Analog & Digital Control System Design

Transfer-function, State-space, & Algebraic Methods

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Analog and Digital

Control System Design:

Transfer-Function, State-Space, and Algebraic Methods

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9 The Inward Approach—Choice of Overall Transfer Functions

9.1 INTRODUCTION

In the design of control systems using the root-locus method or the frequency-domain method, we first choose a configuration and a compensator with open parameters. We then search for parameters such that the resulting overall system will meet design specifications. This approach is essentially a trial-and-error method; therefore, we usually choose the simplest possible feedback configuration (namely, a unity-feedback configuration) and start from the simplest possible compensator—namely, a gain (a compensator of degree 0). If the design objective cannot be met by searching the gain, we then choose a different configuration or a compensator of degree 1 (phase-lead or phase-lag network) and repeat the search. This approach starts from internal compensators and then designs an overall system to meet design specifications; therefore, it may be called the *outward* approach.

In this and the following chapters we shall introduce a different approach, called the *inward* approach. In this approach, we first search for an overall transfer function to meet design specifications, and then choose a configuration and compute the required compensators. Choice of overall transfer functions will be discussed in this chapter. The implementation problem—namely, choosing a configuration and computing the required compensators—will be discussed in the next chapter.

Consider a plant with proper transfer function $G(s) = N(s)/D(s)$ as shown in Figure 9.1. In the inward approach, the first step is to choose an overall transfer function $G_o(s)$ from the reference input r to the plant output y to meet a set of

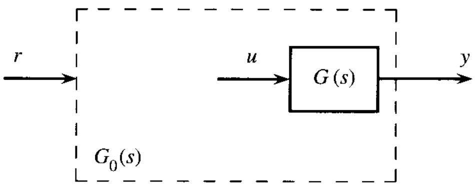


Figure 9.1 Design of control systems.

specifications. We claim that

$$G_o(s) = 1$$

is the best possible system we can design. Indeed if $G_o(s) = 1$, then $y(t) = r(t)$, for $t \geq 0$. Thus the position and velocity errors are zero; the rise time, settling time, and overshoot are all zero. Thus no other $G_o(s)$ can perform better than $G_o(s) = 1$. Note that although $r(t) = y(t)$, the power levels at the reference input and plant output are different. The reference signal may be provided by turning a knob by hand; the plant output $y(t)$ may be the angular position of an antenna with weight over several tons.

Although $G_o(s) = 1$ is the best system, we may not be able to implement it in practice. Recall from Chapter 6 that practical constraints, such as proper compensators, well-posedness, and total stability, do exist in the design of control systems. These constraints impose some limitations in choosing $G_o(s)$. We first discuss this problem.

9.2 IMPLEMENTABLE TRANSFER FUNCTIONS

Consider a plant with transfer function

$$G(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are two polynomials and are assumed to have no common factors. We assume $n = \deg D(s) \geq \deg N(s)$, that is, $G(s)$ is proper and has degree n. An overall transfer function $G_o(s)$ is said to be *implementable* if there exists a configuration such that the transfer function from the reference input r to the plant output y in Figure 9.1 equals $G_o(s)$ and the design meets the following four constraints:

1. All compensators used have proper rational transfer functions.
2. The resulting system is well-posed.
3. The resulting system is totally stable.
4. There is no plant leakage in the sense that all forward paths from r to y pass through the plant.

The first constraint is needed, as discussed in Section 5.4, for building compensators using operational amplifier circuits. If a compensator has an improper transfer function, then it cannot be easily built in practice. The second and third constraints are needed, as discussed in Chapter 6, to avoid amplification of high-frequency noise and to avoid unstable pole-zero cancellations. The fourth constraint implies that all power must pass through the plant and that no compensator be introduced in parallel with the plant. This constraint appears to be reasonable and seems to be met by every configuration in the literature. This constraint is called "no plant leakage" by Horowitz [35].

If an overall transfer function $G_o(s)$ is not implementable, then no matter what configuration is used to implement it, the design will violate at least one of the preceding four constraints. Therefore, in the inward approach, the $G_o(s)$ we choose must be implementable.

The question then is how to tell whether or not a $G_o(s)$ is implementable. It turns out that the answer is very simple.

THEOREM 9.1

Consider a plant with proper transfer function $G(s) = N(s)/D(s)$. Then $G_o(s)$ is implementable if and only if $G_o(s)$ and

$$T(s) := \frac{G_o(s)}{G(s)}$$

are proper and stable. ■

We discuss first the necessity of the theorem. Consider, for example, the configuration shown in Figure 9.2. Noise, which may enter into the intput and output terminals of each block, is not shown. If the closed-loop transfer function from r to y is $G_o(s)$ and if there is no plant leakage, then the *closed-loop* transfer function from r to u is $T(s)$. Well-posedness requires every closed-loop transfer function to be proper, thus $T(s)$ and $G_o(s)$ must be proper. Total stability requires every closed-loop transfer function to be stable, thus $G_o(s)$ and $T(s)$ must be stable. This establishes the necessity of the theorem. The sufficiency of the theorem will be established constructively in the next chapter.

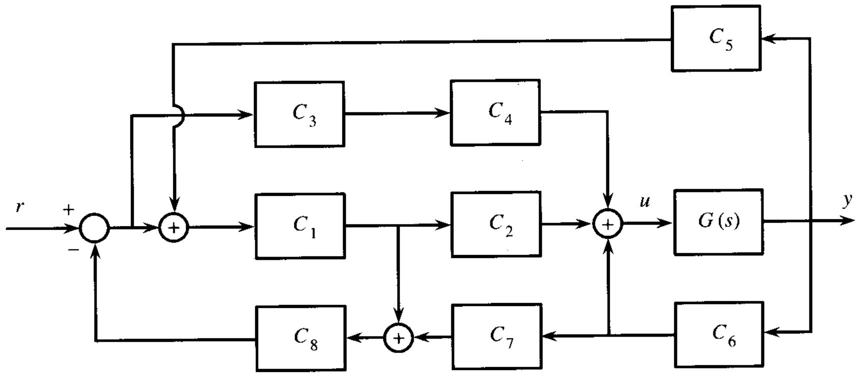


Figure 9.2 Feedback system without plant leakage.

We discuss now the implication of Theorem 9.1. Let us write

$$G(s) = \frac{N(s)}{D(s)} \qquad G_o(s) = \frac{N_o(s)}{D_o(s)} \qquad T(s) = \frac{N_t(s)}{D_t(s)}$$

We assume that the numerator and denominator of each transfer function have no common factors. The equality $G_o(s) = G(s)T(s)$ or

$$\frac{N_o(s)}{D_o(s)} = \frac{N(s)}{D(s)} \cdot \frac{N_t(s)}{D_t(s)}$$

implies

$$\deg D_o(s) - \deg N_o(s) = \deg D(s) - \deg N(s) + (\deg D_t(s) - \deg N_t(s))$$

Thus if $T(s)$ is proper, that is, $\deg D_t(s) \geq \deg N_t(s)$, then we have

$$\deg D_o(s) - \deg N_o(s) \geq \deg D(s) - \deg N(s) \tag{9.1}$$

Conversely, if (9.1) holds, then $\deg D_t(s) \geq \deg N_t(s)$, and $T(s)$ is proper.

Stability of $G_o(s)$ and $T(s)$ requires both $D_o(s)$ and $D_t(s)$ to be Hurwitz. From

$$T(s) = \frac{N_t(s)}{D_t(s)} = \frac{G_o(s)}{G(s)} = \frac{N_o(s)}{D_o(s)} \cdot \frac{D(s)}{N(s)}$$

we see that if $N(s)$ has closed right-half-plane (RHP) roots, and if these roots are not canceled by $N_o(s)$, then $D_t(s)$ cannot be Hurwitz. Therefore, in order for $T(s)$ to be stable, all the closed RHP roots of $N(s)$ must be contained in $N_o(s)$. This establishes the following corollary.

COROLLARY 9.1

Consider a plant with proper transfer function $G(s) = N(s)/D(s)$. Then $G_o(s) = N_o(s)/D_o(s)$ is implementable if and only if

(a) $\deg D_o(s) - \deg N_o(s) \geq \deg D(s) - \deg N(s)$ (pole-zero excess inequality).
(b) All closed RHP zeros of $N(s)$ are retained in $N_o(s)$ (retainment of non-minimum-phase zeros).
(c) $D_o(s)$ is Hurwitz. ■

As was defined in Section 8.3.1, zeros in the closed RHP are called non-minimum-phase zeros. Zeros in the open left half plane are called minimum-phase zeros. Poles in the closed RHP are called *unstable poles*. We see that the non-minimum-phase zeros of $G(s)$ impose constraints on implementable $G_o(s)$ but the unstable poles of $G(s)$ do not. This can be easily explained from the unity-feedback configuration shown in Figure 9.3. Let

$$G(s) = \frac{N(s)}{D(s)} \qquad C(s) = \frac{N_c(s)}{D_c(s)}$$

be respectively the plant transfer function and compensator transfer function. Let

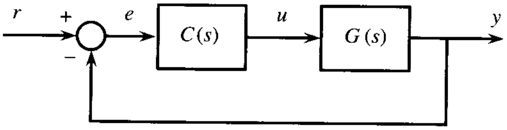


Figure 9.3 Unity-feedback configuration.

$G_o(s) = N_o(s)/D_o(s)$ be the overall transfer function from the reference input r to the plant output y. Then we have

$$G_o(s) = \frac{N_o(s)}{D_o(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{N(s)N_c(s)}{D(s)D_c(s) + N(s)N_c(s)} \tag{9.2}$$

We see that $N(s)$ appears directly as a factor of $N_o(s)$. If a root of $N(s)$ does not appear in $N_o(s)$, the only way to achieve this is to introduce the same root in $D(s)D_c(s) + N(s)N_c(s)$ to cancel it. This cancellation is an unstable pole-zero cancellation if the root of $N(s)$ is in the closed right half s-plane. In this case, the system cannot be totally stable and the cancellation is not permitted. Therefore all non-minimum-phase zeros of $G(s)$ must appear in $N_o(s)$. The poles of $G(s)$ or the roots of $D(s)$ are *shifted* to $D(s)D_c(s) + N(s)N_c(s)$ by feedback, and it is immaterial whether $D(s)$ is Hurwitz or not. Therefore, unstable poles of $G(s)$ do not impose any constraint on $G_o(s)$, but non-minimum-phase zeros of $G(s)$ do. Although the preceding assertion is developed for the unity-feedback system shown in Figure 9.3, it is generally true that, in any feedback configuration without plant leakage, feedback will *shift* the poles of the plant transfer function to new locations but will not affect its zeros. Therefore the non-minimum-phase zeros of $G(s)$ impose constraints on $G_o(s)$ but the unstable poles of $G(s)$ do not.

Example 9.2.1

Consider

$$G(s) = \frac{(s+2)(s-1)}{s(s^2 - 2s + 2)}$$

Then we have

$G_o(s) = 1$ Not implementable, because it violates (a) and (b) in Corollary 9.1.

$G_o(s) = \dfrac{s+2}{(s+3)(s+1)}$ Not implementable, meets (a) and (c) but violates (b).

$G_o(s) = \dfrac{s-1}{s(s+2)}$ Not implementable, meets (a) and (b), violates (c).

$G_o(s) = \dfrac{s-1}{(s+3)(s+1)}$ Implementable.

$$G_o(s) = \frac{s-1}{(s+3)(s+1)^2} \qquad \text{Implementable.}$$

$$G_o(s) = \frac{(2s-3)(s-1)}{(s+2)^3} \qquad \text{Implementable.}$$

$$G_o(s) = \frac{(2s-3)(s-1)(s+1)}{(s+2)^5} \qquad \text{Implementable.}$$

Exercise 9.2.1

Given $G(s) = (s-2)/(s-3)^2$, are the following implementable?

a. $\dfrac{1}{s+1}$ **b.** $\dfrac{s-2}{s+1}$ **c.** $\dfrac{s-2}{(s+1)^2}$ **d.** $\dfrac{(s-2)(s-3)}{(s+1)^3}$

[**Answers:** No, no, yes, yes.]

Exercise 9.2.2

Given $G(s) = (s+1)/(s-3)^2$, are the following implementable?

a. $\dfrac{1}{s+1}$ **b.** $\dfrac{s-2}{s+1}$ **c.** $\dfrac{s-2}{(s+1)^2}$ **d.** $\dfrac{(s-2)s^4}{(s+2)^6}$

[**Answers:** Yes, no, yes, yes.]

From the preceding examples, we see that if the pole-zero excess inequality is met, then all poles and all minimum-phase zeros of $G_o(s)$ can be arbitrarily assigned. To be precise, all poles of $G_o(s)$ can be assigned anywhere inside the open left half s-plane (to insure stability). Other than retaining all non-minimum-phase zeros of $G(s)$, all minimum-phase zeros of $G_o(s)$ can be assigned anywhere in the entire s-plane. In the assignment, if a complex number is assigned as a zero or pole, its complex conjugate must also be assigned. Otherwise, the coefficients of $G_o(s)$ will be complex, and $G_o(s)$ cannot be realized in the real world. Therefore, roughly speaking, if $G_o(s)$ meets the pole-zero excess inequality, its poles and zeros can be arbitrarily assigned.

Consider a plant with transfer function $G(s)$. The problem of designing a system so that its overall transfer function equals a given model with transfer function $G_m(s)$ is called the *model-matching* problem. Now if $G_m(s)$ is not implementable, no matter how hard we try, it is not possible to match $G_m(s)$ without violating the four constraints. On the other hand, if $G_m(s)$ is implementable, it is possible, as will be shown in the next chapter, to match $G_m(s)$. Therefore, the model-matching problem is the same as our implementability problem. In conclusion, in model matching, we can arbitrarily assign poles as well as minimum-phase zeros so long as they meet the pole-zero excess inequality.

To conclude this section, we mention that if G_o is implementable, it does not mean that it can be implemented using *any* configuration. For example, $G_o(s) = 1/(s+1)^2$ is implementable for the plant $G(s) = 1/s(s-1)$. This $G_o(s)$, however, cannot be implemented in the unity-feedback configuration shown in Figure 9.3; it can be implemented using some other configurations, as will be discussed in the next chapter. In conclusion, for any $G(s)$ and any implementable $G_o(s)$, there exists at least one configuration in which $G_s(s)$ can be implemented under the preceding four constraints.

9.2.1 Asymptotic Tracking and Permissible Pole-Zero Cancellation Region

A control system with overall transfer function

$$G_o(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \cdots + \beta_m s^m}{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_n s^n} \tag{9.3}$$

with $\alpha_n > 0$ and $n \geq m$, is said to achieve *asymptotic tracking* if the plant output $y(t)$ tracks eventually the reference input $r(t)$ without an error, that is,

$$\lim_{t\to\infty} |y(t) - r(t)| = 0$$

Clearly if $G_o(s)$ is not stable, it cannot track any reference signal. Therefore, we require $G_o(s)$ to be stable, which in turn requires $\alpha_i > 0$ for all i[1]. Thus, the denominator of $G_o(s)$ cannot have any missing term or a term with a negative coefficient. Now the condition for $G_o(s)$ to achieve asymptotic tracking depends on the type of $r(t)$ to be tracked. The more complicated $r(t)$, the more complicated $G_o(s)$. From Section 6.3.1, we conclude that if $r(t)$ is a step function, the conditions for $G_o(s)$ to achieve tracking are $G_o(s)$ stable and $\alpha_0 = \beta_0$. If $r(t)$ is a ramp function, the conditions are $G_o(s)$ stable, $\alpha_0 = \beta_0$, and $\alpha_1 = \beta_1$. If $r(t) = at^2$, an acceleration function, then the conditions are $G_o(s)$ stable, $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1$, and $\alpha_2 = \beta_2$. If $r(t) = 0$, the only condition for $y(t)$ to track $r(t)$ is $G_o(s)$ stable. In this case, the output may be excited by nonzero initial conditions, which in turn may be excited by noise or disturbance. To bring $y(t)$ to zero is called the *regulating problem.* In conclusion, the conditions for $G_o(s)$ to achieve asymptotic tracking are simple and can be easily met in the design.

Asymptotic tracking is a property of $G_o(s)$ as $t \to \infty$ or a steady-state property of $G_o(s)$. It is not concerned with the manner or the speed at which $y(t)$ approaches $r(t)$. This is the transient performance of $G_o(s)$. The transient performance depends on the location of the poles and zeros of $G_o(s)$. How to choose poles and zeros to meet the specification on transient performance, however, is not a simple problem.

[1]Also, they can all be negative. For convenience, we consider only the positive case.

In choosing an implementable overall transfer function, if a zero of $G(s)$ is not retained in $G_o(s)$, we must introduce a pole to cancel it in implementation. If the zero is a non-minimum-phase zero, the pole that is introduced to cancel it is not stable and the resulting system will not be totally stable. If the zero is minimum phase but has a large imaginary part or is very close to the imaginary axis, then, as was discussed in Section 6.6.2, the pole may excite a response that is very oscillatory or takes a very long time to vanish. Therefore, in practice, not only the non-minimum-phase zeros of $G(s)$ but also those minimum-phase zeros that are close to the imaginary axis should be retained in $G_o(s)$, or the zeros of $G(s)$ lying outside the region C shown in Figures 6.13 or 7.4 should be retained in $G_o(s)$. How to determine such a region, however, is not necessarily simple. See the discussion in Chapter 7.

Exercise 9.2.3

What types of reference signals can the following systems track without an error?

a. $\dfrac{s + 5}{s^3 + 2s^2 + 8s + 5}$

b. $\dfrac{8s + 5}{s^3 + 2s^2 + 8s + 5}$

c. $\dfrac{2s^2 + 9s + 68}{s^3 + 2s^2 + 9s + 68}$

[**Answers:** (a) Step functions. (b) Ramp functions. (c) None, because it is not stable.]

9.3 VARIOUS DESIGN CRITERIA

The performance of a control system is generally specified in terms of the rise time, settling time, overshoot, and steady-state error. Suppose we have designed two systems, one with a better transient performance but a poorer steady-state performance, the other with a poorer transient performance but a better steady-stage performance. The question is: Which system should we use? This difficulty arises from the fact that the criteria consist of more than one factor. In order to make comparisons, the criteria may be modified as

$$\begin{aligned} J := k_1 &\times \text{(Rise time)} + k_2 \times \text{(Settling time)} \\ &+ k_3 \times \text{(Overshoot)} + k_4 \times \text{(Steady-state error)} \end{aligned} \tag{9.4}$$

where the k_i are weighting factors and are chosen according to the relative importance of the rise time, settling time, and so forth. The system that has the smallest J is called the *optimal* system with respect to the criterion J. Although the criterion is reasonable, it is not easy to track analytically. Therefore more trackable criteria are used in engineering.

We define

$$e(t) := r(t) - y(t)$$

It is the error between the reference input and the plant output at time t as shown in Figure 9.4. Because an error exists at every t, we must consider the *total* error in $[0, \infty)$. One way to define the total error is

$$J_1 := \int_0^\infty e(t)\,dt \tag{9.5}$$

This is not a useful criterion, however, because of possible cancellations between positive and negative errors. Thus a small J_1 may not imply a small $e(t)$ for all t. A *better definition of the total error is*

$$J_2 := \int_0^\infty |e(t)|\,dt \tag{9.6}$$

This is called the *integral of absolute error* (IAE). In this case, a small J_2 will imply a small $e(t)$. Other possible definitions are

$$J_3 := \int_0^\infty |e(t)|^2\,dt \tag{9.7}$$

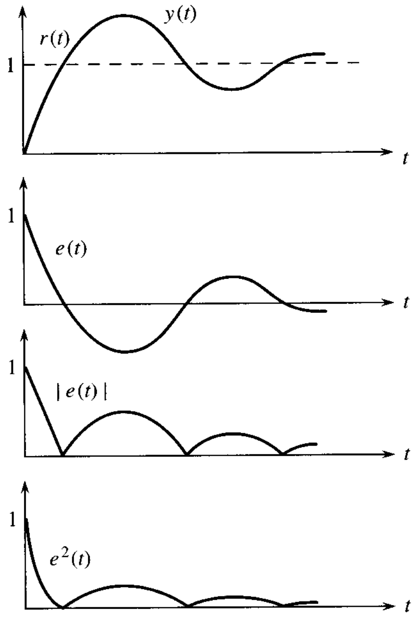


Figure 9.4 Errors.

and

$$J_4 := \int_0^\infty t|e(t)|\, dt \tag{9.8}$$

The former is called the *integral of square error* (ISE) or *quadratic error*, and the latter the *integral of time multiplied by absolute error* (ITAE). The ISE penalizes large errors more heavily than small errors, as is shown in Figure 9.4. Because of the unavoidable large errors at small t due to transient responses, it is reasonable not to put too much weight on those errors. This is achieved by multiplying t with $|e(t)|$. Thus the ITAE puts less weight on $e(t)$ for t small and more weight on $e(t)$ for t large. The total errors defined in J_2, J_3, and J_4 are all reasonable and can be used in design.

Although these criteria are reasonable, they should not be used without considering physical constraints. To illustrate this point, we consider a plant with transfer function $G(s) = (s + 2)/s(s + 3)$. Because $G(s)$ has no non-minimum-phase zero and has a pole-zero excess of 1, $G_o(s) = a/(s + a)$ is implementable for any positive a. We plot in Figure 9.5(a) the responses of $G_o(s)$ due to a unit-step reference input for $a = 1$ (solid line), $a = 10$ (dashed line), and $a = 100$ (dotted line). We see that the larger a is, the smaller J_2, J_3, and J_4 are. In fact, as a approaches infinity, J_2, J_3, and J_4 all approach zero. Therefore an optimal implementable $G_o(s)$ is $a/(s + a)$ with $a = \infty$.

As discussed in Section 6.7, the actuating signal of the plant is usually limited by

$$|u(t)| \leq M \qquad \text{for all } t \geq 0 \tag{9.9}$$

This arises from limited operational ranges of linear models or the physical constraints of devices such as the opening of valves or the rotation of rudders. Clearly, the larger the reference input, the larger the actuating signal. For convenience, the $u(t)$ in (9.9) will be assumed to be excited by a unit-step reference input and the constant M is proportionally scaled. Now we shall check whether this constraint will be met for all a. No matter how $G_o(s)$ is implemented, if there is no plant leakage, the closed-loop transfer function from the reference input r to the actuating signal u is given by

$$T(s) = \frac{G_o(s)}{G(s)} \tag{9.10}$$

If r is a step function, then the actuating signal u equals

$$U(s) = T(s)R(s) = \frac{G_o(s)}{G(s)} \cdot \frac{1}{s} = \frac{a(s + 3)}{(s + 2)(s + a)} \tag{9.11}$$

This response is plotted in Figure 9.5(b) for $a = 1$, 10, and 100. This can be obtained by analysis or by digital computer simulations. For this example, it happens that $|u(t)|_{\max} = u(0) = a$. For $a = 100$, $u(0)$ is outside the range of the plot. We see

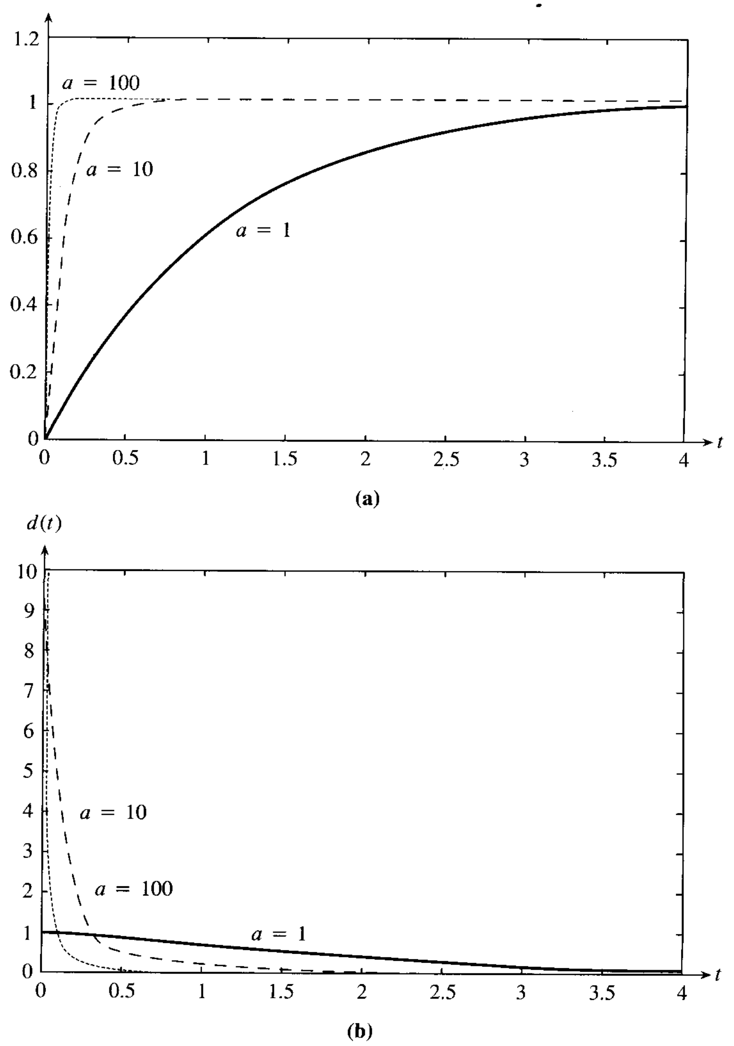


Figure 9.5 (a) Step responses. (b) Actuating signals.

that the larger a is, the larger the magnitude of the actuating signal. Therefore if a is very large, the constraint in (9.9) will be violated.

In conclusion, in using the performance indices in (9.6) to (9.8), we must include the constraint in (9.9). Otherwise we can make these indices as small as desired and the system will always be saturated. Another possible constraint is to limit the bandwidth of resulting overall systems. The reason for limiting the bandwidth is to avoid amplification of high-frequency noise. It is believed that both constraints will lead to comparable results. In this chapter we discuss only the constraint on actuating signals.

9.4 QUADRATIC PERFORMANCE INDICES

In this section we discuss the design of an overall system to minimize the quadratic performance index

$$\int_0^\infty [y(t) - r(t)]^2 \, dt \tag{9.12a}$$

subject to the constraint

$$|u(t)| \leq M \tag{9.12b}$$

for all $t \geq 0$, and for some constant M. Unfortunately, no simple analytical method is available to design such a system. Furthermore, the resulting optimal system may not be linear and time-invariant. If we limit our design to linear time-invariant systems, then (9.12) must be replaced by the following quadratic performance index

$$J = \int_0^\infty [q(y(t) - r(t))^2 + u^2(t)] \, dt \tag{9.13}$$

where q is a weighting factor and is required to be positive. If q is a large positive number, more weight is placed on the error. As q approaches infinity, the contribution of u in (9.13) becomes less significant, and at the extreme, (9.13) reduces to (9.7). In this case, since no penalty is imposed on the actuating signal, the magnitude of the actuating signal may become very large or infinity; hence, the constraint in (9.12b) will be violated. If $q = 0$, then (9.13) reduces to

$$\int_0^\infty u^2(t) \, dt$$

and the optimal system that minimizes the criterion is the one with $u \equiv 0$. From these two extreme cases, we conclude that if q in (9.13) is adequately chosen, then the constraint in (9.12b) will be satisfied. Hence, although we are forced to use the quadratic performance index in (9.13) for mathematical convenience, if q is properly chosen, (9.13) is an acceptable substitution for (9.12).

