Differential Equations in Geophysical Fluid Dynamics

VII. Characteristics of geostrophic current component

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Recap

Original linear NSE

$$if\vec{u} = -g\frac{\partial\eta}{\partial\vec{n}} + A_z\frac{\partial^2\vec{u}}{\partial z^2} \quad (1a)$$

$$A_z\frac{\partial\vec{u}}{\partial z}\bigg|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \quad (1b)$$

$$A_z\frac{\partial\vec{u}}{\partial z}\bigg|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma\vec{u} \quad (1c)$$

where $\vec{u} = u + iv$

Vertical averaged SWE

$$if\vec{u} = -g\frac{\partial\eta}{\partial\vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\gamma}{h}\vec{u} \quad (2)$$

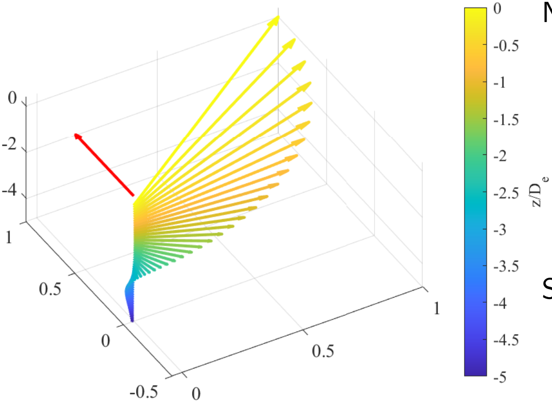
where $\vec{u} = \bar{u} + i\bar{v}$

$$\bar{u} = \frac{1}{h}\int_{-h}^{0} u\,dz$$

$$\bar{v} = \frac{1}{h}\int_{-h}^{0} v\,dz$$

Recap

Surface Ekman current



NSE

$$i f \vec{u} = A_z \frac{\partial^2 \vec{u}}{\partial z^2} \quad (3a)$$

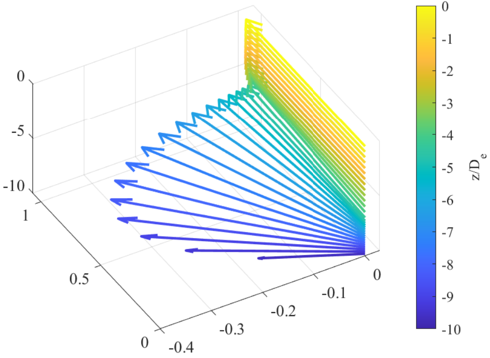
$$A_z \left.\frac{\partial \vec{u}}{\partial z}\right|_{z=0} = \frac{\vec{\tau}^s}{\rho_0} \quad (3b)$$

SWE

$$h\vec{u} = \frac{1}{i}\frac{\vec{\tau}^s}{\rho_0 f} \quad (4)$$

Recap

Bottom Ekman current



NES

$$if\vec{u} = -g\frac{\partial \eta}{\partial \vec{n}} + A_z\frac{\partial^2 \vec{u}}{\partial z^2} \quad (5a)$$

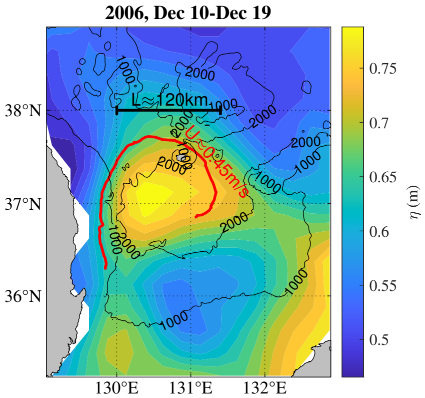
$$A_z\frac{\partial \vec{u}}{\partial z}\bigg|_{z=-h} = \frac{\vec{\tau}^b}{\rho_0} \equiv \gamma\vec{u} \quad (5b)$$

SWE

$$h\vec{u} = -\frac{1}{i}\frac{\vec{\tau}^b}{\rho_0 f} \equiv -\frac{1}{i}\frac{\gamma}{f}\vec{u}_g \quad (6)$$

Recap

Several physical assumptions are mathematically problematic...



