Bayesian inference with INLA - 2.0

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QR code for the slides

https://github.com/JanetVN1201/SA_US_24

Outline

BayesComp group at KAUST

Bayesian inference

Data $\boldsymbol{y}$ (with covariates $\boldsymbol{Z}$), depend on $\boldsymbol{X}$ and $\boldsymbol{\theta}$ such that, $E[Y] = h(\boldsymbol{A}(\boldsymbol{Z})\boldsymbol{X})$.

Bayes' theorem:

$$\begin{aligned} q(\boldsymbol{X}, \boldsymbol{\theta}|\boldsymbol{y}) &\propto L(\boldsymbol{y}|\boldsymbol{X}, \boldsymbol{\theta})p(\boldsymbol{X}, \boldsymbol{\theta}) \\ \text{Posterior} &\propto \text{Likelihood} \times \text{Prior} \end{aligned}$$

Computational aspects

- Analytical methods - conjugacy (pre-computer era)
- Approximate methods - Laplace (can be inaccurate)
- Exact methods - MCMC (very slow for complex models or large data)

Now, due to computing resources approximate methods are gaining popularity - INLA, VB, EP etc.

INLA - 2009 [Rue et al., 2009]
2021+ [Van Niekerk et al., 2023]
HPC [Gaedke-Merzhäuser et al., 2023]

Model definition - GAMM

Suppose we have response data $\boldsymbol{y}_{n\times 1}$ (conditionally independent) with density function $\pi(y|\boldsymbol{X},\boldsymbol{\theta})$ and link function $h(.)$, that is linked to some covariates $\boldsymbol{Z}$ through linear predictors

$$\boldsymbol{\eta}_n = \beta_0 + \boldsymbol{Z}_\beta\boldsymbol{\beta} + \sum f^k(\boldsymbol{Z}_f) = \boldsymbol{A}\boldsymbol{X}$$

The inferential aim is to estimate the latent field $\boldsymbol{X}_m = \{\beta_0, \boldsymbol{\beta}, \boldsymbol{f}\}$, and $\boldsymbol{\theta}$.

GAMM $\rightarrow$ LGM

Assume

$$\boldsymbol{X}|\boldsymbol{\theta} \sim N(\mathbf{0}, \boldsymbol{Q}(\boldsymbol{\theta})^{-1})$$

where $\boldsymbol{Q}(\boldsymbol{\theta})$ is a sparse matrix ($\boldsymbol{X}$ is a GMRF).

$p(\boldsymbol{X}, \boldsymbol{\theta}) = p(\boldsymbol{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})$ and $p(\boldsymbol{\theta})$ can be non-Gaussian.

Why is INLA so accurate and so fast?

- LGM structure
- Sparse precision matrix
- Specialized matrix algebra for sparse matrices
- NEW: VB (low-rank) correction [van Niekerk and Rue, 2024]

Use precision matrix instead of covariance matrix $\rightarrow$ natural occurrence

How common are sparse $\boldsymbol{Q}(\boldsymbol{\theta})$?

Consider an AR(1) model..

AR(1) example

AR(1) example

Ingredients

Likelihood - $\prod_{i=1}^{n} \pi(y_i|\boldsymbol{X}, \boldsymbol{\theta})$
Prior for the latent - $\pi(\boldsymbol{X}|\boldsymbol{\theta})$
Prior for the hyperparameters - $\pi(\boldsymbol{\theta})$
Goal:

- $q(X_j|\boldsymbol{y})$
- $q(\theta_k|\boldsymbol{y})$

Posterior approximations by INLA

$$
\begin{aligned}
\pi(\boldsymbol{X},\boldsymbol{\theta},\boldsymbol{y}) &= \pi(\boldsymbol{\theta})\pi(\boldsymbol{X}|\boldsymbol{\theta})\prod_{i=1}^{n}\pi(y_i|(\boldsymbol{A}\boldsymbol{X})_i,\boldsymbol{\theta}) \\
\tilde{q}(\boldsymbol{\theta}|\boldsymbol{y}) &\propto \left.\frac{\pi(\boldsymbol{X},\boldsymbol{\theta},\boldsymbol{y})}{\pi_G(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{y})}\right|_{\boldsymbol{X}=\boldsymbol{\mu}(\boldsymbol{\theta})} \\
\tilde{q}(\theta_j|\boldsymbol{y}) &= \int \tilde{\pi}(\boldsymbol{\theta}|\boldsymbol{y})d\boldsymbol{\theta}_{-j} \\
\tilde{q}(\boldsymbol{X}_j|\boldsymbol{y}) &= \int \tilde{q}(\boldsymbol{X}_j|\boldsymbol{\theta},\boldsymbol{y})\tilde{q}(\boldsymbol{\theta}|\boldsymbol{y})d\boldsymbol{\theta},
\end{aligned}
$$

$\tilde{q}(\boldsymbol{X}_j|\boldsymbol{\theta},\boldsymbol{y})$ depends on the approximation used, for Gaussian it is straightforward for the Laplace approximation we do another Gaussian approximation to $\tilde{q}(\boldsymbol{X}_{-j}|\boldsymbol{\theta},\boldsymbol{y})$.

