# A structured random effects introdution, the GMRFs

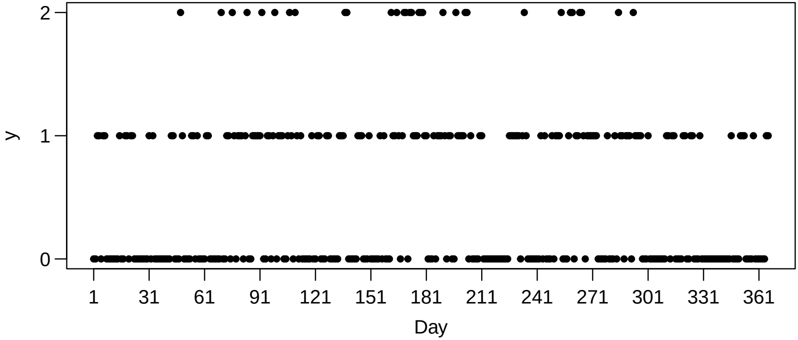
Elias T. Krainski

April 2022

# Motivating examples

# Tokyo example

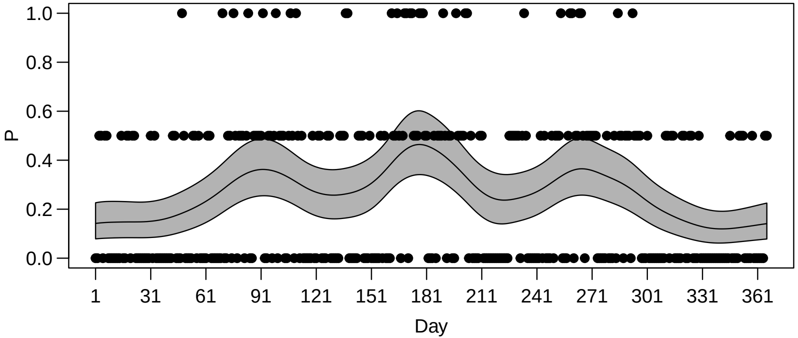
Rainfall over 1 mm in the Tokyo area for each calendar day during two years (1983-84) are registered.




**Problem:** model the probability of rain each day of the year

# Tokyo example (fit)

Rainfall over 1 mm in the Tokyo area for each calendar day during two years (1983-84) are registered.




The probability of rain each day of the year

# Scotland example (from WinBUGS)

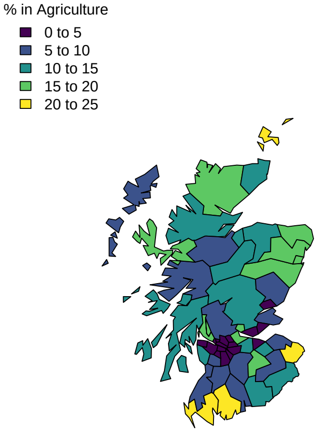
The rates of lip cancer in 56 counties in Scotland have been analysed by Clayton and Kaldor (1987) and Breslow and Clayton (1993). The form of the data includes the observed and expected cases (expected numbers based on the population and its age and sex distribution in the county), a covariate measuring th percentage of the population engaged in agriculture, fishing, or forestry, and the "position" of each county expressed as a list of adjacent counties.

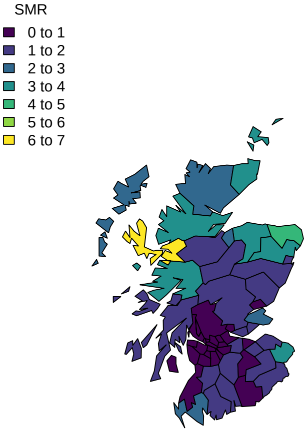
| County | Observed cases $O_i$ | Expected cases $E_i$ | Percentage in agric. $x_i$ | SMR | Adjacent counties |
|---|---|---|---|---|---|
| 1 | 9 | 1.4 | 16 | 652.2 | 5,9,11,19 |
| 2 | 39 | 8.7 | 16 | 450.3 | 7,10 |
| ... | ... | ... | ... | ... | ... |
| 56 | 0 | 1.8 | 10 | 0.0 | 18,24,30,33,45,55 |