9.4.1 Quadratic Optimal Systems

Consider a plant with transfer function

$$G(s) = \frac{N(s)}{D(s)} \tag{9.14}$$

It is assumed that $N(s)$ and $D(s)$ have no common factors and $\deg N(s) \leq \deg D(s) = n$. The design problem is to find an overall transfer function to minimize the quadratic performance index

$$J = \int_0^\infty [q(y(t) - r(t))^2 + u^2(t)] \, dt \tag{9.15}$$

where q is a positive constant, r is the reference signal, y is the output, and u is the actuating signal. Before proceeding, we first discuss the spectral factorization.

Consider the polynomial

$$Q(s) := D(s)D(-s) + qN(s)N(-s) \tag{9.16}$$

It is formed from the denominator and numerator of the plant transfer function and the weighting factor q. It is clear that $Q(s) = Q(-s)$. Hence, if s_1 is a root of $Q(s)$, so is $-s_1$. Since all the coefficients of $Q(s)$ are real by assumption, if s_1 is a root of $Q(s)$, so is its complex conjugate s_1^*. Consequently all the roots of $Q(s)$ are symmetric with respect to the real axis, the imaginary axis, and the origin of the s-plane, as shown in Figure 9.6. We now show that $Q(s)$ has no root on the imaginary axis. Consider

$$\begin{aligned} Q(j\omega) &= D(j\omega)D(-j\omega) + qN(j\omega)N(-j\omega) \\ &= |D(j\omega)|^2 + q|N(j\omega)|^2 \end{aligned} \tag{9.17}$$

The assumption that $D(s)$ and $N(s)$ have no common factors implies that there exists no ω_0 such that $D(j\omega_0) = 0$ and $N(j\omega_0) = 0$. Otherwise $s^2 + \omega_0^2$ would be a common factor of $D(s)$ and $N(s)$. Thus if $q \neq 0$, $Q(j\omega)$ in (9.17) cannot be zero for any ω. Consequently, $Q(s)$ has no root on the imaginary axis. Now we shall divide the roots of $Q(s)$ into two groups, those in the open left half plane and those in the open right half plane. If all the open left-half-plane roots are denoted by $D_o(s)$, then, because of the symmetry property, all the open right-half-plane roots can be denoted by $D_o(-s)$. Thus, we can always factor $Q(s)$ as

$$Q(s) = D(s)D(-s) + qN(s)N(-s) = D_o(s)D_o(-s) \tag{9.18}$$

where $D_o(s)$ is a Hurwitz polynomial. The factorization in (9.18) is called the *spectral factorization*.

With the spectral factorization, we are ready to discuss the optimal overall transfer function. The optimal overall transfer function depends on the reference signal $r(t)$. The more complicated $r(t)$, the more complicated the optimal overall transfer function. We discuss in the following only the case where $r(t)$ is a step function.

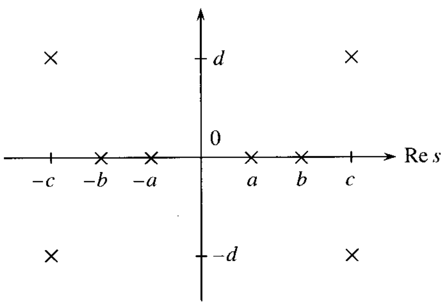


Figure 9.6 Distribution of the roots of $Q(s)$ in (9.16).

Problem Consider a plant with transfer function $G(s) = N(s)/D(s)$, as shown in Figure 9.1, where $N(s)$ and $D(s)$ have no common factors and $\deg N(s) \leq \deg D(s) = n$. Find an *implementable* overall transfer function $G_o(s)$ to minimize the quadratic performance index

$$J = \int_0^\infty [q(y(t) - r(t))^2 + u^2(t)]\,dt$$

where $q > 0$, and $r(t) = 1$ for $t \geq 0$, that is, $r(t)$ is a step-reference signal.

Solution First we compute the spectral factorization:

$$Q(s) := D(s)D(-s) + qN(s)N(-s) = D_o(s)D_o(-s)$$

where $D_o(s)$ is a Hurwitz polynomial. Then the optimal overall transfer function is given by

$$G_o(s) = \frac{qN(0)}{D_o(0)} \cdot \frac{N(s)}{D_o(s)} \tag{9.19}$$

The proof of (9.19) is beyond the scope of this text; its employment, however, is very simple. This is illustrated by the following example.

Example 9.4.1

Consider a plant with transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{1}{s(s + 2)} \tag{9.20}$$

Find $G_o(s)$ to minimize

$$J = \int_0^\infty [9(y(t) - 1)^2 + u^2(t)]\,dt \tag{9.21}$$

Clearly we have $q = 9$,

$$D(s) = s(s + 2) \qquad D(-s) = -s(-s + 2)$$

and

$$N(s) = 1 \qquad N(-s) = 1$$

We compute

$$\begin{aligned} Q(s) &:= D(s)D(-s) + qN(s)N(-s) \\ &= s(s + 2)(-s)(-s + 2) + 9 \cdot 1 \cdot 1 \\ &= -s^2(-s^2 + 4) + 9 = s^4 - 4s^2 + 9 \end{aligned} \tag{9.22}$$

It is an even function of s. If terms with odd powers of s appear in $Q(s)$, an error must have been committed in the computation. Using the formula for computing the roots of quadratic equations, we have

$$s^2 = \frac{4 \pm \sqrt{16 - 4 \cdot 9}}{2} = \frac{4 \pm j\sqrt{20}}{2} = 2 \pm j\sqrt{5} = 3e^{\pm j\theta}$$

with

$$\theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) = 48°$$

Thus the four roots of $Q(s)$ are

$$\begin{array}{ll} \sqrt{3}e^{j\theta/2} = \sqrt{3}e^{j24°} & -\sqrt{3}e^{j24°} = \sqrt{3}e^{j(180°+24°)} = \sqrt{3}e^{j204°} \\ \sqrt{3}e^{-j24°} & -\sqrt{3}e^{-j24°} = \sqrt{3}e^{j(180°-24°)} = \sqrt{3}e^{j156°} \end{array}$$

as shown in Figure 9.7. The two roots in the left column are in the open right half s-plane; the two roots in the right column are in the open left half s-plane. Using the two left-half-plane roots, we form

$$\begin{aligned} D_o(s) &= (s + \sqrt{3}e^{j24°})(s + \sqrt{3}e^{-j24°}) \\ &= s^2 + \sqrt{3}(e^{j24°} + e^{-j24°})s + 3 \\ &= s^2 + 2 \cdot \sqrt{3}(\cos 24°)s + 3 = s^2 + 3.2s + 3 \end{aligned} \tag{9.23}$$

This completes the spectral factorization. Because $q = 9$, $N(0) = 1$, and $D_o(0) = 3$, the optimal system is, using (9.19),

$$G_o(s) = \frac{9 \cdot 1}{3} \cdot \frac{1}{s^2 + 3.2s + 3} = \frac{3}{s^2 + 3.2s + 3} \tag{9.24}$$

This $G_o(s)$ is clearly implementable. Because $G_o(0) = 1$, the optimal system has a zero position error. The implementation of this optimal system will be discussed in the next chapter.

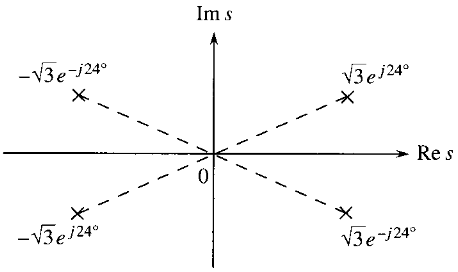


Figure 9.7 Roots of (9.22).

Exercise 9.4.1

Given $G(s) = (s - 1)/s(s + 1)$, find an implementable overall transfer function to minimize

$$J = \int_0^\infty [9(y(t) - 1)^2 + u^2(t)]\,dt \tag{9.25}$$

[**Answer:** $G_o(s) = -3(s - 1)/(s + 3)(s + 1)$.]

9.4.2 Computation of Spectral Factorizations

The design of quadratic optimal systems requires the computation of spectral factorizations. One way to carry out the factorization is to compute all the roots of $Q(s)$ and then group all the left-half-plane roots, as we did in (9.23). This method can be easily carried out if software for solving roots of polynomials is available. For example, if we use PC-MATLAB to carry out the spectral factorization of $Q(s)$ in (9.22), then the commands

```
q=[1 0 -4 0 9];
r=roots(q)
```

yield the following four roots:

```
r= -1.5811+0.7071i
   -1.5811-0.7071i
    1.5811+0.7071i
    1.5811-0.7071i
```

The first and second roots are in the open left half plane and will be used to form $D_o(s)$. The command

```
poly([r(1) r(2)])
```

yields a polynomial of degree 2 with coefficients

```
1.0000   3.1623   3.0000
```

This is $D_o(s)$. Thus the use of a digital computer to carry out spectral factorizations is very simple.

We now introduce a method of carrying out spectral factorizations without solving for roots. Consider the $Q(s)$ in (9.22). It is a polynomial of degree 4. In the spectral factorization of

$$Q(s) = s^4 - 4s^2 + 9 = D_o(s)D_o(-s) \tag{9.26}$$

the degrees of polynomials $D_o(s)$ and $D_o(-s)$ are the same. Therefore, the degree of $D_o(s)$ is half of that of $Q(s)$, or two for this example. Let

$$D_o(s) = b_0 + b_1 s + b_2 s^2 \tag{9.27}$$

where b_i are required to be all positive.[2] If any one of them is zero or negative, then $D_o(s)$ is not Hurwitz. Clearly, we have

$$D_o(-s) = b_0 + b_1(-s) + b_2(-s)^2 = b_0 - b_1 s + b_2 s^2 \tag{9.28}$$

The multiplication of $D_o(s)$ and $D_o(-s)$ yields

$$\begin{aligned} D_o(s)D_o(-s) &= (b_0 + b_1 s + b_2 s^2)(b_0 - b_1 s + b_2 s^2) \\ &= b_0^2 + (2b_0 b_2 - b_1^2)s^2 + b_2^2 s^4 \end{aligned}$$

It is an even function of s. In order to meet (9.26), we equate

$$b_0^2 = 9$$

$$2b_0 b_2 - b_1^2 = -4$$

and

$$b_2^2 = 1$$

Thus we have $b_0 = 3$, $b_2 = 1$ and

$$b_1^2 = 2b_0 b_2 + 4 = 2 \cdot 3 \cdot 1 + 4 = 10$$

which implies $b_1 = \sqrt{10}$. Note that we require all b_i to be positive; therefore, we have taken only the positive part of the square roots. Thus the spectral factorization of (9.26) is

$$D_o(s) = 3 + \sqrt{10}\, s + s^2 = 3 + 3.2s + s^2$$

We see that this procedure is quite simple and can be used if a digital computer and the required software are not available. The preceding result can be stated more generally as follows: If

$$Q(s) = a_0 + a_2 s^2 + a_4 s^4 \tag{9.29}$$

and if

$$D_o(s) = b_0 + b_1 s + b_2 s^2 \tag{9.30}$$

then

$$b_0 = \sqrt{a_0} \qquad b_2 = \sqrt{a_4} \qquad b_1 = \sqrt{(-a_2 + 2b_0 b_2)} \tag{9.31}$$

Note that before computing b_1, we must compute first b_0 and b_2.

Now we shall extend the preceding procedure to a more general case. Consider

$$Q(s) = a_0 + a_2 s^2 + a_4 s^4 + a_6 s^6 \tag{9.32}$$

It is an even polynomial of degree 6. Let

$$D_o(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 \tag{9.33}$$

[2]Also, they can all be negative. For convenience, we consider only the positive case.

Then

$$D_o(-s) = b_0 - b_1 s + b_2 s^2 - b_3 s^3$$

and

$$D_o(s)D_o(-s) = b_0^2 + (2b_0b_2 - b_1^2)s^2 + (b_2^2 - 2b_1b_3)s^4 - b_3^2 s^6 \tag{9.34}$$

Equating (9.32) and (9.34) yields

$$b_0^2 = a_0 \tag{9.35a}$$

$$2b_0b_2 - b_1^2 = a_2 \tag{9.35b}$$

$$b_2^2 - 2b_1b_3 = a_4 \tag{9.35c}$$

and

$$b_3^2 = -a_6 \tag{9.35d}$$

From (9.35a) and (9.35d), we can readily compute $b_0 = \sqrt{a_0}$ and $b_3 = \sqrt{-a_6}$. In other words, the leading and constant coefficients of $D_o(s)$ are simply the square roots of the magnitudes of the leading and constant coefficients of $Q(s)$. Once b_0 and b_3 are computed, there are only two unknowns, b_1 and b_2, in the two equations in (9.35b) and (9.35c). These two equations are not linear and can be solved iteratively as follows. We rewrite them as

$$b_1 = \sqrt{2b_0b_2 - a_2} \tag{9.36a}$$

$$b_2 = \sqrt{a_4 + 2b_1b_3} \tag{9.36b}$$

First we choose an arbitrary b_2—say, $b_2^{(0)}$—and use this $b_2^{(0)}$ to compute b_1 as

$$b_1^{(1)} = \sqrt{2b_0b_2^{(0)} - a_2}$$

We then use this $b_1^{(1)}$ to compute b_2 as

$$b_2^{(1)} = \sqrt{a_4 + 2b_1^{(1)}b_3}$$

If $b_2^{(1)}$ happens to equal $b_2^{(0)}$, then the chosen $b_2^{(0)}$ is the solution of (9.36). Of course, the possibility of having $b_2^{(1)} = b_2^{(0)}$ is extremely small. We then use $b_2^{(1)}$ to compute a new b_1 as

$$b_1^{(2)} = \sqrt{2b_0b_2^{(1)} - a_2}$$

and then a new b_2 as

$$b_2^{(2)} = \sqrt{a_4 + 2b_1^{(2)}b_3}$$

If $b_2^{(2)}$ is still quite different from $b_2^{(1)}$, we repeat the process. It can be shown that the process will converge to the true solutions.[3] This is an iterative method of carrying out the spectral factorization. In application, we may stop the iteration when the difference between two subsequent $b_2^{(i)}$ and $b_2^{(i+1)}$ is smaller than, say, 5%. This is illustrated by an example.

[3] If we compute $b_2 = (a_2 + b_1^2)/2b_0$ and $b_1 = (b_2^2 - a_4)/2b_3$ iteratively, the process will diverge.

Example 9.4.2

Compute the spectral factorization of

$$Q(s) = 25 - 41s^2 + 20s^4 - 4s^6 \tag{9.37}$$

Let

$$D_o(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3$$

Its constant term and leading coefficient are simply the square roots of the corresponding coefficients of $Q(s)$:

$$b_0 = \sqrt{25} = 5 \qquad b_3 = \sqrt{|-4|} = \sqrt{4} = 2$$

The substitution of these into (9.36) yields

$$b_1 = \sqrt{10b_2 + 41}$$
$$b_2 = \sqrt{20 + 4b_1}$$

Now we shall solve these equations iteratively. Arbitrarily, we choose b_2 as $b_2^{(0)} = 0$ and compute

	(0)	(1)	(2)	(3)	(4)	(5)
b_1		6.4	10.42	10.93	10.99	10.999
b_2	0	6.75	7.85	7.98	7.998	7.9998

We see that they converge rapidly to the solutions $b_1 = 11$ and $b_2 = 8$. To verify the convergence, we now choose b_2 as $b_2^{(0)} = 100$ and compute

	(0)	(1)	(2)	(3)	(4)	(5)
b_1		32.26	12.77	11.19	11.02	11.002
b_2	100	12.21	8.43	8.05	8.005	8.0006

They also converge rapidly to the solutions $b_1 = 11$ and $b_2 = 8$.

The preceding iterative procedure can be extended to the general case. The basic idea is the same and will not be repeated.

Exercise 9.4.2

Carry out spectral factorizations for

a. $4s^4 - 9s^2 + 16$

b. $-4s^6 + 10s^4 - 20s^2 + 16$

[**Answers:** $2s^2 + 5s + 4$, $2s^3 + 6.65s^2 + 8.56s + 4$.]

9.4.3 Selection of Weighting Factors

In this subsection we discuss the problem of selecting a weighting factor in the quadratic performance index to meet the constraint $|u(t)| \le M$ for all $t \ge 0$. It is generally true that a larger q yields a larger actuating signal and a faster response. Conversely, a smaller q yields a smaller actuating signal and a slower response. Therefore, by choosing q properly, the constraint on the actuating signal can be met. We use the example in (9.20) to illustrate the procedure.

Consider a plant with transfer function $G(s) = 1/s(s + 2)$. Design an overall system to minimize

$$J = \int_0^{\infty} [q(y(t) - 1)^2 + u^2(t)]dt$$

It is also required that the actuating signal due to a unit-step reference input meet the constraint $|u(t)| \le 3$, for all $t \ge 0$. Arbitrarily, we choose $q = 100$ and compute

$$Q(s) = s(s + 2)(-s)(-s + 2) + 100 \cdot 1 \cdot 1 = s^4 - 4s^2 + 100$$

Its spectral factorization can be computed as, using (9.31),

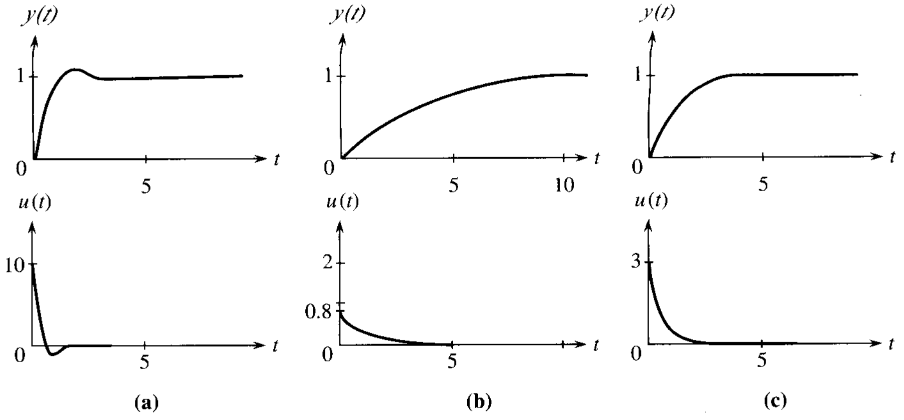
$$D_o(s) = s^2 + \sqrt{24}s + 10 = s^2 + 4.9s + 10$$

Thus the quadratic optimal transfer function is

$$G_o(s) = \frac{Y(s)}{R(s)} = \frac{qN(0)}{D_o(0)} \cdot \frac{N(s)}{D_o(s)} = \frac{100 \cdot 1}{10} \cdot \frac{1}{s^2 + 4.9s + 10}$$
$$= \frac{10}{s^2 + 4.9s + 10}$$

The unit-step response of this system is simulated and plotted in Figure 9.8(a). Its rise time, settling time, and overshoot are 0.92 s, 1.70 s, and 2.13%, respectively. Although the response is quite good, we must check whether or not its actuating signal meets the constraint. No matter what configuration is used to implement $G_o(s)$, if there is no plant leakage, the transfer function from the reference signal r to the actuating signal u is

$$T(s) = \frac{G_o(s)}{G(s)} = \frac{10}{s^2 + 4.9s + 10} \cdot \frac{s(s + 2)}{1} = \frac{10s(s + 2)}{s^2 + 4.9s + 10}$$



	(a)	(b)	(c)
q	= 100	0.64	9
Rise time	= 0.92	6.21	2.01
Settling time	= 1.70	10.15	2.80
Overshoot	= 2.13%	0%	0.09%
$u(0+)$	= 10	0.8	3

Figure 9.8 Responses of quadratic optimal systems.

The unit-step response of $T(s)$ is simulated and also plotted in Figure 9.8(a). We see that $u(0^+) = 10$ and the constraint $|u(t)| \leq 3$ is violated. Because the largest magnitude of $u(t)$ occurs at $t = 0^+$, it can also be computed by using the initial-value theorem (see Appendix A). The response $u(t)$ due to $r(t) = 1$ is

$$U(s) = T(s)R(s) = \frac{10s(s + 2)}{s^2 + 4.9s + 10} \cdot \frac{1}{s}$$

The application of the initial-value theorem yields

$$u(0^+) = \lim_{s\to\infty} sU(s) = \lim_{s\to\infty} sT(s)R(s) = \lim_{s\to\infty} T(s) = 10$$

Thus the constraint $|u(t)| \leq 3$ is not met and the selection of $q = 100$ is not acceptable.[4]

Next we choose $q = 0.64$ and repeat the design. The optimal transfer function is found as

$$G_o(s) = \frac{0.64 \times 1}{0.8} \cdot \frac{1}{s^2 + \sqrt{5.6}\, s + 0.8} = \frac{0.8}{s^2 + 2.4s + 0.8}$$

[4]It is shown by B. Seo [51] that if a plant transfer function is of the form $(b_1 s + b_0)/s(s + a)$, with $b_0 \neq 0$, then the maximum magnitude of the actuating signal of quadratic optimal systems occurs at $t = 0^+$ and $|u(t)| \leq u(0^+) = \sqrt{q}$.

Its unit-step response and the actuating signal are plotted in Figure 9.8(b). The response is fairly slow. Because $|u(t)| \leq u(0^+) = 0.8$ is much smaller than 3, the system can be designed to respond faster. Next we try $q = 9$, and compute

$$Q(s) = s(s + 2)(-s)(-s + 2) + 9 \cdot 1 \cdot 1$$

Its spectral factorization, using (9.31), is

$$D_o(s) = s^2 + \sqrt{10}\, s + 3 = s^2 + 3.2s + 3$$

Thus the optimal transfer function is

$$G_o(s) = \frac{qN(0)}{D_o(0)} \cdot \frac{N(s)}{D_o(s)} = \frac{9 \cdot 1}{3} \cdot \frac{1}{s^2 + 3.2s + 3} = \frac{3}{s^2 + 3.2s + 3} \tag{9.38}$$

and the transfer function from r to u is

$$T(s) = \frac{G_o(s)}{G(s)} = \frac{3s(s + 2)}{s^2 + 3.2s + 3}$$

Their unit-step responses are plotted in Figure 9.8(c). The rise time of $y(t)$ is 2.01 seconds, the settling time is 2.80 seconds, and the overshoot is 0.09%. We also have $|u(t)| \leq u(0^+) = T(\infty) = 3$, for all t. Thus this overall system has the fastest response under the constraint $|u(t)| \leq 3$.

From this example, we see that the weighting factor q is to be chosen by trial and error. We choose an arbitrary q, say $q = q_0$, and carry out the design. After the completion of the design, we then simulate the resulting overall system. If the response is slow or sluggish, we may increase q and repeat the design. In this case, the response will become faster. However, the actuating signal may also become larger and the plant may be saturated. Thus the choice of q is generally reached by a compromise between the speed of response and the constraint on the actuating signal.

Optimality is a fancy word because it means "the best." However, without introducing a performance index, it is meaningless to talk about optimality. Even if a performance index is introduced, if it is not properly chosen, the resulting system may not be satisfactory in practice. For example, the second system in Figure 9.8 is optimal with $q = 0.64$, but it is very slow. Therefore, the choice of a suitable performance index is not necessarily simple.

Exercise 9.4.3

Given a plant with transfer function $G(s) = (s + 2)/s(s - 2)$, find a quadratic optimal system under the constraint that the magnitude of the actuating signal due to a unit step reference input is less than 5.

[**Answer:** $G_o(s) = 5(s + 2)/(s^2 + 7s + 10)$.]

9.5 THREE MORE EXAMPLES

In this section we shall discuss three more examples. Every one of them will be redesigned in latter sections and be compared with quadratic optimal design.

Example 9.5.1

Consider a plant with transfer function [19, 34]

$$G(s) = \frac{2}{s(s^2 + 0.25s + 6.25)} \tag{9.39}$$

Design an overall system to minimize

$$J = \int_0^\infty [q(y(t) - 1)^2 + u^2(t)]dt \tag{9.40}$$

The weighting factor q is to be chosen so that the actuating signal $u(t)$ due to a unit-step reference input meets $|u(t)| \le 10$ for $t \ge 0$. First we choose $q = 9$ and compute

$$\begin{aligned} Q(s) &= D(s)D(-s) + qN(s)N(-s) \\ &= s(s^2 + 0.25s + 6.25) \cdot (-s)(s^2 - 0.25s + 6.25) + 9 \cdot 2 \cdot 2 \qquad (9.41) \\ &= -s^6 - 12.4375s^4 - 39.0625s^2 + 36 \end{aligned}$$

The spectral factorization of (9.41) can be carried out iteratively as discussed in Section 9.4.2 or by solving its roots. As a review, we use both methods in this example. We first use the former method. Let

$$D_o(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3$$

Its constant term and leading coefficient are simply the square roots of the corresponding coefficients of $Q(s)$:

$$b_0 = \sqrt{36} = 6 \qquad b_3 = \sqrt{|-1|} = \sqrt{1} = 1$$

The substitution of these into (9.36) yields

$$b_1 = \sqrt{12b_2 + 39.0625}$$
$$b_2 = \sqrt{2b_1 - 12.4375}$$

Now we shall solve these equations iteratively. Arbitrarily, we choose b_2 as $b_2^{(0)} = 0$ and compute

b_1		6.25	6.49	6.91	7.30	7.53	7.65	7.70	7.73	7.75	7.75
b_2	0	0.25	0.73	1.18	1.47	1.62	1.69	1.72	1.74	1.74	1.75

We see that they converge to the solution $b_1 = 7.75$ and $b_2 = 1.75$. Thus we have $Q(s) = D_o(s)D_o(-s)$ with

$$D_o(s) = s^3 + 1.75s^2 + 7.75s + 6$$

Thus the optimal overall transfer function that minimizes (9.40) with $q = 9$ is

$$\begin{aligned} G_o(s) &= \frac{qN(0)}{D_o(0)} \cdot \frac{N(s)}{D_o(s)} = \frac{9 \cdot 2}{6} \cdot \frac{2}{s^3 + 1.75s^2 + 7.75s + 6} \\ &= \frac{6}{s^3 + 1.75s^2 + 7.75s + 6} \end{aligned} \tag{9.42}$$

For this overall transfer function, it is found by computer simulation that $|u(t)| \leq 3$, for $t \geq 0$. Thus we may choose a larger q. We choose $q = 100$ and compute

$$\begin{aligned} Q(s) &= D(s)D(-s) + 100N(s)N(-s) \\ &= -s^6 - 12.4375s^4 - 39.0625s^2 + 400 \end{aligned}$$

Now we use the second method to carry out the spectral factorization. We use PC-MATLAB to compute its roots. The command

```
r=roots([-1 0 -12.4375 0 -39.0625 0 400])
```

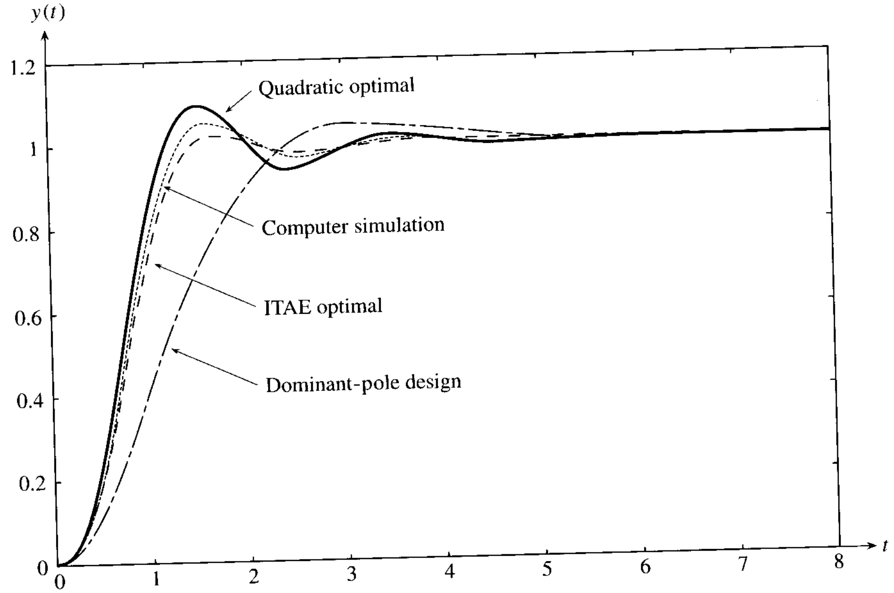


Figure 9.9 Responses of various designs of (9.39).

yields

```
r= -0.9917+3.0249i
   -0.9917-3.0249i
    0.9917+3.0249i
    0.9917-3.0249i
    1.9737
   -1.9737
```

The first, second, and last roots are in the open left half plane. The command

```
poly([r(1) r(2) r(6)])
```

yields [1.000 3.9571 14.0480 20.0000]. Thus we have $D_o(s) = s^3 + 3.957s^2 + 14.048s + 20$ and the quadratic optimal overall transfer function is

$$G_o(s) = \frac{20}{s^3 + 3.957s^2 + 14.048s + 20} \tag{9.43}$$

For this transfer function, the maximum amplitude of the actuating signal due to a unit-step reference input is 10. Thus we cannot choose a larger q. The unit-step response of $G_o(s)$ in (9.43) is plotted in Figure 9.9 with the solid line. The response appears to be quite satisfactory.