INLA 1.0

- $\boldsymbol{X} = \{\boldsymbol{\eta}, \beta_0, \boldsymbol{\beta}, \boldsymbol{f}\}$
- For each j,

$$\tilde{q}(X_j|\boldsymbol{\theta}, \boldsymbol{y}) = \left. \frac{\pi(\boldsymbol{X}, \boldsymbol{\theta}, \boldsymbol{y})}{\pi_G(\boldsymbol{X}_{-j}|X_j, \boldsymbol{\theta}, \boldsymbol{y})} \right|_{\boldsymbol{X}_{-j} = \boldsymbol{\mu}_{-j}}$$

Wishes and dreams

1. How can we get a good and cheap approximation $\tilde{q}(\boldsymbol{X}_j|\boldsymbol{\theta}, \boldsymbol{y})$ using $\pi_G(\boldsymbol{X}|\boldsymbol{\theta}, \boldsymbol{y})$?
   Non-Gaussian? Other family?
2. How can we remove $\boldsymbol{\eta}$ from $\boldsymbol{X}$ and still produce full posteriors of $\boldsymbol{\eta}$?
   Huge data? Prediction? Stability?

Gaussian approximation $\pi_G(\boldsymbol{X}|\boldsymbol{\theta}, \boldsymbol{y})$

Laplace method - fit the best Gaussian at the mode of a curve where the variance is derived from the inverse Hessian at the mode.

$$\begin{aligned}
\log\left(\pi(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{y})\right) &\propto -\frac{1}{2}\boldsymbol{X}^\top \boldsymbol{Q}(\boldsymbol{\theta})\boldsymbol{X} + \sum_{i=1}^{n}\left(b_i \boldsymbol{X}_i - \frac{1}{2}c_i \boldsymbol{X}_i^2\right) \\
&= -\frac{1}{2}\boldsymbol{X}^\top\left(\boldsymbol{Q}(\boldsymbol{\theta}) + \boldsymbol{D}\right)\boldsymbol{X} - \boldsymbol{b}^\top\boldsymbol{X}
\end{aligned}$$

hence

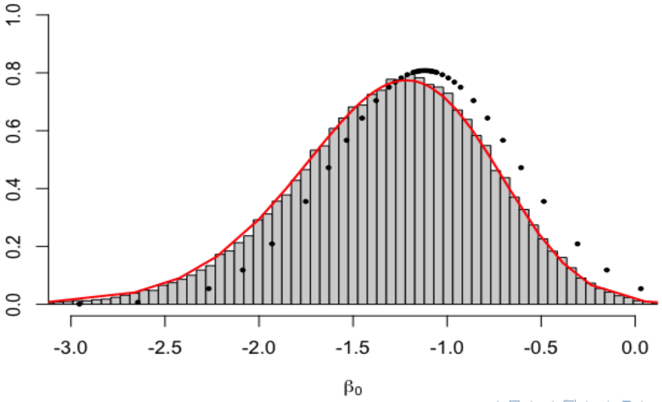
$$\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{y} \sim N(\boldsymbol{\mu}, (\boldsymbol{Q}(\boldsymbol{\theta}) + diag(\boldsymbol{c}))^{-1}) \tag{1}$$

with $(\boldsymbol{Q}(\boldsymbol{\theta}) + \boldsymbol{D})\boldsymbol{\mu} = \boldsymbol{b}$.

Can we do better than this?

Poisson example

$$Y_i|\beta_0 = -1, \beta_1 = -0.5 \quad \sim \quad \text{Poisson}(\exp(\beta_0 + \beta_1 X_i))$$



Corrected Gaussian approximation

Can we find a better Gaussian approximation (closer to LA) that is cheap and will scale well?

Low-Rank VB correction [1]

[1] van Niekerk, J. and Rue, H., 2024. Low-rank Variational Bayes correction to the Laplace method. Journal of Machine Learning Research, 25(62), pp.1-25.

VB from Zellner (1988)[2]

Based on prior information $\mathcal{I}$, data $\boldsymbol{y}$ and parameters $\boldsymbol{\theta}$, define the following:

1. $\pi(\boldsymbol{\theta}|\mathcal{I})$ is the prior model
2. $q(\boldsymbol{\theta}|\mathcal{D})$ is the learned model from the prior information and the data where $\mathcal{D} = \{\mathcal{I}, \boldsymbol{y}\}$
3. $l(\boldsymbol{\theta}|\boldsymbol{y}) = f(\boldsymbol{y}|\boldsymbol{\theta})$ is the likelihood
4. $p(\boldsymbol{y}|\mathcal{I})$ is the marginal model for the data where $p(\boldsymbol{y}|\mathcal{I}) = \int f(\boldsymbol{y}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}|\mathcal{I})d\boldsymbol{\theta}$

The input information in the learning of $\boldsymbol{\theta}$ is given by $\pi(\boldsymbol{\theta}|\mathcal{I})$ and $l(\boldsymbol{\theta}|\boldsymbol{y})$. An information processing rule (IPR) then delivers $q(\boldsymbol{\theta}|\mathcal{D})$ and $p(\boldsymbol{y}|\mathcal{I})$ as output information.

[2] Zellner, A., 1988. Optimal information processing and Bayes's theorem. The American Statistician, 42(4), pp.278-280.

Bayes Rule as an efficient IPR

A stable and efficient IPR would provide the same amount of output information than received through the input information, thus being information conservative. Thus, we learn $q(\boldsymbol{\theta}|\mathcal{D})$ such that it minimizes

$$\begin{aligned} & -\int \left[\log \pi(\boldsymbol{\theta}|\mathcal{I}) + \log l(\boldsymbol{\theta}|\boldsymbol{y})\right] q(\boldsymbol{\theta}|\mathcal{D})d\boldsymbol{\theta} + \int \left[\log q(\boldsymbol{\theta}|\mathcal{D}) + \log p(\boldsymbol{y}|\mathcal{I})\right] q(\boldsymbol{\theta}|\mathcal{D})d\boldsymbol{\theta} \\ = \; & E_{q(\boldsymbol{\theta}|\mathcal{D})}\left[-\log l(\boldsymbol{\theta}|\boldsymbol{y})\right] + \text{KLD}\left[q(\boldsymbol{\theta}|\mathcal{D})||\pi(\boldsymbol{\theta}|\mathcal{I})\right]. \end{aligned} \tag{2}$$

Variational form of Bayes' theorem

Finding the best fit from a certain family $P(\Theta)$,

$$\arg\min_{p \in P(\Theta)} \left(\mathrm{E}_{p(\boldsymbol{\theta})} \left[-\sum_{i=1}^{n} \log f(y_i | \boldsymbol{\theta}) \right] + \mathrm{KLD}(p || \pi) \right) \tag{3}$$

How can we use this?

