Differential Equations in Geophysical Fluid Dynamics

II. Simplification and shallow water equations

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

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Recap

The primitive equations are given by

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial (uu)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}-fv \\
&\qquad -\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right)
\end{aligned}
\tag{1a}
$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+\frac{\partial (vu)}{\partial x}+\frac{\partial (vv)}{\partial y}+\frac{\partial (vw)}{\partial z}+fu= \\
&\qquad -\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(A_h\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right)
\end{aligned}
\tag{1b}
$$

$$
\begin{aligned}
&\frac{\partial w}{\partial t}+\frac{\partial (wu)}{\partial x}+\frac{\partial (wv)}{\partial y}+\frac{\partial (ww)}{\partial z}= \\
&\qquad -\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\left(A_h\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial w}{\partial z}\right)-g
\end{aligned}
\tag{1c}
$$

$$
\frac{\partial \rho}{\partial t}+\frac{\partial (\rho u)}{\partial x}+\frac{\partial (\rho v)}{\partial y}+\frac{\partial (\rho w)}{\partial z}=0 \tag{1d}
$$

Who is easy (linear) and who is difficult (nonlinear)?

Assignment

In the continuity equation (1d), density ρ can be decomposed into constant component ρ_0, so $\rho = \rho_0 + \rho'$. Substituting $\rho = \rho_0 + \rho'$ into (1d) and then, conduct scale analysis and obtain nondimensionalized version of the equation.

1. What is the nondimensional number controlling the equation?
2. In general, $\rho_0 \approx 1025kg\,m^{-3}$ and the order of ρ' is less than $10\,kg\,m^{-3}$. Is the constant density assumption valid in (1d)?

Assignment

Apparently, $\rho' \ll \rho_0$ so constant density assumption could be valid:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

$$\downarrow$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

However, we can easily see that below equations used at the same time in many textbooks:

From constant density assumption

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \tag{2a}$$

$$\boxed{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0} \tag{2b}$$

Where did this one come from?

Boussinesq approximation $\rho' \ll \rho_0$

In continuity equation with $\rho = \rho_0 + \rho'$,

$$\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{\partial \rho'}{\partial t} + \frac{\partial (\rho' u)}{\partial x} + \frac{\partial (\rho' v)}{\partial y} + \frac{\partial (\rho' w)}{\partial z} = 0 \quad (3)$$

Boussinesq approximation $\rho' \ll \rho_0$

Similarly, we can approximate most ρ as ρ_0 in the momentum equations...

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = \\
- \frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right)
\end{aligned}
\tag{4a}
$$

$$
= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial (vw)}{\partial z} + fu = \\
- \frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(A_h \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right)
\end{aligned}
\tag{4b}
$$

$$
= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + v\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0}
$$

Boussinesq approximation $\rho' \ll \rho_0$

But **not all of them**!

$$
\frac{\partial w}{\partial t} + \boxed{\frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial (ww)}{\partial z}} = - \frac{1}{\boxed{\rho}} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left(A_h \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial w}{\partial z}\right) - g \tag{5}
$$

?

$$
= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + w\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0}
$$

Boussinesq approximation $\rho' \ll \rho_0$

So, vertical momentum equation becomes

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = - \frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left(A_h\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial w}{\partial z}\right) - \overbrace{g\frac{\rho}{\rho_0}}^{\text{Reduced gravity}} \quad (6)$$

What if $H/L \ll 1$?

Hydrostatic approximation $H \ll L$

So, vertical momentum equation becomes

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} - g = 0 \tag{7}$$

that is referred to as hydrostatic equation.

