# An overview of the SPDE approach

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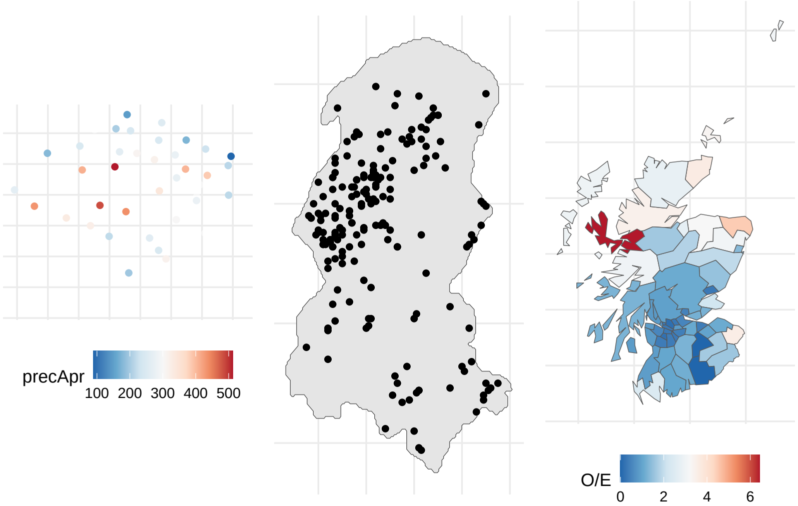
# Outline

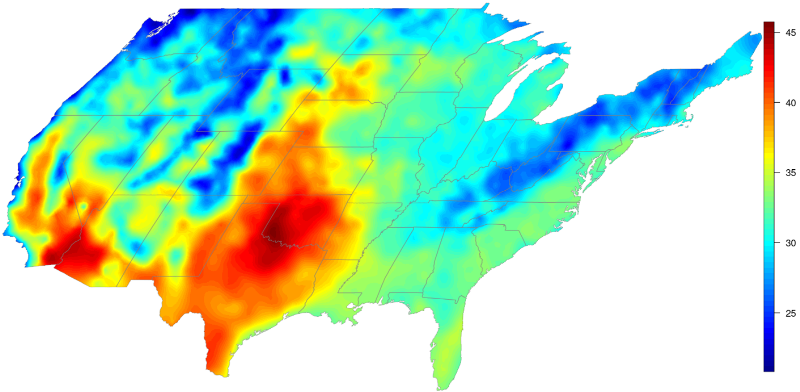
# Introduction

# Spatial statistics data



# A process over space

- (est.) Maximum temperature in US mainland, in 2022-07-20



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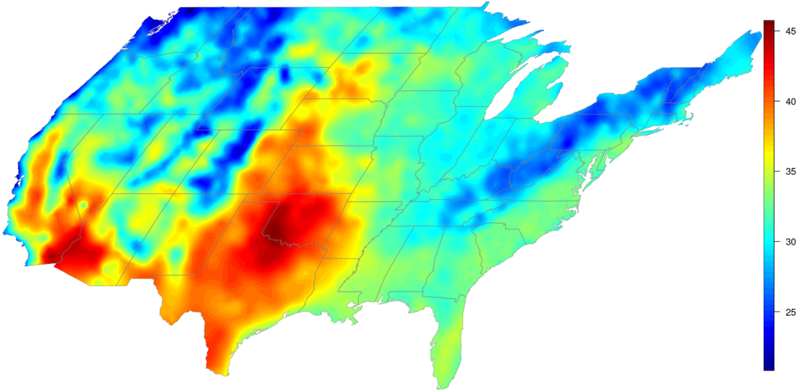


- The spatial domain, S, is continuous.
  - E.g. $S \in \mathbb{R}^2$
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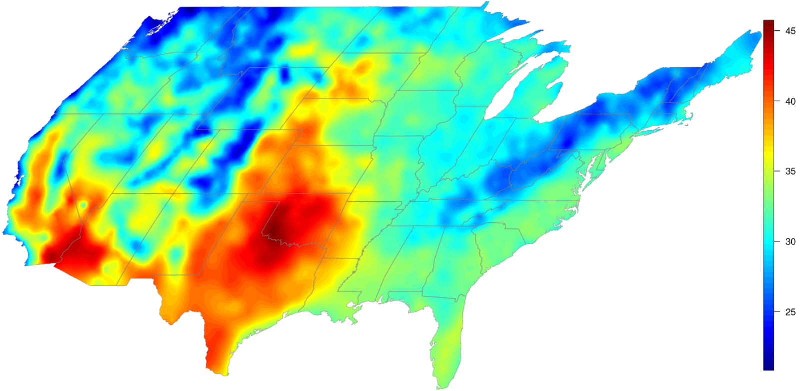