We apply this to the Gaussian approximation in the denominator.
Recall that $(\boldsymbol{Q}(\boldsymbol{\theta}) + \boldsymbol{D})\boldsymbol{\mu} = \boldsymbol{Q}\boldsymbol{\mu} = \boldsymbol{b}$.
Now let's formulate $\boldsymbol{\mu}^* = \boldsymbol{\mu} + \boldsymbol{\delta}$.

$$\arg_{\boldsymbol{\delta}} \min_{p(\boldsymbol{X}|\boldsymbol{y},\boldsymbol{\theta})} \left(\mathrm{E}_{p(\boldsymbol{X}|\boldsymbol{y},\boldsymbol{\theta})} \left[-\sum_{i=1}^{n} \log f(y_i|X_i, \boldsymbol{\theta}) \right] + \mathrm{KLD}(p||\pi) \right)$$

where $\boldsymbol{X}|\boldsymbol{y}, \boldsymbol{\theta} \sim N(\boldsymbol{\mu} + \boldsymbol{\delta}, \boldsymbol{Q}^{-1})$.
But $\boldsymbol{X}$ can be very large...

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But $\boldsymbol{X}$ can be very large...

Implicit mean correction

Recall that $\boldsymbol{Q\mu} = \boldsymbol{b}$.
Now let's formulate $\boldsymbol{Q\mu}^* = \boldsymbol{b} + \boldsymbol{\lambda}$, so that

$$\boldsymbol{\mu}^* = \boldsymbol{\mu} + \boldsymbol{M\lambda}$$

So now we solve for,

$$\arg_{\lambda} \min_{p(\boldsymbol{X}|\boldsymbol{y},\boldsymbol{\theta})} \left(\mathrm{E}_{p(\boldsymbol{X}|\boldsymbol{y},\boldsymbol{\theta})} \left[-\sum_{i=1}^{n} \log f(y_i|\boldsymbol{X}_i, \boldsymbol{\theta}) \right] + \mathrm{KLD}(p||\pi) \right)$$

where $\boldsymbol{X}|\boldsymbol{y}, \boldsymbol{\theta} \sim N(\boldsymbol{\mu}^*, \boldsymbol{Q}^{-1})$.
Low-rank correction $\rightarrow$ Only correct some $b's$, change to all $\mu's$.

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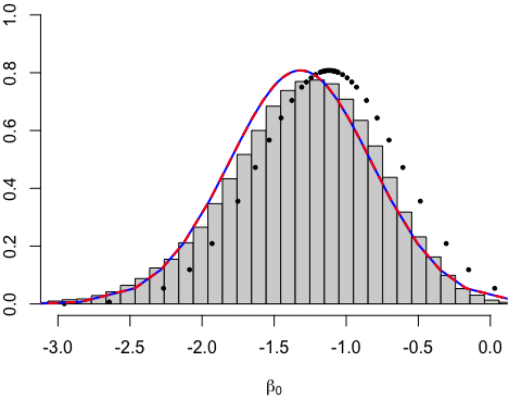
where $\boldsymbol{X}|\boldsymbol{y}, \boldsymbol{\theta} \sim N(\boldsymbol{\mu}^*, \boldsymbol{Q}^{-1})$.
Low-rank correction $\rightarrow$ Only correct some $b's$, change to all $\mu's$.

Example (small data)

$$Y_i|\beta_0 = -1, \beta_1 = -0.5 \quad \sim \quad \text{Poisson}(\exp(\beta_0 + \beta_1 X_i))$$



Results of mean correction

	GA	INLA 1.0	VBC	MCMC
β_0	-1.119	-1.317	-1.316	-1.302
β_1	-0.361	-0.401	-0.401	-0.391

Table: Posterior means from the Gaussian method, INLA, VBC and MCMC

Example (large data)

$$Y_i|\beta_0 = -1, \beta_1 = -0.5 \quad \sim \quad \text{Poisson}(\exp(\beta_0 + \beta_1 X_i))$$

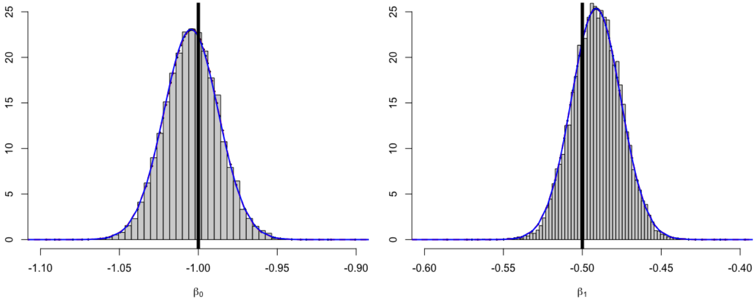


Figure: Marginal posterior of β_0 (center) and β_1 (right) from the Laplace method (points), VBC (solid line) and INLA (broken line) approximations

Results

	GA	INLA 1.0	VBC	MCMC
β_0	-1.004	-1.005	-1.005	-1.005
β_1	-0.491	-0.492	-0.492	-0.492
Time (s)	6.346	37.168	7.344	5779.42

