Spatial SPDE approach overview

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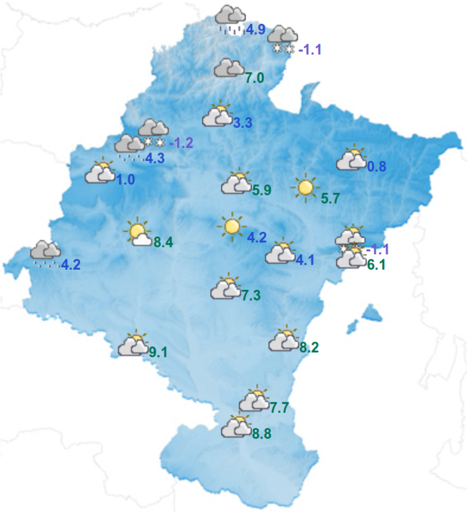
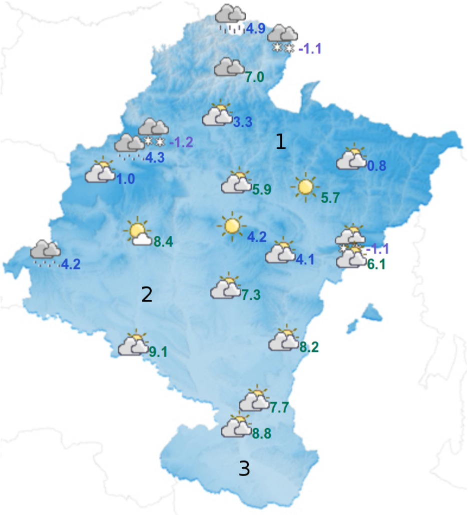
Initial ideas



Aguilar de Codés GN	4.2ºC
Aoiz GN	5.7ºC
Aralar GN	-1.2ºC
Arangoiti GN	-1.1ºC

Interpolation problem



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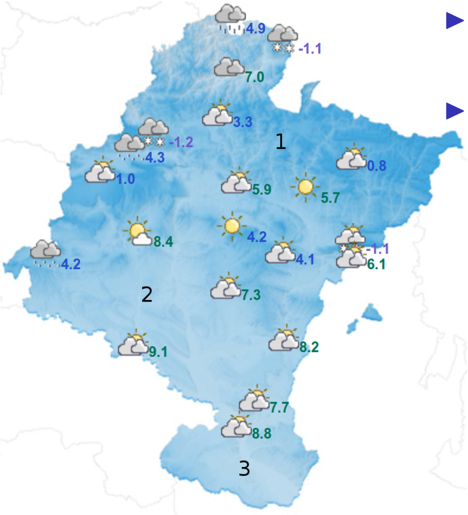


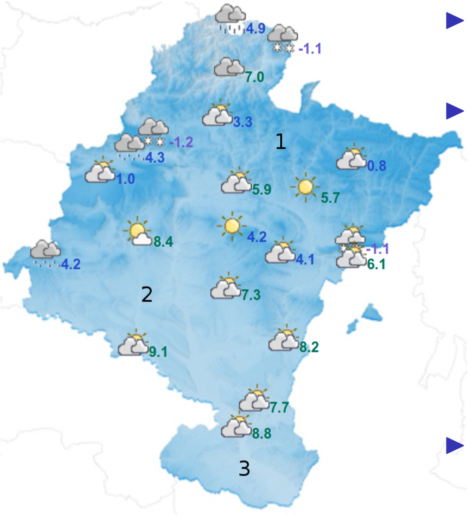
- predict temperature at 1, 2 and 3: t_1, t_2 and t_3.
- naive approach

$$t_1 = \frac{3.3 + 5.9 + 5.7 + 0.8}{4}$$

$$t_2 = \frac{8.4 + 4.2 + 9.1 + 7.3}{4}$$

$$t_3 = 8.8$$

Interpolation problem



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- statistical approach

$$p(\text{new}|\text{obs.}) = \frac{p(\text{new}, \text{obs.})}{p(\text{obs.})}$$

Ideas and assumptions

- Nearby observations may be similar

- Smaller variance between nearby observations
- Bigger variance between far apart observations

- Prediction
  - Closest observations $\rightarrow$ bigger weights
  - Variance is inversely proportional to weights

Multivariate Gaussian distribution

- Consider the joint distribution of $\mathbf{x}^T = \{\mathbf{x}_A^T, \mathbf{x}_B^T\}^T$

$$\begin{pmatrix} \boldsymbol{x}_A \\ \boldsymbol{x}_B \end{pmatrix} \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_B \end{pmatrix}, \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q}_{AA} & \boldsymbol{Q}_{AB} \\ \boldsymbol{Q}_{BA} & \boldsymbol{Q}_{BB} \end{pmatrix}^{-1} \right)$$

