Differential Equations in Geophysical Fluid Dynamics

I. Governing equations, scale analysis, and approximations

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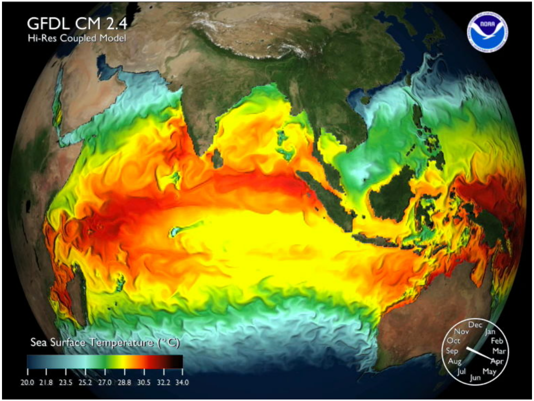
Introduction

What is Geophysical Fluid Dynamics (GFD)?

Fluid dynamics in large scale where rotation of world becomes matter.



https://www.gfdl.noaa.gov/hires_indian_sst-2/

Governing equation

Why are differential equations so important?

Newton's second law

$$\frac{d\vec{u}}{dt} = \sum_i \vec{F}_i \tag{1}$$

that is the origin of (almost) all classic dynamics and already differential equation.

Navier-Stokes equations

Inertia Advection "Coriolis force"

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right) \tag{2a}$$

$$\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial (vw)}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(A_h \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right) \tag{2b}$$

Diffusion (eddy viscosity) Gravity

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial (ww)}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left(A_h \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial w}{\partial z}\right) - g \tag{2c}$$

Pressure gradient

and mass conservation (continuity equation)

Before that, a bit more about **Coriolis force**:
https://www.youtube.com/watch?v=nMPXBAYsWvs

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{3}$$

Scale analysis

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right)$$

Let us define

Constant "order" (including unit)

$$x = L\, x^*,\ y = L\, y^*,\ z = H\, z^*,\ t = T\, t^*,$$
$$u = U\, u^*,\ v = U\, v^*,\ w = W\, w^*,\ P = p\, P^* \tag{4}$$

Nondimensionalized variable

Scale analysis

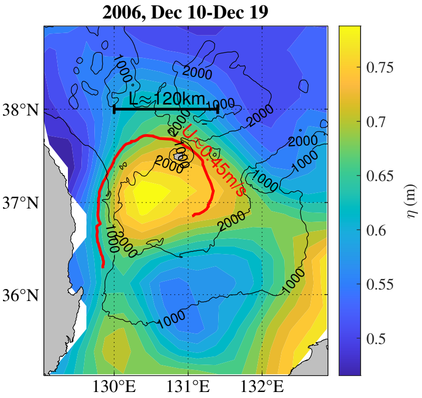
Order (rough size) of each term!

$$
\begin{aligned}
\left(\frac{U}{T}\right)\frac{\partial u^*}{\partial t^*} + \left(\frac{U^2}{L}\right)\left(\frac{\partial(u^*u^*)}{\partial x^*} + \frac{\partial(u^*v^*)}{\partial y^*} + \frac{\partial(u^*w^*)}{\partial z^*}\right) = \\
- \left(\frac{p}{\rho L}\right)\frac{\partial P^*}{\partial x^*} - (fU)\, v^* \\
+ \left(\frac{A_h U}{L^2}\right)\left[\frac{\partial}{\partial x^*}\left(A_h^* \frac{\partial u^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(A_h^* \frac{\partial u^*}{\partial y^*}\right)\right] \\
+ \left(\frac{A_z U}{H^2}\right)\frac{\partial}{\partial z^*}\left(A_z^* \frac{\partial u^*}{\partial z^*}\right)
\end{aligned}
\tag{5}
$$

Scale analysis: application

Ulleung Eddy in the East Sea

Let us assume pressure gradient is important, and what is the most significant term balancing the pressure gradient.



