Differential Equations in Geophysical Fluid Dynamics

III. Forced inertial oscillation and resonance

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Recap

Inertial oscillation problem:
First order linear homogeneous ordinary differential equation

Governing equation:

$$\frac{d\vec{u}}{\partial t} + if\vec{u} = 0 \tag{1a}$$

$$\vec{u}\big|_{t=0} = \vec{U}_0 \tag{1b}$$

Solution:

$$\vec{u} = \vec{U}_0 \, \boxed{e^{-ift}} \tag{2}$$

Oscillation with frequency f
(period $2\pi/f$)

that yields circle trajectory of which radius is $|\vec{U}_0|/f$.

Recap

Linear homogeneous differential equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + if\vec{u}}_{\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + if} = \boxed{0}$$

No forcing term (homogeneous) : $F(t) = 0$

Superposition principle of linear homogeneous differential equation

1. Once $L[u_1] = 0$, $L[Cu_1] = 0$.
2. Once $L[u_1] = 0$ and $L[u_2] = 0$, $L[u_1 + u_2] = 0$
3. Therefore, once $L[u_1] = 0, \ldots, L[u_n] = 0$, $L[C_1 u_2 + \cdots + C_n u_n] = 0$

Assignment

Inertial oscillation problem with bottom friction:
First order linear homogeneous ordinary differential equation

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -\frac{\gamma}{h}\vec{u} \tag{3a}$$

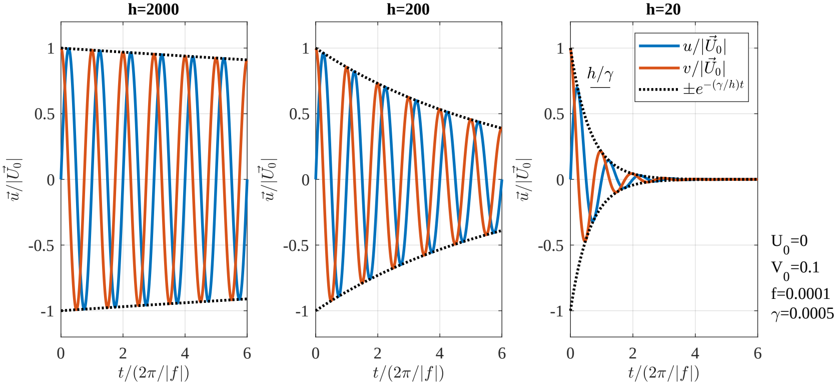
$$\vec{u}|_{t=0} = \vec{U}_0 \tag{3b}$$

Solution to the problem is

$$\vec{u} = \vec{U}_0 \underbrace{e^{-(\gamma/h)t}}_{\text{Exponential decay}} \overbrace{e^{-ift}}^{\text{oscillation } (\sin(ft) \text{ and } \cos(ft))}. \tag{4}$$

Assignment

Inertial oscillation problem with bottom friction:



"h/γ": frictional adjustment time (Csanady, 1981)

Linear momentum equation of shallow water equation

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + \underbrace{A_h\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right)}_{\approx 0 \text{ (laterally \textbf{inviscid})}} + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h}\bar{u} \tag{5a}$$

$$\frac{\partial \bar{v}}{\partial t} - f\bar{u} = -g\frac{\partial \eta}{\partial y} + A_h\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h}\bar{v} \tag{5b}$$

Writing (5) in complex coordinate ((5a)$+i\times$(5b)) yields

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \underbrace{\boxed{-g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h}}}_{\text{Forcing terms!}} - \frac{\gamma}{h}\vec{u}$$

where $\vec{u} = u + iv$, $\partial\eta/\partial\vec{n} = (\partial\eta/\partial x) + i(\partial\eta/\partial y)$, and $\vec{\tau}^s = \tau_x^s + i\tau_y^s$.

Inertial oscillation with wind force

When η is arbitrary given function, we can still stay on the ordinary differential equation problem!

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\gamma}{h}\vec{u} \tag{6}$$

For simplicity, let us consider no sea surface height gradient ($\partial\eta/\partial\vec{n} = 0$) and sinusoidal wind stress $\vec{\tau}^s = \hat{\tau}_0 e^{-iw_0 t}$:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h}e^{-iw_0 t} - \frac{\gamma}{h}\vec{u}} \tag{7}$$

where $\vec{\tau}_0$ and w_0 are constants representing amplitude and frequency of the wind forcing, respectively.

