Differential Equations in Geophysical Fluid Dynamics

XI. Advection-diffusion-reaction equation

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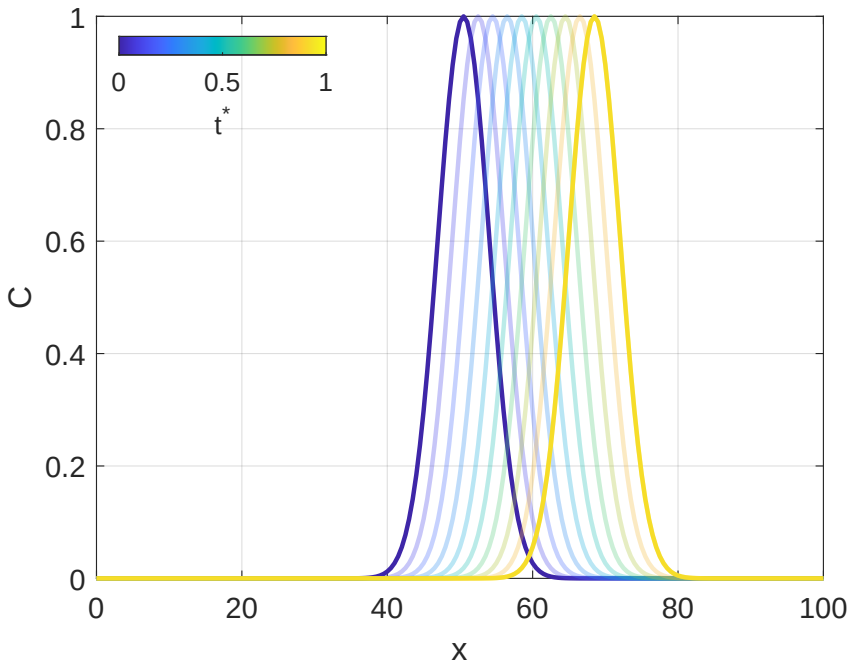
Recap

Now, we know two partial differential equations:

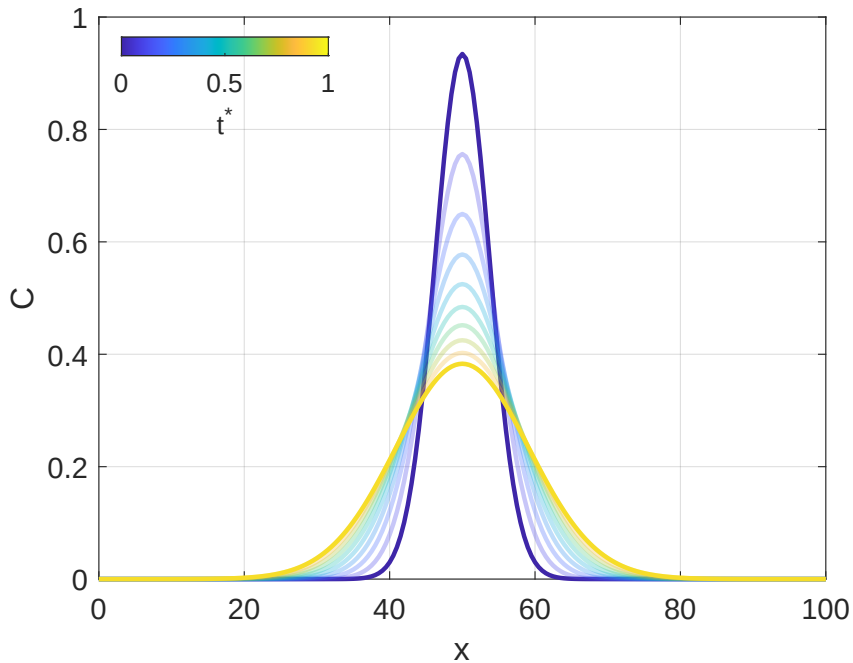
Advection equation

$$\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} = 0 \qquad (1)$$



Diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(A\frac{\partial C}{\partial x}\right) \qquad (2)$$



