

Differential Equations in Geophysical Fluid Dynamics

II. Simplification and shallow water equations

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Recap

The primitive equations are given by

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} - fv \\ - \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(A_h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right) \end{aligned} \quad (1a)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial(vv)}{\partial y} + \frac{\partial(vw)}{\partial z} + fu = \\ - \frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left(A_h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right) \end{aligned} \quad (1b)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial(ww)}{\partial z} = \\ - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left(A_h \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - g \end{aligned} \quad (1c)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (1d)$$

Who is easy (linear) and who is difficult (nonlinear)?

Assignment

In the continuity equation (3), density ρ can be decomposed into constant component ρ_0 , so $\rho = \rho_0 + \rho'$. Substituting $\rho = \rho_0 + \rho'$ into (3) and then, conduct scale analysis and obtain nondimensionalized version of the equation.

1. What is the nondimensional number controlling the equation?
2. In general, $\rho_0 \approx 1025 \text{ kg m}^{-3}$ and the order of ρ' is less than 10 kg m^{-3} . Is the constant density assumption valid in (3)?

Assignment

Apparently, $\rho' \ll \rho_0$ so constant density assumption could be valid:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

However, we can easily see that below equations used at the same time in many textbooks:

From constant density assumption

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2a)$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \quad (2b)$$

Where did this one come from?

Boussinesq approximation $\rho' \ll \rho_0$

In continuity equation with $\rho = \rho_0 + \rho'$,

$$\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{\partial \rho'}{\partial t} + \frac{\partial(\rho' u)}{\partial x} + \frac{\partial(\rho' v)}{\partial y} + \frac{\partial(\rho' w)}{\partial z} = 0 \quad (3)$$

Boussinesq approximation $\rho' \ll \rho_0$

Similarly, we can approximate most ρ as ρ_0 in the momentum equations...

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} - f v = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(A_h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right) \end{aligned} \quad (4a)$$
$$= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + u \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_0$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial(vv)}{\partial y} + \frac{\partial(vw)}{\partial z} + f u = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left(A_h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right) \end{aligned} \quad (4b)$$
$$= u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + v \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_0$$

Boussinesq approximation $\rho' \ll \rho_0$

But **not all of them!**

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial(ww)}{\partial z} = \\ - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left(A_h \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - g \end{aligned}$$

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$$= u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + w \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_0 \quad (5)$$

Boussinesq approximation $\rho' \ll \rho_0$

So, vertical momentum equation becomes

$$\begin{aligned} \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left(A_h \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - \overbrace{g \frac{\rho}{\rho_0}}^{\text{Reduced gravity}} \end{aligned} \quad (6)$$

What if $H/L \ll 1$?

Hydrostatic approximation $H \ll L$

So, vertical momentum equation becomes

$$-\frac{1}{\rho} \frac{\partial P}{\partial z} - g = 0 \quad (7)$$

that is referred to as hydrostatic equation.

Shallow water equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v = \\ - g \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \left(A_h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right) \end{aligned} \quad (8a)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u = \\ - g \frac{\partial \eta}{\partial y} + \frac{\partial}{\partial x} \left(A_h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right) \end{aligned} \quad (8b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (8c)$$

If $H \ll L$, can we just consider just two-dimensional problem?

This question yields “**shallow water equations**”, that is vertically averaged (7) from surface $z = \eta$ to bottom $z = -h$ with boundary conditions:

Shallow water equations

Kinematic boundary conditions

$$w|_{z=\eta} = \frac{\partial \eta}{\partial t} + u|_{z=\eta} \frac{\partial \eta}{\partial x} + v|_{z=\eta} \frac{\partial \eta}{\partial y} \quad (9a)$$

$$w|_{z=-h} = \frac{\partial h}{\partial t} + u|_{z=-h} \frac{\partial h}{\partial x} + v|_{z=-h} \frac{\partial h}{\partial y} \quad (9b)$$

Dynamic boundary condition

$$-A_h \frac{\partial u}{\partial x} \Big|_{z=\eta} \frac{\partial \eta}{\partial x} - A_h \frac{\partial u}{\partial y} \Big|_{z=\eta} \frac{\partial \eta}{\partial y} + A_h \frac{\partial u}{\partial z} \Big|_{z=\eta} = \frac{\tau_x^s}{\rho_0} \quad (10a)$$

$$A_h \frac{\partial u}{\partial x} \Big|_{z=-h} \frac{\partial h}{\partial x} + A_h \frac{\partial u}{\partial y} \Big|_{z=-h} \frac{\partial h}{\partial y} + A_h \frac{\partial u}{\partial z} \Big|_{z=-h} = \frac{\tau_x^b}{\rho_0} \quad (10b)$$

$$-A_h \frac{\partial v}{\partial x} \Big|_{z=\eta} \frac{\partial \eta}{\partial x} - A_h \frac{\partial v}{\partial y} \Big|_{z=\eta} \frac{\partial \eta}{\partial y} + A_h \frac{\partial v}{\partial z} \Big|_{z=\eta} = \frac{\tau_y^s}{\rho_0} \quad (11a)$$

$$A_h \frac{\partial v}{\partial x} \Big|_{z=-h} \frac{\partial h}{\partial x} + A_h \frac{\partial v}{\partial y} \Big|_{z=-h} \frac{\partial h}{\partial y} + A_h \frac{\partial v}{\partial z} \Big|_{z=-h} = \frac{\tau_y^b}{\rho_0} \quad (11b)$$

