Some spacetime models for areal data

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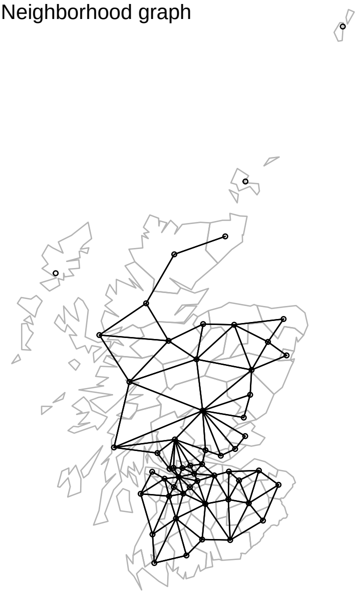
Outline

The pre spacetime work

A graph from the map (Scotland)



Besag: randon walk over areas, Besag (1974)

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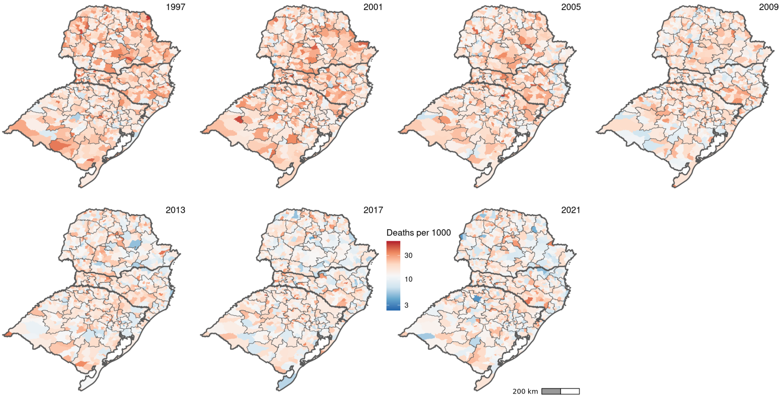
$$\mathbf{R}_{ij} = \begin{cases} n_i & \text{if } i = j \\ -1 & \text{if } j \sim i \\ 0 & \text{otherwise} \end{cases} .$$

The spacetime discrete domain case

$$\mathbf{u} = \{u_{11}, \ldots, u_{n1}, u_{12}, \ldots, u_{n2}, \ldots, u_{1T}, \ldots, u_{nT}\}$$



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- Interaction between fixed and random terms
  - linear trend for each area, Bernardinelli et al. (1995)
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- Interaction between spatial and temporal random effects, Clayton (1996)

Spacetime models

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- The Kronecker product models follows the Clayton's rule
- Combine **Q1** (time) and **Q2** (space) available

Some examples on spacetime interactions

- Precision($\mathbf{u}$) $= \tau \mathbf{H} \otimes \mathbf{R}$, Knorr-Held (2000)
  - $\mathbf{H}$: precision structure matrix over the temporal domain
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- **care** when main effects are in the model
- **super care** when **Q1** and/or **Q2** have rank deficiency
  - e.g. `rw1`, `rw2` and `besag` models
  - if both Q1 and Q2 are intrinsic, Knorr-Held (2000):
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- Proper in time, Martínez-Beneito, López-Quilez, and Botella-Rocamora (2008)
  - example in INLA:

```
f(spatial, model='besag', ...,
group=time, control.group=list(model='ar1'))
```

  - NOTE: Here the constraints are only set for the main model, and not for the group model!

The Knorr-Held (2000) models

- Linear predictor is modeled as

$$\eta_{it} = \text{other effects} + v_t + s_i + d_{it}$$

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  - d_{it} is the space-time effect.
- Decompose: $v_t = \gamma_t + \alpha_t$ and $s_i = \phi_i + \theta_i$
  - zero mean Gaussian distributions, with precision $\tau_\gamma \mathbf{I}$, $\tau_\alpha \mathbf{K}$, $\tau_\phi \mathbf{I}$ and $\tau_\theta \mathbf{R}$, for γ, α, ϕ and θ, respectively.
  - $\mathbf{I}$: identity matrix with required dimention
  - $\mathbf{K}$ and $\mathbf{R}$: temporal and spatial precision structure matrices, from the neighborhood structure

The Knorr-Held (2000) models (cont.)

