# Overview of the spacetime SPDE approach

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# Outline

Ideas and motivation

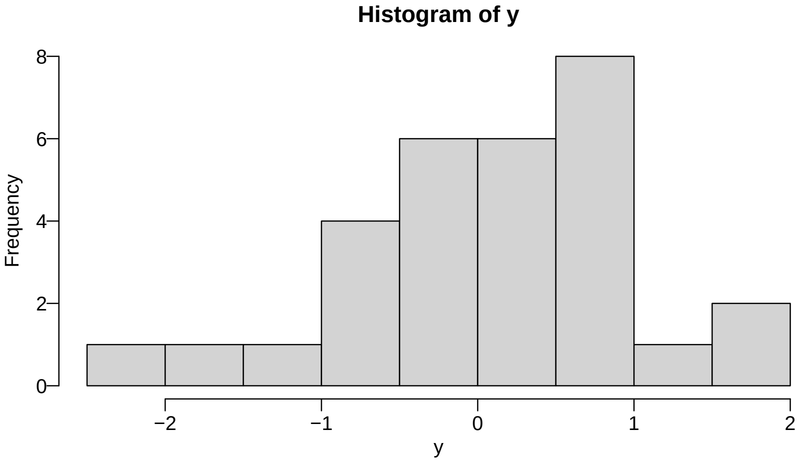
An spacetime SPDE model extension, Lindgren et al. (2024)

