A GMRF overview: a class of structured random effects

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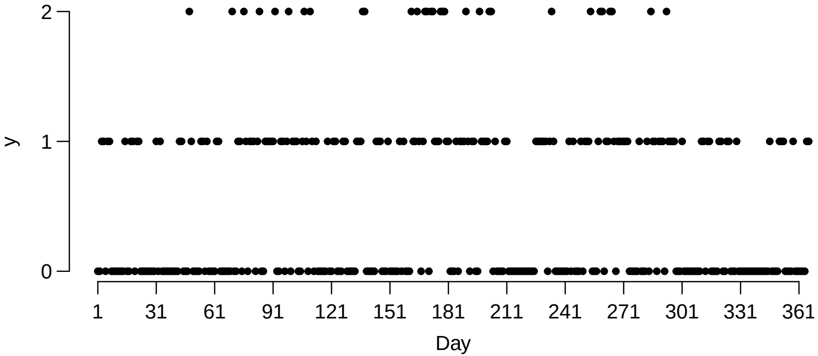
Motivating examples

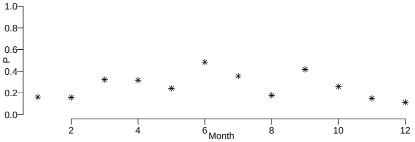
Tokyo example

Rainfall over 1 mm in the Tokyo area for each calendar day during two years (1983-84) are registered.



Problem: model the probability of rain each day of the year

Tokyo example exploration



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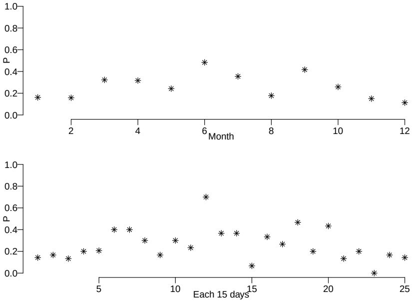

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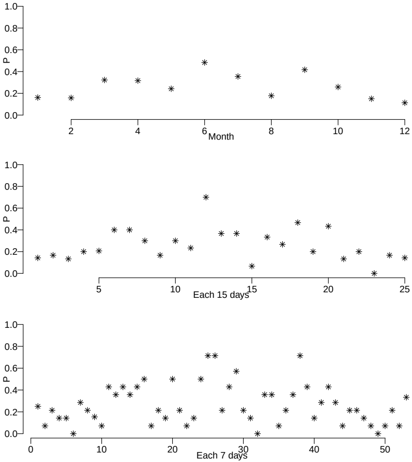

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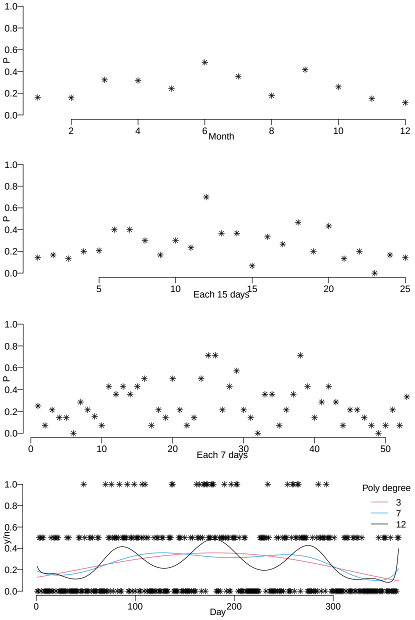

Tokyo example (fit)

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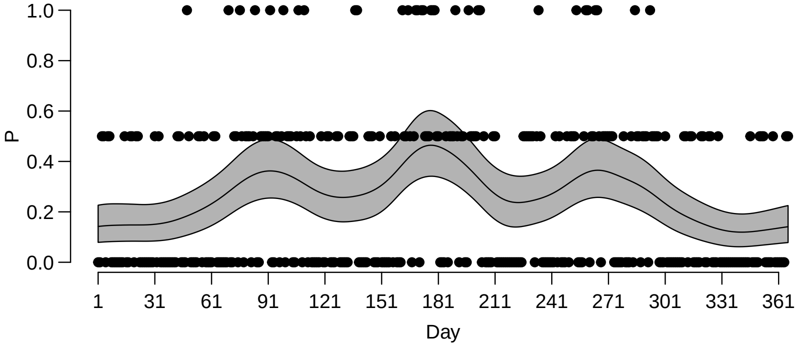