Example 9.5.2

Consider a plant with transfer function

$$G(s) = \frac{s + 3}{s(s - 1)} \tag{9.44}$$

Find an overall transfer function to minimize the quadratic performance index

$$J = \int_0^{\infty} [100(y(t) - 1)^2 + u^2(t)]dt \tag{9.45}$$

where the weighting factor has been chosen as 100. We first compute

$$\begin{aligned} Q(s) &= D(s)D(-s) + qN(s)N(-s) \\ &= s(s - 1)(-s)(-s - 1) + 100(s + 3)(-s + 3) \\ &= s^4 - 101s^2 + 900 \\ &= (s + 9.5459)(s - 9.5459)(s + 3.1427)(s - 3.1427) \end{aligned}$$

where we have used PC-MATLAB to compute the roots of $Q(s)$. Thus we have $Q(s) = D_o(s)D_o(-s)$ with

$$D_o(s) = (s + 9.5459)(s + 3.1427) = s^2 + 12.7s + 30$$

and the quadratic optimal system is given by

$$G_o(s) = \frac{qN(0)}{D_o(0)} \cdot \frac{N(s)}{D_o(s)} = \frac{10(s + 3)}{s^2 + 12.7s + 30} \tag{9.46}$$

Its response due to a unit-step reference input is shown in Figure 9.10(a) with the solid line. The actuating signal due to a unit-step reference input is shown in Figure 9.10(b) with the solid line; it has the property $|u(t)| \leq 10$ for $t \geq 0$.

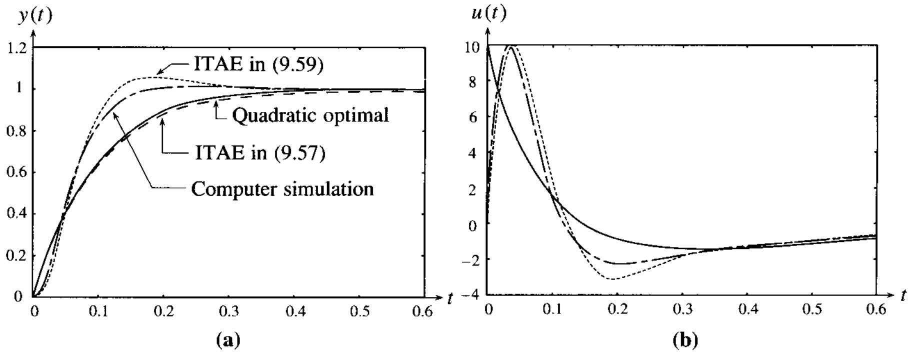


Figure 9.10 Responses of various designs of (9.44).

Example 9.5.3

Consider a plant with transfer function

$$G(s) = \frac{s - 1}{s(s - 2)} \tag{9.47}$$

It has a non-minimum-phase zero. To find the optimal system to minimize the quadratic performance index in (9.45), we compute

$$\begin{aligned} Q(s) &= s(s - 2)(-s)(-s - 2) + 100(s - 1)(-s - 1) = s^4 - 104s^2 + 100 \\ &= (s + 10.1503)(s - 10.1503)(s + 0.9852)(s - 0.9852) \end{aligned}$$

Thus we have $D_o(s) = s^2 + 11.14s + 10$ and

$$G_o(s) = \frac{-10(s - 1)}{s^2 + 11.14s + 10} \tag{9.48}$$

Its unit-step response is shown in Figure 9.11 with the solid line. By computer simulation we also find $|u(t)| \leq 10$ for $t \geq 0$ if the reference input is a unit-step function.

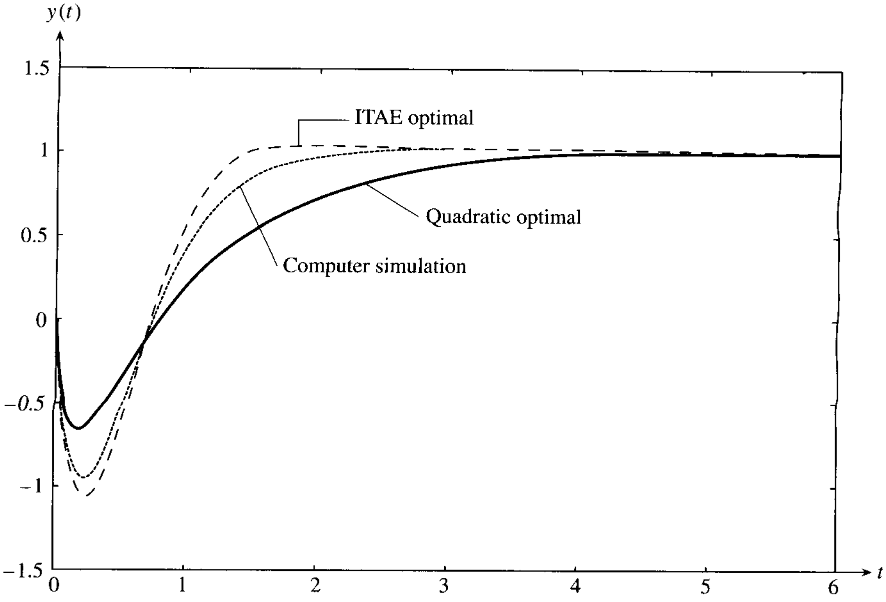


Figure 9.11 Responses of various designs of (9.47).

9.5.1 Symmetric Root Loci[5]

We discuss in this subsection the poles of $G_o(s)$ as a function of q by using the root-locus method. Consider the polynomial

$$D(s)D(-s) + qN(s)N(-s) \tag{9.49}$$

The roots of (9.49) are the zeros of the rational function

$$1 + qG(s)G(-s)$$

or the solution of the equation

$$\frac{-1}{q} = G(s)G(-s) = \frac{N(s)N(-s)}{D(s)D(-s)} \tag{9.50}$$

These equations are similar to (7.11) through (7.13), thus the root-locus method can be directly applied. The root loci of (9.50) for $G(s) = 1/s(s + 2)$ are plotted in

[5]This section may be skipped without loss of continuity.

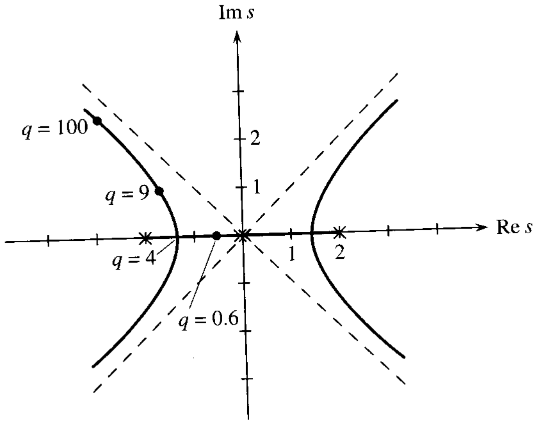


Figure 9.12 Root loci of (9.50).

Figure 9.12. The roots for $q = 0.64$, 4, 9, and 100 are indicated as shown. We see that the root loci are symmetric with respect to the imaginary axis as well as the real axis. Furthermore the root loci will not cross the imaginary axis for $q > 0$. Although the root loci reveal the migration of the poles of the quadratic optimal system, they do not tell us how to pick a specific set of poles to meet the constraint on the actuating signal.

We discuss now the poles of $G_o(s)$ as $q \to \infty$. It is assumed that $G(s)$ has n poles and m zeros and has no non-minimum-phase zeros. Then, as $q \to \infty$, $2m$ root loci of $G(s)G(-s)$ will approach the $2m$ roots of $N(s)N(-s)$ and the remaining $(2n - 2m)$ root loci will approach the $(2n - 2m)$ asymptotes with angles

$$\frac{(2k + 1)\pi}{2n - 2m} \qquad k = 0, 1, 2, \ldots, 2n - 2m - 1$$

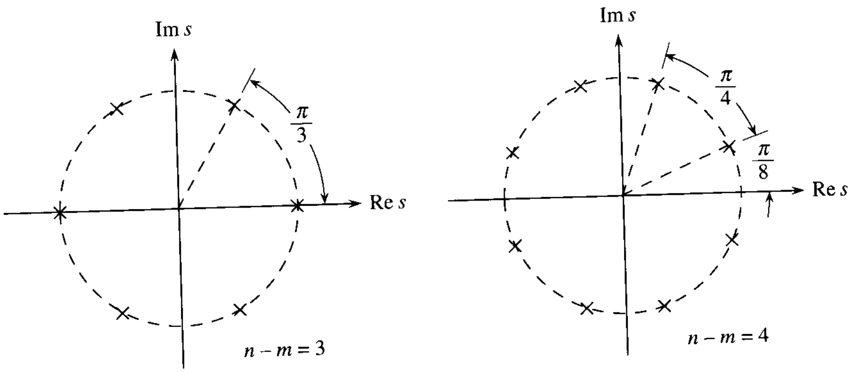


Figure 9.13 Distribution of optimal poles as $q \to \infty$.

(see Section 7.4, in particular, (7.27)). Thus as q approaches infinity, m poles of $G_o(s)$ will cancel m zeros of $G(s)$ and the remaining $(n - m)$ poles of $G_o(s)$ will distribute as shown in Figure 9.13, where we have assumed that the centroid defined in (7.27a) is at the origin. The pole pattern is identical to that of the Butterworth filter [13].

9.6 ITAE OPTIMAL SYSTEMS [33]

In this section we discuss the design of control systems to minimize the integral of time multiplied by absolute error (ITAE) in (9.8). For the quadratic overall system

$$G_o(s) = \frac{1}{s^2 + 2\zeta s + 1}$$

the ITAE, the integral of absolute error (IAE) in (9.6), and the integral of square error (ISE) in (9.7) as a function of the damping ratio ζ are plotted in Figure 9.14. The ITAE has largest changes as ζ varies, and therefore has the best selectivity. The ITAE also yields a system with a faster response than other criteria, therefore Graham and Lathrop [33] chose it as their design criterion. The system that has the smallest ITAE is called the *optimal system in the sense of ITAE* or the *ITAE optimal system.*

Consider the overall transfer function

$$G_o(s) = \frac{\alpha_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2 s^2 + \alpha_1 s + \alpha_0} \tag{9.51}$$

This transfer function contains no zeros. Because $G_o(0) = 1$, if $G_o(s)$ is stable, then the position error is zero, or the plant output will track asymptotically any step-reference input. By analog computer simulation, the denominators of ITAE optimal systems were found to assume the forms listed in Table 9.1. Their poles and unit-step responses, for $\omega_0 = 1$, are plotted in Figures 9.15 and 9.16. We see that the optimal poles are distributed evenly around the neighborhood of the unit circle. We also see that the overshoots of the unit-step responses are fairly large for large n. These systems are called the *ITAE zero-position-error optimal systems.*

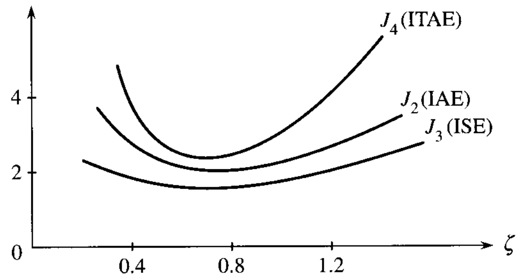


Figure 9.14 Comparison of various design criteria.

Table 9.1 ITAE Zero-Position-Error Optimal Systems

$$s + \omega_0$$
$$s^2 + 1.4\omega_0 s + \omega_0^2$$
$$s^3 + 1.75\omega_0 s^2 + 2.15\omega_0^2 s + \omega_0^3$$
$$s^4 + 2.1\omega_0 s^3 + 3.4\omega_0^2 s^2 + 2.7\omega_0^3 s + \omega_0^4$$
$$s^5 + 2.8\omega_0 s^4 + 5.0\omega_0^2 s^3 + 5.5\omega_0^3 s^2 + 3.4\omega_0^4 s + \omega_0^5$$
$$s^6 + 3.25\omega_0 s^5 + 6.60\omega_0^2 s^4 + 8.60\omega_0^3 s^3 + 7.45\omega_0^4 s^2 + 3.95\omega_0^5 s + \omega_0^6$$
$$s^7 + 4.475\omega_0 s^6 + 10.42\omega_0^2 s^5 + 15.08\omega_0^3 s^4 + 15.54\omega_0^4 s^3 + 10.64\omega_0^5 s^2 + 4.58\omega_0^6 s + \omega_0^7$$
$$s^8 + 5.20\omega_0 s^7 + 12.80\omega_0^2 s^6 + 21.60\omega_0^3 s^5 + 25.75\omega_0^4 s^4 + 22.20\omega_0^5 s^3 + 13.30\omega_0^6 s^2 + 5.15\omega_0^7 s + \omega_0^8$$

We now discuss the optimization of

$$G_o(s) = \frac{\alpha_1 s + \alpha_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2 s^2 + \alpha_1 s + \alpha_0} \tag{9.52}$$

with respect to the ITAE criterion. The transfer function has one zero; their coefficients, however, are constrained so that $G_o(s)$ has zero position error and zero velocity error. This system will track asymptotically any ramp-reference input. By analog computer simulation, the optimal step responses of $G_o(s)$ in (9.52) are found as shown in Figure 9.17. The optimal denominators of $G_o(s)$ in (9.52) are listed in Table 9.2. The systems are called the *ITAE zero-velocity-error optimal systems.*

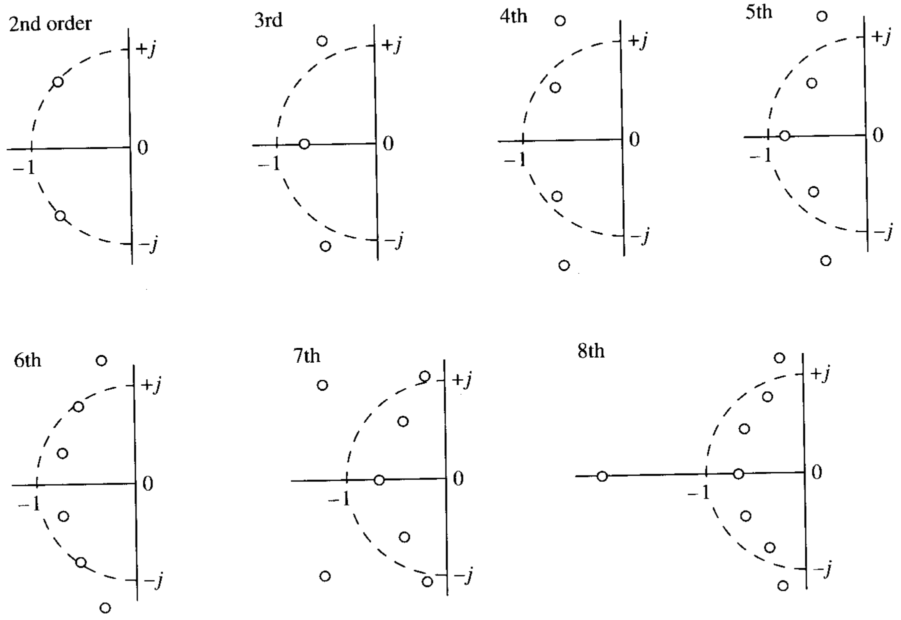


Figure 9.15 Optimal pole locations.

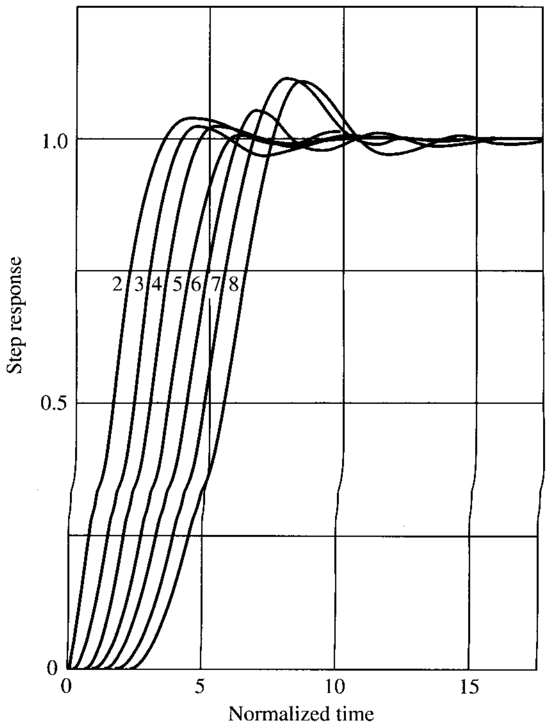


Figure 9.16 Step responses of ITAE optimal systems with zero position error.

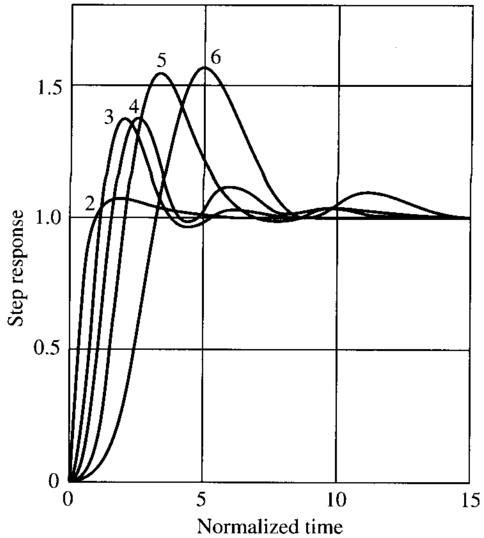


Figure 9.17 Step responses of ITAE optimal systems with zero velocity error.

Table 9.2 ITAE Zero-Velocity-Error Optimal Systems

$s^2 + 3.2\omega_0 s + \omega_0^2$
$s^3 + 1.75\omega_0 s^2 + 3.25\omega_0^2 s + \omega_0^3$
$s^4 + 2.41\omega_0 s^3 + 4.93\omega_0^2 s^2 + 5.14\omega_0^3 s + \omega_0^4$
$s^5 + 2.19\omega_0 s^4 + 6.50\omega_0^2 s^3 + 6.30\omega_0^3 s^2 + 5.24\omega_0^4 s + \omega_0^5$
$s^6 + 6.12\omega_0 s^5 + 13.42\omega_0^2 s^4 + 17.16\omega_0^3 s^3 + 14.14\omega_0^4 s^2 + 6.76\omega_0^5 s + \omega_0^6$

Similarly, for the following overall transfer function

$$G_o(s) = \frac{\alpha_2 s^2 + \alpha_1 s + \alpha_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2 s^2 + \alpha_1 s + \alpha_0} \tag{9.53}$$

the optimal step responses are shown in Figure 9.18 and the optimal denominators are listed in Table 9.3. They are called the *ITAE zero-acceleration-error optimal systems.* We see from Figures 9.16, 9.17, and 9.18 that the optimal step responses for $G_o(s)$ with and without zeros are quite different. It appears that if a system is required to track a more complicated reference input, then the transient performance will be poorer. For example, the system in Figure 9.18 tracks acceleration reference inputs, but its transient response is much worse than the one for the system in Figure 9.16, which can track only step-reference inputs. Therefore a price must be paid if we design a more complex system.

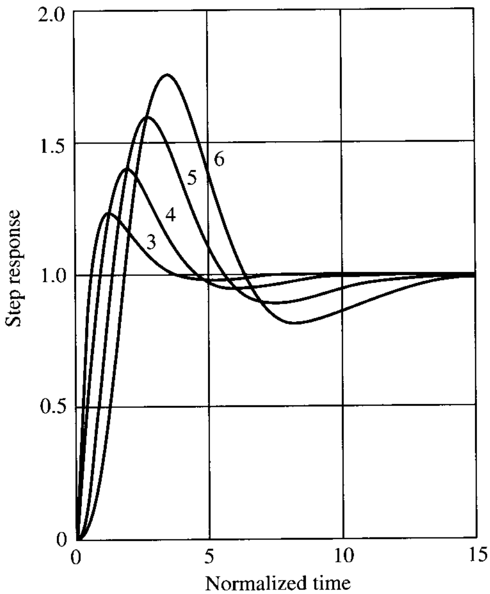


Figure 9.18 Step responses of ITAE optimal systems with zero acceleration error.

Table 9.3 *ITAE Zero-Acceleration-Error Optimal Systems*

$s^3 + 2.97\omega_0 s^2 + 4.94\omega_0^2 s + \omega_0^3$
$s^4 + 3.71\omega_0 s^3 + 7.88\omega_0^2 s^2 + 5.93\omega_0^3 s + \omega_0^4$
$s^5 + 3.81\omega_0 s^4 + 9.94\omega_0^2 s^3 + 13.44\omega_0^3 s^2 + 7.36\omega_0^4 s + \omega_0^5$
$s^6 + 3.93\omega_0 s^5 + 11.68\omega_0^2 s^4 + 18.56\omega_0^3 s^3 + 19.3\omega_0^4 s^2 + 8.06\omega_0^5 s + \omega_0^6$

9.6.1 Applications

In this subsection we discuss how to use Tables 9.1 through 9.3 to design ITAE optimal systems. These tables were developed without considering plant transfer functions. For example, for two different plant transfer functions such as $1/s(s + 2)$ and $1/s(s - 10)$, the optimal transfer function $G_o(s)$ can be chosen as

$$\frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

The actuating signals for both systems, however, will be different. Therefore ω_0 in both systems should be different. We shall use the constraint on the actuating signal as a criterion in choosing ω_0. This will be illustrated in the following examples.

Example 9.6.1

Consider the plant transfer function in (9.20) or

$$G(s) = \frac{1}{s(s + 2)}$$

Find a zero-position-error system to minimize ITAE. It is also required that the actuating signal due to a unit-step reference input satisfy the constraint

$$|u(t)| \leq 3$$

for all t.

The ITAE optimal overall transfer function is chosen from Table 9.1 as

$$G_o(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

It is implementable. Clearly the larger the ω_0, the faster the response. However, the actuating signal will also be larger. Now we shall choose ω_0 to meet $|u(t)| \leq 3$. The transfer function from r to u is

$$T(s) := \frac{U(s)}{R(s)} = \frac{G_o(s)}{G(s)} = \frac{\omega_0^2 s(s + 2)}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

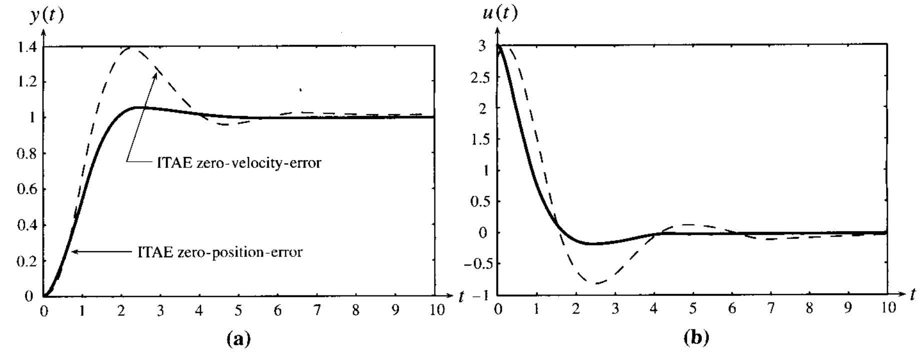


Figure 9.19 Step responses of (9.54) (with solid lines) and (9.55) (with dashed lines).

Consequently, we have

$$U(s) = T(s)R(s) = \frac{\omega_0^2 s(s+2)}{s^2 + 1.4\omega_0 s + \omega_0^2} \cdot \frac{1}{s}$$

By computer simulation, we find that the largest magnitude of $u(t)$ occurs at $t = 0^+$.[6] Thus the largest magnitude of $u(t)$ can be computed by using the initial-value theorem as

$$u_{\max} = u(0^+) = \lim_{s\to\infty} sU(s) = \lim_{s\to\infty} \frac{\omega_0^2 s(s+2)}{s^2 + 1.4\omega_0 s + \omega_0^2} = \omega_0^2$$

In order to meet the constraint $|u(t)| \leq 3$, we set $\omega_0^2 = 3$. Thus the ITAE optimal system is

$$G_o(s) = \frac{3}{s^2 + 1.4 \times \sqrt{3}s + 3} = \frac{3}{s^2 + 2.4s + 3} \tag{9.54}$$

This differs from the quadratic optimal system in (9.38) only in one coefficient. Because they minimize different criteria, there is no reason that they have the same overall transfer function. The unit-step responses of $G_o(s)$ and $T(s)$ are shown in Figure 9.19 with solid lines. They appear to be satisfactory.

Exercise 9.6.1

Consider a plant with transfer function $2/s^2$. Find an optimal system to minimize the ITAE criterion under the constraint $|u(t)| \leq 3$.

[**Answer:** $6/(s^2 + 3.4s + 6)$.]

[6]If the largest magnitude of $u(t)$ does not occur at $t = 0$, then its analytical computation will be complicated. It is easier to find it by computer simulations.

