An spacetime extension for the SPDE approach

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May 2024

Outline

Ideas and motivation

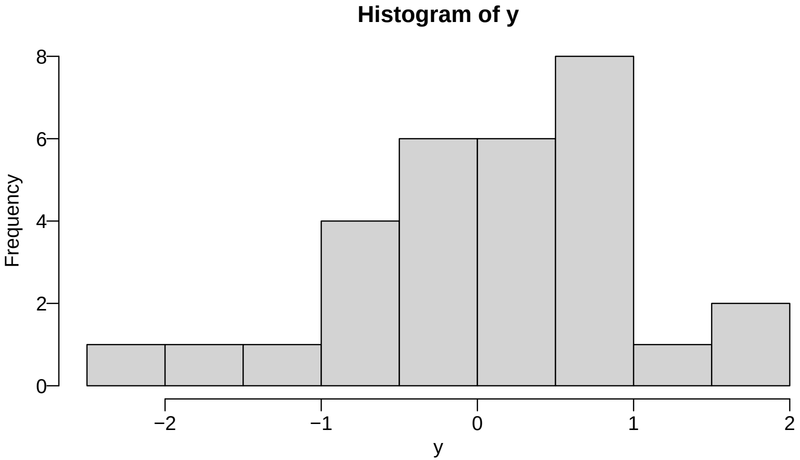
An spacetime SPDE model extension, Lindgren et al. (2024)

Implementation

Examples

Ideas and motivation

Symmetric or assymetric?



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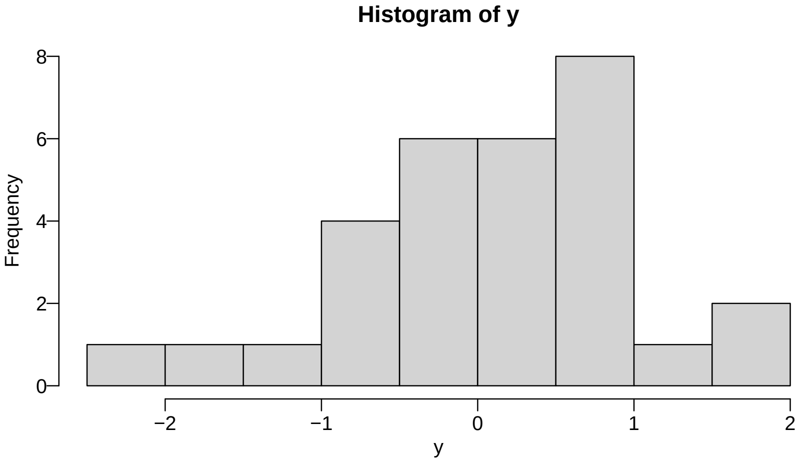
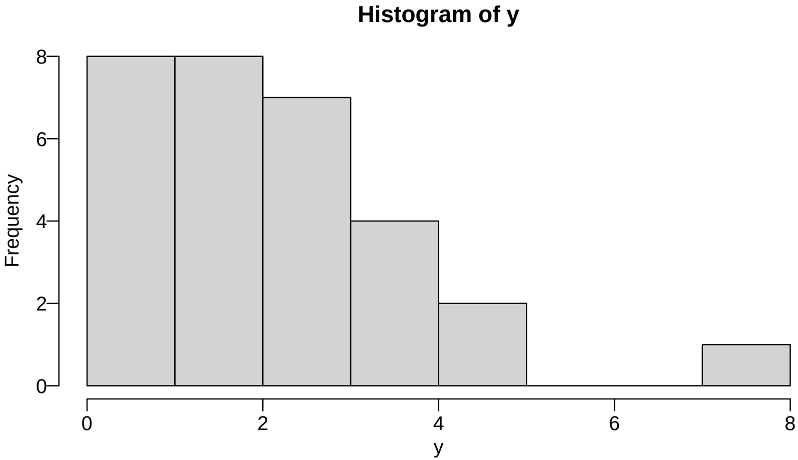



```r
set.seed(1); y <- rnorm(30); hist(y)
```

What causes this to be assymetric?