- The spatial domain, S, is continuous.
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- $u(\mathbf{s})$ is a stochastic process, $\mathbf{s} \in S$

# Real world data: at a finite number of locations

At a finite set of $n$ locations

$u(\mathbf{s}_1), \ldots, u(\mathbf{s}_n)$

# Stochastic process $u(\mathbf{l})$

- Covariance

$$V(\mathbf{l}, \mathbf{l}') = \mathrm{Cov}(u(\mathbf{l}), u(\mathbf{l}'))$$

- Spectral density

$$u(\mathbf{l}) = \int_{-\infty}^{\infty} \mathrm{e}^{i\mathbf{w}\mathbf{l}} du(\mathbf{w})$$

- Kernel convolution

$$u(\mathbf{l}) = \int k(\mathbf{l} - \mathbf{u}) \mathcal{W}(\mathbf{l}) d\mathbf{u}$$

- Stochastic Partial Differential Equation - SPDE

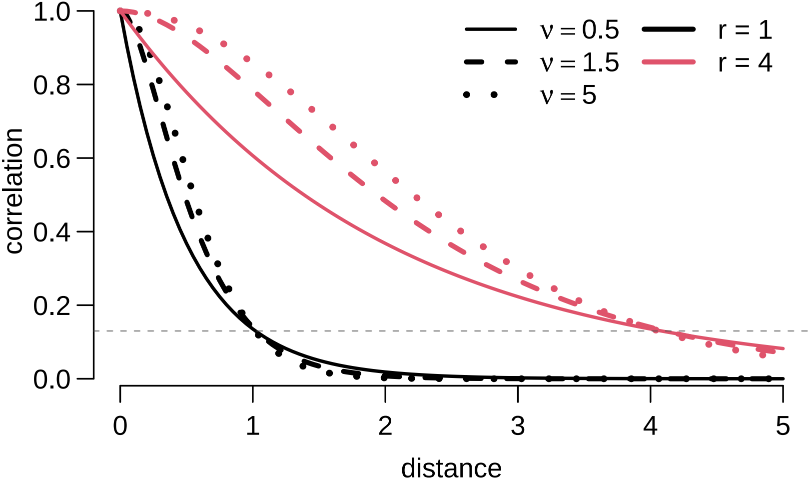
$$\mathcal{L}_1 u(\mathbf{l}) = \mathcal{L}_2 \mathcal{W}(\mathbf{l})$$

- Conditional distributions ( DISCRETE DOMAIN !!! )

$$u_{\mathbf{l}} | u_{\mathbf{neighbourhood}} \sim \mathbb{P}(.)$$

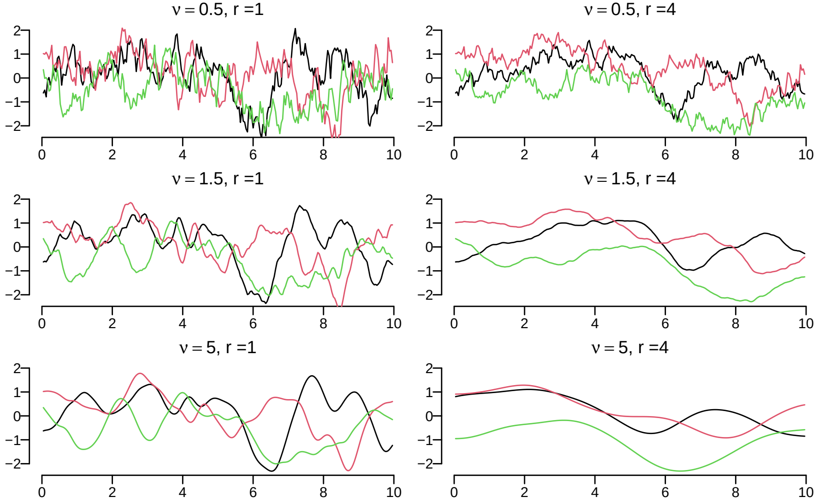
# Whittle-Matérn covariance, Matérn (1960)

