Differential Equations in Geophysical Fluid Dynamics

X. Other heat equation problems

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Recap

How to solve the simplest version of heat equation problem?

$$\frac{\partial \eta}{\partial t} = \gamma \frac{\partial^2 \eta}{\partial x^2} \qquad (1) \qquad\qquad \eta|_{t=0} = \eta_0 \sin(k_0 x) \qquad (2)$$

Assume $\eta = X(x)T(t)$ and substituting it into (1) yields

$$\frac{1}{\gamma}\underbrace{\frac{T'}{T}}_{\text{Function of only } t} = \underbrace{\frac{X''}{X}}_{\text{Function of only } x} = \overbrace{\lambda}^{\text{Must be the same as a constant.}}. \qquad (3)$$

Therefore, we can obtain two ODEs:

$$T' = \gamma \lambda T, \qquad X'' = \lambda X \qquad (4)$$

Note that assuming $X = e^{kx}$ yields $k = \pm\sqrt{\lambda}$.

Recap

1. If $\lambda = 0$, $\eta = A_0 x + B_0$.
2. If $0 < \lambda$ ($\lambda \equiv \lambda^+$;positive), $\eta = e^{\gamma\lambda t}(A_1 e^{\sqrt{\lambda^+}x} + B_1 e^{-\sqrt{\lambda^+}x})$.
3. If $\lambda < 0$ ($\lambda \equiv -\lambda^-$;negative), $\eta = e^{-\gamma\lambda^- t}(A_2\cos(\sqrt{\lambda^- x}) + B_2\sin(\sqrt{\lambda^- x}))$

Therefore, general solution based on superposition principle is

$$\begin{aligned}\eta = A_0 x + B_0 + e^{\gamma\lambda t}(A_1 e^{\sqrt{\lambda^+}x} + B_1 e^{-\sqrt{\lambda^+}x}) \\ + e^{-\gamma\lambda^- t}(A_2\cos(\sqrt{\lambda^- x}) + B_2\sin(\sqrt{\lambda^- x}))\end{aligned} \tag{5}$$

Based on initial condition, $A_0 = B_0 = A_1 = B_1 = A_2 = 0$, $B_2 = \eta_0$, $\sqrt{\lambda^-} = \sqrt{-\lambda} = k_0$. So the particulate solution is given by

$$\boxed{\eta = \eta_0 e^{-\lambda {k_0}^2 t}\sin(k_0 x)} \tag{6}$$

Considering boundary conditions

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} \tag{7a}$$

$$f\bar{u} = -\frac{\gamma}{h}\bar{v} \tag{7b}$$

$$\frac{\partial \eta}{\partial t} + h\frac{\partial \bar{u}}{\partial x} = 0 \tag{7c}$$

$$\bar{u}\big|_{x=0} = 0 \tag{7d}$$

$$\bar{u}\big|_{x=L} = 0 \tag{7e}$$

Closed B.C.

$$\boxed{\frac{\partial \eta}{\partial t} = \gamma' \frac{\partial^2 \eta}{\partial x^2}} \tag{8}$$

Let us consider arbitrary initial condition given by

$$\boxed{\eta\big|_{t=0} = f(x)} \tag{9}$$

with boundary conditions:

$$\boxed{\left.\frac{\partial \eta}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial \eta}{\partial x}\right|_{x=L} = 0.} \tag{10}$$

Q. The easiest initial condition to solve this problem is $f(x) = \eta_0 \cos(k_0 x)$. Why $\cos$ not $\sin$?

Considering boundary conditions

$$\eta = \sum_{n=0}^{\infty} A_n e^{-\gamma(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right) \tag{11}$$

Substituting (11) into the initial condition (9) yields

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \tag{12}$$

Fourier's trick

$$\int_0^L \cos(\frac{m\pi x}{L}) \left[\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)\right] dx$$

$$\sum_{n=0}^{\infty} A_n \boxed{\int_0^L \cos(\frac{m\pi x}{L}) \cos\left(\frac{n\pi x}{L}\right)\, dx} = \int_0^L \cos(\frac{m\pi x}{L}) f(x) dx \tag{13}$$

