# Differential Equations in Geophysical Fluid Dynamics

## XI. Advection-diffusion-reaction equation

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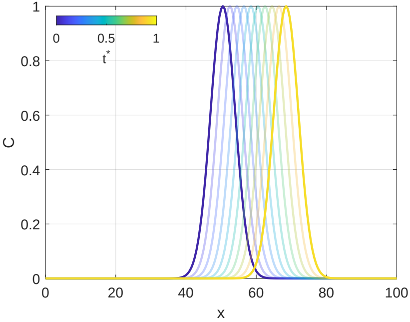
# Recap

Now, we know two partial differential equations:

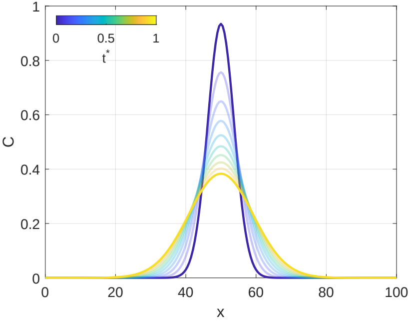
**Advection equation**

$$\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} = 0 \qquad (1)$$




**Diffusion equation**

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(A\frac{\partial C}{\partial x}\right) \qquad (2)$$



# Advection-diffusion equation

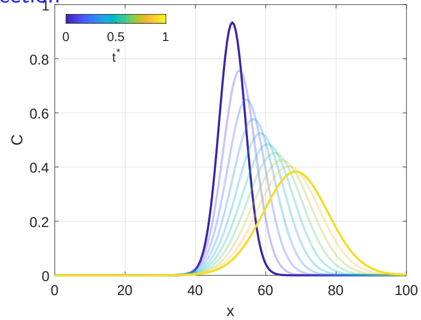
So, we know advection-diffusion equation, that governs transport of almost everything!

**Advection-diffusion equation**

Diffusion

$$\frac{\partial C}{\partial t} + \boxed{\frac{\partial (uC)}{\partial x}} = \boxed{\frac{\partial}{\partial x}\left(A\frac{\partial C}{\partial x}\right)} \tag{3}$$

Advection



# Random-walk and diffusion

We talked Eulerian and Lagrangian descriptions of advection:

**Eulerian**

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0 \qquad (4)$$

**Lagrangian**

$$\frac{dX}{dt} = u, \quad \frac{dC}{dt} = 0 \qquad (5)$$

How about those of diffusion?

**Eulerian**

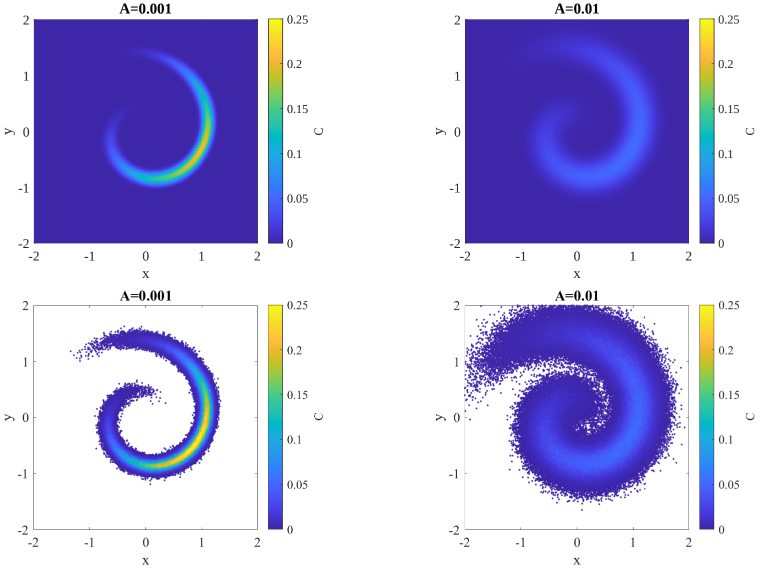
$$\frac{\partial C}{\partial t} = A\frac{\partial^2 C}{\partial x^2} \qquad (6)$$

**Lagrangian**

$$dX = \sqrt{2A}dW$$

$$X^{n+1} = X^n + \sqrt{2A\Delta t}\,N(0,1) \qquad (7)$$

# Advection-diffusion-reaction equation




https://jang-geun.github.io/vis_geo_adv_diff_1.gif

# Advection-diffusion-reaction equation

Governing equation (model) for radioactive decay is given by

$$\frac{dC}{dt} = -aC \tag{8}$$

where $a$ is decay rate. **How do we couple this chemical model to hydrodynamics model?**

Just add advection and diffusion terms!

$$\frac{\partial C}{\partial t} + \underbrace{\boxed{\nabla \cdot (\vec{u}C)}}_{\frac{\partial (uC)}{\partial x} + \frac{\partial (vC)}{\partial y} + \frac{\partial (wC)}{\partial z}} = \underbrace{\boxed{\nabla \cdot A \nabla C}}_{A_h \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{\partial}{\partial z} \left( A_z \frac{\partial C}{\partial z} \right)} - aC \tag{9}$$

# Advection-diffusion-reaction equation

Tons of applications...

Stock et al., 2005; He et al., 2008; Lee et al., 2024; Kim et al., 2016; Shin et al., 2017; Choi et al., 2018; Cheng et al., 2021; Kampouris et al., 2021; Choi et al., 2023; Choi et al., 2025;...

# Assignment

Consider ageophysical $(1 \ll Ro_T)$ linear wind-driven current problem given by

$$\frac{\partial u}{\partial t} = A_z \frac{\partial^2 u}{\partial z^2} \tag{10a}$$

$$A_z \frac{\partial u}{\partial z}\bigg|_{z=0} = \frac{\tau_x^s}{\rho_0} \tag{10b}$$

$$u = 0|_{z=-h} \tag{10c}$$

1. Find steady-state solution $\tilde{u}$ of the problem.
2. Based on the superposition principle, non-steady velocity component, defined as $u = \tilde{u} + u'$, can be decomposed. Find governing equation for $u'$ and solve it with initial condition $u|_{t=0} = f(z)$.
3. If free-slip bottom boundary condition, $\partial u/\partial z|_{z=-h} = 0$, is used instead of (10c), is there steady-state solution?

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