The probability of rain each day of the year

Scotland example (from WinBUGS)

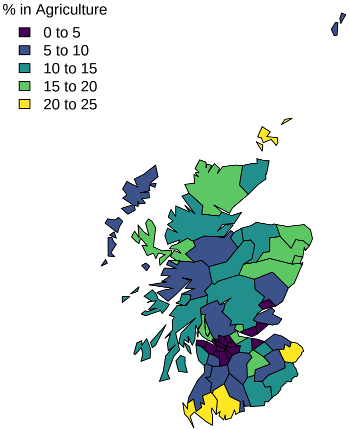
The rates of lip cancer in 56 counties in Scotland have been analysed by Clayton and Kaldor (1987) and Breslow and Clayton (1993). The form of the data includes the observed and expected cases (expected numbers based on the population and its age and sex distribution in the county), a covariate measuring tl percentage of the population engaged in agriculture, fishing, or forestry, and the "position" of each county expressed as a list of adjacent counties.

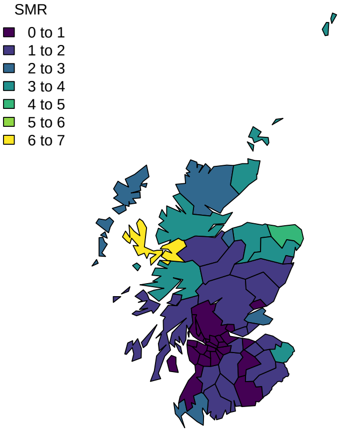
County	Observed cases O_i	Expected cases E_i	Percentage in agric. x_i	SMR	Adjacent counties
1	9	1.4	16	652.2	5,9,11,19
2	39	8.7	16	450.3	7,10
...	...	...	...	...	...
56	0	1.8	10	0.0	18,24,30,33,45,55

$$O_i \sim \text{Poisson}(\mu_i)$$

$$\log \mu_i = \quad \text{og}\ E_i + \alpha_0 + \alpha_1 x_i / 10 + b_i$$

Scotland maps





Scotland data: GLM

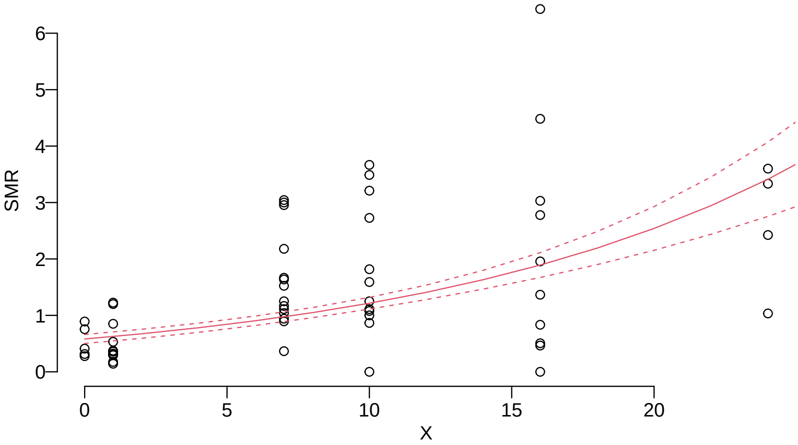
Number of **O**bserved cases as a function of the percentage working in agriculture (**X**)

```
m1 <- glm(O ~ X, poisson, offset=log(E), data=map@data)
summary(m1)
##
## Call:
## glm(formula = O ~ X, family = poisson, data = map@data, offset = log(E))
##
## Deviance Residuals:
##    Min      1Q  Median      3Q     Max
## -4.763  -1.216   0.097   1.336   4.713
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -0.54227    0.06952    -7.8  6.2e-15 ***
## X            0.07373    0.00596    12.4  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 380.73  on 55  degrees of freedom
## Residual deviance: 238.62  on 54  degrees of freedom
## AIC: 450.6
##
## Number of Fisher Scoring iterations: 5
```

