Differential Equations in Geophysical Fluid Dynamics

III. Forced inertial oscillation and resonance

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

Mar, 2025

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography community COKOAA.

Recap

Inertial oscillation problem:
First order linear homogeneous ordinary differential equation

Governing equation:

$$\frac{d\vec{u}}{\partial t} + if\vec{u} = 0 \tag{1a}$$

$$\vec{u}|_{t=0} = \vec{U}_0 \tag{1b}$$

Solution:

$$\vec{u} = \vec{U}_0 \, \boxed{e^{-ift}} \tag{2}$$

Oscillation with frequency f
(period $2\pi/f$)

that yields circle trajectory of which radius is $|\vec{U}_0|/f$.

Recap

Linear homogeneous differential equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + if\vec{u}}_{\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + if} = \underbrace{\boxed{0}}_{\text{No forcing term (homogeneous)}: F(t)=0}$$

Superposition principle of linear homogeneous differential equation

1. Once $L[u_1] = 0$, $L[Cu_1] = 0$.
2. Once $L[u_1] = 0$ and $L[u_2] = 0$, $L[u_1 + u_2] = 0$
3. Therefore, once $L[u_1] = 0, \ldots, L[u_n] = 0$, $L[C_1 u_2 + \cdots + C_n u_n] = 0$

Assignment

Inertial oscillation problem with bottom friction:
First order linear homogeneous ordinary differential equation

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -\frac{\gamma}{h}\vec{u} \tag{3a}$$

$$\vec{u}\big|_{t=0} = \vec{U}_0 \tag{3b}$$

Solution to the problem is

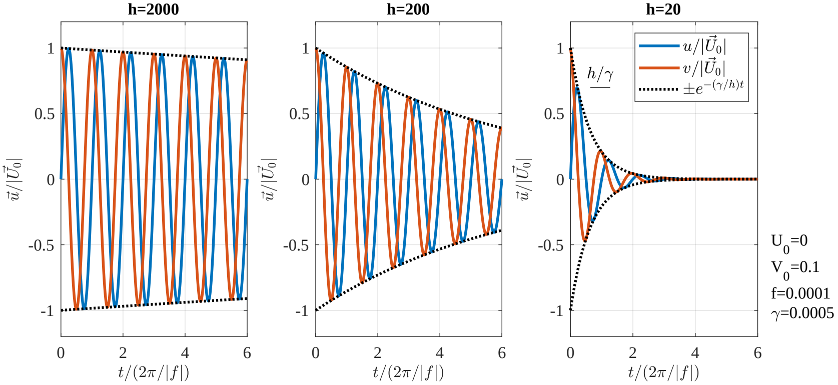
oscillation ($\sin(ft)$ and $\cos(ft)$)

$$\vec{u} = \vec{U}_0 \, e^{-(\gamma/h)t} \, e^{-ift}. \tag{4}$$

Exponential decay

Assignment

Inertial oscillation problem with bottom friction:



"h/γ": frictional adjustment time (Csanady, 1981)

Linear momentum equation of shallow water equation

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + \overbrace{A_h\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right)}^{\approx 0 \text{ (laterally \textbf{inviscid})}} + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h}\bar{u} \qquad \text{(5a)}$$

$$\frac{\partial \bar{v}}{\partial t} - f\bar{u} = -g\frac{\partial \eta}{\partial y} + A_h\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h}\bar{v} \qquad \text{(5b)}$$

Writing (5) in complex coordinate ((5a)+$i\times$(5b)) yields

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \underbrace{\boxed{-g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h}}}_{\text{Forcing terms!}} - \frac{\gamma}{h}\vec{u}$$

where $\vec{u} = u + iv$, $\partial\eta/\partial\vec{n} = (\partial\eta/\partial x) + i(\partial\eta/\partial y)$, and $\vec{\tau}^s = \tau_x^s + i\tau_y^s$.

Inertial oscillation with wind force

When η is arbitrary given function, we can still stay on the ordinary differential equation problem!

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\gamma}{h}\vec{u} \tag{6}$$

For simplicity, let us consider no sea surface height gradient ($\partial\eta/\partial\vec{n} = 0$) and sinusoidal wind stress $\vec{\tau}^s = \hat{\tau}_0 e^{-iw_0 t}$:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h}e^{-iw_0 t} - \frac{\gamma}{h}\vec{u}} \tag{7}$$

where $\vec{\tau}_0$ and w_0 are constants representing amplitude and frequency of the wind forcing, respectively.

Inertial oscillation with wind force

So, we have first order non-homogeneous ordinary differential equation problem:

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t} - \frac{\gamma}{h}\vec{u}. \tag{8a}$$

$$\vec{u}\big|_{t=0} = \vec{U}_0 \tag{8b}$$

$$\vec{u} = \frac{\overbrace{\hat{\tau}_0 e^{-iw_0 t}}^{\vec{\tau}^s}}{\rho_0 h(i(f - w_0) + \gamma/h)} \equiv \vec{u}_p \tag{9}$$

This indicates...