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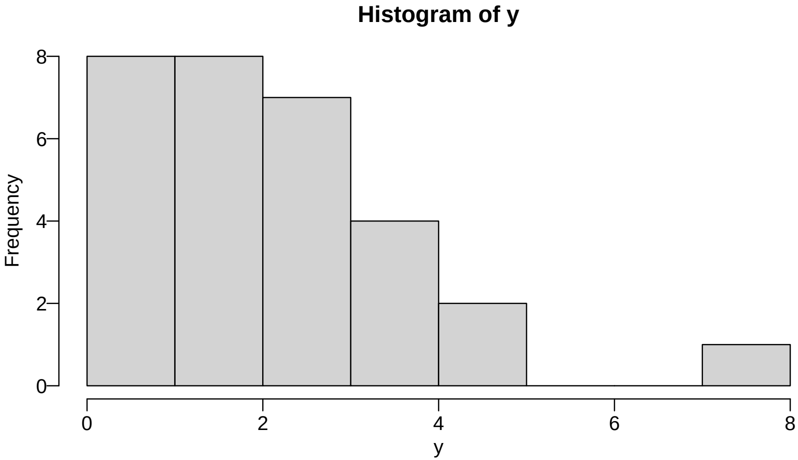


```r
set.seed(1); x = rexp(30); y = 1 + x + rnorm(30); hist(y)
```

Stationary and non-stationary

Stationary

- Covariance is the same over the domain
  - E.g. 1d: does not matter where, but only the lag

$$Cov[u(t_1), u(t_2)] = cov(h), \quad h = |t_1 - t_2|$$

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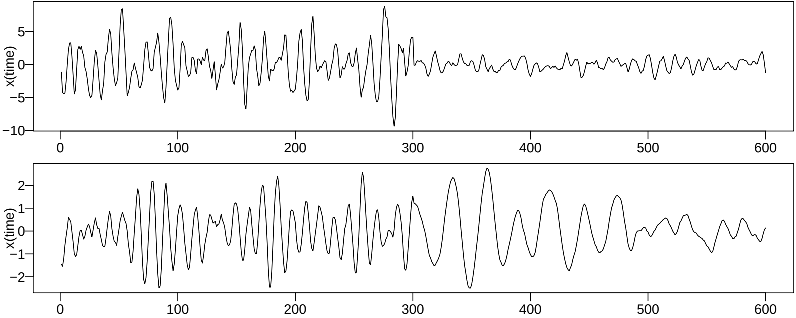
$$Cov[u(t_1), u(t_2)] = cov(h), \quad h = |t_1 - t_2|$$

Non-stationary

- Any model where precision/covariance depends on the location
  - e.g.: 1 dimentional example (only varying over time)



Stationary or non-stationary?

ELSEVIER

Spatial Statistics

Volume 14, Part C, November 2015, Pages 505-531

Does non-stationary spatial data always require non-stationary random fields?

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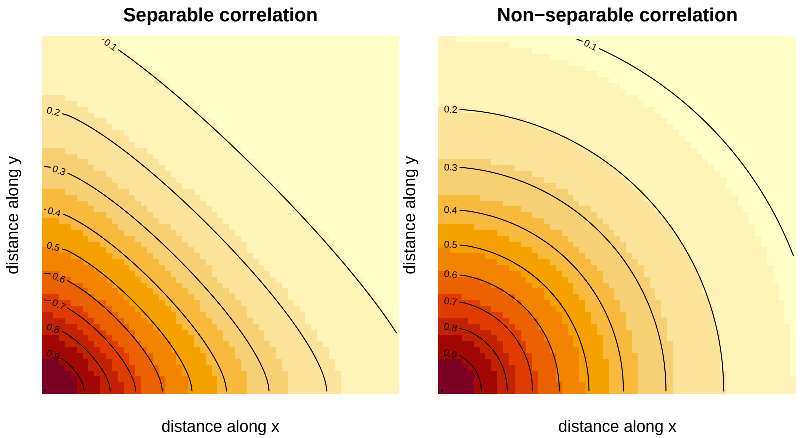
It is hard to beat simple suitable models, Fuglstad et al. (2015).

Separable or non-separable?



