# MATHEMATICS FOR SCIENCE STUDENTS

**An open-source book**


Written, illustrated and typeset (mostly) by

**PELEG BAR SAPIR**

with contributions from others


$$a^b = e^{b\log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha\vec{u}+\beta\vec{v}\right) = \alpha T\left(\vec{u}\right)+\beta T\left(\vec{v}\right)$$

$$A = Q\Lambda Q^{-1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

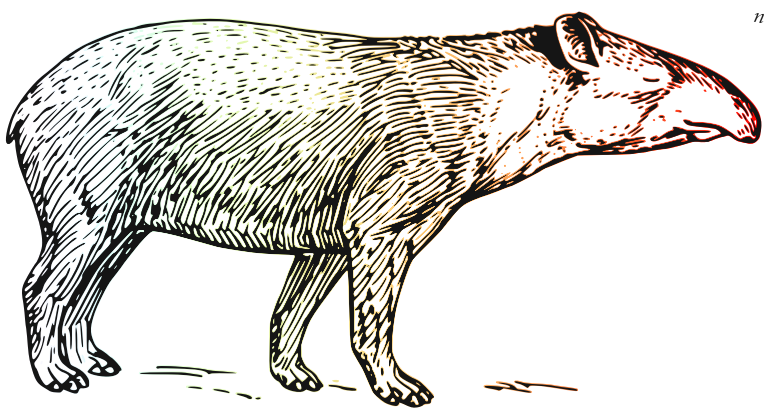
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$e^{\pi i}+1=0$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

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Rights, lefts, etc. will be written here in the future

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# CHAPTER 0

# INTRODUCTION

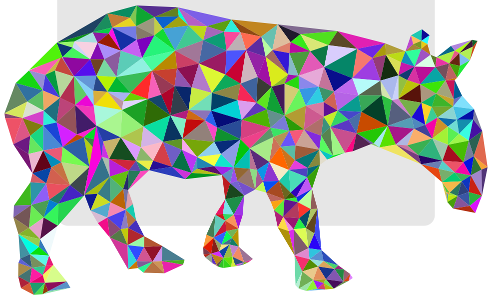
In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

**Note 0.1 In case you are already familiar with the topics**

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

CHAPTER

1



# LINEAR ALGEBRA
(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only

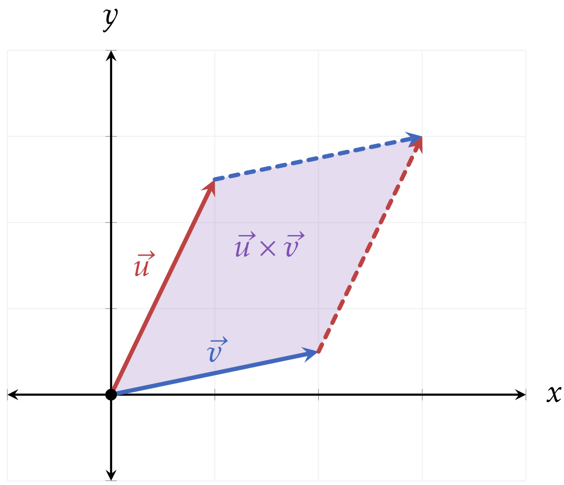



**Figure 1.1** The cross product in $\mathbb{R}^2$ of two vectors $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ as the signed area of the parallogram defined by the vectors.

then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

## 1.1 VECTORS

### 1.1.1 The cross product

Another commonly used product of two vectors is the so-called **cross product**. Unlike the dot product, it is only really valid in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^7$, of which we will focus on $\mathbb{R}^3$ and touch a bit on its uses in $\mathbb{R}^2$. Also in contrast to the dot product, the cross product in $\mathbb{R}^3$ results in a vector rather than a scalar - therefore the product is sometimes known as the **vector product**. The cross product uses the notation $\vec{a} \times \vec{b}$, from which it derives its name.

We start with the definition of the cross product in $\mathbb{R}^2$: the cross product of two vectors $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ is the (signed) area of the parallelogram defined by the two vectors (see Figure 1.1).