$\Sigma_{ij} = \frac{\sigma^2(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)^\nu K_\nu(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)}{\Gamma(\nu + d/2)(4\pi)^{d/2}\kappa^{2\nu}2^{\nu-1}}$. If $d = 2$ and $\nu = 1$: Whittle (1954)

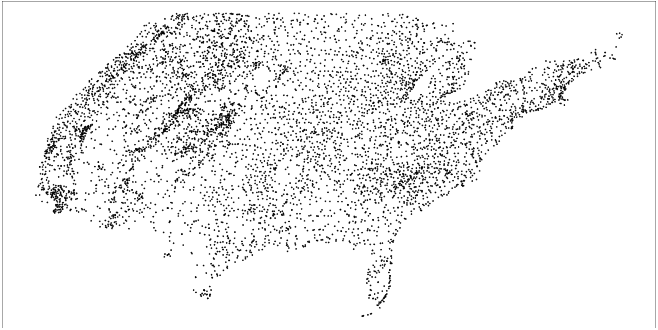



practical range $=$ r $= \sqrt{8\nu}/\kappa$, corr(r) $\approx 0.13$

# Simulations, 1D, $\sigma^2 = 1$



# Consider a set of locations



- Consider the distribution at these locations
  - $\pi(u(\mathbf{s}_1), ..., u(\mathbf{s}_n)|\theta) = \pi(u_1, ..., u_n|\theta) = \pi(\mathbf{u}|\theta)$

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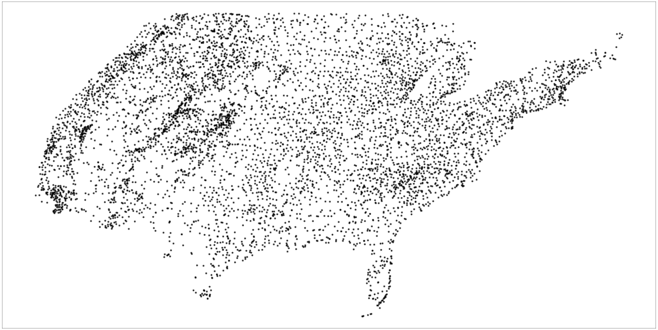

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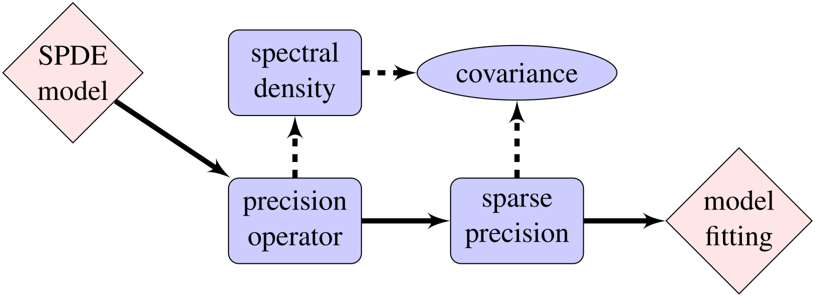
  - $\pi(u(\mathbf{s}_1), ..., u(\mathbf{s}_n)|\theta) = \pi(u_1, ..., u_n|\theta) = \pi(\mathbf{u}|\theta)$
- If it is Gaussian with some covariance $\Sigma(\theta)$

$$\pi(\mathbf{u}|\theta) = (2\pi)^{-n/2}|\Sigma(\theta)|^{-1/2} \exp\left(-\frac{1}{2}\mathbf{u}^T\Sigma(\theta)^{-1}\mathbf{u}\right)$$

# The SPDE approach

# SPDE framework



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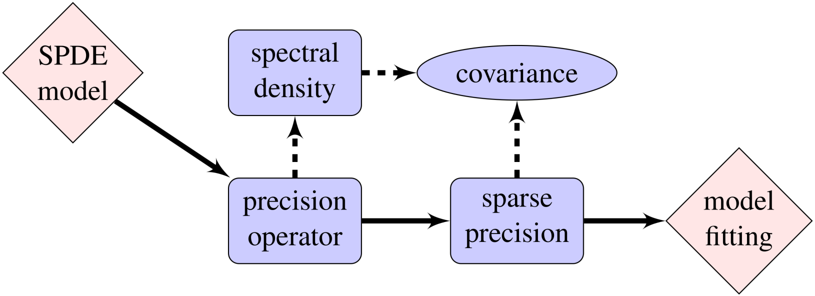