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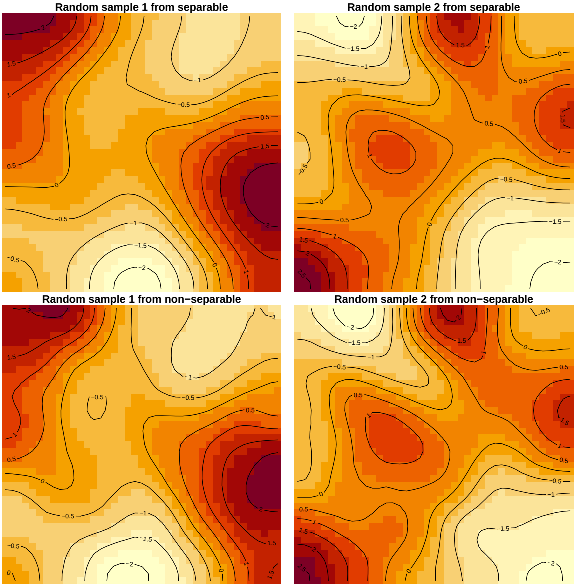
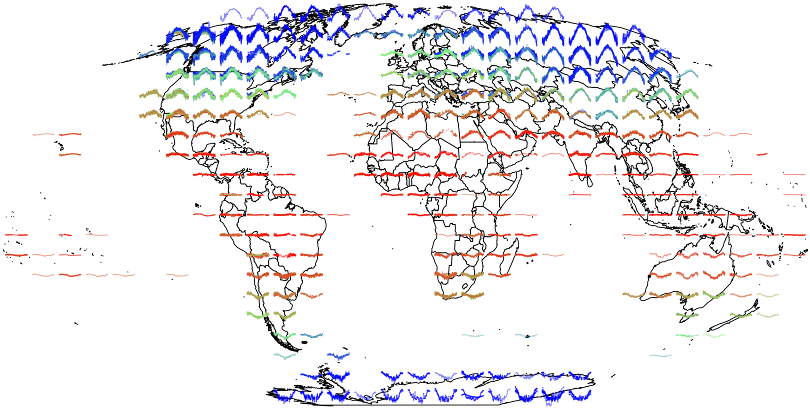


Global temperature, see Lindgren et al. (2024)



Symmetric? Stationary? Separable?

An spacetime SPDE model extension, Lindgren et al. (2024)

The space-time models

Models for processes that on the space-time domain

- Discrete space-time domain
  - $\mathbf{u} = \{u_{11}, ..., u_{n1}, u_{12}, ..., u_{n2}, ..., u_{1T}, ..., u_{nT}\}$
- Continuous space-time domain
  - $\mathbf{u}(\mathbf{s}, t)$, $\mathbf{s} \in \mathbb{R}^d$, $t \in \mathbb{R}$

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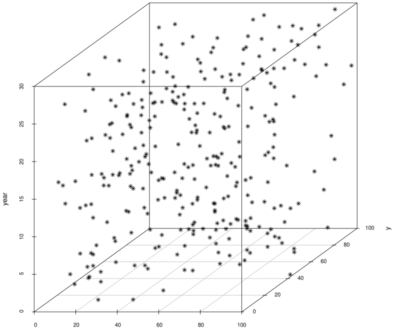
- Continuous space-time domain
  - $\mathbf{u}(\mathbf{s}, t)$, $\mathbf{s} \in \mathbb{R}^d$, $t \in \mathbb{R}$
- Real world (locations)



Separable space-time models

- When covariance/precision can be written as Kronecker product of a purely temporal and purely spatial ones
- **E.g.:** Discrete space-time domain

$$\pi(\mathbf{u}) \propto (|\mathbf{Q1} \otimes \mathbf{Q2}|^{-1})^{1/2} \exp\left(-\frac{1}{2}\mathbf{u}^T\{\mathbf{Q1} \otimes \mathbf{Q2}\}\mathbf{u}\right)$$

  - **Q1** has dimension equals T
  - **Q2** has dimension equals n

Desired properties of a space-time process

- Three desired properties, Stein (2013)
  - **1st:** "any degree of smoothness in space and any (possibly different) degree of smoothness in time"
  - **2nd:** "smoother away from the origin than it is at the origin"
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Why SPDE?