Example 9.6.2

Consider the problem in Example 9.6.1 with the additional requirement that the velocity error be zero. A possible overall transfer function is, from Table 9.2,

$$G_o(s) = \frac{3.2\omega_0 s + \omega_0^2}{s^2 + 3.2\omega_0 s + \omega_0^2}$$

However, this is not implementable because it violates the pole-zero excess inequality. Now we choose from Table 9.2 the transfer function of degree 3:

$$G_o(s) = \frac{3.25\omega_0^2 s + \omega_0^3}{s^3 + 1.75\omega_0 s^2 + 3.25\omega_0^2 s + \omega_0^3}$$

This is implementable and has zero velocity error. Now we choose ω_0 so that the actuating signal due to a unit-step reference input meets $|u(t)| \leq 3$. The transfer function from r to u is

$$T(s) = \frac{G_o(s)}{G(s)} = \frac{(3.25\omega_0^2 s + \omega_0^3)s(s + 2)}{s^3 + 1.75\omega_0 s^2 + 3.25\omega_0^2 s + \omega_0^3}$$

Its unit-step response is shown in Figure 9.19(b) with the dashed line. We see that the largest magnitude of $u(t)$ does not occur at $t = 0^+$. Therefore, the procedure in Example 9.6.1 cannot be used to choose ω_0 for this problem. By computer simulation, we find that if $\omega_0 = 0.928$, then $|u(t)| \leq 3$. For this ω_0, $G_o(s)$ becomes

$$G_o(s) = \frac{2.799s + 0.799}{s^3 + 1.624s^2 + 2.799s + 0.799} \tag{9.55}$$

This is the ITAE zero-velocity-error optimal system. Its unit-step response is plotted in Figure 9.19(a) with the dashed line. It is much more oscillatory than that of the ITAE zero-position-error optimal system. The corresponding actuating signal is plotted in Figure 9.19(b).

Example 9.6.3

Consider the plant transfer function in (9.39), that is,

$$G(s) = \frac{2}{s(s^2 + 0.25s + 6.25)}$$

Find an ITAE zero-position-error optimal system. It is also required that the actuating signal $u(t)$ due to a unit-step reference input meet the constraint $|u(t)| \leq 10$, for $t \geq 0$. We choose from Table 9.1

$$G_o(s) = \frac{\omega_0^3}{s^3 + 1.75\omega_0 s^2 + 2.15\omega_0^2 s + \omega_0^3}$$

By computer simulation, we find that if $\omega_0 = 2.7144$, then $|u(t)| \leq u(0) = 10$ for all $t \geq 0$. Thus the ITAE optimal system is

$$G_o(s) = \frac{20}{s^3 + 4.75s^2 + 15.84s + 20} \tag{9.56}$$

Its unit-step response is plotted in Figure 9.9 with the dashed line. Compared with the quadratic optimal design, the ITAE design has a faster response and a smaller overshoot. Thus for this problem, the ITAE optimal system is more desirable.

Example 9.6.4

Consider the plant transfer function in (9.44) or

$$G(s) = \frac{s + 3}{s(s - 1)}$$

Find an ITAE zero-position-error optimal system. It is also required that the actuating signal $u(t)$ due to a unit-step reference input meet the constraint $|u(t)| \leq 10$, for $t \geq 0$. The pole-zero excess of $G(s)$ is 1 and $G(s)$ has no non-minimim-phase zero; therefore, the ITAE optimal transfer function

$$G_o(s) = \frac{\omega_0}{s + \omega_0} \tag{9.57}$$

is implementable. We find by computer simulation that if $\omega_0 = 10$, then $G_o(s)$ meets the design specifications. Its step response and actuating signal are plotted in Figure 9.10 with the dashed line. They are almost indistinguishable from those of the quadratic optimal system. Because $G_o(s)$ does not contain plant zero $(s + 3)$, its implementation will involve the pole-zero cancellation of $(s + 3)$, as will be discussed in the next chapter. Because it is a stable pole and has a fairly small time constant, its cancellation will not affect seriously the behavior of the overall system, as will be demonstrated in the next chapter.

In addition to (9.57), we may also choose the following ITAE optimal transfer function

$$G_o(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2} \tag{9.58}$$

It has pole-zero excess larger than that of $G(s)$ and is implementable. We find by computer simulation that if $\omega_0 = 24.5$, then the $G_o(s)$ in (9.58) or

$$G_o(s) = \frac{600.25}{s^2 + 34.3s + 600.25} \tag{9.59}$$

meets the design specifications. Its step response and actuating signal are plotted in Figure 9.10 with the dotted lines. The step response is much faster than the ones of (9.57) and the quadratic optimal system. However, it has an overshoot of about 4.6%.

Example 9.6.5

Consider the plant transfer function in (9.47) or

$$G(s) = \frac{s - 1}{s(s - 2)}$$

Find an ITAE zero-position-error optimal system. It is also required that the actuating signal $u(t)$ due to a unit-step reference input meet the constraint $|u(t)| \leq 10$, for $t \geq 0$. This plant transfer function has a non-minimum-phase zero and no ITAE standard form is available to carry out the design. However, we can employ the idea in [34] and use computer simulation to find its ITAE optimal transfer function as [54]

$$G_o(s) = \frac{-10(s - 1)}{s^2 + 5.1s + 10} \tag{9.60}$$

under the constraint $|u(t)| \leq 10$. We mention that the non-minimum-phase zero $(s - 1)$ of $G(s)$ must be retained in $G_o(s)$, otherwise $G_o(s)$ is not implementable. Its step response is plotted in Figure 9.11 with the dashed line. It has a faster response than the one of the quadratic optimal system in (9.48); however, it has a larger undershoot and a larger overshoot. Therefore it is difficult to say which system is better.

9.7 SELECTION BASED ON ENGINEERING JUDGMENT

In the preceding sections, we introduced two criteria for choosing overall transfer functions. The first criterion is the minimization of the quadratic performance index. The main reason for choosing this criterion is that it renders a simple and straightforward procedure to compute the overall transfer function. The second criterion is the minimization of the integral of time multiplied by absolute error (ITAE). It was chosen in [33] because it has the best selectivity. This criterion, however, does not render an analytical method to find the overall transfer function; it is obtained by trial and error and by computer simulation. In this section, we forego the concept of minimization or optimization and select overall transfer functions based on engineering judgment. We require the system to have a zero position error and a *good* transient performance. By a good transient performance, we mean that the rise and settling times are small and the overshoot is also small. Without comparisons, it is not possible to say what is small. Fortunately, we have quadratic and ITAE optimal systems for comparisons. Therefore, we shall try to find an overall system that has a comparable or better transient performance than the quadratic or ITAE optimal system. Whether the transient performance is comparable or better is based on engineering judgment; no mathematical criterion will be used. Consequently, the selection will be subjective and the procedure of selection is purely trial and error.

Example 9.7.1

Consider the plant transfer function in (9.39), or

$$G(s) = \frac{2}{s(s^2 + 0.25s + 6.25)}$$

We use computer simulation to select the following two overall transfer functions

$$G_{o1}(s) = \frac{20}{(s + 2)(s^2 + 2.5s + 10)} = \frac{20}{s^3 + 4.5s^2 + 15s + 20} \tag{9.61}$$

$$G_{o2}(s) = \frac{20}{(s + 10)(s^2 + 2s + 2)} = \frac{20}{s^3 + 12s^2 + 22s + 20} \tag{9.62}$$

The actuating signals of both systems due to a unit-step reference input meet the constraint $|u(t)| \leq 10$ for $t \geq 0$. Their step responses are plotted in Figure 9.9 with, respectively, the dotted line and the dashed-and-dotted line. The step response of $G_{o1}(s)$ lies somewhere between those of the quadratic optimal system and the ITAE optimal system. Therefore, $G_{o1}(s)$ is a viable alternative of the quadratic or ITAE optimal system.

The concept of dominant poles can also be used to select $G_o(s)$. Consider the overall transfer function in $G_{o2}(s)$. It has a pair of complex-conjugate poles at $-1 \pm j1$ and a real pole at -10. Because the response due to the real pole dies out much faster than does the response due to the complex-conjugate poles, the response of $G_{o2}(s)$ is essentially determined or dominated by the complex-conjugate poles. The complex-conjugate poles have the damping ratio 0.707, and consequently the step response has an overshoot of about 5% (see Section 7.2.1), as can be seen from Figure 9.9. However, because the product of the three poles must equal 20 in order to meet the constraint on the actuating signal, if we choose the nondominant pole far away from the imaginary axis, then the complex-conjugate poles cannot be too far away from the origin of the s-plane. Consequently, the time constant of $G_{o2}(s)$ is larger than that of $G_{o1}(s)$ and its step response is slower, as is shown in Figure 9.9. Therefore for this problem, the use of dominant poles does not yield a satisfactory system. We mention that if the complex-conjugate poles are chosen at $-2 \pm j2$, then the system will respond faster. However, the real pole must be chosen as $20/8 = 2.5$ in order to meet the constraint on the actuating signal. In this case, the complex-conjugate poles no longer dominate over the real pole, and the concept of dominant poles cannot be used.

Example 9.7.2

Consider the plant transfer function in (9.44), or

$$G(s) = \frac{s + 3}{s(s - 1)} \tag{9.63}$$

We have designed a quadratic optimal system in (9.46) and two ITAE optimal systems in (9.57) with $\omega_0 = 10$ and (9.59). Their step responses are shown in Figure 9.10. Now using computer simulation, we find that the following

$$G_o(s) = \frac{784}{s^2 + 50.4s + 784} \tag{9.64}$$

has the response shown with the dashed-and-dotted line in Figure 9.10. It is comparable with that of the ITAE optimal system in (9.59) under the same constraint on the actuating signal. Thus, the overall transfer function in (9.64) can also be used, although it is not optimal in any sense.

Example 9.7.3

Consider the plant transfer function in (9.47) or

$$G(s) = \frac{s - 1}{s(s - 2)}$$

We have designed a quadratic optimal system in (9.48) and an ITAE optimal system in (9.60). Their step responses are shown in Figure 9.11. Now we find, by using computer simulation, that the response of

$$G_o(s) = \frac{-10(s - 1)}{(s + \sqrt{10})^2} \tag{9.65}$$

lies somewhere between those of (9.48) and (9.60) under the same constraint on the actuating signal. Therefore, (9.65) can also be chosen as an overall transfer function.

In this section, we have shown by examples that it is possible to use computer simulation to select an overall transfer function whose performance is comparable to that of the quadratic or ITAE optimal system. The method, however, is a trial-and-error method. In the search, we vary the coefficients of the quadratic or ITAE system and see whether or not the performance could be improved. If we do not have the quadratic or ITAE optimal system as a starting point, it would be difficult to find a good system. Therefore, the computer simulation method cannot replace the quadratic design method, nor the standard forms of the ITAE optimal systems. It can be used to complement the two optimal methods.

9.8 SUMMARY AND CONCLUDING REMARKS

This chapter introduced the inward approach to design control systems. In this approach, we first find an overall transfer function to meet design specifications and then implement it. In this chapter, we discussed only the problem of choosing an overall transfer function. The implementation problem is discussed in the next chapter.

The choice of an overall transfer function is not entirely arbitrary; otherwise we may simply choose the overall transfer function as 1. Given a plant transfer function $G(s) = N(s)/D(s)$, an overall transfer function $G_o(s) = N_o(s)/D_o(s)$ is said to be *implementable* if there exists a configuration with no plant leakage such that $G_o(s)$ can be built using only proper compensators. Furthermore, the resulting system is required to be well posed and totally stable—that is, the closed-loop transfer function of every possible input-output pair of the system is proper and stable. The necessary and sufficient conditions for $G_o(s)$ to be implementable are that (1) $G_o(s)$ is stable, (2) $G_o(s)$ contains the non-minimum-phase zeros of $G(s)$, and (3) the pole-zero excess of $G_o(s)$ is equal to or larger than that of $G(s)$. These constraints are not stringent; poles of $G_o(s)$ can be arbitrarily assigned so long as they all lie in the open left half s-plane; other than retaining all zeros outside the region C in Figures 6.13 or 7.4, all other zeros of $G_o(s)$ can be arbitrarily assigned in the entire s-plane.

In this chapter, we discussed how to choose an implementable overall system to minimize the quadratic and ITAE performance indices. In using these performance indices, a constraint on the actuating signal or on the bandwidth of resulting systems must be imposed; otherwise, it is possible to design an overall system to have a performance index as small as desirable and the corresponding actuating signal will approach infinity. The procedure of finding quadratic optimal systems is simple and straightforward; after computing a spectral factorization, the optimal system can be readily obtained from (9.19). Spectral factorizations can be carried out by iteration without computing any roots, or computing all the roots of (9.16) and then grouping the open left half s-plane roots. ITAE optimal systems are obtainable from Tables 9.1 through 9.3. Because the tables are not exhaustive, for some plant transfer functions (for example, those with non-minimum-phase zeros), no standard forms are available to find ITAE optimal systems. In this case, we may resort to computer simulation to find an ITAE optimal system.

In this chapter, we also showed by examples that overall transfer functions that have comparable performance as quadratic or ITAE optimal systems can be obtained by computer simulation without minimizing any mathematical performance index. It is therefore suggested that after obtaining quadratic or ITAE optimal systems, we may change the parameters of the optimal systems to see whether a more desirable system can be obtained. In conclusion, we should make full use of computers to carry out the design.

We give some remarks concerning the quadratic optimal design to conclude this chapter.

1. The quadratic optimal system in (9.19) is reduced from a general formula in Reference [10]. The requirement of implementability is included in (9.19). If no

such requirement is included, the optimal transfer function that minimizes (9.15) with $r(t) = 1$ is

$$\overline{G}_o(s) = \frac{qN(0)}{D_o(0)} \cdot \frac{N_+(s)}{D_o(s)} \tag{9.66}$$

where $N_+(s)$ is $N(s)$ with all its right-half-plane roots reflected into the left half plane. In this case, the resulting overall transfer function may not be implementable. For example, if $G(s) = (s - 1)/s(s + 1)$, then the optimal system that minimizes

$$J = \int_0^\infty [q(y(t) - 1)^2 + u^2(t)]dt \tag{9.67}$$

with $q = 9$ is

$$\overline{G}_o(s) = \frac{3(s + 1)}{s^2 + 4s + 3}$$

which does not retain the non-minimum-phase zero and is not implementable. For this optimal system, J can be computed as $J = 3$. See Chapter 11 of Reference [12] for a discussion of computing J. The *implementable* optimal system that minimizes J in (9.67) is

$$G_o(s) = \frac{-3(s - 1)}{s^2 + 4s + 3}$$

For this implementable $G_o(s)$, J can be computed as $J = 21$. It is considerably larger than the J for $\overline{G}_o(s)$. Although $\overline{G}_o(s)$ has a smaller performance index, it cannot be implemented.

2. If $r(t)$ in (9.15) is a ramp function, that is $r(t) = at$, $t \geq 0$, then the optimal system that minimizes (9.15) is

$$G_o(s) = q(k_1 + k_2 s)\frac{N(s)}{D_o(s)} = \left(1 + \frac{k_2}{k_1}s\right)\frac{qN(0)}{D_o(0)} \cdot \frac{N(s)}{D_o(s)} \tag{9.68}$$

where

$$k_1 = \frac{N(0)}{D_o(0)} \quad \text{and} \quad k_2 = \frac{d}{ds}\left[\frac{N(-s)}{D_o(-s)}\right]_{s=0}$$

or, if $N(s) = N_0 + N_1 s + \cdots + N_m s^m$ and $D_o(s) = D_0 + D_1 s + \cdots + D_n s^n$,

$$k_1 = \frac{N_0}{D_0} \quad \text{and} \quad k_2 = \frac{N_0 D_1 - D_0 N_1}{D_0^2}$$

The optimal system in (9.68) is not implementable because it violates the pole-zero excess inequality. However, if we modify (9.68) as

$$\overline{G}_o(s) = \left(1 + \frac{k_2}{k_1}s\right)\frac{qN(0)}{D_o(0)}\frac{N(s)}{D_o(s) + \epsilon s^{n+1}} \tag{9.69}$$

where $n := \deg D_o(s)$ and ϵ is a very small positive number, then $\overline{G}_o(s)$ will be implementable. Furthermore, for a sufficiently small ϵ, $D_o(s) + \epsilon s^{n+1}$ is Hurwitz, and the frequency response of $\overline{G}_o(s)$ is very close to that of $G_o(s)$ in (9.68). Thus (9.69) is a simple and reasonable modification of (9.68).[7]

3. The quadratic optimal design can be carried out using transfer functions or using state-variable equations. In using state-variable equations, the concepts of controllability and observability are needed. The optimal design requires solving an algebraic Riccati equation and designing a state estimator (see Chapter 11). For the single-variable systems studied in this text, the transfer function approach is simpler and intuitively more transparent. The state-variable approach, however, can be more easily extended to multivariable systems.

PROBLEMS

9.1. Given $G(s) = (s + 2)/(s - 1)$, is $G_o(s) = 1$ implementable? Given $G(s) = (s - 1)/(s + 2)$, is $G_o(s) = 1$ implementable?

9.2. Given $G(s) = (s + 3)(s - 2)/s(s + 2)(s - 3)$, which of the following $G_o(s)$ are implementable?

$$\frac{s-2}{s(s+2)} \qquad \frac{s+3}{(s+2)(s-3)} \qquad \frac{s-2}{(s+2)^2}$$

$$\frac{(s+4)(s-2)}{s^4+4s^2+3s+6} \qquad \frac{s-2}{s^3+4s+2}$$

9.3. Consider a plant with transfer function $G(s) = (s + 3)/s(s - 2)$.

a. Find an implementable overall transfer function that has all poles at -2 and has a zero position error.

b. Find an implementable overall transfer function that has all poles at -2 and has a zero velocity error. Is the choice unique? Do you have to retain $s + 3$ in $G_o(s)$? Find two sets of solutions: One retains $s + 3$ and the other does not.

9.4. Consider a plant with transfer function $G(s) = (s - 3)/s(s - 2)$.

a. Find an implementable overall transfer function that has all poles at -2 and has a zero position error.

b. Find an implementable overall transfer function that has all poles at -2 and has a zero velocity error. Is the choice unique if we require the degree of $G_o(s)$ to be as small as possible?

[7]This modification was suggested by Professor Jong-Lick Lin of Cheng Kung University, Taiwan.

9.5. What types of reference signals will the following $G_o(s)$ track without an error?

a. $G_o(s) = \dfrac{-5s - 2}{-s^2 - 5s - 2}$

b. $G_o(s) = \dfrac{4s^2 + s + 3}{s^5 + 3s^4 + 4s^2 + s + 3}$

c. $G_o(s) = \dfrac{-2s^2 + 154s + 120}{s^4 + 14s^3 + 71s^2 + 154s + 120}$

9.6. Consider two systems. One has a settling time of 10 seconds and an overshoot of 5%, the other has a settling time of 7 seconds and an overshoot of 10%. Is it possible to state which system is better? Now we introduce a performance index as

$$J = k_1 \cdot (\text{Settling time}) + k_2 \cdot (\text{Percentage overshoot})$$

If $k_1 = k_2 = 0.5$, which system is better? If $k_1 = 0.8$ and $k_2 = 0.2$, which system is better?

9.7. Is the function

$$J = \int_0^\infty [q(y(t) - r(t)) + u(t)]dt$$

with $q > 0$ a good performance criterion?

9.8. Consider the design problem in Problem 7.15 or a plant with transfer function $G(s) = -2/s^2$. Design an overall system to minimize the quadratic performance index in (9.15) with $q = 4$. What are its position error and velocity error?

9.9. In Problem 9.8, design a quadratic optimal system that is as fast as possible under the constraint that the actuating signal due to a step-reference input must have a magnitude less than 5.

9.10. Plot the poles of $G_o(s)$ as a function of q in Problem 9.9.

9.11. Consider the design problem in Problem 7.14 or a plant with transfer function

$$G(s) = \frac{0.015}{s^2 + 0.11s + 0.3}$$

Design an overall system to minimize the quadratic performance index in (9.15) with $q = 9$. Is the position error of the optimal system zero? Is the index of the optimal system finite?

9.12. Consider the design problem in Problem 7.12 or a plant with transfer function

$$G(s) = \frac{4(s + 0.05)}{s(s + 2)(s - 1.2)}$$

Find a quadratic optimal system with $q = 100$. Carry out the spectral factorization by using the iterative method discussed in Section 9.4.2.

9.13. Let $Q(s) = D_o(s)D_o(-s)$ with

$$Q(s) = a_0 + a_2 s^2 + a_4 s^4 + \cdots + a_{2n} s^{2n}$$

and

$$D_o(s) = b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n$$

Show

$$\begin{aligned} a_0 &= b_0^2 \\ a_2 &= 2b_0 b_2 - b_1^2 \\ a_4 &= 2b_0 b_4 - 2b_1 b_3 + b_2^2 \\ &\vdots \\ a_{2n} &= 2b_0 b_{2n} - 2b_1 b_{2n-1} + 2b_2 b_{2n-2} - \cdots + (-1)^n b_n^2 \end{aligned}$$

where $b_i = 0$, for $i > n$.

9.14. The depth of a submarine can be maintained automatically by a control system, as discussed in Problem 7.8. The transfer function of the submarine from the stern angle θ to the actual depth y can be approximated as

$$G(s) = \frac{10(s + 2)^2}{(s + 10)(s^2 + 0.1)}$$

Find an overall system to minimize the performance index

$$J = \int_0^{\infty} [(y(t) - 1)^2 + \theta^2]dt$$

9.15. Consider a plant with transfer function $s/(s^2 - 1)$. Design an overall system to minimize the quadratic performance index in (9.15) with $q = 1$. Does the optimal system have zero position error? If not, modify the overall system to yield a zero position error.

9.16. Consider a plant with transfer function $G(s) = 1/s(s + 1)$. Find an implementable transfer function to minimize the ITAE criterion and to have zero position error. It is also required that the actuating signal due to a unit-step reference input have a magnitude less than 10.

9.17. Repeat Problem 9.16 with the exception that the overall system is required to have a zero velocity error.

9.18. Repeat Problem 9.16 for $G(s) = 1/s(s - 1)$.

9.19. Repeat Problem 9.17 for $G(s) = 1/s(s - 1)$.

9.20. Find an ITAE zero-position-error optimal system for the plant given in Problem 9.8. The magnitude of the actuating signal is required to be no larger than the one in Problem 9.8.

9.21. Find an ITAE zero-position-error optimal system for the plant in Problem 9.11. The real part of the poles of the optimal system is required to equal that in Problem 9.11.

9.22. Is it possible to obtain an ITAE optimal system for the plant in Problem 9.12 from Table 9.1 or 9.2? If yes, what will happen to the plant zero?

9.23. Repeat Problem 9.22 for the plant in Problem 9.14.

9.24. **a.** Consider a plant with transfer function $G(s) = (s + 4)/s(s + 1)$. Design an ITAE zero-position-error optimal system of degree 1. It is required that the actuating signal due to a unit-step reference input have a magnitude less than 10.

b. Consider a plant with transfer function $G(s) = (s + 4)/s(s + 1)$. Design an ITAE zero-position-error optimal system of degree 2. It is required that the actuating signal due to a unit-step reference input have a magnitude less than 10.

c. Compare their unit-step responses.

9.25. Consider the generator-motor set in Figure 6.1. Its transfer function is assumed to be

$$G(s) = \frac{300}{s^4 + 184s^3 + 760s^2 + 162s}$$

It is a type 1 transfer function. Design a quadratic optimal system with $q = 25$. Design an ITAE optimal system with $u(0^+) = 5$. Plot their poles. Are there many differences?

9.26. Consider a plant with transfer function $1/s^2$. Find an optimal system with zero velocity error to minimize the ITAE criterion under the constraint $|u(t)| \leq 6$. [Answer: $(6s + 2.5)/(s^3 + 2.38s^2 + 6s + 2.5)$.]

9.27. If software for computing step responses is available, adjust the coefficients of the quadratic optimal system in Problem 9.8, 9.11, 9.12, 9.14, or 9.15 to see whether a comparable or better transient performance can be obtained.

10 Implementation—Linear Algebraic Method

10.1 INTRODUCTION

The first step in the design of control systems using the inward approach is to find an overall transfer function to meet design specifications. This step was discussed in Chapter 9. Now we discuss the second step—namely, implementation of the chosen overall transfer function. In other words, given a plant transfer function $G(s)$ and an implementable $G_o(s)$, we shall find a feedback configuration without plant leakage and compute compensators so that the transfer function of the resulting system equals $G_o(s)$. The compensators used must be proper and the resulting system must be well posed and totally stable.

The preceding problem can also be stated as follows: Given a plant $G(s)$ and given a model $G_o(s)$, design an overall system so that the overall transfer function equals or matches $G_o(s)$. Thus the problem can also be called the *model-matching problem*. In the model-matching problem, we match not only poles but also zeros; therefore, it can also be called the *pole-and-zero placement problem*. There is a closely related problem, called the *pole-placement problem*. In the pole-placement problem, we match or control only poles of resulting overall systems; zeros are not specified. In this chapter, we study both the model-matching and pole-placement problems.

This chapter introduces three control configurations. They are the unity-feedback, two-parameter, and plant input/output feedback configurations. The unity-feedback configuration can be used to achieve any pole placement but not any model matching. The other two configurations, however, can be used to achieve any model matching. In addition to model matching and pole placement, this chapter also studies robust tracking and disturbance rejection.

The idea used in this chapter is very simple. The design is carried out by matching coefficients of compensators with desired polynomials. *If the denominator $D(s)$ and numerator $N(s)$ of a plant transfer function have common factors, then it is not possible to achieve any pole placement or any model matching. Therefore, we require $D(s)$ and $N(s)$ to have no common factors or to be coprime. Under this assumption,* the conditions of achieving matching depend on the degree of compensators. The larger the degree, the more parameters we have for matching. If the degree of compensators is large enough, matching is always possible. The design procedures in this chapter are essentially developed from these concepts and conditions.

10.2 UNITY-FEEDBACK CONFIGURATION—MODEL MATCHING

We discuss in this section the implementation of an implementable $G_o(s)$ by using the unity-feedback configuration shown in Figure 10.1. Let $G(s)$ and $C(s)$ be respectively the transfer function of the plant and compensator. If the overall transfer function from r to y is $G_o(s)$, then we have

$$G_o(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} \tag{10.1}$$

which implies

$$G_o(s) + C(s)G(s)G_o(s) = C(s)G(s)$$

and

$$C(s)G(s)(1 - G_o(s)) = G_o(s)$$

Thus, the compensator can be computed from

$$C(s) = \frac{G_o(s)}{G(s)(1 - G_o(s))} \tag{10.2}$$

We use examples to illustrate its computation and discuss the issues that arise in its implementation.

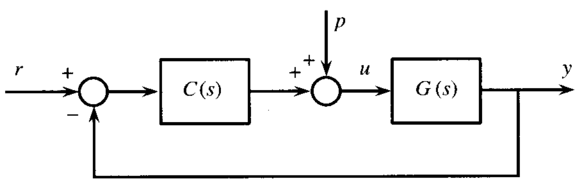


Figure 10.1 Unity-feedback system.