Table: Posterior means from the Laplace method, INLA 1.0, VBC and MCMC

Conditional posterior of η_i

In order to calculate $\tilde{\pi}(\eta_i|\boldsymbol{y})$, we first calculate $\tilde{\pi}(\eta_i|\boldsymbol{\theta}, \boldsymbol{y})$. We postulate a Gaussian density for $\eta_i|\boldsymbol{\theta}, \boldsymbol{y}$ such that $\tilde{\pi}(\eta_i|\boldsymbol{\theta}, \boldsymbol{y}) = \pi_G(\eta_i|\boldsymbol{\theta}, \boldsymbol{y})$, with mean

$$E(\boldsymbol{\eta}|\boldsymbol{\theta}, \boldsymbol{y}) = \boldsymbol{A}E(\boldsymbol{X}|\boldsymbol{\theta}, \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{\mu}(\boldsymbol{\theta})$$

and covariance matrix

$$\text{Cov}(\boldsymbol{\eta}|\boldsymbol{\theta}, \boldsymbol{y}) = \boldsymbol{A}\text{Cov}(\boldsymbol{X}|\boldsymbol{\theta}, \boldsymbol{y})\boldsymbol{A}^\top,$$

VB corrected marginal posterior of η_i

$$\begin{aligned}
\eta_j|\boldsymbol{\theta},\boldsymbol{y} &\sim N(\mu_j(\boldsymbol{\theta}),\sigma_j^2(\boldsymbol{\theta})) \\
\mu_j(\boldsymbol{\theta}) &= (\boldsymbol{A}\boldsymbol{\mu}^*(\boldsymbol{\theta}))_j \\
\tilde{\pi}(\eta_j|\boldsymbol{y}) &\approx \sum_{k=1}^{K}\pi_G(\eta_j|\boldsymbol{\theta}_k,\boldsymbol{y})\tilde{\pi}(\boldsymbol{\theta}_k|\boldsymbol{y})\delta_k.
\end{aligned}$$

Universal tools in INLA

- Model selection metrics - WAIC, DIC
- Cross validation (1 and group) and model-based clustering
- Prediction of unobserved areas or new profiles
- Mean or quantile models
- Joint models
- Multiple imputation
- Coregionalization models
- etc..... ask at `https://groups.google.com/g/r-inla-discussion-group?pli=1` or e-mail help@r-inla.org

Tokyo example

The Tokyo dataset in the R INLA library contains information on the number of times the daily rainfall measurements in Tokyo was more than $1mm$ on a specific day t for two consecutive years. In order to model the annual rainfall pattern, a stochastic spline model with fixed precision is used to smooth the data.

$$
\begin{aligned}
y_i|\mathcal{X} &\sim Bin\left(n_i, p_i = \frac{\exp(\alpha_i)}{1+\exp(\alpha_i)}\right) \\
(\alpha_{i+1} - 2\alpha_i + \alpha_{i-1})|\tau &\overset{\text{iid}}{\sim} N(0, \tau^{-1}),
\end{aligned}
$$

where $i = 1, 2, \dots, 366$ on a torus, and $n_{60} = 1$ else $n_i = 2$.

Results

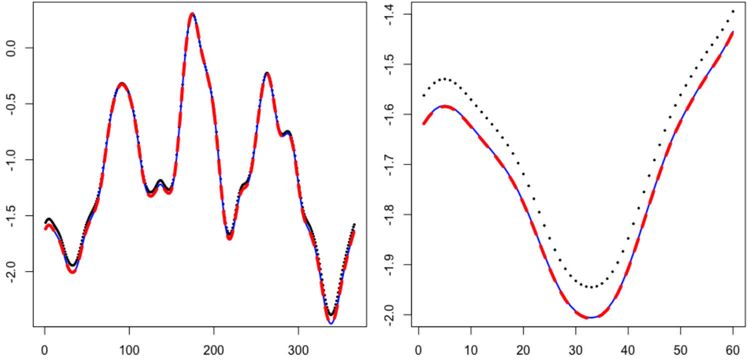


Figure: Posterior mean of α (left) (zoomed for the first two months (right)) from the Laplace method (points), VBC (solid line) and INLA (broken line)

Cox proportional hazards model

We simulate survival data for n patients using the following very simple Cox proportional hazards model

$$h_i(t) = h_0(t)\exp(\beta x_i) = 1.2t^{0.2}\exp\left(0.1x_i\right), \quad 1 = 1, 2, ..., n,$$

where x is a scaled and centered continuous covariate, and the baseline hazard, $h_0(t)$ is estimated using a scaled random walk order one model with 50 bins. We also consider four different values of n which are $n = 10^2$, to 10^5.

Cox proportional hazards model

n	Augmented size	classic INLA (s)	modern INLA (s)
10^2	1 327	1.6	0.1
10^3	12 657	1.3	0.4
10^4	131 807	10.2	2.3
10^5	1 302 413	113.3	22.5

Table: Results from simulation of Cox proportional hazards model

Flexible Besag model[3]

Instead of one precision for the entire area, we define multiple precision parameters, $\tau_1, \tau_2, ..., \tau_P$, to account for covariance non-stationarity. The conditional density for the spatial effect of area i is

$$x_i | \boldsymbol{x}_{-i}, \tau_1, \ldots, \tau_P \sim N\left(\frac{1}{2} \sum_{\substack{i \text{ in sub-region } k \\ j \text{ in sub-region } l \\ i \sim j}} (\tau_l + \tau_k) \tau_{x_i}^{-1} x_j, \tau_{x_i}^{-1}\right),$$

and

$$\tau_{x_i} = \frac{1}{2}\left(n_i \tau_k + \sum_l n_{il} \tau_l\right).$$

[3] Abdul-Fattah, E., Krainski, E., Van Niekerk, J. and Rue, H., 2024. Non-stationary Bayesian spatial model for disease mapping based on sub-regions. Statistical Methods in Medical Research, p.09622802241244613.

