Some spacetime models for areal data

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Outline

The pre spacetime work

Besag: randon walk over areas, Besag (1974)

$$\pi(x_i|\mathbf{x}_{-i},\tau) \sim N(\frac{1}{n_i}\sum_{j\sim i} x_j, \frac{1}{n_i\tau})$$

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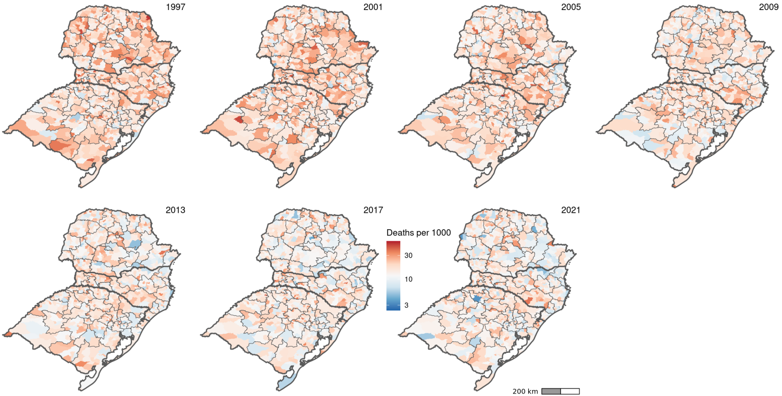
&= \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\sum_{j\sim i}^{n}(x_i - x_j)^2\right) = \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\mathbf{x}^T\mathbf{R}\mathbf{x}\right)
\end{aligned}$$

$$\mathbf{R}_{ij} = \begin{cases} n_i & \text{if } i = j \\ -1 & \text{if } j \sim i \\ 0 & \text{otherwise} \end{cases} .$$

The Scotland graph

The discrete domain case

$$\mathbf{u} = \{u_{11}, \ldots, u_{n1}, u_{12}, \ldots, u_{n2}, \ldots, u_{1T}, \ldots, u_{nT}\}$$



Spatially structured trends

- interaction between fixed and random terms
  - linear trend for each area, Bernardinelli et al. (1995)
  - quadratic term in the same way Assunção, Reis, and Oliveira (2001)
  - B-splines with spatially structured coefficients, macnabD:2001, and macnabD:2002

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- Combine models for variation over space with models for variation over time,
- space-time interaction of the main spatial and temporal effects, Gilks, Richardson, and Spiegelhalter (1996)

Spacetime models

Kronecker product models

- Consider the random vector indexed as follows $\mathbf{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$
- Assume

$$\pi(\mathbf{x}) \propto (|\mathbf{Q1} \otimes \mathbf{Q2}|^{-1})^{1/2} \exp\left(-\frac{1}{2}\mathbf{x}^T\{\mathbf{Q1} \otimes \mathbf{Q2}\}\mathbf{x}\right)$$

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- Common models: Precision($\mathbf{u}$) $= \tau\mathbf{H} \otimes \mathbf{R}$, Knorr-Held (2000)
  - **H**: precision structure matrix over the temporal domain
  - **R**: precision structure matrix over the spatial domain
  - This defines a precision/covariance separable model for **u**

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f(spatial, model='besag', ...,
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- **care** when main effects are in the model
- **super care** when **Q1** and/or **Q2** have rank deficiency
  - e.g. `rw1`, `rw2` and `besag` models
  - if both Q1 and Q2 are intrinsic: use other approach!
    - see `inla.knmodels()`, Knorr-Held (2000)

The Knorr-Held (2000) models

linear predictor is modeled as

$$\eta_{it} = \text{other effects} + v_t + s_i + d_{it} \tag{1}$$

where v_t is a temporal effect common for all areas, s_i is a spatial effect common for all times and d_{it} is the space-time effect.

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- $v_t = \gamma_t + \alpha_t$ and $s_i = \phi_i + \theta_i$, assuming zero mean Gaussian distributions wit precision $\tau_\gamma \mathbf{I}$, $\tau_\alpha \mathbf{K}$, $\tau_\phi \mathbf{I}$ and $\tau_\theta \mathbf{R}$, for γ, α, ϕ and θ, respectively.
- The $\mathbf{I}$ matrix is the identity matrix with the needed dimension, and $\mathbf{K}$ and $\mathbf{R}$ matrices are the temporal and spatial precision structure matrices defined from the neighborhood structure

The Knorr-Held (2000) models (cont.)

