Differential Equations in Geophysical Fluid Dynamics

IV. Generalization of forcing term

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

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Recap

Wind-forced linear momentum equation:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + if\vec{u}} = \frac{\vec{\tau}^s}{\rho_0 h} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t} \qquad (1)$$

$\left(\frac{1}{fT}\right) \frac{\partial \vec{u}^*}{\partial t^*} + i\vec{u}^*$ where $1/(fT) \equiv Ro_T$
is the temporal Rossby number.

Solution to the problem is given by

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i (f - w_0)}. \qquad (2)$$

Recap

$$\frac{\partial \vec{u}}{\partial t} + \boxed{if\vec{u} = \frac{\vec{\tau}^s}{\rho_0 h}} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}$$

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i (f - w_0)}$$

1. case $w_0 \ll f$ (low frequency or short period forcing)

$$\vec{u} \approx \frac{\vec{\tau}^s}{i\rho_0 h f} \equiv \vec{u}_e \tag{3}$$

that satisfies

$$\boxed{if\vec{u} \approx \frac{\vec{\tau}^s}{\rho_0 h}}. \tag{4}$$

Negligible inertia

Recap

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + if\vec{u} \boxed{= \frac{\vec{\tau}^s}{\rho_0 h}} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}$$

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i(f - w_0)}$$

2. case $w_0 \gg f$ (high frequency or long period forcing)

$$\vec{u} \approx \frac{\vec{\tau}^s}{(-iw_0)\rho_0 h} \tag{5}$$

that satisfies

$$\boxed{\frac{\partial \vec{u}}{\partial t} \approx \frac{\vec{\tau}^s}{\rho_0 h}}. \tag{6}$$

https://www.youtube.com/watch?v=w1UsKanMatM
https://www.youtube.com/watch?v=nJphsM4obOk

Negligible Coriolis force

Recap

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}^s}{\rho_0 h} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-ift}$$

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 hi(\boxed{f - w_0})}$$

As $w_0 \to f$,
magnitude of $\vec{u}$ increases!

3. case $w_0 \approx f$ (frequency close to the system's natural frequency)

$$\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h} \boxed{t} e^{-ift} \qquad (7)$$

Amplitude linearly increase with time!

that is what we call "resonance".

Introduction

Why have so many people used sinusoidal (or monochromatic) forcing?

1. Stommel's (1948) wind driven circulation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + A\frac{\partial\psi}{\partial x} = B\sin(k_0 y) \tag{8}$$

2. Cushman-Roisin's (2011) upwelling problem

$$\frac{\partial^3\psi}{\partial t^3} - A\frac{\partial^3\psi}{\partial t\partial x^2} + B\frac{\partial\psi}{\partial t} = 0, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = C\cos(w_0 t) \tag{9}$$

3. Csanady's (1978) steady coastal trapped wave problem

$$-A\frac{\partial\psi}{\partial y} = \frac{\partial^2\psi}{\partial x^2}, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = B\cos(k_0 y) \tag{10}$$

Superposition principle: Key characteristics of linear differential equations

Problem with additional sinusoidal forcing term

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u}}_{\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + (if + \frac{\gamma}{h})} = \overbrace{\frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}}^{\equiv f_0} + \overbrace{\frac{\hat{\tau}_1}{\rho_0 h} e^{-iw_1 t}}^{\equiv f_1} \qquad (11)$$

Superposition principle of linear non-homogeneous differential equation

1. Once $L[u_0] = f_0$ and $L[u_1] = f_1$,
   $L[c_0 u_0 + c_1 u_1] = c_0 f_0 + c_1 f_1$.
2. Therefore, once $L[u_0] = f_0$, $L[u_0 + C u_1] = f_0$ where
   $L[u_1] = 0$ (case of $f_1 = 0$).