Example 10.2.1

Consider a plant with transfer function $1/s(s + 2)$. The optimal system that minimizes the ITAE criterion and meets the constraint $|u(t)| \leq 3$ was computed in (9.54) as $3/(s^2 + 2.4s + 3)$. If we implement this $G_o(s)$ in Figure 10.1, then the compensator is

$$C(s) = \frac{\dfrac{3}{s^2 + 2.4s + 3}}{\dfrac{1}{s(s+2)} \cdot \left(1 - \dfrac{3}{s^2 + 2.4s + 3}\right)} = \frac{\dfrac{3}{s^2 + 2.4s + 3}}{\dfrac{1}{s(s+2)} \cdot \dfrac{s^2 + 2.4s}{s^2 + 2.4s + 3}} = \frac{3(s+2)}{s + 2.4} \tag{10.3}$$

It is a proper compensator. This implementation has a pole-zero cancellation. Because the cancelled pole $(s + 2)$ is a stable pole, the system is totally stable. The condition for the unity-feedback configuration in Figure 10.1 to be well posed is $1 + G(\infty)C(\infty) \neq 0$. This is the case for this implementation; therefore, the system is well posed. In fact, if $G(s)$ is strictly proper and if $C(s)$ is proper, then the unity-feedback configuration is always well posed.

This implementation has the cancellation of the stable pole $(s + 2)$. This canceled pole is dictated by the plant, and the designer has no control over it. If this cancellation is acceptable, then the design is completed. In a latter section, we shall discuss a different implementation where the designer has freedom in choosing canceled poles.

Example 10.2.2

Consider a plant with transfer function $2/(s + 1)(s - 1)$. If $G_o(s) = 2/(s^2 + 2s + 2)$, then

$$C(s) = \frac{\dfrac{2}{s^2 + 2s + 2}}{\dfrac{2}{(s+1)(s-1)} \cdot \left(1 - \dfrac{2}{s^2 + 2s + 2}\right)} = \frac{\dfrac{2}{s^2 + 2s + 2}}{\dfrac{2}{(s+1)(s-1)} \cdot \dfrac{s^2 + 2s}{s^2 + 2s + 2}} = \frac{(s+1)(s-1)}{s(s+2)}$$

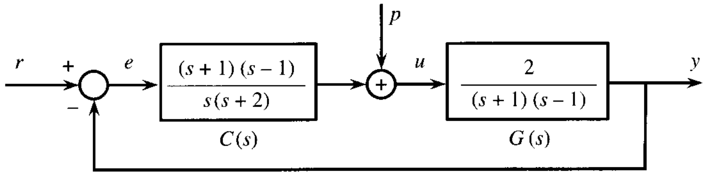


Figure 10.2 Unity-feedback system.

The compensator is proper. The implementation is plotted in Figure 10.2. The implementation has a stable pole-zero cancellation and an unstable pole-zero cancellation between $C(s)$ and $G(s)$.

As discussed in Chapter 6, noise and/or disturbance may enter a control system at every terminal. Therefore we require every control system to be totally stable. We compute the closed-loop transfer function from p to y shown in Figure 10.2:

$$G_{yp}(s) := \frac{Y(s)}{P(s)} = \frac{\dfrac{2}{(s+1)(s-1)}}{1 + \dfrac{(s+1)(s-1)}{s(s+2)} \cdot \dfrac{2}{(s+1)(s-1)}}$$

$$= \frac{\dfrac{2}{(s+1)(s-1)}}{\dfrac{s(s+2)+2}{s(s+2)}} = \frac{2s(s+2)}{(s-1)(s+1)(s^2+2s+2)}$$

It is unstable. Thus the output will approach infinity if there is any nonzero, no matter how small, disturbance. Consequently, the system is not totally stable, and the implementation is not acceptable.

Exercise 10.2.1

Consider a plant with transfer function $1/s^2$. Implement $G_o(s) = 6/(s^2 + 3.4s + 6)$ in the unity-feedback configuration. Is the implementation acceptable?

[**Answer:** $C(s) = 6s/(s + 3.4)$, unacceptable.]

These examples show that the implementation of $G_o(s)$ in the unity-feedback configuration will generally involve pole-zero cancellations. The canceled poles are determined by the plant transfer function, and we have no control over them. In general, if $G(s)$ has open right-half-plane poles or two or more poles at $s = 0$, then the unity-feedback configuration cannot be used to implement any $G_o(s)$. Thus, the unity-feedback configuration cannot, in general, be used in model matching.

10.3 UNITY-FEEDBACK CONFIGURATION—POLE PLACEMENT BY MATCHING COEFFICIENTS

Although the unity-feedback configuration cannot be used in *every* model matching, it can be used to achieve arbitrary pole placement. In pole placement, we assign only poles and leave zeros unspecified. We first use an example to illustrate the basic idea and to discuss the issues involved.

Example 10.3.1

Consider a plant with transfer function

$$G(s) = \frac{1}{s(s+2)}$$

and consider the unity-feedback configuration shown in Figure 10.1. If the compensator $C(s)$ is a gain of k (a transfer function of degree 0), then the overall transfer function can be computed as

$$G_o(s) = \frac{kG(s)}{1 + kG(s)} = \frac{k}{s^2 + 2s + k}$$

This $G_o(s)$ has two poles. These two poles cannot be arbitrarily assigned by choosing a value for k. For example, if we assign the two poles at -2 and -3, then the denominator of $G_o(s)$ must equal

$$s^2 + 2s + k = (s+2)(s+3) = s^2 + 5s + 6$$

Clearly, there is no k to meet the equation. Therefore, if the compensator is of degree 0, it is not possible to achieve arbitrary pole placement.[1]

Next let the compensator be proper and of degree 1 or

$$C(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

with $A_1 \neq 0$. Then the overall transfer function can be computed as

$$\begin{aligned} G_o(s) &= \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{B_0 + B_1 s}{s(s+2)(A_1 s + A_0) + B_1 s + B_0} \\ &= \frac{B_1 s + B_0}{A_1 s^3 + (2A_1 + A_0)s^2 + (2A_0 + B_1)s + B_0} \end{aligned}$$

This $G_o(s)$ has three poles. We show that all these three poles can be arbitrarily assigned by choosing a suitable $C(s)$. Let the denominator of $G_o(s)$ be

$$D_o(s) = s^3 + F_2 s^2 + F_1 s + F_0$$

[1]The root loci of this problem are plotted in Figure 7.5. If $C(s) = k$, we can assign the two poles only along the root loci.

where F_i are entirely arbitrary. Now we equate the denominator of $G_o(s)$ with $D_o(s)$ or

$$A_1s^3 + (2A_1 + A_0)s^2 + (2A_0 + B_1)s + B_0 = s^3 + F_2s^2 + F_1s + F_0$$

Matching the coefficients of like power of s yields

$$A_1 = 1 \qquad 2A_1 + A_0 = F_2 \qquad 2A_0 + B_1 = F_1 \qquad B_0 = F_0$$

which imply

$$A_1 = 1 \qquad A_0 = F_2 - 2A_1 \qquad B_1 = F_1 - 2F_2 + 4A_1 \qquad B_0 = F_0$$

For example, if we assign the three poles $G_o(s)$ as -2 and $-2 \pm 2j$, then $D_o(s)$ becomes

$$\begin{aligned} D_o(s) &= s^3 + F_2s^2 + F_1s + F_0 = (s + 2)(s + 2 + 2j)(s + 2 - 2j) \\ &= s^3 + 6s^2 + 16s + 16 \end{aligned}$$

We mention that if a complex pole is assigned in $D_o(s)$, its complex conjugate must also be assigned. Otherwise, $D_o(s)$ will have complex coefficients. For this set of poles, we have

$$A_1 = 1 \qquad A_0 = 6 - 2 = 4 \qquad B_1 = 16 - 2 \cdot 6 + 4 = 8 \qquad B_0 = 16$$

and the compensator is

$$C(s) = \frac{8s + 16}{s + 4}$$

This compensator will place the poles of $G_o(s)$ at -2 and $-2 \pm 2j$. To verify this, we compute

$$\begin{aligned} G_o(s) &= \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\dfrac{8s + 16}{s + 4} \cdot \dfrac{1}{s(s + 2)}}{1 + \dfrac{8s + 16}{s + 4} \cdot \dfrac{1}{s(s + 2)}} \\ &= \frac{8s + 16}{s(s + 2)(s + 4) + 8s + 16} = \frac{8s + 16}{s^3 + 6s^2 + 16s + 16} \end{aligned}$$

Indeed $G_o(s)$ has poles at -2 and $-2 \pm 2j$. Note that the compensator also introduces zero $(8s + 16)$ into $G_o(s)$. The zero is obtained from solving a set of equations, and we have no control over it. Thus, pole placement is different from pole-and-zero placement or model matching.

This example shows the basic idea of pole placement in the unity-feedback configuration. It is achieved by matching coefficients. In the following, we shall extend the procedure to the general case and also establish the condition for achieving pole placement. Consider the unity-feedback system shown in Figure 10.1. Let

$$G(s) = \frac{N(s)}{D(s)} \qquad C(s) = \frac{B(s)}{A(s)} \qquad G_o(s) = \frac{N_o(s)}{D_o(s)}$$

and $\deg N(s) \leq \deg D(s) = n$. The substitution of these into (10.1) yields

$$G_o(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\frac{B(s)}{A(s)} \cdot \frac{N(s)}{D(s)}}{1 + \frac{B(s)}{A(s)} \cdot \frac{N(s)}{D(s)}}$$

which becomes

$$G_o(s) = \frac{N_o(s)}{D_o(s)} = \frac{B(s)N(s)}{A(s)D(s) + B(s)N(s)} \tag{10.4}$$

Given $G(s)$, if there exists a proper compensator $C(s) = B(s)/A(s)$ so that all poles of $G_o(s)$ can be arbitrarily assigned, the design is said to achieve arbitrary pole placement. In the placement, if a complex number is assigned as a pole, its complex conjugate must also be assigned. From (10.4), we see that the pole-placement problem is equivalent to solving

$$A(s)D(s) + B(s)N(s) = D_o(s) \tag{10.5}$$

This polynomial equation is called a *Diophantine equation.* In the equation, $D(s)$ and $N(s)$ are given, the roots of $D_o(s)$ are the poles of the overall system to be assigned, and $A(s)$ and $B(s)$ are unknown polynomials to be solved. Note that $B(s)$ also appears in the numerator $N_o(s)$ of $G_o(s)$. Because $B(s)$ is obtained from solving (10.5), we have no direct control over it and, consequently, no control over the zeros of $G_o(s)$. Note that before solving (10.5), we don't know what $G_o(s)$ will be; therefore $C(s) = B(s)/A(s)$ cannot be computed from (10.2).

10.3.1 Diophantine Equations

The crux of pole placement is solving the Diophantine equation in (10.5) or

$$A(s)D(s) + B(s)N(s) = D_o(s)$$

In this equation, $D(s)$ and $N(s)$ are given, $D_o(s)$ is to be chosen by the designer. The questions are: Under what conditions will solutions $A(s)$ and $B(s)$ exist? and will the compensator $C(s) = B(s)/A(s)$ be proper? First we show that if $D(s)$ and $N(s)$ have common factors, then $D_o(s)$ cannot be arbitrarily chosen or, equivalently, arbitrary pole placement is not possible. For example, if $D(s)$ and $N(s)$ both contain the factor $(s - 2)$ or $D(s) = (s - 2)\bar{D}(s)$ and $N(s) = (s - 2)\bar{N}(s)$, then (10.5) becomes

$$A(s)D(s) + B(s)N(s) = (s - 2)[A(s)\bar{D}(s) + B(s)\bar{N}(s)] = D_o(s)$$

This implies that $D_o(s)$ must contain the same common factor $(s - 2)$. Thus, if $N(s)$ and $D(s)$ have common factors, then not every root of $D_o(s)$ can be arbitrarily assigned. Therefore we assume from now on that $D(s)$ and $N(s)$ have no common factors.

Because $G(s)$ and $C(s)$ are proper, we have deg $N(s) \leq$ deg $D(s) = n$ and deg $B(s) \leq$ deg $A(s) = m$, where deg stands for "the degree of." Thus, $D_o(s)$ in (10.5) has degree $n + m$ or, equivalently, the unity-feedback system in Figure 10.1 has $(n + m)$ number of poles. We develop in the following the conditions under which all $(n + m)$ number of poles can be arbitrarily assigned.

If deg $C(s) = 0$ (that is, $C(s) = k$, where k is a real number), then from the root-locus method, we see immediately that it is not possible to achieve *arbitrary* pole placement. We can assign poles only along the root loci. If the degree of $C(s)$ is 1, that is

$$C(s) = \frac{B(s)}{A(s)} = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

then we have four adjustable parameters for pole placement. Thus, the larger the degree of the compensator, the more parameters we have for pole placement. Therefore, if the degree of the compensator is sufficiently large, it is possible to achieve arbitrary pole placement.

Conventionally, the Diophantine equation is solved directly by using polynomials and the solution is expressed as a general solution. The general solution, however, is not convenient for our application. See Problem 10.19 and Reference [41]. In our application, we require deg $B(s) \leq$ deg $A(s)$ to insure properness of compensators. We also require the degree of compensators to be as small as possible. Instead of solving (10.5) directly, we shall transform it into a set of linear algebraic equations. We write

$$D(s) := D_0 + D_1 s + D_2 s^2 + \cdots + D_n s^n \qquad D_n \neq 0 \tag{10.6a}$$

$$N(s) := N_0 + N_1 s + N_2 s^2 + \cdots + N_n s^n \tag{10.6b}$$

$$A(s) := A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m \tag{10.7a}$$

and

$$B(s) := B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m \tag{10.7b}$$

where D_i, N_i, A_i, B_i are all real numbers, not necessarily nonzero. Because deg $D_o(s) = n + m$, we can express $D_o(s)$ as

$$D_o(s) = F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m} \tag{10.8}$$

The substitution of these into (10.5) yields

$$\begin{aligned}(A_0 &+ A_1 s + \cdots + A_m s^m)(D_0 + D_1 s + \cdots + D_n s^n) \\ &+ (B_0 + B_1 s + \cdots + B_m s^m)(N_0 + N_1 s + \cdots + N_n s^n) \\ &= F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m}\end{aligned}$$

which becomes, after grouping the coefficients associated with the same powers of s,

$$(A_0D_0 + B_0N_0) + (A_0D_1 + B_0N_1 + A_1D_0 + B_1N_0)s + \cdots + (A_mD_n + B_mN_n)s^{n+m} = F_0 + F_1s + F_2s^2 + \cdots + F_{n+m}s^{n+m}$$

Matching the coefficients of like powers of s yields

$$\begin{aligned} A_0D_0 + B_0N_0 &= F_0 \\ A_0D_1 + B_0N_1 + A_1D_0 + B_1N_0 &= F_1 \\ &\vdots \\ A_mD_n + B_mN_n &= F_{n+m} \end{aligned}$$

There are a total of $(n + m + 1)$ equations. These equations can be arranged in matrix form as

$$\mathbf{S_m c_m} := \left[\begin{array}{cc|cc|c|cc} D_0 & N_0 & 0 & 0 & & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ D_n & N_n & D_{n-1} & N_{n-1} & & D_0 & N_0 \\ 0 & 0 & D_n & N_n & & D_1 & N_1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & D_n & N_n \end{array}\right] \left[\begin{array}{c} A_0 \\ B_0 \\ \hline A_1 \\ B_1 \\ \hline \vdots \\ \hline A_m \\ B_m \end{array}\right] = \left[\begin{array}{c} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n+m} \end{array}\right] \tag{10.9}$$

This is a set of $(n + m + 1)$ linear algebraic equations. The matrix $\mathbf{S_m}$ has $(n + m + 1)$ rows and $2(m + 1)$ columns and is formed from the coefficients of $D(s)$ and $N(s)$. The first two columns are simply the coefficients of $D(s)$ and $N(s)$ arranged in ascending order. The next two columns are the first two columns shifted down by one position. We repeat the process until we have $(m + 1)$ sets of coefficients. We see that solving the Diophantine equation in (10.5) has now been transformed into solving the linear algebraic equation in (10.9).

As discussed in Appendix B, the equation in (10.9) has a solution for any F_i or, equivalently, for any $D_o(s)$ if and only if the matrix $\mathbf{S_m}$ has a full row rank. A necessary condition for $\mathbf{S_m}$ to have a full row rank is that $\mathbf{S_m}$ has more columns than rows or an equal number of columns and rows:

$$n + m + 1 \leq 2(m + 1) \qquad \text{or} \qquad n - 1 \leq m \tag{10.10}$$

Thus in order to achieve arbitrary pole placement, the degree of compensators in the unity-feedback configuration must be $n - 1$ or higher. If the degree is less than $n - 1$, it may be possible to assign *some* set of poles but not *every* set of poles. Therefore, we assume from now on that $m \geq n - 1$.

With $m \geq n - 1$, it is shown in Reference [15] that the matrix $\mathbf{S_m}$ has a full row rank if and only if $D(s)$ and $N(s)$ are coprime or have no common factors. We mention that if $m = n - 1$, the matrix $\mathbf{S_m}$ is a square matrix of order $2n$. In this case, for every $D_o(s)$, the solution of (10.9) is unique. If $m \geq n$, then (10.9) has more unknowns than equations and solutions of (10.9) are not unique.

Next we discuss the condition for the compensator to be proper or deg $B(s) \leq$ deg $A(s)$. We consider first $G(s)$ strictly proper and then $G(s)$ biproper. If $G(s)$ is strictly proper, then $N_n = 0$ and the last equation of (10.9) becomes

$$A_m D_n + B_m N_n = A_m D_n = F_{n+m}$$

which implies

$$A_m = \frac{F_{n+m}}{D_n} \tag{10.11}$$

Thus if $F_{n+m} \neq 0$, then $A_m \neq 0$ and the compensator $C(s) = B(s)/A(s)$ is proper. Note that if $m = n - 1$, the solution of (10.9) is unique and for any desired poles, there is a unique proper compensator to achieve the design. If $m \geq n$, then the solution of (10.9) is not unique, and some parameters of the compensator can be used, in addition to arbitrary pole placement, to achieve other design objective, as will be discussed later.

If $G(s)$ is biproper and if $m = n - 1$, then $\mathbf{S_m}$ in (10.9) is a square matrix and the solution of (10.9) is unique. In this case, there is no guarantee that $A_{n-1} \neq 0$ and the compensator may become improper. See Reference [15, p. 463.]. If $m \geq n$, then solutions of (10.9) are not unique and we can always find a strictly proper compensator to achieve pole placement. The preceding discussion is summarized as theorems.

THEOREM 10.1

Consider the unity-feedback system shown in Figure 10.1 with a strictly proper plant transfer function $G(s) = N(s)/D(s)$ with deg $N(s) <$ deg $D(s) = n$. It is assumed that $N(s)$ and $D(s)$ are coprime. If $m \geq n - 1$, then for *any* polynomial $D_o(s)$ of degree $(n + m)$, a proper compensator $C(s) = B(s)/A(s)$ of degree m exists to achieve the design. If $m = n - 1$, the compensator is unique. If $m \geq n$, the compensators are not unique and some of the coefficients of the compensators can be used to achieve other design objectives. Furthermore, the compensator can be determined from the linear algebraic equation in (10.9). ■

THEOREM 10.2

Consider the unity-feedback system shown in Figure 10.1 with a biproper plant transfer function $G(s) = N(s)/D(s)$ with deg $N(s) =$ deg $D(s) = n$. It is assumed that $N(s)$ and $D(s)$ are coprime. If $m \geq n$, then for *any* polynomial $D_o(s)$ of degree $(n + m)$, a proper compensator $C(s) = B(s)/A(s)$ of degree m exists to achieve the design. If $m = n$, and if the compensator is chosen to be strictly proper, then the compensator is unique. If $m \geq n + 1$, compensators are not unique and some of the coefficients of the compensators can be used to achieve other design objectives. Furthermore, the compensator can be determined from the linear algebraic equation in (10.9). ■

Example 10.3.2

Consider a plant with transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{s-2}{(s+1)(s-1)} = \frac{-2+s+0\cdot s^2}{-1+0\cdot s+s^2} \tag{10.12}$$

Clearly $D(s)$ and $N(s)$ have no common factor and $n = 2$. Let

$$C(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

It is a compensator of degree $m = n - 1 = 1$. Arbitrarily, we choose the three poles of the overall system as -3, $-2 + j1$ and $-2 - j1$. Then we have

$$D_o(s) := (s+3)(s+2-j1)(s+2+j1) = 15 + 17s + 7s^2 + s^3$$

We form the linear algebraic equation in (10.9) as

$$\left[\begin{array}{cc|cc} -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} A_0 \\ B_0 \\ \hline A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 7 \\ 1 \end{bmatrix} \tag{10.13}$$

Its solution can easily be obtained as

$$A_0 = \frac{79}{3} \qquad A_1 = 1 \qquad B_0 = -\frac{62}{3} \qquad B_1 = -\frac{58}{3}$$

Thus the compensator is

$$C(s) = \frac{-\dfrac{62}{3} - \dfrac{58}{3}s}{\dfrac{79}{3} + s} = \frac{-(58s+62)}{3s+79} = \frac{-(19.3s+20.7)}{s+26.3}. \tag{10.14}$$

and the resulting overall system is, from (10.4),

$$G_o(s) = \frac{B(s)N(s)}{D_o(s)} = \frac{\left(-\dfrac{62}{3} - \dfrac{58}{3}s\right)(s-2)}{s^3 + 7s^2 + 17s + 15} = \frac{-(58s+62)(s-2)}{3(s^3+7s^2+17s+15)}$$

Note that the zero $58s + 62$ is solved from the Diophantine equation and we have no control over it. Because $G_o(0) = 124/45$, if we apply a unit-step reference input, the output will approach $124/45 = 2.76$. See (4.25). Thus the output of this overall system will not track asymptotically step-reference inputs.

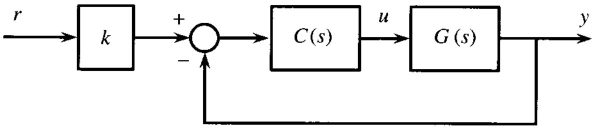


Figure 10.3 Unity-feedback system with a precompensator.

The design in Example 10.3.2 achieves pole placement but not tracking of step-reference inputs. This problem, however, can be easily corrected by introducing a constant gain k as shown in Figure 10.3. If we choose k so that $kG_o(0) = k \times 124/45 = 1$ or $k = 45/124$, then the plant output in Figure 10.3 will track any step-reference input. We call the constant gain in Figure 10.3 a *precompensator.* In practice, the precompensator may be incorporated into the reference input r by calibration or by resetting. For example, in temperature control, if r_0, which corresponds to 70°, yields a steady-state temperature of 67° and r_1 yields a steady-state temperature of 70°. We can simply change the scale so that r_0 corresponds to 67° and r_1 corresponds to 70°. By so doing, no steady-state error will be introduced in tracking step-reference inputs.

Example 10.3.3

Consider a plant with transfer function $1/s^2$. Find a compensator in Figure 10.1 so that the resulting system has all poles at $s = -2$. This plant transfer function has degree 2. If we choose a compensator of degree $m = n - 1 = 2 - 1 = 1$, then we can achieve arbitrary pole placement. Clearly we have

$$D_o(s) = (s + 2)^3 = s^3 + 6s^2 + 12s + 8$$

From the coefficients of

$$\frac{1}{s^2} = \frac{1 + 0 \cdot s + 0 \cdot s^2}{0 + 0 \cdot s + 1 \cdot s^2}$$

and $D_o(s)$, we form

$$\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} A_0 \\ B_0 \\ \hdashline A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 6 \\ 1 \end{bmatrix}$$

Its solution is

$$A_0 = 6 \qquad A_1 = 1 \qquad B_0 = 8 \qquad B_1 = 12$$

Thus the compensator is

$$C(s) = \frac{12s + 8}{s + 6}$$

and the resulting overall system is

$$G_o(s) = \frac{B(s)N(s)}{D_o(s)} = \frac{12s + 8}{s^3 + 6s^2 + 12s + 8} \tag{10.15}$$

Note that the zero $12s + 8$ is solved from the Diophantine equation and we have no control over it. Because the constant and s terms of the numerator and denominator of $G_o(s)$ are the same, the overall system has zero position error and zero velocity error. See Section 6.3.1. Thus, the system will track asymptotically any ramp-reference input. For this problem, there is no need to introduce a precompensator as in Figure 10.3. The reason is that the plant transfer function is of type 2 or has double poles at $s = 0$. In this case, the unity-feedback system, if it is stable, will automatically have zero velocity error.

From the preceding two examples, we see that arbitrary pole placement in the unity-feedback configuration can be used to achieve asymptotic tracking. If a plant transfer function is of type 0, we need to introduce a precompensator to achieve tracking of step-reference inputs. In practice, we can simply reset the reference input rather than introduce the precompensator. If $G(s)$ is of type 1, after pole placement, the unity-feedback system will automatically track step-reference inputs. If $G(s)$ is of type 2, the unity-feedback system will track ramp-reference inputs. If $G(s)$ is of type 3, the unity-feedback system will track acceleration-reference inputs. In pole placement, some zeros will be introduced. These zeros will affect the transient response of the system. Therefore, it is important to check the response of the resulting system before the system is actually built in practice.

To conclude this section, we mention that the algebraic equation in (10.9) can be solved by using MATLAB. For example, to solve (10.13), we type

```
a=[-1 -2 0 0;0 1 -1 -2;1 0 0 1;0 0 1 0];b=[15;17;7;1];
a\b
```

Then MATLAB will yield

```
 26.3333
-20.6667
  1.0000
-19.3333
```

This yields the compensator in (10.14).

Exercise 10.3.1

Redesign the problem in Example 10.3.1 by solving (10.9).

Exercise 10.3.2

Consider a plant with transfer function

$$G(s) = \frac{1}{s^2 - 1}$$

Design a proper compensator $C(s)$ and a gain k such that the overall system in Figure 10.3 has all poles located at -2 and will track asymptotically step-reference inputs.

[**Answers:** $C(s) = (13s + 14)/(s + 6)$, $k = 4/7$.]

10.3.2 Pole Placement with Robust Tracking

Consider the design problem in Example 10.3.2, that is, given $G(s) = (s - 2)/(s + 1)(s - 1)$ in (10.12), if we use the compensator in (10.14) and a precompensator $k = 45/124$, then the resulting overall system in Figure 10.3 will track any step-reference input. As discussed in Section 6.4, the transfer function of a plant may *change due to changes of load, aging, wearing or external perturbation such as wind* gust on an antenna. Now suppose, after implementation, the plant transfer function changes to $\overline{G}(s) = (s - 2.1)/(s + 1)(s - 0.9)$, the question is: Will the system in Figure 10.3 with the perturbed transfer function still track any step-reference input without an error?