Inertial oscillation with wind force

So, we have first order non-homogeneous ordinary differential equation problem:

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t} - \frac{\gamma}{h}\vec{u}. \tag{8a}$$

$$\vec{u}\big|_{t=0} = \vec{U}_0 \tag{8b}$$

$$\vec{u} = \frac{\overbrace{\hat{\tau}_0 e^{-iw_0 t}}^{\vec{\tau}^s}}{\rho_0 h(i(f - w_0) + \gamma/h)} \equiv \vec{u}_p \tag{9}$$

This indicates...

Inertial oscillation with wind force

If the solution is too complicated to get some wisdom,
try simplifying it by **taking limits** (or by **thinking specific cases**).

1. $(\gamma/h, w_0) \ll f$
2. $(\gamma/h, f) \ll w_0$
3. (w_0 and f) $\ll \gamma/h$
4. $\gamma/h \ll w_0 \approx f$

Summary

Solution to the problem considering initial condition is given by

Component associated with initial condition

$$\vec{u} = \boxed{\left(\vec{U}_0 - \frac{\hat{\tau}_0}{\rho_0(f - w_0) + \gamma/h}\right) e^{-(if+\gamma/h)t}} + \boxed{\frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h(i(f - w_0) + \gamma/h)}} \tag{10}$$

Component associated with forcing

What is time scale during which the influence of the initial condition exists?

Summary

1. Frictionless assumption cannot be global, valid for finite time ($t \ll h/\gamma$).
2. For $w_0 \ll f$ (low frequency forcing), Ekman transport, wind stress balanced by Coriolis force, becomes predominant.
3. For $f \ll w_0$ (high frequency forcing), currents are accelerated in the direction of wind stress balanced by inertia.
4. For $w_0 \approx f$, resonance appears and current response to wind stress is maximized.
5. In this forced problem, period of forcing represents the time scale of phenomenon.

Advanced topic

A generalization of forcing term: Fourier series and convolution

Inertial oscillation with wind force

Why have so many people used sinusoidal (or monochromatic) forcing?

1. Stommel's (1948) wind driven circulation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + A\frac{\partial\psi}{\partial x} = B\sin(k_0 y) \tag{11}$$

2. Cushman-Roisin's (2011) upwelling problem

$$\frac{\partial^3\psi}{\partial t^3} - A\frac{\partial^3\psi}{\partial t\partial x^2} + B\frac{\partial\psi}{\partial t} = 0, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = C\cos(w_0 t) \tag{12}$$

3. Csanady's (1978) steady coastal trapped wave problem

$$-A\frac{\partial\psi}{\partial y} = \frac{\partial^2\psi}{\partial x^2}, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = B\cos(k_0 y) \tag{13}$$

Superposition principle: Key characteristics of linear differential equations

Problem with additional sinusoidal forcing term

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u}}_{\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + (if + \frac{\gamma}{h})} = \overbrace{\frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}}^{\equiv f_0} + \overbrace{\frac{\hat{\tau}_1}{\rho_0 h} e^{-iw_1 t}}^{\equiv f_1} \qquad (14)$$

Superposition principle of linear non-homogeneous differential equation

1. Once $L[u_0] = f_0$ and $L[u_1] = f_1$, $L[c_1 u_1 + c_2 u_2] = c_1 f_0 + c_2 f_1$.
2. Therefore, once $L[u_0] = f_0$, $L[u_0 + C u_1] = f_0$ where $L[u_1] = 0$ (case of $f_1 = 0$).

Generalization to problem with arbitrary forcing term

Arbitrary function of t
(any time series of wind stress)

$$\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u} = \frac{\boxed{\vec{\tau}^s(t)}}{\rho_0 h} \equiv \frac{1}{\rho_0 h}\boxed{\sum_{n=-\infty}^{\infty} \hat{\tau}_n e^{-iw_n t}} \tag{15}$$

Fourier series

Note that

$$\hat{\tau}(w) = \mathcal{F}(\vec{\tau}(t)) \tag{16}$$

where $\mathcal{F}$ indicates Fourier operator. Particular solution to the problem is

