Differential Equations in Geophysical Fluid Dynamics

XII. Wind-driven circulation: Introduction and Sverdrup balance

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Introduction

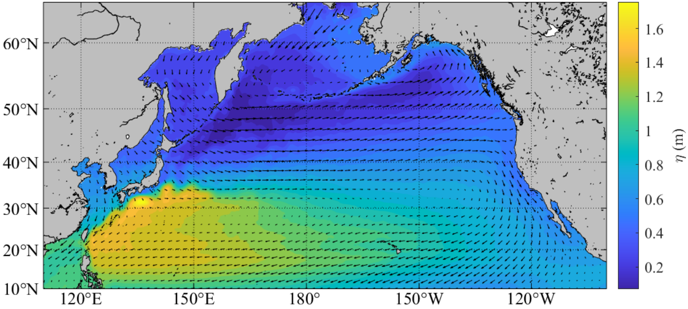


Figure: Climatologial mean wind stress and sea surface height fields.

Governing equation

Let us consider steady, linear, and lateral inviscid shallow water equations (Stommel, 1948):

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + \nabla \cdot (A_h \nabla \bar{u}) + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h}\bar{u} \quad (1a)$$

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f\bar{u} = -g\frac{\partial \eta}{\partial y} + \nabla \cdot (A_h \nabla \bar{v}) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h}\bar{v} \quad (1b)$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0. \quad (1c)$$

We will use the vorticity equation. Taking curl of momentum equations yields

$$\underbrace{\frac{\partial f}{\partial y}\bar{v}}_{\text{Planetary } \beta\text{-term}} = \underbrace{\frac{1}{\rho_0 h}\left(\frac{\partial \tau_y^s}{\partial x} - \frac{\partial \tau_x^s}{\partial y}\right)}_{\nabla \times \vec{\tau}^s\text{: Wind stress curl}} \overbrace{- \frac{\gamma}{h}\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right)}^{-(\gamma/h)\nabla \times \vec{u}\text{: Bottom stress curl}} \quad (2)$$

Governing equation

Using stream functions, $\bar{v} = \partial\psi/\partial x$ and $\bar{u} = -\partial\psi/\partial y$, (2) can be written as

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s - \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \qquad (3)$$

where $\beta = \partial f/\partial y$.

Note that $f = 2\Omega \sin\theta$ where θ is latitude. There are two approximation: "f-plane ($f \approx f_0$)" and "β-plane ($f \approx f_0 + \beta_0 y$)" where f_0 and β_0 are constants.

This is based on the **Taylor expansion** (Pedlosky, 1987; Verkley, 1990).

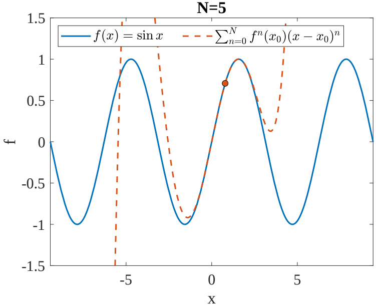
f- and β-planes

Taylor expansion

Arbitrary function $f(x)$ near $x = a$ can be expressed as infinite sum of polynomials:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{4}$$

where $f^{(n)} = \partial^n f/\partial x^n$, f differentiated n times.



https://jang-geun.github.io/vis_taylor_expansion.gif

f- and β-planes

Taylor expansion of the Coriolis frequency f near θ_0 yields

$$\begin{aligned} f &= 2\Omega \sin\theta \\ &\approx 2\Omega \left(\sin\theta_0 + (\cos\theta_0)(\theta-\theta_0) - (\sin\theta_0)(\theta-\theta_0)^2 \cdots\right) \\ &= 2\Omega \left(\sin\theta_0 + (\cos\theta_0)\frac{y}{R} - (\sin\theta_0)\left(\frac{y}{R}\right)^2 \cdots\right) \end{aligned} \tag{5}$$

If $\sin\theta_0 \approx \cos\theta_0$ (mid-latitude) and $y/R < 1$ (so smaller length scale than R),

$$f \approx \underbrace{\boxed{2\Omega\sin\theta_0}}_{f_0} + \overbrace{\boxed{\frac{2\Omega\cos\theta_0}{R}}}^{\beta_0} y \tag{6}$$