In order to answer this question, we compute the overall transfer function with $G(s)$ replaced by $\overline{G}(s)$:

$$\begin{aligned} \overline{G}_o(s) &= \frac{45}{124} \cdot \frac{\dfrac{s - 2.1}{(s + 1)(s - 0.9)} \cdot \dfrac{-(58s + 62)}{3s + 79}}{1 + \dfrac{s - 2.1}{(s + 1)(s - 0.9)} \cdot \dfrac{-(58s + 62)}{3s + 79}} \\ &= \frac{45}{124} \cdot \frac{-58s^2 + 59.8s + 130.2}{3s^3 + 21.3s^2 + 65s + 59.1} \end{aligned} \tag{10.16}$$

Because $\overline{G}_o(0) = (45 \times 130.2)/(124 \times 59.1) = 0.799 \neq 1$, the system no longer tracks asymptotically step-reference inputs. If the reference input is 1, the output approaches 0.799 and the tracking error is about 20%. This type of design is called *nonrobust tracking* because plant parameter perturbations destroy the tracking property.

Now we shall redesign the system so that the tracking property will not be destroyed by plant parameter perturbations. This is achieved by increasing the degree of the compensator by one and then using the extra parameters to design a type 1 compensator. The original compensator in (10.14) has degree 1. We increase its degree and consider

$$C(s) = \frac{B_0 + B_1 s + B_2 s^2}{A_0 + A_1 s + A_2 s^2} \tag{10.17}$$

Both the plant and compensator have degree 2, therefore the unity-feedback system in Figure 10.1 has four poles. We assign the four poles arbitrarily as -3, -3, $-2 \pm j$, then we have

$$\begin{aligned} D_o(s) &= (s + 3)^2(s + 2 - j1)(s + 2 + j1) \\ &= s^4 + 10s^3 + 38s^2 + 66s + 45 \end{aligned} \tag{10.18}$$

The compensator that achieves this pole placement can be solved from the following linear algebraic equation

$$\left[\begin{array}{cc|cc|cc} -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 45 \\ 66 \\ 38 \\ 10 \\ 1 \end{bmatrix} \tag{10.19}$$

This equation has six unknowns and five equations. After deleting the first column, the remaining square matrix of order 5 in (10.19) still has a full row rank. Therefore A_0 can be arbitrarily assigned. (See Appendix B). If we assign it as zero, then the compensator in (10.17) has a pole at $s = 0$ or becomes type 1. With $A_0 = 0$, the solution of (10.19) can be computed as $B_0 = -22.5$, $A_1 = 68.83$, $B_1 = -78.67$, $A_2 = 1$, and $B_2 = -58.83$. Therefore, the compensator in (10.17) becomes

$$C(s) = \frac{-(58.83s^2 + 78.67s + 22.5)}{s(s + 68.83)} \tag{10.20}$$

and the overall transfer function is

$$\begin{aligned} G_o(s) &= \frac{\dfrac{-(58.83s^2 + 78.67s + 22.5)}{s(s + 68.83)} \cdot \dfrac{s - 2}{(s + 1)(s - 1)}}{1 + \dfrac{-(58.83s^2 + 78.67s + 22.5)}{s(s + 68.83)} \cdot \dfrac{s - 2}{(s + 1)(s - 1)}} \\ &= \frac{-58.83s^3 + 38.99s^2 + 134.84s + 45}{s^4 + 10s^3 + 37.99s^2 + 66.01s + 45} \end{aligned} \tag{10.21}$$

We see that, other than truncation errors, the denominator of $G_o(s)$ practically equals $D_o(s)$ in (10.18). Thus the compensator in (10.20) achieves the pole placement. Because $G_o(0) = 45/45 = 1$, the unity-feedback system achieves asymptotic tracking of step-reference inputs. Note that there is no need to introduce a precompensator as in Figure 10.3 in this design, because the compensator is designed to be of type 1.

Now we show that even if the plant transfer function changes to $\overline{G}(s) = (s - 2.1)/(s + 1)(s - 0.9)$, the overall system will still achieve tracking. With the perturbed $\overline{G}(s)$, the overall transfer function becomes

$$\overline{G}_o(s) = \frac{\dfrac{-(58.83s^2 + 78.67s + 22.5)}{s(s + 68.83)} \cdot \dfrac{s - 2.1}{(s + 1)(s - 0.9)}}{1 + \dfrac{-(58.83s^2 + 78.67s + 22.5)}{s(s + 68.83)} \cdot \dfrac{s - 2.1}{(s + 1)(s - 0.9)}} = \frac{-58.83s^3 + 44.873s^2 + 142.707s + 47.25}{s^4 + 10.1s^3 + 50.856s^2 + 80.76s + 47.25} \tag{10.22}$$

We first check the stability of $\overline{G}_o(s)$. If $\overline{G}_o(s)$ is not stable, it cannot track any signal. The application of the Routh test to the denominator of $\overline{G}_o(s)$ yields

	s^4	1	50.856	47.25	
1/10.1	s^3	10.1	80.76		[0 42.86 47.25]
10.1/42.86	s^2	42.86	47.25		[0 69.63]
	s	69.63			
	1	47.25			

All entries in the Routh table are positive, therefore $\overline{G}_o(s)$ is stable. Because $\overline{G}_o(0) = 47.25/47.25 = 1$, the overall system with the perturbed plant transfer function still tracks asymptotically any step-reference input. In fact, because the compensator is of type 1, no matter how large the changes in the coefficients of the plant transfer function, so long as the unity-feedback system remains stable, the system will track asymptotically any step-reference input. Therefore the tracking property of this design is *robust.*

Exercise 10.3.3

Given a plant with transfer function $1/(s - 1)$, design a compensator of degree 0 so that the pole of the unity-feedback system in Figure 10.3 is -2. Find also a precompensator so that the overall system will track asymptotically any step-reference input. If the plant transfer function changes to $1/(s - 1.1)$, is the unity-feedback system still stable? Will it still track any step-reference input without an error?

[**Answers:** 3, 2/3, yes, no.]

Exercise 10.3.4

Given a plant with transfer function $1/(s - 1)$, design a compensator of degree 1 so that the poles of the unity-feedback system in Figure 10.3 are -2 and -2. In this design, do you have freedom in choosing some of the coefficients of the compensator? Can you choose compensator coefficients so that the system in Figure 10.3 will track asymptotically step-reference inputs with $k = 1$? In this design, will the system remain stable and track any step-reference input after the plant transfer function changes to $1/(s - 1.1)$?

[**Answers:** $[(5 - \alpha)s + (4 + \alpha)]/(s + \alpha)$; yes; yes by choosing $\alpha = 0$; yes.]

To conclude this subsection, we remark on the choice of poles in pole placement. Clearly, the poles chosen must be stable. Furthermore, they should not be very close to the imaginary axis. As a guide, we may place them evenly inside the region C shown in Figure 6.13. Pole placement, however, will introduce some zeros into the resulting system. These zeros will also affect the response of the system. (See Figure 2.16.) There is no way to predict where these zeros will be located; therefore, it would be difficult to predict from the poles chosen what the final response of the resulting system will be. It is therefore advisable to simulate the resulting system before it is actually built.

10.3.3 Pole Placement and Model Matching

In this subsection, we give some remarks regarding the design based on pole placement and the design based on model matching. We use an example to illustrate the issues. Consider a plant with transfer function $G(s) = (s + 3)/s(s - 1)$. Its quadratic optimal system was computed in (9.46) as $G_o(s) = 10(s + 3)/(s^2 + 12.7s + 30)$ under the constraint that the actuating signal due to a unit-step reference input meets $|u(t)| \leq 10$ for all $t \geq 0$. Now we shall redesign the problem by using the method of pole placement. In other words, given $G(s) = (s + 3)/s(s - 1)$, we shall find a compensator of degree 1 in the unity-feedback configuration in Figure 10.1 so that the resulting system has a set of three desired poles. Because the poles of the quadratic optimal system or, equivalently, the roots of $(s^2 + 12.7s + 30)$ are -9.56 and -3.14, we choose the three poles as -9.56, -3.14 and -10. Thus we have

$$D_o(s) = (s^2 + 12.7s + 30)(s + 10) = s^3 + 22.7s^2 + 157s + 300$$

and the compensator $C(s) = (B_0 + B_1 s)/(A_0 + A_1 s)$ to achieve this set of pole placement can be computed from

$$\left[\begin{array}{cc:cc} 0 & 3 & 0 & 0 \\ -1 & 1 & 0 & 3 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 300 \\ 157 \\ 22.7 \\ 1 \end{bmatrix}$$

as

$$C(s) = \frac{20.175s + 100}{s + 3.525}$$

Thus, from (10.4), the overall transfer function from r to y is

$$G_o(s) = \frac{(20.175s + 100)(s + 3)}{s^3 + 22.7s^2 + 157s + 300}$$

and the transfer function from r to u is

$$T(s) := \frac{U(s)}{R(s)} = \frac{G_o(s)}{G(s)} = \frac{(20.175s + 100)s(s - 1)}{s^3 + 22.7s^2 + 157s + 300}$$

The unit-step responses of $G_o(s)$ and $T(s)$ are plotted in Figure 10.4 with solid lines. For comparison, the corresponding responses of the quadratic optimal system are also plotted with the dashed lines. We see that for the set of poles chosen, the resulting system has a larger overshoot than that of the quadratic optimal system. Furthermore, the actuating signal does not meet the constraint $|u(t)| \leq 10$.

It is conceivable that a set of poles can be chosen in pole placement so that the resulting system has a comparable response as one obtained by model matching. The problem is that, in pole placement, there is no way to predict the response of the resulting system from the set of poles chosen, because the response also depends on the zeros which are yet to be solved from the Diophantine equation. Therefore, pole placement design should consist of the following steps: (1) choose a set of poles, (2) compute the required compensator, (3) compute the resulting overall transfer function, and (4) check the response of the resulting system and check whether the actuating signal meets the constraint. If the design is not satisfactory, we go back to the first step and repeat the design. In model matching, we can choose an overall system to meet design specifications. Only after a satisfactory overall system is chosen do we compute the required compensator. Therefore, it is easier and simpler to obtain a good control system by model matching than by pole placement.

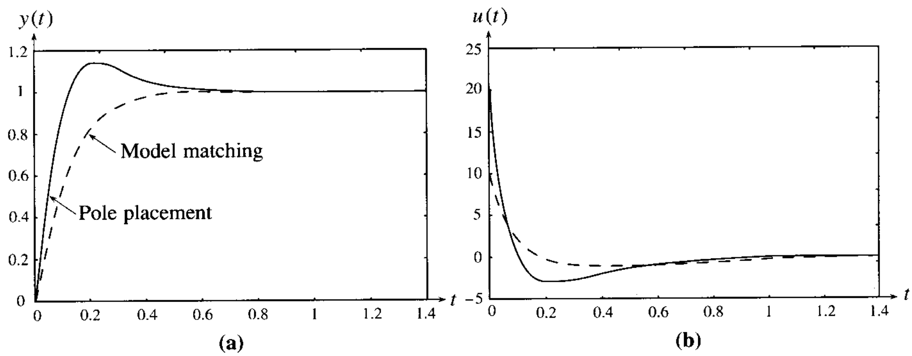


Figure 10.4 (a) Unit-step response. (b) Actuating signal.

10.4 TWO-PARAMETER COMPENSATORS

Although the unity-feedback configuration can be used to achieve arbitrary pole placement, it generally cannot be used to achieve model matching. In this section, we introduce a configuration, called the *two-parameter* configuration, that can be used to implement *any* implementable overall transfer function.

The actuating signal in Figure 10.1 is of the form, in the Laplace transform domain,

$$U(s) = C(s)(R(s) - Y(s)) = C(s)R(s) - C(s)Y(s) \tag{10.23}$$

That is, the *same* compensator is applied to the reference input and plant output to generate the actuating signal. Now we shall generalize it to

$$U(s) = C_1(s)R(s) - C_2(s)Y(s) \tag{10.24}$$

as shown in Figure 10.5(a). This is the most general form of compensators. We call $C_1(s)$ *feedforward* compensator and $C_2(s)$ *feedback* compensator. Let $C_1(s)$ and $C_2(s)$ be

$$C_1(s) = \frac{L(s)}{A_1(s)} \qquad C_2(s) = \frac{M(s)}{A_2(s)}$$

where $L(s)$, $M(s)$, $A_1(s)$, and $A_2(s)$ are polynomials. In general, $A_1(s)$ and $A_2(s)$ need not be the same. It turns out that even if they are chosen to be the same, the two compensators can be used to achieve any model matching. Furthermore, simple and straightforward design procedures can be developed. Therefore we assume $A_1(s) = A_2(s) = A(s)$ and the compensators become

$$C_1(s) = \frac{L(s)}{A(s)} \qquad C_2(s) = \frac{M(s)}{A(s)} \tag{10.25}$$

and the configuration in Figure 10.5(a) becomes the one in Figure 10.5(b). If $A(s)$, which is yet to be designed, contains unstable roots, the signal at the output of $L(s)/A(s)$ will grow to infinity and the system cannot be totally stable. Therefore the configuration in Figure 10.5(b) cannot be used in actual implementation. If we move $L(s)/A(s)$ into the feedback loop, then the configuration becomes the one shown in Figure 10.5(c). This configuration is also not satisfactory for two reasons: First, if $L(s)$ contains unstable roots, the design will involve unstable pole-zero cancellations and the system cannot be totally stable. Second, because the two compensators $L(s)/A(s)$ and $M(s)/L(s)$ have different denominators, if they are implemented using operational amplifier circuits, they will use twice as many integrators as the one to be discussed immediately. See Section 5.6.1. Therefore the configuration in Figure 10.5(c) should not be used. If we move $M(s)/L(s)$ outside the loop, then the resulting system is as shown in Figure 10.5(d). This configuration should not be used for the same reasons as for the configuration in Figure 10.5(c). Therefore, the three configurations in Figures 10.5(b), (c), and (d) will not be used in actual implementation.

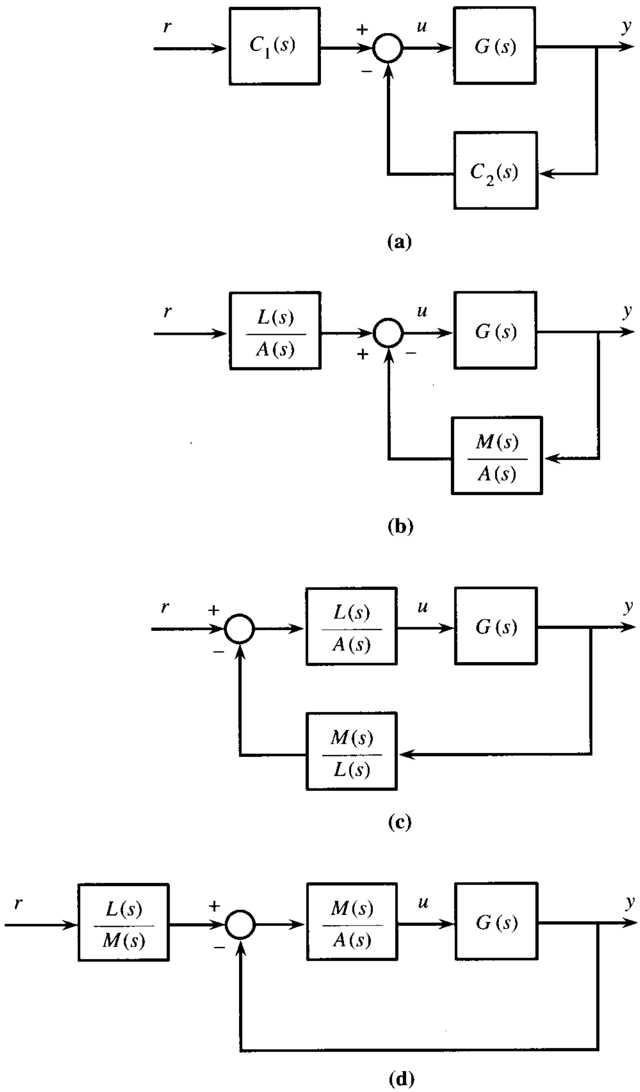


Figure 10.5 Various two-parameter configurations.

We substitute (10.25) into (10.24) and rewrite it into matrix form as

$$U(s) = \frac{L(s)}{A(s)} R(s) - \frac{M(s)}{A(s)} Y(s) = \begin{bmatrix} \frac{L(s)}{A(s)} & -\frac{M(s)}{A(s)} \end{bmatrix} \begin{bmatrix} R(s) \\ Y(s) \end{bmatrix} \tag{10.26}$$

Thus the compensator

$$\mathbf{C}(s) := [C_1(s) \quad -C_2(s)] = \begin{bmatrix} \frac{L(s)}{A(s)} & -\frac{M(s)}{A(s)} \end{bmatrix} = A^{-1}(s)[L(s) \quad -M(s)] \tag{10.27}$$

is a 1×2 rational matrix; it has two inputs r and y and one output u, and can be plotted as shown in Figure 10.6. The minus sign in (10.27) is introduced to take care of the negative feedback in Figure 10.6. Mathematically, this configuration is no different from the ones in Figure 10.5. However, if we implement $\mathbf{C}(s)$ in (10.27) as a unit, then the problem of possible unstable pole-zero cancellation will not arise. Furthermore, its implementation will use the minimum number of integrators. Therefore, the configuration in Figure 10.6 will be used exclusively in implementation. We call the compensator in Figure 10.6 a two-parameter compensator [63]. The configurations in Figure 10.5 are called two-degree-of-freedom structures in [36].

We can use the procedure in Section 5.6.1 to implement a two-parameter compensator as a unit. For example, consider

$$[C_1(s) \quad -C_2(s)] = \begin{bmatrix} \dfrac{10(s+30)^2}{s(s-15.2)} & -\dfrac{88.9s^2+140s+9000}{s(s-15.2)} \end{bmatrix}$$

We first expand it as

$$[C_1(s) \quad -C_2(s)] = [10 \quad -88.9] + \begin{bmatrix} \dfrac{752s+9000}{s^2-15.2s+0} & \dfrac{-1491.28s-9000}{s^2-15.2s+0} \end{bmatrix}$$

From its coefficients and (5.44), we can obtain the following two-dimensional state-variable equation realization

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 15.2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 752 & -1491.28 \\ 9000 & -9000 \end{bmatrix} \begin{bmatrix} r(t) \\ y(t) \end{bmatrix}$$

$$u(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [10 \quad -88.9] \begin{bmatrix} r(t) \\ y(t) \end{bmatrix}$$

From this equation, we can easily plot a basic block diagram as shown in Figure 10.7. It consists of two integrators. The block diagram can be built using operational amplifier circuits. We note that the adder in Figure 10.6 does not correspond to any adder in Figure 10.7.

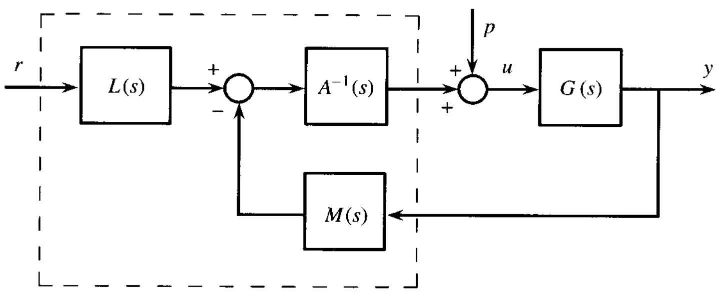


Figure 10.6 Two-parameter feedback configuration.

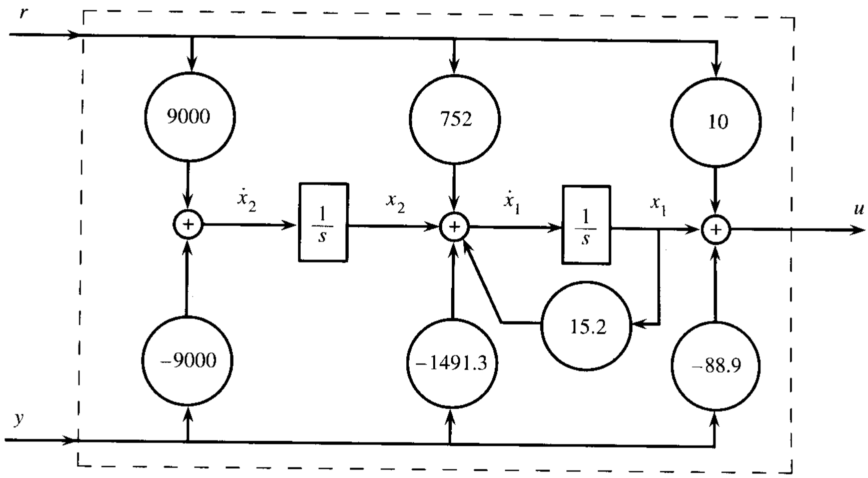


Figure 10.7 Basic block diagram.

Exercise 10.4.1

Find a minimal realization of

$$[C_1(s) \quad -C_2(s)] = \begin{bmatrix} \dfrac{10(s+30)}{s+5.025} & -\dfrac{38.7s+300}{s+5.025} \end{bmatrix}$$

and then draw a basic block diagram for it.

[**Answer:** $\dot{x}(t) = -5.025x(t) + [249.75 \quad -105.5325]\begin{bmatrix} r(t) \\ y(t) \end{bmatrix}$

$$u(t) = x(t) + [10 \quad -38.7]\begin{bmatrix} r(t) \\ y(t) \end{bmatrix}.]$$

10.4.1 Two-Parameter Configuration—Model Matching

In this subsection, we present a procedure to implement any implementable $G_o(s)$ in the two-parameter configuration. From Mason's formula, the transfer function from r to y in Figure 10.6 is

$$\frac{Y(s)}{R(s)} = \frac{L(s)A^{-1}(s)G(s)}{1 + A^{-1}(s)M(s)G(s)}$$

which becomes, after substituting $G(s) = N(s)/D(s)$ and multiplying by $A(s)D(s)/A(s)D(s)$

$$\frac{Y(s)}{R(s)} = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)} \tag{10.28}$$

Now we show that this can be used to achieve any model matching. For convenience, we discuss only the case where $G(s)$ is strictly proper.

Problem Given $G(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are coprime, $\deg N(s) < \deg D(s) = n$, and given an implementable $G_o(s) = N_o(s)/D_o(s)$, find proper compensators $L(s)/A(s)$ and $M(s)/A(s)$ such that

$$G_o(s) = \frac{N_o(s)}{D_o(s)} = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)} \tag{10.29}$$

Procedure:

Step 1: Compute

$$\frac{G_o(s)}{N(s)} = \frac{N_o(s)}{D_o(s)N(s)} =: \frac{N_p(s)}{D_p(s)} \tag{10.30}$$

where $N_p(s)$ and $D_p(s)$ are coprime. Since $N_o(s)$ and $D_o(s)$ are coprime by assumption, common factors may exist only between $N_o(s)$ and $N(s)$. Cancel all common factors between them and denote the rest $N_p(s)$ and $D_p(s)$. Note that if $N_o(s) = N(s)$, then $D_p(s) = D_o(s)$ and $N_p(s) = 1$. Using (10.30), we rewrite (10.29) as

$$G_o(s) = \frac{N_p(s)N(s)}{D_p(s)} = \frac{L(s)N(s)}{A(s)D(s) + M(s)N(s)} \tag{10.31}$$

From this equation, one might be tempted to set $L(s) = N_p(s)$ and to solve for $A(s)$ and $M(s)$ from $D_p(s) = A(s)D(s) + M(s)N(s)$. Unfortunately, the resulting compensators are generally not proper. Therefore, some more manipulation is needed.

Step 2: Introduce an arbitrary Hurwitz polynomial $\bar{D}_p(s)$ so that the degree of $D_p(s)\bar{D}_p(s)$ is at least $2n - 1$. In other words, if $\deg D_p(s) := p$, then the degree of $\bar{D}_p(s)$ must be at least $2n - 1 - p$. Because the polynomial $\bar{D}_p(s)$ will be canceled in the design, its roots should be chosen to lie inside an acceptable pole-zero cancellation region.

Step 3: Rewrite (10.31) as

$$G_o(s) = \frac{N(s)N_p(s)}{D_p(s)} = \frac{N(s)[N_p(s)\bar{D}_p(s)]}{D_p(s)\bar{D}_p(s)} = \frac{N(s)L(s)}{A(s)D(s) + M(s)N(s)} \tag{10.32}$$

Now we set

$$L(s) = N_p(s)\bar{D}_p(s) \tag{10.33}$$

and solve $A(s)$ and $M(s)$ from

$$A(s)D(s) + M(s)N(s) = D_p(s)\bar{D}_p(s) =: F(s) \tag{10.34}$$

If we write

$$A(s) = A_0 + A_1 s + \cdots + A_m s^m \tag{10.35a}$$

$$M(s) = M_0 + M_1 s + \cdots + M_m s^m \tag{10.35b}$$

and

$$F(s) := D_p(s)\overline{D}_p(s) = F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m} \tag{10.36}$$

with $m \geq n - 1$, then $A(s)$ and $M(s)$ in (10.34) can be solved from the following linear algebraic equation:

$$\left[\begin{array}{cc|cc|c|cc} D_0 & N_0 & 0 & 0 & & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ D_n & N_n & D_{n-1} & N_{n-1} & & D_0 & N_0 \\ 0 & 0 & D_n & N_n & & D_1 & N_1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & D_n & N_n \end{array}\right] \left[\begin{array}{c} A_0 \\ M_0 \\ \hline A_1 \\ M_1 \\ \hline \vdots \\ \hline A_m \\ M_m \end{array}\right] = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n+m} \end{bmatrix} \tag{10.37}$$

The solution and (10.33) yield the compensators $L(s)/A(s)$ and $M(s)/A(s)$. This completes the design.

Now we show that the compensators are proper. Equation (10.37) becomes (10.9) if M_i is replaced by B_i. Thus, Theorem 10.1 is directly applicable to (10.37). Because $\deg N(s) < \deg D(s)$ and $\deg F(s) \geq 2n - 1$, Theorem 10.1 implies the existence of $M(s)$ and $A(s)$ in (10.37) or (10.34) with $\deg M(s) \leq \deg A(s)$. Thus, the compensator $M(s)/A(s)$ is proper. Furthermore, (10.34) implies

$$\deg A(s) = \deg [D_p(s)\overline{D}_p(s)] - \deg D(s) = \deg F(s) - n$$

Now we show $\deg L(s) \leq \deg A(s)$. Applying the pole-zero excess inequality of $G_o(s)$ to (10.32) and using (10.33), we have

$$\deg [D_p(s)\overline{D}_p(s)] - (\deg N(s) + \deg L(s)) \geq \deg D(s) - \deg N(s)$$

which implies

$$\deg L(s) \leq \deg [D_p(s)\overline{D}_p(s)] - \deg D(s) = \deg A(s)$$

Thus the compensator $L(s)/A(s)$ is also proper.

The design always involves the pole-zero cancellation of $\overline{D}_p(s)$. The polynomial $\overline{D}_p(s)$, however, is chosen by the designer. Thus if $\overline{D}_p(s)$ is chosen to be Hurwitz or to have its roots lying inside the region C shown in Figure 6.13, then the two-parameter system in Figure 10.6 is totally stable. The condition for the two-parameter configuration to be well posed is $1 + G(\infty)C_2(\infty) \neq 0$ where $C_2(s) = M(s)/A(s)$. This condition is always met if $G(s)$ is strictly proper and $C_2(s)$ is proper. Thus the system is well posed. The configuration clearly has no plant leakage. Thus this design meets all the constraints discussed in Chapter 9. In conclusion, the two-parameter configuration in Figure 10.6 can be used to implement any implementable overall transfer function.

