Differential Equations in Geophysical Fluid Dynamics

XIX. Wave in rotation: Rossby wave

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Jan, 2026

This seminar is supported by mathematics community EM (maintained by Prof. Gunhee Cho) and oceanography commuinity COKOAA.

Recap

We discussed wave over f-plane ($f = f_0$), so-called the Poincaré wave, and geostrophic adjustment governed by

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial \eta}{\partial x} \qquad \text{(1a)}$$

$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} = 0 \qquad \text{(1b)}$$

$$\frac{\partial \eta}{\partial t} + h\frac{\partial \bar{u}}{\partial x} = 0 \qquad \text{(1c)}$$

$$\frac{\partial^3 \eta}{\partial t^3} - gh\frac{\partial}{\partial t}\frac{\partial^2 \eta}{\partial x^2} + f^2\frac{\partial \eta}{\partial t} = 0 \qquad (2)$$

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} - gh\frac{\partial^2 \eta}{\partial x^2} + f^2\eta = q(x)} \qquad (3)$$

where

$$q(x) = \left(\frac{\partial^3 \eta}{\partial t^3} - gh\frac{\partial}{\partial t}\frac{\partial^2 \eta}{\partial x^2} + f^2\frac{\partial \eta}{\partial t}\right)\bigg|_{t=0} \qquad (4)$$

Governing equation

Let us consider wave over β-plane, valid for larger spatial scales, governed by

β-plane: $f(y) = f_0 + \beta y$

$$\frac{\partial \bar{u}}{\partial t} - \boxed{f}\,\bar{v} = -g\frac{\partial \eta}{\partial x} \tag{5a}$$

$$\frac{\partial \bar{v}}{\partial t} + \boxed{f}\,\bar{u} = -g\frac{\partial \eta}{\partial y} \tag{5b}$$

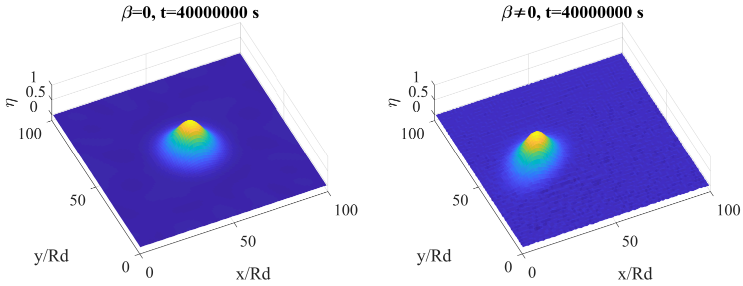
$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0. \tag{5c}$$

Substituting (5a) and (5b) into (5c) yields

$$\frac{\partial \eta}{\partial t} \boxed{-\beta\frac{gh}{f^2}}\,\frac{\partial \eta}{\partial x} = 0 \tag{6}$$

Constant (phase) velocity!

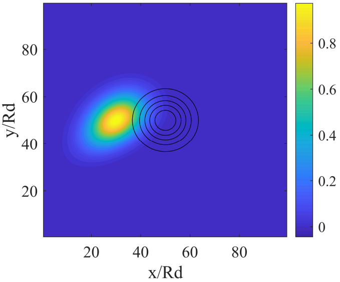
Rossby wave in long wave limit



https://jang-geun.github.io/vis_rossby_big.gif

Rossby wave in long wave limit

Prediction based on analytical solution to (6), left panel, and numerical solution to (5), right panel, resolving inertia.



Rossby wave perturbed by inertia

Weak but exist!
How to consider presence of small terms?
"**Perturbation theory!**"

$$\boxed{\frac{\partial \bar{u}}{\partial t}} - (f_0 + \beta y)\bar{v} = -g\frac{\partial \eta}{\partial x} \tag{7a}$$

$$\boxed{\frac{\partial \bar{v}}{\partial t}} + (f_0 + \beta y)\bar{u} = -g\frac{\partial \eta}{\partial y} \tag{7b}$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0. \tag{7c}$$

Let us consider a decomposition $\bar{u} = \bar{u}_0 + \bar{u}_1$ and $\bar{v} = \bar{v}_0 + \bar{v}_1$, where

$\bar{u}_0$, $\bar{v}_0$: zeroth order (predominant) components
$\bar{u}_1$, $\bar{v}_1$: first order (small) components

so $(\bar{u}_1, \bar{v}_1) \ll (\bar{u}_0, \bar{v}_0)$ will be assumed.