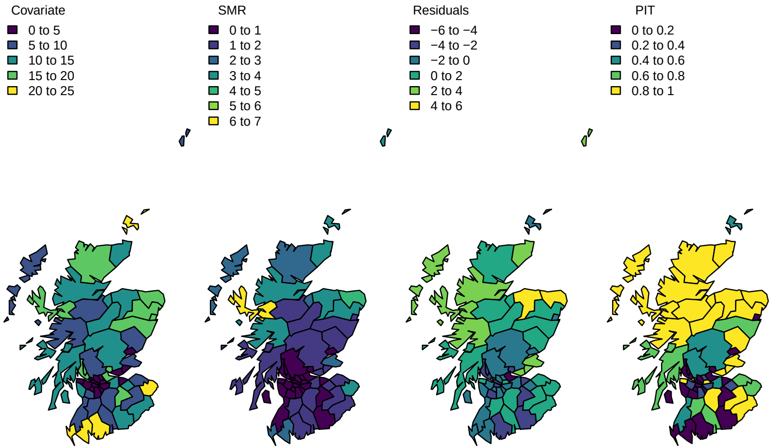
Scotland data: model 1

```
r1 <- inla(O ~ X, family='poisson', offset=log(E), data=map@data,
           control.compute=list(cpo=TRUE))
summary(r1)
##
## Call:
##    c("inla.core(formula = formula, family = family, contrasts = contrasts,
##    ", " data = data, quantiles = quantiles, E = E, offset = offset, ", "
##    scale = scale, weights = weights, Ntrials = Ntrials, strata = strata,
##    ", " lp.scale = lp.scale, link.covariates = link.covariates, verbose =
##    verbose, ", " lincomb = lincomb, selection = selection, control.compute
##    = control.compute, ", " control.predictor = control.predictor,
##    control.family = control.family, ", " control.inla = control.inla,
##    control.fixed = control.fixed, ", " control.mode = control.mode,
##    control.expert = control.expert, ", " control.hazard = control.hazard,
##    control.lincomb = control.lincomb, ", " control.update =
##    control.update, control.lp.scale = control.lp.scale, ", "
##    control.pardiso = control.pardiso, only.hyperparam = only.hyperparam,
##    ", " inla.call = inla.call, inla.arg = inla.arg, num.threads =
##    num.threads, ", " blas.num.threads = blas.num.threads, keep = keep,
##    working.directory = working.directory, ", " silent = silent, inla.mode
##    = inla.mode, safe = FALSE, debug = debug, ", " .parent.frame =
##    .parent.frame)")
## Time used:
##     Pre = 0.773, Running = 0.2, Post = 0.0153, Total = 0.988
## Fixed effects:
##               mean    sd 0.025quant 0.5quant 0.975quant mode kld
## (Intercept) -0.542 0.070     -0.680   -0.541     -0.408   NA   0
## X            0.074 0.006      0.062    0.074      0.085   NA   0
##
## Marginal log-Likelihood:  -234.10
## CPO, PIT is computed
## Posterior summaries for the linear predictor and the fitted values are computed
## (Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')
```

The fitted covariate effect



After the covariate effect, is there something left?



Structured random effects

Smoothed probability over time

- Temporally smooth probability of rain
  - is different for each day but similar for nearby days
    - p_i is similar to p_{i+1}
    - assume $\text{logit}(p_i) = x_i$

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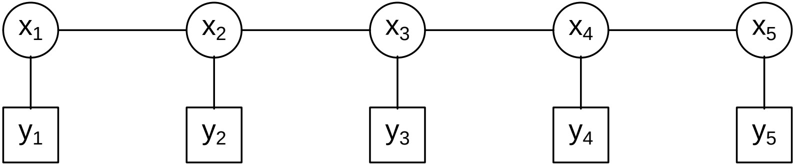

- dependence on x
- y conditionally independent given x
  - y_i conditional on x_i is independent of y_{i-1} and of y_{i+1}

The RW1 prior

- It seems natural to borrow strength over time.
  - **x**: smoothing over time
  - ***R**andon **W**alk* - RW of first order: `rw1`
  - Gaussian distribution for the successive differences (**R** esparse)

