Differential Equations in Geophysical Fluid Dynamics

XVI. Wave: surface gravity wave and wave equation

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Recap

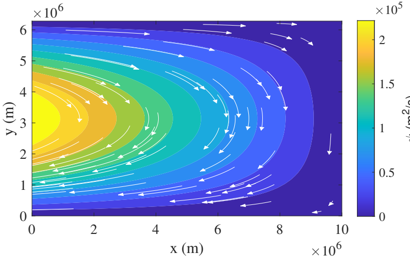
We discussed wind driven circulation governed by

Sverdrup (1947):

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_x^s}{\rho_0 h} \qquad \text{(1a)}$$

$$f\bar{u} = -g\frac{\partial \eta}{\partial y} + \frac{\tau_y^s}{\rho_0 h} \qquad \text{(1b)}$$

$$h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0 \qquad \text{(1c)}$$



Recap

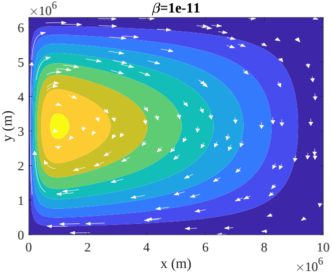
We discussed wind driven circulation governed by

Stommel (1948):

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h}\bar{u} \quad \text{(2a)}$$

$$f\bar{u} = -g\frac{\partial \eta}{\partial y} + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h}\bar{v} \quad \text{(2b)}$$

$$h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0 \quad \text{(2c)}$$



Recap

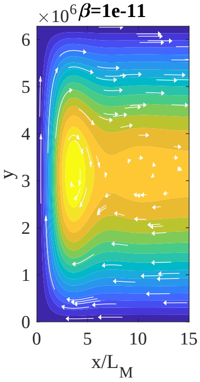
We discussed wind driven circulation governed by

Munk (1950):

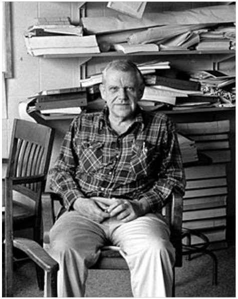
$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_x^s}{\rho_0 h} \quad \text{(3a)}$$

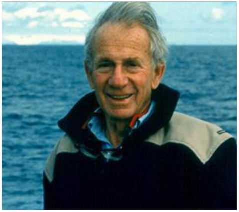
$$f\bar{u} = -g\frac{\partial \eta}{\partial y} + \frac{\tau_y^s}{\rho_0 h} + A_h\frac{\partial^2 \bar{v}}{\partial x^2} \quad \text{(3b)}$$

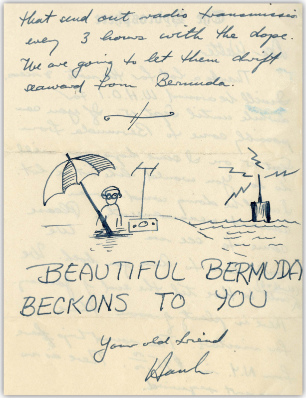
$$h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0 \quad \text{(3c)}$$



A few asides







that send out radio transmissions
every 3 hours with the dope.
We are going to let them drift
seaward from Bermuda.

BEAUTIFUL BERMUDA
BECKONS TO YOU

Your old friend
Hank

Henry Stommel, Walter Munk, and part of a letter from Stommel to Munk (left to right).

Introduction

But all assumed "steady-state ($\partial/\partial t \approx 0$)"!

The fluid surfaces found in nature keep fluctuating ($\partial/\partial t \neq 0$), resulting in what we refer to as "**waves**".

Niagara River, New York, photo by J.-G. Choi.

Hydrostatic surface gravity wave

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + A_h \nabla^2 \bar{u} + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h}\bar{u} \tag{4a}$$

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f\bar{u} = -g\frac{\partial \eta}{\partial y} + A_h \nabla^2 \bar{v} + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h}\bar{v} \tag{4b}$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0 \tag{4c}$$

Substituting $\partial(4a)/\partial x$ and $\partial(4b)/\partial y$ into $\partial(4c)/\partial t$ yields

$$\frac{\partial^2 \eta}{\partial t^2} = gh\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) \tag{5}$$

that is referred to as the "**wave equation**".

