Differential Equations in Geophysical Fluid Dynamics

XIV. Wind-driven circulation: Munk circulation problem

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Recap

Dominant balance changes as choice of length scale relative to $L_S = \gamma/(h\beta)$:

O(1) for $x = L_S x^*$

$$\boxed{\frac{\gamma}{h}\left(\frac{\partial^2\psi}{\partial x^2}} + \frac{\partial^2\psi}{\partial y^2}\right) + \boxed{\beta\frac{\partial\psi}{\partial x}} = \boxed{-\frac{\tau_0\pi}{\rho_0 h L_y}\sin(\pi y/L_y)} \tag{1a}$$

O(1) for $x = Lx^*$

that can be simplified to

$$\beta\frac{\partial\psi_I}{\partial x} = -\frac{\tau_0\pi}{\rho_0 h L_y}\sin(\pi y/L_y) \tag{2a}$$

$$\boxed{\frac{\partial^2\psi_B}{\partial x^2} + \beta\frac{\partial\psi_B}{\partial x} = 0} \tag{2b}$$

$$\psi|_{x=0} = 0 \quad \rightarrow \quad \boxed{\psi_B|_{x=0} = -\psi_I|_{x=0}} \tag{2c}$$

"ψ_B forced by boundary condition depending on ψ_I"

Recap

Let us consider further simplified one-dimensional ($\partial/\partial y \approx 0$) problem (Kämpf, 2009):

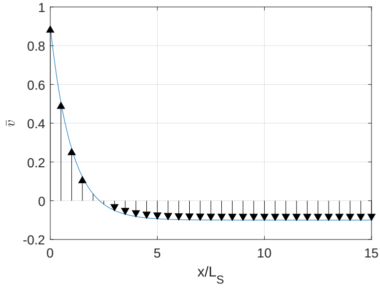
$$\frac{\gamma}{h}\frac{\partial \bar{v}}{\partial x} + \beta \bar{v} = -\overbrace{\boxed{F_0}}^{(\tau_0 \pi/(\rho_0 h L_y))\sin(\pi y_0/L_y)}. \qquad (3)$$

Solution to the problem is given by

$$\bar{v} = V_B e^{-x/L_S} - \underbrace{\boxed{V_I}}_{= F_0/\beta} \qquad (4)$$

V_I: interior flows caused by Sverdrup balance.

V_B: undetermined constant implying boundary current intensity.



But... does $L_S \approx 10^4\, m$ make sense?

Munk wind-driven circulation

Stommel (1948)

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_x^s}{\rho_0 h} \boxed{- \frac{\gamma}{h}\bar{u}} \quad (5)$$

$$f\bar{u} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_y^s}{\rho_0 h} \boxed{- \frac{\gamma}{h}\bar{v}} \quad (6)$$

$$\downarrow$$

$$\boxed{\frac{\gamma}{h}\nabla^2\psi} + \beta\frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \times \vec{\tau}^s \quad (7)$$

$$\frac{\gamma}{h}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

Munk (1950)

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_x^s}{\rho_0 h} \boxed{+ A_h \nabla^2 \bar{u}} \quad (8)$$

$$f\bar{u} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_y^s}{\rho_0 h} \boxed{+ A_h \nabla^2 \bar{v}} \quad (9)$$

$$\downarrow$$

$$\boxed{- A_h \nabla^4 \psi} + \beta\frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \times \vec{\tau}^s \quad (10)$$

$$A_h\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^2 \psi}{\partial y^4}\right)$$

"Biharmonic operator"

Munk wind-driven circulation

We will talk about

$$-f\bar{v} = -g\frac{\partial \eta}{\partial x} + \frac{\tau_x^s}{\rho_0 h} \tag{11a}$$

$$f\bar{u} = -g\frac{\partial \eta}{\partial y} + \frac{\tau_y^s}{\rho_0 h} + A_h\frac{\partial^2 \bar{v}}{\partial x^2} \tag{11b}$$

$$\downarrow$$

$$\boxed{A_h\frac{\partial^4 \psi}{\partial x^4} - \beta\frac{\partial \psi}{\partial x} = -\frac{1}{\rho_0 h}\nabla \times \vec{\tau}^s} \tag{12}$$

and the boundary conditions are given by

$$\psi|_{x=0} = 0,\ \ \psi|_{x=L_x} = 0,\ \ \left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0,\ \ \left.\frac{\partial \psi}{\partial x}\right|_{x=L_x} = 0. \tag{13a}$$

$\bar{v}|_{x=0} = 0,\ \ \bar{v}|_{x=L_x} = 0$ **"No-slip conditions"**

Scale analysis

Using $x = Lx^*$, $y = Ly^*$, and $\psi = \Psi\psi^*$, the governing equation is nondimensionalized to

$$\epsilon^3 \frac{\partial^4 \psi^*}{\partial x^{*4}} \overbrace{\boxed{- \frac{\partial \psi^*}{\partial x^*} = -\nabla \times \vec{\tau}_s^*}}^{O(1)\text{: big “Sverdrup balance”}} \tag{14}$$

where $\epsilon = L_M/L \ll 1$ and $L_M = (A_h/\beta)^{1/3}$. The order of the forcing term is set to be identical to that of the beta term.