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- The log of the (joint) distribution for **x** is

$$\log(\pi(\mathbf{x}|\tau)) \propto -\frac{\tau}{2}\sum_{i=2}^{n}(x_i - x_{i-1})^2 = -\frac{\tau}{2}\mathbf{x}'\mathbf{R}\mathbf{x},$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 1 \end{bmatrix}$$

The cyclic RW1

- 1st of January is similar to December, 31: cyclic random walk

$$
\begin{aligned}
\pi(\mathbf{x}|\theta) &\propto \exp\left\{-\frac{\theta}{2}\left[(x_1 - x_n)^2 + \sum_{i=2}^{n}(x_i - x_{i-1})^2\right]\right\} \\
&= \exp\left\{-\frac{\theta}{2}\mathbf{x}^T\mathbf{R}\mathbf{x}\right\}
\end{aligned}
$$

where, now,

$$
\mathbf{R} = \begin{bmatrix}
2 & -1 & & & & & & -1 \\
-1 & 2 & -1 & & & & & \\
 & -1 & 2 & -1 & & & & \\
 & & & & \ddots & & & \\
 & & & & & -1 & 2 & -1 \\
 & & & & & & -1 & 2 & -1 \\
-1 & & & & & & -1 & 2
\end{bmatrix}
$$

- Cyclic second order is analogous.

Tokyo example: the model

- y_i assume values 0, 1 or 2, for $i = 1, ..., n$
  - assuming conditional independence, thus

$$y_i | p_i \sim \mathrm{Binomial}(n_i, p_i)$$

  - link function (logit)

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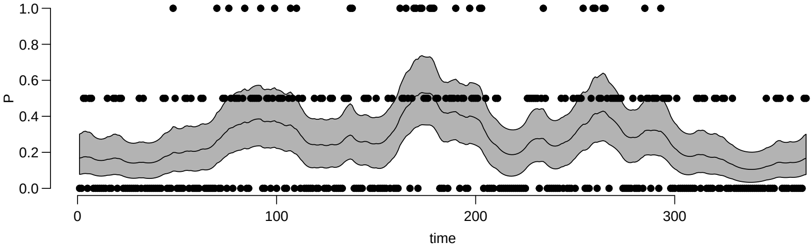
- $\mathbf{x}$ is Latent Gaussian (field)
- τ: local precision parameter

Model fit in INLA

$$
\begin{array}{rcll}
y_i|x_i & \sim & \text{Binomial}(2, p_i) & \rightarrow \text{likelihood} \\
\mathbf{x}|\tau & \sim & N(\mathbf{0}, (\tau\mathbf{R})^{-}) & \rightarrow \text{latent field, GMRF} \\
\tau & \sim & p(\tau) & \rightarrow \text{prior distribution}
\end{array}
$$

```
head(Tokyo, 5)
##   y n time   P
## 1 0 2    1 0.0
## 2 0 2    2 0.0
## 3 1 2    3 0.5
## 4 1 2    4 0.5
## 5 0 2    5 0.0
model <- y ~ f(time, model='rw1', cyclic=TRUE)
result <- inla(model, family='binomial',
               data=Tokyo, Ntrials=n,
               control.compute=list(cpo=TRUE))
```

Result for the time series



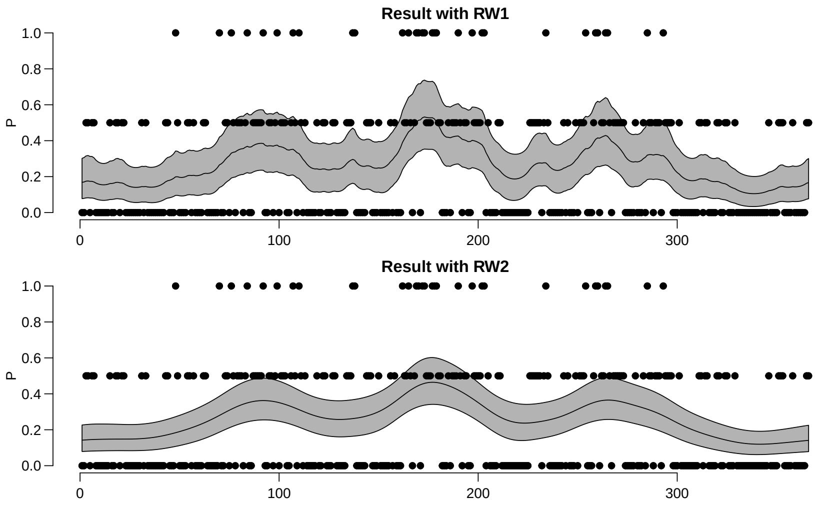
Smoothing more

Gaussian distribution for the second order differences (`rw2`)