- Let $\mathbf{R} = \tilde{\mathbf{G}} - \mathbf{G}$, where $\mathbf{G}$ is the spatial neighborhood structure, defined as

$$\mathbf{G}_{i,j} = \begin{cases} 1 & \text{if} \quad j \sim i \\ 0 & \text{otherwise,} \end{cases}$$

  where $j \sim i$ means "j neighbor to i", $\tilde{\mathbf{G}}$ is the diagonal matrix, having its diagonal as the row sum of $\mathbf{G}$.

- The precision for the distribution of d (considering the type IV interaction) is given as

$$\tau_d \mathbf{K} \otimes \mathbf{R} = \tau_d (\tilde{\mathbf{H}} - \mathbf{H}) \otimes (\tilde{\mathbf{G}} - \mathbf{G}) = \\ \tau_d (\tilde{\mathbf{H}} \otimes \tilde{\mathbf{G}} - \tilde{\mathbf{H}} \otimes \mathbf{G} - \mathbf{H} \otimes \tilde{\mathbf{G}} + \mathbf{H} \otimes \mathbf{G}) \qquad (2)$$

Continuous domain attempts

Continuous domain case

Areal count data

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- Relative risk model: $r_i = \exp(f_i + u_i = \eta_i)$
  - y_i cases at area i given the expected number of cases E_i

$$y_i \sim \mathrm{Poisson}(E_i r_i)$$

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$$u_i = \int_{\mathbb{A}_i} u(\mathbf{s})\partial\mathbf{s} \approx \tilde{u}_i$$

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$$\text{Jensen's inequality: } \exp\left(\tilde{u}_i = \frac{\sum_j a_j u_j}{\sum_j a_j}\right) \leq \frac{\sum_j a_j \exp(u_j)}{\sum a_j} = \hat{u}_i$$

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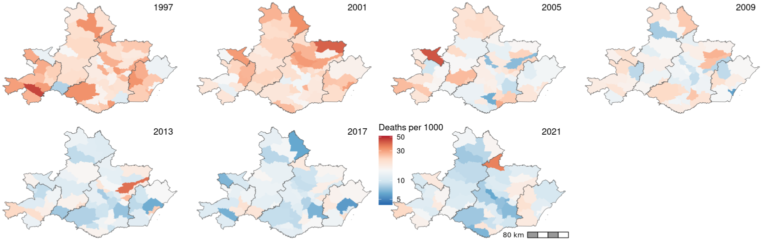
The right thing to do: Integral in the risk scale

$$u_i = \int_{\mathbb{A}_i} \exp(u(\mathbf{s}))\partial\mathbf{s} \approx \hat{u}_i$$

Application (work in progress.)



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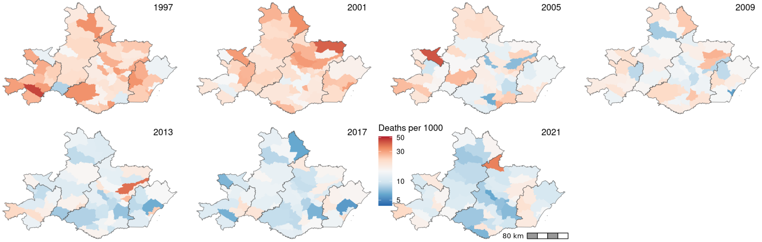

$$y_{it} \sim \mathrm{Poisson}(n_{it}\lambda_{it}), \quad \log(\lambda_{it}) = \eta_{it}, \quad i \in \{1, ..., n\}, \ t \in \{1, ..., T\}$$

Models considered

- $M0$, a model without spatio-temporal effect
  - $\eta_{ij} = \alpha + v_t + r_i$
- Add a spatio-temporal areal effect
  - considering the Knorr-Held's type 1, 2, 3 and 4
  - a proper spatio-temporal interaction
- Proper models for **v** and **r** plus spatio-temporal continuous domain models: MA (102), MB (121), MC (202) and MD (220)