Inertial oscillation with wind force

If the solution is too complicated to get some wisdom,
try simplifying it by **taking limits** (or by **thinking specific cases**).

1. $(\gamma/h, w_0) \ll f$
2. $(\gamma/h, f) \ll w_0$
3. (w_0 and f)$\ll \gamma/h$
4. $\gamma/h \ll w_0 \approx f$

Summary

Solution to the problem considering initial condition is given by

Component associated with initial condition

$$\vec{u} = \boxed{\left(\vec{U}_0 - \frac{\hat{\tau}_0}{\rho_0(f - w_0) + \gamma/h}\right) e^{-(if+\gamma/h)t}} + \boxed{\frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h(i(f - w_0) + \gamma/h)}} \tag{10}$$

Component associated with forcing

What is time scale during which the influence of the initial condition exists?

Summary

1. Frictionless assumption cannot be global, valid for finite time ($t \ll h/\gamma$).
2. For $w_0 \ll f$ (low frequency forcing), Ekman transport, wind stress balanced by Coriolis force, becomes predominant.
3. For $f \ll w_0$ (high frequency forcing), currents are accelerated in the direction of wind stress balanced by inertia.
4. For $w_0 \approx f$, resonance appears and current response to wind stress is maximized.
5. In this forced problem, period of forcing represents the time scale of phenomenon.

Advanced topic

A generalization of forcing term: Fourier series and convolution

Inertial oscillation with wind force

Why have so many people used sinusoidal (or monochromatic) forcing?

1. Stommel's (1948) wind driven circulation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + A\frac{\partial \psi}{\partial x} = B\sin(k_0 y) \tag{11}$$

2. Cushman-Roisin's (2011) upwelling problem

$$\frac{\partial^3 \psi}{\partial t^3} - A\frac{\partial^3 \psi}{\partial t \partial x^2} + B\frac{\partial \psi}{\partial t} = 0, \quad \left.\frac{\partial \psi}{\partial x}\right|_{x=0} = C\cos(w_0 t) \tag{12}$$

3. Csanady's (1978) steady coastal trapped wave problem

$$-A\frac{\partial \psi}{\partial y} = \frac{\partial^2 \psi}{\partial x^2}, \quad \left.\frac{\partial \psi}{\partial x}\right|_{x=0} = B\cos(k_0 y) \tag{13}$$

Superposition principle: Key characteristics of linear differential equations

Problem with additional sinusoidal forcing term

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u}}_{\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + (if + \frac{\gamma}{h})} = \overbrace{\frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}}^{\equiv f_0} + \overbrace{\frac{\hat{\tau}_1}{\rho_0 h} e^{-iw_1 t}}^{\equiv f_1} \tag{14}$$

Superposition principle of linear non-homogeneous differential equation

1. Once $L[u_0] = f_0$ and $L[u_1] = f_1$,
   $L[c_1 u_1 + c_2 u_2] = c_1 f_0 + c_2 f_1$.
2. Therefore, once $L[u_0] = f_0$, $L[u_0 + C u_1] = f_0$ where
   $L[u_1] = 0$ (case of $f_1 = 0$).

Generalization to problem with arbitrary forcing term

Arbitrary function of t
(any time series of wind stress)

$$\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h} \right) \vec{u} = \frac{\vec{\tau}^s(t)}{\rho_0 h} \equiv \frac{1}{\rho_0 h} \sum_{n=-\infty}^{\infty} \hat{\tau}_n e^{-iw_n t} \tag{15}$$

Fourier series

Note that

$$\hat{\tau}(w) = \mathcal{F}(\vec{\tau}(t)) \tag{16}$$

where $\mathcal{F}$ indicates Fourier operator. Particular solution to the problem is

Transfer function ($\equiv \hat{g}(w_n)$)

$$\vec{u} = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{\rho_0 h(i(f - w_n) + \gamma/h)} \hat{\tau}_n}_{\hat{u}(w_n)} e^{-iw_n t}. \tag{17}$$

Generalization to problem with arbitrary forcing term

Let us consider problem without initial condition ($\lim_{t \to -\infty} \vec{u} = 0$)

$$\vec{u} = \frac{1}{\rho_0 h} \sum_{n=-\infty}^{\infty} \frac{1}{(i(f - w_n) + \gamma/h)} \hat{\tau}_n e^{-i w_n t}. \tag{18}$$

Corresponding solution (Lentz and Winant, 1986; Austin and Barth, 2002) obtained by the integrating factor is given by

$$\vec{u} = \frac{1}{\rho_0 h} \int_{-\infty}^{t} g(\tilde{t} - t) \vec{\tau}(\tilde{t})\, d\tilde{t}. \tag{19}$$