Contraction prior: Non-stationary $\rightarrow$ stationary

The joint PC prior for $\boldsymbol{\theta} = \log \boldsymbol{\tau}$ can be derived as a convolution of the PC prior for τ from the Besag model, as follows

$$\pi(\boldsymbol{\theta}) = 2^{-(P+2)/2}\pi^{-P/2}\lambda\sigma^{-P}\exp\left(-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{1}\overline{\theta})^T\tilde{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{\theta}-\overline{\theta}\mathbf{1})-\overline{\theta}/2-\lambda e^{-\overline{\theta}/2}\right),$$

This prior contracts

$$\tau_1, \tau_2, \ldots, \tau_P \quad \rightarrow \quad \tau$$

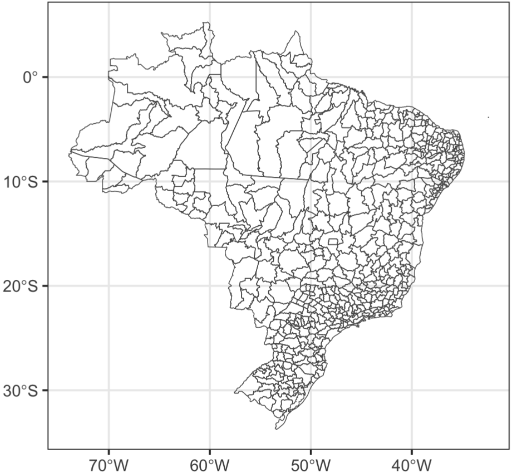
Dengue risk in Brazil

We analyze the effects of hydrometeorological hazards on dengue risk in Brazil. To test the spatial variations in the spread of the virus in different sub-regions of Brazil, we fit dengue counts with a Poisson regression model as follows,

$$\boldsymbol{y} \sim \text{Poisson}(Ee^{\boldsymbol{\eta}}), \quad \boldsymbol{\eta} = \mathbf{1}^T \mu + \boldsymbol{\alpha}$$

where $\boldsymbol{y}$ is the observed counts in November of dengue cases, E is the expected number of counts , $\boldsymbol{\eta}$ is the linear predictor, μ is the overall intercept, and $\boldsymbol{\alpha}$ is the Besag or flexible Besag model over space.

Dengue risk in Brazil



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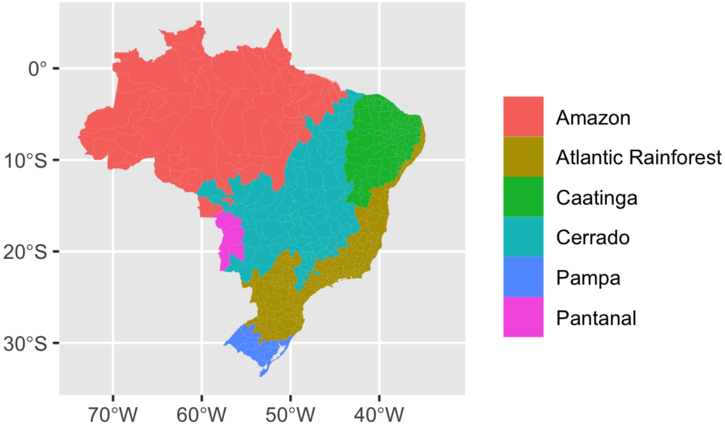

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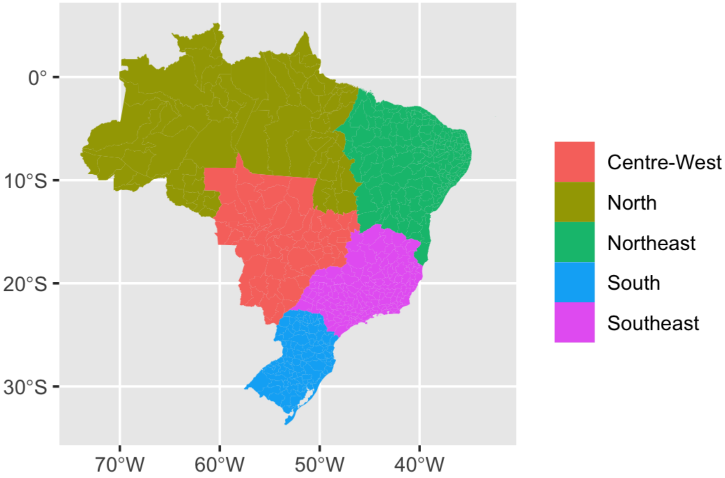
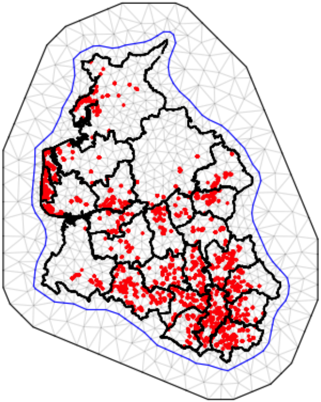


Spatial survival example

Consider the R dataset `Leuk` that features the survival times of 1043 patients with acute myeloid leukemia (AML) in Northwest England between 1982 to 1998.



Cox spatial model

$$h(t, \boldsymbol{s}) = h_0(t) \exp(\beta \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{s})),$$

with

$$\eta_i(s) = \beta_0 + \beta_1 \text{Age}_i + \beta_2 \text{WBC}_i + \beta_3 \text{TPI}_i + u(s).$$

which implies a latent field of size $m = 39158$.

