Ecosystem modeling

I. The most basic equations for biology: exponential growth and decay

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

July, 2025

Introduction

Modeling

Any activity to produce model, a simplified representation of a real system.

Mathematics is the language of science! The equations are perfect tool to describe the nature.

We will talk about:

- How to formulate ecosystem (building equation)
- How to use the formulation (using/solving equation)

Governing equation

$$\underbrace{\frac{\partial P}{\partial t}}_{\text{Temporal change of subject } P} = \sum \underbrace{S_o}_{\text{Source}} - \sum \underbrace{S_i}_{\text{Sink}} \tag{1}$$

representing that the temporal change of a subject (e.g., phytoplankton) is sum of source (e.g., growth) minus sum of sink S_i (e.g., mortality).

Good a priori: to assume that the source and sink are proportional to the amount of (chemical or biological) substances:

$$\sum S_o = UP, \qquad \sum S_i = \sigma P \tag{2}$$

where U (growth rate) and σ (mortality rate) are the proportional coefficients.

Governing equation

If both coefficients are constant, (1) becomes

$$\boxed{\frac{dP}{dt} = U_{net}P} \qquad (3)$$

where $U_{net} = U - \sigma$ representing “net” growth rate.

Imagine phytoplankton that double in a certain time T_d (doubling time), that can be formulated by

$$P(t) = P_0 2^{t/T_d} = P_0 \left(e^{\ln 2}\right)^{t/T_d} = P_0 e^{U_{net}t} \qquad (4)$$

where P_0 is initial concentration and $U_{net} = (\ln 2)/T_d$[1]. This is the solution to (3).

[1] This is relationship between the net growth rate and doubling time.

Numerical solution using finite difference method

Solving equations based on repeated calculations using computers.

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} \approx \frac{P^{n+1} - P^n}{\Delta t} \tag{5}$$

Therefore, the governing equation can be approximated into following form:

$$\frac{dP}{dt} = U_{net} P \quad \to \quad \frac{P^{n+1} - P^n}{\Delta t} = U_{net} P^n$$

$$\therefore P^{n+1} = P^n + \Delta t U_{net} P^n \tag{6}$$

So, we can compute P in next step (P^{n+1}) using previously known information (P^n). And then, it can be repeated to get the next step P^{n+2} using the obtained P^{n+1}.

Lab 1

Numerically solve

$$\frac{dP}{dt} = U_{net}P \tag{7}$$

using the discretized equation $P^{n+1} = P^n + \Delta t U_{net} P^n$ with $U_{net} = 1\, d^{-1}$, $P_0 = P^1 = 2.0\, cell\, L^{-1}$, and $\Delta t = 2.0\, d^{-1}$.

1. Simulate time series of P from $t = 0$ to $t = 10\, d$.
2. Compare the numerical solution with the analytical solution (4).
3. Conduct sensitivity experiments for Δt. Try $U_{net} < 0$ with a long time step Δt. Model will behave incorrectly!

Lab 2

In most cases, the coefficients (U and σ) are not constant. We can model that the mortality rate is proportional to P, so $\sigma = \sigma' P$. Substituting it into (3) yields

$$\frac{dP}{dt} = U_{net} P - \sigma' P^2. \tag{8}$$

1. Numerically solve (8).
2. Compare it with the solution of exponential equation.