

Differential Equations in Geophysical Fluid Dynamics

IX. Frictional spin-down and heat equation

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Recap

A knowledge from Taylor-Proudman theorem:

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} + \nabla \cdot (A_h \nabla \bar{u}) + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h} \bar{u} \quad (1a)$$

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} + \nabla \cdot (A_h \nabla \bar{v}) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h} \bar{v} \quad (1b)$$

$$\frac{\partial \eta}{\partial t} + h \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0 \quad (1c)$$

Substituting (1a) and (1b) into (1c) yields


$$\frac{\partial \eta}{\partial t} = 0 \quad (2)$$

that means no change in η . So, pure barotropic fluid geostrophic current component in f -plane cannot change sea surface height (η).

Considering bottom friction

What if we consider bottom friction?

$$\frac{\partial \bar{u}}{\partial t} + \vec{u} \cdot \nabla \bar{u} - f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} + \nabla \cdot (A_h \nabla \bar{u}) + \frac{\tau_x^s}{\rho_0 h} - \frac{\gamma}{h} \bar{u} \quad (3a)$$

Bottom friction 

$$\frac{\partial \bar{v}}{\partial t} + \vec{u} \cdot \nabla \bar{v} + f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} + \nabla \cdot (A_h \nabla \bar{v}) + \frac{\tau_y^s}{\rho_0 h} - \frac{\gamma}{h} \bar{v} \quad (3b)$$

$$\frac{\partial \eta}{\partial t} + h \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0 \quad (3c)$$

Let us “approximately” solve the equations for η . Decompose velocities into geostrophic and Ekman components, $\bar{u} = \bar{u}_g + \bar{u}_e$ and $\bar{v} = \bar{v}_g + \bar{v}_e$, and then assume $\bar{u}_e \ll \bar{u}_g$ and $\bar{v}_e \ll \bar{v}_g$.

Governing equation

$$-f_0(\bar{v}_g + \bar{v}_e) = -g \frac{\partial \eta}{\partial x} - \frac{\gamma}{h}(\bar{u}_g + \bar{u}_e)$$

$$-f_0(U_g \bar{v}_g^* + U_e \bar{v}_e^*) = -(f U_g) \frac{\partial \eta^*}{\partial x^*} - \frac{\gamma}{h}(U_g \bar{u}_g^* + U_e \bar{u}_e^*)$$

$$-\left(\bar{v}_g^* + \frac{U_e}{U_g} \bar{v}_e^*\right) = -\frac{\partial \eta^*}{\partial x^*} - \frac{\gamma}{f_0 h} \left(\bar{u}_g^* + \frac{U_e}{U_g} \bar{u}_e^*\right)$$

Let us choose $U_e/U_g \approx \gamma/(f_0 h) \approx \epsilon$ where ϵ indicates arbitrary small nondimensional number much less than one ($\epsilon \ll 1$).

The diagram shows the governing equation with asymptotic scaling annotations. A blue arrow labeled $O(1)$: big points to the \bar{v}_g^* term in the left-hand side. A red arrow labeled $O(\epsilon)$: small points to the $\epsilon \bar{v}_e^*$ term in the left-hand side. A blue box highlights the $-\frac{\partial \eta^*}{\partial x^*}$ term. A red box highlights the $\epsilon \bar{u}_g^*$ term. A black arrow labeled $O(\epsilon^2)$: very small points to the $\epsilon^2 \bar{u}_e^*$ term. The equation is labeled (4).

$$-\left(\bar{v}_g^* + \epsilon \bar{v}_e^*\right) = -\frac{\partial \eta^*}{\partial x^*} - \left(\epsilon \bar{u}_g^* + \epsilon^2 \bar{u}_e^*\right) \quad (4)$$

Governing equation

$$-f_0(\bar{v}_g + \bar{v}_e) = -g \frac{\partial \eta}{\partial x} - \frac{\gamma}{h}(\bar{u}_g + \bar{u}_e)$$

O(1) balance (blue arrows pointing to \bar{v}_g and $-g \frac{\partial \eta}{\partial x}$)
O(ϵ) balance (red arrows pointing to \bar{v}_e and \bar{u}_g)

$$\therefore \bar{v}_g = \frac{g}{f_0} \frac{\partial \eta}{\partial x}, \quad \bar{v}_e = \frac{\gamma}{f_0 h} \bar{u}_g \quad (5)$$

This is based on the “fundamental theorem of perturbation theory”.

In the same manner, v -momentum equation yields

$$\bar{u}_g = -\frac{g}{f_0} \frac{\partial \eta}{\partial y}, \quad \bar{u}_e = -\frac{\gamma}{f_0 h} \bar{v}_g \quad (6)$$

Governing equation

Substituting (5) and (6) into the continuity equation (3c) yields

$$\begin{aligned}
 \frac{\partial \eta}{\partial t} &= -h \left(\frac{\partial \bar{u}_g}{\partial x} + \frac{\partial \bar{v}_g}{\partial y} + \frac{\partial \bar{u}_e}{\partial x} + \frac{\partial \bar{v}_e}{\partial y} \right) \quad \begin{array}{l} \xrightarrow{\text{= } \nabla \cdot \vec{u}_g = 0} \\ \xleftarrow{\text{= } \nabla \times \vec{u}_g} \end{array} \\
 &= -h \left(\frac{\partial \bar{u}_e}{\partial x} + \frac{\partial \bar{v}_e}{\partial y} \right) = \frac{\gamma}{f_0} \left(\frac{\partial \bar{v}_g}{\partial x} - \frac{\partial \bar{u}_g}{\partial y} \right) \\
 &\quad \begin{array}{l} \xrightarrow{\text{= } \nabla \cdot \vec{u}_e} \\ \xleftarrow{\text{= } \nabla^2 \vec{u}_e} \end{array} \\
 &= \frac{\gamma g}{f_0^2} \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) \\
 &\quad \begin{array}{l} \xleftarrow{\equiv \gamma'} \end{array} \\
 &\boxed{\therefore \frac{\partial \eta}{\partial t} = \gamma' \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right)} \quad (7)
 \end{aligned}$$

where $\gamma' = \gamma g / f_0^2$. This is referred to as the “heat equation”.