General values for the parameters:

$$A_h \approx 10^2\, m^2\, s^{-1},\ A_z \approx 1\, m^2\, s^{-1},\ \text{and}\ f \approx 10^{-4}\, s^{-1}.$$

A bit advanced topics for the Ekman current

How to estimate the Ekman depth (or A_z) for surface layer?
Simple parameterization given by

$$D_e = \kappa \frac{u^*}{f} \tag{7}$$

where $u^* = \sqrt{\left|\vec{\tau}^s\right| / \rho_0}$ and $\kappa = 0.1 - 0.4$ (Csanady, 1981; Cushman-Roisin and Beckers, 2011).
Inverse modeling approach using simple curve-fitting (Cole et al., 2017).
Considering time-depndency (inertia) and varying vertical eddy viscosity A_z?
Wenegrat and McPhaden, 2016; Elipot and Gille, 2009; Constantin, Paldor, and Dritschel, 2020; Lilly and Elipot, 2021
Relation with mixed layer depth
Brink, 2023 (see Section 3.5)

Governing equation

$$\frac{\partial u}{\partial t} + \vec{u}\cdot\nabla u - f_0 v = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \nabla\cdot(A_h \nabla u) + \frac{\partial}{\partial z}\left(A_z \frac{\partial u}{\partial z}\right) \tag{8a}$$

$$\frac{\partial v}{\partial t} + \vec{u}\cdot\nabla v + f_0 u = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \nabla\cdot(A_h \nabla v) + \frac{\partial}{\partial z}\left(A_z \frac{\partial v}{\partial z}\right) \tag{8b}$$

$$\frac{\partial w}{\partial t} + \vec{u}\cdot\nabla w = -\frac{1}{\boxed{\rho_0}}\frac{\partial P}{\partial z} + + \nabla\cdot(A_h \nabla w) + \frac{\partial}{\partial z}\left(A_z \frac{\partial w}{\partial z}\right) - g \tag{8c}$$

Constant density

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8d}$$

Taylor-Proudman theorem (Proudman, 1916; Taylor, 1917)

For flows governed by (7), velocities are depth-independent:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0. \tag{9}$$

Taylor-Proudman theorem

$$v = \frac{g}{f_0}\frac{\partial \eta}{\partial x} \tag{10a}$$

$$u = -\frac{g}{f_0}\frac{\partial \eta}{\partial y} \tag{10b}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\int_{-h}^{\eta} u\,dz\right) + \frac{\partial}{\partial y}\left(\int_{-h}^{\eta} v\,dz\right) = 0 \tag{10c}$$

Substituting (9a) and (9b) into (9c) yields

$$\underbrace{\boxed{\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}}}_{\equiv \frac{dH}{dt}} = 0 \tag{11}$$

where $H = \eta + h$ representing height of water column and note that $\partial h/\partial t = 0$. What does (10) mean?

Eulerian and Lagrangian description

There are two ways to observe a object: Lagrangian and Eulerian. To be specific, mathematical descriptions for passive tracer transport are given by

Eulerian description

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = 0 \tag{12}$$

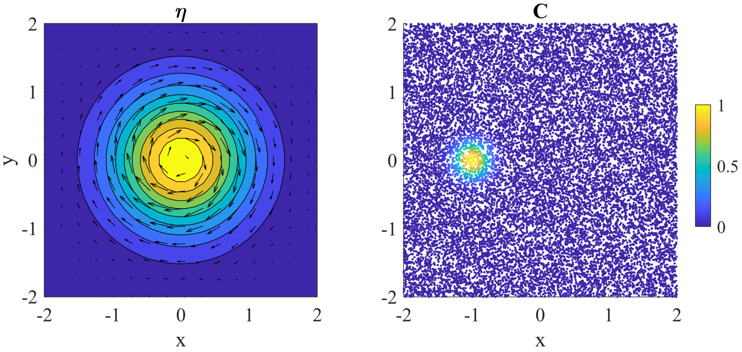
Lagrangian description

$$\frac{dC}{dt} = 0 \quad (13a) \qquad \frac{dX}{dt} = u \quad (13b) \qquad \frac{dY}{dt} = v. \quad (13c)$$

The object is not different—just a different point of view!