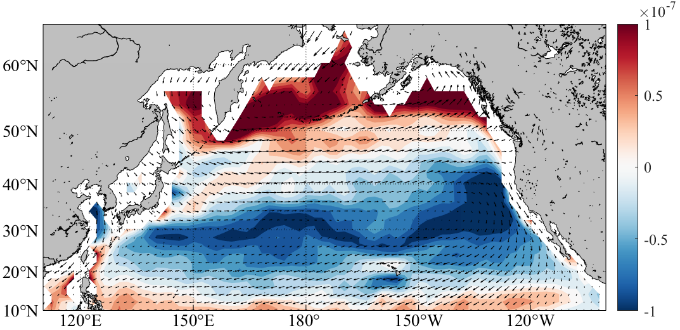
Note that the second β-term ($\beta_0 y$) still depends on y/R, so it can be negligible for “much smaller” length scale.

Wind-driven circulation over f-plane

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s - \frac{\gamma}{h} \Bigg(\overbrace{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}}^{\nabla \times \vec{u}} \Bigg) \qquad (7)$$

$$\boxed{\therefore \quad \nabla \times \vec{u} = \frac{1}{\rho_0 \gamma} \nabla \times \vec{\tau}^s}$$

The curl of ocean currents ($\nabla \times \vec{u}$) is proportional to the curl of wind stress ($\nabla \times \vec{\tau}^s$) in the same direction.



Wind-driven circulation over β-plane

$$\overbrace{\beta_0 \frac{\partial \psi}{\partial x}}^{(\beta_0 U)\frac{\partial \psi^*}{\partial x^*}} + \overbrace{\frac{\gamma}{h}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)}^{\left(\frac{\gamma U}{hL}\right)\frac{\partial^2 \psi^*}{\partial {x^*}^2}} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s \tag{8}$$

For large scale ocean ($\gamma U/(hL) \ll \beta_0 U$ so $\gamma/(h\beta_0) \ll L$), the bottom stress curl becomes negligible.

$$\beta_0 \underbrace{\frac{\partial \psi}{\partial x}}_{\bar{v}} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s \tag{9}$$

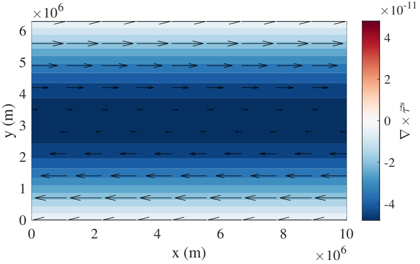
that is referred to as Sverdrup balance equation (Sverdrup, 1947).

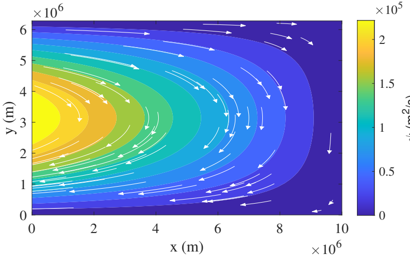
Sverdrup balance

Solution to (9) can be easily obtained by integrating (9) with respect to x over the domain:

$$\psi = \frac{1}{\rho_0 h \beta_0} \int_0^L \nabla \times \vec{\tau}^s \, dx \tag{10}$$

Below are the solution with idealized wind stress $(\tau_x^s, \tau_y^s) = (-\tau_0 \cos(\pi y / L_y), 0)$, and a boundary condition $\psi|_{x=L_x} = 0$.





Summary

Governing equation to the Stommel's wind-driven circulation is

Sverdrup balance

$$\boxed{\beta \frac{\partial \psi}{\partial x}} = \boxed{\frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s} \boxed{- \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)} \tag{11}$$

f-plane circulation

1. Based on the Taylor expansion, $f = 2\Omega \sin \theta$ can be appoimated to $f \approx f_0 + \beta y$ (β-plane approximation).
2. For relatively small scale, β-term becomes negligible so $f \approx f_0$.
3. On the f-plane, wind stress curl is balanced by bottom stress curl.
4. On the β-plane at large length scales, it is dominantly balanced by the planetary β-term.

Assignment

Shows that the potential vorticity can be approximated to

$$q = \frac{\xi + f}{h + \eta} = \frac{\xi + f}{h} \frac{1}{1 + \eta/h} \approx \frac{1}{h}\left(\xi + f - \frac{f}{h}\eta\right) \qquad (12)$$

when $Ro = \xi/f \ll 1$ and $\eta/h \approx Ro \ll 1$. Ignore terms of order Ro^s and below. Use Taylor series $1/(1+x) = 1 - x + x^2 - x^3 \cdots$.

References I

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