

Differential Equations in Geophysical Fluid Dynamics

III. Inertial oscillation

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Recap

The primitive equations are given by

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} - fv \\ - \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(A_h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right) \end{aligned} \quad (1a)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial(vv)}{\partial y} + \frac{\partial(vw)}{\partial z} + fu = \\ - \frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left(A_h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right) \end{aligned} \quad (1b)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial(wv)}{\partial z} = \\ - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left(A_h \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - g \end{aligned} \quad (1c)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (1d)$$

Recap: Boussinesq approximation $\rho' \ll \rho_0$

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(A_h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right) \end{aligned} \quad (2a)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left(A_h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right) \end{aligned} \quad (2b)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left(A_h \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - g \frac{\rho}{\rho_0} \end{aligned} \quad (2c)$$

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \end{aligned} \quad (2d)$$

Recap: hydrostatic approximation $H \ll L$

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(A_h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial u}{\partial z} \right) \end{aligned} \quad (3a)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left(A_h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial v}{\partial z} \right) \end{aligned} \quad (3b)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \\ - \frac{1}{\rho_0} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left(A_h \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(A_h \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - g \frac{\rho}{\rho_0} \end{aligned} \quad (3c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \quad (3d)$$

Recap: shallow water equations

What is the simplest approximated solution high-school student can solve?

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} - f \bar{v} = \\ -g \frac{\partial \eta}{\partial x} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{u}}{\partial y} \right) \right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \end{aligned} \quad (4a)$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + f \bar{u} = \\ -g \frac{\partial \eta}{\partial y} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{v}}{\partial y} \right) \right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \end{aligned} \quad (4b)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \quad (4c)$$

Recap: shallow water equations

What is the simplest approximated solution high-school student can solve?

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} - f \bar{v} = \\ -g \frac{\partial \eta}{\partial x} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{u}}{\partial y} \right) \right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \end{aligned} \quad (5a)$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + f \bar{u} = \\ -g \frac{\partial \eta}{\partial y} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{v}}{\partial y} \right) \right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \end{aligned} \quad (5b)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \quad (5c)$$

Inertial motion in fixed coordinate

This is not trivial and has physical meaning (uniform linear motion)!

$$\frac{d\bar{u}}{dt} = 0 \quad (6a)$$

$$\frac{d\bar{v}}{dt} = 0 \quad (6b)$$

$$\bar{u}|_{t=0} = U_0, \quad \bar{v}|_{t=0} = V_0 \quad (6c)$$

Solutions to the equations are given by

$$\bar{u} = U_0, \quad \bar{v} = V_0 \quad (7)$$

\bar{u} and \bar{v} can be considered as velocity components of a water parcel, so equations for its position are $dX/dt = \bar{u}$ and $dY/dt = \bar{v}$.

$$X = U_0 t + X_0, \quad Y = V_0 t + Y_0 \quad (8)$$

Inertial motion in rotating coordinate

What if we consider Coriolis force (rotation of coordinate)?

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} - f \bar{v} = \\ -g \frac{\partial \eta}{\partial x} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{u}}{\partial y} \right) \right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \end{aligned} \quad (9a)$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + f \bar{u} = \\ -g \frac{\partial \eta}{\partial y} + \frac{A_h}{h} \left(\frac{\partial}{\partial x} \left(h \frac{\partial \bar{v}}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \bar{v}}{\partial y} \right) \right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \end{aligned} \quad (9b)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \quad (9c)$$

Inertial motion in rotating coordinate

$$\frac{d\bar{u}}{dt} - f\bar{v} = 0 \quad (10a)$$

$$\frac{d\bar{v}}{dt} + f\bar{u} = 0 \quad (10b)$$

$$\bar{u}|_{t=0} = U_0, \quad \bar{v}|_{t=0} = V_0 \quad (10c)$$

Solution to the equations are given by

$$\bar{u} = V_0 \sin(f t) + U_0 \cos(f t) \quad (11a)$$

$$\bar{v} = V_0 \cos(f t) - U_0 \sin(f t) \quad (11b)$$

Oscillation with frequency f (period $2\pi/f$)

This is what we call "inertial oscillation".