The value of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \left\| \vec{u} \right\| \left\| \vec{v} \right\| \sin\left(\theta\right), \tag{1.1.1}$$

where $\theta$ is the angle between the vectors. This is extremely similar to the scalar product, and we can use this fact to find how to calculate the cross product from vectors in column form: if we replace $\vec{u}$ by a vector orthogonal to it, denoted by $\vec{u}_{\perp}$, the cross product is then

$$\vec{u}\times\vec{v}=\left\|\vec{u}_{\perp}\right\|\left\|\vec{v}\right\|\sin\left(\theta+\frac{\pi}{2}\right), \tag{1.1.2}$$

since the angle between $\vec{u}_{\perp}$ and $\vec{v}$ is $\frac{\pi}{2}$ more than that between $\vec{u}$ and $\vec{v}$. Using the fact that $\sin\left(\theta+\frac{\pi}{2}\right)=\cos(\theta)$, we get the equality

$$\begin{aligned}\vec{u}\times\vec{v}&=\left\|\vec{u}_{\perp}\right\|\left\|\vec{v}\right\|\sin\left(\theta+\frac{\pi}{2}\right)\\&=\left\|\vec{u}_{\perp}\right\|\left\|\vec{v}\right\|\cos(\theta)\\&=\vec{u}_{\perp}\cdot\vec{v}.\end{aligned} \tag{1.1.3}$$

In $\mathbb{R}^2$, any vector $\vec{u}=\begin{bmatrix}a\\b\end{bmatrix}$ has two vectors orthogonal to it: $\begin{bmatrix}-b\\a\end{bmatrix}$ and $\begin{bmatrix}b\\-a\end{bmatrix}$. Choosing the former gives

$$\vec{u}\times\vec{v}=\begin{bmatrix}-b\\a\end{bmatrix}\cdot\begin{bmatrix}c\\d\end{bmatrix}=-bc+ad, \tag{1.1.4}$$

while choosing the latter gives

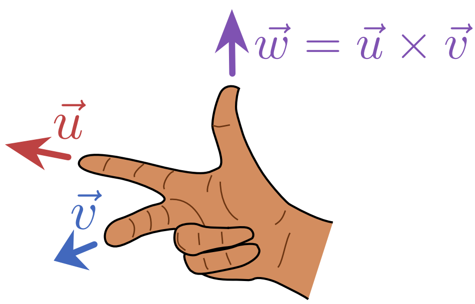
$$\vec{u}\times\vec{v}=\begin{bmatrix}b\\-a\end{bmatrix}\cdot\begin{bmatrix}c\\d\end{bmatrix}=bc-ad. \tag{1.1.5}$$

These two forms are the opposite of each other - i.e. if one yields the value 4, the other yields the value $-4$. We will see which one is used in a moment.

On to $\mathbb{R}^3$: geometrically, the cross product of two vectors $\vec{u},\vec{v}\in\mathbb{R}^3$ is defined as a **vector** $\vec{w}\in\mathbb{R}^3$ which is **orthogonal to both** $\vec{u}$ and $\vec{v}$, and with norm of the same magnitude as the product would have in $\mathbb{R}^2$, i.e.

$$\left\|\vec{w}\right\|=\left\|\vec{u}\right\|\left\|\vec{v}\right\|\sin(\theta). \tag{1.1.6}$$

The direction of $\vec{u}\times\vec{v}$ is determined by the **right-hand rule**: using a person's right hand, when $\vec{u}$ points in the direction of their index finger and $\vec{v}$ points in the direction of their middle finger, then vector $\vec{w}=\vec{u}\times\vec{v}$ points in the direction of their thumb:



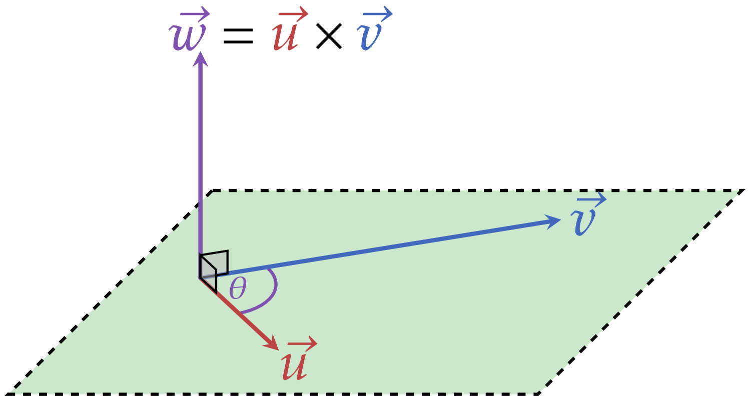



**Figure 1.2** The cross product of the vectors $\vec{u}$ and $\vec{v}$ relative to the plane spanned by the two vectors.

