# Differential Equations in Geophysical Fluid Dynamics

## I. Governing equations, scale analysis, and approximations

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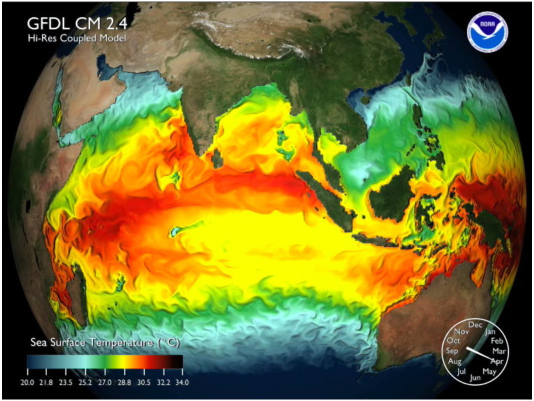
# Introduction

What is Geophysical Fluid Dynamics (GFD)?

Fluid dynamics in large scale where rotation of world becomes matter.




https://www.gfdl.noaa.gov/hires_indian_sst-2/

# Governing equation

Why are differential equations so important?

## Newton's second law

$$\frac{d\vec{u}}{dt} = \sum_i \vec{F}_i \tag{1}$$

that is the origin of (almost) all classic dynamics and already differential equation.

# Navier-Stokes equations

Inertia Advection "Coriolis force"

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right) \tag{2a}$$

$$\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial (vw)}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(A_h \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right) \tag{2b}$$

Diffusion (eddy viscosity) Gravity

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial (ww)}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left(A_h \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial w}{\partial z}\right) - g \tag{2c}$$

Pressure gradient

# and mass conservation (continuity equation)

Before that, a bit more about **Coriolis force**:
https://www.youtube.com/watch?v=nMPXBAYsWvs

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{3}$$

# Scale analysis

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = - \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right)$$

Let us define

Constant "order" (including unit)

$$x = L\, x^*,\ y = L\, y^*,\ z = H\, z^*,\ t = T\, t^*,$$
$$u = U\, u^*,\ v = U\, v^*,\ w = W\, w^*,\ P = p\, P^* \tag{4}$$

Nondimensionalized variable

# Scale analysis

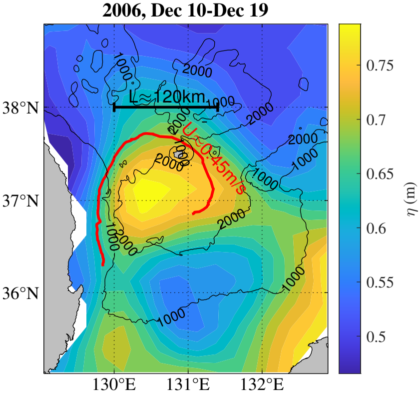
Order (rough size) of each term!

$$\left(\frac{U}{T}\right)\frac{\partial u^*}{\partial t^*} + \left(\frac{U^2}{L}\right)\left(\frac{\partial(u^*u^*)}{\partial x^*} + \frac{\partial(u^*v^*)}{\partial y^*} + \frac{\partial(u^*w^*)}{\partial z^*}\right) =$$
$$- \left(\frac{p}{\rho L}\right)\frac{\partial P^*}{\partial x^*} - (fU)\, v^*$$
$$+ \left(\frac{A_h U}{L^2}\right)\left[\frac{\partial}{\partial x^*}\left(A_h^* \frac{\partial u^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(A_h^* \frac{\partial u^*}{\partial y^*}\right)\right]$$
$$+ \left(\frac{A_z U}{H^2}\right)\frac{\partial}{\partial z^*}\left(A_z^* \frac{\partial u^*}{\partial z^*}\right) \quad (5)$$

# Scale analysis: application

## Ulleung Eddy in the East Sea

Let us assume pressure gradient is important, and what is the most significant term balancing the pressure gradient.




