Differential Equations in Geophysical Fluid Dynamics

VI. Vertical structure of Ekman and geostrophic current component

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Apr, 2025

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography commuinity COKOAA.

Recap

Generalization of forcing

Arbitrary function of t
(any time series of wind stress)

$$\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u} = \frac{\boxed{\vec{\tau}^s(t)}}{\rho_0 h} \equiv \frac{1}{\rho_0 h} \boxed{\sum_{n=-\infty}^{\infty} \hat{\tau}_n e^{-iw_n t}} \qquad (1)$$

Fourier series

Solution to the problem is given by

$$\vec{u} = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{\rho_0 h(i(f - w_n) + \gamma/h)} \hat{\tau}_n}_{\hat{u}(w_n)} e^{-iw_n t}.$$

$$= \frac{1}{\rho_0 h} \boxed{\int_{-\infty}^{t} g(\tilde{t} - t)\vec{\tau}(\tilde{t})\, d\tilde{t}}. \qquad (2)$$

Convolution theorem

where $g(t^\star) = e^{(if+\gamma/h)t^\star}$.

Recap

Superposition principle of nonhomogeneous differential equation

$$\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u} = \underbrace{-g\frac{\partial \eta}{\partial \vec{n}}}_{f_1(t)} + \underbrace{\frac{\vec{\tau}^s}{\rho_0 h}}_{f_2(t)} \tag{3}$$

Solution to (3) is $\vec{u}_1 + \vec{u}_2$ where

$$\frac{\partial \vec{u}_1}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u}_1 = -g\frac{\partial \eta}{\partial \vec{n}} \tag{4a}$$

$$\frac{\partial \vec{u}_2}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u}_2 = \frac{\vec{\tau}^s}{\rho_0 h}. \tag{4b}$$

Each represents pressure-driven ($\vec{u}_1$) and wind-driven current component ($\vec{u}_2$) and is the basis of current component decomposition.

Current component in geophysical scales

Assume $Ro_T = 1/(fT) \ll 1$

$$\cancel{\frac{\partial \vec{u}}{\partial t}} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\vec{\tau}^b}{\rho_0 h} \tag{5}$$

$$\therefore \vec{u} = \underbrace{\boxed{\frac{1}{i}\left(-\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}}\right)}}_{\equiv \vec{u}_g \text{ (geostrophic current)}} + \underbrace{\boxed{\frac{1}{i}\frac{\vec{\tau}^s}{\rho_0 h f}}}_{\equiv \vec{u}_e^s \text{ (surface Ekman current)}} \underbrace{\boxed{- \frac{1}{i}\frac{\vec{\tau}^b}{\rho_0 h f}}}_{\equiv \vec{u}_e^b \text{ (bottom Ekman current)}} \tag{6}$$

But do not forget that this is vertical averaged by the definition:

$$\vec{u} = \bar{u} + i\bar{v}$$

$$\text{where} \quad \bar{u} = \frac{1}{h}\underbrace{\boxed{\int_{-h}^{0} u\,dz}}_{\text{Referred to as ``transfport''}} \quad \text{and} \quad \bar{v} = \frac{1}{h}\underbrace{\boxed{\int_{-h}^{0} v\,dz}}_{\text{Referred to as ``transfport''}} \tag{7}$$

Current component in geophysical scales

Original linear NSE

$$if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} + A_z\frac{\partial^2 \vec{u}}{\partial z^2} \quad (8a)$$

$$A_z\frac{\partial \vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \quad (8b)$$

$$A_z\frac{\partial \vec{u}}{\partial z}\bigg|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma\vec{u} \quad (8c)$$

where $\vec{u} = u + iv$

Vertical averaged SWE

$$if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\gamma}{h}\vec{u} \quad (9)$$

where $\vec{u} = \bar{u} + i\bar{v}$

$$\bar{u} = \frac{1}{h}\int_{-h}^{0} u\,dz$$

$$\bar{v} = \frac{1}{h}\int_{-h}^{0} v\,dz$$

We need to study original equation to talk about vertical structure of current component.