Considering boundary conditions

Orthogonality of sinusoidal functions

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L/2 & n = m \\ 0 & n \neq m \end{cases} \tag{14a}$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L/2 & n = m \neq 0 \\ L & n = m = 0 \\ 0 & n \neq m \end{cases} \tag{14b}$$

$$\int_0^L \sin\left(\frac{n\pi x}{L_x}\right)\cos\left(\frac{m\pi x}{L_x}\right)\,dx = 0 \tag{14c}$$

$$\sum_{n=0}^{\infty} A_n \boxed{\int_0^L \cos(\frac{m\pi x}{L})\cos\left(\frac{n\pi x}{L}\right)\,dx} = A_m\frac{L}{2} \tag{15}$$

Only one mode of which $m = n(\neq 0)$ (nonzero) survive, so it can be solved for $A_m = A_n$!

Considering boundary conditions

Consequently, the particulate solution is given by

$$\eta = \sum_{n=0}^{\infty} A_n e^{-\gamma(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right) \tag{16}$$

where

$$A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_0^L f(x)dx & n = 0 \\ \dfrac{2}{L}\displaystyle\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)dx & n \neq 0. \end{cases} \tag{17}$$

Other heat equation problem

Heat equation analogy of coastal trapped wave (Csanady, 1978)

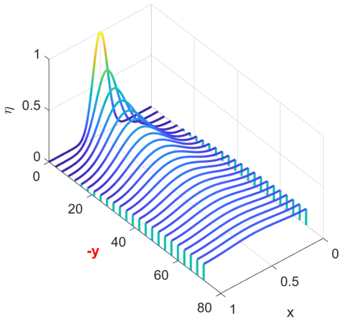
$$-f_0\bar{v} = -g\frac{\partial \eta}{\partial x} \qquad (18)$$

$$f_0\bar{u} = -g\frac{\partial \eta}{\partial y} - \frac{\gamma}{h}\bar{v} \qquad (19)$$

$$\frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0 \qquad (20)$$

$$h = \alpha x + h_0 \qquad (21)$$

that can be approximately written as



$$\boxed{\frac{\partial \psi}{\partial y} = -\gamma' \frac{\partial^2 \psi}{\partial x^2}} \qquad (22)$$

$$\psi|_{y=0} = f(x)$$

Other heat equation problem

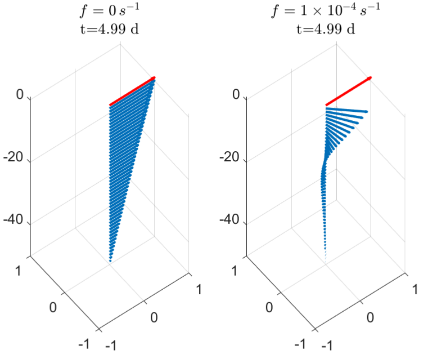
Inertia-Ekman current (Elipot and Gille, 2009; Wenegrat and McPhaden, 2016)

$$\boxed{\frac{\partial \vec{u}}{\partial t} + if\vec{u} = A_z \frac{\partial^2 \vec{u}}{\partial z^2}} \qquad (23a)$$

$$\vec{u}\big|_{t=0} = 0 \qquad (23b)$$

$$A_z \frac{\partial \vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \qquad (24)$$

$$\vec{u}\big|_{z=-h} = 0 \qquad (25)$$



`https://jang-geun.github.io/inertia_ekman.gif`

Assignment

1. Solve system (7) for $\bar{u}$ to obtain one equation. Shows that the result is
$$\frac{\partial \bar{u}}{\partial t} = \gamma' \frac{\partial^2 \bar{u}}{\partial x^2} \tag{26}$$
that is still the heat equation.
2. Solve (23) with boundary conditions (7) and an arbitrary initial condition $\bar{u}|_{t=0} = g(x)$.

References I

Csanady, G. T. (1978). "The arrested topographic wave". In: *Journal of Physical Oceanography* 8.1, pp. 47–62.

Elipot, S. and S. T. Gille (2009). "Ekman layers in the Southern Ocean: Spectral models and observations, vertical viscosity and boundary layer depth". In: *Ocean Science* 5.2, pp. 115–139.

Wenegrat, J. O. and M. J. McPhaden (2016). "A simple analytical model of the diurnal Ekman layer". In: *Journal of Physical Oceanography* 46.9, pp. 2877–2894.