- A solution to a linear non-fractional SPDE is a Markov process, Rozanov (1977)
  - Reciprocal of its spectral density is a polynomial
    - 1st and 2nd desired properties follows
- Allow efficient computations
  - Replace Stein's third desired property by
    - fitting process computationally efficient
- More in Vergara, Allard, and Desassis (2022)

Space-time SPDE models

- SDE in time **then** SPDE in space

$$
\begin{aligned}
\left(\gamma_t \frac{\partial}{\partial t} + a^2\right) u(\mathbf{s}, \delta t) = \xi(\mathbf{s}, \delta t) \\
(\kappa^2 - \Delta)^{\alpha_e/2} \xi(\mathbf{s}, \delta t) = \mathcal{W}(\mathbf{s}, \delta t).
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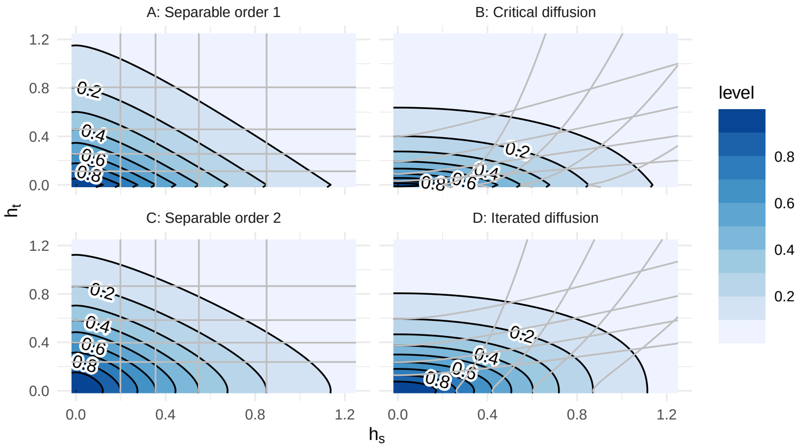
Table 1: Four specific models on 2-dimensional spatial domains.

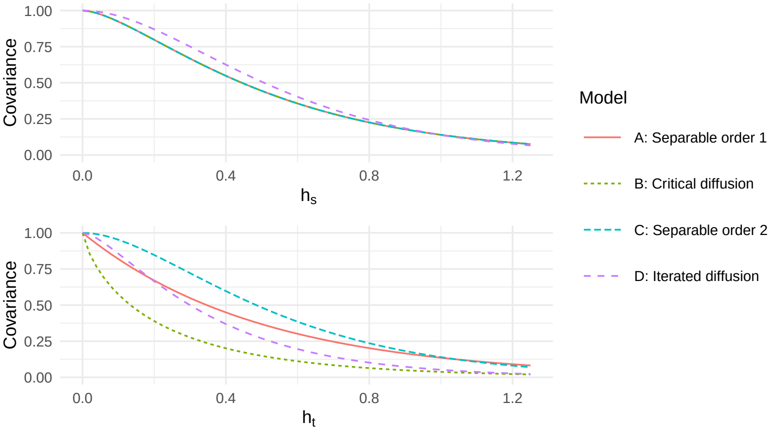
Model	α_t	α_s	α_e	Type	ν_t	ν_s	nonsep
A:(1,0,2)	1	0	2	Separable order 1	1/2	1	0.0
B:(1,2,1)	1	2	1	Critical diffusion	1/2	1	0.5
C:(2,0,2)	2	0	2	Separable order 2	3/2	1	0.0
D:(2,2,0)	2	2	0	Iterated diffusion	1	2	1.0

Separability is a function of α_t, α_s, α_e and d.