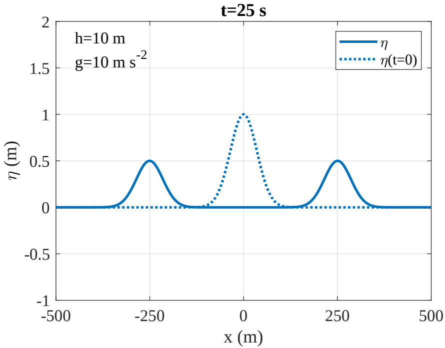
Wave equation

Let us consider one-dimensional problem ($\partial/\partial y \approx 0$ or $\bar{v} \approx 0$) and then conduct scale analysis ($t = Tt^*$, $x = Lx^*$, and $\eta = \phi\eta^*$):

$$\frac{\partial^2 \eta}{\partial t^2} = gh \frac{\partial^2 \eta}{\partial x^2}. \tag{6}$$

Unit of velocity!

$$\frac{L}{T} \sim \pm\sqrt{gh} \tag{7}$$



https://jang-geun.github.io/vis_wave1d.gif

Wave equation

Assume $\eta = \eta_0 e^{i(kx - wt)}$ and then substituting it to the governing equation yields

Wave equation

$$\frac{\partial^2 \eta}{\partial t^2} = A \frac{\partial^2 \eta}{\partial x^2}$$

Unit of velocity!

$$w = \pm\sqrt{A}k, \quad \left(\boxed{\frac{w}{k}} = \pm\sqrt{A} \right)$$

$$\therefore \eta = C_1 e^{\boxed{i}\,(k(x - \sqrt{A}t))} + C_2 e^{\boxed{i}\,(k(x + \sqrt{A}t))}$$

Heat equation

$$\frac{\partial \eta}{\partial t} = A \frac{\partial^2 \eta}{\partial x^2}$$

$$w = -iAk^2$$

$$\therefore \eta = C_1 e^{\boxed{ikx}} e^{\boxed{-Ak^2 t}}$$

The equation showing relationship between wavenumber k and frequency w is referred to as "**dispersion relation**". This is already a solution for the differential equation.

Wave equation

Do not forget where we come from:

$$\frac{\partial \bar{u}}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{8a}$$

$$\frac{\partial \eta}{\partial t} + h\frac{\partial \bar{u}}{\partial x} = 0 \tag{8b}$$

As we assumed $\eta = \eta_0 e^{i(kx-wt)}$, we can assume $\bar{u} = \bar{u}_0 e^{i(kx-wt)}$.[1]
Substituting these in to system (8) yields:

$$-iw\bar{u}_0 e^{i(kx-wt)} + gik\eta_0 e^{i(kx-wt)} = 0 \tag{9a}$$

$$-iw\eta_0 e^{i(kx-wt)} + hik\bar{u}_0 e^{i(kx-wt)} = 0 \tag{9b}$$

$$\underbrace{\begin{bmatrix} -w & gk \\ hk & -w \end{bmatrix}}_{\vec{M}} \begin{bmatrix} \bar{u}_0 \\ \eta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

[1] We can easily see that the k and w in $\bar{u}$ should be identical to those in η based on the equation.

Energy conservation of wave equation

The wave equation satisfies that

$$\frac{\partial}{\partial t}\left(\underbrace{\frac{1}{2}h\bar{u}^2}_{\text{Kinetic energy } (KE)} + \underbrace{\frac{1}{2}g\eta^2}_{\text{Potential energy } (PE)}\right) + gh\frac{\partial(\bar{u}\eta)}{\partial x} = 0 \tag{11}$$

Integrating over spaces with closed boundary conditions yields

$$\frac{\partial}{\partial t}\left(\underbrace{\int_0^L E\,dx}_{E \text{ is conserved!}}\right) = 0 \tag{12}$$

where $E = KE + PE$. Using $\bar{u} = u_0 e^{i(kx-wt)}$, $\eta = \eta_0 e^{i(kx-wt)}$, and dispersion relation $w/k = \pm\sqrt{gh}$, (11) can be written as

$$\frac{\partial E}{\partial t} \underbrace{\pm \sqrt{gh}}_{E \text{ is transported by wave!}} \frac{\partial E}{\partial x} = 0. \tag{13}$$

Conclusion

1. By substituting $\eta = \eta_0 e^{i(kx - wt)}$ into the governing equation of the wave problem, we can obtain relation between wavenumber k and frequency w, so-called dispersion relation. In case of the surface gravity wave, it is given by

$$w^2 - Ak^2 = 0 \tag{14}$$

2. $w/k (\equiv c)$ is the wave velocity that can be estimated by the dispersion relation (e.g., $c = \pm\sqrt{gh}$).
3. Energy is conserved in the wave problems and transported by the waves.

Assignment

Govering equation of Rossby wave in low frequency limit is given by

$$\frac{\partial \eta}{\partial t} - g\frac{gh}{f^2}\frac{\partial \eta}{\partial x} = 0 \tag{15}$$

Using wave-like solution $\eta = \eta_0 e^{i(kx-wt)}$, find (1) dispersion relation and (2) wave velocity.

References I

- Munk, Walter H (1950). “On the wind-driven ocean circulation”. In: *Journal of Atmospheric Sciences* 7.2, pp. 80–93.
- Stommel, Henry (1948). “The westward intensification of wind-driven ocean currents”. In: *Eos, Transactions American Geophysical Union* 29.2, pp. 202–206.
- Sverdrup, Harald Ulrich (1947). “Wind-driven currents in a baroclinic ocean; with application to the equatorial currents of the eastern Pacific”. In: *Proceedings of the National Academy of Sciences* 33.11, pp. 318–326.