Differential Equations in Geophysical Fluid Dynamics

VIII. Vorticity equation and high order variables

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

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Recap

$$\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u - f_0 v = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + \nabla \cdot (A_h \nabla u) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right) \quad (1a)$$

$$\frac{\partial v}{\partial t} + \vec{u} \cdot \nabla v + f_0 u = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + \nabla \cdot (A_h \nabla v) + \frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right) \quad (1b)$$

$$\frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w = -\frac{1}{\boxed{\rho_0}} \frac{\partial P}{\partial z} + + \nabla \cdot (A_h \nabla w) + \frac{\partial}{\partial z}\left(A_z \frac{\partial w}{\partial z}\right) - g \quad (1c)$$

Constant density

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1d)$$

Taylor-Proudman theorem (Proudman, 1916; Taylor, 1917)

For flows governed by (7), velocities are depth-independent:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0. \quad (2)$$

Recap

$$v = \frac{g}{f_0}\frac{\partial \eta}{\partial x}, \quad u = -\frac{g}{f_0}\frac{\partial \eta}{\partial y} \tag{3a}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\int_{-h}^{\eta} u\, dz\right) + \frac{\partial}{\partial y}\left(\int_{-h}^{\eta} v\, dz\right) = 0 \tag{3b}$$

Substituting (3a) into (9c) yields

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = 0 \qquad (H = \eta + h) \tag{4}$$

No change

that is referred to as "advection equation"[1]. In Lagrangian aspect, it can be rewritten as system of ordinary differential equations

$$\frac{dX}{dt} = u, \quad \frac{dY}{dt} = v, \quad \boxed{\frac{dH}{dt} = 0} \tag{5}$$

[1]A.k.a., the transport equation and first-order wave equation.

Vorticity conservation equation

Shallow water equation for inviscid fluid

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + A_h \nabla^2 \bar{u} + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \quad \text{(6a)}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = -g\frac{\partial \eta}{\partial y} + A_h \nabla^2 \bar{v} + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \quad \text{(6b)}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial(\bar{u}(\eta + h))}{\partial x} + \frac{\partial(\bar{v}(\eta + h))}{\partial y} = 0 \quad \text{(6c)}$$

Taking curl (a.k.a., cross-differentiation) of momentum equations, $\partial(6b)/\partial x - \partial(6a)/\partial y$, and then rearranging the equation with (6c) yields

Vorticity conservation equation

$$\frac{\partial q}{\partial t} + \bar{u}\frac{\partial q}{\partial x} + \bar{v}\frac{\partial q}{\partial y} = 0$$
$$\left(\text{where } q = \frac{\xi + f}{\eta + h} \text{ and } \xi = \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) \tag{7}$$

that is referred to as "(potential) vorticity conservation equation"[2], indicating that the quantity q (potential vorticity) is not changed in Lagrangian aspect.

In q, ξ indicates "relative vorticity" representing rotation tendency of the velocity fields ($\bar{u}$, $\bar{v}$).

[2]A detailed derivation is provided in Section 1.3.2 of Choi and Kim (2024).

Vector operators: gradient

Gradient

For a scalar field $f(x, y, \cdots)$, gradient of f, ∇f, is given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \cdots \right) \tag{8}$$

that is a vector field of which x-, y-, and z-directional components are $\partial f / \partial x$, $\partial f / \partial y$, and $\partial f / \partial z$, respectively.

Vector operators: divergence

Divergence

For a vector field $\vec{u} = (u, v, \cdots)$, divergence of $\vec{u}$, $\nabla \cdot \vec{u}$, is given by

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \cdots \tag{9}$$

that is a scalar field describing divergence intensity (positive), or convergence intensity (negative), of the velocity field.

Vector operators: curl

Curl

For a 2D vector field $\vec{u} = (u, v)$, curl of $\vec{u}$, $\nabla \times \vec{u}$, is defined by

$$\nabla \times \vec{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u & v \end{vmatrix} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi \tag{10}$$

For a 3D vector field $\vec{u} = (u, v, w)$, it is given by

$$\begin{aligned} \nabla \times \vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &= \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{\xi_{yz}} \vec{i} - \underbrace{\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)}_{\xi_{xz}} \vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\xi_{xy}} \vec{k} \end{aligned} \tag{11}$$

Vector operators in shallow water equations

Gradient of sea surface height η, "$\nabla\eta$"

$$\cdots - fv = -g \boxed{\frac{\partial \eta}{\partial x}} \cdots, \qquad \cdots + fu = -g \boxed{\frac{\partial \eta}{\partial y}} \cdots \tag{12a}$$

$$\frac{\partial \eta}{\partial t} = -\left(\boxed{\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}} \right) \tag{12b}$$

Divergence of transport $\vec{U}$, "$\nabla \cdot \vec{U}$"

where $U = \int_{-h}^{\eta} u \, dz$ and $V = \int_{-h}^{\eta} u \, dz$.