Rossby wave perturbed by inertia

Consider an environment where $Ro \ll 1$, $\bar{u}_0 = U\bar{u}_0^*$ and following scales for velocity components:

Ro times smaller than the zeroth order component

$$\bar{u}_0 = U\bar{u}_0^*, \quad \bar{u}_1 = \boxed{Ro}\, U\bar{u}_1^*$$
$$\bar{v}_0 = U\bar{v}_0^*, \quad \bar{v}_1 = \boxed{Ro}\, U\bar{v}_1^* \tag{8}$$

Governing equations in nondimensional version is given by

$$Ro\frac{\partial \bar{u}^*}{\partial t^*} - \bar{v}^* - Roy^*\bar{v}^* = -\frac{\partial \eta^*}{\partial x^*}$$
$$Ro\frac{\partial \bar{v}^*}{\partial t^*} + \bar{u}^* + Roy^*\bar{u}^* = -\frac{\partial \eta^*}{\partial y^*} \tag{9}$$

where order of pressure gradient is set to fU and that of β term is set to $RofU$.

Rossby wave perturbed by inertia

Substituting (8)[1] into (9) and assuming balance between terms having same order yields

O(1) balance:

$$-f_0\bar{v}_0 = -g\frac{\partial \eta}{\partial x}, \qquad \therefore \bar{v}_0 = \frac{g}{f_0}\frac{\partial \eta}{\partial x} \tag{10a}$$

$$f_0\bar{u}_0 = -g\frac{\partial \eta}{\partial y}, \qquad \therefore \bar{u}_0 = -\frac{g}{f_0}\frac{\partial \eta}{\partial y} \tag{10b}$$

O(Ro) balance:

$$\frac{\partial \bar{u}_0}{\partial t} - f_0\bar{v}_1 - \beta y\bar{v}_0 = 0, \qquad \therefore \bar{v}_1 = \frac{1}{f_0}\left(\frac{\partial \bar{u}_0}{\partial t} - \beta y\bar{v}_0\right) \tag{11a}$$

$$\frac{\partial \bar{v}_0}{\partial t} + f_0\bar{u}_1 + \beta y\bar{u}_0 = 0, \qquad \therefore \bar{u}_1 = -\frac{1}{f_0}\left(\frac{\partial \bar{v}_0}{\partial t} + \beta y\bar{u}_0\right) \tag{11b}$$

[1] Note that $\bar{u}^* = \bar{u}_0^* + Ro\bar{u}_1^*$ and $\bar{v}^* = \bar{v}_0^* + Ro\bar{v}_1^*$

Rossby wave perturbed by inertia

Substituting (10) and (11) into (7c) yields

$$\frac{\partial \eta}{\partial t} + h\left(\overbrace{\frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial y}}^{=0} + \frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{v}_1}{\partial y}\right) = 0$$

$$\boxed{\therefore \frac{\partial \eta}{\partial t} - \beta \frac{gh}{{f_0}^2}\frac{\partial \eta}{\partial x} - \frac{gh}{{f_0}^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) = 0} \qquad (12)$$

whose dispersion relation, using $\eta = \eta_0 e^{i(kx - wt)}$, is given by

$$w = -\beta Rd^2 \frac{k}{1 + Rd^2(k^2 + l^2)} \qquad (13)$$

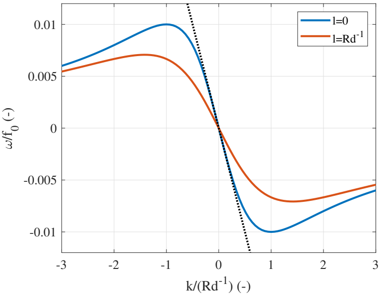
where $Rd = \sqrt{gh}/f_0$ indicating Rossby radius of deformation.

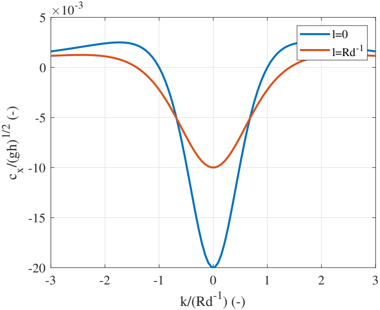
Rossby wave perturbed by inertia

Phase and group velocities are, respectively, given by

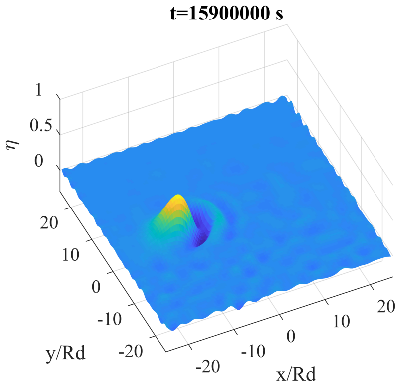
$$c_x = \frac{w}{k} = \beta Rd^2 \frac{1}{1 + Rd^2(k^2 + l^2)} \tag{14}$$

$$c_{g_x} = \frac{\partial w}{\partial k} = -\beta Rd^2 \frac{1 + Rd^2(l^2 - k^2)}{(1 + Rd^2(k^2 + l^2))^2} \tag{15}$$





Rossby wave perturbed by inertia



https://jang-geun.github.io/vis_rossby_small.gif

Conclusion

1. Over β-plane, a large bump drifters westward as Rossby wave.
2. It is dispersive due to presence of inertia, and mostly non-dispersive in long wave limit.