Implementation

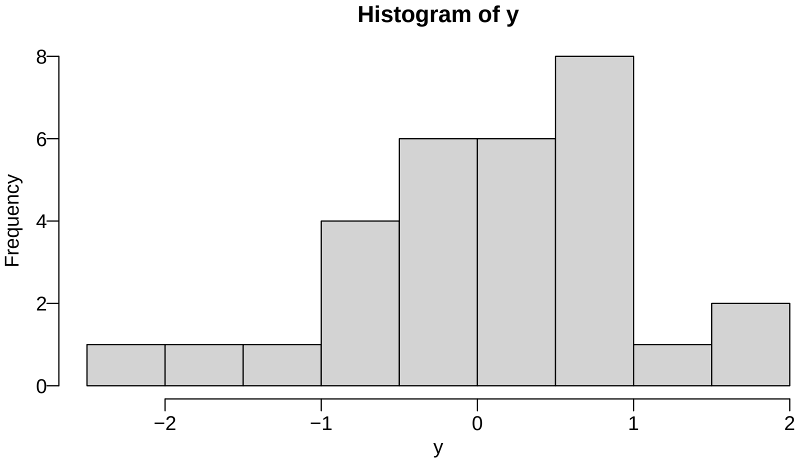
Examples

# Ideas and motivation

# Symmetric or assymetric?

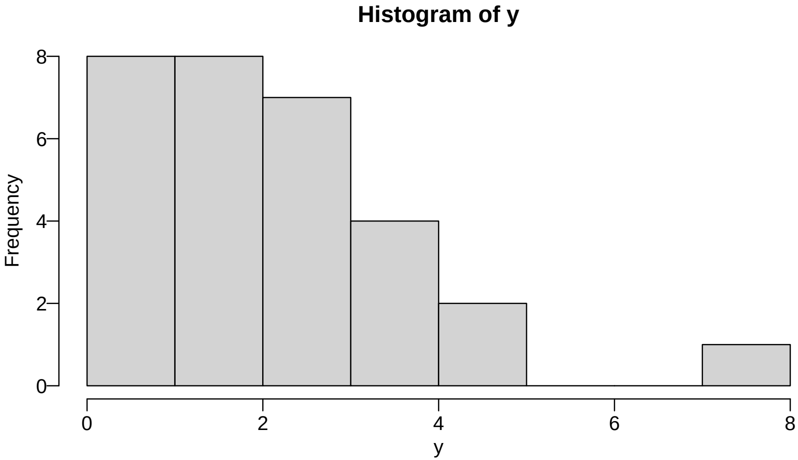


# Symmetric or assymetric?




```r
set.seed(1); y <- rnorm(30); hist(y)
```

# What causes this to be assymetric?



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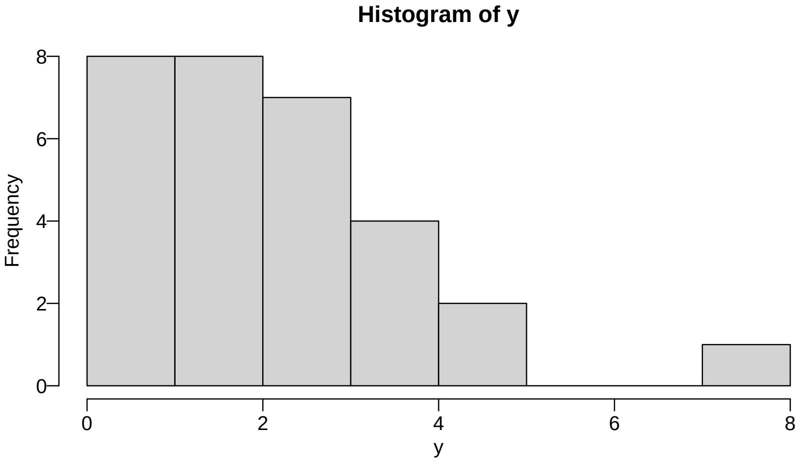


```r
set.seed(1); x = rexp(30); y = 1 + x + rnorm(30); hist(y)
```

# Stationary and non-stationary

## Stationary

- Covariance is the same over the domain
  - E.g. 1d: does not matter where, but only the lag

$$Cov[u(t_1), u(t_2)] = cov(h), \quad h = |t_1 - t_2|$$

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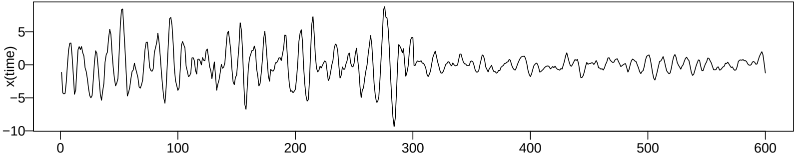
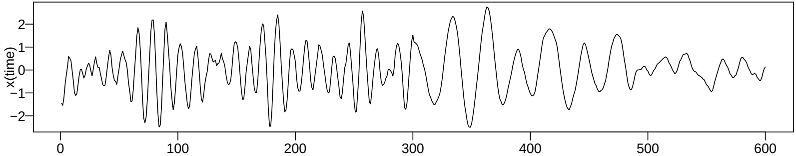
$$Cov[u(t_1), u(t_2)] = cov(h), \quad h = |t_1 - t_2|$$

## Non-stationary

- Any model where precision/covariance depends on the location
  - e.g.: 1 dimentional example (only varying over time)



## Stationary or non-stationary?

Spatial Statistics

Volume 14, Part C, November 2015, Pages 505-531


# Does non-stationary spatial data always require non-stationary random fields?

Geir-Arne Fuglstad [a], Daniel Simpson [a], Finn Lindgren [b], Håvard Rue [a]

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## Does non-stationary spatial data always require non-stationary random fields?

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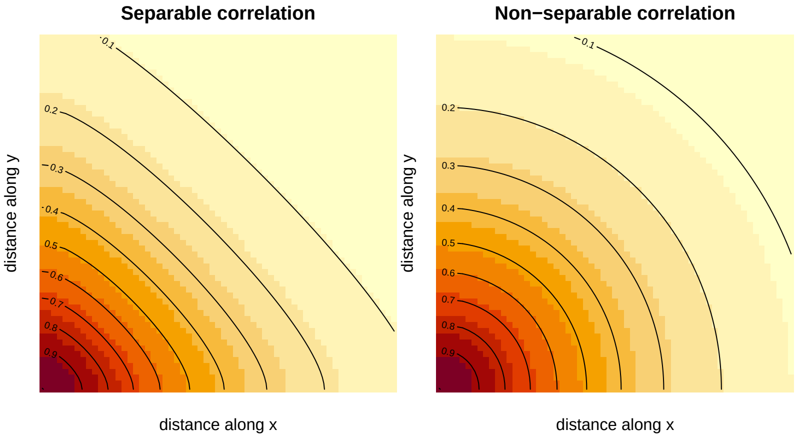
It is hard to beat simple suitable models, Fuglstad et al. (2015).