$$O_i \sim \text{Poisson}(\mu_i)$$

$$\log \mu_i = \quad \text{og}\ E_i + \alpha_0 + \alpha_1 x_i / 10 + b_i$$

# Scotland maps



# Scotland data: GLM

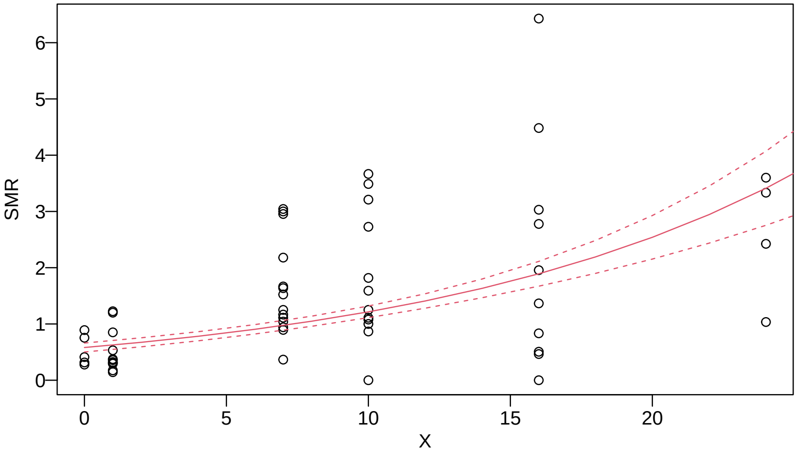
Number of **O**bserved cases as a function of the percentage working in agriculture (**X**)

```
m1 <- glm(O ~ X, poisson, offset=log(E), data=map@data)
summary(m1)
##
## Call:
## glm(formula = O ~ X, family = poisson, data = map@data, offset = log(E))
##
## Deviance Residuals:
##    Min      1Q  Median      3Q     Max
## -4.763  -1.216   0.097   1.336   4.713
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -0.54227    0.06952    -7.8  6.2e-15 ***
## X            0.07373    0.00596    12.4  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 380.73  on 55  degrees of freedom
## Residual deviance: 238.62  on 54  degrees of freedom
## AIC: 450.6
##
## Number of Fisher Scoring iterations: 5
```