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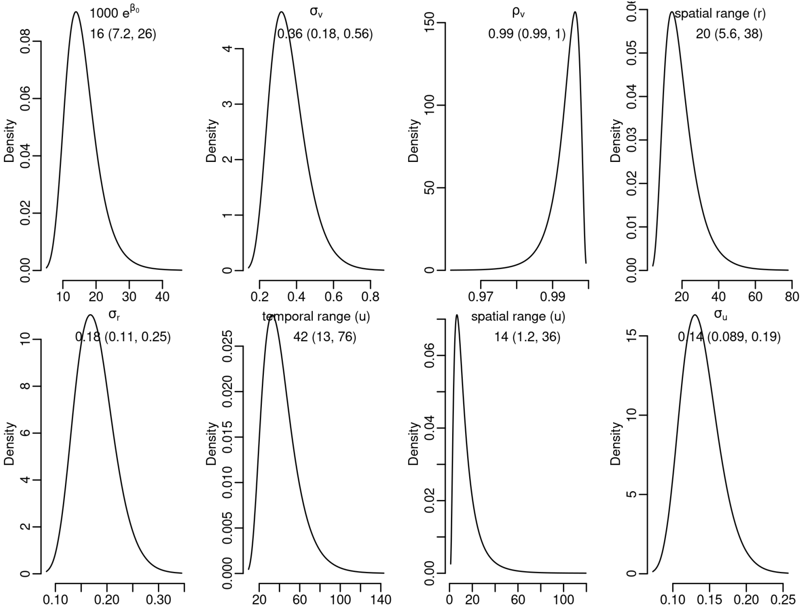
DIC, WAIC, minus the sum of the log score (LPO) and its cross-validated version (LCPO)

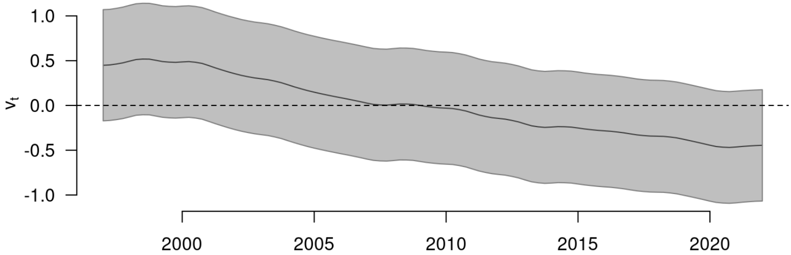
	M0	T1	T2	T3	T4	T4p	M102	M121	M202	M220
DIC	62.7	14.6	11.9	23.6	11.2	0.0	9.1	9.6	11.4	12.3
WAIC	66.9	8.6	17.6	28.7	17.0	0.0	12.5	12.6	15.7	16.4
LCPO	26.6	15.1	4.6	16.3	3.7	0.0	1.5	1.8	2.8	3.1
LPO	143.4	0.0	63.8	23.3	63.4	16.2	68.7	67.8	72.2	74.1

$*$ differences with respect to the lowest

Posterior for the model parameters with MD

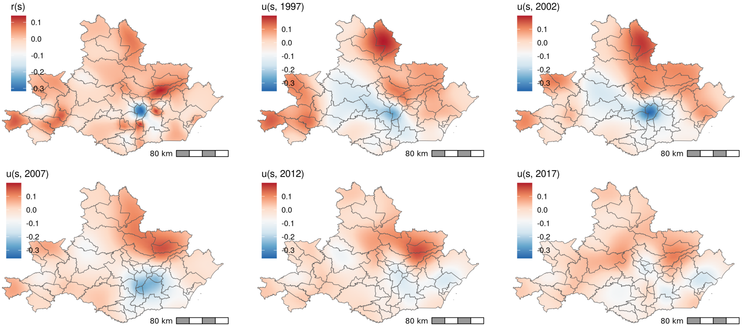


Time effect



- **v** could be replaced by a linear trend in this case

Spatial and spatio-temporal (with MB)



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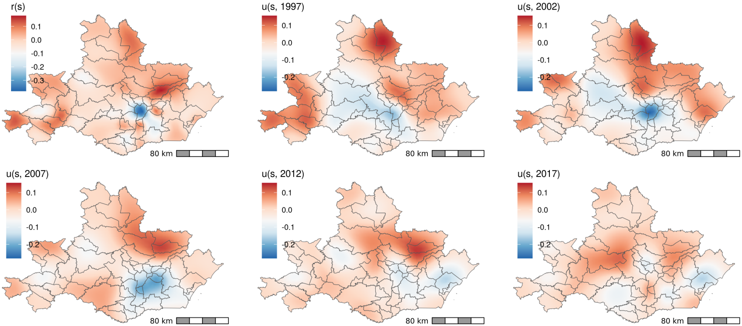

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