Example 10.4.1

Consider the plant with transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{s+3}{s(s-1)}$$

studied in (9.44). Its ITAE optimal system was found in (9.59) as

$$G_o(s) = \frac{600.25}{s^2 + 34.3s + 600.25} \tag{10.38}$$

Note that the zero $(s + 3)$ of $G(s)$ does not appear in $G_o(s)$, thus the design will involve the pole-zero cancellation of $s + 3$. Now we implement $G_o(s)$ by using the two-parameter configuration in Figure 10.6. We compute

$$\frac{G_o(s)}{N(s)} = \frac{600.25}{(s^2 + 34.3s + 600.25)(s+3)} =: \frac{N_p(s)}{D_p(s)}$$

Next we choose $\bar{D}_p(s)$ so that the degree of $D_p(s)\bar{D}_p(s)$ is at least $2n - 1 = 3$. Because the degree of $D_p(s)$ is 3, the degree of $\bar{D}_p(s)$ can be chosen as 0. We choose $\bar{D}_p(s) = 1$. Thus we have

$$L(s) = N_p(s)\bar{D}_p(s) = 600.25 \times 1 = 600.25$$

The polynomials $A(s) = A_0 + A_1 s$ and $M(s) = M_0 + M_1 s$ can be solved from $A(s)D(s) + M(s)N(s) = D_p(s)\bar{D}_p(s)$ with

$$\begin{aligned} F(s) := D_p(s)\bar{D}_p(s) &= (s^2 + 34.3s + 600.25)(s+3) \\ &= s^3 + 37.3s^2 + 703.15s + 1800.75 \end{aligned}$$

or from the following linear algebraic equation:

$$\left[\begin{array}{cc:cc} D_0 & N_0 & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 \\ D_2 & N_2 & D_1 & N_1 \\ 0 & 0 & D_2 & N_2 \end{array}\right]\begin{bmatrix} A_0 \\ M_0 \\ A_1 \\ M_1 \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 3 & 0 & 0 \\ -1 & 1 & 0 & 3 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]\begin{bmatrix} A_0 \\ M_0 \\ A_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} 1800.75 \\ 703.15 \\ 37.3 \\ 1 \end{bmatrix}$$

The solution is $A(s) = A_0 + A_1 s = 3 + s$ and $M(s) = M_0 + M_1 s = 600.25 + 35.3s$. Thus the compensator is

$$[C_1(s) \quad -C_2(s)] = \begin{bmatrix} \dfrac{600.25}{s+3} & -\dfrac{35.3s + 600.25}{s+3} \end{bmatrix} \tag{10.39}$$

This completes the design.

Example 10.4.2

Consider the same plant transfer function in the preceding example. Now we shall implement its quadratic optimal transfer function $G_o(s) = 10(s + 3)/(s^2 + 12.7s + 30)$ developed in (9.46). First we compute

$$\frac{G_o(s)}{N(s)} = \frac{10(s + 3)}{(s^2 + 12.7s + 30)(s + 3)} = \frac{10}{s^2 + 12.7s + 30} =: \frac{N_p(s)}{D_p(s)}$$

Because the degree of $D_p(s)$ is 2, we must introduce $\bar{D}_p(s)$ of degree at least 1 so that the degree of $D_p(s)\bar{D}_p(s)$ is at least $2n - 1 = 3$. Arbitrarily, we choose

$$\bar{D}_p(s) = s + 3 \tag{10.40}$$

(This issue will be discussed further in the next section.) Thus we have

$$L(s) = N_p(s)\bar{D}_p(s) = 10(s + 3)$$

and

$$\begin{aligned} F(s) &= D_p(s)\bar{D}_p(s) = (s^2 + 12.7s + 30)(s + 3) \\ &= s^3 + 15.7s^2 + 68.1s + 90 \end{aligned}$$

The polynomials $A(s)$ and $M(s)$ can be solved from

$$\left[\begin{array}{cc|cc} 0 & 3 & 0 & 0 \\ -1 & 1 & 0 & 3 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} A_0 \\ M_0 \\ \hdashline A_1 \\ M_1 \end{array}\right] = \begin{bmatrix} 90 \\ 68.1 \\ 15.7 \\ 1 \end{bmatrix} \tag{10.41}$$

as $A(s) = A_1 s + A_0 = s + 3$ and $M(s) = M_1 s + M_0 = 13.7s + 30$. Thus, the compensator is

$$\begin{aligned} [C_1(s) \quad -C_2(s)] &= \begin{bmatrix} \dfrac{10(s + 3)}{s + 3} & -\dfrac{13.7s + 30}{s + 3} \end{bmatrix} \\ &= \begin{bmatrix} 10 & -\dfrac{13.7s + 30}{s + 3} \end{bmatrix} \end{aligned} \tag{10.42}$$

This completes the design. Note that $C_1(s)$ reduces to 10 because $\bar{D}_p(s)$ was chosen as $s + 3$. For different $\bar{D}_p(s)$, $C_1(s)$ is not a constant, as is seen in the next section.

Example 10.4.3

Consider a plant with transfer function $G(s) = 1/s(s + 2)$. Implement its ITAE optimal system $G_o(s) = 3/(s^2 + 2.4s + 3)$. This $G_o(s)$ was implemented by using the unity-feedback system in Example 10.2.1. The design had the pole-zero cancellation of $s + 2$, which was dictated by the given plant transfer function. Now we implement $G_o(s)$ in the two-parameter configuration and show that the designer has the freedom of choosing canceled poles. First we compute

$$\frac{G_o(s)}{N(s)} = \frac{3}{(s^2 + 2.4s + 3) \cdot 1} =: \frac{N_p(s)}{D_p(s)}$$

In this example, we have $N_p(s) = N_o(s) = 3$ and $D_p(s) = D_o(s) = s^2 + 2.4s + 3$. We choose a polynomial $\bar{D}_p(s)$ of degree 1 so that the degree of $D_p(s)\bar{D}_p(s)$ is $2n - 1 = 3$. Arbitrarily, we choose $\bar{D}_p(s) = s + 10$. Then we have $L(s) = N_p(s)\bar{D}_p(s) = 3(s + 10)$ and

$$F(s) = D_p(s)\bar{D}_p(s) = (s^2 + 2.4s + 3)(s + 10) = s^3 + 12.4s^2 + 27s + 30$$

The polynomials $A(s)$ and $M(s)$ can be solved from

$$\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} A_0 \\ M_0 \\ \hdashline A_1 \\ M_1 \end{array}\right] = \left[\begin{array}{c} 30 \\ 27 \\ 12.4 \\ 1 \end{array}\right]$$

as $A(s) = A_1 s + A_0 = s + 10.4$ and $M(s) = M_1 s + M_0 = 6.2s + 30$. This completes the design.

Although this problem can be implemented in the unity-feedback and two-parameter configurations, the former involves the pole-zero cancellation of $(s + 2)$, which is dictated by the plant; the latter involves the cancellation of $(s + 10)$, which is chosen by the designer. Therefore, the two-parameter configuration is more flexible and may be preferable to the unity-feedback configuration in achieving model matching.

Exercise 10.4.2

Given $G(s) = 1/s(s - 1)$, implement

a. $G_o(s) = 4/(s^2 + 2.8s + 4)$

b. $G_o(s) = (13s + 8)/(s^3 + 3.5s^2 + 13s + 8)$

in the two-parameter configuration. All canceled poles are to be chosen at $s = -4$.

[**Answers:** (a) $L(s)/A(s) = 4(s + 4)/(s + 7.8)$, $M(s)/A(s) = (23s + 16)/(s + 7.8)$. (b) $L(s)/A(s) = (13s + 8)/(s + 4.5)$, $M(s)/A(s) = (17.5s + 8)/(s + 4.5)$.]

10.5 EFFECT OF $\bar{D}_p(s)$ ON DISTURBANCE REJECTION AND ROBUSTNESS[2]

In the two-parameter configuration, we must introduce a Hurwitz polynomial $\bar{D}_p(s)$ in (10.32) or

$$G_o(s) = \frac{N(s)N_p(s)\bar{D}_p(s)}{D_p(s)\bar{D}_p(s)} = \frac{N(s)L(s)}{A(s)D(s) + M(s)N(s)}$$

to insure that the resulting compensators are proper. Because $\bar{D}_p(s)$ is completely canceled in $G_o(s)$, the tracking of $r(t)$ by the plant output $y(t)$ is not affected by the choice of $\bar{D}_p(s)$. Neither is the actuating signal affected by $\bar{D}_p(s)$, because the transfer function from r to u is

$$T(s) := \frac{U(s)}{R(s)} = \frac{G_o(s)}{G(s)} = \frac{N(s)N_p(s)\bar{D}_p(s)D(s)}{N(s)D_p(s)\bar{D}_p(s)} = \frac{N_p(s)D(s)}{D_p(s)} \tag{10.43}$$

where $\bar{D}_p(s)$ does not appear directly or indirectly. Therefore the choice of $\bar{D}_p(s)$ does not affect the tracking property of the overall system and the magnitude of the actuating signal.

Although $\bar{D}_p(s)$ does not appear in $G_o(s)$, it will appear in the closed-loop transfer functions of some input/output pairs. We compute the transfer function from the disturbance input p to the plant output y in Figure 10.6:

$$\begin{aligned} H(s) := \frac{Y(s)}{P(s)} &= \frac{G(s)}{1 + G(s)M(s)A^{-1}(s)} \\ &= \frac{N(s)A(s)}{A(s)D(s) + M(s)N(s)} = \frac{N(s)A(s)}{D_p(s)\bar{D}_p(s)} \end{aligned} \tag{10.44}$$

We see that $\bar{D}_p(s)$ appears directly in $H(s)$; it also affects $A(s)$ through the Diophantine equation in (10.34). Therefore the choice of $\bar{D}_p(s)$ will affect the disturbance rejection property of the system. This problem will be studied in this section by using examples.

In this section, we also study the effect of $\bar{D}_p(s)$ on the stability range of the overall system. As discussed in Section 6.4, the plant transfer function $G(s)$ may change due to changes of the load, power supplies, or other reasons. Therefore it is of practical interest to see how much the coefficients of $G(s)$ may change before the overall system becomes unstable. The larger the region in which the coefficients of $G(s)$ are permitted to change, the more *robust* the overall system is. In the following examples, we also study this problem.

[2]May be skipped without loss of continuity.

Example 10.5.1

Consider a plant with transfer function $G(s) = (s + 3)/s(s - 1)$. Implement its quadratic optimal system $G_o(s) = 10(s + 3)/(s^2 + 12.7s + 30)$ in the two-parameter configuration. As shown in Example 10.4.2, we must choose $\bar{D}_p(s)$ of degree 1 to achieve the design. If $\bar{D}_p(s)$ is chosen as $(s + 3)$, then the compensator was computed in (10.42). For this $\bar{D}_p(s)$ and compensator, the transfer function from p to y can be computed as

$$H(s) = \frac{N(s)A(s)}{D_p(s)\bar{D}_p(s)} = \frac{(s + 3)(s + 3)}{(s^2 + 12.7s + 30)(s + 3)} = \frac{s + 3}{s^2 + 12.7s + 30} \tag{10.45}$$

If $\bar{D}_p(s)$ is chosen as $(s + 30)$, using the same procedure as in Example 10.4.2, we can compute the compensator as

$$[C_1(s) \quad -C_2(s)] = \begin{bmatrix} \dfrac{10(s + 30)}{s + 5.025} & -\dfrac{38.675s + 300}{s + 5.025} \end{bmatrix} \tag{10.46}$$

For this $\bar{D}_p(s)$ and compensator, we have

$$H(s) = \frac{(s + 3)(s + 5.025)}{(s^2 + 12.7s + 30)(s + 30)} \tag{10.47}$$

Next we choose $\bar{D}_p(s) = s + 300$, and compute the compensator as

$$[C_1(s) \quad -C_2(s)] = \begin{bmatrix} \dfrac{10(s + 300)}{s + 25.275} & -\dfrac{288.425s + 3000}{s + 25.275} \end{bmatrix} \tag{10.48}$$

For this $\bar{D}_p(s)$ and compensator, we have

$$H(s) = \frac{(s + 3)(s + 25.275)}{(s^2 + 12.7s + 30)(s + 300)} \tag{10.49}$$

Now we assume the disturbance p to be a unit-step function and compute the plant outputs for the three cases in (10.45), (10.47), and (10.49). The results are plotted in Figure 10.8 with the solid line for $\bar{D}_p(s) = s + 3$, the dashed line for $\bar{D}_p(s) = s + 30$, and the dotted line for $\bar{D}_p(s) = s + 300$. We see that the system with $\bar{D}_p(s) = s + 300$ attenuates the disturbance most. We plot in Figure 10.9 the amplitude characteristics of the three $H(s)$. The one corresponding to $\bar{D}_p(s) = s + 300$ again has the best attenuation property for all ω. Therefore, we conclude that, for this example, the faster the root of $\bar{D}_p(s)$, the better the disturbance rejection property.

Now we study the robustness property of the system. First we consider the case $\bar{D}_p(s) = s + 300$ with the compensator in (10.48). Suppose that after the imple-

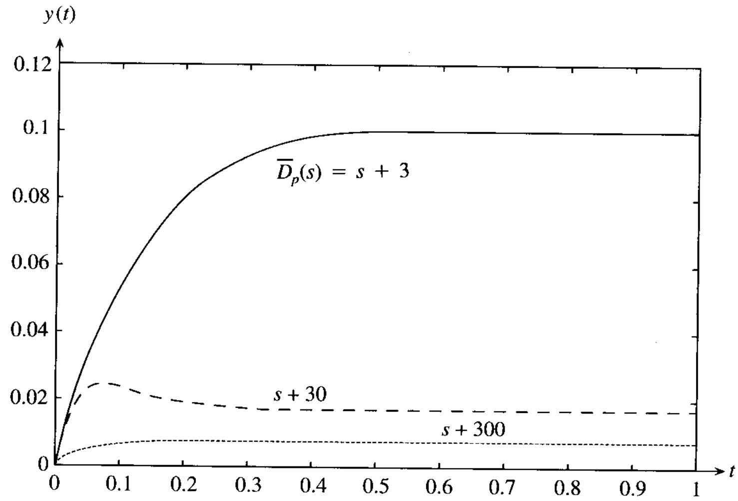


Figure 10.8 Effect of canceled poles on disturbance rejection (time domain).

mentation, the plant transfer function $G(s)$ changes to

$$\overline{G}(s) = \frac{\overline{N}(s)}{\overline{D}(s)} = \frac{s + 3 + \epsilon_2}{s(s - 1 + \epsilon_1)} \tag{10.50}$$

With this perturbed transfer function, the transfer function from r to y becomes

$$\overline{G}_o(s) = \frac{L(s)\overline{N}(s)}{A(s)\overline{D}(s) + M(s)\overline{N}(s)}$$

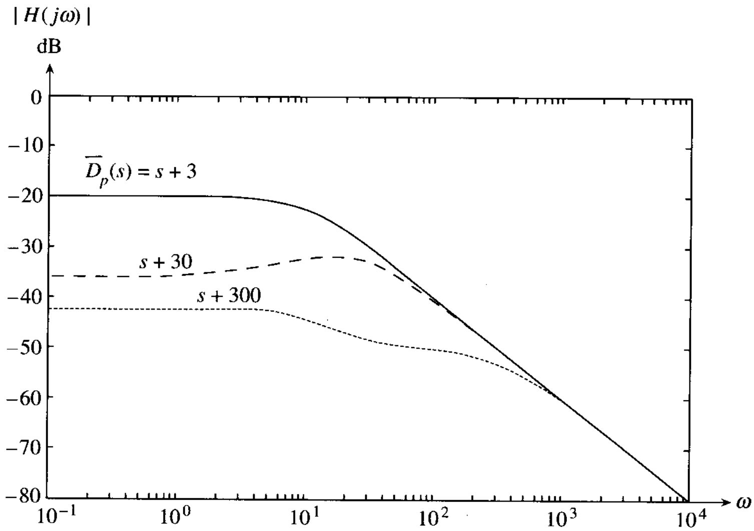


Figure 10.9 Effect of canceled poles on disturbance rejection (frequency domain).

The perturbed overall system is stable if

$$\begin{aligned} A(s)\overline{D}(s) + M(s)\overline{N}(s) &= (s + 25.275) \cdot s(s - 1 + \epsilon_1) \\ &\quad + (288.425s + 3000)(s + 3 + \epsilon_2) \\ &= s^3 + (312.7 + \epsilon_1)s^2 \\ &\quad + (3840 + 25.275\epsilon_1 + 288.45\epsilon_2)s + 3000(3 + \epsilon_2) \end{aligned} \tag{10.51}$$

is a Hurwitz polynomial. The application of the Routh test to (10.51) yields the following stability conditions:

$$312.7 + \epsilon_1 > 0 \qquad 3 + \epsilon_2 > 0 \tag{10.52a}$$

and

$$3840 + 25.275\epsilon_1 + 288.425\epsilon_2 - \frac{3000(3 + \epsilon_2)}{312.7 + \epsilon_1} > 0 \tag{10.52b}$$

(See Exercise 4.6.3.) These conditions can be simplified to

$$\epsilon_2 > -3 \qquad \text{if } \epsilon_1 > -117.7$$

and

$$\epsilon_2 > -\frac{1191768 + 11743.4925\epsilon_1 + 25.275\epsilon_1^2}{87190.4975 + 288.425\epsilon_1} \qquad \text{if } -302.29 < \epsilon_1 < -117.7$$

It is plotted in Figure 10.10 with the dotted line.

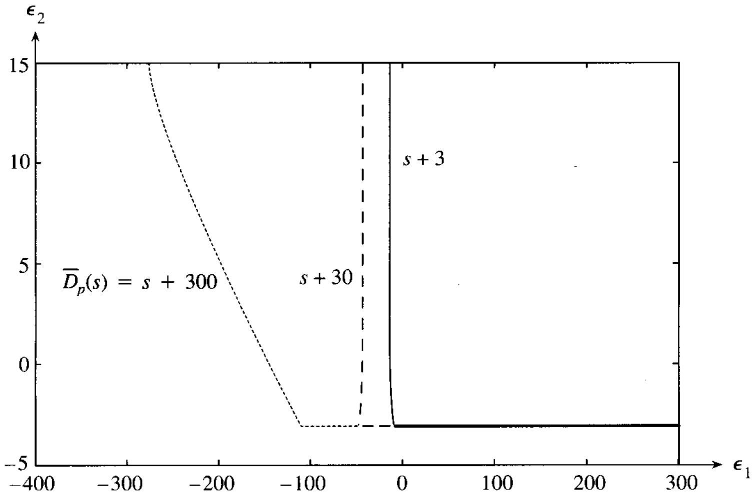


Figure 10.10 Effect of canceled poles on stability range.

In order to make comparisons, we repeat the computation for the cases $\overline{D}_p(s) = s + 30$ and $\overline{D}_p(s) = s + 3$. Their stability regions are also plotted in Figure 10.10 respectively with the dashed line and solid line. We see that the region corresponding to $\overline{D}_p(s) = s + 300$ is the largest and the one corresponding to $\overline{D}_p(s) = s + 3$ is the smallest. Thus we conclude that for this problem, the faster the root of $\overline{D}_p(s)$, the more robust the resulting system is.

Example 10.5.2

Consider a plant with transfer function $G(s) = N(s)/D(s) = (s - 1)/s(s - 2)$. Implement its quadratic optimal system $G_o(s) = -10(s - 1)/(s^2 + 11.14s + 10)$ in the two-parameter configuration. First, we compute

$$\frac{G_o(s)}{N(s)} = \frac{-10(s - 1)}{(s^2 + 11.14s + 10)(s - 1)} = \frac{-10}{s^2 + 11.14s + 10} =: \frac{N_p(s)}{D_p(s)} \tag{10.53}$$

Because the degree of $D_p(s)$ is 2, which is smaller than $2n - 1 = 3$, we must choose a Hurwitz polynomial $\overline{D}_p(s)$ of degree at least 1. We choose

$$\overline{D}_p(s) = s + \beta$$

Then we have

$$\begin{aligned} D_p(s)\overline{D}_p(s) &= (s^2 + 11.14s + 10)(s + \beta) \\ &= s^3 + (11.14 + \beta)s^2 + (10 + 11.14\beta)s + 10\beta \end{aligned}$$

and

$$L(s) = N_p(s)\overline{D}_p(s) = -10(s + \beta)$$

The polynomials $A(s)$ and $M(s)$ can be solved from

$$\left[\begin{array}{cc:cc} 0 & -1 & 0 & 0 \\ -2 & 1 & 0 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} A_0 \\ M_0 \\ \hdashline A_1 \\ M_1 \end{array}\right] = \begin{bmatrix} 10\beta \\ 10 + 11.14\beta \\ 11.14 + \beta \\ 1 \end{bmatrix}$$

This can be solved directly. It can also be solved by computing

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -2 & 1 & 0 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & -1 & -2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 4 \end{bmatrix}$$

Thus we have

$$\begin{bmatrix} A_0 \\ M_0 \\ A_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 10\beta \\ 10 + 11.14\beta \\ 11.14 + \beta \\ 1 \end{bmatrix} = \begin{bmatrix} -23.14 - 22.14\beta \\ -10\beta \\ 1 \\ 36.28 + 23.14\beta \end{bmatrix}$$

and the compensator is

$$\begin{aligned}[C_1(s) \quad -C_2(s)] &= \begin{bmatrix} \dfrac{L(s)}{A(s)} & -\dfrac{M(s)}{A(s)} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{-10(s + \beta)}{s - 23.14 - 22.14\beta} & -\dfrac{(36.28 + 23.14\beta)s - 10\beta}{s - 23.14 - 22.14\beta} \end{bmatrix} \end{aligned} \tag{10.54}$$

This completes the implementation of the quadratic optimal system. Note that no matter what value β assumes, as long as it is positive, $\bar{D}_p(s) = s + \beta$ will not affect the tracking property of the overall system. Neither will it affect the magnitude of the actuating signal.

Now we study the effect of $\bar{D}_p(s)$ on disturbance rejection. The transfer function from the disturbance p to the plant output y is

$$H(s) := \frac{Y(s)}{P(s)} = \frac{N(s)A(s)}{D_p(s)\bar{D}_p(s)} = \frac{(s - 1)(s - 23.14 - 22.14\beta)}{(s^2 + 11.14s + 10)(s + \beta)} \tag{10.55}$$

Let the disturbance be a unit-step function. We compute the unit-step responses of (10.55) on a personal computer and plot the results in Figure 10.11 for $\bar{D}_p(s) =$

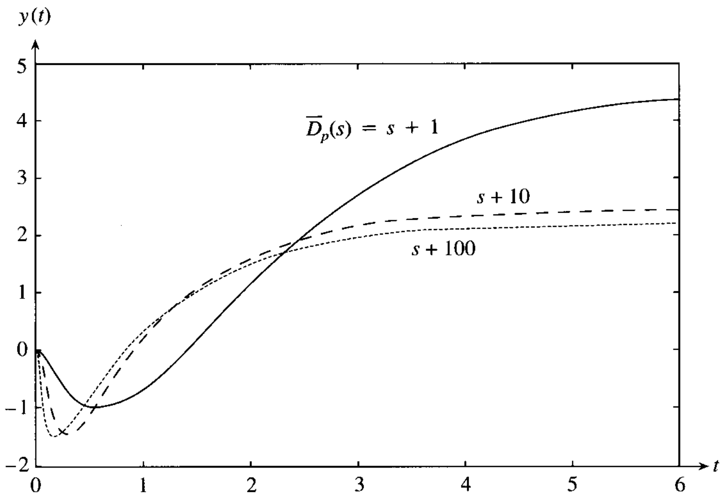


Figure 10.11 Effect of canceled poles on disturbance rejection (time domain).

$s + 1$ (solid line), $\overline{D}_p(s) = s + 10$ (dashed line), and $\overline{D}_p(s) = s + 100$ (dotted line). We see that the choice of $\overline{D}_p(s)$ does affect the disturbance rejection property of the system. Although the one corresponding to $\overline{D}_p(s) = s + 100$ has the smallest steady-state value, its undershoot is the largest. We plot in Figure 10.12 the amplitude characteristics of $H(s)$ for $\beta = 1$, 10, and 100 respectively with the solid line, dashed line, and dotted line. The one corresponding to $\beta = 100$ has the largest attenuation for small ω, but it has less attenuation for $\omega \geq 2$. Therefore, for this example, the choice of $\overline{D}_p(s)$ is not as clear-cut as in the preceding example. To have a small steady-state effect, we should choose a large β. If the frequency spectrum of disturbance lies mainly between 2 and 1000 radians per second, then we should choose a small β.

Now we study the effect of $\overline{D}_p(s)$ on the robustness of the overall system. Suppose after the implementation of $G_o(s)$, the plant transfer function $G(s)$ changes to

$$\overline{G}(s) = \frac{\overline{N}(s)}{\overline{D}(s)} = \frac{s - 1 + \epsilon_2}{s(s - 2 + \epsilon_1)} \tag{10.56}$$

With this plant transfer function and the compensators in (10.54), the overall transfer function becomes

$$\overline{G}_o(s) = \frac{L(s)\overline{N}(s)}{A(s)\overline{D}(s) + M(s)\overline{N}(s)}$$

with

$$\begin{aligned} &A(s)\overline{D}(s) + M(s)\overline{N}(s) \\ &= (s - 23.14 - 22.14\beta) \cdot s(s - 2 + \epsilon_1) \\ &\quad + [(36.28 + 23.14\beta)s - 10\beta] \cdot (s - 1 + \epsilon_2) \end{aligned} \tag{10.57}$$

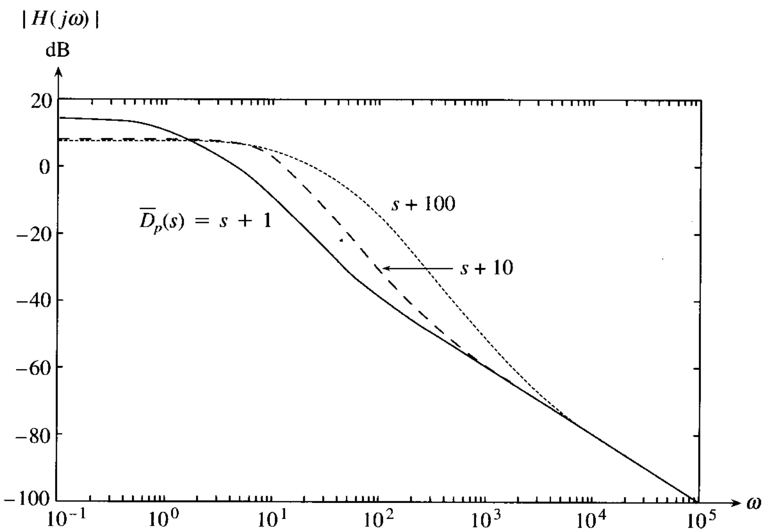


Figure 10.12 Effect of canceled poles on disturbance rejection (frequency domain).