Shallow water equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv = -g\frac{\partial \eta}{\partial x}+\frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right) \tag{8a}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu = -g\frac{\partial \eta}{\partial y}+\frac{\partial}{\partial x}\left(A_h\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right) \tag{8b}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{8c}$$

If $H \ll L$, can we just consider just two-dimensional problem? This question yields "**shallow water equations**", that is vertically averaged (7) from surface $z=\eta$ to bottom $z=-h$ with boundary conditions:

Shallow water equations

Kinematic boundary conditions

$$w|_{z=\eta} = \frac{\partial \eta}{\partial t} + u|_{z=\eta}\frac{\partial \eta}{\partial x} + v|_{z=\eta}\frac{\partial \eta}{\partial y} \tag{9a}$$

$$w|_{z=-h} = \frac{\partial h}{\partial t} + u|_{z=-h}\frac{\partial h}{\partial x} + v|_{z=-h}\frac{\partial h}{\partial y} \tag{9b}$$

Dynamic boundary condition

$$-A_h \left.\frac{\partial u}{\partial x}\right|_{z=\eta}\frac{\partial \eta}{\partial x} - A_h \left.\frac{\partial u}{\partial y}\right|_{z=\eta}\frac{\partial \eta}{\partial y} + A_h \left.\frac{\partial u}{\partial z}\right|_{z=\eta} = \frac{\tau_x^s}{\rho_0} \tag{10a}$$

$$A_h \left.\frac{\partial u}{\partial x}\right|_{z=-h}\frac{\partial h}{\partial x} + A_h \left.\frac{\partial u}{\partial y}\right|_{z=-h}\frac{\partial h}{\partial y} + A_h \left.\frac{\partial u}{\partial z}\right|_{z=-h} = \frac{\tau_x^b}{\rho_0} \tag{10b}$$

$$-A_h \left.\frac{\partial v}{\partial x}\right|_{z=\eta}\frac{\partial \eta}{\partial x} - A_h \left.\frac{\partial v}{\partial y}\right|_{z=\eta}\frac{\partial \eta}{\partial y} + A_h \left.\frac{\partial v}{\partial z}\right|_{z=\eta} = \frac{\tau_y^s}{\rho_0} \tag{11a}$$

$$A_h \left.\frac{\partial v}{\partial x}\right|_{z=-h}\frac{\partial h}{\partial x} + A_h \left.\frac{\partial v}{\partial y}\right|_{z=-h}\frac{\partial h}{\partial y} + A_h \left.\frac{\partial v}{\partial z}\right|_{z=-h} = \frac{\tau_y^b}{\rho_0} \tag{11b}$$

Shallow water equations

The strategy is to convert equations for 3D variables ($u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$), to those for vertically averaged 2D variables: $\bar{u}(x, y)$, $\bar{v}(x, y)$, and $\eta(x, y)$ where

$$\bar{u} = \frac{1}{\eta + h} \int_h^\eta u \, dz \approx \frac{1}{h} \int_h^\eta u \, dz \approx \frac{1}{h} \int_h^0 u \, dz \tag{12a}$$

$$\bar{v} = \frac{1}{\eta + h} \int_h^\eta v \, dz \approx \frac{1}{h} \int_h^\eta v \, dz \approx \frac{1}{h} \int_h^0 v \, dz \tag{12b}$$

Note that $\eta(x, y)$ and $h(x, y)$, so do not forget to use Leibniz integral rule!

Shallow water equations

After a lot of calculations (maybe with some approximations), we get the form of shallow water equation frequently used as a start point of many oceanographic problems:

$$\frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right) \text{ with (10)}$$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = \\ - g\frac{\partial \eta}{\partial x} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y}\right)\right) + \boxed{\frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h}} \end{aligned} \tag{13a}$$

$$\frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right) \text{ with (11)}$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = \\ - g\frac{\partial \eta}{\partial y} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right)\right) + \boxed{\frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h}} \end{aligned} \tag{13b}$$

$$\boxed{\frac{\partial \eta}{\partial t}} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \tag{13c}$$

$$\frac{\partial w}{\partial z} \text{ with (9)}$$

Shallow water equations

Spoiler Alert!

