MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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$$a^b = e^{b\log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$

$$A = Q\Lambda Q^{-1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

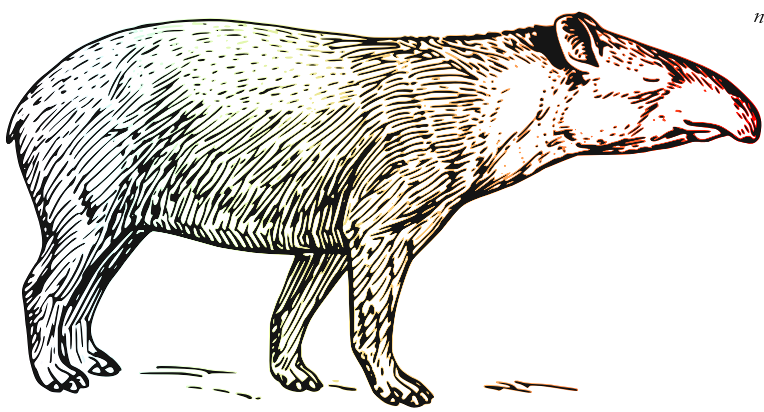
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$e^{\pi i} + 1 = 0$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



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HERE BE TABLE

CHAPTER 0

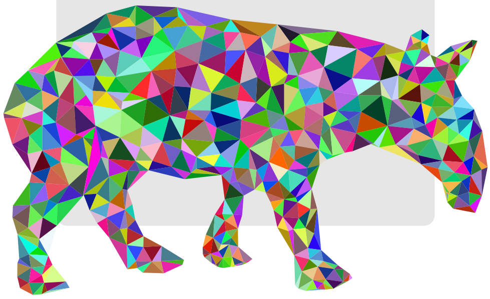
INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

CHAPTER 1



LINEAR ALGEBRA
(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

1.1 COVECTORS

1.1.1 Basics

In the matrix section we learned that any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by the product of a matrix with m rows and n columns and a vector in $\mathbb{R}^n$ (in this order). The special case were $m = 1$ - i.e. a matrix with a single row - has some interesting properties which we will explore in this section.

We start with a definition: given a space $\mathbb{R}^n$, a linear transformation of the type $T : \mathbb{R}^n \to \mathbb{R}$ is called a **dual vector** and is represented by a row of n real numbers (refered to as a **row vector**):

$$\underline{v} = \left[v_1,\ v_2,\ \cdots,\ v_n\right]. \tag{1.1.1}$$

(note that the components of $\underline{v}$ are deoted with a lower-index notation, as opposed to the components of vectors)

The set of all dual vectors for a given space $\mathbb{R}^n$ is called the **dual space** of $\mathbb{R}^n$.

1.1.2 Inner and outer products

The application of a dual vector $\underline{u}$ on a vector $\vec{v}$ is equal to the product of a matrix row representing $\underline{i}$ with a matrix column representing $\vec{v}$ - i.e. it is simply the inner product of $\underline{u}$ with $\vec{v}$:

$$\underline{u}\left(\vec{v}\right) = \underline{u} \cdot \vec{v} = \left[u_1,\ u_2,\ \cdots,\ u_n\right]\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix} = \sum_{k=1}^{n} u_k v^k = u_k v^k. \tag{1.1.2}$$

However, by just using this method we run into a problem: since the inner product is commutative we expect that $\underline{u} \cdot \vec{v} = \vec{v} \cdot \underline{u}$, but the latter gives

$$\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}\left[u_1,\ u_2,\ \cdots,\ u_n\right].$$

This represents a product of an $n \times 1$ matrix with a $1 \times n$ matrix, which should be an $m \times m$ matrix. Indeed, we actually call this the **outer product**. Since for $n > 1$ an $n \times n$ matrix isn't equal any scalar, we have an issue.

To circumvent this problem we define the inner product of a vector and a dual vector such that the object on the left is always represented by a row vector, and the object on the right is always represented by a column vector. So if we want to calculate $\underline{u} \cdot \vec{v}$ everything is ok, but if we want to calculate $\vec{v} \cdot \underline{u}$ we need to **transpose** both representations:

$$\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}^\top = \begin{bmatrix} v_1, \ v_2, \ \cdots, \ v_n \end{bmatrix}, \quad \begin{bmatrix} u_1, \ u_2, \ \cdots, \ u_n \end{bmatrix}^\top = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^n \end{bmatrix}. \tag{1.1.3}$$

1.1.3 Geometric visualization

1.1.4 Basis sets and duality

1.1.5 Covariant vs. contravariant vectors