Spacetime smoothing rates

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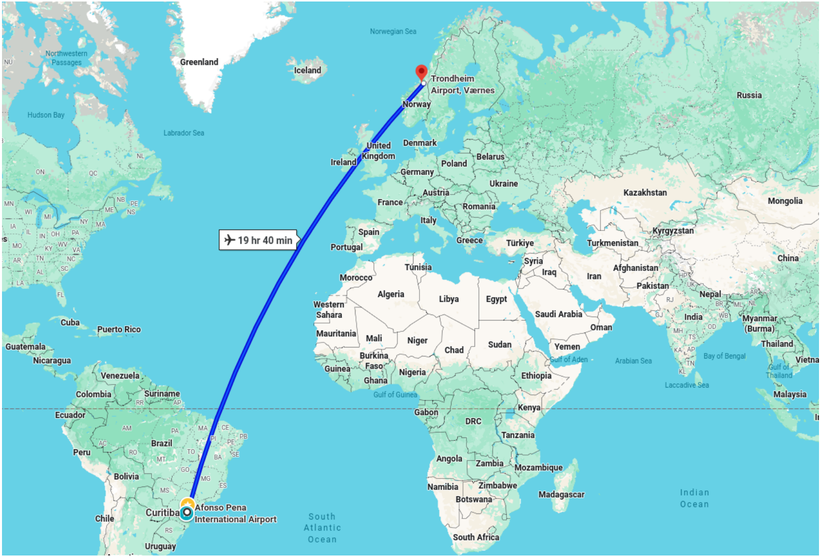
Where

I am from Brazil, meet professor Håvard Rue in Norway



Data and parameters

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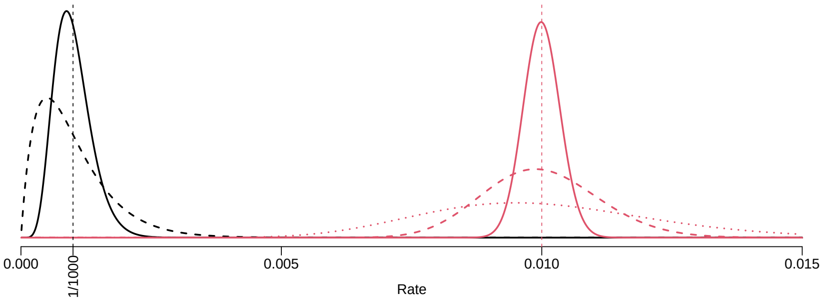
Rate and uncertainty



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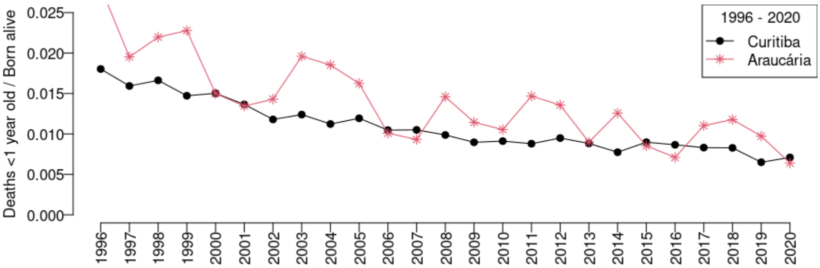
- Infant mortality rate (per thousand) (por mil)

	2018	2019	2020
Curitiba	8.276	6.497	7.097
[illegible]	[illegible]	[illegible]	[illegible]

Visualize the observed rates



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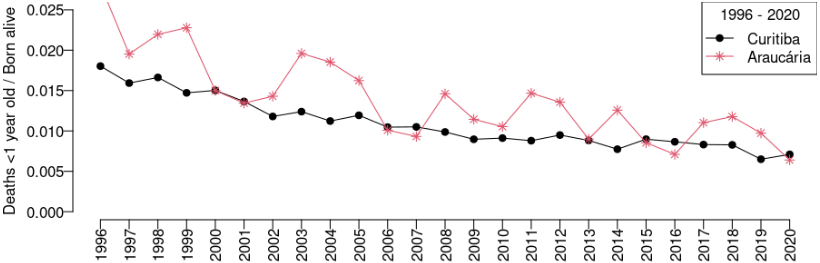
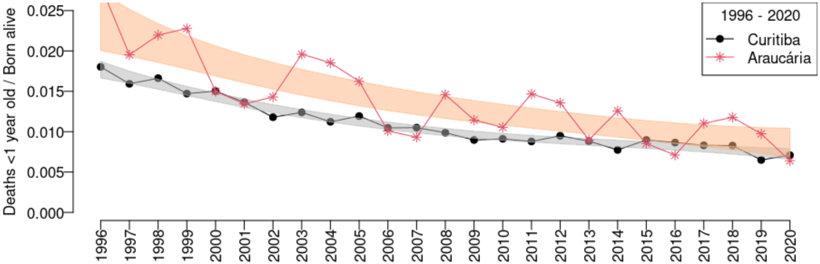