General values for the parameters:

$$A_h \approx 10^2\, m^2\, s^{-1},\ A_z \approx 1\, m^2\, s^{-1},\ \text{and}\ f \approx 10^{-4}\, s^{-1}.$$

# Scale analysis: application

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv = \\ -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right)$$

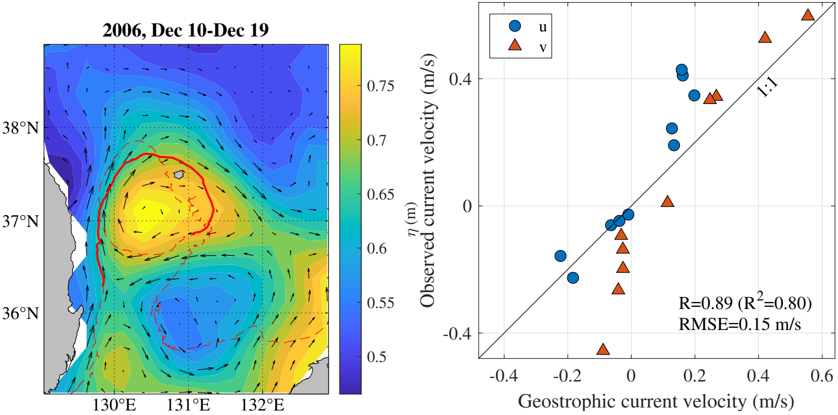
$$\downarrow$$

$$-fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{6}$$

that is what we call "geostrophic balance". But is it too simple to consider complexity of the nature (and original equations)?

# Scale analysis: application

Nope, at least locally when our choice of the scales ("assumptions") are correct.




(a) (b)

# Nondimensionalization

Dividing (5) by order of the term we interested (e.g., Coriolis; $fU$) yields

$Ro_T$: temporal Rossby number

$Ro$: Rossby number

$$
\begin{aligned}
\boxed{\left(\frac{1}{fT}\right)} \frac{\partial u^*}{\partial t^*} + \boxed{\left(\frac{U}{fL}\right)} \left(\frac{\partial (u^* u^*)}{\partial x^*} + \frac{\partial (u^* v^*)}{\partial y^*} + \frac{\partial (u^* w^*)}{\partial z^*}\right) = \\
- \boxed{\left(\frac{p}{\rho L f U}\right)} \frac{\partial P^*}{\partial x^*} - v^* \\
+ \boxed{\left(\frac{A_h}{fL^2}\right)} \left[\frac{\partial}{\partial x^*}\left(A_h^* \frac{\partial u^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(A_h^* \frac{\partial u^*}{\partial y^*}\right)\right] \\
+ \boxed{\left(\frac{A_z}{fH^2}\right)} \frac{\partial}{\partial z^*}\left(A_z^* \frac{\partial u^*}{\partial z^*}\right)
\end{aligned}
\tag{7}
$$

$Ek_h$:Lateral Ekman number

$Ek$:(Vertical) Ekman number

Blue indicate order relative to the other term's order, what we call nondimensional number.

# Summary

1. Scale analysis, applicable to any equations, is easy and **powerful**.
2. **Stupidly simplified equations work** unless the assumptions are valid.
3. Further profound discussions about the scale analysis are in Price (2005; available at `https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2024/10/DA_2024_A2.pdf`).

# Assignment

In the continuity equation (3), density $\rho$ can be decomposed into constant component $\rho_0$, so $\rho = \rho_0 + \rho'$. Substituting $\rho = \rho_0 + \rho'$ into (3) and then, conduct scale analysis and obtain nondimensionalized version of the equation.

1. What is the nondimensional number controlling the equation?
2. In general, $\rho_0 \approx 1025 kg\, m^{-3}$ and the order of $\rho'$ is less than $10\, kg\, m^{-3}$. Is the constant density assumption valid in (3)?