- Let $\mathbf{R} = \tilde{\mathbf{G}} - \mathbf{G}$, where
  - $\mathbf{G}$ is the spatial neighborhood structure

$$\mathbf{G}_{i,j} = \begin{cases} 1 & \text{if} \quad j \sim i \\ 0 & \text{otherwise,} \end{cases}$$

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- Considering the type IV interaction:

$$\begin{aligned} \tau_d \mathbf{K} \otimes \mathbf{R} &= \tau_d(\tilde{\mathbf{H}} - \mathbf{H}) \otimes (\tilde{\mathbf{G}} - \mathbf{G}) \\ &= \tau_d(\tilde{\mathbf{H}} \otimes \tilde{\mathbf{G}} - \tilde{\mathbf{H}} \otimes \mathbf{G} - \mathbf{H} \otimes \tilde{\mathbf{G}} + \mathbf{H} \otimes \mathbf{G}) \end{aligned}$$

Continuous domain attempts

Areal count data

- Relative risk model: $r_i = \exp(f_i + u_i = \eta_i)$
  - y_i cases at area i given the expected number of cases E_i

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$$\text{Jensen's inequality: } \exp\left(\tilde{u}_i = \frac{\sum_j a_j u_j}{\sum_j a_j}\right) \leq \frac{\sum_j a_j \exp(u_j)}{\sum a_j} = \hat{u}_i$$

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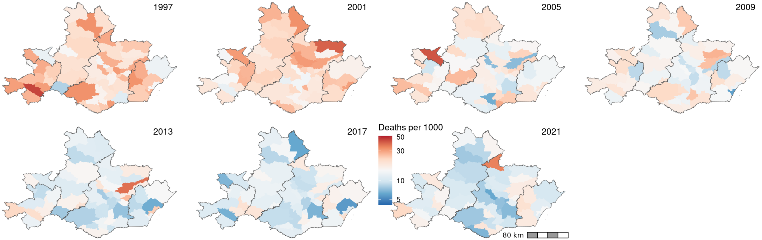
The right thing to do: Integral in the risk scale

$$u_i = \int_{\mathbb{A}_i} \exp(u(\mathbf{s}))\partial\mathbf{s} \approx \hat{u}_i$$

Application (work in progress.)



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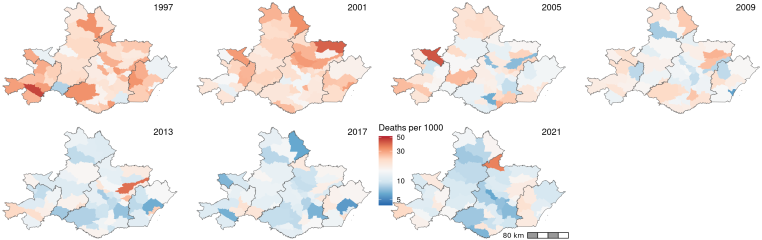

$$y_{it} \sim \mathrm{Poisson}(n_{it}\lambda_{it}), \quad \log(\lambda_{it}) = \eta_{it}, \quad i \in \{1, ..., n\}, \; t \in \{1, ..., T\}$$

Models considered

- $M0$, a model without spatio-temporal effect
  - $\eta_{ij} = \alpha + v_t + r_i$
- Add a spatio-temporal areal effect
  - considering the Knorr-Held's type 1, 2, 3 and 4
  - a proper spatio-temporal interaction
- Proper models for **v** and **r** plus spatio-temporal continuous domain models: MA (102), MB (121), MC (202) and MD (220)