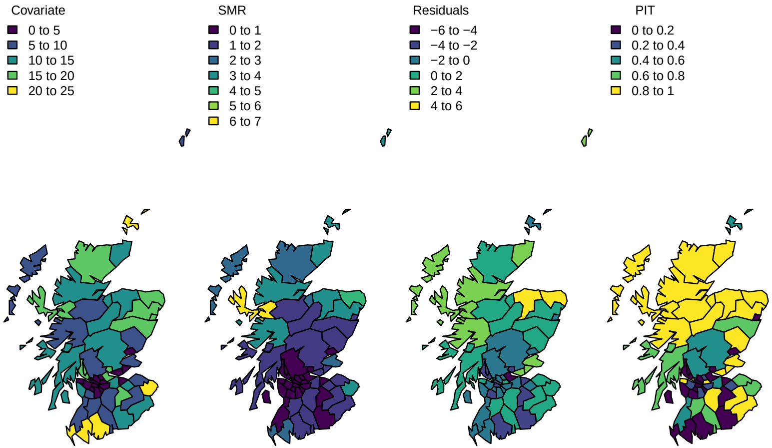
# Scotland data: model 1

```
r1 <- inla(O ~ X, family='poisson', offset=log(E), data=map@data,
          control.compute=list(cpo=TRUE))
summary(r1)
##
## Call:
##    c("inla.core(formula = formula, family = family, contrasts = contrasts,
##    ", " data = data, quantiles = quantiles, E = E, offset = offset, ", "
##    scale = scale, weights = weights, Ntrials = Ntrials, strata = strata,
##    ", " lp.scale = lp.scale, link.covariates = link.covariates, verbose =
##    verbose, ", " lincomb = lincomb, selection = selection, control.compute
##    = control.compute, ", " control.predictor = control.predictor,
##    control.family = control.family, ", " control.inla = control.inla,
##    control.fixed = control.fixed, ", " control.mode = control.mode,
##    control.expert = control.expert, ", " control.hazard = control.hazard,
##    control.lincomb = control.lincomb, ", " control.update =
##    control.update, control.lp.scale = control.lp.scale, ", "
##    control.pardiso = control.pardiso, only.hyperparam = only.hyperparam,
##    ", " inla.call = inla.call, inla.arg = inla.arg, num.threads =
##    num.threads, ", " blas.num.threads = blas.num.threads, keep = keep,
##    working.directory = working.directory, ", " silent = silent, inla.mode
##    = inla.mode, safe = FALSE, debug = debug, ", " .parent.frame =
##    .parent.frame)")
## Time used:
##     Pre = 0.692, Running = 0.208, Post = 0.0168, Total = 0.916
## Fixed effects:
##               mean    sd 0.025quant 0.5quant 0.975quant mode kld
## (Intercept) -0.542 0.070     -0.680   -0.541     -0.408   NA   0
## X            0.074 0.006      0.062    0.074      0.085   NA   0
##
## Marginal log-Likelihood:  -234.10
## CPO, PIT is computed
## Posterior summaries for the linear predictor and the fitted values are computed
## (Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')
```

# The fitted covariate effect



# After the covariate effect, is there something left?



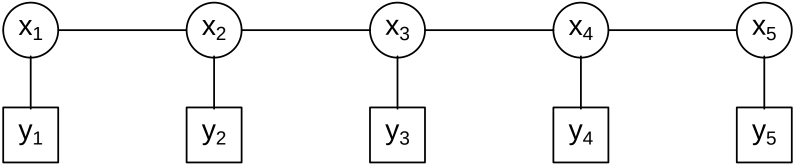
# Structured random effects

# Smoothed probability over time

- Temporally smooth probability of rain
  - is different for each day but similar for nearby days
    - $p_i$ is similar to $p_{i+1}$
    - assume $\text{logit}(p_i) = x_i$

# Smoothed probability over time

- Temporally smooth probability of rain
  - is different for each day but similar for nearby days
    - $p_i$ is similar to $p_{i+1}$
    - assume $\text{logit}(p_i) = x_i$




- dependence on $x$
- $y$ conditionally independent given $x$
  - $y_i$ conditional on $x_i$ is independent of $y_{i-1}$ and of $y_{i+1}$

# The RW1 prior

- It seems natural to borrow strength over time.
  - **x**: smoothing over time
  - ***R**andon **W**alk* - RW of first order: `rw1`
  - Gaussian distribution for the successive differences (**R** esparse)

$$\Delta_i = x_i - x_{i-1} \sim N(0, \tau^{-1})$$

# The RW1 prior

- It seems natural to borrow strength over time.
  - **x**: smoothing over time
  - ***R**andon **W**alk* - RW of first order: `rw1`
  - Gaussian distribution for the successive differences (**R** esparse)

$$\Delta_i = x_i - x_{i-1} \sim N(0, \tau^{-1})$$

- The log of the (joint) distribution for **x** is

$$\log(\pi(\mathbf{x}|\tau)) \propto -\frac{\tau}{2}\sum_{i=2}^{n}(x_i - x_{i-1})^2 = -\frac{\tau}{2}\mathbf{x}'\mathbf{R}\mathbf{x},$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 1 \end{bmatrix}$$