- Rates vary more in “small areas” (less poputation)

Fitted rates

- Considering temporal "neighborhood"



Spatial neighborhood

Old survival in Europe, Ribeiro et al. (2016)

Probability (75-84 years old in 2001 survive until 2011)

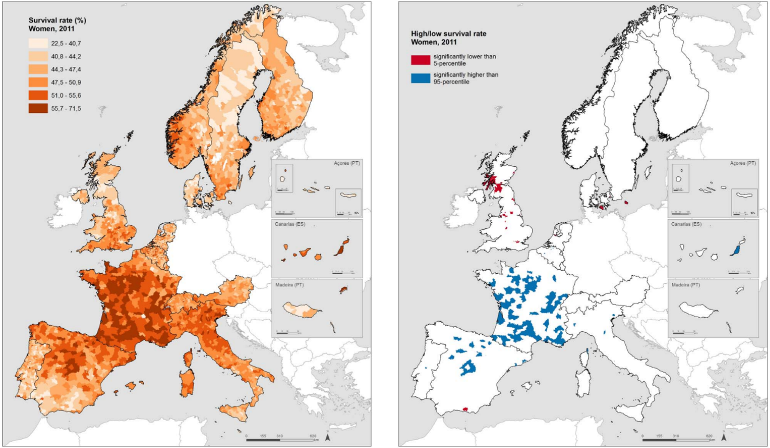


Figure 2 Spatial distribution of the 10-year survival rates across small areas of Europe in 2001 and 2011 (women). (A) Survival rates; (B) areas of high (above 95th centile) and low (below 5th centile) survival.

Tuberculosis in Portugal, Apolinário et al. (2017)

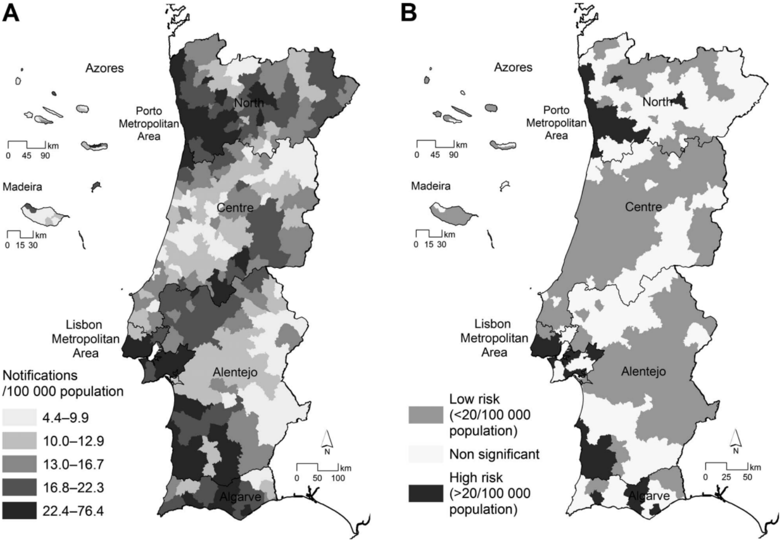


Figure Spatial distribution of **A)** tuberculosis-standardised notification rates and **B)** location of municipalities with higher/lower notification rates, Portugal, 2010–2014.

Wasting, Stunting and Underweight: shared risk, Kinyoki et al. (2016)