Eulerian and Lagrangian description

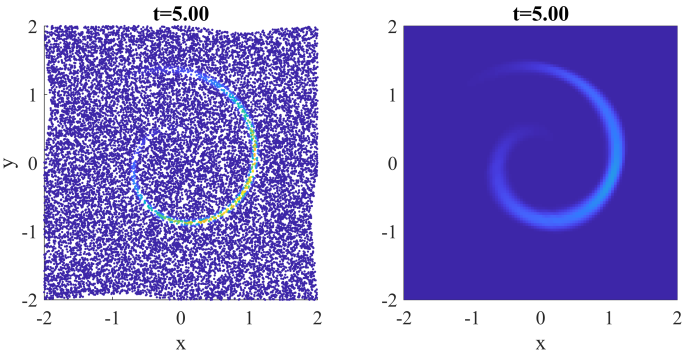
A simple numerical experiment



https://jang-geun.github.io/vis_geo_advection1.gif

Eulerian and Lagrangian description

A simple numerical experiment



https://jang-geun.github.io/vis_geo_advection2.gif

Eulerian and Lagrangian description

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = 0 \qquad (H = \eta + h) \tag{14}$$

that is

$$\frac{dH}{dt} = 0 \qquad \therefore H = H\big|_{t=0} \tag{15}$$

and represents H is not changed in the Lagrangian aspect (trajectory).
For constant depth, (13) simplifies to

$$\frac{\partial \eta}{\partial t} + \underbrace{u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}}_{\left(-\frac{g}{f_0}\frac{\partial \eta}{\partial y}\right)\frac{\partial \eta}{\partial x} + \left(\frac{g}{f_0}\frac{\partial \eta}{\partial x}\right)\frac{\partial \eta}{\partial y} = 0} = 0, \qquad \therefore \frac{\partial \eta}{\partial t} = 0$$

Hamilton system

$$\frac{dX}{dt} = -\frac{\partial \mathcal{H}}{\partial Y} \tag{16a}$$

$$\frac{dY}{dt} = \frac{\partial \mathcal{H}}{\partial X} \tag{16b}$$

In case of our transport problem, $H = (g/f_0)\eta$. This is the so-called "Hamiltonian system". Once (15) and $\mathcal{H}$ and $\partial\mathcal{H}/\partial t = 0$ (time-independent $\mathcal{H}$),

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + u\frac{\partial \mathcal{H}}{\partial x} + v\frac{\partial \mathcal{H}}{\partial y} = 0 \tag{17}$$

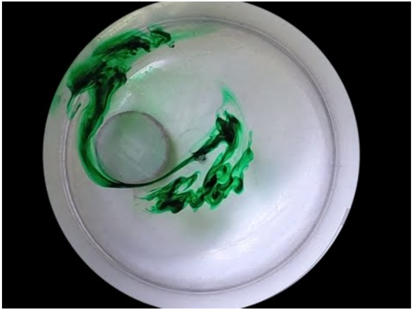
so $\mathcal{H}$ is not changed in the Lagrangian aspect.

Taylor column

If $\eta \ll h$, (13) simplifies to

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0. \tag{18}$$

This means that h is not changed in the Lagrangian aspect (flows follow isobath) and can be considered as the governing equation of the "Taylor column" phenomenon.



https://www.youtube.com/watch?v=7GGfsW7gOLI

Summary

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = 0$$

is $dH/dt = 0$ in the Lagrangian aspect and means that H experienced by a water mass (following trajectory) is not changed.

For flows governed by geostrophic balance over f-plane with homogeneous density,

1. Flows is depth-independent "barotropic" current (Taylor-Proudman theorem).
2. In the Lagrangian aspect, height of water column $(\eta + h)$ is not changed.
3. If $\eta \ll h$, flows follow isobath (Taylor column).

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