

Differential Equations in Geophysical Fluid Dynamics

X. Other heat equation problems

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Recap

How to solve the simplest version of heat equation problem?

$$\frac{\partial \eta}{\partial t} = \gamma \frac{\partial^2 \eta}{\partial x^2} \quad (1)$$

$$\eta|_{t=0} = \eta_0 \sin(k_0 x) \quad (2)$$

Assume $\eta = X(x)T(t)$ and substituting it into (1) yields

$$\frac{1}{\gamma} \frac{T'}{T} = \frac{X''}{X} = \lambda. \quad (3)$$

Diagram annotations:
- A red line from the text "Must be the same as a constant." points to the λ term.
- A blue line from the text "Function of only t " points to the $\frac{1}{\gamma} \frac{T'}{T}$ term.
- A blue line from the text "Function of only x " points to the $\frac{X''}{X}$ term.

Therefore, we can obtain two ODEs:

$$T' = \gamma \lambda T, \quad X'' = \lambda X \quad (4)$$

Note that assuming $X = e^{kx}$ yields $k = \pm\sqrt{\lambda}$.

Recap

1. If $\lambda = 0$, $\eta = A_0x + B_0$.
2. If $0 < \lambda$ ($\lambda \equiv \lambda^+$; positive), $\eta = e^{\gamma\lambda t}(A_1e^{\sqrt{\lambda^+}x} + B_1e^{-\sqrt{\lambda^+}x})$.
3. If $\lambda < 0$ ($\lambda \equiv -\lambda^-$; negative),
 $\eta = e^{-\gamma\lambda^-t}(A_2\cos(\sqrt{\lambda^-}x) + B_2\sin(\sqrt{\lambda^-}x))$

Therefore, general solution based on superposition principle is

$$\eta = A_0x + B_0 + e^{\gamma\lambda t}(A_1e^{\sqrt{\lambda^+}x} + B_1e^{-\sqrt{\lambda^+}x}) + e^{-\gamma\lambda^-t}(A_2\cos(\sqrt{\lambda^-}x) + B_2\sin(\sqrt{\lambda^-}x)) \quad (5)$$

Based on initial condition, $A_0 = B_0 = A_1 = B_1 = A_2 = 0$, $B_2 = \eta_0$, $\sqrt{\lambda^-} = \sqrt{-\lambda} = k_0$. So the particulate solution is given by

$$\boxed{\eta = \eta_0 e^{-\lambda k_0^2 t} \sin(k_0 x)} \quad (6)$$

Considering boundary conditions

$$-f\bar{v} = -g\frac{\partial\eta}{\partial x} \quad (7a)$$

$$f\bar{u} = -\frac{\gamma}{h}\bar{v} \quad (7b)$$

$$\frac{\partial\eta}{\partial t} + h\frac{\partial\bar{u}}{\partial x} = 0 \quad (7c)$$

$$\bar{u}|_{x=0} = 0 \quad (7d)$$

$$\bar{u}|_{x=L} = 0 \quad (7e)$$

Closed B.C.

$$\frac{\partial\eta}{\partial t} = \gamma' \frac{\partial^2\eta}{\partial x^2} \quad (8)$$

Let us consider arbitrary initial condition given by

$$\eta|_{t=0} = f(x) \quad (9)$$

with boundary conditions:

$$\left. \frac{\partial\eta}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial\eta}{\partial x} \right|_{x=L} = 0. \quad (10)$$

Q. The easiest initial condition to solve this problem is $f(x) = \eta_0 \cos(k_0 x)$. Why cos not sin?

Considering boundary conditions

$$\eta = \sum_{n=0}^{\infty} A_n e^{-\gamma(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right) \quad (11)$$

Substituting (11) into the initial condition (9) yields

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \quad (12)$$

Fourier's trick

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right) \left[\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \right] dx$$

$$\sum_{n=0}^{\infty} A_n \left[\int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \right] = \int_0^L \cos\left(\frac{m\pi x}{L}\right) f(x) dx \quad (13)$$

Considering boundary conditions

Orthogonality of sinusoidal functions

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L/2 & n = m \\ 0 & n \neq m \end{cases} \quad (14a)$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L/2 & n = m \neq 0 \\ L & n = m = 0 \\ 0 & n \neq m \end{cases} \quad (14b)$$

$$\int_0^L \sin\left(\frac{n\pi x}{L_x}\right) \cos\left(\frac{m\pi x}{L_x}\right) dx = 0 \quad (14c)$$

$$\sum_{n=0}^{\infty} A_n \left[\int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \right] = A_m \frac{L}{2} \quad (15)$$

Only one mode of which $m = n (\neq 0)$ (nonzero) survive, so it can be solved for $A_m = A_n$!