$$\Delta_i^2 = x_i - 2x_{i-1} + x_{i-2} \sim N(0, \tau^{-1})$$

```
model2 <- y ~ f(time, model='rw2', cyclic=TRUE)
result2 <- inla(model2, family='binomial',
                data=Tokyo, Ntrials=n,
                control.compute=list(cpo=TRUE))
```

Both results for the time series



Smooth areal dependent risk

- Doing similar over areas (discrete spatial domain)

$$y_i \sim \mathrm{Poisson}(E_i r_i)$$
$$\log(r_i) = \alpha + \beta X_i + x_i$$

- $x_i | x_j$, j the index for the neighbours of i
- **x** is a *Gaussian Markov Random Field* - GMRF, Rue and Held (2005)

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- We recommend to use the `bym2` model, Riebler et al. (2016)

Besag: randon walk over areas, Besag (1974)

$$\pi(x_i|\mathbf{x}_{-i},\tau) \sim N(\frac{1}{n_i}\sum_{j\sim i} x_j, \frac{1}{n_i\tau})$$

where $j \sim i$ means j *neighbour of* i. This gives:

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$$\pi(\mathbf{x}|\tau) \quad \propto \quad \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\sum_{i}^{n}(x_i - \frac{1}{n_i}\sum_{j\sim i} x_j)^2\right)$$

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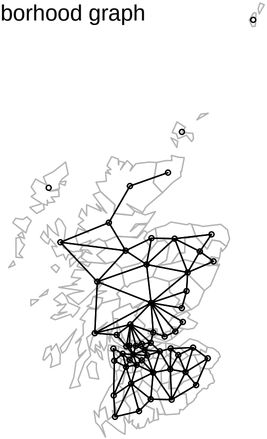
&= \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\sum_{j\sim i}^{n}(x_i - x_j)^2\right) = \tau^{\frac{n-1}{2}} \exp\left(-\frac{\tau}{2}\mathbf{x}^T\mathbf{R}\mathbf{x}\right)
\end{aligned}$$

$$\mathbf{R}_{ij} = \begin{cases} n_i & \text{if } i = j \\ -1 & \text{if } j \sim i \\ 0 & \text{otherwise} \end{cases} .$$

The Scotland graph

Scotland map

Neighborhood graph



Kronecker product models

- Consider the random vector indexed as follows
  $\mathbf{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$
- Assume

$$\pi(\mathbf{x}) \propto (|\mathbf{Q1} \otimes \mathbf{Q2}|^{-1})^{1/2} \exp\left(-\frac{1}{2}\mathbf{x}^T\{\mathbf{Q1} \otimes \mathbf{Q2}\}\mathbf{x}\right)$$

  where
  - **Q1** has dimension equals T
  - **Q2** has dimension equals n

Spacetime interactions

- The Kronecker product models follows the Clayton's rule
- Combine **Q1** (time) and **Q2** (space) available

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- Example, Martínez-Beneito, López-Quilez, and Botella-Rocamora (2008):