# Separable or non-separable?



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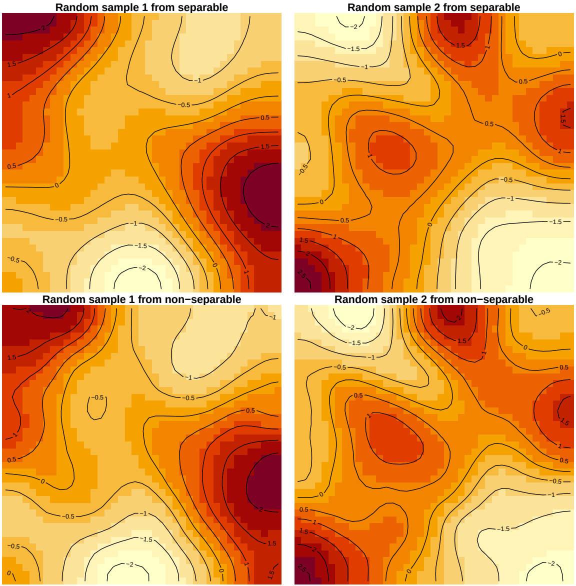
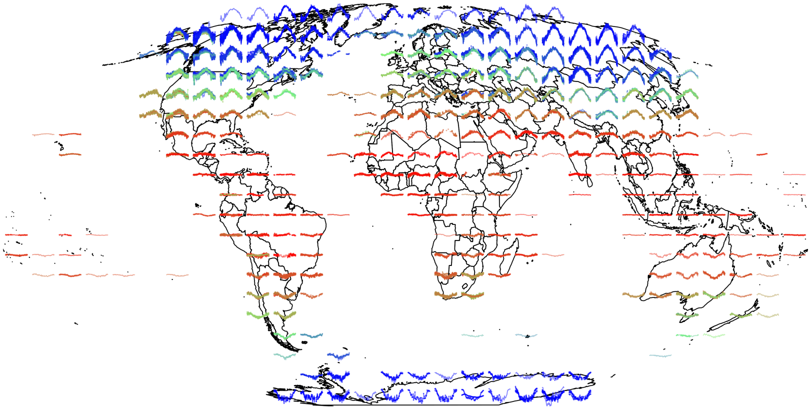


# Global temperature, see Lindgren et al. (2024)



Symmetric? Stationary? Separable?

# An spacetime SPDE model extension, Lindgren et al. (2024)

# The space-time models

Models for processes that on the space-time domain

- Discrete space-time domain
  - $\mathbf{u} = \{u_{11}, ..., u_{n1}, u_{12}, ..., u_{n2}, ..., u_{1T}, ..., u_{nT}\}$
- Continuous space-time domain
  - $\mathbf{u}(\mathbf{s}, t)$, $\mathbf{s} \in \mathbb{R}^d$, $t \in \mathbb{R}$

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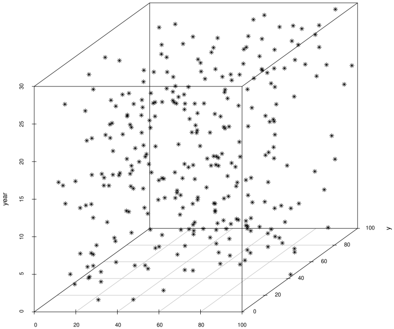
  - $\mathbf{u}(\mathbf{s}, t)$, $\mathbf{s} \in \mathbb{R}^d$, $t \in \mathbb{R}$
- Real world (locations)



