Differential Equations in Geophysical Fluid Dynamics

IV. Generalization of forcing term

Jang-Geun Choi

Center for Ocean Engineering
University of New Hampshire

Apr, 2025

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography commuinity COKOAA.

Recap

Wind-forced linear momentum equation:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + if\vec{u}} = \frac{\vec{\tau}^s}{\rho_0 h} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t} \tag{1}$$

$\left(\frac{1}{fT}\right)\frac{\partial \vec{u}^*}{\partial t^*} + i\vec{u}^*$ where $1/(fT) \equiv Ro_T$ is the temporal Rossby number.

Solution to the problem is given by

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i (f - w_0)}. \tag{2}$$

Recap

$$\frac{\partial \vec{u}}{\partial t} + \boxed{if\vec{u} = \frac{\vec{\tau}^s}{\rho_0 h}} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}$$

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i(f - w_0)}$$

1. case $w_0 \ll f$ (low frequency or short period forcing)

$$\vec{u} \approx \frac{\vec{\tau}^s}{i\rho_0 hf} \equiv \vec{u}_e \tag{3}$$

that satisfies

$$\boxed{if\vec{u} \approx \frac{\vec{\tau}^s}{\rho_0 h}}. \tag{4}$$

Negligible inertia

Recap

$$\boxed{\frac{\partial \vec{u}}{\partial t}} + if\vec{u} \boxed{= \frac{\vec{\tau}^s}{\rho_0 h}} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}$$

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i (f - w_0)}$$

2. case $w_0 \gg f$ (high frequency or long period forcing)

$$\vec{u} \approx \frac{\vec{\tau}^s}{(-iw_0)\rho_0 h} \tag{5}$$

that satisfies

$$\boxed{\frac{\partial \vec{u}}{\partial t} \approx \frac{\vec{\tau}^s}{\rho_0 h}}. \tag{6}$$

Negligible Coriolis force

Recap

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}^s}{\rho_0 h} = \frac{\hat{\tau}_0}{\rho_0 h} e^{-ift}$$

$$\vec{u} = \frac{\hat{\tau}_0 e^{-iw_0 t}}{\rho_0 h i(\boxed{f - w_0})}$$

As $w_0 \to f$,
magnitude of $\vec{u}$ increases!

3. case $w_0 \approx f$ (frequency close to the system's natural frequency)

$$\vec{u} = \frac{\hat{\tau}_0}{\rho_0 h} \boxed{t}\, e^{-ift} \tag{7}$$

Amplitude linearly increase with time!

that is what we call "resonance".

Introduction

Why have so many people used sinusoidal (or monochromatic) forcing?

1. Stommel's (1948) wind driven circulation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + A\frac{\partial\psi}{\partial x} = B\sin(k_0 y) \tag{8}$$

2. Cushman-Roisin's (2011) upwelling problem

$$\frac{\partial^3\psi}{\partial t^3} - A\frac{\partial^3\psi}{\partial t\partial x^2} + B\frac{\partial\psi}{\partial t} = 0, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = C\cos(w_0 t) \tag{9}$$

3. Csanady's (1978) steady coastal trapped wave problem

$$-A\frac{\partial\psi}{\partial y} = \frac{\partial^2\psi}{\partial x^2}, \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = B\cos(k_0 y) \tag{10}$$

Superposition principle:
Key characteristics of linear differential equations

Problem with additional sinusoidal forcing term

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u}}_{\equiv L[\vec{u}] \text{ where } L = \frac{\partial}{\partial t} + (if + \frac{\gamma}{h})} = \overbrace{\frac{\hat{\tau}_0}{\rho_0 h} e^{-iw_0 t}}^{\equiv f_0} + \overbrace{\frac{\hat{\tau}_1}{\rho_0 h} e^{-iw_1 t}}^{\equiv f_1} \tag{11}$$

Superposition principle of linear non-homogeneous differential equation

1. Once $L[u_0] = f_0$ and $L[u_1] = f_1$,
   $L[c_1 u_1 + c_2 u_2] = c_1 f_0 + c_2 f_1$.
2. Therefore, once $L[u_0] = f_0$, $L[u_0 + C u_1] = f_0$ where
   $L[u_1] = 0$ (case of $f_1 = 0$).

Generalization to problem with arbitrary forcing term

Arbitrary function of t
(any time series of wind stress)

$$\frac{\partial \vec{u}}{\partial t} + \left(if + \frac{\gamma}{h}\right)\vec{u} = \frac{\vec{\tau}^s(t)}{\rho_0 h} \equiv \frac{1}{\rho_0 h}\sum_{n=-\infty}^{\infty} \hat{\tau}_n e^{-iw_n t} \tag{12}$$

Fourier series

Note that

$$\hat{\tau}(w) = \mathcal{F}(\vec{\tau}(t)) \tag{13}$$

where $\mathcal{F}$ indicates Fourier operator. Particular solution to the problem is

Transfer function ($\equiv \hat{g}(w_n)$)

$$\vec{u} = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{\rho_0 h(i(f - w_n) + \gamma/h)}\hat{\tau}_n}_{\hat{u}(w_n)} e^{-iw_n t}. \tag{14}$$

Generalization to problem with arbitrary forcing term

Let us consider problem without initial condition ($\lim_{t\to-\infty}\vec{u}=0$)

$$\vec{u}=\frac{1}{\rho_0 h}\boxed{\sum_{n=-\infty}^{\infty}\frac{1}{(i(f-w_n)+\gamma/h)}\hat{\tau}_n e^{-iw_n t}}. \tag{15}$$