# The cyclic RW1

- 1st of January is similar to December, 31: cyclic random walk

$$\begin{aligned}\pi(\mathbf{x}|\theta) &\propto \exp\left\{-\frac{\theta}{2}\left[(x_1 - x_n)^2 + \sum_{i=2}^{n}(x_i - x_{i-1})^2\right]\right\} \\ &= \exp\left\{-\frac{\theta}{2}\mathbf{x}^T\mathbf{R}\mathbf{x}\right\}\end{aligned}$$

where, now,

$$\mathbf{R} = \begin{bmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & & \ddots & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{bmatrix}$$

- Cyclic second order is analogous.

# Tokyo example: the model

- $y_i$ assume values 0, 1 or 2, for $i = 1, ..., n$
  - assuming conditional independence, thus

$$y_i | p_i \sim \mathrm{Binomial}(n_i, p_i)$$

  - link function (logit)

$$p_i = 1/(1 + \exp(-x_i))$$

# Tokyo example: the model

- $y_i$ assume values 0, 1 or 2, for $i = 1, ..., n$
  - assuming conditional independence, thus

$$y_i | p_i \sim \text{Binomial}(n_i, p_i)$$

  - link function (logit)

$$p_i = 1/(1 + \exp(-x_i))$$

- $\mathbf{x}$ is a *Gaussian Markov Random Field* - GMRF, Rue and Held (2005)

# Tokyo example: the model

- $y_i$ assume values 0, 1 or 2, for $i = 1, ..., n$
  - assuming conditional independence, thus

$$y_i | p_i \sim \mathrm{Binomial}(n_i, p_i)$$

  - link function (logit)

$$p_i = 1/(1 + \exp(-x_i))$$

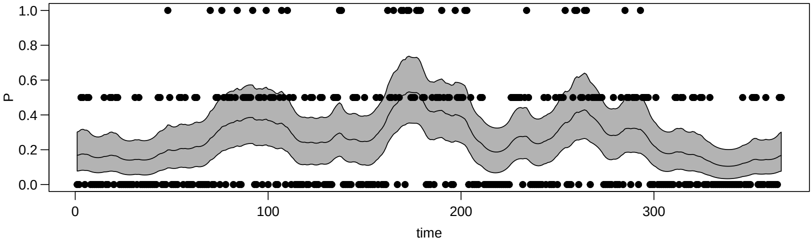
- $\mathbf{x}$ is a *Gaussian Markov Random Field* - GMRF, Rue and Held (2005)
- $\tau$: local precision parameter

# Model fit in INLA

$$\begin{aligned} y_i | x_i &\sim \text{Binomial}(2, p_i) && \rightarrow \text{likelihood} \\ \mathbf{x} | \tau &\sim N(\mathbf{0}, (\tau \mathbf{R})^{-}) && \rightarrow \text{latent field, GMRF} \\ \tau &\sim p(\tau) && \rightarrow \text{prior distribution} \end{aligned}$$

```
head(Tokyo, 5)
##   y n time   P
## 1 0 2    1 0.0
## 2 0 2    2 0.0
## 3 1 2    3 0.5
## 4 1 2    4 0.5
## 5 0 2    5 0.0
model <- y ~ f(time, model='rw1', cyclic=TRUE)
result <- inla(model, family='binomial',
               data=Tokyo, Ntrials=n,
               control.compute=list(cpo=TRUE))
```