Inertial motion in rotating coordinate

Governing equations for the position of water mass are

$$\frac{dX}{dt} = \bar{u} \quad (12a)$$

$$\frac{dY}{dt} = \bar{v} \quad (12b)$$

$$X|_{t=0} = X_0 \quad Y|_{t=0} = Y_0 \quad (12c)$$

and the solutions are given by

$$X = \frac{U_0}{f} \sin(ft) - \frac{V_0}{f} \cos(ft) + X_0 + \frac{V_0}{f} \quad (13a)$$

$$Y = \frac{V_0}{f} \sin(ft) + \frac{U_0}{f} \cos(ft) + Y_0 - \frac{U_0}{f} \quad (13b)$$

Inertial oscillation

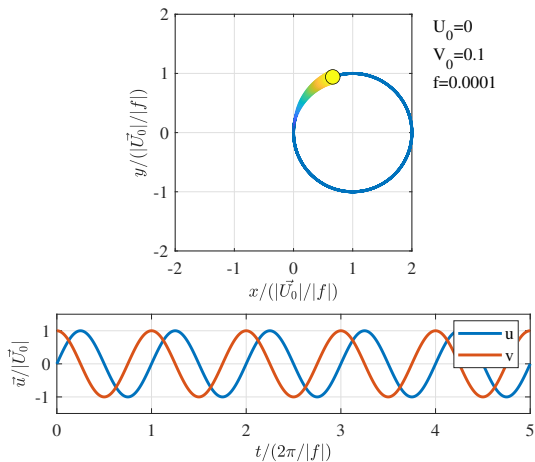
that can be rewritten as

$$\left(X - \left(X_0 + \frac{V_0}{f}\right)\right)^2 + \left(Y - \left(Y_0 - \frac{U_0}{f}\right)\right)^2 = \left(\frac{\sqrt{(U_0^2 + V_0^2)}}{f}\right)^2. \quad (14)$$

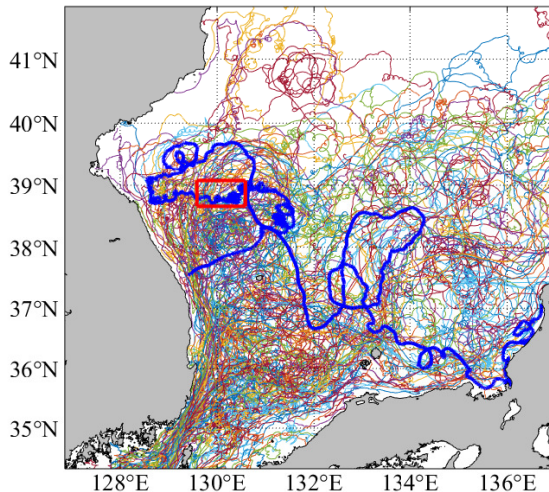
Radius of water mass trajectory

This is equation of a circle!

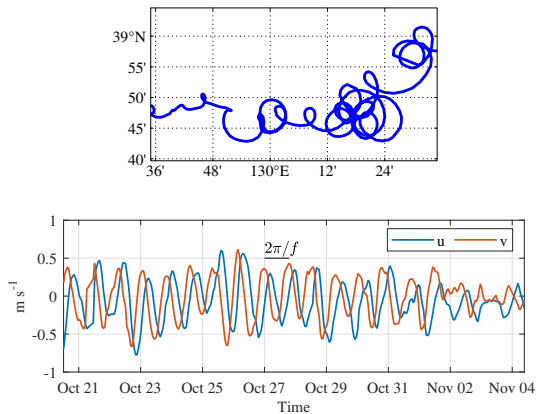
Inertial oscillation



Inertial oscillation



Inertial oscillation



<https://jmlilly.net/talks/lilly16-osu.html#63>

Summary

In rotating coordinate, not uniform linear motion but inerial oscillation.

Governing equation:

$$\frac{d\vec{u}}{dt} + if\vec{u} = 0 \quad (15a)$$

$$\vec{u}|_{t=0} = \vec{U}_0 \quad (15b)$$

Solution:

$$\vec{u} = \vec{U}_0 e^{-ift} \quad (16)$$

Oscillation with frequency f
(period $2\pi/f$)

that yields circle trajectory of which radius is $|\vec{U}_0|/f$.

Assignment

Let us consider one more term, bottom stress (friction; $\tau^b/(\rho_0 h)$ in (5)). The governing equations are given by

$$\frac{d\bar{u}}{dt} - f\bar{v} = -\frac{\tau_x^b}{\rho_0 h} = -\frac{\gamma}{h}\bar{u} \quad (17a)$$

$$\frac{d\bar{v}}{dt} + f\bar{u} = -\frac{\tau_y^b}{\rho_0 h} = -\frac{\gamma}{h}\bar{v} \quad (17b)$$

$$\bar{u}|_{t=0} = U_0, \quad \bar{v}|_{t=0} = V_0 \quad (17c)$$

where bottom stress is modeled by linear friction bottom boundary condition and γ is linear friction coefficient, that is a constant.

1. Find two important time scales governing the equations.
2. Analytically solve above equations.
3. Under presence of friction, What kind of condition is required for inertial oscillations to be observed?