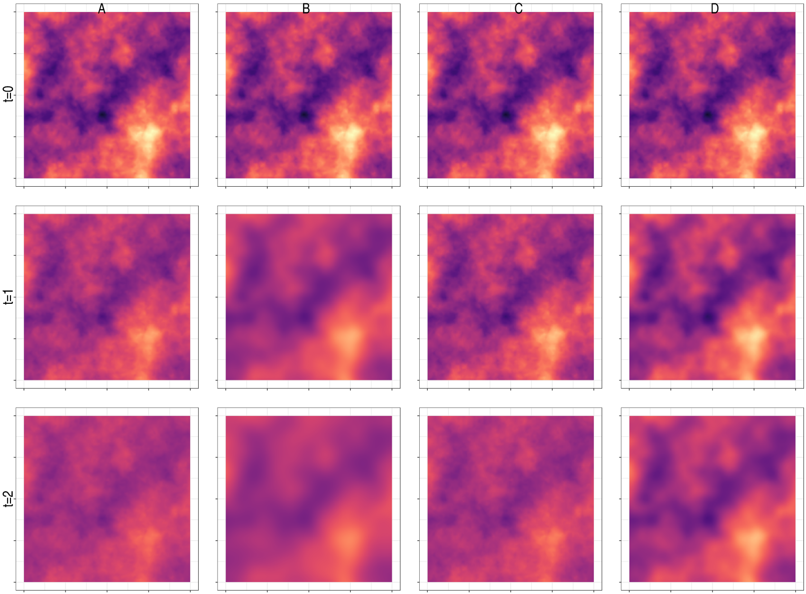
Space-time correlations



Marginal correlations



Example on predictions



Precision matrix

- A sum of Kronecker products

$$\mathbf{Q}_u(\gamma_s, \gamma_t, \gamma_s) = \gamma_e^2 \sum_{k=0}^{2\alpha_t} \gamma_t^k \mathbf{J}_{\alpha_t, k/2} \otimes \mathbf{K}_{\alpha_s}(\gamma_s)$$

where the $\mathbf{J}_{\alpha_t, k/2}$ matrices are from the temporal discretization.

Interpretable parametrization

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- The parameters in the SPDE are local
- It is easier to consider the marginal parameters
- $\sigma = \frac{C(\alpha_t, \alpha_s, \alpha_e, d)}{\gamma_e^2 \gamma_t \gamma_s^{2\alpha - d}}$
- $r_s = \sqrt{8\nu}/\gamma_s^2$
- $r_t = \frac{\gamma_t \sqrt{8(\alpha_t - 0.5)}}{\gamma_s^{\alpha_s}}$

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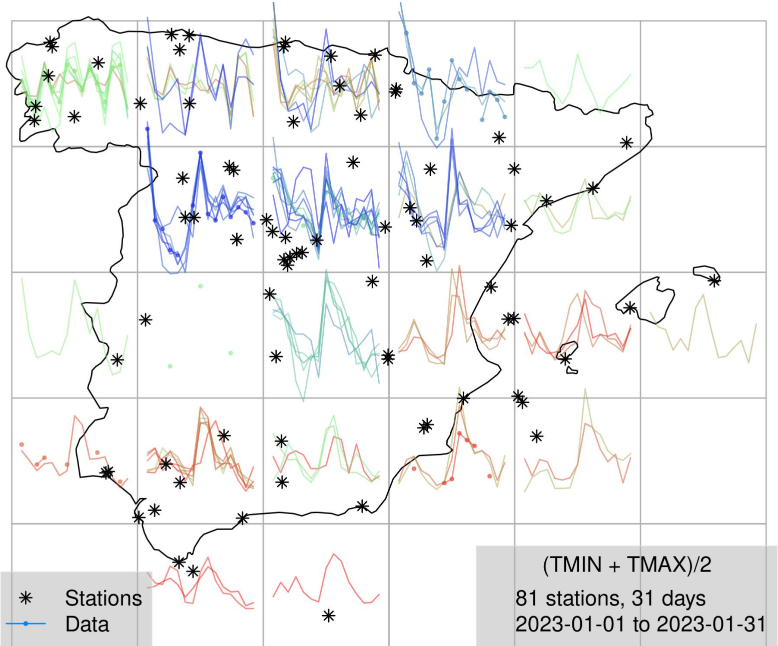
- **INLAspacetime** package in
  - https://github.com/eliaskrainski/INLAspacetime

Examples

Daily temperature data in Spain, January 2023



The data in the long format

```
str(dataf)
## 'data.frame':    2511 obs. of  6 variables:
##  $ iloc: int  1 2 3 4 5 6 7 8 9 10 ...
##  $ xloc: num  442 168 367 578 925 ...
##  $ yloc: num  4474 4311 4059 4795 4584 ...
##  $ elev: num  0.667 0.185 0.007 0.251 0.004 ...
##  $ time: int  1 1 1 1 1 1 1 1 1 1 ...
##  $ resp: num  11.3 15.1 13.8 16.9 12.2 ...
```

Elevation (in km) to be used as covariate

Model definition, with INLAspacetime

Define the space-time model with `INLAspacetime`

```
stmodel <- stModel.define(
  smesh = smesh, # spatial mesh
  tmesh = tmesh, # temporal mesh
  model = "121", # model (\alpha_t, \alpha_s, \alpha_e)
  control.priors = list(
    prs = c(100, 0.05), # P(spatial range < 100) = 0.05
    prt = c(2, 0.05),   # P(temporal range < 1) = 0.05
    psigma = c(4, 0.05)), # P(sigma_u > 4) = 0.05
  constr = TRUE)
```