# Result for the time series



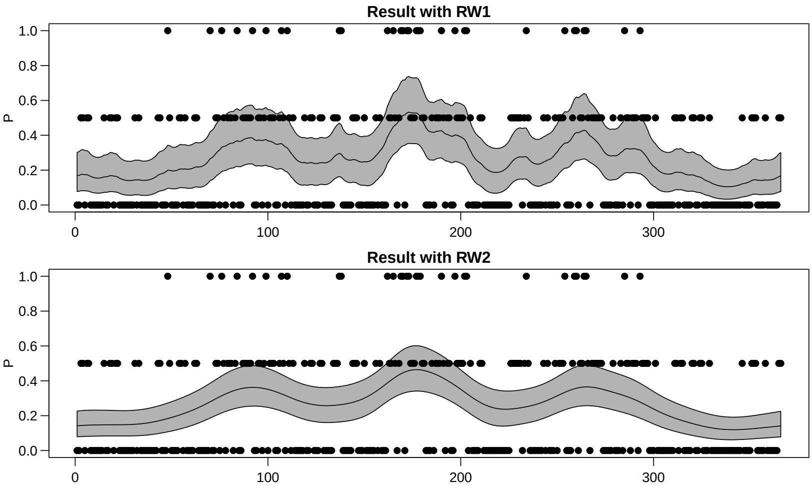
# Smoothing more

Gaussian distribution for the second order differences (`rw2`)

$$\Delta_i^2 = x_i - 2x_{i-1} + x_{i-2} \sim N(0, \tau^{-1})$$

```
model2 <- y ~ f(time, model='rw2', cyclic=TRUE)
result2 <- inla(model2, family='binomial',
                data=Tokyo, Ntrials=n,
                control.compute=list(cpo=TRUE))
```

# Both results for the time series



# Smooth areal dependent risk

$$y_i \sim \mathrm{Poisson}(E_i r_i)$$

$$\log(r_i) = \alpha + \beta X_i + s_i$$

where $s_i$ may be spatial smooth

$s_i | s_j$, $j$ the index for the neighbours of $i$

# Besag: randon walk over areas, Besag (1974)

$$\pi(x_i|\mathbf{x}_{-i},\tau) \sim N(\frac{1}{n_i}\sum_{j\sim i} x_j, \frac{1}{n_i\tau})$$

where $j \sim i$ means $j$ *neighbour of* $i$. This gives:

# Besag: randon walk over areas, Besag (1974)

$$\pi(x_i|\mathbf{x}_{-i},\tau) \sim N(\frac{1}{n_i}\sum_{j\sim i} x_j, \frac{1}{n_i\tau})$$

where $j \sim i$ means $j$ *neighbour of* $i$. This gives:

$$\pi(\mathbf{x}|\tau) \quad \propto \quad \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\sum_{i}^{n}(x_i - \frac{1}{n_i}\sum_{j\sim i} x_j)^2\right)$$

# Besag: randon walk over areas, Besag (1974)

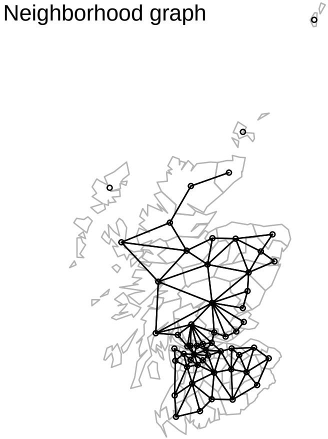
$$\pi(x_i|\mathbf{x}_{-i},\tau) \sim N(\frac{1}{n_i}\sum_{j\sim i} x_j, \frac{1}{n_i\tau})$$

where $j \sim i$ means $j$ *neighbour of* $i$. This gives:

$$\begin{aligned}
\pi(\mathbf{x}|\tau) &\propto \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\sum_{i}^{n}(x_i - \frac{1}{n_i}\sum_{j\sim i} x_j)^2\right) \\
&= \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\sum_{j\sim i}^{n}(x_i - x_j)^2\right) = \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\mathbf{x}^T\mathbf{R}\mathbf{x}\right)
\end{aligned}$$

$$\mathbf{R}_{ij} = \begin{cases} n_i & \text{if } i = j \\ -1 & \text{if } j \sim i \\ 0 & \text{otherwise} \end{cases}.$$