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f(spatial, model='besag', ...,
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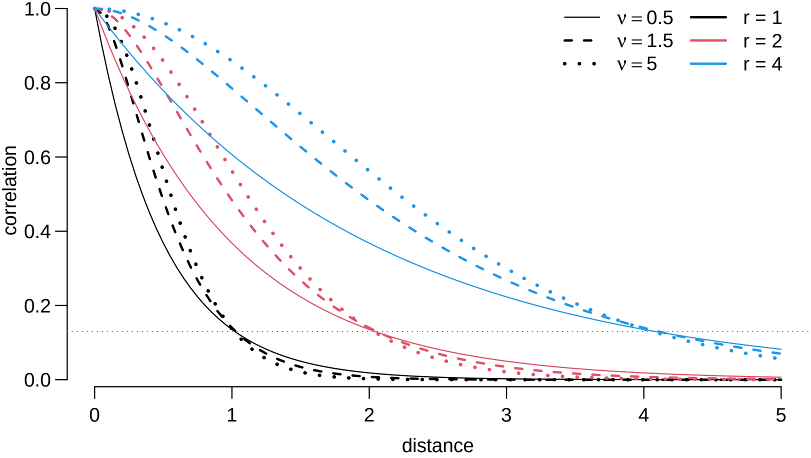
```

- **care** when main effects are in the model
- **super care** when **Q1** and/or **Q2** have rank deficiency
  - e.g. `rw1`, `rw2` and `besag` models
  - if both Q1 and Q2 are intrinsic: use other approach!
    - see `inla.knmodels()`, Knorr-Held (2000)

The SPDE modeling approach

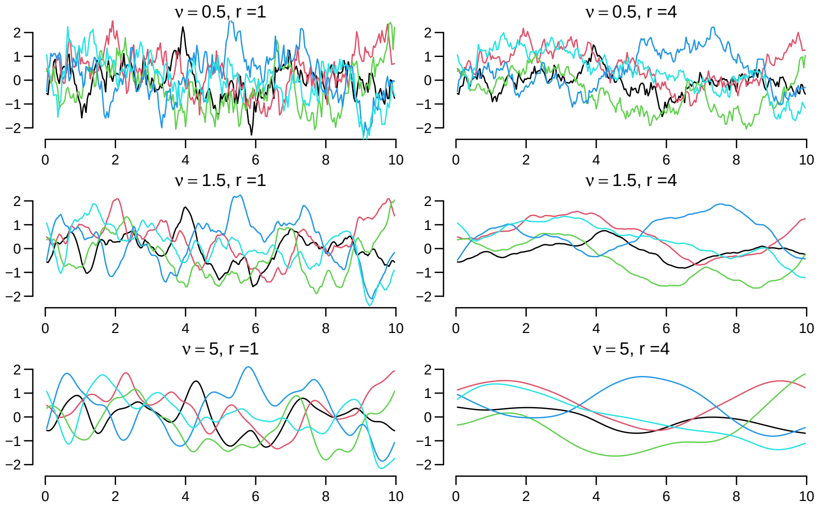
The Matérn covariance

$$\Sigma_{ij} = \sigma_x^2 \frac{2^{1-\nu} K_\nu(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)}{\Gamma(\nu)(\kappa\|\mathbf{s}_i - \mathbf{s}_j\|)^{-\nu}}, \text{ practical range} = \mathrm{r} = \sqrt{8\nu}/\kappa$$



$$\text{corr}\left(\sqrt{8\nu}/\kappa\right) \approx 0.13$$

Simulations, 1D, $\sigma_x^2 = 1$



The SPDE approach

- Fields with Matérn covariance are solutions to the following Stochastic Partial Differential Equation (SPDE)

$$(\kappa^2 - \Delta)^{\alpha/2}\xi(\mathbf{s}) = \tau\mathcal{W}(\mathbf{s})$$

  - $\kappa > 0$: scale parameter
  - $\alpha = \nu + d/2$: smoothness
  - Δ is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

- See Whittle (1954) and Lindgren, Rue, and Lindström (2011)

Regular grid, $d = 2$

- $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
- $\mathbf{C} = \mathbf{I}$, $\mathbf{G}$ = Laplacian (4 neighbours)
  - Laplacian-local pattern:

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

  - $\mathbf{Q}_{1,\kappa}$-local pattern

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$$\begin{bmatrix} & -1 & \\ -1 & 4+\kappa^2 & -1 \\ & -1 & \end{bmatrix}$$

- κ is a scale parameter
  - $\rightarrow$ Sparse precision $\mathbf{Q}$ {**!!!**}
  - remember: $(\kappa^2 - \Delta)^{\alpha/2}\xi(\mathbf{s}) = \tau\mathcal{W}(\mathbf{s})$