# Separable space-time models

- When covariance/precision can be written as Kronecker product of a purely temporal and purely spatial ones
- **E.g.:** Discrete space-time domain

$$\pi(\mathbf{u}) \propto (|\mathbf{Q1} \otimes \mathbf{Q2}|^{-1})^{1/2} \exp\left(-\frac{1}{2}\mathbf{u}^T\{\mathbf{Q1} \otimes \mathbf{Q2}\}\mathbf{u}\right)$$

  - $\mathbf{Q1}$ has dimension equals $T$
  - $\mathbf{Q2}$ has dimension equals $n$

# Space-time SPDE models

- SDE in time **then** SPDE in space

$$\left(\gamma_t \frac{\partial}{\partial t} + a^2\right) u(\mathbf{s}, \delta t) = \xi(\mathbf{s}, \delta t)$$
$$(\kappa^2 - \Delta)^{\alpha_e/2} \xi(\mathbf{s}, \delta t) = \mathcal{W}(\mathbf{s}, \delta t).$$

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- SPDE in time **and** in space

$$\left(\gamma_t \frac{\partial}{\partial t} + (\gamma_s^2 - \Delta)^{\alpha_s/2}\right) u(\mathbf{s}, \delta t) = \xi(\mathbf{s}, \delta t)$$
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# A class of SPDE models, Lindgren et al. (2024)

- Spatial operator $L_s = \gamma_s^2 - \Delta$

$$
\begin{aligned}
\left(\gamma_t \frac{\partial}{\partial t} + L_s^{\alpha_s/2}\right)^{\alpha_t} u(\mathbf{s}, \delta t) &= \xi(\mathbf{s}, \delta t) \\
L_s^{\alpha_e/2} \xi(\mathbf{s}, \delta t) &= \mathcal{W}(\mathbf{s}, \delta t).
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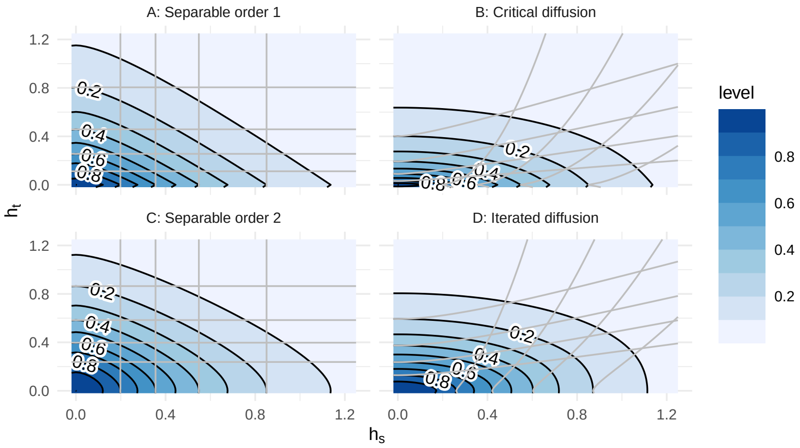
**Table 1:** Four specific models on 2-dimensional spatial domains.

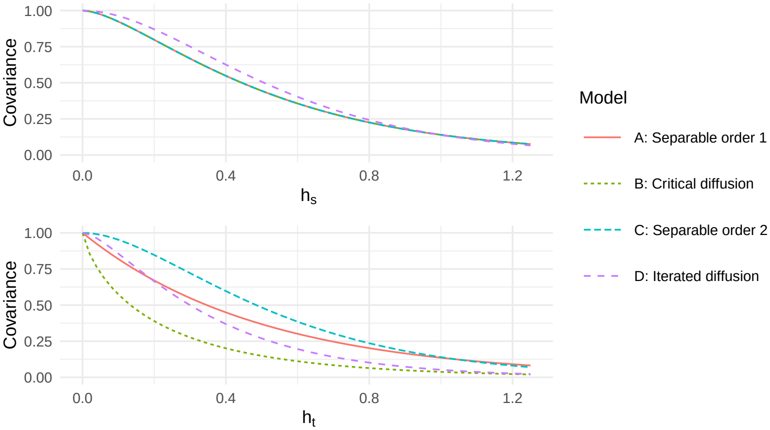
| Model | $\alpha_t$ | $\alpha_s$ | $\alpha_e$ | Type | $\nu_t$ | $\nu_s$ | nonsep |
|---|---|---|---|---|---|---|---|
| A:(1,0,2) | 1 | 0 | 2 | Separable order 1 | 1/2 | 1 | 0.0 |
| B:(1,2,1) | 1 | 2 | 1 | Critical diffusion | 1/2 | 1 | 0.5 |
| C:(2,0,2) | 2 | 0 | 2 | Separable order 2 | 3/2 | 1 | 0.0 |
| D:(2,2,0) | 2 | 2 | 0 | Iterated diffusion | 1 | 2 | 1.0 |