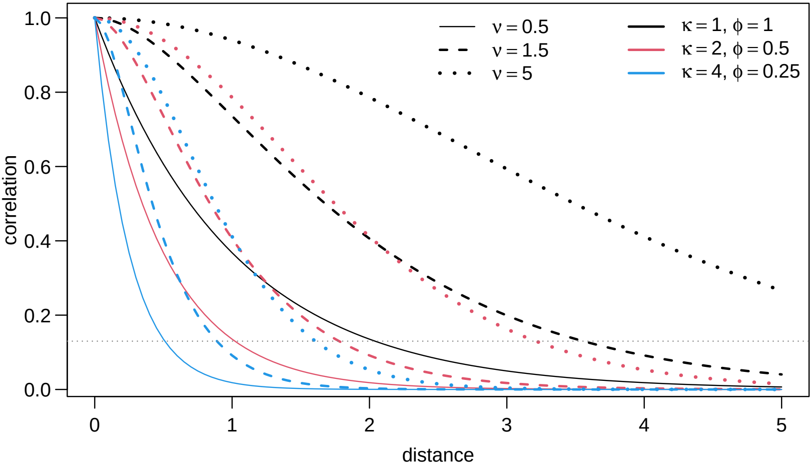
# The Scotland graph

# The SPDE modeling approach

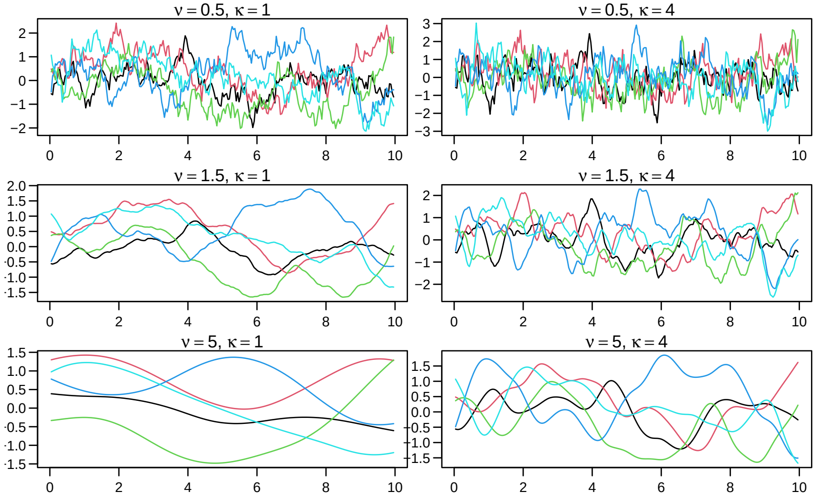
# The Matérn covariance

$$\Sigma_{ij} = \sigma_x^2 \frac{2^{1-\nu} K_\nu(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)}{\Gamma(\nu)(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)^{-\nu}},\ \kappa = 1/\phi$$




corr $((8\nu)^{1/2}/\kappa) \approx 0.13$

# Simulations, 1D, $\sigma_x^2 = 1$



# The Stochastic Partial Differential Approach - SPDE

- Fields with Matérn covariance are solutions to (SPDE)

$$(\kappa^2 - \Delta)^{\alpha/2}\xi(\mathbf{s}) = \tau\mathcal{W}(\mathbf{s})$$

  - $\kappa > 0$: scale parameter
  - $\alpha = \nu + d/2$: smoothness
  - $\Delta$ is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

- See Whittle (1954) and Lindgren, Rue, and Lindström (2011)

# Regular grid, $d = 2$

- $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
- $\mathbf{C} = \mathbf{I}$, $\mathbf{G}$ = Laplacian (4 neighbours)
  - Laplacian-local pattern:

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

  - $\mathbf{Q}_{1,\kappa}$-local pattern

$$\begin{bmatrix} & -1 & \\ -1 & 4+\kappa^2 & -1 \\ & -1 & \end{bmatrix}$$

# Regular grid, $d = 2$

- $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
- $\mathbf{C} = \mathbf{I}$, $\mathbf{G} =$ Laplacian (4 neighbours)
  - Laplacian-local pattern:

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

  - $\mathbf{Q}_{1,\kappa}$-local pattern

$$\begin{bmatrix} & -1 & \\ -1 & 4+\kappa^2 & -1 \\ & -1 & \end{bmatrix}$$

- $\kappa$ is a scale parameter
  - $\rightarrow$ Sparse precision $\mathbf{Q}$ {**!!!**}
  - remember: $(\kappa^2 - \Delta)^{\alpha/2}\xi(\mathbf{s}) = \tau\mathcal{W}(\mathbf{s})$
  - $\rightarrow \quad (\mathbf{Q}_{1,\kappa})^{1/2}\xi =$ independent noise
  - 'effective' range (0.139) $\approx \sqrt{8\nu/\kappa}$

# Important fact: role of $\alpha$

- Bigger $\alpha \rightarrow \mathbf{Q}$ less sparse $\rightarrow$ smoother
  - $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
  - $\alpha = 2$: $\mathbf{Q}_{2,\kappa} = \mathbf{K}_\kappa\mathbf{C}^{-1}\mathbf{K}_\kappa$
  - $\alpha = 3, 4, \ldots$: $\mathbf{Q}_{\alpha,\kappa} = \mathbf{K}_\kappa\mathbf{C}^{-1}\mathbf{Q}_{\alpha-2,\kappa}\mathbf{C}^{-1}\mathbf{K}_\kappa$

# Important fact: role of $\alpha$

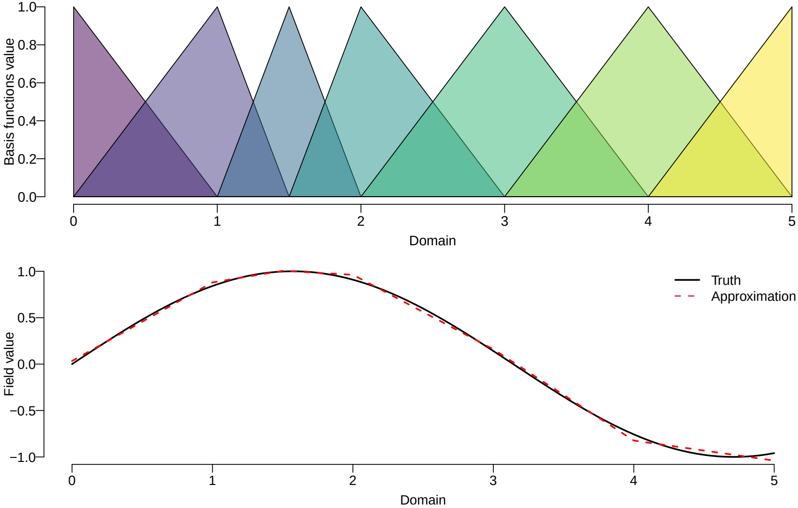
- Bigger $\alpha \rightarrow \mathbf{Q}$ less sparse $\rightarrow$ smoother
  - $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
  - $\alpha = 2$: $\mathbf{Q}_{2,\kappa} = \mathbf{K}_\kappa \mathbf{C}^{-1} \mathbf{K}_\kappa$
  - $\alpha = 3, 4, \ldots$: $\mathbf{Q}_{\alpha,\kappa} = \mathbf{K}_\kappa \mathbf{C}^{-1} \mathbf{Q}_{\alpha-2,\kappa} \mathbf{C}^{-1} \mathbf{K}_\kappa$
- $d = 1$ and $\kappa = 0$, the model at the knots is
  - $\alpha = 1$: like RW1
  - $\alpha = 2$: like RW2