Advection-diffusion equation

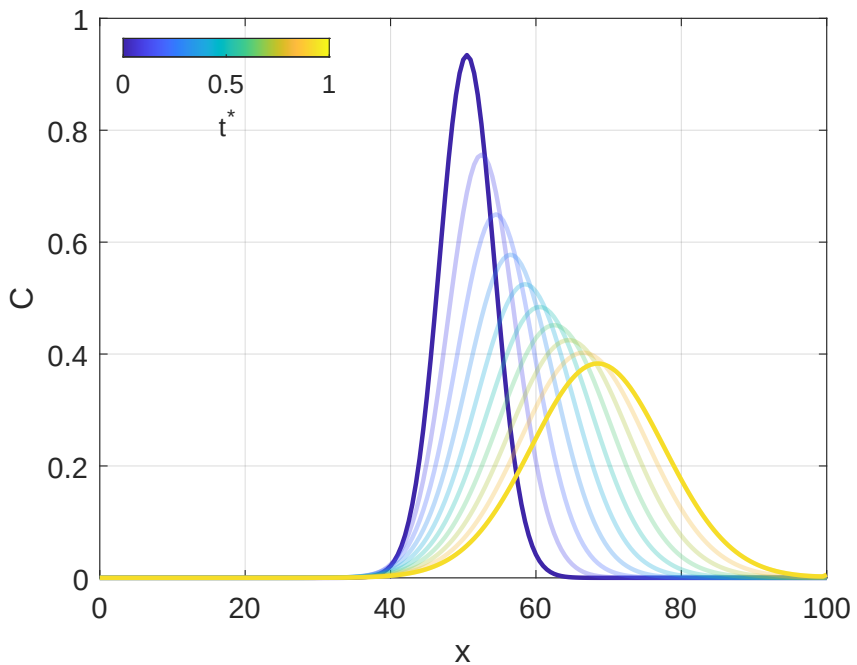
So, we know advection-diffusion equation, that governs transport of almost everything!

Advection-diffusion equation

Diffusion

$$\frac{\partial C}{\partial t} + \nabla \cdot (\vec{u} C) = \frac{\partial}{\partial x}\left(A \frac{\partial C}{\partial x} \right) \tag{3}$$

Advection



Random-walk and diffusion

We talked Eulerian and Lagrangian descriptions of advection:

Eulerian

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0 \qquad (4)$$

Lagrangian

$$\frac{dX}{dt} = u, \quad \frac{dC}{dt} = 0 \qquad (5)$$

How about those of diffusion?

Eulerian

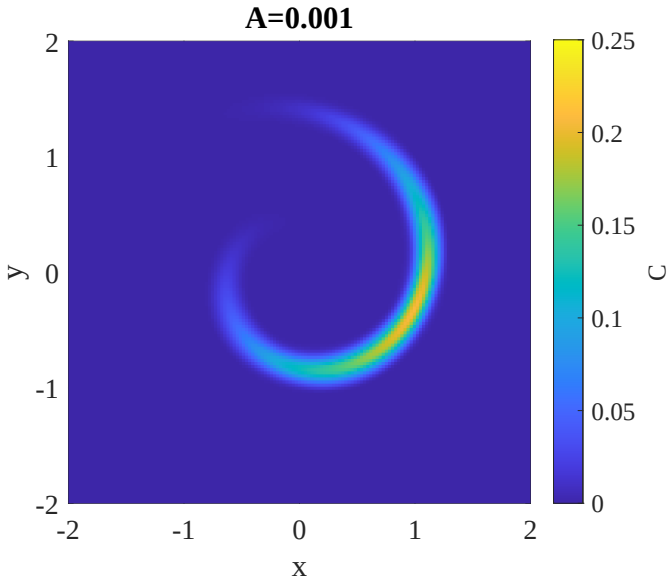
$$\frac{\partial C}{\partial t} = A\frac{\partial^2 C}{\partial x^2} \qquad (6)$$