Separability is a function of $\alpha_t$, $\alpha_s$, $\alpha_e$ and $d$.

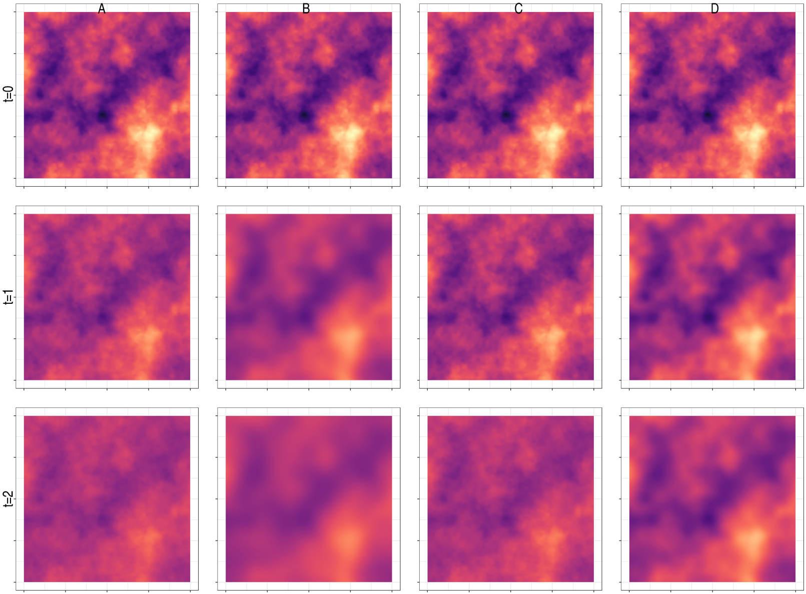
# Space-time correlations



# Marginal correlations



# Example on predictions



# Precision matrix

- A sum of Kronecker products

$$\mathbf{Q}_u(\gamma_s, \gamma_t, \gamma_s) = \gamma_e^2 \sum_{k=0}^{2\alpha_t} \gamma_t^k \mathbf{J}_{\alpha_t, k/2} \otimes \mathbf{K}_{\alpha_s}(\gamma_s)$$

where the $\mathbf{J}_{\alpha_t, k/2}$ matrices are from the temporal discretization.

# Interpretable parametrization

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- It is easier to consider the marginal parameters
- $\sigma = \frac{C(\alpha_t,\alpha_s,\alpha_e,d)}{\gamma_e^2\gamma_t\gamma_s^{2\alpha-d}}$
- $r_s = \sqrt{8\nu}/\gamma_s^2$
- $r_t = \frac{\gamma_t\sqrt{8(\alpha_t-0.5)}}{\gamma_s^{\alpha_s}}$

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  - to implement new random effect models
  - define functions to compute $\mathbf{Q}$
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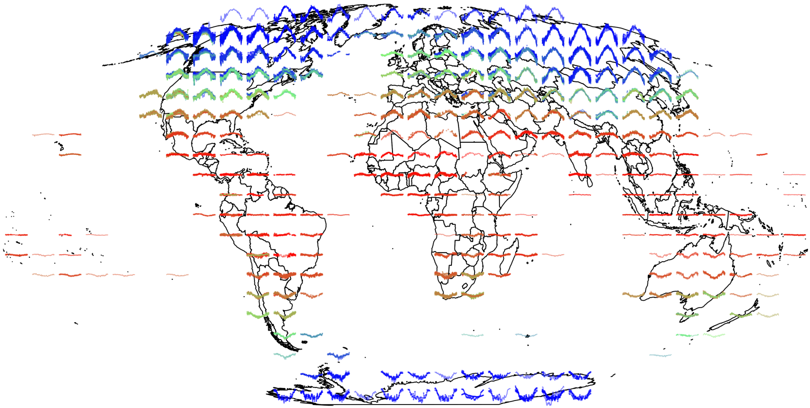
  - to implement new random effect models
  - define functions to compute $\mathbf{Q}$
  - `cgeneric` takes full advantage of parallel computations
- **INLAspacetime** package in
  - https://github.com/eliaskrainski/INLAspacetime