# Important fact: role of $\alpha$

- Bigger $\alpha \rightarrow \mathbf{Q}$ less sparse $\rightarrow$ smoother
  - $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
  - $\alpha = 2$: $\mathbf{Q}_{2,\kappa} = \mathbf{K}_\kappa\mathbf{C}^{-1}\mathbf{K}_\kappa$
  - $\alpha = 3, 4, ...$: $\mathbf{Q}_{\alpha,\kappa} = \mathbf{K}_\kappa\mathbf{C}^{-1}\mathbf{Q}_{\alpha-2,\kappa}\mathbf{C}^{-1}\mathbf{K}_\kappa$
- $d = 1$ and $\kappa = 0$, the model at the knots is
  - $\alpha = 1$: like RW1
  - $\alpha = 2$: like RW2
- $d = 2$ (equivalent model at the mesh nodes)
  - $\alpha = 2$, Whittle (1954).
  - $\alpha = 1$, Besag (1974)
  - $\alpha = 1$ & $\kappa = 0$: intrinsic

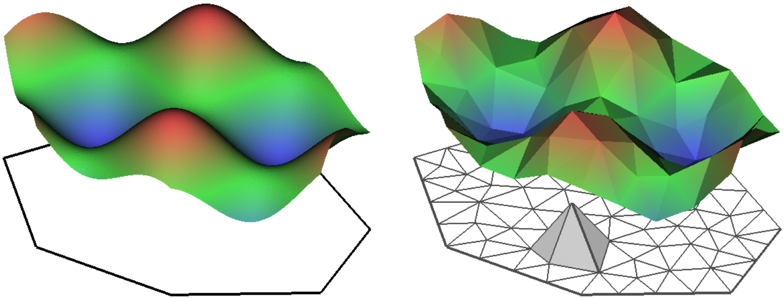
# Continous approximation: 1d case

- $\xi(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s}) w_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0)\xi(\mathbf{s}_0)$,
  - $\psi_k$: basis functions evaluated at data locations $\mathbf{s}$
  - $w_k$: weights, the process at the discretization points $\mathbf{s}_0$



# Continous approximation: 2d case

- $\xi(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s}) w_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0)\xi(\mathbf{s}_0)$,
  - $\psi_k$: basis functions evaluated at data locations $\mathbf{s}$
  - $w_k$: weights, the process at the discretization points $\mathbf{s}_0$



# SPDE as a random effect in INLA

- The (1d, 2d, ...) SPDE models: "continuous domain random effects"
- A new term to be considered in the linear predictor
- A the random effect on knots/nodes
- Needs to be projected to the observation location
- Some extra work ...

$$\eta = \mathbf{X}\beta + \mathbf{Au}$$

- **X** is *n* (observation) times *p* (covariates)
- **Z** is *n* (observation) times *m* (knots/nodes)
- see more on Krainski et al. (2018)

# References

# References


Besag, J. 1974. "Spatial Interaction and the Statistical Analysis of Lattice Systems." *JRSS-B* 36 (2): 192–236.

Krainski, E. T., V. Gómez-Rubio, H. Bakka, A. Lenzi, D. Castro-Camilio, D. Simpson, F. Lindgren, and H. Rue. 2018. *Advanced Spatial Modeling with Stochastic Partial Differential Equations Using R and INLA*. New York: Chapman; Hall/CRC. https://doi.org/10.1201/9780429031892.

Lindgren, F., H. Rue, and J. Lindström. 2011. "An Explicit Link Between Gaussian Fields and Gaussian Markov Random Fields: The Stochastic Partial Differential Equation Approach (with Discussion)." *JRSS-B* 73 (4): 423–98.

Rue, H., and L. Held. 2005. *Gaussian Markov Random Fields: Theory and Applications*. Monographs on Statistics & Applied Probability. Boca Raton: Chapman; Hall.

Whittle, P. 1954. "On Stationary Processes in the Plane." *Biometrika* 41 (3/4): 434–49.