

Differential Equations in Geophysical Fluid Dynamics

XVIII. Geostrophic adjustment

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Recap

Governing equations of the Poincaré wave are given by

$$\frac{\partial \bar{u}}{\partial t} - fv = -g \frac{\partial \eta}{\partial x} \quad (1a)$$

$$\frac{\partial \bar{v}}{\partial t} + fu = 0 \quad (1b)$$

$$\boxed{\frac{\partial^3 \eta}{\partial t^3} - gh \frac{\partial}{\partial t} \frac{\partial^2 \eta}{\partial x^2} + f^2 \frac{\partial \eta}{\partial t} = 0}$$

$$\frac{\partial \eta}{\partial t} + h \frac{\partial \bar{u}}{\partial x} = 0 \quad (1c) \quad (2)$$

Using $\eta = \eta_0 e^{i(kx-wt)}$, dispersion relation of the Poincaré wave can be obtained:

$$w(w^2 - ghk^2 - f^2) = 0$$

$$w = \pm \sqrt{ghk^2 + f^2}, \quad \left(c = \frac{w}{k} = \pm \sqrt{gh + (f/k)^2} \right) \quad (3)$$

wave velocity c depends on k !

“Dispersive wave”

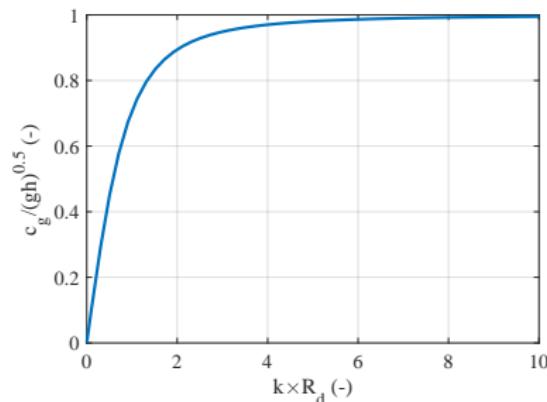
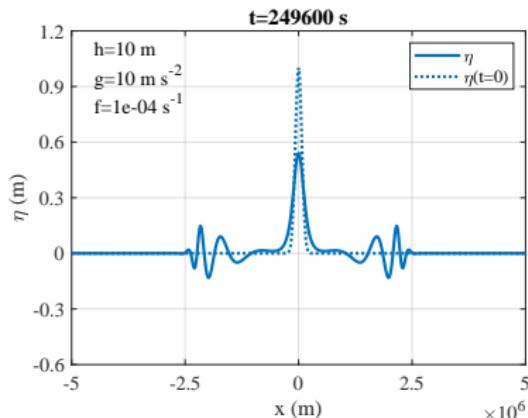
Recap

Note that what we see is not a single wave but wave groups whose speed is defined and given by

$$c_g = \frac{\partial w}{\partial k} = \pm \frac{ghk}{\sqrt{ghk^2 + f^2}} = \pm \sqrt{gh} \frac{k^2}{k^2 + (1/Rd)^2} \quad (4)$$

where $Rd = \sqrt{gh}/f$ referred to as the “Rossby radius of deformation”, and energy is transported by the wave group:

$$\frac{\partial E}{\partial t} + c_g \frac{\partial E}{\partial x} = 0 \quad (5)$$



Governing equation

Let us consider linear inviscid shallow water equation over f -plane, that is identical to the equations for the Poincaré wave:

$$\frac{\partial^3 \eta}{\partial t^3} - gh \frac{\partial}{\partial t} \frac{\partial^2 \eta}{\partial x^2} + f^2 \frac{\partial \eta}{\partial t} = 0, \quad (6)$$

$\left(w (w^2 - g h k^2 - f^2 = 0) \right)$

Note that $w^2 - g h k^2 - f^2 = 0$ "or" $w = 0$

Be aware that all terms in (6) are differentiated, so it can be integrated in time:

$$\frac{\partial^2 \eta}{\partial t^2} - gh \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta = \left(\frac{\partial^2 \eta}{\partial t^2} - gh \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta \right) \Big|_{t=0} \quad (7)$$

\uparrow
 $Ro = 1/(fT) \ll 1$

Governing equation

Therefore, the governing equation of geostrophic adjustment (Rossby, 1938) is given by

$$-gh \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta = \left(-gh \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta \right) \Big|_{t=0} \quad (8)$$

Let us consider one of the simplest initial condition (Gill et al., 1986):

$$\eta|_{t=0} = \eta_0 \operatorname{sgn}(x) = \begin{cases} -\eta_0, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \eta_0, & \text{if } x > 0 \end{cases} \quad (9)$$

with boundary conditions:

