

# Differential Equations in Geophysical Fluid Dynamics

## VII. Characteristics of geostrophic current component

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# Recap

## Original linear NSE

$$if\vec{u} = -g\frac{\partial\eta}{\partial\vec{n}} + A_z\frac{\partial^2\vec{u}}{\partial z^2} \quad (1a)$$

$$A_z\frac{\partial\vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \quad (1b)$$

$$A_z\frac{\partial\vec{u}}{\partial z}\bigg|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma\vec{u} \quad (1c)$$

where  $\vec{u} = \textcolor{red}{u} + i\textcolor{red}{v}$

## Vertical averaged SWE

$$if\vec{u} = -g\frac{\partial\eta}{\partial\vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\gamma}{h}\vec{u} \quad (2)$$

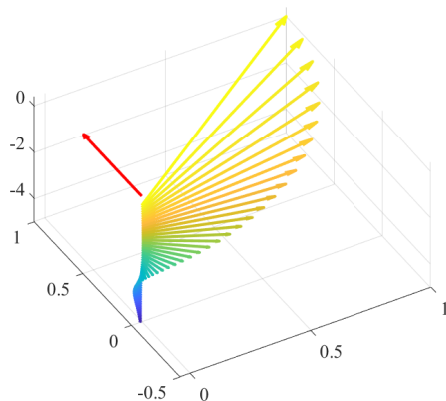
where  $\vec{u} = \textcolor{red}{\bar{u}} + i\textcolor{red}{\bar{v}}$

$$\bar{u} = \frac{1}{h} \int_{-h}^0 u \, dz$$

$$\bar{v} = \frac{1}{h} \int_{-h}^0 v \, dz$$

# Recap

## Surface Ekman current



NSE

$$if\vec{u} = A_z \frac{\partial^2 \vec{u}}{\partial z^2} \quad (3a)$$

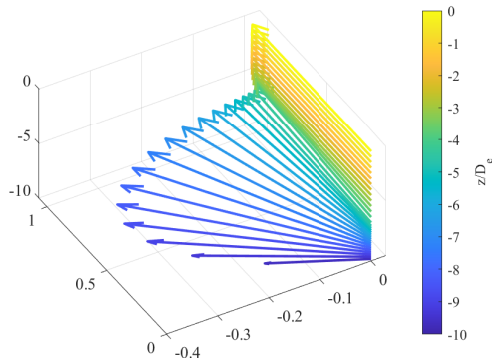
$$\frac{z}{D} A_z \frac{\partial \vec{u}}{\partial z} \Big|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \quad (3b)$$

SWE

$$h\vec{u} = \frac{1}{i} \frac{\vec{\tau}^s}{\rho_0 f} \quad (4)$$

# Recap

## Bottom Ekman current



NES

$$if\vec{u} = -g\frac{\partial\eta}{\partial\vec{n}} + A_z\frac{\partial^2\vec{u}}{\partial z^2} \quad (5a)$$

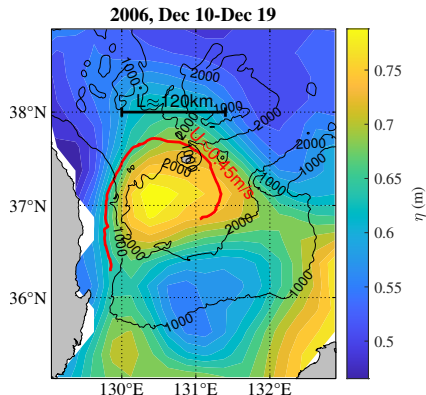
$$A_z\frac{\partial\vec{u}}{\partial z}\Big|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma\vec{u} \quad (5b)$$

SWE

$$h\vec{u} = -\frac{1}{i}\frac{\vec{\tau}^b}{\rho_0 f} \equiv -\frac{1}{i}\frac{\gamma}{f}\vec{u}_g \quad (6)$$

# Recap

Several physical assumptions are mathematically problematic...



General values for the parameters:

$$A_h \approx 10^2 \text{ m}^2 \text{ s}^{-1}, A_z \approx 1 \text{ m}^2 \text{ s}^{-1}, \text{ and } f \approx 10^{-4} \text{ s}^{-1}.$$

## A bit advanced topics for the Ekman current

### How to estimate the Ekman depth (or $A_z$ ) for surface layer?

Simple parameterization given by

$$D_e = \kappa \frac{u^*}{f} \quad (7)$$

where  $u^* = \sqrt{|\vec{\tau}^s|/\rho_0}$  and  $\kappa = 0.1 - 0.4$  (Csanady, [1981](#); Cushman-Roisin and Beckers, [2011](#)).

Inverse modeling approach using simple curve-fitting (Cole et al., [2017](#)).

### Considering time-dependency (inertia) and varying vertical eddy viscosity $A_z$ ?

Wenegrat and McPhaden, [2016](#); Elipot and Gille, [2009](#);  
Constantin, Paldor, and Dritschel, [2020](#); Lilly and Elipot, [2021](#)

## Governing equation

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u - f_0 v = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + \nabla \cdot (A_h \nabla u) + \frac{\partial}{\partial z} \left( A_z \frac{\partial u}{\partial z} \right) \quad (8a)$$

$$\frac{\partial v}{\partial t} + \vec{u} \cdot \nabla v + f_0 u = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + \nabla \cdot (A_h \nabla v) + \frac{\partial}{\partial z} \left( A_z \frac{\partial v}{\partial z} \right) \quad (8b)$$

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w = -\frac{1}{\rho_0} \frac{\partial P}{\partial z} + \nabla \cdot (A_h \nabla w) + \frac{\partial}{\partial z} \left( A_z \frac{\partial w}{\partial z} \right) - g \quad (8c)$$