  - $\rightarrow \quad (\mathbf{Q}_{1,\kappa})^{1/2}\xi$ = independent noise
  - 'effective' range (0.139) $\approx \sqrt{8\nu/\kappa}$

Important fact: role of α

- Bigger $\alpha \to \mathbf{Q}$ less sparse $\to$ smoother
  - $\alpha = 1$: $\mathbf{Q}_{1,\kappa} = \mathbf{K}_\kappa = \kappa^2\mathbf{C} + \mathbf{G}$
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  - $\alpha = 3, 4, \ldots$: $\mathbf{Q}_{\alpha,\kappa} = \mathbf{K}_\kappa\mathbf{C}^{-1}\mathbf{Q}_{\alpha-2,\kappa}\mathbf{C}^{-1}\mathbf{K}_\kappa$

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  - $\alpha = 1$: like RW1
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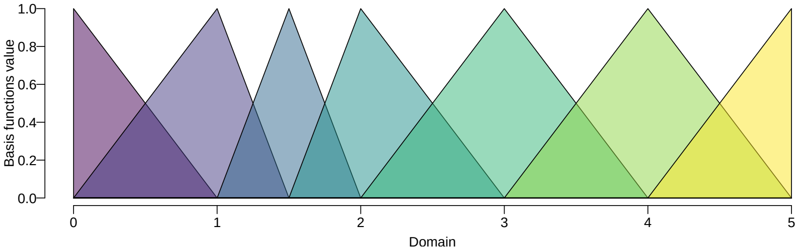
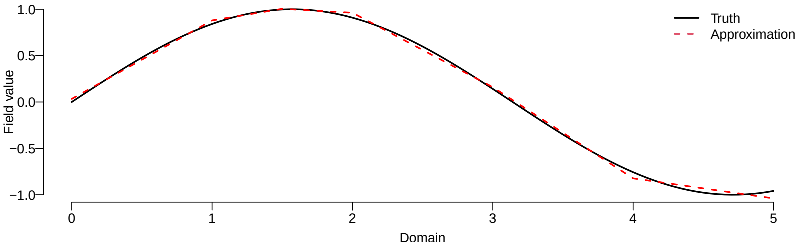
- $d = 2$ (equivalent model at the mesh nodes)
  - $\alpha = 2$, Whittle (1954).
  - $\alpha = 1$, Besag (1974)
  - $\alpha = 1$ & $\kappa = 0$: intrinsic

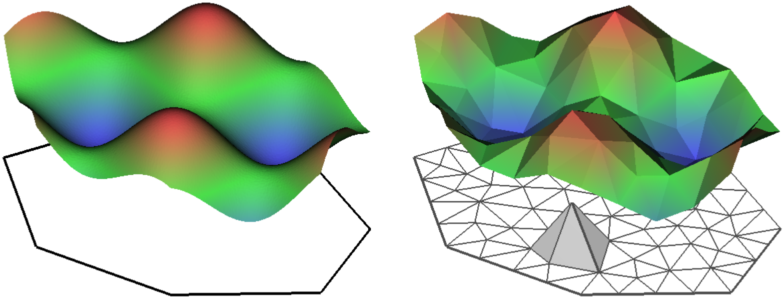
Continous approximation: 1d case



- $\xi(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s})\xi_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0)\xi(\mathbf{s}_0)$,
  - ψ_k: basis functions evaluated at data locations $\mathbf{s}$
  - ξ_k: the process at the discretization points $\mathbf{s}_0$



Continous approximation: 2d case



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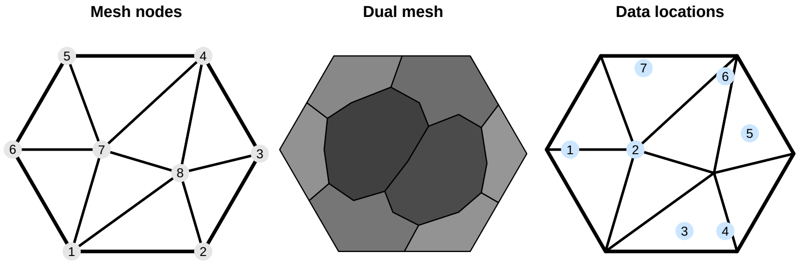
SPDE as a random effect in INLA

- The (1d, 2d, ...) SPDE models are processes on a continuous domain
- Used as a random effect on a set of knots/nodes
  - Needs to be projected to the observation location

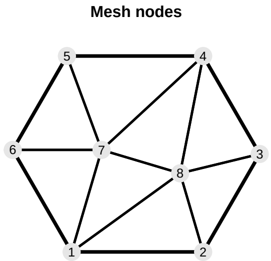
$$\eta = \mathbf{X}\beta + \mathbf{A}\xi$$

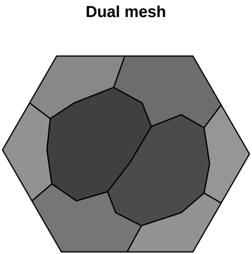
  - **X** is *n* (observation) times *p* (covariates)