# Examples

# Daily temperature data, Lindgren et al. (2024)



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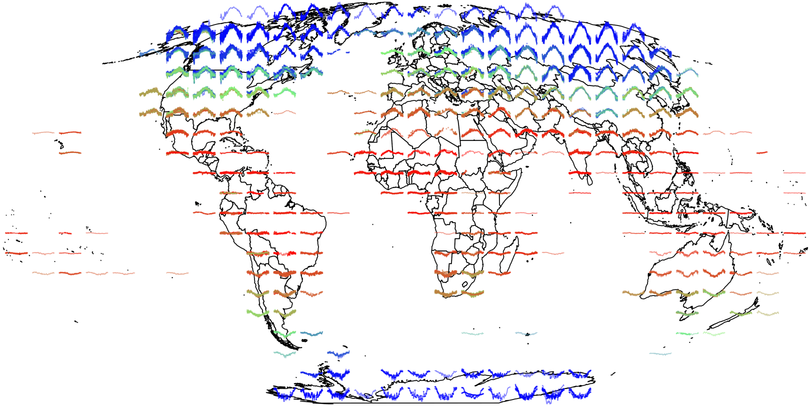
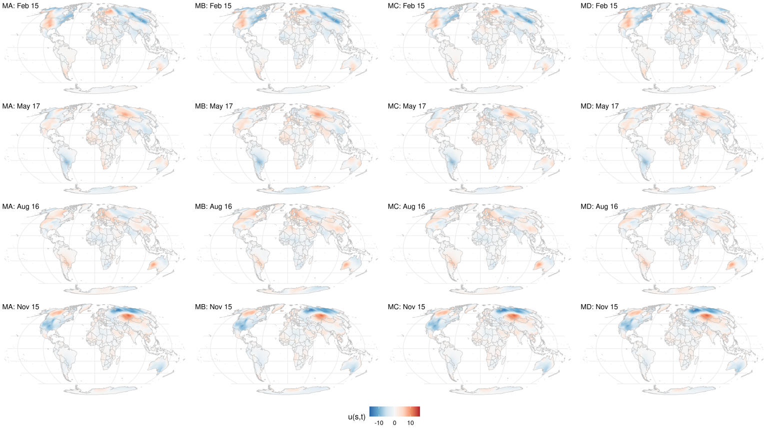


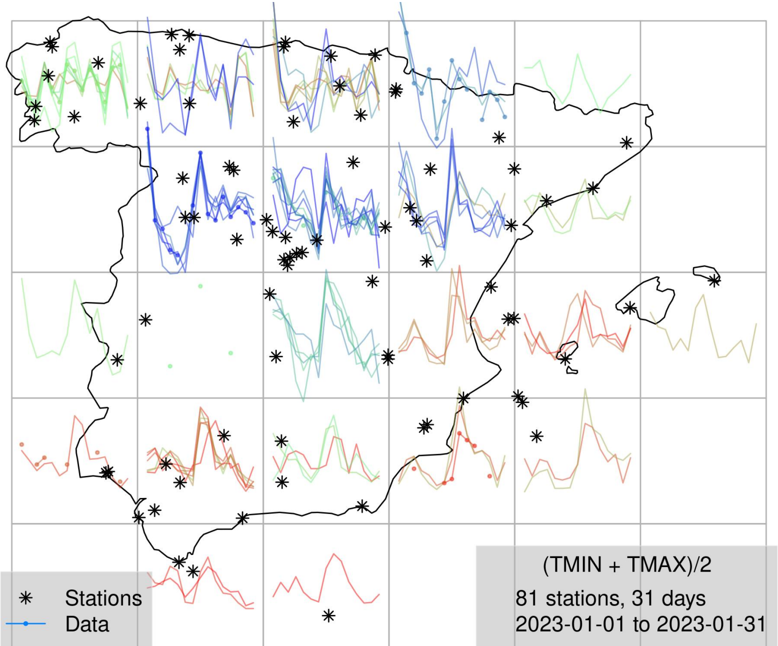
$$\boldsymbol{y} = \boldsymbol{1}\mu + \boldsymbol{E}\alpha + \boldsymbol{B}\boldsymbol{b} + \boldsymbol{A}_v\boldsymbol{v} + \boldsymbol{A}_u\boldsymbol{u} + \boldsymbol{e}$$