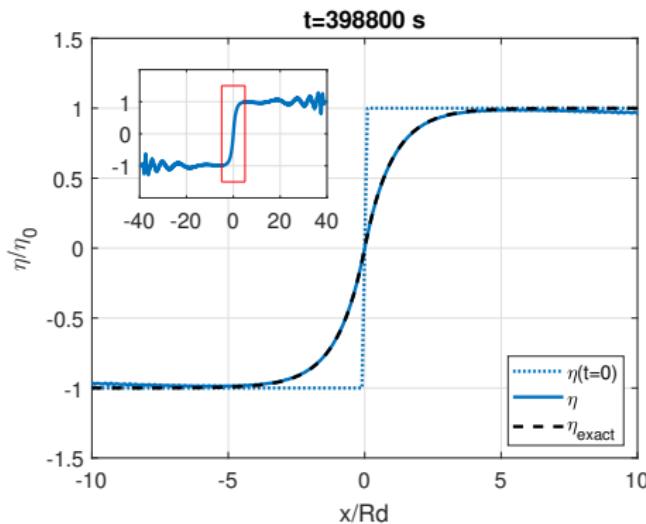
$$\lim_{x \rightarrow \infty} \eta = \eta_0 \quad (10a)$$

$$\lim_{x \rightarrow -\infty} \eta = -\eta_0 \quad (10b)$$

Geostrophic adjustment

Solution to the problem is given by

$$\eta = \begin{cases} \eta_0(e^{x/Rd} - 1), & \text{if } x < 0 \\ \eta_0(1 - e^{-x/Rd}) & \text{if } x > 0 \end{cases} \quad (11)$$



Geostrophic adjustment

Let us consider another, further simple, initial and boundary conditions (Choi and Kim, 2024):

$$\eta|_{t=0} = \eta_0 e^{-x/Rd}, \quad \bar{u}|_{t=0} = 0, \quad \bar{v}|_{t=0} = 0 \quad (12a)$$

$$\bar{u}|_{x=0} = 0 \quad \left(\leftrightarrow \frac{\partial \eta}{\partial x} \Big|_{x=0} = 0 \right), \quad \lim_{x \rightarrow \infty} \eta = 0 \quad (12b)$$

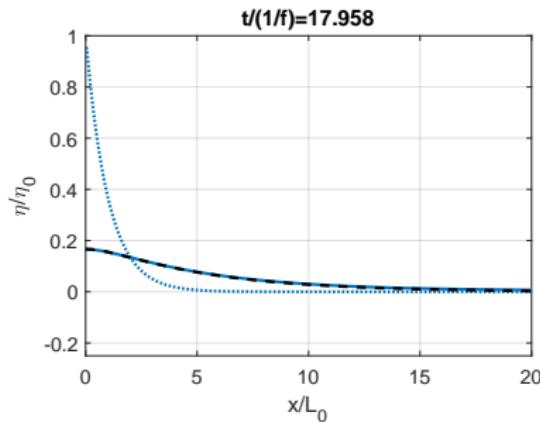
Solution to the problem is given by

$$\eta = \eta_0 \left(\frac{Rd/L_0}{(Rd/L_0)^2 - 1} e^{-x/Rd} - \frac{1}{(Rd/L_0)^2 - 1} e^{-x/L_0} \right) \quad (13)$$

We can consider two limit: large ($Rd/L_0 \ll 1$ so $Rd/L_0 \rightarrow 0$) and small ($1 \ll Rd/L_0$ so $Rd/L_0 \rightarrow \infty$) initial bump, relative to Rd .

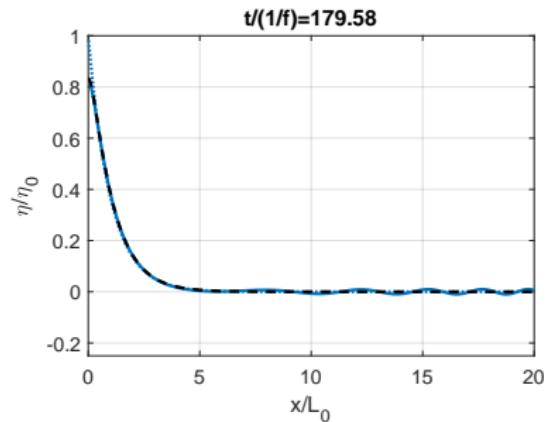
Geostrophic adjustment

Small bump ($L_0 = Rd/5$)



https://jang-geun.github.io/vis_geo_adj_small.gif

Large bump ($L_0 = 5Rd$)



https://jang-geun.github.io/vis_geo_adj_large.gif

Further reading

The geostrophic adjustment problem can be generalized to

$$-gh \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta = \left(-gh \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta \right) \Big|_{t=0} = q_0(x) \quad (14)$$

where q_0 is given arbitrary function in x . If $q_0(x) \approx f^2 \eta|_{t=0}$, solution to the problem is given by

$$\eta = \int_{-\infty}^{\infty} G(\tilde{x} - x) \eta|_{t=0} d\tilde{x} \quad (15)$$

where

$$G(x^*) = \frac{1}{2Rd} e^{-|x^*|/Rd} \quad (16)$$

that was used to discuss energetics of the geotrophic adjustment (Middleton, 1987).

Conclusion

1. High wavenumber modes, whose wave length is smaller than Rd , mostly propagates as the Poincaré wave.
2. Low wavenumber modes remains without propagation as geostrophic adjustment.

References I

-  Choi, Jang-Geun and Deoksu Kim (2024). 해양학을 위한 지구 물리 유체 역학. BOOKK.
-  Gill, AE et al. (1986). "Rossby adjustment over a step". In: *Journal of Marine Research*.
-  Middleton, John F (1987). "Energetics of linear geostrophic adjustment". In: *Journal of physical oceanography* 17.6, pp. 735–740.
-  Rossby, Carl-Gustav (1938). "On the mutual adjustment of pressure and velocity distributions in certain simple current systems, II". In: *Journal of Marine Research*.