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- The conditional distribution of $\mathbf{x}_B$ is

$$\boldsymbol{x}_B | \boldsymbol{x}_A \sim N(\boldsymbol{m}, \boldsymbol{V}) \text{ where}$$

$$\begin{aligned} \boldsymbol{m} &= \boldsymbol{\mu}_B - \Sigma_{BA}\Sigma_{AA}^{-1}(\boldsymbol{x}_A - \boldsymbol{\mu}_A) \\ &= \boldsymbol{\mu}_B - \boldsymbol{Q}_{BB}^{-1}\boldsymbol{Q}_{BA}(\boldsymbol{x}_A - \boldsymbol{\mu}_A) \\ \boldsymbol{V} &= \Sigma_{BB} - \Sigma_{BA}\Sigma_{AA}^{-1}\Sigma_{AB} = \boldsymbol{Q}_{BB}^{-1} \end{aligned}$$

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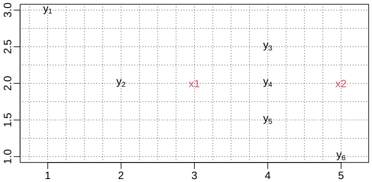
- Kriging: Let $\mathbf{x}_A$ observations and $\mathbf{x}_B$ to be predicted
    - prediction: a weighted average
        - weights: $-\Sigma_{BA}\Sigma_{AA}^{-1} = \boldsymbol{Q}_{BB}^{-1}\boldsymbol{Q}_{BA}$

Example with some locations



- Exponential spatial correlation:

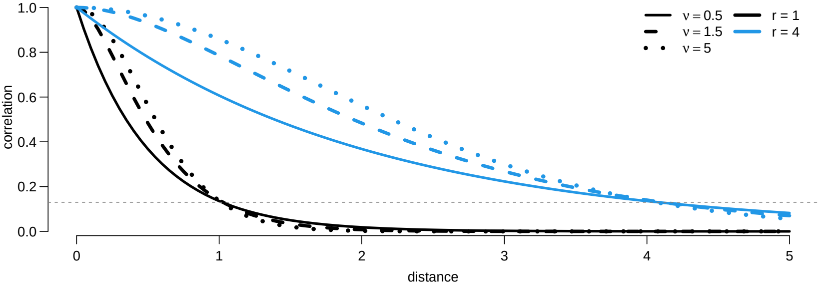
$$\rho(h) = \mathrm{e}^{-\frac{h}{\phi}}$$

- Kriginig weights, for $\phi = \{3\}$:

1	2	3	4	5	6
0.0177	0.4278	0.1856	0.1819	0.2052	-0.0351
0.0009	-0.0479	0.2724	0.1942	0.0657	0.4172

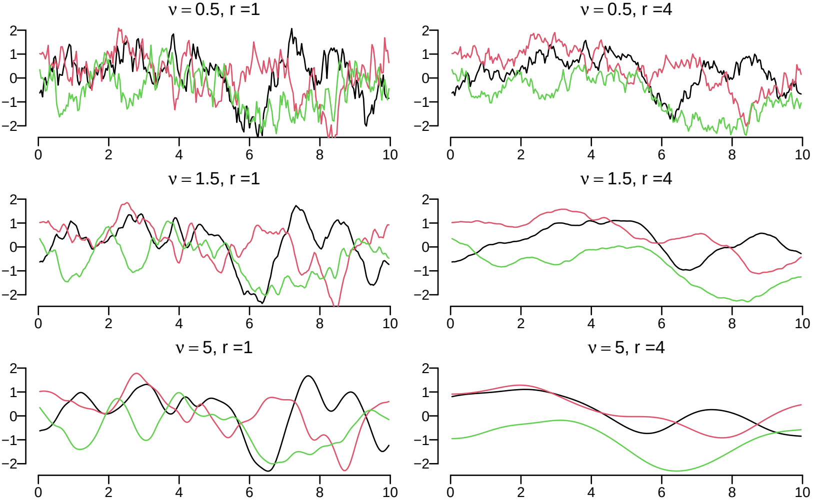
The Matérn covariance, Matérn (1960)

$$\Sigma_{ij} = \sigma^2 \frac{2^{1-\nu} K_\nu(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)}{\Gamma(\nu)(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)^{-\nu}}, \ \nu = 1\text{: Whittle (1954)}$$