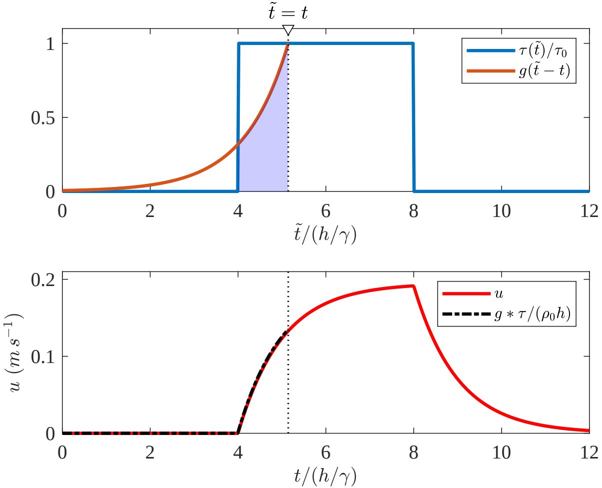
$\equiv g * \vec{\tau}$ (Convolution!)

where $g(t^\star) = e^{(if + \gamma/h)t^\star}$.

If solution is unique, they must be same. This is end up with "**convolution theorem**":

$$\hat{g}(w)\hat{\tau}(w) = \mathcal{F}(g(t) * \vec{\tau}(t)) \tag{20}$$

Generalization to problem with arbitrary forcing term



The slap model

Pollard and Millard (1970):

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}}{\rho_0 h_1} - \gamma^\star \vec{u} \tag{21}$$

where $\gamma^\star$ represents linear damping coefficient, that can be determined by calibration.

This is a theoretical model that can still be used for publication (Whitt and Thomas, 2015; Gough et al., 2016; Wang et al., 2019; Zhang et al., 2023)!

Summary

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}(t)}{\rho_0 h_1} - \gamma^\star \vec{u}$$

where $\vec{\tau}(t)$ is an arbitrary function of t. Solution to the problem is given by

$$\begin{aligned}\vec{u} &= \frac{1}{\rho_0 h} \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{(i(f-w_n)+\gamma/h)} \hat{\tau}_n e^{-iw_n t}} \\ &= \frac{1}{\rho_0 h} \boxed{\int_{-\infty}^{t} g(\tilde{t}-t)\vec{\tau}(\tilde{t})\, d\tilde{t}}\,.\end{aligned} \tag{22}$$

where $g(t^\star) = e^{(if+\gamma/h)t^\star}$. This implies convolution theorem. The contribution of past wind stress to the present current decays exponentially over time by friction term.

Assignment

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} - \frac{\gamma}{h}\vec{u} \tag{23a}$$

$$\vec{u}|_{t=0} = \vec{U}_0 \tag{23b}$$

$$\frac{d\vec{X}}{dt} = \vec{u} \tag{24}$$

where $-g\partial\eta/\partial\vec{n}$ is arbitrary constants and $\vec{X} = X + iY$. X and Y represent x- and y-position of an object, respectively.

1. Solve differential equation (23) for $\vec{u}$. What is physical meaning of particular solution component?
2. Solve (24) using $\vec{u}$ from (23) and constant f assumption.

References I

Austin, Jay A and John A Barth (2002). "Variation in the position of the upwelling front on the Oregon shelf". In: *Journal of Geophysical Research: Oceans* 107.C11, pp. 1–15.

Csanady, Gabriel Tibor (1981). "Circulation in the coastal ocean". In: *Advances in geophysics*. Vol. 23. Elsevier, pp. 101–183.

Gough, Matt K et al. (2016). "Resonant near-surface inertial oscillations in the northeastern Gulf of Mexico". In: *Journal of Geophysical Research: Oceans* 121.4, pp. 2163–2182.

Lentz, SJ and CD Winant (1986). "Subinertial currents on the southern Califoinia shelf". In: *Journal of Physical Oceanography* 16.11, pp. 1737–1750.

Pollard, Raymond T and RC Millard (1970). "Comparison between observed and simulated wind-generated inertial oscillations". In: *Deep Sea Research and Oceanographic Abstracts*. Vol. 17. 4. Elsevier, pp. 813–821.

References II

Wang, Pengcheng et al. (2019). “Modulation of near-inertial oscillations by low-frequency current variations on the inner scotian shelf”. In: *Journal of Physical Oceanography* 49.2, pp. 329–352.

Whitt, Daniel B and Leif N Thomas (2015). “Resonant generation and energetics of wind-forced near-inertial motions in a geostrophic flow”. In: *Journal of Physical Oceanography* 45.1, pp. 181–208.

Zhang, Yimin et al. (2023). “Spatial and seasonal variations of near-inertial kinetic energy in the upper Ross Sea and the controlling factors”. In: *Frontiers in Marine Science* 10, p. 1173900.