ECE484 - HW Ch2 Solutions

2.1-1 Both $\varphi(t)$ and $w_0(t)$ are periodic.

The average power of $\varphi(t)$ is $P_g = \frac{1}{T}\int_0^T \varphi^2(t)\,dt = \frac{1}{\pi}\int_0^\pi \left(e^{-t/2}\right)^2 dt = \frac{1-e^{-\pi}}{\pi}$.

The average power of $w_0(t)$ is $P_g = \frac{1}{T_0}\int_o^{T_0} w_o^2(t)\,dt = \frac{1}{T_0}\int_0^{T_0} 1\cdot dt = 1$.

2.1-7

$$\begin{aligned}
P_g &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} g(t)g^*(t)\,dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=m}^{n}\sum_{r=m}^{n} D_k D^*{}_r e^{j(\omega_k-\omega_r)t}\,dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=m}^{n}\sum_{r=m,r\neq k}^{n} D_k D^*{}_r e^{j(\omega_k-\omega_r)t}\,dt + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=m}^{n}|D_k|^2\,dt
\end{aligned}$$

The integrals of the cross-product terms (when $k\neq r$) are finite because the integrands (functions to be integrated) are periodic signals (made up of sinusoids). These terms, when divided by $T\to\infty$, yield zero. The remaining terms ($k=r$) yield

$$P_g = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=m}^{n}|D_k|^2\,dt = \sum_{k=m}^{n}|D_k|^2$$

2.3-1

$$g_2(t) = g(t-1)+g_1(t-1),\quad g_3(t) = g(t-1)+g_1(t+1),\quad g_4(t) = g(t-0.5)+g_1(t+0.5)$$

The signal $g_5(t)$ can be obtained by (i) delaying $g(t)$ by 1 second (replace t with $t-1$), (ii) then time-expanding by a factor 2 (replace t with $t/2$), (iii) then multiplying by 1.5. Thus $g_5(t) = 1.5g(\frac{t}{2}-1)$.

2.3-5

$$E_{-g} = \int_{-\infty}^{\infty}[-g(t)]^2\,dt = \int_{-\infty}^{\infty} g^2(t)\,dt = E_g,\quad E_{g(-t)} = \int_{-\infty}^{\infty}[g(-t)]^2\,dt = \int_{-\infty}^{\infty} g^2(x)\,dx = E_g$$

$$E_{g(t-T)} = \int_{-\infty}^{\infty}[g(t-T)]^2\,dt = \int_{-\infty}^{\infty} g^2(x)\,dx = E_g,\quad E_{g(at)} = \int_{-\infty}^{\infty}[g(at)]^2\,dt = \frac{1}{a}\int_{-\infty}^{\infty} g^2(x)\,dx = E_g/a$$

$$E_{g(at-b)} = \int_{-\infty}^{\infty}[g(at-b)]^2\,dt = \frac{1}{a}\int_{-\infty}^{\infty} g^2(x)\,dx = E_g/a,\quad E_{g(t/a)} = \int_{-\infty}^{\infty}[g(t/a)]^2\,dt = a\int_{-\infty}^{\infty} g^2(x)\,dt = aE_g$$

$$E_{ag(t)} = \int_{-\infty}^{\infty}[ag(t)]^2\,dt = a^2\int_{-\infty}^{\infty} g^2(t)\,dt = a^2E_g$$

2.5-1

$$|\mathbf{e}|^2 = |\mathbf{g}|^2 + c^2|\mathbf{x}|^2 - 2c\mathbf{g}\cdot\mathbf{x}$$

To minimize error, set $\frac{d|\mathbf{e}|^2}{dc} = 0$:

$$2c|\mathbf{x}|^2 - 2\mathbf{g}\cdot\mathbf{x} = 0$$

$$c = \frac{\mathbf{g}\cdot\mathbf{x}}{|\mathbf{x}|^2} = \frac{<\mathbf{g},\mathbf{x}>}{|\mathbf{x}|^2}$$

2.5-5

(a) If $x(t)$ and $y(t)$ are orthogonal, then we can show that the energy of $x(t) \pm y(t)$ is $E_x + E_y$.

$$\begin{aligned}\int |x(t) \pm y(t)|^2\,dt &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \pm \int_{-\infty}^{\infty} x(t)y^*(t)\,dt \pm \int_{-\infty}^{\infty} x^*(t)y(t)\,dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt\end{aligned}$$

The last result follows from the fact that because of orthogonality, the two integrals of the cross products $x(t)y^*(t)$ and $x^*(t)y(t)$ are zero [see Eq. (2.40)]. Thus the energy of $x(t) + y(t)$ is equal to that of $x(t) - y(t)$ if $x(t)$ and $y(t)$ are orthogonal.