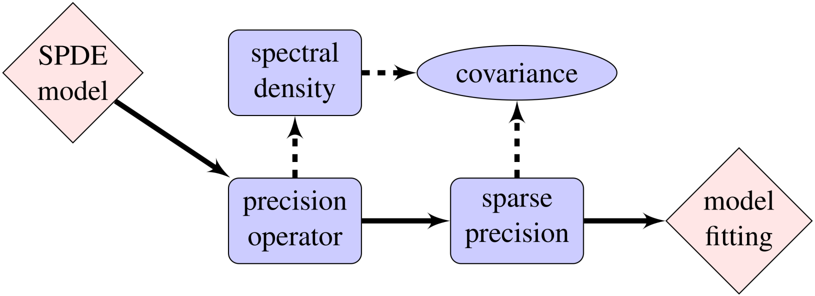


corr $(r) \approx 0.13$

Simulations, 1D, $\sigma^2 = 1$



SPDE framework



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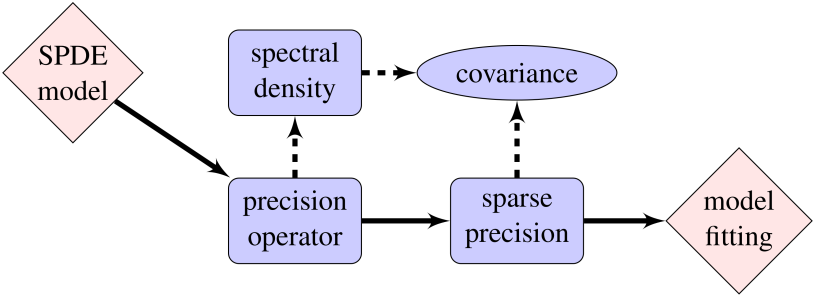

- It avoids specifying covariance!
  - Simpson, Lindgren, and Rue (2011)
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The Matérn's SPDE

- Whittle (1954), Whittle (1963):
  - Fields with Matérn covariance are solutions to the following Stochastic Partial Differential Equation (SPDE)

$$\tau(\kappa^2 - \Delta)^{\alpha/2} u(\mathbf{s}) = \mathcal{W}(\mathbf{s})$$

  - $\kappa > 0$: scale parameter
  - $\alpha = \nu + d/2$: smoothness
  - Δ is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

Lindgren, Rue, and Lindström (2011)

- Discretization
  - sparse precision matrix:
  - $\mathbf{Q}_\alpha(\tau, \kappa)$, for $\alpha \in \{1, 2, ...\}$.

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  - $\alpha = 2, 3, 4, ...$: $\tau \mathbf{K}_1(\kappa)\mathbf{C}^{-1}\mathbf{K}_{\alpha-2}(\kappa)\mathbf{C}^{-1}\mathbf{K}_1(\kappa)$

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- Equivalent models (discretized): Whittle (1954), Besag (1974), Besag (1981), Besag and Kooperberg (1995) and Besag and Mondal (2005).

$\mathbf{Q}_\alpha(\tau, \kappa)$: grid and piecewise linear basis

- $\alpha = 1$: $\tau \mathbf{K}_1(\kappa) = \tau(\kappa^2 \mathbf{C} + \mathbf{G})$
  - d=1, $u_1, u_2, \ldots, u_n$, two neighbours

$$\tau^2 \begin{bmatrix} 1+\kappa^2 & -1 & & & \\ -1 & 2+\kappa^2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2+\kappa^2 & -1 \\ & & & -1 & 1+\kappa^2 \end{bmatrix}$$

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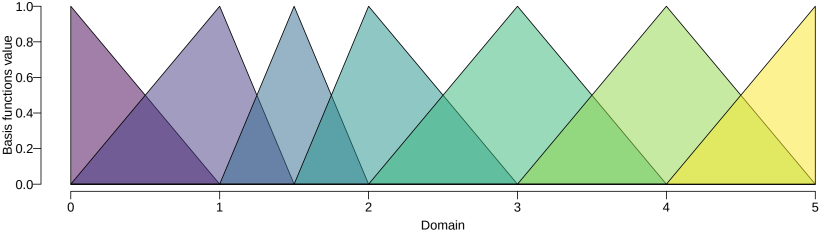
  - $d = 2$, $\mathbf{C} = \mathbf{I}$, $\mathbf{G} =$ Laplacian (4 neighbours)

Piecewise linear basis, Finite Element Method (FEM): 1d



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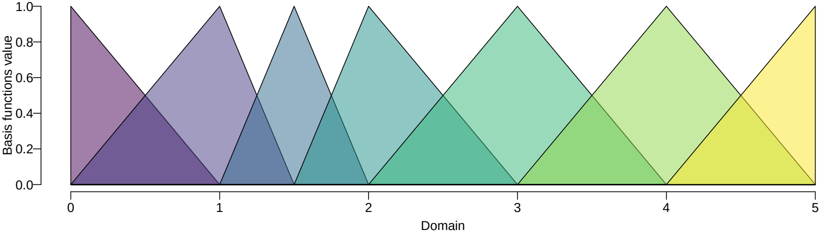
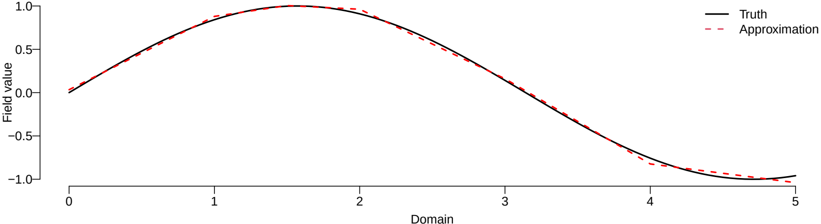