Why it is still important, regardless of **big** limitation from assuming $\rho = \rho_0$ (e.g., no stratification)?

Governing equations for reduced gravity model:

$$
\begin{aligned}
&\frac{\partial \bar{u}^{(1)}}{\partial t} + \bar{u}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial x} + \bar{v}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial y} - f \bar{v}^{(1)} = \\
&\qquad - g' \frac{\partial h'}{\partial x} + \frac{A_h}{h_1} \left(\frac{\partial}{\partial x} \left(h_1 \frac{\partial \bar{u}^{(1)}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h_1 \frac{\partial \bar{u}^{(1)}}{\partial y} \right) \right) + \frac{\tau_x^s}{\rho_0 h_1}
\end{aligned} \tag{14a}
$$

$$
\begin{aligned}
&\frac{\partial \bar{v}^{(1)}}{\partial t} + \bar{u}^{(1)} \frac{\partial \bar{v}^{(1)}}{\partial x} + \bar{v}^{(1)} \frac{\partial \bar{v}^{(1)}}{\partial y} + f \bar{u}^{(1)} = \\
&\qquad - g' \frac{\partial h'}{\partial y} + \frac{A_h}{h_1} \left(\frac{\partial}{\partial x} \left(h_1 \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h_1 \frac{\partial \bar{v}}{\partial y} \right) \right) + \frac{\tau_y^s}{\rho_0 h_1}
\end{aligned} \tag{14b}
$$

$$
\frac{\partial h'}{\partial t} + \frac{\partial ((h' + h_1) \bar{u})}{\partial x} + \frac{\partial ((h' + h_1) \bar{v})}{\partial y} = 0 \tag{14c}
$$

Can you see differences? At some aspect, it is simpler.

Summary

1. Boussinesq approximation $\rho' \ll \rho_0$ makes $\rho \approx \rho_0$ except for ρ associated with g.
2. Hydrostatic approximation $H \ll L$ let us consider only two terms (pressure gradient and gravity) in vertical momentum equation.
3. Shallow water equations are vertically averaged (or integrated) version of the Navier-Stokes and continuity equations.

Now, we are ready to talk about the ocean phenomenon using the shallow water equations!

Assignment

In many ocean's problems, Navier-Stokes and continuity equations can be simplified into linear system:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + A_h\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + A_z\frac{\partial^2 u}{\partial z^2} \tag{15a}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + A_h\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + A_z\frac{\partial^2 u}{\partial z^2} \tag{15b}$$

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial z} = -g \tag{15c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{15d}$$

and the boundary conditions are given by

$$w\big|_{z=0} = \frac{\partial \eta}{\partial t} \tag{16a}$$

$$w\big|_{z=-h} = 0 \tag{16b}$$

$$A_z \left. \frac{\partial u}{\partial z} \right|_{z=0} = \frac{\tau_x^s}{\rho_0 h}, \qquad A_z \left. \frac{\partial v}{\partial z} \right|_{z=0} = \frac{\tau_y^s}{\rho_0 h} \tag{16c}$$

$$A_z \left. \frac{\partial u}{\partial z} \right|_{z=-h} = \frac{\tau_x^b}{\rho_0 h} \approx \frac{\gamma}{h} \bar{u}, \qquad A_z \left. \frac{\partial v}{\partial z} \right|_{z=-h} = \frac{\tau_y^b}{\rho_0 h} \approx \frac{\gamma}{h} \bar{v} \tag{16d}$$

1. What kind of assumptions were mane in (15) and (16)?
2. By vertical averaging ($\int_{-h}^{0} dz$) (15) with the boundary conditions (16), obtain governing equations for $\bar{u}$, $\bar{v}$, and η where

$$\bar{u} = \frac{1}{h} \int_{-h}^{0} u \, dz, \quad \bar{v} = \frac{1}{h} \int_{-h}^{0} v \, dz. \tag{17}$$

 that will be the linear shallow water equations.