forward-and-backward-algorithm-in-hidden-markov-model/

Backward algorithm

HMM

Remember: Even if we have not explicitly stated, beginning and end are also states

What is strange in this HMM compared to the ones we have been using?



HMM

Backward algorithm

If our subsequence was TAGA, which is the probability that we have ended at that each stage?

Time Step, t / State, l	0	1 (T)	2 (A)	3 (G)	4 (A)
γ_0	-	-	-	-	**0**
γ_1	-	-	-	-	**0**
γ_2	-	-	-	-	**0**
γ_3	-	-	-	-	**0.6**
γ_4	-	-	-	-	**0.9**
γ_5	-	-	-	-	**0**

Backward algorithm

HMM

Algorithm: Backward algorithm

Initialisation ($i = L$): $b_k(L) = a_{k0}$ for all k.

Recursion ($i = L-1, \ldots, 1$): $b_k(i) = \sum_l a_{kl} e_l(x_{i+1}) b_l(i+1)$.

Termination: $P(x) = \sum_l a_{0l} e_l(x_1) b_l(1)$.

HMM

The occasionally dishonest casino problem

Baum–Welch algorithm

Probability of ending at position i in state k

Probability that I change from k to l

Probability of emission of symbol x at position i+1 using state l

$$P(\pi_i = k, \pi_{i+1} = l | X, HMM) = \frac{f_k(i) a_{kl} e_l(x_{i+1}) b_l(i+1)}{P(X)}$$

Probability of the sequence from i+1 to L if I start at i+1 at state l

Probability of a transition from state k at position i to state l in position i+1

HMM

The occasionally dishonest casino problem

Baum–Welch algorithm

We can do this for each position. Adding over all the positions, we have an estimate of *"how often would occur a transition from state k to l if the parameter a_{kl} that we are using was the one that generated the data"*

$$A_{kl} = \sum_{i}^{L} P(\pi_i = k, \pi_{i+1} = l | X, HMM)$$

$$a_{kl}^{(t+1)} = \frac{A_{kl}^{t}}{\sum_{l'}^{K} A_{kl'}^{t}}$$

"Given that I come from state k, which is the probability of moving to any other state"

Posterior probability of akl(t+1) given akl(t) and the observed data

When the state was not hidden, this was the **maximum likelihood** estimate of the transition akl!

HMM

The occasionally dishonest casino problem

Baum–Welch algorithm

How can we estimate the emission probabilities?

x 1,4,5,1,3,2,1,6,6,4,1,6,6,6,6,6,2,3,4,5,1,4,3,6,2,1,3,5,6,6,6,6,6,6,6,1,2,3,1,1,1,1,2,3,4,5,5,4

Pick one of the symbols (for example, b = 6)

The index i of this position is 8

$$E_k(b) = \frac{1}{P(x)} \sum_{\{i|x_i = b\}} f_k(i) b_k(i)$$

Probability that we are at state k in position i

For all the positions where we observe 6

New emission probabilities

$$e_k^{t+1}(b) = \frac{E_k(b)}{\sum_{b'} E_k(b')}$$

HMM

Baum-Welch algorithm

1) Propose a set of transition ($a_{..}$) and emission ($b_{.}(.)$) matrices
2) Until change of likelihood $\mathrm{P(X|HMM)}$ is not smaller than threshold, do
3) Using the forward and backward algorithm, estimate the new $\mathrm{A}_{..}$ and $\mathrm{B}_{..}$ variables
4) Estimate the new $\mathrm{a}_{..}$ and $\mathrm{b}_{.}(.)$ matrices
5) Go to 2)

The Baum-Welch is a special case of the EM algorithm

- Two steps
  - Expectation: missing data are estimated given the observed data and current estimate of the model parameters.
  - Maximization: the likelihood function is maximized under the assumption that the missing data are known.

How does a phenotype change over a phylogeny?

Identifying Hidden Rate Changes in the Evolution of a Binary Morphological Character: The Evolution of Plant Habit in Campanulid Angiosperms
JEREMY M. BEAULIEU1,∗, BRIAN C. O'MEARA2, AND MICHAEL J. DONOGHUE1

Model adequacy is a different concern than model fit: the latter determines which of a set of models is least bad (or "best") for the data, whereas the former determines whether a single model adequately describes the data. For example, a model that says humans, chimps, and mice all diverged simultaneously 70 million years ago is a better fit to the data than one that puts that divergence 700 million years ago, but neither model adequately describes the data. Adequacy is often evaluated by simulating under the focal model to see if it generates data indistinguishable from the empirical data using one or more measures.