# Differential Equations in Geophysical Fluid Dynamics

## IX. Frictional spin-down and heat equation

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Apr, 2025

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography commuinity COKOAA.

# Recap

A knowledge from Taylor-Proudman theorem:

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} + \nabla \cdot (A_h \nabla \bar{u}) + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h} \bar{u} \tag{1a}$$

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} + \nabla \cdot (A_h \nabla \bar{v}) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h} \bar{v} \tag{1b}$$

$$\frac{\partial \eta}{\partial t} + h \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0 \tag{1c}$$

Substituting (1a) and (1b) into (1c) yields

$$\frac{\partial \eta}{\partial t} = 0 \tag{2}$$

that means no change in $\eta$. So, pure barotropic fluid geostrophic current component in $f$-plane cannot change sea surface height ($\eta$).

# Considering bottom friction

What if we consider bottom friction?

Bottom friction

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} + \nabla \cdot (A_h \nabla \bar{u}) + \frac{\tau_x^s}{\rho_0 h} \boxed{- \frac{\gamma}{h} \bar{u}} \quad \text{(3a)}$$

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} + \nabla \cdot (A_h \nabla \bar{v}) + \frac{\tau_y^s}{\rho_0 h} \boxed{- \frac{\gamma}{h} \bar{v}} \quad \text{(3b)}$$

$$\frac{\partial \eta}{\partial t} + h \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0 \quad \text{(3c)}$$

Let us "approximately" solve the equations for $\eta$. Decompose velocities into geostrophic and Ekman components, $\bar{u} = \bar{u}_g + \bar{u}_e$ and $\bar{v} = \bar{v}_g + \bar{v}_e$, and then assume $\bar{u}_e \ll \bar{u}_g$ and $\bar{v}_e \ll \bar{v}_g$.

# Governing equation

$$-f_0(\bar{v}_g + \bar{v}_e) = -g\frac{\partial \eta}{\partial x} - \frac{\gamma}{h}(\bar{u}_g + \bar{u}_e)$$

$$-f_0(U_g\bar{v}_g^* + U_e\bar{v}_e^*) = -(fU_g)\frac{\partial \eta^*}{\partial x^*} - \frac{\gamma}{h}(U_g\bar{u}_g^* + U_e\bar{u}_e^*)$$

$$-\left(\bar{v}_g^* + \frac{U_e}{U_g}\bar{v}_e^*\right) = -\frac{\partial \eta^*}{\partial x^*} - \frac{\gamma}{f_0 h}\left(\bar{u}_g^* + \frac{U_e}{U_g}\bar{u}_e^*\right)$$

Let us choose $U_e/U_g \approx \gamma/(f_0 h) \approx \epsilon$ where $\epsilon$ indicates arbitrary small nondimensional number much less than one ($\epsilon \ll 1$).

$O(1)$: big

$O(\epsilon^2)$: very small

$$-\left(\boxed{\bar{v}_g^*} + \boxed{\epsilon\bar{v}_e^*}\right) = \boxed{-\frac{\partial \eta^*}{\partial x^*}} - \left(\boxed{\epsilon\bar{u}_g^*} + \epsilon^2\bar{u}_e^*\right) \qquad (4)$$

$O(\epsilon)$: small

# Governing equation

$O(1)$ balance

$$-f_0(\boxed{\bar{v}_g} + \boxed{\bar{v}_e}) = \boxed{-g\frac{\partial \eta}{\partial x}} - \frac{\gamma}{h}(\boxed{\bar{u}_g} + \bar{u}_e)$$

$O(\epsilon)$ balance

$$\therefore \boxed{\bar{v}_g = \frac{g}{f_0}\frac{\partial \eta}{\partial x}}, \quad \boxed{\bar{v}_e = \frac{\gamma}{f_0 h}\bar{u}_g} \tag{5}$$

This is based on the “fundamental theorem of perturbation theory”.

In the same manner, $v$-momentum equation yields

$$\bar{u}_g = -\frac{g}{f_0}\frac{\partial \eta}{\partial y}, \quad \bar{u}_e = -\frac{\gamma}{f_0 h}\bar{v}_g \tag{6}$$

# Governing equation

Substituting (5) and (6) into the continuity equation (3c) yields

$$
\begin{aligned}
\frac{\partial \eta}{\partial t} &= -h\left(\underbrace{\frac{\partial \bar{u}_g}{\partial x} + \frac{\partial \bar{v}_g}{\partial y}}_{= \nabla \cdot \vec{u}_g = 0} + \frac{\partial \bar{u}_e}{\partial x} + \frac{\partial \bar{v}_e}{\partial y}\right) \\
&= -h\left(\underbrace{\frac{\partial \bar{u}_e}{\partial x} + \frac{\partial \bar{v}_e}{\partial y}}_{= \nabla \cdot \vec{u}_e}\right) = \frac{\gamma}{f_0}\left(\underbrace{\frac{\partial \bar{v}_g}{\partial x} - \frac{\partial \bar{u}_g}{\partial y}}_{= \nabla \times \vec{u}_g}\right) \\
&= \underbrace{\frac{\gamma g}{{f_0}^2}}_{\equiv \gamma'}\left(\underbrace{\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}}_{= \nabla^2 \vec{u}_e}\right)
\end{aligned}
$$

$$
\boxed{\therefore \frac{\partial \eta}{\partial t} = \gamma' \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)} \tag{7}
$$

where $\gamma' = \gamma g / {f_0}^2$. This is referred to as the "heat equation".