Transfer function ($\equiv \hat{g}(w_n)$)

$$\vec{u} = \sum_{n=-\infty}^{\infty} \underbrace{\boxed{\frac{1}{\rho_0 h(i(f - w_n) + \gamma/h)}}\,\hat{\tau}_n}_{\hat{u}(w_n)}\, e^{-iw_n t}. \tag{17}$$

https://jang-geun.github.io/vis_fourier_transform.gif

Generalization to problem with arbitrary forcing term

Let us consider problem without initial condition ($\lim_{t\to-\infty} \vec{u} = 0$)

$$\vec{u} = \frac{1}{\rho_0 h} \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{(i(f - w_n) + \gamma/h)} \hat{\tau}_n e^{-iw_n t}}\,. \tag{18}$$

Corresponding solution (Lentz and Winant, 1986; Austin and Barth, 2002) obtained by the integrating factor is given by

$$\vec{u} = \frac{1}{\rho_0 h} \boxed{\int_{-\infty}^{t} g(\tilde{t} - t)\vec{\tau}(\tilde{t})\, d\tilde{t}}\,. \tag{19}$$

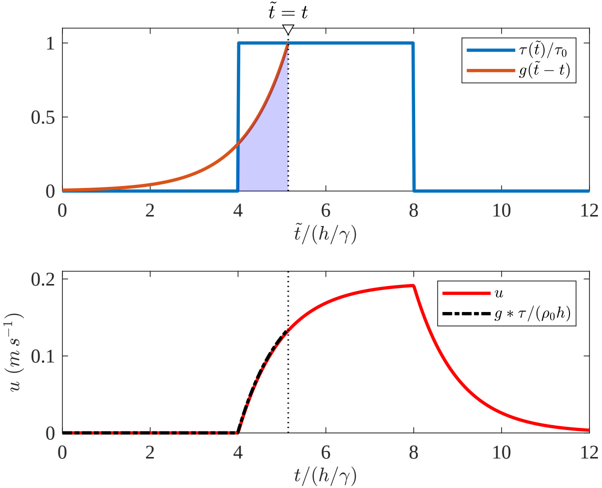
$\equiv g * \vec{\tau}$ (Convolution!)

where $g(t^\star) = e^{(if+\gamma/h)t^\star}$.

If solution is unique, they must be same. This is end up with "**convolution theorem**":

$$\hat{g}(w)\hat{\tau}(w) = \mathcal{F}(g(t) * \vec{\tau}(t)) \tag{20}$$

Generalization to problem with arbitrary forcing term



https://jang-geun.github.io/vis_convolution.gif

The slap model

Pollard and Millard (1970):

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}}{\rho_0 h_1} - \gamma^{\star}\vec{u} \tag{21}$$

where $\gamma^{\star}$ represents linear damping coefficient, that can be determined by calibration.

This is a theoretical model that can still be used for publication (Whitt and Thomas, 2015; Gough et al., 2016; Wang et al., 2019; Zhang et al., 2023)!

Summary

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}(t)}{\rho_0 h_1} - \gamma^\star \vec{u}$$

where $\vec{\tau}(t)$ is an arbitrary function of t. Solution to the problem is given by

$$\begin{aligned}\vec{u} &= \frac{1}{\rho_0 h} \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{(i(f-w_n)+\gamma/h)} \hat{\tau}_n e^{-iw_n t}} \\ &= \frac{1}{\rho_0 h} \boxed{\int_{-\infty}^{t} g(\tilde{t}-t)\vec{\tau}(\tilde{t})\, d\tilde{t}}\,.\end{aligned} \tag{22}$$

where $g(t^\star) = e^{(if+\gamma/h)t^\star}$. This implies convolution theorem. The contribution of past wind stress to the present current decays exponentially over time by friction term.

Assignment

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} - \frac{\gamma}{h}\vec{u} \tag{23a}$$

$$\vec{u}\big|_{t=0} = \vec{U}_0 \tag{23b}$$

$$\frac{d\vec{X}}{dt} = \vec{u} \tag{24}$$

where $-g\partial\eta/\partial\vec{n}$ is arbitrary constants and $\vec{X} = X + iY$. X and Y represent x- and y-position of an object, respectively.

1. Solve differential equation (23) for $\vec{u}$. What is physical meaning of particular solution component?
2. Solve (24) using $\vec{u}$ from (23) and constant f assumption.

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