Shallow water equations

The strategy is to convert equations for 3D variables ($u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$), to those for vertically averaged 2D variables: $\bar{u}(x, y)$, $\bar{v}(x, y)$, and $\eta(x, y)$ where

$$\bar{u} = \frac{1}{\eta + h} \int_h^\eta u \, dz \approx \frac{1}{h} \int_h^\eta u \, dz \approx \frac{1}{h} \int_h^0 u \, dz \quad (12a)$$

$$\bar{v} = \frac{1}{\eta + h} \int_h^\eta v \, dz \approx \frac{1}{h} \int_h^\eta v \, dz \approx \frac{1}{h} \int_h^0 v \, dz \quad (12b)$$

Note that $\eta(x, y)$ and $h(x, y)$, so do not forget to use Leibniz integral rule!

Shallow water equations

After a lot of calculations (maybe with some approximations), we get the form of shallow water equation frequently used as a start point of many oceanographic problems:

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} - f \bar{v} = & \underbrace{\frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right)}_{\text{with (9)}} \\ & - g \frac{\partial \eta}{\partial x} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{u}}{\partial y} \right) \right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \end{aligned} \quad (13a)$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + f \bar{u} = & \underbrace{\frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right)}_{\text{with (10)}} \\ & - g \frac{\partial \eta}{\partial y} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{v}}{\partial y} \right) \right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \end{aligned} \quad (13b)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \quad (13c)$$

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 $\frac{\partial w}{\partial z}$ with (8)

Shallow water equations

Spoiler Alert!

Why it is still important, regardless of **big** limitation from assuming $\rho = \rho_0$ (e.g., no stratification)?

Governing equations for reduced gravity model:

$$\begin{aligned} \frac{\partial \bar{u}^{(1)}}{\partial t} + \bar{u}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial x} + \bar{v}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial y} - f \bar{v}^{(1)} = \\ -g' \frac{\partial h'}{\partial x} + \frac{A_h}{h_1} \left(\frac{\partial}{\partial x} \left(h_1 \frac{\partial \bar{u}^{(1)}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h_1 \frac{\partial \bar{u}^{(1)}}{\partial y} \right) \right) + \frac{\tau_x^s}{\rho_0 h_1} \end{aligned} \quad (14a)$$

$$\begin{aligned} \frac{\partial \bar{v}^{(1)}}{\partial t} + \bar{u}^{(1)} \frac{\partial \bar{v}^{(1)}}{\partial x} + \bar{v}^{(1)} \frac{\partial \bar{v}^{(1)}}{\partial y} + f \bar{u}^{(1)} = \\ -g' \frac{\partial h'}{\partial y} + \frac{A_h}{h_1} \left(\frac{\partial}{\partial x} \left(h_1 \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h_1 \frac{\partial \bar{v}}{\partial y} \right) \right) + \frac{\tau_y^s}{\rho_0 h_1} \end{aligned} \quad (14b)$$

$$\frac{\partial h'}{\partial t} + \frac{\partial((h' + h_1)\bar{u})}{\partial x} + \frac{\partial((h' + h_1)\bar{v})}{\partial y} = 0 \quad (14c)$$

Can you see differences? At some aspect, it is simpler.

Summary

1. Boussinesq approximation $\rho' \ll \rho_0$ makes $\rho \approx \rho_0$ except for ρ associated with g .
2. Hydrostatic approximation $H \ll L$ let us consider only two terms (pressure gradient and gravity) in vertical momentum equation.
3. Shallow water equations are vertically averaged (or integrated) version of the Navier-Stokes and continuity equations.

Now, we are ready to talk about the ocean phenomenon using the shallow water equations!

Assignment

In many ocean's problems, Navier-Stokes and continuity equations can be simplified into linear system:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + A_h \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + A_z \frac{\partial^2 u}{\partial z^2} \quad (15a)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + A_h \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + A_z \frac{\partial^2 v}{\partial z^2} \quad (15b)$$

$$-\frac{1}{\rho_0} \frac{\partial P}{\partial z} = -g \quad (15c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (15d)$$

and the boundary conditions are given by

$$w|_{z=0} = \frac{\partial \eta}{\partial t} \quad (16a)$$

$$w|_{z=-h} = 0 \quad (16b)$$

$$A_z \left. \frac{\partial u}{\partial z} \right|_{z=0} = \frac{\tau_x^s}{\rho_0 h}, \quad A_z \left. \frac{\partial v}{\partial z} \right|_{z=0} = \frac{\tau_y^s}{\rho_0 h} \quad (16c)$$

$$A_z \left. \frac{\partial u}{\partial z} \right|_{z=-h} = \frac{\tau_x^b}{\rho_0 h} \approx \frac{\gamma}{h} \bar{u}, \quad A_z \left. \frac{\partial v}{\partial z} \right|_{z=-h} = \frac{\tau_y^b}{\rho_0 h} \approx \frac{\gamma}{h} \bar{v} \quad (16d)$$

1. What kind of assumptions were made in (14) and (15)?
2. By vertical averaging ($\int_{-h}^0 dz$) (14) with the boundary conditions (15), obtain governing equations for \bar{u} , \bar{v} , and η where

$$\bar{u} = \frac{1}{h} \int_{-h}^0 u dz, \quad \bar{v} = \frac{1}{h} \int_{-h}^0 v dz. \quad (17)$$

that will be the linear shallow water equations.