# Governing equation

For simplicity, consider one-dimensional problem.

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} \qquad (8a)$$

$$f\bar{u} = -\frac{\gamma}{h}\bar{v} \qquad (8b)$$

$$\frac{\partial \eta}{\partial t} + h\frac{\partial \bar{u}}{\partial x} = 0 \qquad (8c)$$

$$\boxed{\frac{\partial \eta}{\partial t} = \gamma' \frac{\partial^2 \eta}{\partial x^2}} \qquad (9)$$

Let us consider an initial condition given by

$$\boxed{\eta|_{t=0} = \eta_0 \sin(k_0 x)} \qquad (10)$$

without boundary conditions.

Key approach is to assume $\eta = X(x)T(t)$ where $X$ and $T$ are arbitrary function depends on only $x$ and $t$, respectively.

# Solution to the heat equation

Solution to the initial value problem is given by

$$\eta = \eta_0 e^{-wt} \sin(k_0 x) \tag{11}$$

where $w = \gamma' {k_0}^2 > 0$ representing decaying rate of the amplitude. Based on the superposition principle of linear non-homogeneous differential equation, the problem can be generalized to

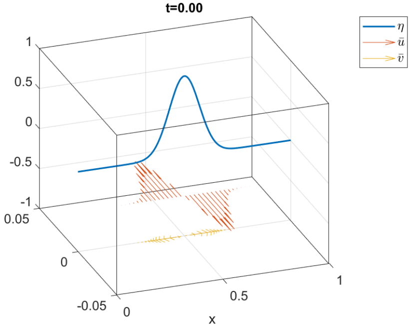
$$\frac{\partial \eta}{\partial t} = \gamma' \frac{\partial^2 \eta}{\partial x^2} \tag{12a}$$

$$\eta|_{t=0} = f(x) \equiv \sum_{n=-\infty}^{\infty} \hat{\eta}_n e^{ik_n x} \tag{12b}$$

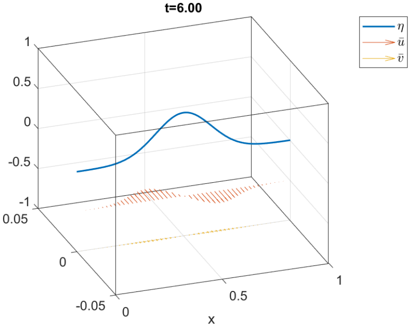
and the solution is given by

$$\eta = \sum_{n=-\infty}^{\infty} \hat{\eta}_n e^{-w_n t} e^{ik_n x} \quad \text{where} \quad w_n = \gamma' {k_n}^2. \tag{13}$$

# Solution to the heat equation



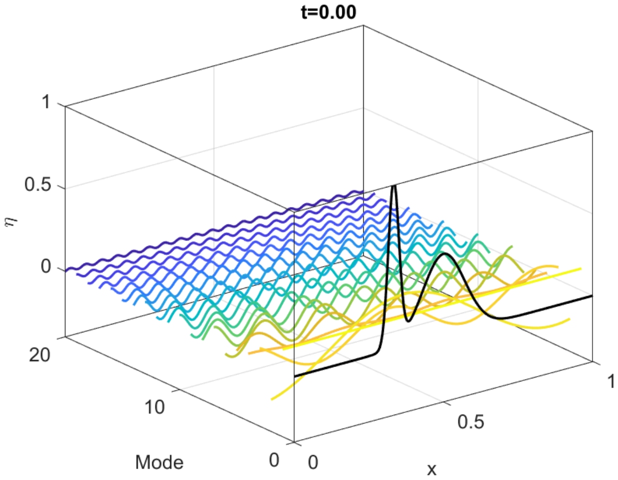

(a)




(b)

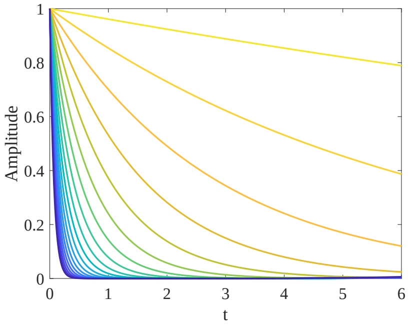
https://jang-geun.github.io/vis_friction_spindown.gif

# Dispersion relation of heat equation

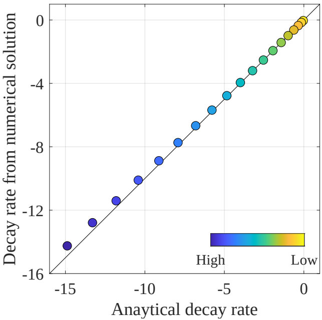



https://jang-geun.github.io/vis_heateq_dispersion.gif

# Dispersion relation of heat equation




(c)




(d)

# Summary

Governing equation to the frictional spin-down is the heat equation given by

$$\frac{\partial \eta}{\partial t} = \gamma' \frac{\partial^2 \eta}{\partial x^2} \tag{14}$$

that describes "diffusion". The solution to the equation as initial value problem is

$$\eta = \sum_{n=-\infty}^{\infty} \hat{\eta}_n e^{-w_n t} e^{i k_n x} \quad \text{where} \quad w_n = \gamma' {k_n}^2. \tag{15}$$

so the diffusion is equivalent to faster exponential decay of high-wavenumber modes.