Differential Equations in Geophysical Fluid Dynamics

XIV. Wind-driven circulation: approximated solution to Stommel wind-driven circulation problem

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Recap

Stommel's wind-driven circulation problem is given by

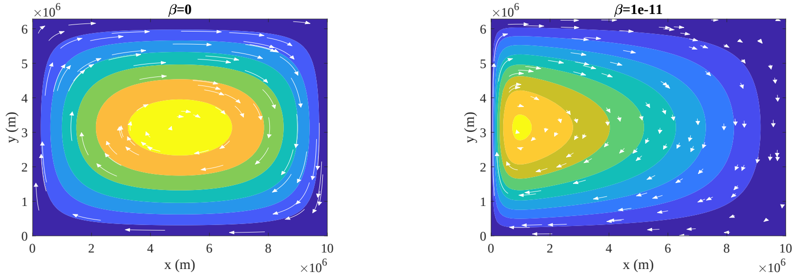
$-(\gamma/h)\nabla \times \vec{u}$: Bottom stress curl

$$\frac{\gamma}{h}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \beta\frac{\partial\psi}{\partial x} = -\frac{\tau_0\pi}{\rho_0 h L_y}\sin(\pi y/L_y) \tag{1a}$$

$\beta\bar{v}$: Planetary β-term

Wind stress curl

$$\psi|_{x=0} = 0, \quad \psi|_{x=L_x} = 0, \quad \psi|_{y=0} = 0, \quad \psi|_{y=L_y} = 0. \tag{1b}$$

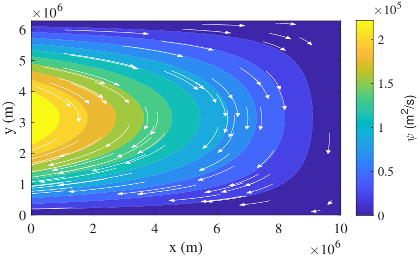


Scale analysis

Let us nondimensionalize the governing equation using $x = Lx^*$, $y = Ly^*$, and $\psi = \Psi\psi^*$:

$$\underbrace{\epsilon\left(\frac{\partial^2\psi^*}{\partial x^{*2}} + \frac{\partial^2\psi^*}{\partial y^{*2}}\right)}_{O(\epsilon)\text{: small}} + \overbrace{\boxed{\frac{\partial\psi^*}{\partial x^*} = \nabla \times \vec{\tau}_s^{\,*}}}^{O(1)\text{: big “Sverdrup balance”}} \tag{2}$$

where $\epsilon = L_S/L \ll 1$ and $L_S = \gamma/(h\beta)$. The order of the forcing term is set to be identical to that of the beta term.



Scale analysis

What if we set $x = L_S x^*$ representing narrow western boundary region?

O(1): big

$$\boxed{\frac{\partial^2\psi^*}{\partial x^{*2}}} + \epsilon^2 \frac{\partial^2\psi^*}{\partial y^{*2}} + \boxed{\frac{\partial\psi^*}{\partial x^*}} = \epsilon\nabla\times\vec{\tau}_s^{\,*} \tag{3}$$

$O(\epsilon)$: small

$O(\epsilon^2)$: very small

So, for the narrow western boundary, the governing equation can be simplified to

$$\frac{\partial^2\psi}{\partial x^{*2}} + \frac{\partial\psi^*}{\partial x^*} = 0 \tag{4}$$

$$\left(\frac{\gamma}{h}\frac{\partial^2\psi}{\partial x^2} + \beta\frac{\partial\psi}{\partial x} = 0\right)$$

Simplified problem

Consider two components of the steam function: interior component (ψ_I) and boundary component (ψ_B), so $\psi = \psi_I + \psi_B$.

Based on scaling analysis above, simplified governing equations for each component are given by

$$\beta \frac{\partial \psi_I}{\partial x} = -\frac{\tau_0 \pi}{\rho_0 h L_y} \sin(\pi y / L_y) \tag{5}$$

$$\frac{\gamma}{h} \frac{\partial^2 \psi_B}{\partial x^2} + \beta \frac{\partial \psi_B}{\partial x} = 0 \tag{6}$$

and the boundary conditions, also simplified, are given by

$$\psi|_{x=0} = 0 \qquad \lim_{x \to \infty} \psi = \psi_I. \tag{7}$$

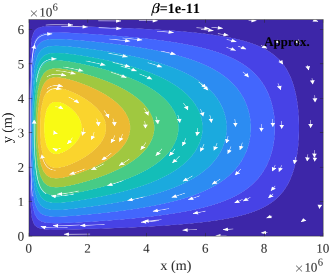
Simplified problem

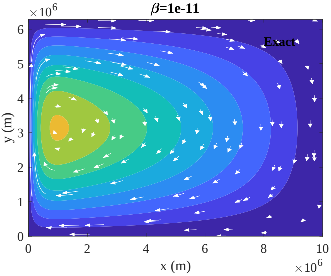
Solution to the simplified problem is given by

$$\psi = \psi_I + \psi_B \tag{8a}$$

$$\psi_I = \left(-\frac{\tau_0 \pi}{\rho_0 L_y h \beta} \sin(\pi y / L_y) \right) (x - L_x) \tag{8b}$$

$$\psi_B = -\psi_I e^{x/L_S} \tag{8c}$$





Summary

For the narrow western boundary region, governing equation is simplified to

$$\frac{\gamma}{h}\frac{\partial^2\psi_B}{\partial x^2} + \beta\frac{\partial\psi_B}{\partial x} = 0 \tag{9a}$$

$$\psi_B|_{x=0} = -\psi_I \tag{9b}$$

$$\lim_{x\to\infty}\psi_B = 0 \tag{9c}$$

This implies that the boundary current is forced by interior flow, rather than regional wind stress forcing.

Solution to the problem is given by

$$\psi_B = -\psi_I e^{x/L_S} \tag{10}$$

where $L_S = \gamma/(h\beta)$ representing scale of width of western boundary current.