

Ecosystem modeling

I. The most basic equations for biology: exponential growth and decay

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Introduction

Modeling

Any activity to produce model, a simplified representation of a real system.

Mathematics is the language of science! The equations are perfect tool to describe the nature.

We will talk about:

- ▶ How to formulate ecosystem (building equation)
- ▶ How to use the formulation (using/solving equation)

Governing equation

$$\begin{array}{c} \text{Temporal change of subject } P \\ \downarrow \\ \boxed{\frac{dP}{dt}} = \sum \boxed{S_o} - \sum \boxed{S_i} \\ \begin{array}{cc} \text{Source} & \text{Sink} \end{array} \end{array} \quad (1)$$

representing that the temporal change of a subject (e.g., phytoplankton) is sum of source (e.g., growth) minus sum of sink S_i (e.g., mortality).

Good a priori: to assume that the source and sink are proportional to the amount of (chemical or biological) substances:

$$\sum S_o = UP, \quad \sum S_i = \sigma P \quad (2)$$

where U (growth rate) and σ (mortality rate) are the proportional coefficients.

Governing equation

If both coefficients are constant, (1) becomes

$$\boxed{\frac{dP}{dt} = U_{net}P} \quad (3)$$

where $U_{net} = U - \sigma$ representing “net” growth rate.

Imagine phytoplankton that double in a certain time T_d (doubling time), that can be formulated by

$$P(t) = P_0 2^{t/T_d} = P_0 \left(e^{\ln 2} \right)^{t/T_d} = P_0 e^{U_{net}t} \quad (4)$$

where P_0 is initial concentration and $U_{net} = (\ln 2)/T_d$ ¹. This is the solution to (3).

¹This is relationship between the net growth rate and doubling time. 

Numerical solution using finite difference method

Solving equations based on repeated calculations using computers.

$$\frac{dP}{dt} = \lim_{\Delta t \rightarrow 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} \approx \frac{P^{n+1} - P^n}{\Delta t} \quad (5)$$

Therefore, the governing equation can be approximated into following form:

$$\frac{dP}{dt} = U_{net}P \quad \rightarrow \quad \frac{P^{n+1} - P^n}{\Delta t} = U_{net}P^n$$

$$\therefore P^{n+1} = P^n + \Delta t U_{net} P^n \quad (6)$$

So, we can compute P in next step (P^{n+1}) using previously known information (P^n). And then, it can be repeated to get the next step P^{n+2} using the obtained P^{n+1} .

Lab 1

Numerically solve

$$\frac{dP}{dt} = U_{net}P \quad (7)$$

using the discretized equation $P^{n+1} = P^n + \Delta t U_{net} P^n$ with $U_{net} = 1 \text{ d}^{-1}$, $P_0 = P^1 = 2.0 \text{ cell L}^{-1}$, and $\Delta t = 2.0 \text{ d}^{-1}$.

1. Simulate time series of P from $t = 0$ to $t = 10 \text{ d}$.
2. Compare the numerical solution with the analytical solution (4).
3. Conduct sensitivity experiments for Δt . Try $U_{net} < 0$ with a long time step Δt . Model will behave incorrectly!

Lab 2

In most cases, the coefficients (U and σ) are not constant. We can model that the mortality rate is proportional to P , so $\sigma = \sigma' P$. Substituting it into (3) yields

$$\frac{dP}{dt} = U_{net}P - \sigma' P^2. \quad (8)$$

1. Numerically solve (8).
2. Compare it with the solution of exponential equation.