We compute its stability ranges for the three cases with $\overline{D}_p(s) = s + 1$, $\overline{D}_p(s) = s + 10$, and $\overline{D}_p(s) = s + 100$. If $\overline{D}_p(s) = s + 1$, (10.57) becomes

$$\begin{aligned} A(s)\overline{D}(s) + M(s)\overline{N}(s) &= (s - 45.28) \cdot s(s - 2 + \epsilon_1) \\ &\quad + (59.42s - 10)(s - 1 + \epsilon_2) \\ &= s^3 + (12.14 + \epsilon_1)s^2 \\ &\quad + (21.14 - 45.28\epsilon_1 + 59.42\epsilon_2)s + 10(1 - \epsilon_2) \end{aligned}$$

It is Hurwitz under the following three conditions

$$12.14 + \epsilon_1 > 0 \qquad 1 - \epsilon_2 > 0$$

and

$$(21.14 - 45.28\epsilon_1 + 59.42\epsilon_2) - \frac{10(1 - \epsilon_2)}{(12.14 + \epsilon_1)} > 0$$

(See Exercise 4.6.3.) These conditions can be simplified as

$$\frac{-246.6396 + 528.5592\epsilon_1 + 45.28\epsilon_1^2}{731.3588 + 59.42\epsilon_1} < \epsilon_2 < 1 \qquad \text{for } -12.14 < \epsilon_1 < 1.78$$

From these inequalities, the stability range of ϵ_1 and ϵ_2 can be plotted as shown in Figure 10.13 with the solid line. We repeat the computation for $\overline{D}_p(s) = s + 10$ and $\overline{D}_p(s) = s + 100$ and plot the results in Figure 10.13, respectively, with the

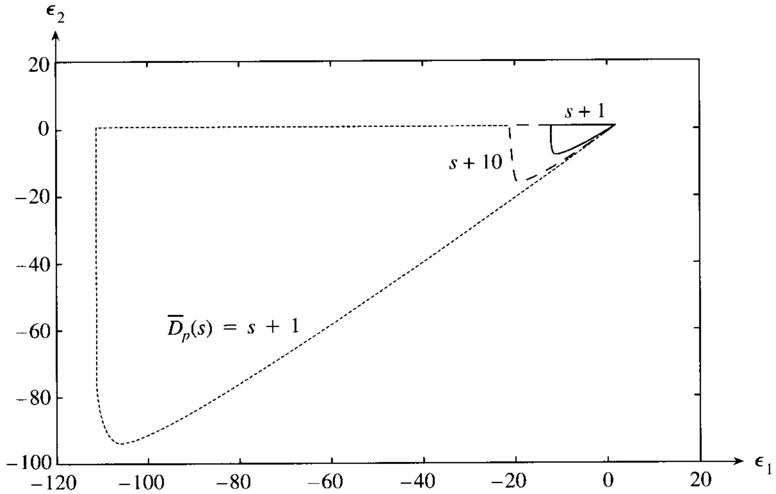


Figure 10.13 Effect of canceled poles on stability range.

dashed line and dotted line. We see that, roughly speaking, the larger β, the larger the stability range. (Note that in the neighborhood of $\epsilon_1 = 0$ and $\epsilon_2 = 0$, the stability region corresponding to $\bar{D}_p(s) = s + 100$ does not include completely the regions corresponding to $\bar{D}_p(s) = s + 10$ and $\bar{D}_p(s) = s + 1$.) Therefore we conclude that the faster the root of $\bar{D}_p(s)$, the more robust the overall system.

From the preceding two examples, we conclude that the choice of $\bar{D}_p(s)$ in the two-parameter configuration does affect the disturbance rejection and robustness properties of the resulting system. For the system in Example 10.5.1, which has no non-minimum-phase zeros, the faster the root of $\bar{D}_p(s)$, the better the step disturbance rejection and the more robust the resulting system. For the system in Example 10.5.2, which has a non-minimum-phase zero, the choice of $\bar{D}_p(s)$ is no longer clear-cut. In conclusion, in using the two-parameter configuration to achieve model matching, although the choice of $\bar{D}_p(s)$ does not affect the tracking property of the system and the magnitude of the actuating signal, it does affect the disturbance rejection and robustness properties of the system. Therefore, we should utilize this freedom in the design. No general rule of choosing $\bar{D}_p(s)$ seems available at present. However, we may always choose it by trial and error.

10.5.1 Model Matching and Disturbance Rejection

A system is said to achieve step *disturbance rejection* if the plant output excited by any step disturbance eventually vanishes, that is,

$$\lim_{t\to\infty} y_p(t) = 0 \tag{10.58}$$

where $y_p(t)$ is the plant output excited by the disturbance $p(t)$ shown in Figure 10.6. In the examples in the preceding section, we showed that by choosing the root of $\bar{D}_p(s)$ appropriately, the effect of step disturbances can be reduced. However, no matter where the root is chosen, the steady-state effect of step disturbances can never be completely eliminated. Now we shall show that by increasing the degree of $\bar{D}_p(s)$, step disturbances can be completely eliminated as $t\to\infty$.

The transfer function from p to y in Figure 10.6 is

$$H(s) = \frac{N(s)A(s)}{D_p(s)\bar{D}_p(s)} \tag{10.59}$$

If p is a step function with magnitude a, then

$$Y_p(s) = H(s)P(s) = \frac{N(s)A(s)}{D_p(s)\bar{D}_p(s)} \cdot \frac{a}{s} \tag{10.60}$$

The application of the final-value theorem to (10.60) yields

$$\lim_{t\to\infty} y_p(t) = \lim_{s\to 0} s \cdot \frac{N(s)A(s)}{D_p(s)\bar{D}_p(s)} \cdot \frac{a}{s} = \frac{aN(0)A(0)}{D_o(0)\bar{D}_p(0)} \tag{10.61}$$

This becomes zero for any a if and only if $N(0) = 0$ or $A(0) = 0$. Note that $D_p(0) \neq 0$ and $\bar{D}_p(0) \neq 0$ because $D_p(s)$ and $\bar{D}_p(s)$ are Hurwitz. The constant $N(0)$ is given and is often nonzero. Therefore, the only way to achieve disturbance rejection is to design $A(s)$ with $A(0) = 0$. Recall that $A(s)$ is to be solved from the Diophantine equation in (10.34) or the linear algebraic equation in (10.37). If the degree of $\bar{D}_p(s)$ is chosen so that the degree of $D_p(s)\bar{D}_p(s)$ is $2n - 1$, where $n = \deg D(s)$, then the solution $A(s)$ is unique and we have no control over $A(0)$. However, if we increase the degree of $\bar{D}_p(s)$, then solutions $A(s)$ are no longer unique and we may have the freedom of choosing $A(0)$. This will be illustrated by an example.

Example 10.5.3

Consider a plant with transfer function $G(s) = (s + 3)/s(s - 1)$. Implement its quadratic optimal system $G_o(s) = 10(s + 3)/(s^2 + 12.7s + 30)$. This was implemented in Example 10.5.1 by choosing the degree of $\bar{D}_p(s)$ as 1. Now we shall increase the degree of $\bar{D}_p(s)$ to 2 and repeat the design. First we compute

$$\frac{G_o(s)}{N(s)} = \frac{10}{s^2 + 12.7s + 30} =: \frac{N_p(s)}{D_p(s)}$$

Arbitrarily, we choose

$$\bar{D}_p(s) = (s + 30)^2 \tag{10.62}$$

Then we have

$$\begin{aligned} D_p(s)\bar{D}_p(s) &= (s^2 + 12.7s + 30)(s + 30)^2 \\ &= s^4 + 72.7s^3 + 1692s^2 + 13{,}230s + 27{,}000 \end{aligned}$$

and

$$L(s) = N_p(s)\bar{D}_p(s) = 10(s + 30)^2$$

The polynomials $A(s) = A_0 + A_1 s + A_2 s^2$ and $M(s) = M_0 + M_1 s + M_2 s^2$ can be solved from

$$\left[\begin{array}{cc:cc:cc} 0 & 3 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} A_0 \\ M_0 \\ \hdashline A_1 \\ M_1 \\ \hdashline A_2 \\ M_2 \end{array}\right] = \left[\begin{array}{r} 27{,}000 \\ 13{,}230 \\ 1692 \\ 72.7 \\ 1 \end{array}\right] \tag{10.63}$$

This has 5 equations and 6 unknowns. Because the first column of the 5×6 matrix is linearly dependent on the remaining columns, A_0 can be arbitrarily assigned, in particular, assigned as 0. With $A_0 = 0$, the solution of (10.63) can be computed as

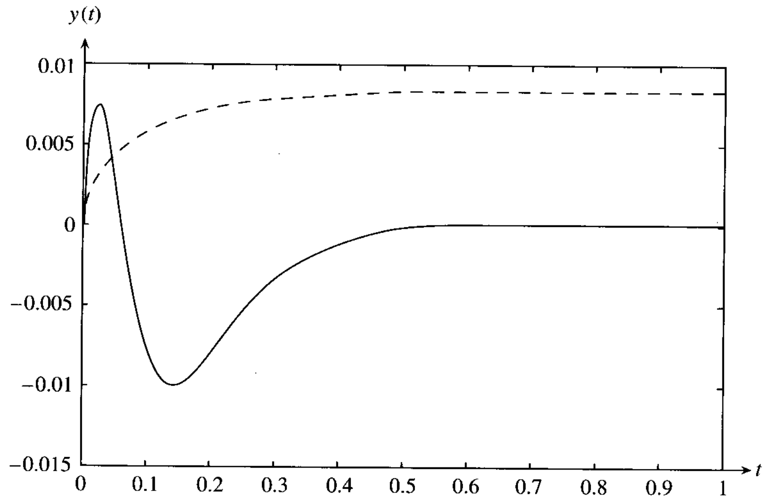


Figure 10.14 Disturbance rejection.

$A_1 = -15.2$, $A_2 = 1$, $M_0 = 9000$, $M_1 = 1410$, and $M_2 = 88.9$. Thus the compensator is

$$[C_1(s) \quad -C_2(s)] = \left[\frac{10(s+30)^2}{s(s-15.2)} \quad -\frac{88.9s^2 + 1410s + 9000}{s(s-15.2)}\right] \tag{10.64}$$

This completes the design. With this compensator, the transfer function from r to y is still $G_o(s) = 10(s+3)/(s^2 + 12.7s + 30)$. Therefore, the tracking property of the system remains unchanged. Now we compute the transfer function from the disturbance p to the plant output y:

$$H(s) := \frac{Y(s)}{P(s)} = \frac{N(s)A(s)}{D_p(s)\overline{D}_p(s)} = \frac{(s+3)s(s-15.2)}{(s^2 + 12.7s + 30)(s+30)^2} \tag{10.65}$$

Because $H(0) = 0$, if the disturbance is a step function, the excited plant output will approach zero as $t \to \infty$, as shown in Figure 10.14 with the solid line. As a comparison, we also show in Figure 10.14 with the dashed line the plant output due to a step disturbance for the design using $\overline{D}_p(s) = s + 300$. We see that by increasing the degree of $\overline{D}_p(s)$, it is possible to achieve step disturbance rejection. In actual design, we may try several $\overline{D}_p(s)$ of degree 2 and then choose one which suppresses most disturbances. In conclusion, by increasing the degree of $\overline{D}_p(s)$, we may achieve disturbance rejection.

Exercise 10.5.1

Repeat the preceding example by choosing $\overline{D}_p(s) = (s+300)^2$.

[**Answer:** $H(s) = s(s+3)(s-2200)/(s^2 + 12.7s + 30)(s+300)^2$.]

10.6 PLANT INPUT/OUTPUT FEEDBACK CONFIGURATION

Consider the configuration shown in Figure 10.15(a) in which $G(s)$ is the plant transfer function and $C_1(s)$, $C_2(s)$, and $C_0(s)$ are proper compensators. This configuration introduces feedback from the plant input and output; therefore, it is called the *plant input/output feedback configuration* or *plant I/O feedback configuration* for short. This configuration can be used to implement *any* implementable $G_o(s)$. Instead of discussing the general case, we discuss only the case where

$$\deg D_o(s) - \deg N_o(s) = \deg D(s) - \deg N(s) \tag{10.66}$$

In other words, the pole-zero excess of $G_o(s)$ equals that of $G(s)$. In this case, we can always assume $C_0(s) = 1$ and

$$C_1(s) = \frac{L(s)}{A(s)} \qquad C_2(s) = \frac{M(s)}{A(s)} \tag{10.67}$$

and the plant I/O feedback configuration can be simplified as shown in Figure 10.15(b). Note that $A(s)$, $L(s)$, and $M(s)$ in Figure 10.15(b) are different from those in the two-parameter configuration in Figure 10.6. The two compensators enclosed by the dashed line can be considered as a two-input, one-output compensator and must be implemented as a unit as discussed in Section 10.4. The configuration has two loops, one with loop gain $-L(s)/A(s)$, the other $-G(s)M(s)/A(s)$. Thus its

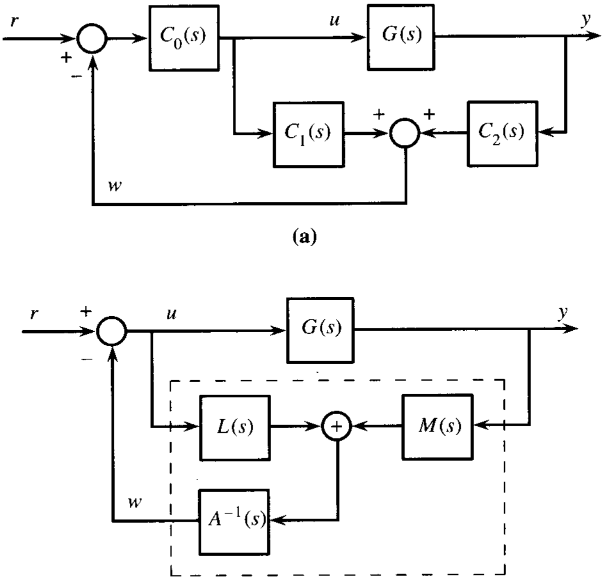


Figure 10.15 Plant I/O feedback system.

characteristic function equals

$$\Delta(s) = 1 - \left(-\frac{L(s)}{A(s)} - G(s)\frac{M(s)}{A(s)}\right) = 1 + \frac{L(s)}{A(s)} + \frac{N(s)M(s)}{D(s)A(s)} \tag{10.68}$$

and the transfer function from r to y equals, using Mason's formula,

$$G_o(s) = \frac{G(s)}{\Delta(s)} = \frac{N(s)A(s)}{A(s)D(s) + L(s)D(s) + M(s)N(s)} \tag{10.69}$$

Now we develop a procedure to implement $G_o(s)$.

Problem Given $G(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are coprime and $\deg N(s) \le \deg D(s)$. Implement an implementable $G_o(s) = N_o(s)/D_o(s)$ with $\deg D_o(s) - \deg N_o(s) = \deg D(s) - \deg N(s)$.

Procedure:

Step 1: Compute

$$\frac{G_o(s)}{N(s)} = \frac{N_o(s)}{D_o(s)N(s)} =: \frac{N_p(s)}{D_p(s)} \tag{10.70}$$

where $N_p(s)$ and $D_p(s)$ have no common factors.

Step 2: If $\deg N_p(s) =: \bar{m} < n - 1$, introduce an arbitrary Hurwitz polynomial $\bar{A}(s)$ of degree $n - 1 - \bar{m}$. If $\bar{m} \ge n - 1$, set $\bar{A}(s) = 1$

Step 3: Rewrite (10.70) and equate it with (10.69):

$$\begin{aligned} G_o(s) &= \frac{N(s)N_p(s)}{D_p(s)} = \frac{N(s)N_p(s)\bar{A}(s)}{D_p(s)\bar{A}(s)} \\ &= \frac{N(s)A(s)}{A(s)D(s) + L(s)D(s) + M(s)N(s)} \end{aligned} \tag{10.71}$$

From this equation, we have

$$\begin{aligned} A(s) &= N_p(s)\bar{A}(s) \\ A(s)D(s) + L(s)D(s) + M(s)N(s) &= D_p(s)\bar{A}(s) \end{aligned} \tag{10.72}$$

which becomes, after substituting (10.72)

$$\begin{aligned} L(s)D(s) + M(s)N(s) &= D_p(s)\bar{A}(s) - A(s)D(s) \\ &= D_p(s)\bar{A}(s) - N_p(s)\bar{A}(s)D(s) \\ &= \bar{A}(s)(D_p(s) - N_p(s)D(s)) =: F(s) \end{aligned} \tag{10.73}$$

This Diophantine equation can be used to solve $L(s)$ and $M(s)$. Because of the introduction of $\bar{A}(s)$, $A(s)$ in (10.72) has a degree at least $n - 1$ and the degrees of $L(s)$ and $M(s)$ are insured to be at most equal to that of $A(s)$. Therefore, the resulting compensators $L(s)/A(s)$ and $M(s)/A(s)$ are proper. Equation (10.73) is similar to (10.34), therefore it can also be solved using a linear algebraic equation. For example, if $\deg A(s) = n - 1$, then deg

$F(s) = 2n - 1$. Let

$$L(s) = L_0 + L_1 s + \cdots + L_{n-1}s^{n-1} \tag{10.74a}$$

$$M(s) = M_0 + M_1 s + \cdots + M_{n-1}s^{n-1} \tag{10.74b}$$

and

$$F(s) = F_0 + F_1 s + F_2 s^2 + \cdots + F_{2n-1}s^{2n-1} \tag{10.74c}$$

Then $L(s)$ and $M(s)$ can be solved from the following linear algebraic equation:

$$\left[\begin{array}{cc:cc:c:cc} D_0 & N_0 & 0 & 0 & & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ D_n & N_n & D_{n-1} & N_{n-1} & & D_0 & N_0 \\ 0 & 0 & D_n & N_n & & D_1 & N_1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & D_n & N_n \end{array}\right] \left[\begin{array}{c} L_0 \\ M_0 \\ \hdashline L_1 \\ M_1 \\ \hdashline \vdots \\ \hdashline L_{n-1} \\ M_{n-1} \end{array}\right] = \left[\begin{array}{c} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{2n-1} \end{array}\right] \tag{10.75}$$

This is illustrated by an example.

Example 10.6.1

Consider $G(s) = (s + 3)/s(s - 1)$. Implement its quadratic optimal system $G_o(s) = 10(s + 3)/(s^2 + 12.7s + 30)$. This problem was implemented in the two-parameter configuration in Example 10.4.2. Now we shall implement it in the plant I/O feedback configuration shown in Figure 10.15(b). First we compute

$$\frac{G_o(s)}{N(s)} = \frac{10(s + 3)}{(s^2 + 12.7s + 30)(s + 3)} = \frac{10}{s^2 + 12.7s + 30} =: \frac{N_p(s)}{D_p(s)}$$

Because the degree of $N_p(s)$ is 0, we must introduce a Hurwitz polynomial $\bar{A}(s)$ of degree at least $n - 1 - \deg N_p(s) = 1$. Arbitrarily, we choose

$$\bar{A}(s) = s + 3 \tag{10.76}$$

Then we have

$$A(s) = N_p(s)\bar{A}(s) = 10(s + 3)$$

and, from (10.73),

$$\begin{aligned} F(s) &= \bar{A}(s)(D_p(s) - N_p(s)D(s)) = (s + 3)[s^2 + 12.7s + 30 - 10s(s - 1)] \\ &= -9s^3 - 4.3s^2 + 98.1s + 90 \end{aligned}$$

The polynomials $L(s)$ and $M(s)$ can be solved from

$$\begin{bmatrix} 0 & 3 & \vdots & 0 & 0 \\ -1 & 1 & \vdots & 0 & 3 \\ 1 & 0 & \vdots & -1 & 1 \\ 0 & 0 & \vdots & 1 & 0 \end{bmatrix} \begin{bmatrix} L_0 \\ M_0 \\ \hdashline L_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} 90 \\ 98.1 \\ -4.3 \\ -9 \end{bmatrix} \tag{10.77}$$

as $L(s) = L_1 s + L_0 = -9s - 27$ and $M(s) = M_1 s + M_0 = 13.7s + 30$. Thus the compensator is

$$C_1(s) = \frac{L(s)}{A(s)} = \frac{-9s - 27}{10(s+3)} = \frac{-9}{10} \tag{10.78a}$$

$$C_2(s) = \frac{M(s)}{A(s)} = \frac{13.7s + 30}{10(s+3)} \tag{10.78b}$$

This completes the design. Note that $C_1(s)$ reduces to a constant because $\bar{A}(s)$ was chosen as $s + 3$. Different $\bar{A}(s)$ will yield nonconstant $C_1(s)$.

We see that the design using the plant I/O feedback configuration is quite similar to that of the two-parameter configuration. Because $\bar{A}(s)$ is completely canceled in $G_o(s)$, the choice of $\bar{A}(s)$ will not affect the tracking property and actuating signal of the system. However, its choice may affect disturbance rejection and stability robustness of the resulting system. The idea is similar to the two-parameter case and will not be repeated.

Exercise 10.6.1

Consider a plant with transfer function $G(s) = 1/s(s-1)$. Find compensators in the plant I/O feedback configuration to yield (a) $G_o(s) = 4/(s^2 + 2.8s + 4)$ and (b) $G_o(s) = (13s + 8)/(s^3 + 3.5s^2 + 13s + 8)$. All canceled poles are to be chosen at $s = -4$.

[**Answers:** (a) $L(s)/A(s) = (-3s - 8.2)/4(s+4)$, $M(s)/A(s) = (23s + 16)/4(s+4)$. (b) $L(s)/A(s) = (-12s - 3.5)/(13s + 8)$, $M(s)/A(s) = (17.5s + 8)/(13s + 8)$.]

10.7 SUMMARY AND CONCLUDING REMARKS

This chapter discusses the problem of implementing overall transfer functions. We discuss first the unity-feedback configuration. Generally, the unity-feedback configuration cannot be used to achieve model matching. However, it can always be used to achieve pole placement. If the degree of $G(s)$ is n, the minimum degree of compensators to achieve arbitrary pole placement is $n - 1$ if $G(s)$ is strictly proper, or n if $G(s)$ is biproper. If we increase the degree of compensators, then the unity-feedback configuration can be used to achieve pole placement and robust tracking.

The two-parameter configuration can be used to achieve any model matching. In this configuration, generally we have freedom in choosing canceled poles. The choice of these poles will not affect the tracking property of the system and the magnitude of the actuating signal. Therefore, these canceled poles can be chosen to suppress the effect of disturbance and to increase the stability robustness of the system. If we increase the degree of compensators, then it is possible to achieve model matching and disturbance rejection.

Finally we introduced the plant input/output feedback configuration. This configuration is developed from state estimator (or observer) and state feedback (or controller) in state-variable equations. See Chapter 11. The configuration can also be used to achieve any model matching. For a comparison of the two-parameter configuration and the plant I/O feedback configuration, see References [16, 44].

In this chapter all compensators for pole placement and model matching are obtained by solving sets of linear algebraic equations. Thus the method is referred to as the *linear algebraic method.*

We now compare the inward approach and the outward approach. We introduced the root-locus method and the frequency-domain method in the outward approach. In the root-locus method, we try to shift the poles of overall systems to the desired pole region. The region is developed from a quadratic transfer function with a constant numerator. Therefore, if an overall transfer function is not of the form, even if the poles are shifted into the region, there is no guarantee that the resulting system has the desired performance. In the frequency-domain method, because the relationship among the phase margin, gain margin, and time response is not exact, even if the design meets the requirement on the phase and gain margins, there is no guarantee that the time response of the resulting system will be satisfactory. Furthermore, if a plant has open right-half-plane poles, the frequency-domain method is rarely used. The constraint on actuating signals is not considered in the root-locus method, nor in the frequency-domain method.

In the inward approach, we first choose an overall transfer function; it can be chosen to minimize the quadratic or ITAE performance index or simply by engineering judgment. The constraint on actuating signals can be included in the choice. Once an overall transfer function is chosen, we may implement it in the unity-feedback configuration. If it cannot be so implemented, we can definitely implement it in the two-parameter or plant input/output feedback configuration. In the implementation, we may also choose canceled poles to improve disturbance rejection property and to increase stability robustness property of the resulting system. Thus, the inward approach appears to be more general and more versatile than the outward approach. Therefore, the inward approach should be a viable alternative in the design of control systems.

We give a brief history about various design methods to conclude this chapter. The earliest systematic method to design feedback systems was developed by Bode in 1945 [7]. It is carried out by using frequency plots, which can be obtained by measurement. Thus, the method is very useful to systems whose mathematical equations are difficult to develop. The method, however, is difficult to employ if a system, such as an aircraft, has unstable poles. In order to overcome the unstable poles of aircrafts, Evans proposed the root-locus method in 1950 [27]. The method has since been widely used in practice. The inward approach was first discussed by Truxal in 1955 [57]. He called the method synthesis through pole-zero configuration. The conditions in Corollary 9.1 were developed for the unity-feedback configuration. In spite of its importance, the method was mentioned only in a small number of control texts. ITAE optimal systems were developed by Graham and Lathrop [33] in 1953. Newton and colleagues [48] and Chang [10] were among the earliest to develop quadratic optimal systems by using transfer functions.

The development of implementable transfer functions for *any* control configuration was attempted in [12]. The conditions of proper compensators and no plant leakage, which was coined by Horowitz [36], were employed. Total stability was not considered. Although the condition of well-posedness was implicitly used, the concept was not fully understood and the proof was incomplete. It was found in [14] that, without imposing well-posedness, the plant input/output feedback configuration can be used to implement $G_o(s) = 1$ using exclusively proper compensators. It was a clear violation of physical constraints. Thus the well-posedness condition was explicitly used in [15, 16] to design control systems. By requiring proper compensators, total stability, well-posedness and no plant leakage, the necessity of the implementability conditions follows immediately for *any* control configuration. Although these constraints were intuitively apparent, it took many years to be fully understood and be stated without any ambiguity. A similar problem was studied by Youla, Bongiorno, and Lu [68], where the conditions for $G_o(s)$ to be implementable in the single-loop configuration in Figure P10.3 by using stable proper compensators are established. The conditions involve the interlacing of real poles of $G_o(s)$ and $G(s)$, and the problem is referred to as *strong stabilization.*

The computation of compensators by solving linear algebraic equations was first suggested by Shipley [55] in 1963. Conditions for the existence of compensators were not discussed. The coprimeness condition was used in 1969 [11] to establish the existence of compensators in the plant input/output feedback system and to establish its relationship with state feedback [66] and state estimators [45]. The two-parameter configuration seems to be first introduced by Aström [2] in 1980 and was employed in [3]. The design procedure discussed in this text follows [21].

Most results in Chapter 10 are available for multivariable systems, systems with more than one input or more than one output. The results were developed from the polynomial fractional approach, which was developed mainly by Rosenbrock [53], Wolovich [65], Kučera [41], and Callier and Desoer [9]. The polynomial fractional method was translated into linear algebraic equations in [14, 15]. Using elementary results in linear algebra, it was possible to develop most results in the polynomial fractional approach in [15]. More importantly, minimum-degree compensators can now be easily obtained by solving linear algebraic equations. This method of computing compensators is therefore called the linear algebraic method.