Governing equation

For simplicity, consider one-dimensional problem.

$$-f\bar{v} = -g\frac{\partial\eta}{\partial x} \quad (8a)$$

$$f\bar{u} = -\frac{\gamma}{h}\bar{v} \quad (8b)$$

$$\frac{\partial\eta}{\partial t} + h\frac{\partial\bar{u}}{\partial x} = 0 \quad (8c)$$

$$\boxed{\frac{\partial\eta}{\partial t} = \gamma' \frac{\partial^2\eta}{\partial x^2}} \quad (9)$$

Let us consider an initial condition given by

$$\boxed{\eta|_{t=0} = \eta_0 \sin(k_0 x)} \quad (10)$$

without boundary conditions.

Key approach is to assume $\eta = X(x)T(t)$ where X and T are arbitrary function depends on only x and t , respectively.

Solution to the heat equation

Solution to the initial value problem is given by

$$\eta = \eta_0 e^{-wt} \sin(k_0 x) \quad (11)$$

where $w = \gamma' k_0^2 > 0$ representing decaying rate of the amplitude. Based on the superposition principle of linear non-homogeneous differential equation, the problem can be generalized to

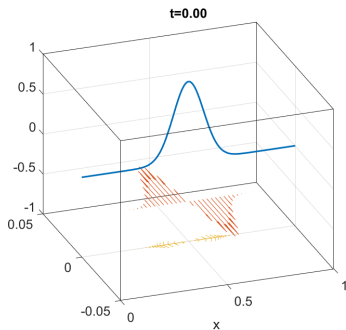
$$\frac{\partial \eta}{\partial t} = \gamma' \frac{\partial^2 \eta}{\partial x^2} \quad (12a)$$

$$\eta|_{t=0} = f(x) \equiv \sum_{n=-\infty}^{\infty} \hat{\eta}_n e^{ik_n x} \quad (12b)$$

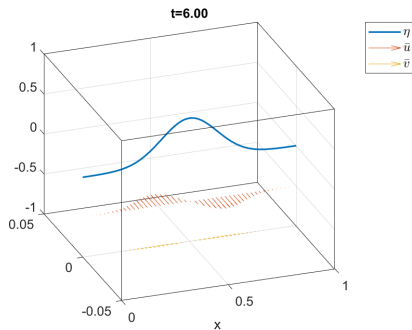
and the solution is given by

$$\eta = \sum_{n=-\infty}^{\infty} \hat{\eta}_n e^{-w_n t} e^{ik_n x} \quad \text{where} \quad w_n = \gamma' k_n^2. \quad (13)$$

Solution to the heat equation



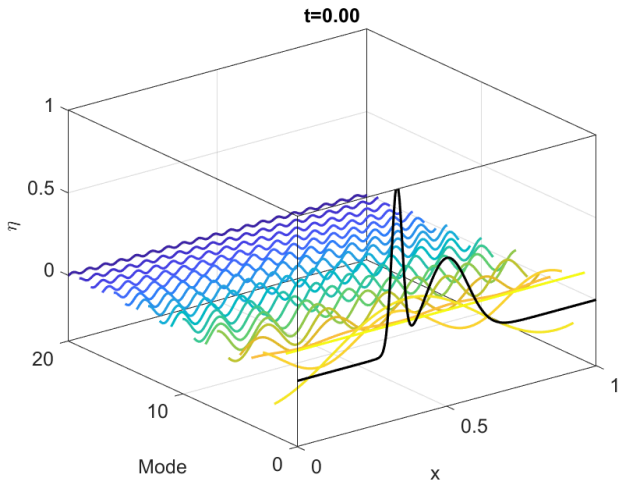
(a)



(b)

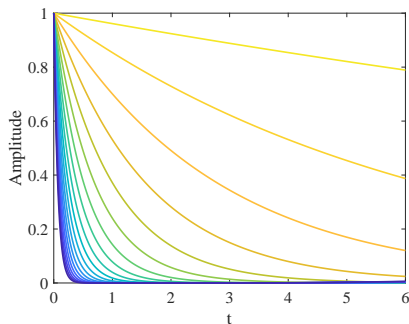
https://jang-geun.github.io/vis_friction_spindown.gif

Dispersion relation of heat equation

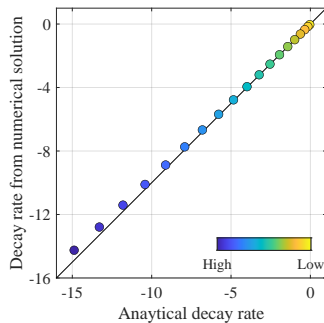


https://jang-geun.github.io/vis_heateq_dispersion.gif

Dispersion relation of heat equation



(c)



(d)

Summary

Governing equation to the frictional spin-down is the **heat equation** given by

$$\frac{\partial \eta}{\partial t} = \gamma' \frac{\partial^2 \eta}{\partial x^2} \quad (14)$$

that describes **"diffusion"**. The solution to the equation as initial value problem is

$$\eta = \sum_{n=-\infty}^{\infty} \hat{\eta}_n e^{-w_n t} e^{ik_n x} \quad \text{where} \quad w_n = \gamma' k_n^2. \quad (15)$$

so the diffusion is equivalent to faster exponential decay of high-wavenumber modes.