Choosing $x = L_M x^*$ yields

$$\overbrace{\boxed{\frac{\partial^4 \psi^*}{\partial x^{*4}} - \frac{\partial \psi^*}{\partial x^*}}}^{O(1)\text{: big}} = \epsilon \nabla \times \vec{\tau}_s^* \tag{15}$$

that can be considered as dominant balance for the western boundary current.

Simplified problem

So, the simplified governing equations are given by

$$\beta \frac{\partial \psi_I}{\partial x} = -\frac{\tau_0 \pi}{\rho_0 h L_y} \sin(\pi y / L_y) \tag{16a}$$

$$-A_h \frac{\partial^4 \psi_B}{\partial x^4} + \beta \frac{\partial \psi_B}{\partial x} = 0 \tag{16b}$$

$$\psi\big|_{x=0} = 0, \quad \left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0 \tag{16c}$$

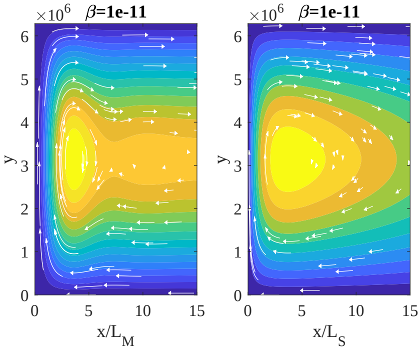
$$\lim_{x \to \infty} \psi = \psi_I \tag{16d}$$

Note that $\psi = \psi_I + \psi_B$. The interior component ψ_I is identical to that of Stommel wind-driven component (Sverdrup solution).

Simplified problem

Solution to the problem is given by

$$\psi_B = -\left.\psi_I\right|_{x=0} e^{-x/(2L_M)} \left(\cos\left(\frac{\sqrt{3}x}{2L_M}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}x}{2L_M}\right)\right) \tag{17}$$



Western boundary current as compensational flow

$$\beta\frac{\partial\psi}{\partial x} + \overset{\frac{\gamma}{h}\nabla^2\psi - A_h\nabla^4\psi\cdots}{O(\epsilon)} = -F_0 \tag{18}$$

Note that we are considering closed ocean with steady state:

$$\frac{\partial\eta}{\partial t} + h\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y}\right) = 0 \tag{19}$$

$$\bar{u}|_{x=0} = 0, \quad \bar{u}|_{x=L_x} = 0, \quad \bar{v}|_{y=0} = 0, \quad \bar{v}|_{y=L_y} = 0. \tag{20}$$

Integrating (19) over entire subdomain, from $y = 0$ to $y = y_0$, yields[1]

$$\boxed{\int_0^{L_x} \bar{v}(x, y_0)\, dx = 0} \tag{21}$$

so southward current (Sverdrup flow) should be compensated by northward current at some region (western boundary)!

[1]This is the "**divergence theorem**".

Remote wind forcing

$$\beta \frac{\partial \psi}{\partial x} + O(\epsilon) = -F_0(x) \tag{22}$$

$$\beta \frac{\partial \psi_I}{\partial x} = -F_0(x) \quad \rightarrow \quad \psi_I = -\int_x^{L_x} F_0(x)\, dx \tag{23a}$$

$$\beta \frac{\partial \psi_B}{\partial x} + O(\epsilon) = 0 \tag{23b}$$

Depends on ψ_B

$$\psi|_{x=0} = 0 \quad \rightarrow \quad \psi_B|_{x=0} = -\,\psi_I|_{x=0}$$

$$\rightarrow \quad \boxed{\psi_B|_{x=0} = \int_0^{L_x} F_0(x)\, dx} \tag{23c}$$

Forced by integrated wind stress curl over the domain!

Summary

1. Planetary β-effect let sea surface height tilted (as a advection term of ψ), that cause equatorward interior flow (Sverdrup, 1947).
2. As a consequence of continuity, the equatorward interior flow is compensated by a strong poleward current near the narrow western boundary that need to be dissipative.
3. The western boundary current can be attributed to the balance between β-term and the other terms, excluding the wind stress curl, such as bottom friction (Stommel, 1948), lateral viscosity (Munk, 1950), and even partly nonlinear momentum advection (Pedlosky, 1987; Pedlosky, 1965).

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