Constant density

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (8d)$$

Taylor-Proudman theorem (Proudman, 1916; Taylor, 1917)

For flows governed by (7), velocities are depth-independent:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0. \quad (9)$$

# Taylor-Proudman theorem

$$v = \frac{g}{f_0} \frac{\partial \eta}{\partial x} \quad (10a)$$

$$u = -\frac{g}{f_0} \frac{\partial \eta}{\partial y} \quad (10b)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left( \int_{-h}^{\eta} u \, dz \right) + \frac{\partial}{\partial y} \left( \int_{-h}^{\eta} v \, dz \right) = 0 \quad (10c)$$

Substituting (10a) and (10b) into (10c) yields

$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = 0 \quad (11)$$

$\xleftarrow{\quad} \equiv \frac{dH}{dt}$

where  $H = \eta + h$  representing height of water column and note that  $\partial h / \partial t = 0$ . What does (11) mean?



# Eulerian and Lagrangian description

There are two ways to observe a object: Lagrangian and Eulerian.  
To be specific, mathematical descriptions for passive tracer transport are given by

## Eulerian description

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = 0 \quad (12)$$

## Lagrangian description

$$\frac{dC}{dt} = 0 \quad (13a)$$

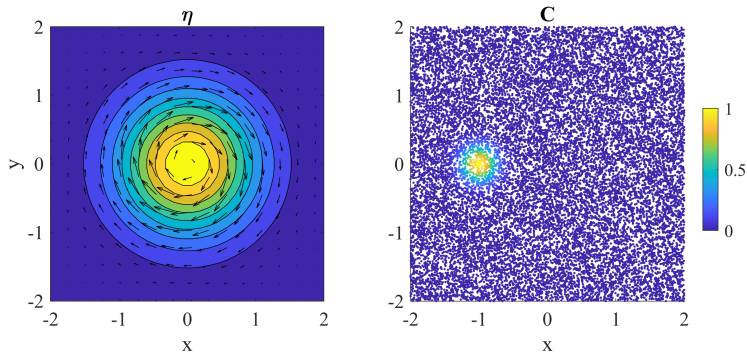
$$\frac{dX}{dt} = u \quad (13b)$$

$$\frac{dY}{dt} = v. \quad (13c)$$

The object is not different—just a different point of view!

# Eulerian and Lagrangian description

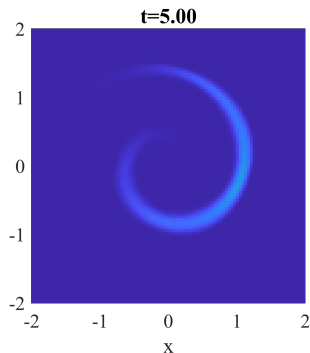
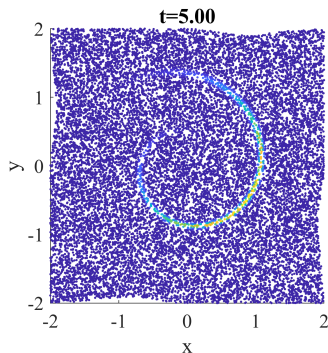
A simple numerical experiment



[https://jang-geun.github.io/vis\\_geo\\_advection1.gif](https://jang-geun.github.io/vis_geo_advection1.gif)

# Eulerian and Lagrangian description

A simple numerical experiment



[https://jang-geun.github.io/vis\\_geo\\_advection2.gif](https://jang-geun.github.io/vis_geo_advection2.gif)

# Eulerian and Lagrangian description

$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = 0 \quad (14)$$
$$(H = \eta + h)$$

that is

$$\frac{dH}{dt} = 0 \quad \therefore H = H|_{t=0} \quad (15)$$

and represents  $H$  is not changed in the Lagrangian aspect (trajectory).

For constant depth, (14) simplifies to

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} = 0, \quad \therefore \frac{\partial \eta}{\partial t} = 0$$
$$\left( -\frac{g}{f_0} \frac{\partial \eta}{\partial y} \right) \frac{\partial \eta}{\partial x} + \left( \frac{g}{f_0} \frac{\partial \eta}{\partial x} \right) \frac{\partial \eta}{\partial y} = 0$$

# Hamilton system

$$\frac{dX}{dt} = -\frac{\partial \mathcal{H}}{\partial Y} \quad (16a)$$

$$\frac{dY}{dt} = \frac{\partial \mathcal{H}}{\partial X} \quad (16b)$$

In case of our transport problem,  $\mathcal{H} = (g/f_0)\eta$ . This is the so-called “Hamiltonian system”. Once (16) and  $\partial \mathcal{H} / \partial t = 0$  (time-independent  $\mathcal{H}$ ),

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + u \frac{\partial \mathcal{H}}{\partial x} + v \frac{\partial \mathcal{H}}{\partial y} = 0 \quad (17)$$

so  $\mathcal{H}$  is not changed in the Lagrangian aspect.

## Taylor column

If  $\eta \ll h$ , (14) simplifies to

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} = 0. \quad (18)$$

This means that  $h$  is not changed in the Lagrangian aspect (flows follow isobath) and can be considered as the governing equation of the “Taylor column” phenomenon.



<https://www.youtube.com/watch?v=7GGfsW7gOLI>

# Summary






$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} = 0$$

is  $dH/dt = 0$  in the Lagrangian aspect and means that  $H$  experienced by a water mass (following trajectory) is not changed.

For flows governed by geostrophic balance over  $f$ -plane with homogeneous density,





1. Flows is depth-independent “barotropic” current (Taylor-Proudman theorem).
2. In the Lagrangian aspect, height of water column ( $\eta + h$ ) is not changed.
3. If  $\eta \ll h$ , flows follow isobath (Taylor column).

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