- It avoids specifying covariance!
  - D. Simpson, Lindgren, and Rue (2011)
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# The Matérn's SPDE

- Whittle (1954), Whittle (1963):
  - Fields with Matérn covariance are solutions to the following Stochastic Partial Differential Equation (SPDE)

$$\tau(\kappa^2 - \Delta)^{\alpha/2} u(\mathbf{s}) = \mathcal{W}(\mathbf{s})$$

  - $\kappa > 0$: scale parameter
  - $\alpha = \nu + d/2$: smoothness
  - $\Delta$ is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

# Lindgren, Rue, and Lindström (2011)

- Discretization
  - sparse precision matrix:
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- Equivalent models (discretized): Whittle (1954), Besag (1974), Besag (1981), Besag and Kooperberg (1995) and Besag and Mondal (2005).

## $\mathbf{Q}_\alpha(\tau, \kappa)$: grid and piecewise linear basis

- $\alpha = 1$: $\tau \mathbf{K}_1(\kappa) = \tau(\kappa^2 \mathbf{C} + \mathbf{G})$
  - d=1, $u_1, u_2, ..., u_n$, two neighbours

$$\tau^2 \begin{bmatrix} 1+\kappa^2 & -1 & & & \\ -1 & 2+\kappa^2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2+\kappa^2 & -1 \\ & & & -1 & 1+\kappa^2 \end{bmatrix}$$

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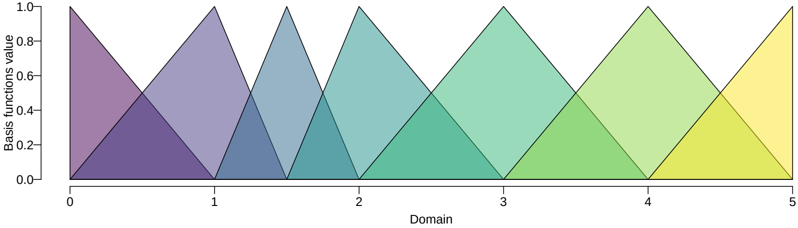
  - $d = 2$, $\mathbf{C} = \mathbf{I}$, $\mathbf{G}$ = Laplacian (4 neighbours)

# Piecewise linear basis, Finite Element Method (FEM): 1d



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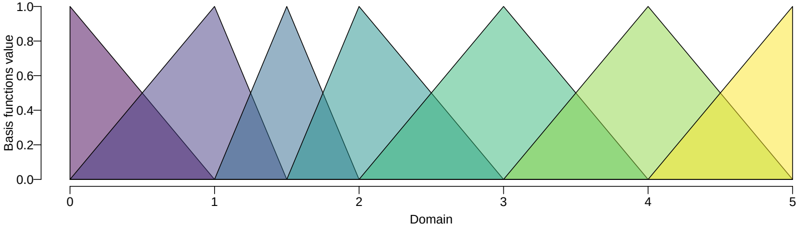
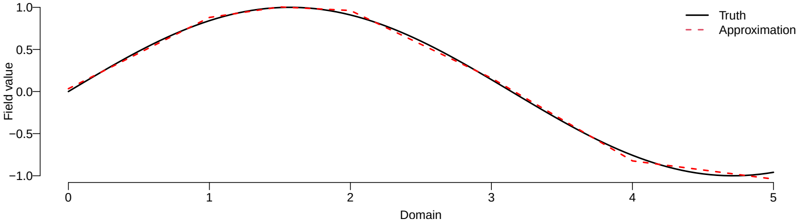



- $u(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s}) u_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0) u(\mathbf{s}_0)$,
  - $\psi_k$: basis functions evaluated at data locations $\mathbf{s}$
  - $u_k$: the process at the discretization points $\mathbf{s}_0$



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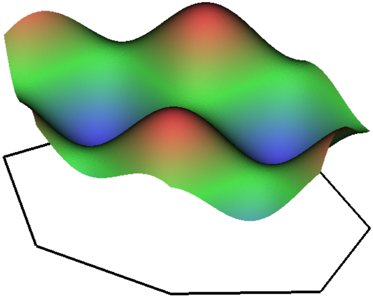