  - **A** is *n* (observation) times *m* (knots/nodes)
- see more on Krainski et al. (2018)

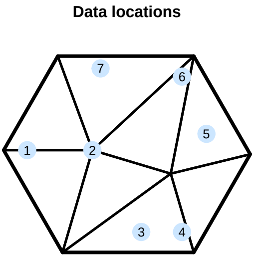
The Finite Element Method matrices

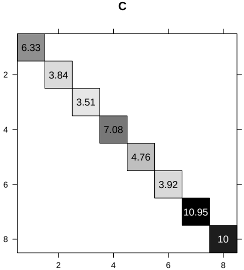


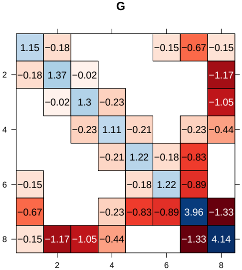
The Finite Element Method matrices

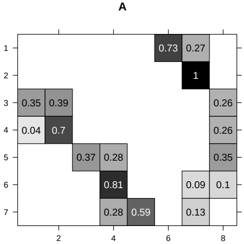












Spacetime models

- SDE in time **then** SPDE in space

$$
\begin{aligned}
\tau\left(\gamma\frac{\partial}{\partial t}+\varphi\right)u(\mathbf{s},\delta t) &= \xi(\mathbf{s},\delta t) \\
(\kappa^2-\Delta)^{\alpha/2}\xi(\mathbf{s},\delta t) &= \mathcal{W}(\mathbf{s},\delta t).
\end{aligned}
$$

Spacetime models

- SDE in time **then** SPDE in space

$$\tau \left(\gamma \frac{\partial}{\partial t} + \varphi \right) u(\mathbf{s}, \delta t) = \xi(\mathbf{s}, \delta t)$$
$$(\kappa^2 - \Delta)^{\alpha/2} \xi(\mathbf{s}, \delta t) = \mathcal{W}(\mathbf{s}, \delta t).$$

- SPDE in time **and** in space

$$\tau \left(\gamma \frac{\partial}{\partial t} + (\kappa^2 - \Delta)^{\alpha_s/2} \right) u(\mathbf{s}, \delta t) = \xi(\mathbf{s}, \delta t)$$
$$(\kappa^2 - \Delta)^{\alpha_e/2}) \xi(\mathbf{s}, \delta t) = \mathcal{W}(\mathbf{s}, \delta t).$$

Spacetime models

- SDE in time **then** SPDE in space

$$\tau\left(\gamma\frac{\partial}{\partial t}+\varphi\right)u(\mathbf{s},\delta t)=\xi(\mathbf{s},\delta t)$$
$$(\kappa^2-\Delta)^{\alpha/2}\xi(\mathbf{s},\delta t)=\mathcal{W}(\mathbf{s},\delta t).$$

- SPDE in time **and** in space

$$\tau\left(\gamma\frac{\partial}{\partial t}+(\kappa^2-\Delta)^{\alpha_s/2}\right)u(\mathbf{s},\delta t)=\xi(\mathbf{s},\delta t)$$
$$(\kappa^2-\Delta)^{\alpha_e/2})\xi(\mathbf{s},\delta t)=\mathcal{W}(\mathbf{s},\delta t).$$

- More next Monday...

References

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