

Differential Equations in Geophysical Fluid Dynamics

XIII. Wind-driven circulation: Stommel wind-driven circulation

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Recap

Stommel's wind-driven circulation problem in vorticity equation form is given by

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s - \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \quad (1)$$

$-(\gamma/h)\nabla \times \vec{u}$: Bottom stress curl

Wind stress curl

$\beta \bar{v}$: Planetary β -term

where $\beta = \partial f / \partial y$ can be approximated to constant ($f \approx f_0$ if $y/R \ll 1$; f -plane) or linear polynomial ($f_0 + \beta_0 y$ if $y/R < 1$; β -plane).

Recap

Over the f -plane where $f \approx f_0$ so $\beta \approx 0$, the governing equation can be simplified to

$-(\gamma/h)\nabla \times \vec{u}$: Bottom stress curl

$$0 = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s - \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \quad (2)$$

that can be rewritten as $\nabla \times \vec{u} = (1/(\rho_0 \gamma)) \nabla \times \vec{\tau}^s$ implying curl of ocean current is proportional to the wind stress curl (rotates in same direction).

Note that the momentum equation we used is

Geostrophic

$$if\vec{u} = -g \frac{\partial \eta}{\partial \vec{n}} + \frac{\vec{\tau}^s}{\rho_0 h} - \frac{\gamma}{h} \vec{u} \quad (3)$$

Surface Ekman

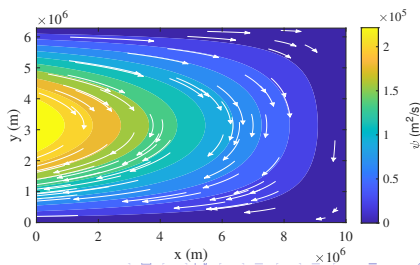
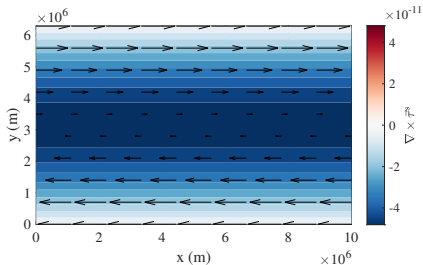
Bottom Ekman

Recap

Over the β -plane where $f_0 + \beta_0 y$, if length scale is large enough to ignore bottom frictional stress curl, the governing equation can be simplified to

$$\beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s - \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \quad (4)$$

that is referred to as **Sverdrup balance** (Sverdrup, 1947). This is easy to solve and good enough “interior flow” but can consider only one boundary condition.



Governing equations

Stommel (1948) discussed

$$\frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s \quad (5)$$

with simple forcing (wind stress fields), that idealized wind stress pattern, given by

$$\vec{\tau}^s = (\tau_x^s, \tau_y^s) = (-\tau_0 \cos(\pi y/L_y), 0). \quad (6)$$

So, substituting (6) into (5) yields

$$\boxed{\frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \beta \frac{\partial \psi}{\partial x} = -\frac{\tau_0 L_y}{\rho_0 \gamma \pi} \sin(\pi y/L_y)} \quad (7)$$

The boundary conditions are given by

$$\boxed{\psi|_{x=0} = 0, \quad \psi|_{x=L_x} = 0, \quad \psi|_{y=0} = 0, \quad \psi|_{y=L_y} = 0.} \quad (8)$$

Stommel's wind-driven circulation

Solution to (7) with boundary conditions (8) is given by

$$\psi = \frac{\tau_0 L_y}{\rho_0 \gamma \pi} \left(1 - \frac{(e^{k^- L_x} - 1)e^{k^+ x} + (1 - e^{k^+ L_x})e^{k^- x}}{e^{k^- L_x} - e^{k^+ L_x}} \right) \quad (9a)$$

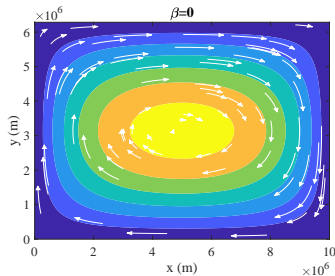
$$k^+ = \frac{-h\beta/\gamma + \sqrt{(h\beta/\gamma)^2 + 4(\pi/L_y)^2}}{2} \quad (9b)$$

$$k^- = \frac{-h\beta/\gamma - \sqrt{(h\beta/\gamma)^2 + 4(\pi/L_y)^2}}{2} \quad (9c)$$

Stommel's wind-driven circulation

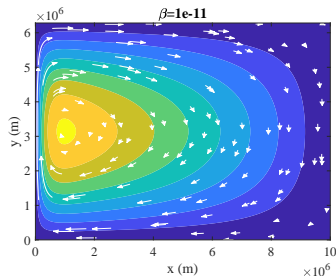
Solution without β -term

$$\frac{\gamma}{h} \nabla^2 \psi = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s$$



Solution with β -term

$$\frac{\gamma}{h} \nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0 h} \nabla \times \vec{\tau}^s$$



Consideration of the β -term tilts the stream function (and sea surface height) westward. This is referred to as “ β -effect” yielding western boundary intensification.

Summary



The diagram shows the Stommel equation (10) with terms grouped into three color-coded boxes and labeled with arrows:

- Advection:** A blue arrow points from the label to the $c \frac{\partial \psi}{\partial t}$ term and the $-\beta \frac{\partial \psi}{\partial x}$ term, which are enclosed in a light purple box.
- Diffusion:** A blue arrow points from the label to the $\frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)$ term, which is enclosed in a light blue box.
- Forcing:** A red arrow points from the label to the $\frac{\tau_0 L_y}{\rho_0 \gamma \pi} \sin(\pi y / L_y)$ term, which is enclosed in a light red box.

$$c \frac{\partial \psi}{\partial t} - \beta \frac{\partial \psi}{\partial x} = \frac{\gamma}{h} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{\tau_0 L_y}{\rho_0 \gamma \pi} \sin(\pi y / L_y) \quad (10)$$

1. Note that the governing equation of Stommel's wind-driven circulation problem is the “steady” advection-diffusion equation with forcing term.
2. The planetary β -term plays a role in advecting ψ toward the negative x -direction (westward).
3. Due to the β -term, the stream function (sea surface height) becomes asymmetric: steep along the narrow west coast and gentle in the other region. As a result, western boundary (geostrophic) currents are intensified.

References I

-  Stommel, Henry (1948). "The westward intensification of wind-driven ocean currents". In: *Eos, Transactions American Geophysical Union* 29.2, pp. 202–206.
-  Sverdrup, Harald Ulrich (1947). "Wind-driven currents in a baroclinic ocean; with application to the equatorial currents of the eastern Pacific". In: *Proceedings of the National Academy of Sciences* 33.11, pp. 318–326.