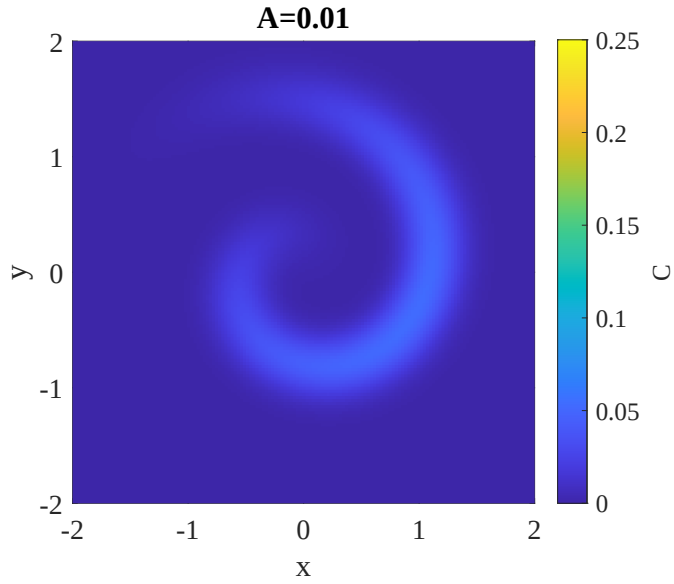
Lagrangian

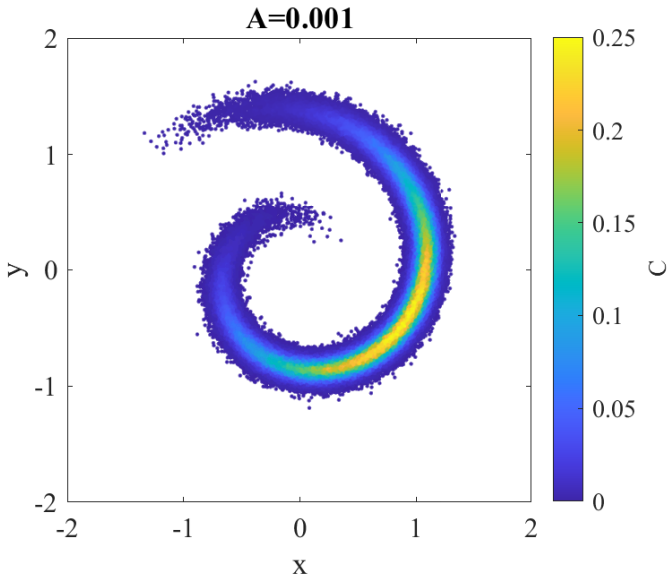
$$dX = \sqrt{2A}dW$$

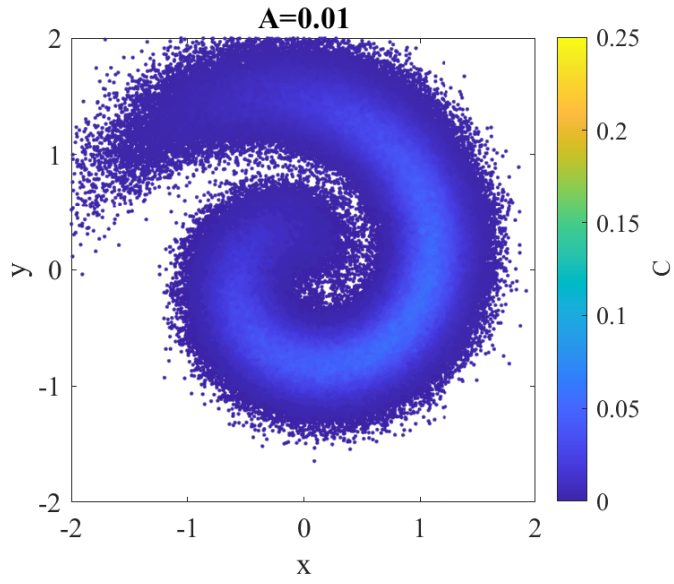
$$X^{n+1} = X^n + \sqrt{2A\Delta t}\, N(0,1) \qquad (7)$$

Advection-diffusion-reaction equation









https://jang-geun.github.io/vis_geo_adv_diff_1.gif

Advection-diffusion-reaction equation

Governing equation (model) for radioactive decay is given by

$$\frac{dC}{dt} = -aC \tag{8}$$

where a is decay rate. **How do we couple this chemical model to hydrodynamics model?**

Just add advection and diffusion terms!

$$\frac{\partial C}{\partial t} + \nabla \cdot (\vec{u}C) = \nabla \cdot A\nabla C - aC \tag{9}$$

$$\frac{\partial (uC)}{\partial x} + \frac{\partial (vC)}{\partial y} + \frac{\partial (wC)}{\partial z}$$

$$A_h \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial C}{\partial z} \right)$$

Advection-diffusion-reaction equation

Tons of applications...

Stock et al., 2005; He et al., 2008; Lee et al., 2024; Kim et al., 2016; Shin et al., 2017; Choi et al., 2018; Cheng et al., 2021; Kampouris et al., 2021; Choi et al., 2023; Choi et al., 2025;...

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