Results

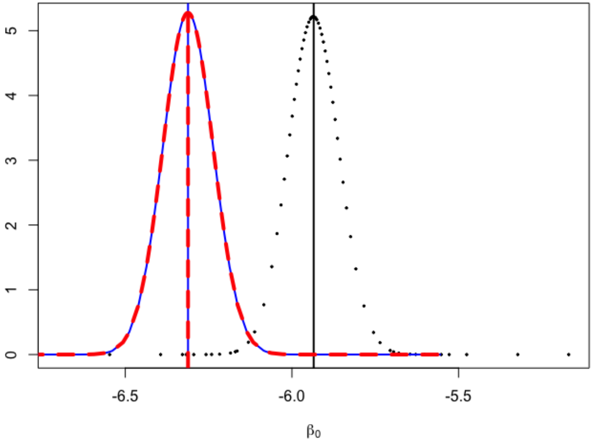
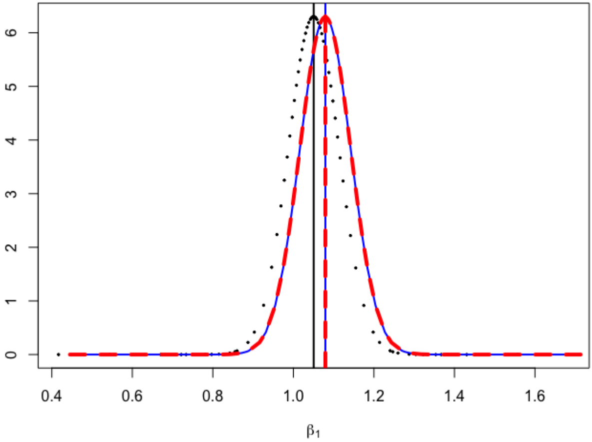


Figure: Marginal posteriors from the Laplace method (points), VBC (solid line) and INLA (broken line)

Results



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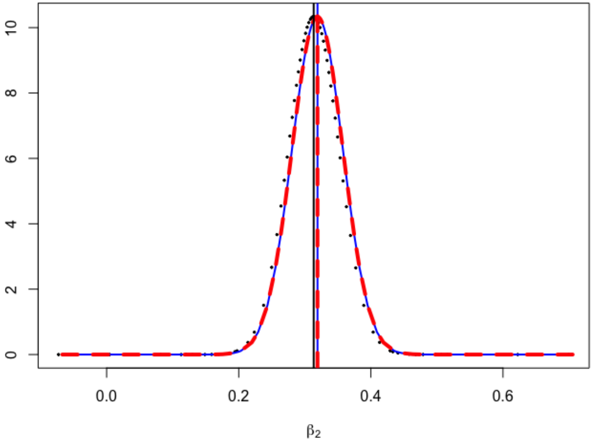

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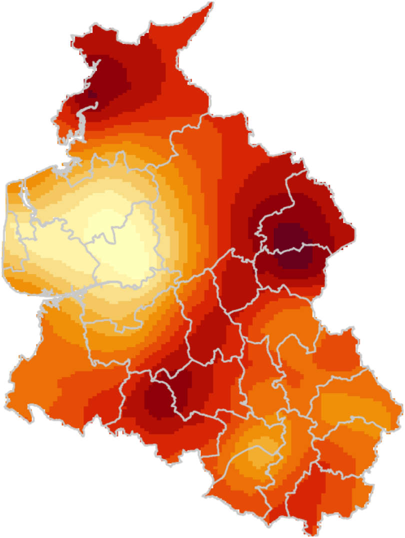
Matern field



The Matérn's SPDE

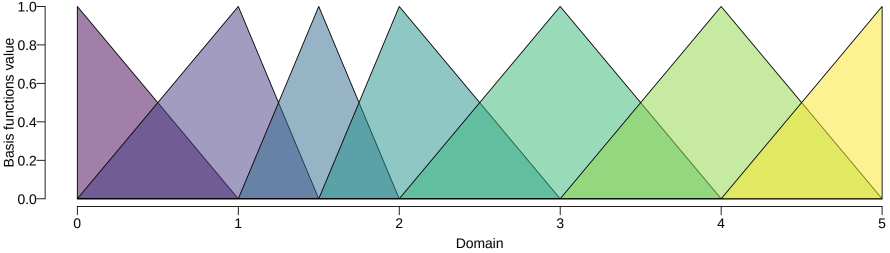
- Whittle (1954), Whittle (1963):
  - Fields with Matérn covariance are solutions to the following Stochastic Partial Differential Equation (SPDE)

$$\tau(\kappa^2 - \Delta)^{\alpha/2} u(\mathbf{s}) = \mathcal{W}(\mathbf{s})$$

  - $\kappa > 0$: scale parameter
  - $\alpha = \nu + d/2$: smoothness
  - Δ is the Laplacian

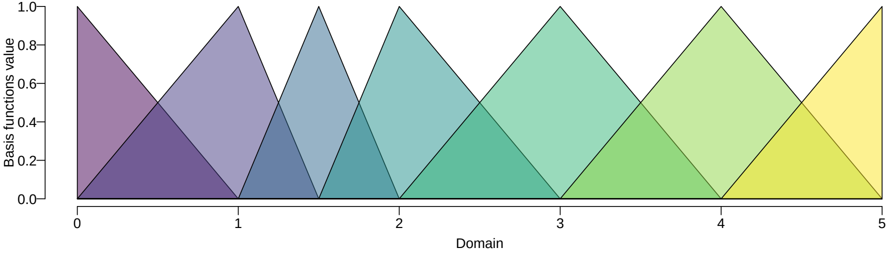
$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial s_i^2}$$