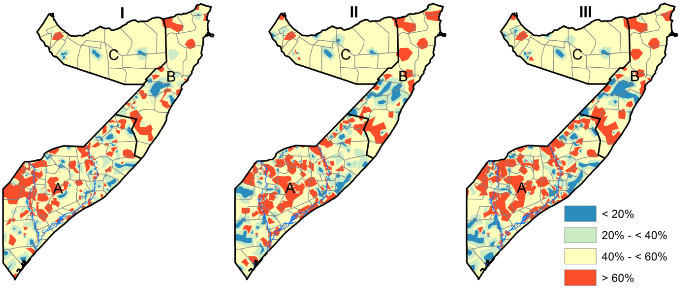


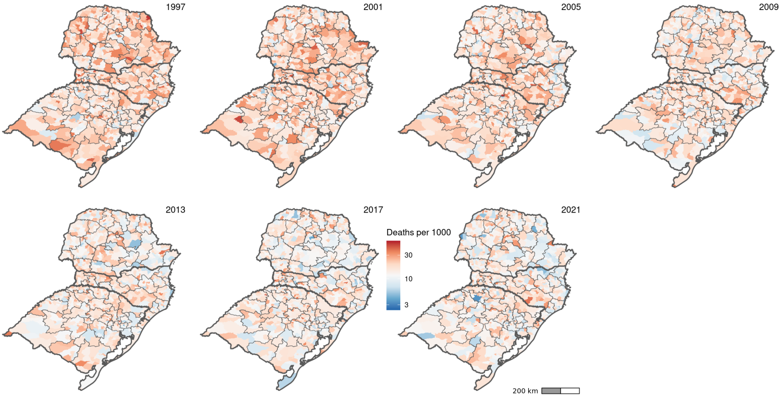
Figure 4 Estimated shared components classified at 95% credible level among children aged 6–59 months using the marginal probabilities calculated using the quintile correction (QC) method.[35] I=Wasting and Stunting, II=Stunting and Underweight, III=Wasting and Underweight. South Central (A), North East (B) and North West (C).

Kinyoki DK, *et al. BMJ Open* 2016;**6**:e009854. doi:10.1136/bmjopen-2015-009854

Spacetime

The spacetime discrete domain case

$$\mathbf{u} = \{u_{11}, \ldots, u_{n1}, u_{12}, \ldots, u_{n2}, \ldots, u_{1T}, \ldots, u_{nT}\}$$



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  - linear trend for each area, Bernardinelli et al. (1995)
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- Interaction between spatial and temporal random effects, Clayton (1996)

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  - **Q1** has dimension equals T (from a model over time)
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- The Kronecker product models follows the Clayton's rule
- Combine **Q1** (time) and **Q2** (space) available

Some examples on spacetime interactions

- Precision($\mathbf{u}$) $= \tau \mathbf{H} \otimes \mathbf{R}$, Knorr-Held (2000)
  - $\mathbf{H}$: precision structure matrix over the temporal domain
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- **care** when main effects are in the model
- **super care** when **Q1** and/or **Q2** have rank deficiency
  - e.g. `rw1`, `rw2` and `besag` models
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- Proper in time, Martínez-Beneito, López-Quilez, and Botella-Rocamora (2008)
  - example in INLA:

```
f(spatial, model='besag', ...,
group=time, control.group=list(model='ar1'))
```

  - NOTE: Here the constraints are only set for the main model, and not for the group model!

The Knorr-Held (2000) models

- Linear predictor is modeled as

$$\eta_{it} = \text{other effects} + v_t + s_i + d_{it}$$

  - v_t is a temporal effect common for all areas,
  - s_i is a spatial effect common for all times and
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  - v_t is a temporal effect common for all areas,
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  - d_{it} is the space-time effect.
- Decompose: $v_t = \gamma_t + \alpha_t$ and $s_i = \phi_i + \theta_i$
  - zero mean Gaussian distributions, with precision $\tau_\gamma \mathbf{I}$, $\tau_\alpha \mathbf{K}$, $\tau_\phi \mathbf{I}$ and $\tau_\theta \mathbf{R}$, for γ, α, ϕ and θ, respectively.
  - $\mathbf{I}$: identity matrix with required dimention
  - $\mathbf{K}$ and $\mathbf{R}$: temporal and spatial precision structure matrices, from the neighborhood structure

The Knorr-Held (2000) models (cont.)

- Let $\mathbf{R} = \tilde{\mathbf{G}} - \mathbf{G}$, where
  - $\mathbf{G}$ is the spatial neighborhood structure

$$\mathbf{G}_{i,j} = \begin{cases} 1 & \text{if} \quad j \sim i \\ 0 & \text{otherwise,} \end{cases}$$

  - $j \sim i$ means "j neighbor to i"
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  - $j \sim i$ means "j neighbor to i"
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- Considering the type IV interaction:

$$\begin{aligned} \tau_d \mathbf{K} \otimes \mathbf{R} &= \tau_d (\tilde{\mathbf{H}} - \mathbf{H}) \otimes (\tilde{\mathbf{G}} - \mathbf{G}) \\ &= \tau_d (\tilde{\mathbf{H}} \otimes \tilde{\mathbf{G}} - \tilde{\mathbf{H}} \otimes \mathbf{G} - \mathbf{H} \otimes \tilde{\mathbf{G}} + \mathbf{H} \otimes \mathbf{G}) \end{aligned}$$

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