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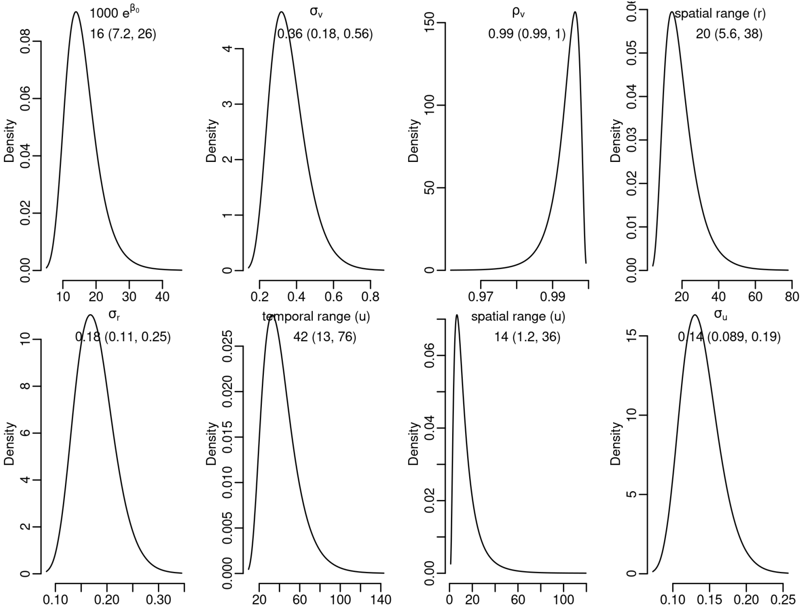
DIC, WAIC, minus the sum of the log score (LPO) and its cross-validated version (LCPO)

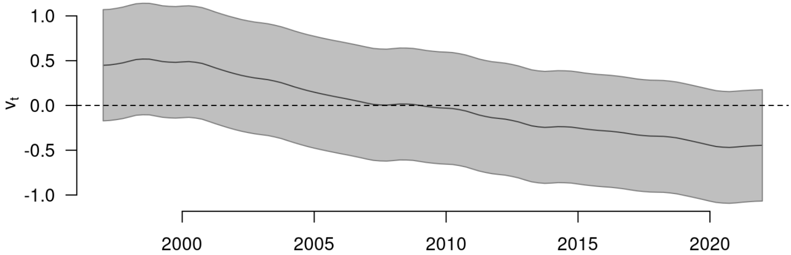
	M0	T1	T2	T3	T4	T4p	M102	M121	M202	M220
DIC	62.7	14.6	11.9	23.6	11.2	0.0	9.1	9.6	11.4	12.3
WAIC	66.9	8.6	17.6	28.7	17.0	0.0	12.5	12.6	15.7	16.4
LCPO	26.6	15.1	4.6	16.3	3.7	0.0	1.5	1.8	2.8	3.1
LPO	143.4	0.0	63.8	23.3	63.4	16.2	68.7	67.8	72.2	74.1

$*$ differences with respect to the lowest

Posterior for the model parameters with MD

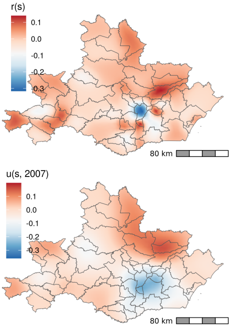


Time effect



- **v** could be replaced by a linear trend in this case

Spatial and spatio-temporal (with MB)



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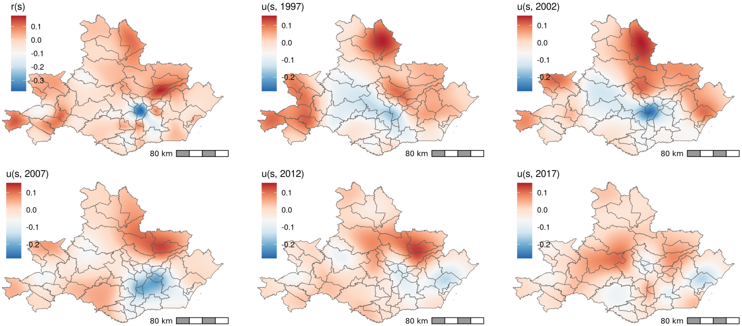

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