General values for the parameters:

$$A_h \approx 10^2\, m^2\, s^{-1},\ A_z \approx 1\, m^2\, s^{-1},\ \text{and}\ f \approx 10^{-4}\, s^{-1}.$$

Scale analysis: application

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right)$$

$$\downarrow$$

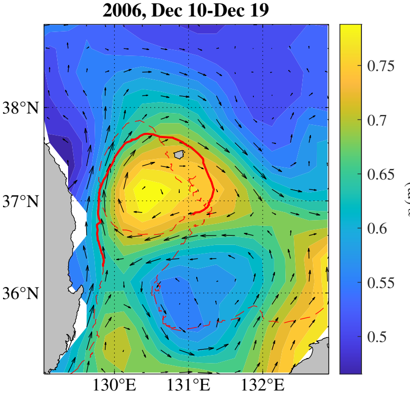
$$-fv = -\frac{1}{\boxed{\rho_0}}\frac{\partial P}{\partial x} \tag{6}$$

Assume constant density $\rho_0 = 1025\,kg\,m^{-3}$
(that is Boussinesq approximation,
will be discussed next time.)

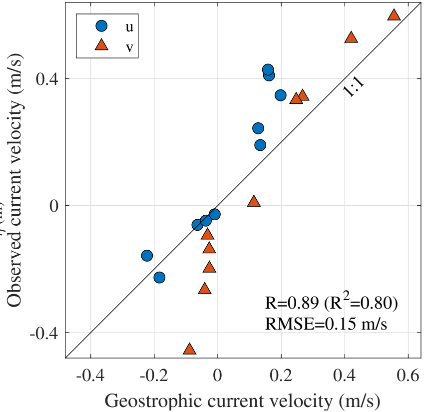
that is what we call "geostrophic balance". But is it too simple to consider complexity of the nature (and original equations)?

Scale analysis: application

Nope, at least locally when our choice of the scales ("assumptions") are correct.



(a)



(b)

Nondimensionalization

Dividing (5) by order of the term we interested (e.g., Coriolis; fU) yields

$$
\overbrace{\boxed{\left(\frac{1}{fT}\right)}}^{Ro_T\text{: temporal Rossby number}}\frac{\partial u^*}{\partial t^*} + \overbrace{\boxed{\left(\frac{U}{fL}\right)}}^{Ro\text{: Rossby number}}\left(\frac{\partial(u^*u^*)}{\partial x^*} + \frac{\partial(u^*v^*)}{\partial y^*} + \frac{\partial(u^*w^*)}{\partial z^*}\right) =
$$

$$
- \boxed{\left(\frac{p}{\rho L f U}\right)}\frac{\partial P^*}{\partial x^*} - v^*
$$

$$
+ \underbrace{\boxed{\left(\frac{A_h}{fL^2}\right)}}_{Ek_h\text{:Lateral Ekman number}}\left[\frac{\partial}{\partial x^*}\left(A_h^*\frac{\partial u^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(A_h^*\frac{\partial u^*}{\partial y^*}\right)\right]
$$

$$
+ \underbrace{\boxed{\left(\frac{A_z}{fH^2}\right)}}_{Ek\text{:(Vertical) Ekman number}}\frac{\partial}{\partial z^*}\left(A_z^*\frac{\partial u^*}{\partial z^*}\right) \tag{7}
$$

Blue indicate order relative to the other term's order, what we call nondimensional number.

Summary

1. Scale analysis, applicable to any equations, is easy and **powerful**.
2. **Stupidly simplified equations work** unless the assumptions are valid.
3. Further profound discussions about the scale analysis are in Price (2005; available at `https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2024/10/DA_2024_A2.pdf`).

Assignment

In the continuity equation (3), density ρ can be decomposed into constant component ρ_0, so $\rho = \rho_0 + \rho'$. Substituting $\rho = \rho_0 + \rho'$ into (3) and then, conduct scale analysis and obtain nondimensionalized version of the equation.

1. What is the nondimensional number controlling the equation?
2. In general, $\rho_0 \approx 1025 kg\, m^{-3}$ and the order of ρ' is less than $10\, kg\, m^{-3}$. Is the constant density assumption valid in (3)?