Differential Equations in Geophysical Fluid Dynamics

III. Inertial oscillation

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Mar, 2025

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography communinity COKOAA.

Recap

The primitive equations are given by

$$\begin{aligned}\frac{\partial u}{\partial t}+\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}-fv\\ -\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right)\end{aligned} \tag{1a}$$

$$\begin{aligned}\frac{\partial v}{\partial t}+\frac{\partial(vu)}{\partial x}+\frac{\partial(vv)}{\partial y}+\frac{\partial(vw)}{\partial z}+fu=\\ -\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(A_h\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right)\end{aligned} \tag{1b}$$

$$\begin{aligned}\frac{\partial w}{\partial t}+\frac{\partial(wu)}{\partial x}+\frac{\partial(wv)}{\partial y}+\frac{\partial(ww)}{\partial z}=\\ -\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\left(A_h\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_h\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial w}{\partial z}\right)-g\end{aligned} \tag{1c}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{1d}$$

Recap: Boussinesq approximation $\rho' \ll \rho_0$

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv \\
- \frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right)
\end{aligned} \tag{2a}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = \\
- \frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(A_h\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right)
\end{aligned} \tag{2b}
$$

$$
\begin{aligned}
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \\
- \frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left(A_h\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial w}{\partial z}\right) - g\frac{\rho}{\rho_0}
\end{aligned} \tag{2c}
$$

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \tag{2d}
$$

Recap: hydrostatic approximation $H \ll L$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv \\ -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right) \tag{3a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = \\ -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(A_h\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right) \tag{3b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \\ -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left(A_h\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial w}{\partial z}\right) - g\frac{\rho}{\rho_0} \tag{3c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \tag{3d}$$

Recap: shallow water equations

What is the simplest approximated solution high-school student can solve?

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = \\ - g\frac{\partial \eta}{\partial x} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y}\right)\right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h}\end{aligned} \tag{4a}$$

$$\begin{aligned}\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = \\ - g\frac{\partial \eta}{\partial y} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right)\right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h}\end{aligned} \tag{4b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \tag{4c}$$

Recap: shallow water equations

What is the simplest approximated solution high-school student can solve?

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = - g\frac{\partial \eta}{\partial x} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y}\right)\right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \tag{5a}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = - g\frac{\partial \eta}{\partial y} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right)\right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \tag{5b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \tag{5c}$$

Inertial motion in fixed coordinate

This is not trivial and has physical meaning (uniform linear motion)!

$$\frac{d\bar{u}}{dt} = 0 \tag{6a}$$

$$\frac{d\bar{v}}{dt} = 0 \tag{6b}$$

$$\bar{u}|_{t=0} = U_0, \quad \bar{v}|_{t=0} = V_0 \tag{6c}$$

Solutions to the equations are given by

$$\bar{u} = U_0, \quad \bar{v} = V_0 \tag{7}$$

$\bar{u}$ and $\bar{v}$ can be considered as velocity components of a water parcel, so equations for its position are $dX/dt = \bar{u}$ and $dY/dt = \bar{v}$.

$$X = U_0 t + X_0, \quad Y = V_0 t + Y_0 \tag{8}$$

Inertial motion in rotating coordinate

What if we consider Coriolis force (rotation of coordinate)?

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = \\ - g\frac{\partial \eta}{\partial x} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y}\right)\right) + \frac{\tau_x^s}{\rho_0 h} - \frac{\tau_x^b}{\rho_0 h} \tag{9a}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = \\ - g\frac{\partial \eta}{\partial y} + \frac{A_h}{h}\left(\frac{\partial}{\partial x}\left(h\frac{\partial \bar{v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right)\right) + \frac{\tau_y^s}{\rho_0 h} - \frac{\tau_y^b}{\rho_0 h} \tag{9b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((\eta + h)\bar{u})}{\partial x} + \frac{\partial((\eta + h)\bar{v})}{\partial y} = 0 \tag{9c}$$

Inertial motion in rotating coordinate

$$\frac{d\bar{u}}{dt} - f\bar{v} = 0 \tag{10a}$$

$$\frac{d\bar{v}}{dt} + f\bar{u} = 0 \tag{10b}$$

$$\bar{u}\big|_{t=0} = U_0, \quad \bar{v}\big|_{t=0} = V_0 \tag{10c}$$

Solution to the equations are given by

$$\bar{u} = V_0 \sin(f\,t) + U_0 \cos(f\,t) \tag{11a}$$

$$\bar{v} = V_0 \cos(f\,t) - U_0 \sin(f\,t) \tag{11b}$$

Oscillation with frequency f (period $2\pi/f$)

This is what we call "inertial oscillation".