Define the right-hand-side, `inlabru`

```
M <- ~ Intercept(1) + elev +
  field(list(space = cbind(xloc, yloc),
             time = time),
        model=stmodel)
```

Model fit with inlabru

Set the PC-prior for the likelihood precision

```
lkprec <- list(
  prec = list(prior = "pcprec",
              param = c(4, 0.05)))
```

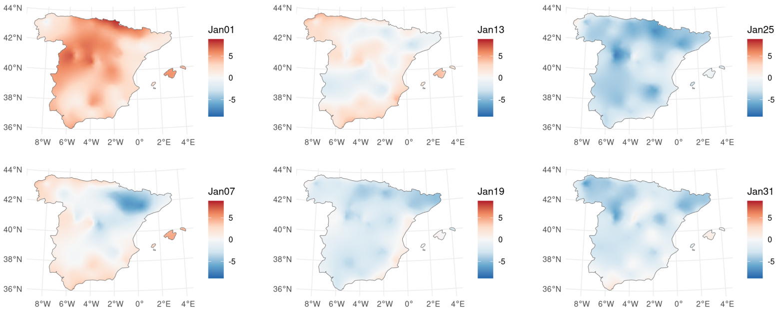
The likelihood setup in inlabru

```
lhood <- like(
  formula = resp ~ .,
  family = "gaussian",
  control.family = list(
    hyper = lkprec),
  data = dataf)
```

Fit

```
result <- bru(M, lhood)
```

Posterior mean of $u(.)$, for some days



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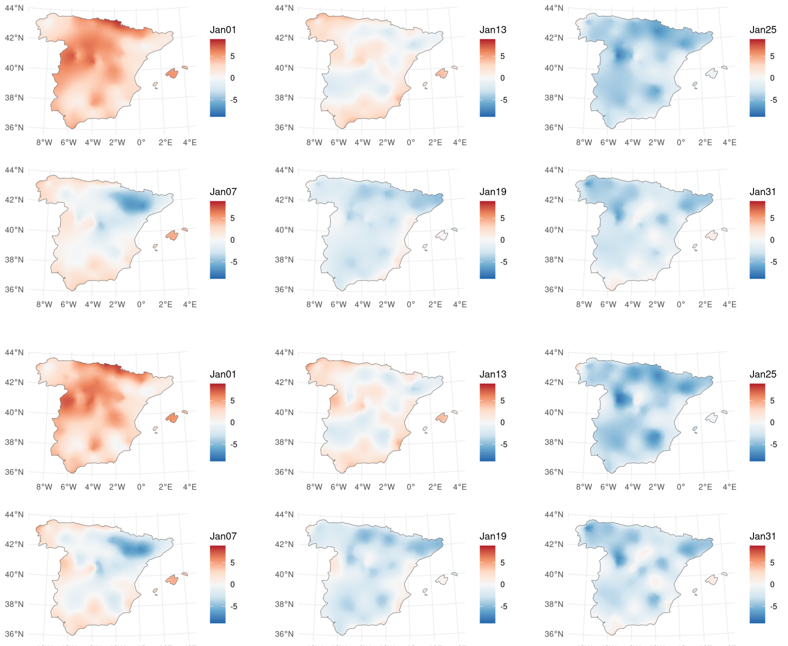
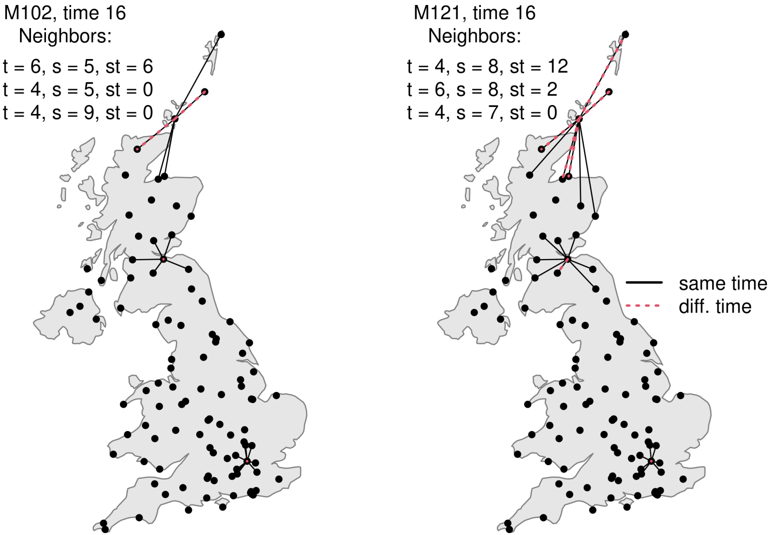