How to solve the SPDE? FEM



- $u(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s}) u_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0) u(\mathbf{s}_0)$,
  - ψ_k: basis functions evaluated at data locations $\mathbf{s}$
  - u_k: the process at the discretization points $\mathbf{s}_0$

How to solve the SPDE? FEM



- $u(\mathbf{s}) \approx \sum_{k=1}^{m} \psi_k(\mathbf{s}) u_k = \mathbf{A}(\mathbf{s}, \mathbf{s}_0) u(\mathbf{s}_0)$,
  - ψ_k: basis functions evaluated at data locations $\mathbf{s}$
  - u_k: the process at the discretization points $\mathbf{s}_0$

Lindgren, Rue, and Lindström (2011)[4] I

- Discretization
  - sparse precision matrix:
  - $\mathbf{Q}_\alpha(\tau, \kappa)$, for $\alpha \in \{1, 2, ...\}$.
- α
  - $\alpha = 1$: $\tau \mathbf{K}_1(\kappa) = \tau(\kappa^2 \mathbf{C} + \mathbf{G})$
  - $\alpha = 2$: $\tau(\kappa^4 \mathbf{C} + 2\kappa^2 \mathbf{G} + \mathbf{G}\mathbf{C}^{-1}\mathbf{G})$
  - $\alpha = 2, 3, 4, ...$: $\tau \mathbf{K}_1(\kappa)\mathbf{C}^{-1}\mathbf{K}_{\alpha-2}(\kappa)\mathbf{C}^{-1}\mathbf{K}_1(\kappa)$

Equivalent models (discretized): Whittle (1954), Besag (1974), Besag (1981), Besag and Kooperberg (1995) and Besag and Mondal (2005).

- $\alpha = 1$: $\tau \mathbf{K}_1(\kappa) = \tau(\kappa^2 \mathbf{C} + \mathbf{G})$

Lindgren, Rue, and Lindström (2011)[5] II

- d=1, $u_1, u_2, \ldots, u_n$, two neighbours

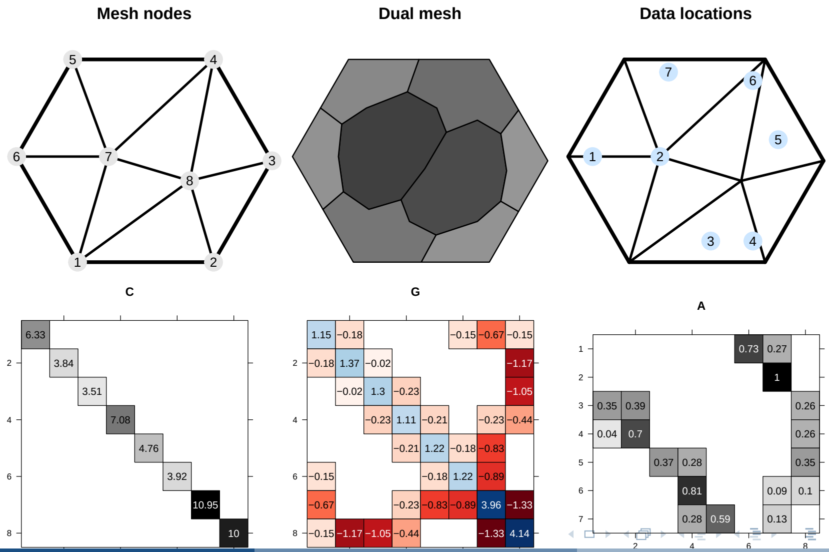
$$\tau^2 \begin{bmatrix} 1+\kappa^2 & -1 & & & \\ -1 & 2+\kappa^2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2+\kappa^2 & -1 \\ & & & -1 & 1+\kappa^2 \end{bmatrix}$$

- $d = 2$, $\mathbf{C} = \mathbf{I}$, $\mathbf{G}$ = Laplacian (4 neighbours)

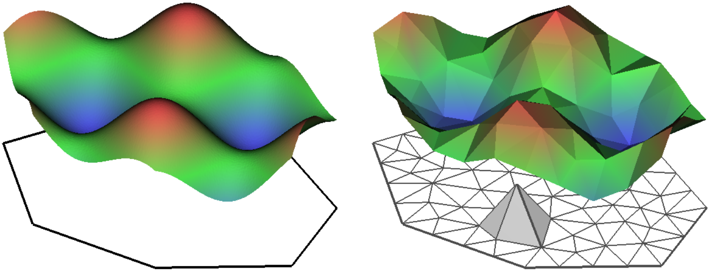
[4] Lindgren, F., Rue, H. and Lindström, J., 2011. An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach. Journal of the Royal Statistical Society Series B: Statistical Methodology, 73(4), pp.423-498.

[5] Lindgren, F., Rue, H. and Lindström, J., 2011. An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach. Journal of the Royal Statistical Society Series B: Statistical Methodology, 73(4), pp.423-498.