(b) We can use a similar argument to show that the energy of $c_1x(t) + c_2y(t)$ is equal to that of $c_1x(t) - c_2y(t)$ if $x(t)$ and $y(t)$ are orthogonal. This energy is given by $|c_1|^2E_x + |c_2|^2E_y$.

(c) If $z(t) = x(t) \pm y(t)$, then it follows from **part (a)** in the preceding derivation that

$$E_z = E_x + E_y \pm (E_{xy} + E_{yx})$$

2.6-1 We shall use Eq. (2.51) to compute ρ_n for each of the four cases. Let us first compute the energies of all the signals:

$$E_x = \int_0^1 \sin^2 2\pi t \, dt = 0.5$$

In the same way, we find $E_{g_1} = E_{g_2} = E_{g_3} = E_{g_4} = 0.5$.

From Eq. (2.51), the correlation coefficients for four cases are found as follows:

(1) $\frac{1}{\sqrt{(0.5)(0.5)}} \int_0^1 \sin 2\pi t \sin 4\pi t \, dt = 0$

(2) $\frac{1}{\sqrt{(0.5)(0.5)}} \int (\sin 2\pi t)(-\sin 2\pi t)\, dt = -1$

(3) $\frac{1}{\sqrt{(0.5)(0.5)}} \int_0^1 0.707 \sin 2\pi t \, dt = 0$

(4) $\frac{1}{\sqrt{(0.5)(0.5)}} \left[\int_0^{0.5} 0.707 \sin 2\pi t \, dt - \int_{0.5}^1 0.707 \sin 2\pi t \, dt\right] = 2.828/\pi = 0.9$

Signals $x(t)$ and $g_2(t)$ provide the maximum protection against noise.

2.7-2

(a) We can choose the normalized $x(t), g_1(t), g_3(t)$ as the first three orthonormal bases for the set of signals, since all the correlations between these signals are zeros. Therefore,

$$\begin{aligned} \phi_1(t) &= \sqrt{2}x(t) \\ \phi_2(t) &= \sqrt{2}g_1(t) \\ \phi_3(t) &= \sqrt{2}g_3(t). \end{aligned}$$

Signal $g_2(t)$ is the negative of signal $x(t)$. Therefore, $g_2(t) = -\frac{1}{\sqrt{2}}\phi_1(t)$.

To represent $g_4(t)$, we need an additional basis function $\phi_4(t)$. We can represent $g_4(t)$ as $g_4(t) = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t) + c_4\phi_4(t)$, where,

$$
\begin{aligned}
c_1 &= \int_0^1 g_4(t)\phi_1^*(t)\,dt = \sqrt{2}\int_0^1 g_4(t)x(t)\,dt = \frac{2}{\pi} \\
c_2 &= \int_0^1 g_4(t)\phi_2^*(t)\,dt = \sqrt{2}\int_0^1 g_4(t)g_1(t)\,dt = \sqrt{2}\left[\int_0^{0.5} 0.707\sin 4\pi t\,dt - \int_{0.5}^1 0.707\sin 4\pi t\,dt\right] = 0 \\
c_3 &= \int_0^1 g_4(t)\phi_3^*(t)\,dt = \sqrt{2}\int_0^1 g_4(t)g_3(t)\,dt = \sqrt{2}\,(0.707)\int_0^1 g_4(t)\,dt = 0
\end{aligned}
$$

Therefore,

$$c_4\phi_4(t) = g_4(t) - c_1\phi_1(t) = g_4(t) - \frac{2\sqrt{2}}{\pi}x(t)$$

The energy of the signal is

$$
\begin{aligned}
E_{c_4\phi_4} &= \int_0^1 \left[g_4(t) - \frac{2\sqrt{2}}{\pi}x(t)\right]^2 dt = \int_0^1 \left[g_4^2(t) + \frac{8}{\pi^2}x^2(t)^2 - \frac{4\sqrt{2}}{\pi}g_4(t)x(t)\right] dt \\
&= \frac{1}{2} + \frac{4}{\pi^2} - \frac{4\sqrt{2}}{\pi}\cdot\frac{\sqrt{2}}{\pi} = \frac{1}{2} + \frac{4}{\pi^2} - \frac{8}{\pi^2} = \frac{1}{2} - \frac{4}{\pi^2} = 0.0946
\end{aligned}
$$