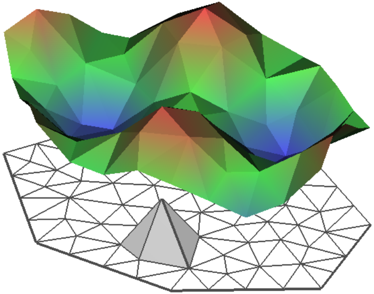

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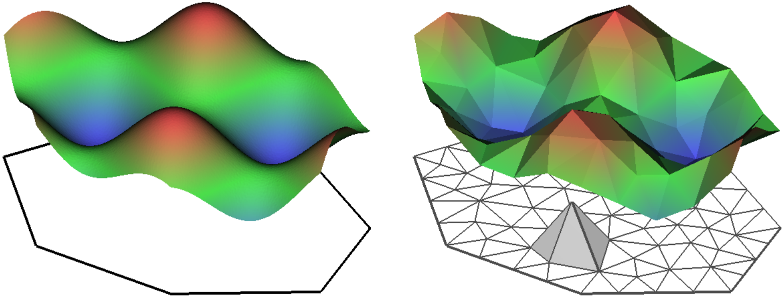

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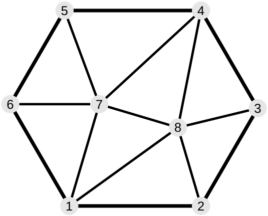
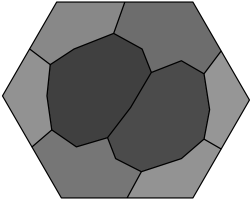
# Piecewise linear basis, FEM matrices

**Mesh nodes**




**Dual mesh**



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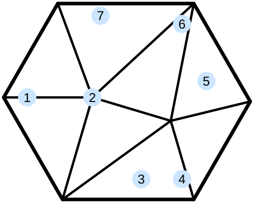

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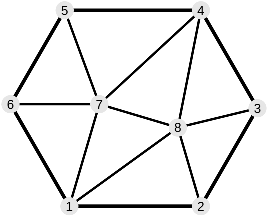


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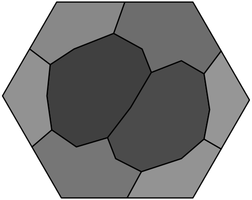

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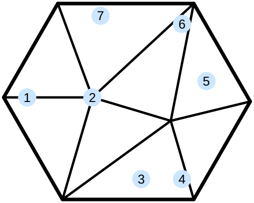
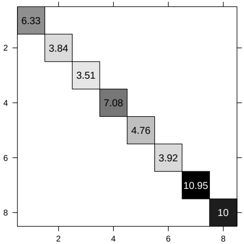
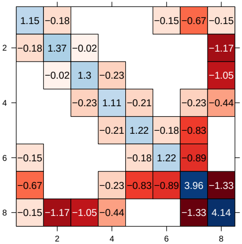



**C**




**G**

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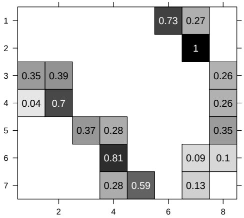

# The log Gaussian Cox process model

# Point pattern and intensity

- Given a set of locations on a domain $\mathcal{D}$
- One interest is to estimate the intensity function
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  - $\lambda(\mathbf{l})$ is a random variable
- Log Gaussian Cox Process (LGCP)
  - $\log(\lambda(.)) = \text{u}(.)$ is a Gaussian process - GP, Møller, Syversveen, and Waagepetersen (1998) .
  - $u(.|\theta)$, $\theta$ are GP parameters

# LGCP inference

- The log-likelihood function:

$$l(\Lambda, \theta|\mathcal{Y}) = c - \int_{\mathcal{D}} \lambda(l) \partial l + \sum_{i=1}^{n} \log(\lambda(\mathbf{l}_i))$$

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- The log-likelihood function direct approximation

$$\begin{aligned} l(\Lambda, \theta|\mathcal{Y}) &\approx c - \textstyle\sum_{j=1}^{m} w_j \lambda(l) + \sum_{i=1}^{n} \log(\lambda(\mathbf{l}_i)) \\ &= c - \textstyle\sum_{j=1}^{m} w_j \exp(\eta(l)) + \sum_{i=1}^{n} \eta(\mathbf{l}_i) \end{aligned}$$

  approximated with $m$ integration points.

- SPDE approach for easier computations, D. P. Simpson et al. (2016)

- more complex Point Process models using INLA in inlabru

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