- $u(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s}) u_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0) u(\mathbf{s}_0)$,
  - ψ_k: basis functions evaluated at data locations $\mathbf{s}$
  - u_k: the process at the discretization points $\mathbf{s}_0$



Piecewise linear basis, Finite Element Method: 2d

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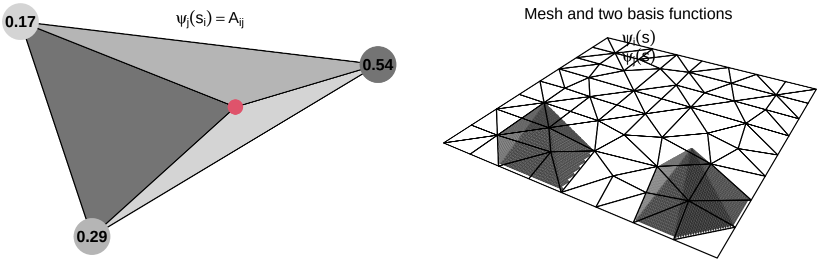
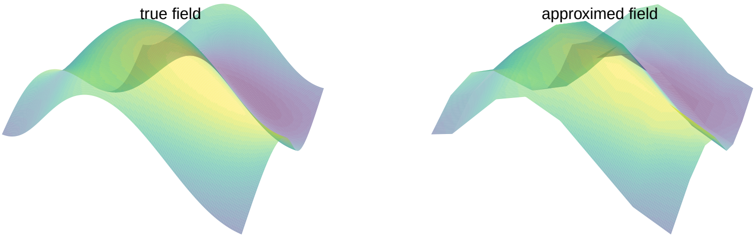
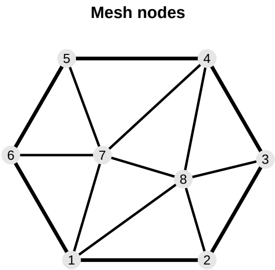
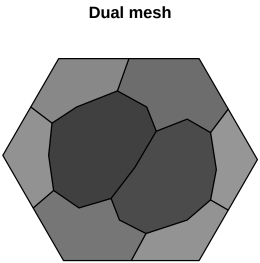
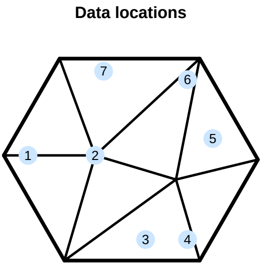




Piecewise linear basis, FEM matrices







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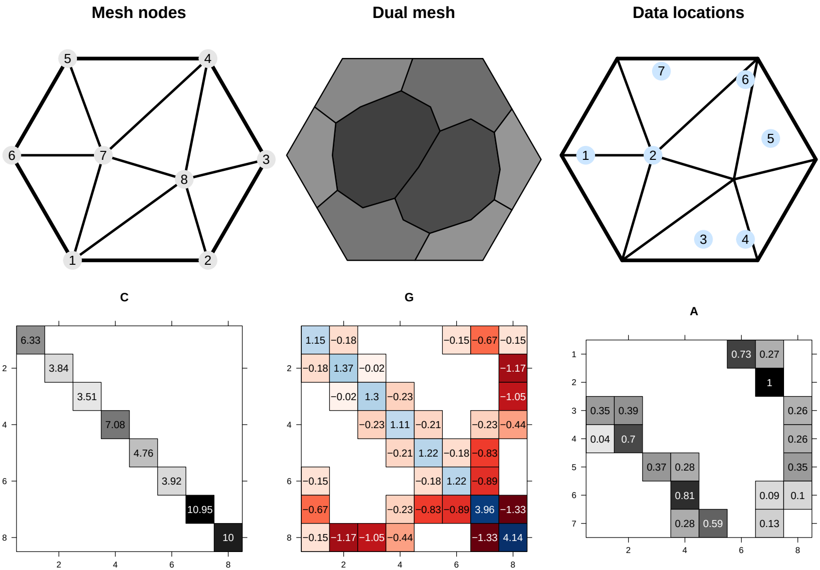

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