Therefore,

$$\phi_4(t) = \frac{g_4(t) - \frac{2\sqrt{2}}{\pi}x(t)}{\sqrt{E_{c_4\phi_4}}} = \frac{g_4(t) - \frac{2\sqrt{2}}{\pi}x(t)}{0.3076}$$

(b)

$$
\begin{aligned}
x_1(t) &= [\ \tfrac{1}{\sqrt{2}}\ \ 0\ \ 0\ \ 0\ ]^T \\
g_1(t) &= [\ 0\ \ \tfrac{1}{\sqrt{2}}\ \ 0\ \ 0\ ]^T \\
g_2(t) &= [\ -\tfrac{1}{\sqrt{2}}\ \ 0\ \ 0\ \ 0\ ]^T \\
g_3(t) &= [\ 0\ \ 0\ \ \tfrac{1}{\sqrt{2}}\ \ 0\ ]^T \\
g_4(t) &= [\ \tfrac{2}{\pi}\ \ 0\ \ 0\ \ 0.3076\ ]^T
\end{aligned}
$$

2.8-2

(a) $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Because of even symmetry, all sine terms are zero.

$$
\begin{aligned}
g(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right) \\
a_0 &= 0 \text{ (by inspection of its lack of dc)} \\
a_n &= \frac{4}{4}\left[\int_0^1 \cos\left(\frac{n\pi}{2}t\right) dt - \int_1^2 \cos\left(\frac{n\pi}{2}t\right) dt\right] \\
&= \frac{4}{n\pi} \sin \frac{n\pi}{2}
\end{aligned}
$$

Therefore, the Fourier series for $g(t)$ is

$$
g(t) = \frac{4}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \frac{1}{7}\cos\frac{7\pi t}{2} + \cdots\right)
$$

Here $b_n = 0$, and we allow C_n to take negative values. Figure S2.8-2(a) shows the plot of C_n.

(b) $T_0 = 10\pi$, $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$. Because of even symmetry, all the sine terms are zero.

$$
\begin{aligned}
g(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right) \\
a_0 &= \frac{1}{5} \qquad \text{(by inspection)} \\
a_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \cos\left(\frac{n}{5}t\right) dt \\
&= \frac{1}{5\pi}\left(\frac{5}{n}\right)\sin\left(\frac{n}{5}t\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right) \\
b_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \sin\left(\frac{n}{5}t\right) dt \\
&= 0 \qquad \text{(integrand is an odd function of } t\text{)}
\end{aligned}
$$

Here $b_n = 0$, and we allow C_n to take negative values. Note that $C_n = a_n$ for $n = 0, 1, 2, 3, \ldots$. Fig. S2.8-2(b) shows the plot of C_n.

(c) $T_0 = 2\pi$, $\omega_0 = 1$, and

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

with

$$a_0 = 0.5 \quad \textbf{(by inspection of the dc or average)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi} \cos nt\, dt = 0, \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi} \sin nt\, dt = -\frac{1}{\pi n}$$

and

$$\begin{aligned} g(t) &= 0.5 - \frac{1}{\pi}\left(\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t + \cdots\right) \\ &= 0.5 + \frac{1}{\pi}\left[\cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2t + \frac{\pi}{2}\right) + \frac{1}{3}\cos\left(3t + \frac{\pi}{2}\right) + \cdots\right] \end{aligned}$$

The cosine terms vanish because when 0.5 (the dc component) is subtracted from $g(t)$, the remaining function has odd symmetry. Hence, the Fourier series would contain dc and sine terms only. Figure S2.8-2(c) shows the plots of C_n and θ_n.

(d) $T_0 = \pi$, $\omega_0 = 2$ and $g(t) = \frac{4}{\pi}t$.

$a_0 = 0$ (by inspection)

$a_n = 0 \quad (n > 0)$ (because of odd symmetry)

$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi} t \sin 2nt\, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$\begin{aligned} g(t) &= \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots \\ &= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdots \end{aligned}$$

Figure S2.8-2(d) shows the plots of C_n and θ_n.