Piecewise linear basis, FEM matrices

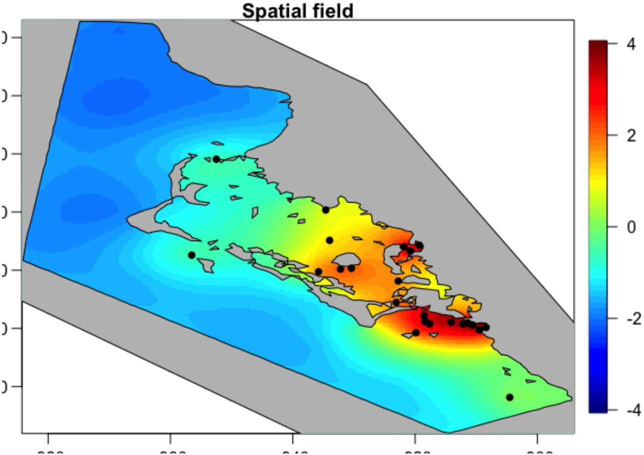


FEM in 3D

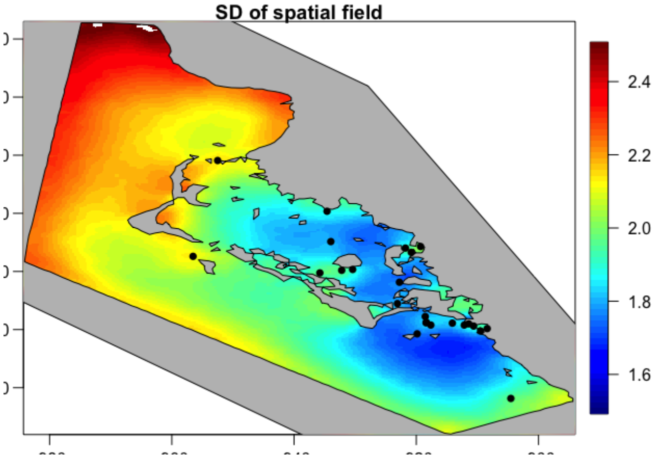


Non-stationary Matern field based on physical barriers

Now the distance is not Euclidean... Construct the mesh with boundaries



Non-stationary Matern field based on physical barriers



cs-fMRI model[6]

Functional magnetic resonance imaging (fMRI) is a noninvasive neuro-imaging technique used to localize regions of specific brain activity during certain tasks.
For T timepoints and N vertices per hemisphere resulting in data $\boldsymbol{y}_{TN\times 1}$ with the latent Gaussian model as follows:

$$
\begin{aligned}
\boldsymbol{y}|\beta, \boldsymbol{b}, \boldsymbol{\theta} &\sim N(\boldsymbol{\mu}_y, \boldsymbol{V}), \quad \boldsymbol{\mu}_y = \sum_{k=0}^{K} \boldsymbol{X}_k \beta_k + \sum_{j=1}^{J} \boldsymbol{Z}_j \boldsymbol{b}_j \\
\beta_k &= \boldsymbol{\Psi}_k \boldsymbol{w}_k \quad \text{(SPDE prior on } \beta_k\text{)} \\
\boldsymbol{w}_k|\boldsymbol{\theta} &\sim N(\boldsymbol{0}, \boldsymbol{Q}^{-1}_{\tau_k,\kappa_k}) \\
\boldsymbol{b}_j &\sim N(\boldsymbol{0}, \delta \boldsymbol{I}) \quad \text{(Diffuse priors for } \boldsymbol{b}_j\text{)} \\
\boldsymbol{\theta} &\sim \pi(\boldsymbol{\theta}),
\end{aligned}
$$

where we have K task signals and J nuisance signals.

[6] Van Niekerk, J., Krainski, E., Rustand, D. and Rue, H., 2023. A new avenue for Bayesian inference with INLA. Computational Statistics & Data Analysis, 181, p.107692.

cs-fMRI model

The data consists of a 3.5-min fMRI for each subject, consisting of 284 volumes, where each subject performs 5 different motor tasks interceded with a 3 second visual cue. Each hemisphere of the brain contained 32492 surface vertices. From these, 5000 are resampled to use for the analysis. This results in a response data vector $\boldsymbol{y}$ of size **2 523 624**, with an SPDE model defined on a mesh with 8795 triangles.

The inference based on the modern formulation of INLA was computed in 148 seconds.

cs-fMRI model

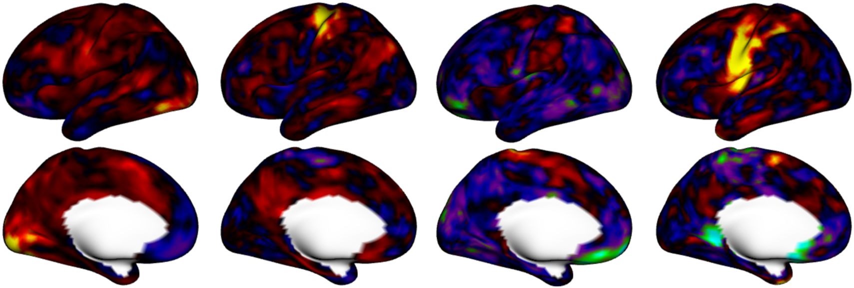


Figure: Activation areas for the different tasks in the left hemisphere - visual cue, right hand motor, right foot motor, tongue motor task (from left to right)

Further details

`www.r-inla.org`

New default setting in INLA (VB) (previously inla.mode = "experimental")

- INLA can fit many different statistical models and complex models can be built using multiple "building blocks"/random effects.
- Remove the linear predictors from the latent field $\rightarrow$ accurate posterior inference with VB correction (I - VB - LA)
- New applications that aren't feasible with INLA 1.0

Gaedke-Merzhäuser, L., van Niekerk, J., Schenk, O., and Rue, H. (2023).
Parallelized integrated nested Laplace approximations for fast Bayesian inference.
Statistics and Computing, 33(1):25.

Rue, H., Martino, S., and Chopin, N. (2009).
Approximate Bayesian inference for latent Gaussian models by using integrated nested laplace approximations.
Journal of the Royal Statistical Society: Series B (Statistical Methodology), 71(2):319–392.

Van Niekerk, J., Krainski, E., Rustand, D., and Rue, H. (2023).
A new avenue for Bayesian inference with INLA.
Computational Statistics & Data Analysis, 181:107692.

van Niekerk, J. and Rue, H. (2024).
Low-rank variational Bayes correction to the Laplace method.
Journal of Machine Learning Research, 25(62):1–25.


Thank you • شكرا
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