elevation + seasonal,latitude + slow space-time + fast space-time

# Posterior mean of $u(.)$, for some days, four models



# Daily temperature data in Spain, January 2023



# The data in the `long` format

```
str(dataf)
## 'data.frame':    2511 obs. of  6 variables:
##  $ iloc: int  1 2 3 4 5 6 7 8 9 10 ...
##  $ xloc: num  442 168 367 578 925 ...
##  $ yloc: num  4474 4311 4059 4795 4584 ...
##  $ elev: num  0.667 0.185 0.007 0.251 0.004 ...
##  $ time: int  1 1 1 1 1 1 1 1 1 1 ...
##  $ resp: num  11.3 15.1 13.8 16.9 12.2 ...
```

Elevation (in km) to be used as covariate

# Model definition, with INLAspacetime

### Define the space-time model with `INLAspacetime`

```
stmodel <- stModel.define(
  smesh = smesh, # spatial mesh
  tmesh = tmesh, # temporal mesh
  model = "121", # model (\alpha_t, \alpha_s, \alpha_e)
  control.priors = list(
    prs = c(100, 0.05), # P(spatial range < 100) = 0.05
    prt = c(2, 0.05),   # P(temporal range < 1) = 0.05
    psigma = c(4, 0.05)), # P(sigma_u > 4) = 0.05
  constr = TRUE)
```

### Define the right-hand-side, `inlabru`

```
M <- ~ Intercept(1) + elev +
  field(list(space = cbind(xloc, yloc),
             time = time),
        model=stmodel)
```

# Model fit with inlabru

## Set the PC-prior for the likelihood precision

```
lkprec <- list(
  prec = list(prior = "pcprec",
              param = c(4, 0.05)))
```

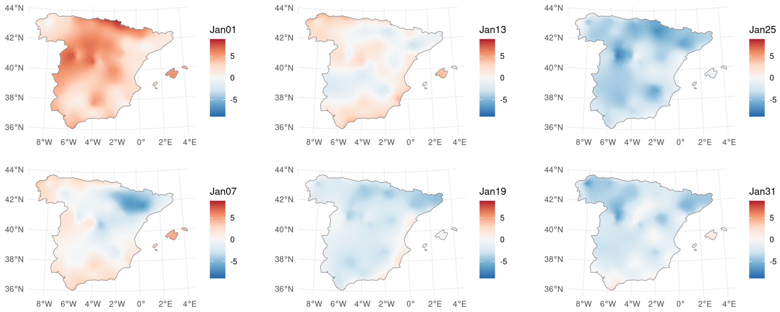
## The likelihood setup in inlabru

```
lhood <- like(
  formula = resp ~ .,
  family = "gaussian",
  control.family = list(
    hyper = lkprec),
  data = dataf)
```