Governing equation

Geostrophic balance

$$A_z \frac{\partial^2 \vec{u}}{\partial z^2} - i f \vec{u} = g \frac{\partial \eta}{\partial \vec{n}} \tag{10a}$$

Ekman balance

$$A_z \left. \frac{\partial \vec{u}}{\partial z} \right|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \tag{10b}$$

We will try several boundary conditions (Ekman, 1905; Welander, 1957; Kim et al., 2023):

$$\left. \vec{u} \right|_{z \to -\infty} = 0 \tag{11a}$$

$$\left. \vec{u} \right|_{z=-h} = 0 \tag{11b}$$

$$A_z \left. \frac{\partial \vec{u}}{\partial z} \right|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma \left. \vec{u} \right|_{z=-h} . \tag{11c}$$

Vertical structure of current components

$$A_z \frac{\partial^2 \vec{u}}{\partial z^2} - if\vec{u} = g\frac{\partial \eta}{\partial \vec{n}} \tag{12a}$$

$$A_z \frac{\partial \vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \tag{12b}$$

$$\vec{u}\big|_{z\to-\infty} = 0 \tag{12c}$$

Solution to the problem is given by

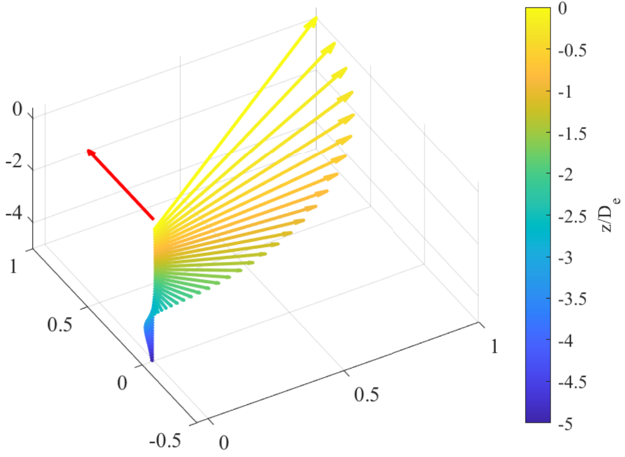
Angle relative to direction of wind stress (θ)

$$\begin{aligned}\vec{u} &= \frac{2\vec{\tau}^s}{\rho_0 f D_e(1+i)} e^{z/D_e} e^{iz/D_e} \\ &= \frac{\sqrt{2}\,|\vec{\tau}|}{f\rho_0 De} e^{z/D_e} e^{i(\theta\ -\pi/4+z/D_e\ )} \equiv \vec{u}_e^s\end{aligned} \tag{13}$$

where $\vec{\tau} = |\vec{\tau}|\,(\cos\theta + i\sin\theta) = |\vec{\tau}|\,e^{i\theta}$ and $\partial\eta/\partial\vec{n}$ must be zero, unless bottom boundary condition cannot be satisfied (ill-posed). $D_e = \sqrt{(2A_z/f)}$ is the Ekman depth.

Vertical structure of current components

Surface Ekman current component



Considering finite depth with no slip condition

$$A_z \frac{\partial^2 \vec{u}}{\partial z^2} - if\vec{u} = g\frac{\partial \eta}{\partial \vec{n}} \tag{14a}$$

$$A_z \frac{\partial \vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \tag{14b}$$

$$\vec{u}|_{z=-h} = 0 \tag{14c}$$

Solution to the problem is given by

$$\vec{u} = \frac{2\vec{\tau}^s}{\rho_0 f D_e(1+i)} \frac{\sinh[j(h+z)]}{\cosh(jh)} + i\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}} - i\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}}\frac{\cosh(jz)}{\cosh(jh)} \tag{15}$$

where $j = (1+i)/D_e$, $\sinh x = (e^x - e^{-x})/2$, and $\cosh x = (e^x + e^{-x})/2$.

Considering finite depth with no slip condition

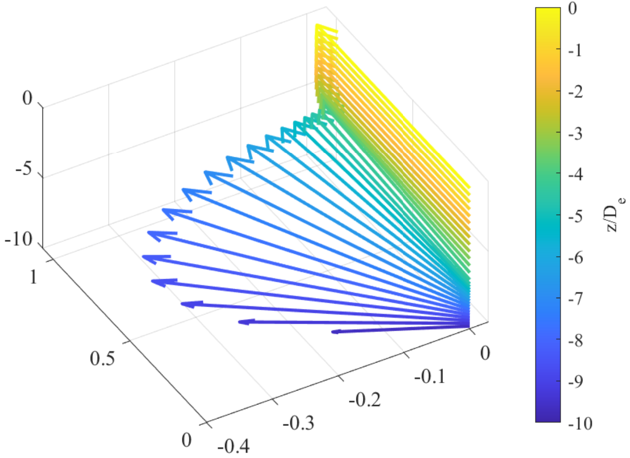
In the limit where $1 \ll h/D_e$, velocity can be expressed as superposition of surface Ekman current, (interior) geostrophic current, and bottom Ekman current components:

$$
\begin{aligned}
\vec{u} \approx\ & \underbrace{\frac{2\vec{\tau}^s}{\rho_0 f D_e(1+i)} e^{z/D_e} e^{iz/D_e}}_{\equiv \vec{u}_e^s} + \overbrace{i\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}}}^{\equiv \vec{u}_g} \\
& \underbrace{- i\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}} e^{-(z+h)/D_e} e^{-i(z+h)/D_e}}_{\equiv \vec{u}_e^b} \\
=\ & \frac{\sqrt{2}\,|\vec{\tau}|}{f\rho_0 De} e^{z/D_e} e^{i(\theta-\pi/4 + z/D_e)} + \vec{u}_g \\
& - |\vec{u}_g|\, e^{-(z+h)/D_e} e^{i(\theta_g - (z+h)/D_e)}
\end{aligned} \tag{16}
$$

Opposite rotation with surface Ekman current

Considering finite depth with no slip condition

Geostrophic current and its bottom Ekman current components



Considering finite depth with no slip condition

In the limit where $h/D_e \ll 1$,

$$\vec{u} \approx \frac{\vec{\tau}^s}{\rho_0 A_z}(z+h) \tag{17}$$

that is the solution to

$$A_z \frac{\partial^2 \vec{u}}{\partial z^2} - \cancel{if\vec{u}} = 0 \tag{18a}$$

$$A_z \frac{\partial \vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \tag{18b}$$

$$\vec{u}\big|_{z=-h} = 0 \tag{18c}$$

so represents ageostrophic environment where is no geostrophic current and wind-driven current is not governed by the Ekman balance.

Considering finite depth with linear drag condition

$$A_z \frac{\partial^2 \vec{u}}{\partial z^2} - if\vec{u} = g\frac{\partial \eta}{\partial \vec{n}} \tag{19a}$$

$$A_z \frac{\partial \vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \tag{19b}$$

$$A_z \frac{\partial \vec{u}}{\partial z}\bigg|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma \left.\vec{u}\right|_{z=-h}. \tag{19c}$$

Solution to the problem, discussed by Kim et al., 2023, is given by

$$\begin{aligned}\vec{u} =& \frac{\vec{\tau}^s}{\rho_0 f D_e(1+i)} \frac{j(A_z/\gamma)\cosh[j(z+h)] + \sinh[j(z+h)]}{j(A_z/\gamma)\sinh(jh) + \cosh(jh)} \\ &+ i\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}} - i\frac{g}{f}\frac{\partial \eta}{\partial \vec{n}}\frac{\cosh(jz)}{j(A_z/\gamma)\sinh(jh) + \cosh(jh)}\end{aligned} \tag{20}$$

In the limit where $\gamma \to \infty$, (20) becomes identical to (15) using no slip condition. In case of $\gamma = 0$ (free slip condition), terms associated bottom friction disappear.

Conclusion

1. Linear momentum equations in geostrophic scale (assuming steady state) have two major current components: geostrophic and Ekman current component.
2. Ekman current components are concentrated at the surface and bottom boundary layers and have rotating spiral vertical structure.
3. Transports (vertical averaged velocity components) of Ekman current are consistent with those based on the shallow water equation.

Assignment

The simplest problem for bottom Ekman current is given by

$$A_z \frac{\partial^2 \vec{u}}{\partial z^2} - if\vec{u} = g\frac{\partial \eta}{\partial \vec{n}} \tag{21a}$$

$$\left.\vec{u}\right|_{z\to\infty} = 0 \tag{21b}$$

$$\left. A_z \frac{\partial \vec{u}}{\partial z}\right|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma \left.\vec{u}\right|_{z=-h}. \tag{21c}$$

1. Find solution to the problem.
2. Shows that the solution to (21) corresponds to the bottom Ekman current term in solution to (20) with $1 \ll h/D_e$.
3. Shows that the solution to (21) with $\gamma \to \infty$ ($fD_e \ll \gamma$) corresponds to the bottom Ekman current term in (16).

References

Ekman, Vagn Walfrid (1905). “On the influence of the earth’s rotation on ocean-currents.”. In: *Arkiv för Matematik, Astronomy Och Fysik*.

Kim, Deoksu et al. (2023). “Upwelling processes driven by contributions from wind and current in the Southwest East Sea (Japan Sea)”. In: *Frontiers in Marine Science* 10, p. 1165366.

Welander, Pierre (1957). “Wind action on a shallow sea: some generalizations of Ekman’s theory”. In: *Tellus* 9.1, pp. 45–52.