The cross product is **anti-commutative**, i.e. changing the order of the vectors results in inverting the product:

$$\vec{u} \times \vec{v} = -\left(\vec{v} \times \vec{u}\right).$$

When the vectors are given as column vectors, $\vec{u} = \begin{bmatrix} u^x \\ u^y \\ u^z \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v^x \\ v^y \\ v^z \end{bmatrix}$, the resulting cross product is

$$\vec{u} \times \vec{v} = \begin{bmatrix} u^y v^z - u^z v^y \\ u^z v^x - u^x v^z \\ u^x v^y - u^y v^x \end{bmatrix}. \tag{1.1.7}$$

Another way of writing the cross product is by using the **Levi-Civita** symbol $\varepsilon_{ijk}$, which is a sort of a generalization of the Kronecker delta to three indices $i, j, k \in (1, 2, 3)$. To understand the Levi-Civita symbol we first must understand the idea of **permutations**.

Given a tuple of $n$ different objects which we label $1, 2, \dots, n$ - a permutation of the tuple is a re-arrangement of the tuple's objects. For example, given the tuple $A = (1, 2, 3, 4)$, we can rearrange its elements to form the tuple $B = (2, 4, 3, 1)$:

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rest of permutations

The Levi-Civita symbol is defined as follows: given the three indices $i, j$ and $k$,

- If they form an **even permutation** of $(1, 2, 3)$ then $\varepsilon = 1$.
- If they form an **odd permutation** of $(1, 2, 3)$ then $\varepsilon = -1$.

- If any two of the indices (or all) are equal then $\varepsilon = 0$.

For example: $\varepsilon_{231} = 1$ since $(2,3,1)$ is an even permutation of $(1,2,3)$. $\varepsilon_{132}$=-1 since $(1,3,2)$ is an odd permutation of $(1,2,3)$. $\varepsilon_{122} = \varepsilon_{113} = \varepsilon_{333} = \varepsilon_{121}$ since in all these cases there are at least two equal indices.

Finally we get to the use of the Levi-Civita symbol in the definition of the cross product: if we use an orthonormal basis set $B = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ which is ordered according to the right-hand rule (for example the standard basis set), then the cross product of the two vectors

$$\vec{a} = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix},$$

is

$$\vec{c} = \vec{a} \times \vec{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a^j b^k. \tag{1.1.8}$$

This probably looks somewhat daunting, so let's take it step-by-step: the component $i$ of the resulting vector $\vec{c} = \vec{a} \times \vec{b}$ is

$$c^i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a^j b^k. \tag{1.1.9}$$

So for example the $x$-component $(i = 1)$ is

$$\begin{aligned}
c^x &= \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} a^j b^k \\
&= \sum_{j=1}^{3} \left( \varepsilon_{1j1} a^j b^1 + \varepsilon_{1j2} a^j b^1 + \varepsilon_{1j3} a^j b^1 \right) \\
&= \cancel{\varepsilon_{111}} a^1 b^1 + \cancel{\varepsilon_{112}} a^1 b^1 + \cancel{\varepsilon_{113}} a^1 b^1 + \cancel{\varepsilon_{121}} a^2 b^1 + \cancel{\varepsilon_{122}} a^2 b^1 + \varepsilon_{123} a^2 b^1 + \cancel{\varepsilon_{131}} a^3 b^1 + \varepsilon_{132} a^3 b^1 + \cancel{\varepsilon_{133}} a^3 b^1 \\
&= a^2 b^1 - a^3 b^1.
\end{aligned}$$