Differential Equations in Geophysical Fluid Dynamics

XVII. Wave in rotation: Poincare wave

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Oct, 2025

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography communinity COKOAA.

Recap

$$\frac{\partial \bar{u}}{\partial t} = -g\frac{\partial \eta}{\partial x} \qquad (1a)$$

$$\frac{\partial \bar{v}}{\partial t} = -g\frac{\partial \eta}{\partial y} \qquad (1b)$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0 \qquad (1c)$$

$$\downarrow$$

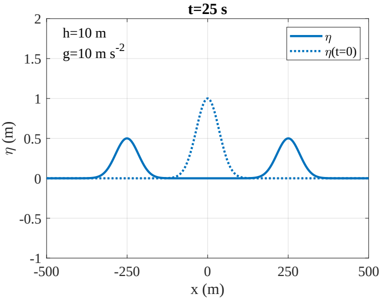
$$\frac{\partial^2 \eta}{\partial t^2} = gh\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) \qquad (2)$$

"Wave equation"

$$\eta = \eta_0 e^{i(kx - wt)} \qquad (3)$$

Substituting (3) into (2) yields dispersion relation.

$$\frac{w}{k} = \pm\sqrt{gh} \qquad (4)$$



Two-dimensional wave equation

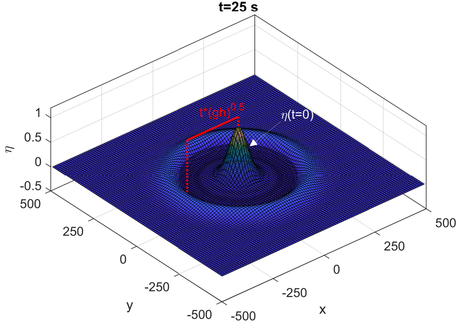
$$\frac{\partial^2 \eta}{\partial t^2} = gh\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) \quad (5)$$

Assuming $\eta = \eta_0 e^{i(kx+ly-wt)}$ and substituting it into (5) yields

$$w^2 = gh(k^2 + l^2) \quad \leftrightarrow \quad \frac{w}{K} = \pm\sqrt{gh} \quad (6)$$

where $K = \sqrt{k^2 + l^2}$.

https://jang-geun.github.io/vis_wave2d.gif



Surface gravity wave over f-plane

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} + A_h \nabla^2 \bar{u} + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h} \bar{u} \tag{7a}$$

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} + A_h \nabla^2 \bar{v} + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h} \bar{v} \tag{7b}$$

$$\frac{\partial \eta}{\partial t} + h \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0 \tag{7c}$$

Substituting $\partial(7a)/\partial x$ and $\partial(7b)/\partial y$ into $\partial(7c)/\partial t$ yields

$$\frac{\partial^2 \eta}{\partial t^2} - gh \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) + hf \, \boxed{\xi} = 0 \tag{8}$$

$= \dfrac{\partial \bar{v}}{\partial x} - \dfrac{\partial \bar{u}}{\partial y}$, relative vorticity!

Surface gravity wave over f-plane

Taking curl of momentum equations (∂(7b)$/\partial x$-∂(7a)$/\partial y$) and then substituting (7c) into the equation yields

$$\frac{\partial \xi}{\partial t} - \frac{f}{h}\frac{\partial \eta}{\partial t} = 0, \qquad \text{so} \qquad \frac{\partial}{\partial t}\left(\xi - \frac{f}{h}\eta\right) = 0\,. \tag{9}$$

Linear potential vorticity equation

Substituting (9) into ∂(8)$/\partial t$ yields

$$\boxed{\frac{\partial^3 \eta}{\partial t^3} - gh\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) + f^2\frac{\partial \eta}{\partial t} = 0} \tag{10}$$

Let us start from one-dimensional problem ($\partial/\partial y \approx 0$).

Dispersion relation

Assuming $\eta = \eta_0 e^{i(kx - wt)}$ and substituting into the governing equation (10) yields dispersion relation:

$$w(w^2 - ghk^2 - f^2) = 0 \tag{11}$$

For $w \neq 0$[1], $w^2 - ghk^2 - f^2 = 0$ becomes the dispersion relation of the surface gravity wave over f-plane (so-called Poincare wave).