## Fit

```
result <- bru(M, lhood)
```

# Posterior mean of $u(.)$, for some days



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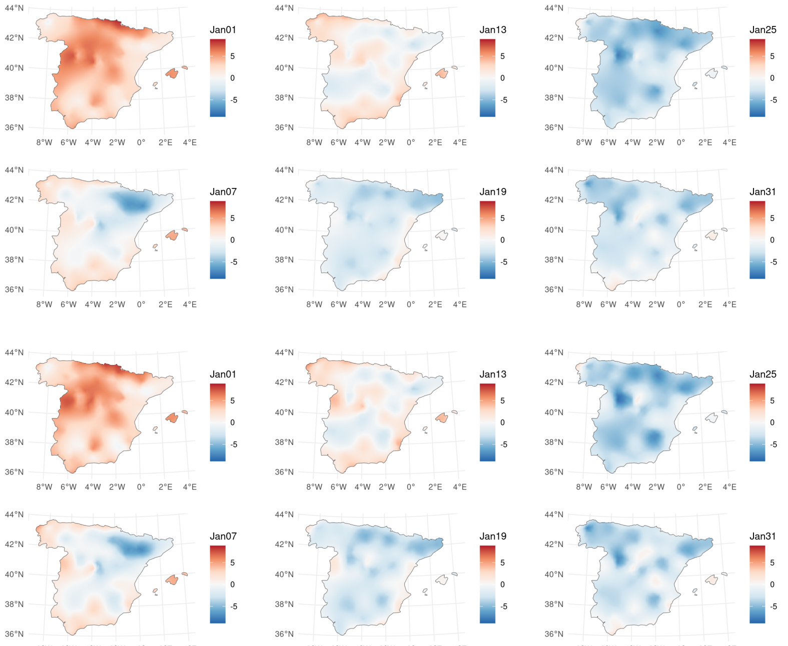
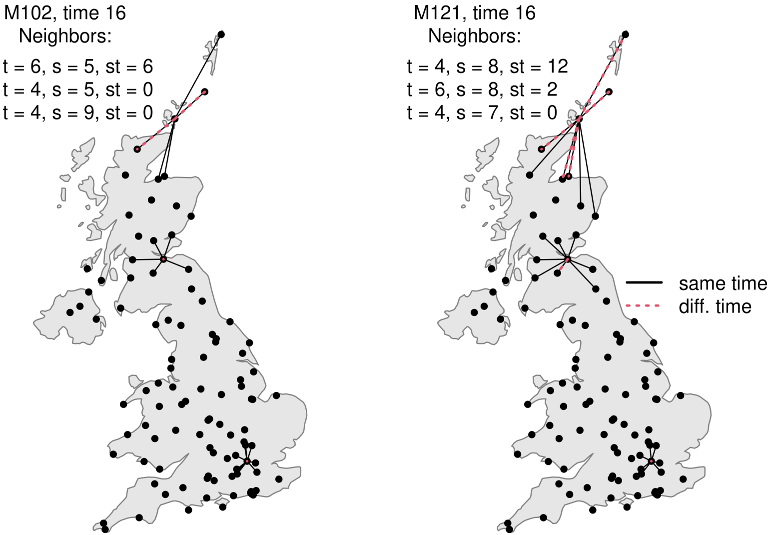


# UK example: Daily wind speed, `inla.gcv(..., m = 10)`



# References


Fuglstad, G-A, D. Simpson, F. Lindgren, and H. Rue. 2015. "Does Non-Stationary Spatial Data Always Require Non-Stationary Random Fields." *Spatial Statistics* 14: 505–31.

Lindgren, F., H. Bakka, D. Bolin, E. T. Krainski, and H. Rue. 2024. "The Diffusion-Based Extension of the Matérn Field to Space-Time (Invited Article with Discussion)." *SORT* 48 (1): 3–66. https://raco.cat/index.php/SORT/article/view/428665.

Rue, H., S. Martino, and N. Chopin. 2009. "Approximate Bayesian Inference for Latent Gaussian Models Using Integrated Nested Laplace Approximations (with Discussion)." *Journal of the Royal Statistical Society, Series B* 71 (2): 319–92.

van Niekerk, J., E. T. Krainski, D. Rustand, and H. Rue. 2023. "A New Avenue for Bayesian Inference with INLA." https://doi.org/10.1016/j.csda.2023.107692.