

Differential Equations in Geophysical Fluid Dynamics

III. Forced inertial oscillation and resonance

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Recap

Inertial oscillation problem:

First order linear homogeneous ordinary differential equation

Governing equation:

$$\frac{d\vec{u}}{dt} + if\vec{u} = 0 \quad (1a)$$

$$\vec{u}|_{t=0} = \vec{U}_0 \quad (1b)$$

Solution:

$$\vec{u} = \vec{U}_0 e^{-ift} \quad (2)$$

Oscillation with frequency f
(period $2\pi/f$)

that yields circle trajectory of which radius is $|\vec{U}_0|/f$.

Recap

Linear homogeneous differential equation

$$\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + if$$

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = 0$$

No forcing term (homogeneous)
: $F(t) = 0$

Superposition principle of linear homogeneous differential equation

1. Once $L[u_1] = 0$, $L[Cu_1] = 0$.
2. Once $L[u_1] = 0$ and $L[u_2] = 0$, $L[u_1 + u_2] = 0$
3. Therefore, once $L[u_1] = 0, \dots, L[u_n] = 0$,
 $L[C_1u_1 + \dots + C_nu_n] = 0$

Assignment

Inertial oscillation problem with bottom friction:

First order linear homogeneous ordinary differential equation

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -\frac{\gamma}{h}\vec{u} \quad (3a)$$

$$\vec{u}|_{t=0} = \vec{U}_0 \quad (3b)$$

Solution to the problem is

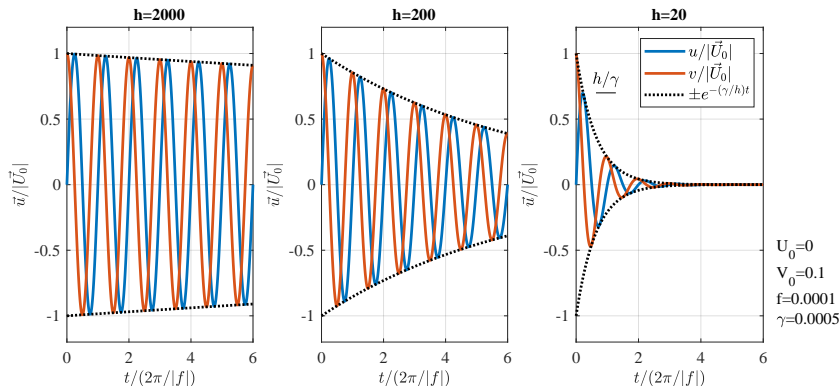
$$\vec{u} = \vec{U}_0 e^{-(\gamma/h)t} e^{-ift}. \quad (4)$$

Diagram illustrating the solution components:

- Exponential decay** (blue box): $e^{-(\gamma/h)t}$
- oscillation ($\sin(ft)$ and $\cos(ft)$)** (red box): e^{-ift}

Assignment

Inertial oscillation problem with bottom friction:



“ h/γ ”: frictional adjustment time (Csanady, [1981](#))

Linear momentum equation of shallow water equation

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + A_h \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h} \bar{u} \quad (5a)$$

≈ 0 (laterally **inviscid**)

$$\frac{\partial \bar{v}}{\partial t} - f\bar{u} = -g\frac{\partial \eta}{\partial y} + A_h \left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h} \bar{v} \quad (5b)$$

Writing (5) in complex coordinate ((5a)+i×(5b)) yields

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \underbrace{-g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h}}_{\text{Forcing terms!}} - \frac{\gamma}{h} \vec{u}$$

where $\vec{u} = u + iv$, $\partial \eta / \partial \vec{n} = (\partial \eta / \partial x) + i(\partial \eta / \partial y)$, and

$$\vec{\tau}^s = \tau_x^s + i\tau_y^s.$$

Inertial oscillation with wind force

When η is arbitrary given function, we can still stay on the ordinary differential equation problem!

For simplicity, let us consider no sea surface height gradient ($\partial\eta/\partial\vec{n} = 0$) and sinusoidal wind stress $\vec{\tau}^s = \hat{\tau}_0 e^{-i w_0 t}$:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + i f \vec{u} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-i w_0 t} - \frac{\gamma}{h} \vec{u}} \quad (6)$$

where $\vec{\tau}_0$ and w_0 are constants representing amplitude and frequency of the wind forcing, respectively.

Inertial oscillation with wind force

So, we have first order non-homogeneous ordinary differential equation problem:

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t} - \frac{\gamma}{h} \vec{u}. \quad (7a)$$

$$\vec{u}|_{t=0} = \vec{U}_0 \quad (7b)$$

$$\vec{u} = \frac{\overbrace{\hat{\tau}_0 e^{-iw_0 t}}^{\vec{\tau}^s}}{\rho_0 h(i(f - w_0) + \gamma/h)} \equiv \vec{u}_p \quad (8)$$

This indicates...

Inertial oscillation with wind force

If the solution is too complicated to get some wisdom, try simplifying it by **taking limits** (or by **thinking specific cases**).

1. $(\gamma/h, w_0) \ll f$
2. $(\gamma/h, f) \ll w_0$
3. $(w_0 \text{ and } f) \ll \gamma/h$
4. $\gamma/h \ll w_0 \approx f$

Summary

Solution to the problem considering initial condition is given by

Component associated with initial condition

$$\vec{u} = \left(\vec{U}_0 - \frac{\hat{\tau}_0}{\rho_0(f - w_0) + \gamma/h} \right) e^{-(if + \gamma/h)t} + \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h (i(f - w_0) + \gamma/h)} \quad (9)$$

Component associated with forcing

What is time scale during which the influence of the initial condition exists?

Summary

1. Frictionless assumption cannot be global, valid for finite time ($t \ll h/\gamma$).
2. For $w_0 \ll f$ (low frequency forcing), Ekman transport, wind stress balanced by Coriolis force, becomes predominant.
3. For $f \ll w_0$ (high frequency forcing), currents are accelerated in the direction of wind stress balanced by inertia.
4. For $w_0 \approx f$, resonance appears and current response to wind stress is maximized.
5. In this forced problem, period of forcing represents the time scale of phenomenon.

Assignment

$$\frac{\partial \vec{u}}{\partial t} + i f \vec{u} = -g \frac{\partial \eta}{\partial \vec{n}} - \frac{\gamma}{h} \vec{u} \quad (10a)$$

$$\vec{u}|_{t=0} = \vec{U}_0 \quad (10b)$$

$$\frac{d\vec{X}}{dt} = \vec{u} \quad (11)$$

where $-g\partial\eta/\partial\vec{n}$ is arbitrary constants and $\vec{X} = X + iY$. X and Y represent x - and y -position of an object, respectively.

1. Solve differential equation (10) for \vec{u} . What is physical meaning of particular solution component?
2. Solve (11) using \vec{u} from (10) and constant f assumption.

References I



Csanady, G. T. (1981). "Circulation in the coastal ocean". In: *Advances in geophysics*. Vol. 23. Elsevier, pp. 101–183.