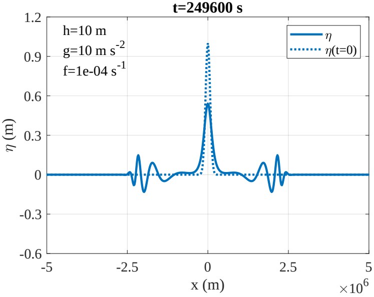
It can be solved for w:

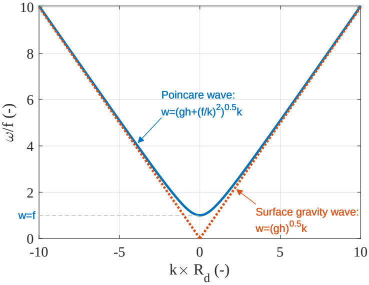
$$w = \pm\sqrt{ghk^2 + f^2}, \quad \underbrace{\left(c = \frac{w}{k} = \pm\sqrt{gh + (f/\boxed{k})^2}\right)}_{\text{wave velocity } c \text{ depends on } k!} \tag{12}$$

"**Dispersive wave**"

[1] Think about its physical meaning...

Dispersion relation





https://jang-geun.github.io/vis_poincare1d.gif

Something is wrong...

Wave group

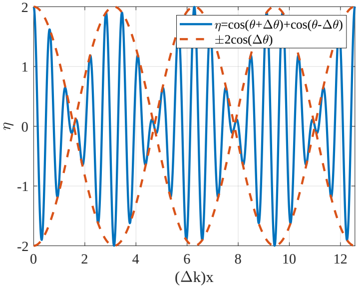
Let us two wave modes, which has tiny difference in wavenumber and frequency, given by

$$\eta = \eta_0 \left[\cos((k+\Delta k)x - (w+\Delta w)t) + \cos((k-\Delta k)x - (w-\Delta w)t)\right] \tag{13}$$

that can be rewritten as[2]

Shape of wave group!

$$\eta = 2\eta_0 \cos(kx - wt)\, \cos(\Delta kx - \Delta wt) \tag{14}$$



Speed of the wave group is $\Delta w / \Delta k$. For $\Delta k \to 0$, group velocity c_g is given by

$$\boxed{c_g = \frac{\partial w}{\partial k}} \tag{15}$$

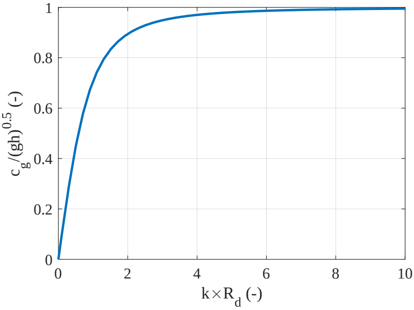
[2] Note that $\cos A + \cos B = 2\cos((A+B)/2)\cos((A-B)/2)$

Wave group

Based on (12), group velocity of the Poincare wave is given by

$$c_g = \frac{\partial w}{\partial k} = \pm \frac{ghk}{\sqrt{ghk^2 + f^2}} = \pm \sqrt{gh \left(\frac{k^2}{k^2 + (1/Rd)^2} \right)} \qquad (16)$$

where $Rd = \sqrt{gh}/f$ referred to as the "**Rossby radius of deformation**".



Low wavenumber group ($k^2 \ll 1/Rd^2$)?

$$c_g \approx \sqrt{gh}(kR_d)$$

High wavenumber group ($1/Rd^2 \ll k^2$)?

$$c_g \approx \sqrt{gh}$$

Energy transport of dispersive equation

Similarly with governing equation of the surface gravity wave, (7) can be rewritten as

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{17}$$

where

$$E = \frac{1}{2}h(\bar{u}^2 + \bar{v}^2) + \frac{1}{2}g\eta^2, \qquad F = gh\bar{u}\eta \tag{18}$$

Based on the wave-like basis and dispersion relation (12), below "averaged" quantities can be obtained:

$$\bar{E} = \frac{1}{2}\left(\frac{1}{2}h(\bar{u}_0^2 + \bar{v}_0^2) + \frac{1}{2}g\eta_0^2\right), \qquad \bar{F} = \underbrace{\boxed{c_g}}_{(12)}\,\bar{E} \tag{19}$$

so

$$\frac{\partial \bar{E}}{\partial t} + \underbrace{\boxed{c_g}}_{\text{Energy transported by waves (wave group), not a wave}}\frac{\partial \bar{E}}{\partial x} = 0 \quad \text{but...?} \tag{20}$$

Conclusion

1. Surface gravity wave + Coriolis = Poincare wave.
2. Dispersion relation of the Poincare wave is given by $w^2 - ghk^2 - f^2 = 0$ so w depends on k. This is what we call dispersive wave. For $k \to 0$, it becomes inertia oscillation. For $k \to \infty$, it becomes surface gravity wave.
3. Energy is transpored by not single wave mode, but by wave group whose velocity given by $\partial w / \partial k$.