Differential Equations in Geophysical Fluid Dynamics

XX. Wave in rotation: Kelvin wave

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

Feb, 2026

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography communinity COKOAA.

Recap

In the long wave (big) limit...

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial \eta}{\partial x} \qquad \text{(1a)}$$

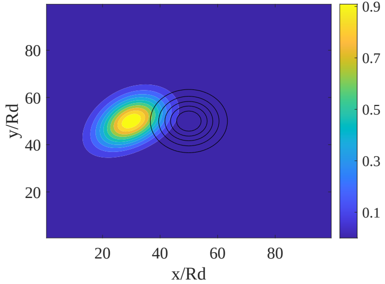
$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} = -g\frac{\partial \eta}{\partial y} \qquad \text{(1b)}$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0. \qquad \text{(1c)}$$

$$\downarrow$$

$$\frac{\partial \eta}{\partial t} \boxed{- \beta\frac{gh}{f^2}} \frac{\partial \eta}{\partial x} = 0 \qquad \text{(2)}$$

Phase velocity!
(nondispersive wave)



Recap

Weak consideration of inertia via perturbation (e.g., $\bar{u} = \bar{u}_0 + \bar{u}_1$):

Big

$$\frac{\partial(\bar{u}_0 + \bar{u}_1)}{\partial t} - (f_0 + \beta y)(\bar{v}_0 + \bar{v}_1) = -g\frac{\partial \eta}{\partial x} \tag{3}$$

Small

O(1) balance (big of big):

$$-f_0 \bar{v}_0 = -g\frac{\partial \eta}{\partial x} \tag{4}$$

O(Ro) balance (small of big):

$$\frac{\partial \bar{u}_0}{\partial t} - f_0 \bar{v}_1 + \beta y \bar{v}_0 = 0 \tag{5}$$

Substituting (4) and (5) into continuity yields

$$\boxed{\frac{\partial \eta}{\partial t} - \beta \frac{gh}{{f_0}^2}\frac{\partial \eta}{\partial x} - \frac{gh}{{f_0}^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) = 0} \tag{6}$$

Topographic Rossby wave

Let us consider f-plane with a slope $h(y) = -\alpha y + h_0$:

$$-f_0\bar{v} = -g\frac{\partial \eta}{\partial x} \qquad (7a)$$

$$f_0\bar{u} = -g\frac{\partial \eta}{\partial y} \qquad (7b)$$

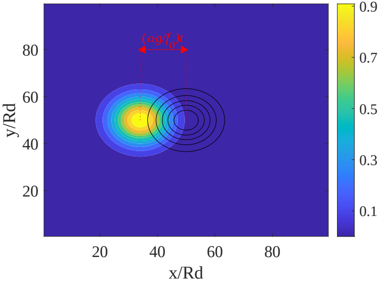
$$\frac{\partial \eta}{\partial t} + h\frac{\partial \bar{u}}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0 \qquad (7c)$$

$$\downarrow$$

$$\frac{\partial \eta}{\partial t} \boxed{-\frac{\alpha g}{f_0}} \frac{\partial \eta}{\partial x} = 0 \qquad (8)$$

Phase velocity!
(nondispersive wave)

so-called topographic Rossby wave.



Topographic Rossby wave perturbed by inertia

$$\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} \tag{9a}$$

$$\frac{\partial \bar{v}}{\partial t} + f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} \tag{9b}$$

$h = -\alpha y + h_0$

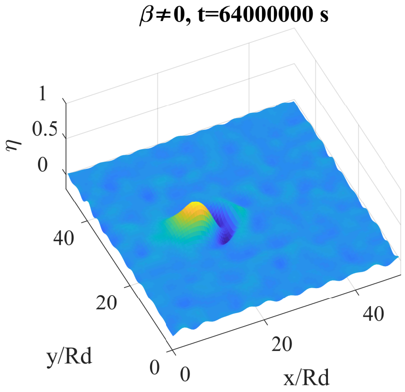
$$\frac{\partial \eta}{\partial t} + h \frac{\partial \bar{u}}{\partial x} + \frac{\partial (h \bar{v})}{\partial y} = 0 \tag{9c}$$

By assuming $Ro \ll 1$ and $\alpha L \ll h_0$, based on $O(1)$ and $O(Ro)$ balances, equation for η is given by

$$\frac{\partial \eta}{\partial t} - \frac{\alpha g}{f_0} \frac{\partial \eta}{\partial x} - \frac{g h_0}{{f_0}^2} \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) = 0 \tag{10}$$

HW: Obtain (10) from (9) using the approximations and perturbation theory.

Topographic Rossby wave perturbed by inertia



https://jang-geun.github.io/vis_topo_rossby_small.gif

Governing equation

Near the western boundary where $\bar{u} \approx 0$, governing equations can be simplified to

$$-f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} \tag{11a}$$

$$\frac{\partial \bar{v}}{\partial t} = -g \frac{\partial \eta}{\partial y} \tag{11b}$$

$$\frac{\partial \eta}{\partial t} + h \frac{\partial \bar{v}}{\partial y} = 0. \tag{11c}$$

Note that (11b) and (11c) can be rewritten as

$$\frac{\partial^2 \eta}{\partial t^2} = gh \frac{\partial^2 \eta}{\partial y^2} \tag{12}$$

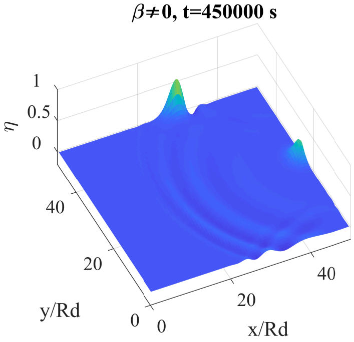
that is identical to the wave equation! But it is constrained by geostrophic balance (11a), that can be rewritten as

$$f_0 \frac{\partial \eta}{\partial y} = -\frac{\partial^2 \eta}{\partial t \partial x} \tag{13}$$

Kelvin wave

Solution to the problem, for a single mode near western boundary, is given by

$$\eta = \eta_0 e^{-x/Rd} e^{il(y+\sqrt{gh}t)} \tag{14}$$



https://jang-geun.github.io/vis_kelvin.gif

Equatorial Kelvin wave

Near the equator where $\bar{v} \approx 0$

$$\frac{\partial \bar{u}}{\partial t} = -g \frac{\partial \eta}{\partial x} \tag{15a}$$

$$\beta y \bar{u} = -g \frac{\partial \eta}{\partial y} \tag{15b}$$

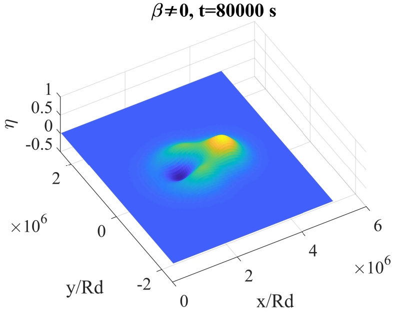
$$\frac{\partial \eta}{\partial t} + h \frac{\partial \bar{u}}{\partial x} = 0 \tag{15c}$$

that yields

$$\frac{\partial^2 \eta}{\partial x^2} = gh \frac{\partial^2 \eta}{\partial x^2} \tag{16a}$$

$$\beta y \frac{\partial \eta}{\partial x} = \frac{\partial^2 \eta}{\partial t \partial y} \tag{16b}$$

Conclusion



https://jang-geun.github.io/vis_equatorial_kelvin.gif