Generalization to problem with arbitrary forcing term

Arbitrary function of t
(any time series of wind stress)

$$\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u} = \frac{\boxed{\vec{\tau}^s(t)}}{\rho_0 h} \equiv \frac{1}{\rho_0 h}\boxed{\sum_{n=-\infty}^{\infty} \hat{\tau}_n e^{-iw_n t}} \tag{12}$$

Fourier series

Note that

$$\hat{\tau}(w) = \mathcal{F}(\vec{\tau}(t)) \tag{13}$$

where $\mathcal{F}$ indicates Fourier operator. Particular solution to the problem is

Transfer function ($\equiv \hat{g}(w_n)$)

$$\vec{u} = \sum_{n=-\infty}^{\infty} \underbrace{\boxed{\frac{1}{\rho_0 h(i(f - w_n) + \gamma/h)}}\,\hat{\tau}_n}_{\hat{u}(w_n)}\, e^{-iw_n t}. \tag{14}$$

https://jang-geun.github.io/vis_fourier_transform.gif

Generalization to problem with arbitrary forcing term

Let us consider problem without initial condition ($\lim_{t\to-\infty}\vec{u}=0$)

$$\vec{u}=\frac{1}{\rho_0 h}\sum_{n=-\infty}^{\infty}\frac{1}{(i(f-w_n)+\gamma/h)}\hat{\tau}_n e^{-iw_n t}. \tag{15}$$

Corresponding solution (Lentz and Winant, 1986; Austin and Barth, 2002) obtained by the integrating factor is given by

$$\vec{u}=\frac{1}{\rho_0 h}\int_{-\infty}^{t} g(\tilde{t}-t)\vec{\tau}(\tilde{t})\,d\tilde{t}. \tag{16}$$

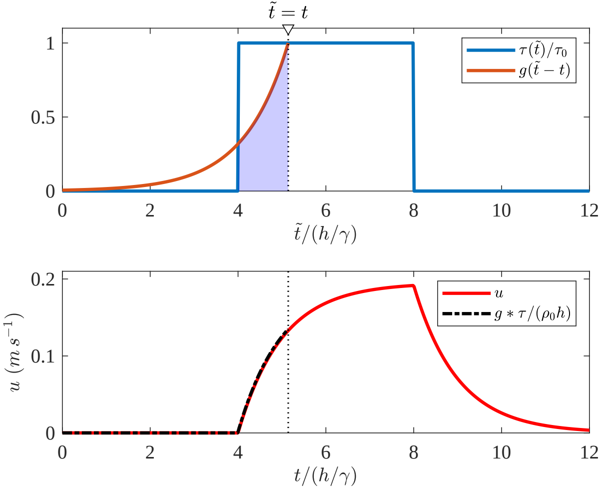
$\equiv g * \vec{\tau}$ (Convolution!)

where $g(t^{\star})=e^{(if+\gamma/h)t^{\star}}$.

If solution is unique, they must be same. This is end up with "**convolution theorem**":

$$\hat{g}(w)\hat{\tau}(w)=\mathcal{F}(g(t)*\vec{\tau}(t)) \tag{17}$$

Generalization to problem with arbitrary forcing term



https://jang-geun.github.io/vis_convolution.gif

The slab model

Pollard and Millard (1970):

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}}{\rho_0 h_1} - \gamma^\star \vec{u} \tag{18}$$

where $\gamma^\star$ represents linear damping coefficient, that can be determined by calibration.

This is a theoretical model that can still be used for publication (Whitt and Thomas, 2015; Gough et al., 2016; Wang et al., 2019; Zhang et al., 2023)!

Summary

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}(t)}{\rho_0 h_1} - \gamma^\star \vec{u}$$

where $\vec{\tau}(t)$ is an arbitrary function of t. Solution to the problem is given by

$$\begin{aligned} \vec{u} &= \frac{1}{\rho_0 h} \sum_{n=-\infty}^{\infty} \frac{1}{(i(f - w_n) + \gamma/h)} \hat{\tau}_n e^{-iw_n t} \\ &= \frac{1}{\rho_0 h} \int_{-\infty}^{t} g(\tilde{t} - t)\vec{\tau}(\tilde{t})\, d\tilde{t} \,. \end{aligned} \tag{19}$$

where $g(t^\star) = e^{(if+\gamma/h)t^\star}$. This implies convolution theorem. The contribution of past wind stress to the present current decays exponentially over time by friction term.

Assignment

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} - \frac{\gamma}{h}\vec{u} \tag{20a}$$

$$\vec{u}|_{t=0} = \vec{U}_0 \tag{20b}$$

$$\frac{d\vec{X}}{dt} = \vec{u} \tag{21}$$

where $-g\partial\eta/\partial\vec{n}$ is arbitrary constants and $\vec{X} = X + iY$. X and Y represent x- and y-position of an object, respectively.

1. Solve differential equation (20) for $\vec{u}$. What is physical meaning of particular solution component?
2. Solve (21) using $\vec{u}$ from (20) and constant f assumption.

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