Inertial motion in rotating coordinate

Governing equations for the position of water mass are

$$\frac{dX}{dt} = \bar{u} \tag{12a}$$

$$\frac{dY}{dt} = \bar{v} \tag{12b}$$

$$X|_{t=0} = X_0 \qquad Y|_{t=0} = Y_0 \tag{12c}$$

and the solutions are given by

$$X = \frac{U_0}{f}\sin(ft) - \frac{V_0}{f}\cos(ft) + X_0 + \frac{V_0}{f} \tag{13a}$$

$$Y = \frac{V_0}{f}\sin(ft) + \frac{U_0}{f}\cos(ft) + Y_0 - \frac{U_0}{f} \tag{13b}$$

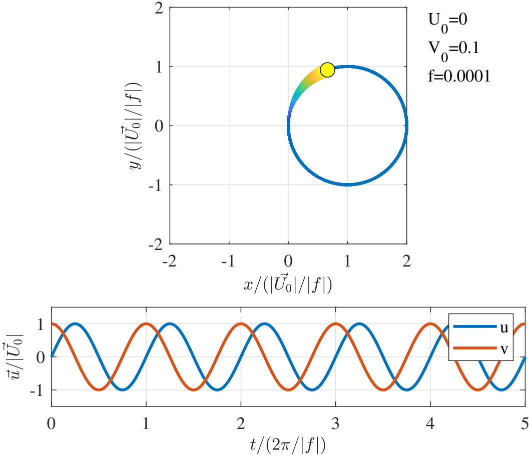
Inertial oscillation

that can be rewritten as

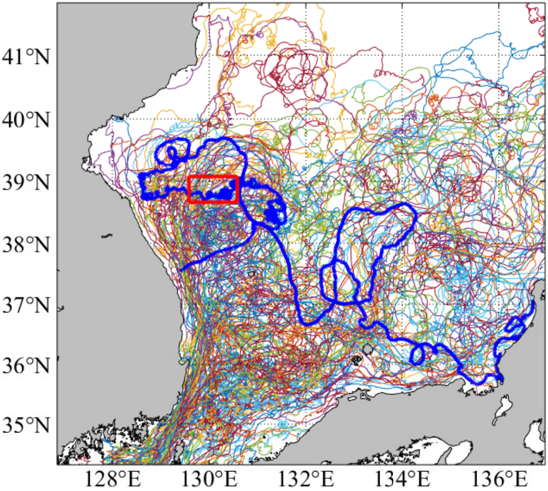
$$\left(X-\left(X_0+\frac{V_0}{f}\right)\right)^2+\left(X-\left(Y_0-\frac{U_0}{f}\right)\right)^2=\left(\underbrace{\boxed{\frac{\sqrt{({U_0}^2+{V_0}^2)}}{f}}}_{\text{Radius of water mass trajectory}}\right)^2 . \tag{14}$$

This is equation of a circle!

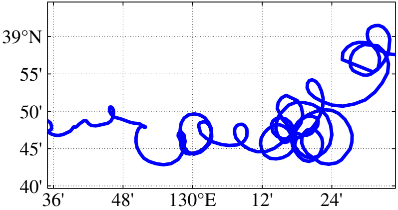
Inertial oscillation

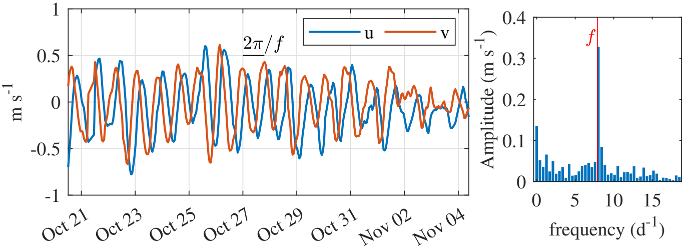


Inertial oscillation



Inertial oscillation





https://jmlilly.net/talks/lilly16-osu.html#63

Summary

In rotating coordinate, not uniform linear motion but inerial oscillation.

Governing equation:

$$\frac{d\vec{u}}{\partial t} + if\vec{u} = 0 \tag{15a}$$

$$\vec{u}\big|_{t=0} = \vec{U}_0 \tag{15b}$$

Solution:

$$\vec{u} = \vec{U}_0 \, \boxed{e^{-ift}} \tag{16}$$

Oscillation with frequency f (period $2\pi/f$)

that yields circle trajectory of which radius is $|\vec{U}_0|/f$.

Assignment

Let us consider one more term, bottom stress (friction; $\tau^b/(\rho_0 h)$ in (5)). The governing equations are given by

$$\frac{d\bar{u}}{dt} - f\bar{v} = -\frac{\tau_x^b}{\rho_0 h} = -\frac{\gamma}{h}\bar{u} \tag{17a}$$

$$\frac{d\bar{v}}{dt} + f\bar{u} = -\frac{\tau_y^b}{\rho_0 h} = -\frac{\gamma}{h}\bar{v} \tag{17b}$$

$$\bar{u}\big|_{t=0} = U_0, \quad \bar{v}\big|_{t=0} = V_0 \tag{17c}$$

where bottom stress is modeled by linear friction bottom boundary condition and γ is linear friction coefficient, that is a constant.

1. Find two important time scales governing the equations.
2. Analytically solve above equations.
3. Under presence of friction, What kind of condition is required for inertial oscillations to be observed?