UK example: Daily wind speed, `inla.gcv(..., m = 10)`



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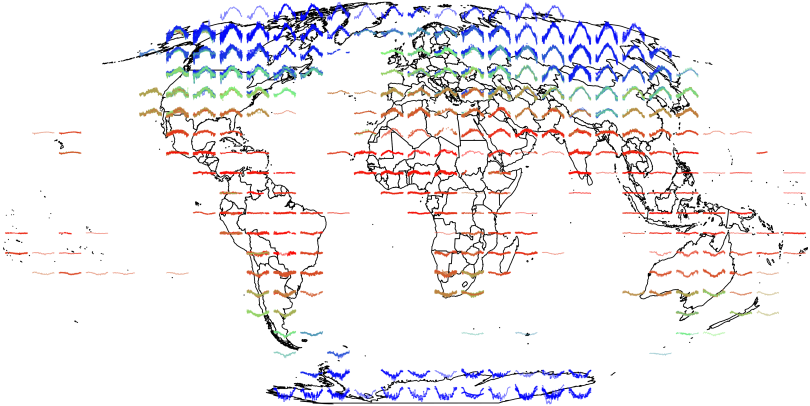

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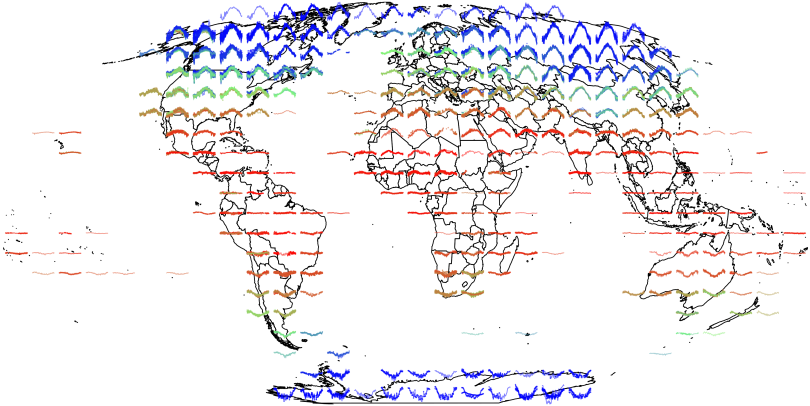
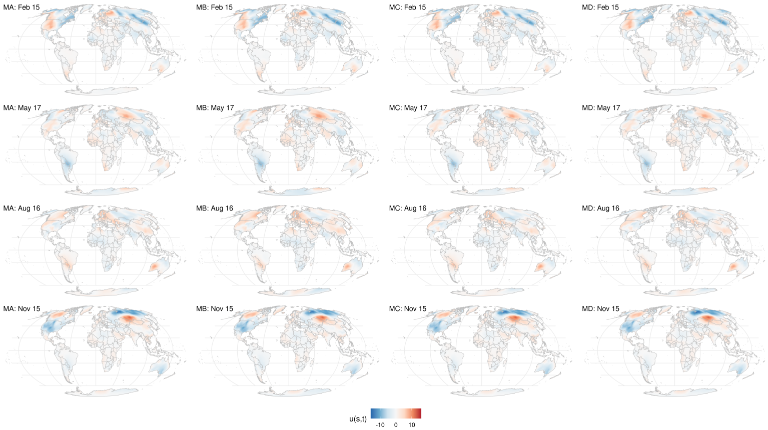


$$\boldsymbol{y} = \mathbf{1}\mu + \boldsymbol{E}\alpha + \boldsymbol{B}\boldsymbol{b} + \boldsymbol{A}_v\boldsymbol{v} + \boldsymbol{A}_u\boldsymbol{u} + \boldsymbol{e}$$

elevation + seasonal,latitude + slow space-time + fast space-time

Posterior mean of $u(.)$, for some days, four models



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