Corresponding solution (Lentz and Winant, 1986; Austin and Barth, 2002) obtained by the integrating factor is given by

$$\vec{u}=\frac{1}{\rho_0 h}\boxed{\int_{-\infty}^{t} g(\tilde{t}-t)\vec{\tau}(\tilde{t})\,d\tilde{t}}. \tag{16}$$

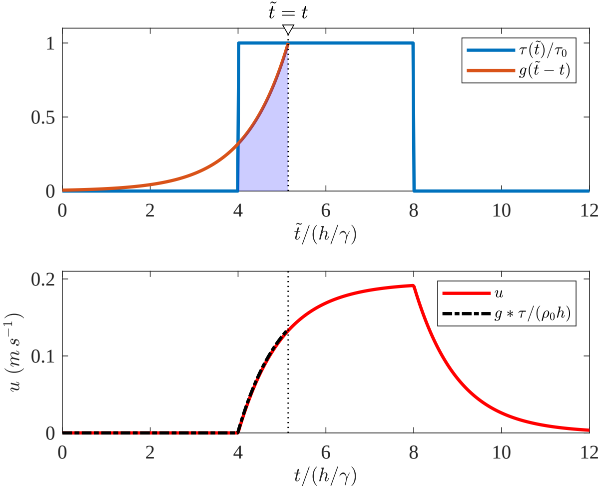
$\equiv g*\vec{\tau}$ (Convolution!)

where $g(t^{\star})=e^{(if+\gamma/h)t^{\star}}$.

If solution is unique, they must be same. This is end up with "**convolution theorem**":

$$\hat{g}(w)\hat{\tau}(w)=\mathcal{F}(g(t)*\vec{\tau}(t)) \tag{17}$$

Generalization to problem with arbitrary forcing term



https://jang-geun.github.io/vis_convolution.gif

The slab model

Pollard and Millard (1970):

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}}{\rho_0 h_1} - \gamma^{\star}\vec{u} \tag{18}$$

where $\gamma^{\star}$ represents linear damping coefficient, that can be determined by calibration.

This is a theoretical model that can still be used for publication (Whitt and Thomas, 2015; Gough et al., 2016; Wang et al., 2019; Zhang et al., 2023)!

Summary

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = \frac{\vec{\tau}(t)}{\rho_0 h_1} - \gamma^\star \vec{u}$$

where $\vec{\tau}(t)$ is an arbitrary function of t. Solution to the problem is given by

$$\begin{aligned}\vec{u} &= \frac{1}{\rho_0 h} \sum_{n=-\infty}^{\infty} \frac{1}{(i(f - w_n) + \gamma/h)} \hat{\tau}_n e^{-iw_n t} \\ &= \frac{1}{\rho_0 h} \int_{-\infty}^{t} g(\tilde{t} - t)\vec{\tau}(\tilde{t})\, d\tilde{t} \,.\end{aligned} \tag{19}$$

where $g(t^\star) = e^{(if+\gamma/h)t^\star}$. This implies convolution theorem. The contribution of past wind stress to the present current decays exponentially over time by friction term.

Assignment

$$\frac{\partial \vec{u}}{\partial t} + if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} - \frac{\gamma}{h}\vec{u} \tag{20a}$$

$$\vec{u}|_{t=0} = \vec{U}_0 \tag{20b}$$

$$\frac{d\vec{X}}{dt} = \vec{u} \tag{21}$$

where $-g\partial\eta/\partial\vec{n}$ is arbitrary constants and $\vec{X} = X + iY$. X and Y represent x- and y-position of an object, respectively.

1. Solve differential equation (23) for $\vec{u}$. What is physical meaning of particular solution component?
2. Solve (24) using $\vec{u}$ from (23) and constant f assumption.

References I

- Austin, J. A. and J. A. Barth (2002). "Variation in the position of the upwelling front on the Oregon shelf". In: *Journal of Geophysical Research: Oceans* 107.C11, pp. 1–15.
- Gough, M. K. et al. (2016). "Resonant near-surface inertial oscillations in the northeastern Gulf of Mexico". In: *Journal of Geophysical Research: Oceans* 121.4, pp. 2163–2182.
- Lentz, S. J. and C. D. Winant (1986). "Subinertial currents on the southern Califoinia shelf". In: *Journal of Physical Oceanography* 16.11, pp. 1737–1750.
- Pollard, R. T. and R. C. Millard (1970). "Comparison between observed and simulated wind-generated inertial oscillations". In: *Deep Sea Research and Oceanographic Abstracts*. Vol. 17. 4. Elsevier, pp. 813–821.
- Wang, P. et al. (2019). "Modulation of near-inertial oscillations by low-frequency current variations on the inner scotian shelf". In: *Journal of Physical Oceanography* 49.2, pp. 329–352.

References II

Whitt, D. B. and L. N. Thomas (2015). "Resonant generation and energetics of wind-forced near-inertial motions in a geostrophic flow". In: *Journal of Physical Oceanography* 45.1, pp. 181–208.

Zhang, Y. et al. (2023). "Spatial and seasonal variations of near-inertial kinetic energy in the upper Ross Sea and the controlling factors". In: *Frontiers in Marine Science* 10, p. 1173900.