Differential Equations in Geophysical Fluid Dynamics

XIII. Wind-driven circulation: Stommel wind-driven circulation

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Recap

Stommel's wind-driven circulation problem in vorticity equation form is given by

$$\underbrace{\beta \frac{\partial \psi}{\partial x}}_{\beta \bar{v}\text{: Planetary } \beta\text{-term}} = \underbrace{\frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s}_{\text{Wind stress curl}} \overbrace{- \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)}^{-(\gamma/h)\nabla \times \vec{u}\text{: Bottom stress curl}} \tag{1}$$

where $\beta = \partial f / \partial y$ can be approximated to constant ($f \approx f_0$ if $y/R \ll 1$; f-plane) or linear polynomial ($f_0 + \beta_0 y$ if $y/R < 1$; β-plane).

Recap

Over the f-plane where $f \approx f_0$ so $\beta \approx 0$, the governing equation can be simplified to

$$0 = \boxed{\frac{1}{\rho_0 h}\nabla \times \vec{\tau}^s} \; \overbrace{\boxed{-\frac{\gamma}{h}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)}}^{-(\gamma/h)\nabla\times\vec{u}\text{: Bottom stress curl}} \tag{2}$$

that can be rewritten as $\nabla \times \vec{u} = (1/(\rho_0\gamma))\nabla \times \vec{\tau}^s$ implying curl of ocean current is proportional to the wind stress curl (rotates in same direction).

Note that the momentum equation we used is

$$\boxed{if\vec{u}} = \boxed{-g\frac{\partial \eta}{\partial \vec{n}}} \boxed{+\frac{\vec{\tau}^s}{\rho_0 h}} \boxed{-\frac{\gamma}{h}\vec{u}} \tag{3}$$

Geostrophic

Surface Ekman

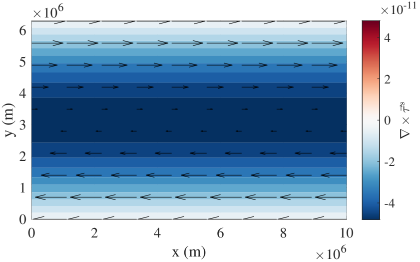
Bottom Ekman

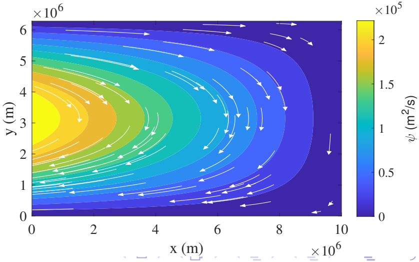
Recap

Over the β-plane where $f_0 + \beta_0 y$, if length scale is large enough to ignore bottom frictional stress curl, the governing equation can be simplified to

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s - \frac{\gamma}{h}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{4}$$

that is referred to as **Sverdrup balance** (Sverdrup, 1947). This is easy to solve and good enough "interior flow" but can consider only one boundary condition.





Governing equations

Stommel (1948) discussed

$$\frac{\gamma}{h}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\beta\frac{\partial\psi}{\partial x}=\frac{1}{\rho_0 h}\nabla\times\vec{\tau}^s \tag{5}$$

with simple forcing (wind stress fields), that idealized wind stress pattern, given by

$$\vec{\tau}^s=(\tau_x^s,\tau_y^s)=\left(-\tau_0\cos(\pi y/L_y),0\right). \tag{6}$$

So, substituting (6) into (5) yields

$$\boxed{\frac{\gamma}{h}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\beta\frac{\partial\psi}{\partial x}=-\frac{\tau_0\pi}{\rho_0 h L_y}\sin(\pi y/L_y)} \tag{7}$$

The boundary conditions are given by

$$\boxed{\psi|_{x=0}=0,\quad \psi|_{x=L_x}=0,\quad \psi|_{y=0}=0,\quad \psi|_{y=L_y}=0.} \tag{8}$$

Stommel's wind-driven circulation

Solution to (7) with boundary conditions (8) is given by

$$\psi = \frac{\tau_0 L_y}{\rho_0 \gamma \pi}\left(1 - \frac{(e^{k^- L_x} - 1)e^{k^+ x} + (1 - e^{k^+ L_x})e^{k^- x}}{e^{k^- L_x} - e^{k^+ L_x}}\right) \tag{9a}$$

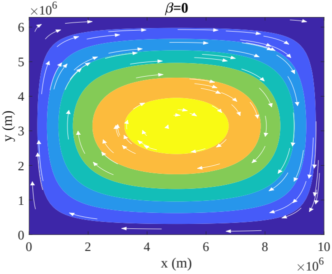
$$k^+ = \frac{-h\beta/\gamma + \sqrt{(h\beta/\gamma)^2 + 4(\pi/L_y)^2}}{2} \tag{9b}$$

$$k^- = \frac{-h\beta/\gamma - \sqrt{(h\beta/\gamma)^2 + 4(\pi/L_y)^2}}{2} \tag{9c}$$

Stommel's wind-driven circulation

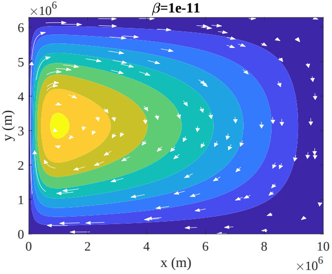
Solution without β-term

$$\frac{\gamma}{h}\nabla^2\psi = \frac{1}{\rho_0 h}\nabla\times\vec{\tau}^s$$



Solution with β-term

$$\frac{\gamma}{h}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = \frac{1}{\rho_0 h}\nabla\times\vec{\tau}^s$$



Consideration of the β-term tilts the stream function (and sea surface height) westward. This is referred to as "β-effect" yielding western boundary intensification.

Summary

$$c\frac{\partial\psi}{\partial t}\ \overbrace{-\beta\frac{\partial\psi}{\partial x}}^{\text{Advection}} = \underbrace{\frac{\gamma}{h}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)}_{\text{Diffusion}} + \overbrace{\frac{\tau_0 L_y}{\rho_0\gamma\pi}\sin(\pi y/L_y)}^{\text{Forcing}} \qquad (10)$$

1. Note that the governing equation of Stommel's wind-driven circulation problem is the "steady" advection-diffusion equation with forcing term.
2. The planetary β-term plays a role in advecting ψ toward the negative x-direction (westward).
3. Due to the β-term, the stream function (sea surface height) becomes asymmetric: steep along the narrow west coast and gentle in the other region. As a result, western boundary (geostrophic) currents are intensified.

References I

Stommel, Henry (1948). “The westward intensification of wind-driven ocean currents”. In: *Eos, Transactions American Geophysical Union* 29.2, pp. 202–206.

Sverdrup, Harald Ulrich (1947). “Wind-driven currents in a baroclinic ocean; with application to the equatorial currents of the eastern Pacific”. In: *Proceedings of the National Academy of Sciences* 33.11, pp. 318–326.