Relative vorticity with constant is given by

Laplacian of η,
"$\nabla \cdot \nabla\eta \equiv \nabla^2\eta \equiv \Delta\eta$"

$$\xi = \boxed{\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}} \approx \frac{\partial \bar{v}_g}{\partial x} - \frac{\partial \bar{u}_g}{\partial y} \approx \frac{g}{f_0} \left(\boxed{\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}} \right) \tag{13}$$

Curl of $\vec{u}$, "$\nabla \times \vec{u}$"

Vorticity transport/balance equation

Shallow water equation with several constant parameters

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = -g\frac{\partial \eta}{\partial x} + A_h \nabla^2 \bar{u} + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \quad \text{(14a)}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = -g\frac{\partial \eta}{\partial y} + A_h \nabla^2 \bar{v} + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \quad \text{(14b)}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial(\bar{u}(\eta + h))}{\partial x} + \frac{\partial(\bar{v}(\eta + h))}{\partial y} = 0 \quad \text{(14c)}$$

The vorticity equation corresponding to equations above is given by

Vorticity transport/balance equation

$$\frac{\partial(\xi+f)}{\partial t}+\overbrace{\vec{u}\cdot\nabla(\xi+f)}^{\bar{u}\frac{\partial(\xi+f)}{\partial x}+\bar{v}\frac{\partial(\xi+f)}{\partial y}}+(f+\xi)\underbrace{\nabla\cdot\vec{u}}_{\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}}=A_h\overbrace{\nabla^2\xi}^{\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}}+\frac{1}{\rho_0 h}\underbrace{\nabla\times\vec{\tau}^s}_{\frac{\partial\tau_s^y}{\partial x}-\frac{\partial\tau_s^x}{\partial y}}-\frac{\gamma}{h}\xi \tag{15}$$

If $\eta \ll h$, $\partial\eta/\partial t \approx 0$ (rigid lid approx.; those yield $\nabla\cdot\vec{u}\approx 0$), and $f\approx 0$, it reduces to (relative) "vorticity transport equation":

$$\frac{d\xi}{dt}+\overbrace{\vec{u}\cdot\nabla\xi}^{\text{Advection}}=\overbrace{A_h\nabla^2\xi}^{\text{Diffusion}}+\overbrace{\frac{1}{\rho_0 h}\nabla\times\vec{\tau}^s}^{\text{Forcing}}-\underbrace{\frac{\gamma}{h}\xi}_{\text{Reaction (decay)}} \tag{16}$$

that is forced advection-diffusion-reaction equation. Note that the advection is nonlinear!

Vorticity equation with stream function

Stream function is defined as

$$\bar{v} = \frac{\partial \psi}{\partial x}, \quad \bar{u} = -\frac{\partial \psi}{\partial y}. \tag{17}$$

Substituting (17) to (15) yields

$$\begin{aligned} &\frac{\partial(\nabla^2\psi + f)}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial(\nabla^2\psi + f)}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial(\nabla^2\psi + f)}{\partial y} \\ &= A_h \underbrace{\boxed{\nabla^4\psi}}_{\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2 \partial y^2} + \frac{\partial^4\psi}{\partial y^4}} + \frac{1}{\rho_0 h}\nabla \times \vec{\tau}^s - \frac{\gamma}{h}\nabla^2\psi \end{aligned} \tag{18}$$

"Biharmonic operator"

but why do we do this? Why do we use the stream function, that makes equation more complex?

Ertel's vorticity

Inviscid Navier-Stokes equations with Boussinesq approximation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} \tag{19a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} \tag{19b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} \tag{19c}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \tag{19d}$$

can be rewritten to Ertel's vorticity[3], Π, conservation equation:

$$\begin{gathered}\frac{\partial \Pi}{\partial t} + u\frac{\partial \Pi}{\partial x} + v\frac{\partial \Pi}{\partial x} + w\frac{\partial \Pi}{\partial z} = 0 \\ \left(\text{where } \Pi = -\frac{\partial \rho}{\partial x}\frac{\xi_{yz}}{\rho_0} - \frac{\partial \rho}{\partial y}\frac{\xi_{xz}}{\rho_0} - \frac{\partial \rho}{\partial z}\frac{\xi_{xy} + f}{\rho_0}\right)\end{gathered} \tag{20}$$

[3] A detailed derivation is in Section 5.3 of Choi and Kim (2024)

Summary

Inviscid shallow water equations can be rewritten as

$$\frac{\partial q}{\partial t} + \bar{u}\frac{\partial q}{\partial x} + \bar{v}\frac{\partial q}{\partial y} = 0.$$
$$\left(\text{where } q = \frac{\xi + f}{\eta + h} \text{ and } \xi = \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) \tag{21}$$

Inviscid Navier-stokes equations can be rewritten as

$$\frac{\partial \Pi}{\partial t} + u\frac{\partial \Pi}{\partial x} + v\frac{\partial \Pi}{\partial x} + w\frac{\partial \Pi}{\partial z} = 0$$
$$\left(\Pi = -\frac{\partial \rho}{\partial x}\frac{\xi_{yz}}{\rho_0} - \frac{\partial \rho}{\partial y}\frac{\xi_{xz}}{\rho_0} - \frac{\partial \rho}{\partial z}\frac{\xi_{xy} + f}{\rho_0}\right) \tag{22}$$

So, those quantities are conservative in Lagrangian aspect.

References I

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