Considering boundary conditions

Consequently, the particulate solution is given by

$$\eta = \sum_{n=0}^{\infty} A_n e^{-\gamma(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right) \quad (16)$$

where

$$A_n = \begin{cases} \frac{1}{L} \int_0^L f(x) dx & n = 0 \\ \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x) dx & n \neq 0. \end{cases} \quad (17)$$

Other heat equation problem

Heat equation analogy of coastal trapped wave (Csanady, 1978)

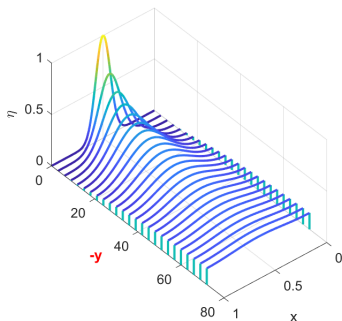
$$-f_0 \bar{v} = -g \frac{\partial \eta}{\partial x} \quad (18)$$

$$f_0 \bar{u} = -g \frac{\partial \eta}{\partial y} - \frac{\gamma}{h} \bar{v} \quad (19)$$

that can be approximately written as

$$\frac{\partial(h\bar{u})}{\partial x} + \frac{\partial(h\bar{v})}{\partial y} = 0 \quad (20)$$

$$h = \alpha x + h_0 \quad (21)$$



$$\boxed{\frac{\partial \psi}{\partial y} = -\gamma' \frac{\partial^2 \psi}{\partial x^2}} \quad (22)$$

$$\psi|_{y=0} = f(x)$$

Other heat equation problem

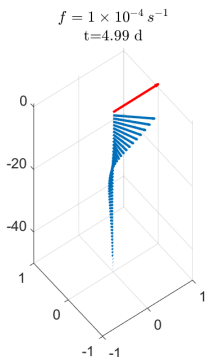
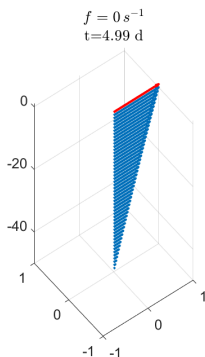
Inertia-Ekman current (Elipot and Gille, 2009; Wenegrat and McPhaden, 2016)

$$\boxed{\frac{\partial \vec{u}}{\partial t} + i f \vec{u} = A_z \frac{\partial^2 \vec{u}}{\partial z^2}} \quad (23a)$$

$$\vec{u}|_{t=0} = 0 \quad (23b)$$

$$A_z \frac{\partial \vec{u}}{\partial z} \Big|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \quad (24)$$

$$\vec{u}|_{z=-h} = 0 \quad (25)$$



https://jang-geun.github.io/vis_inertia_ekman.gif

Assignment




1. Solve system (7) for \bar{u} to obtain one equation. Shows that the result is

$$\frac{\partial \bar{u}}{\partial t} = \gamma' \frac{\partial^2 \bar{u}}{\partial x^2} \quad (26)$$

that is still the heat equation.

2. Solve (23) with boundary conditions (7) and an arbitrary initial condition $\bar{u}|_{t=0} = g(x)$.

References I

-  Csanady, G. T. (1978). “The arrested topographic wave”. In: *Journal of Physical Oceanography* 8.1, pp. 47–62.
-  Elipot, S. and S. T. Gille (2009). “Ekman layers in the Southern Ocean: Spectral models and observations, vertical viscosity and boundary layer depth”. In: *Ocean Science* 5.2, pp. 115–139.
-  Wenegrat, J. O. and M. J. McPhaden (2016). “A simple analytical model of the diurnal Ekman layer”. In: *Journal of Physical Oceanography* 46.9, pp. 2877–2894.