(e) $T_0 = 3$, $\omega_0 = 2\pi/3$, and

$$a_0 = \frac{1}{3}\int_0^1 t\,dt = \frac{1}{6}$$

$$a_n = \frac{2}{3}\int_0^1 t\cos\frac{2n\pi}{3}t\,dt = \frac{3}{2\pi^2 n^2}\left[\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3}\right]$$

$$b_n = \frac{2}{3}\int_0^1 t\sin\frac{2n\pi}{3}t\,dt = \frac{3}{2\pi^2 n^2}\left[\sin\frac{2\pi n}{3} - \frac{2\pi n}{3}\cos\frac{2\pi n}{3}\right]$$

Therefore, $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

and

$$\theta_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

Figure S2.8-2(e) shows the plots of C_n and θ_n.

(f) $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 0.5$ (by inspection of the dc value). There is even symmetry, and $b_n = 0$.

$$a_n = \frac{4}{6}\int_0^3 g(t)\cos\frac{n\pi}{3}\,dt = \frac{2}{3}\left[\int_0^1 \cos\frac{n\pi}{3}\,dt + \int_1^2 (2-t)\cos\frac{n\pi}{3}t\,dt\right] = \frac{6}{\pi^2 n^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]$$

$$g(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3}t + \frac{1}{49}\cos\frac{7\pi}{3}t + \cdots\right)$$

Observe that even harmonics vanish. This is because if the dc (0.5) is subtracted from $g(t)$, the resulting function has half-wave symmetry. Figure S2.8-2(f) shows the plot of C_n.

2.9-1 See Fig. S2.9-1.

(a) $T_0 = 4, \omega_0 = \pi/2$. Also $D_0 = 0$ (by inspection):

$$D_n = \frac{1}{2\pi}\int_{-1}^{1} e^{-j(n\pi/2)t}\,dt - \int_{1}^{3} e^{-j(n\pi/2)t}\,dt = \frac{2}{\pi n}\sin\frac{n\pi}{2}, \qquad |n| \geq 1$$

(b) $T_0 = 10\pi$, $\omega_0 = 2\pi/10\pi = 1/5$. Also $D_0 = 1/5$ (by inspection):

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{n}{5}t}$$

where

$$D_n = \frac{1}{10\pi}\int_{\pi}^{\pi} e^{-j\frac{n}{5}t}\,dt = \frac{j}{2\pi n}\left(-2j\sin\frac{n\pi}{5}\right) = \frac{1}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$

(c)

$$g(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{jnt}$$

where, by inspection,

$$D_0 = 0.5$$

$$D_n = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{t}{2\pi} e^{-jnt}\,dt = \frac{j}{2\pi n}$$

so that

$$|D_n| = \frac{1}{2\pi n}$$

and

$$\angle D_n = \begin{cases} \frac{\pi}{2}, & n > 0 \\ -\frac{\pi}{2}, & n < 0 \end{cases}$$

(d) $T_0 = \pi$, $\omega_0 = 2$ and $D_n = 0$,

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$$

where

$$D_n = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4t}{\pi} e^{-j2nt}\,dt = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

(e) $T_0 = 3, \omega_0 = \frac{2\pi}{3}$. Also, $D_0 = 1/6$ (by inspection):

$$g(t) = \sum_{n=\infty}^{\infty} D_n \, e^{j2\pi nt/3}$$

where

$$D_n = \frac{1}{3}\int_0^1 t\, e^{-j2\pi nt/3}\, dt = \frac{3}{4\pi^2 n^2}\left[e^{-j2\pi n/3}\left(\frac{j2\pi n}{3}+1\right)-1\right]$$

Therefore

$$|D_n| = \frac{3}{4\pi^2 n^2}\left[\sqrt{2+\frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

and

$$\angle D_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6,\ \omega_0 = \pi/3\ D_0 = 0.5$

$$g(t) = 0.5 + \sum_{n=-\infty}^{\infty} D_n e^{j\pi nt/3}$$

$$D_n = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-j\pi nt/3}\, dt + \int_{-1}^{1} e^{-j\pi nt/3}\, dt + \int_1^2 (-t+2)e^{-j\pi nt/3}\, dt\right] = \frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3}\right)$$

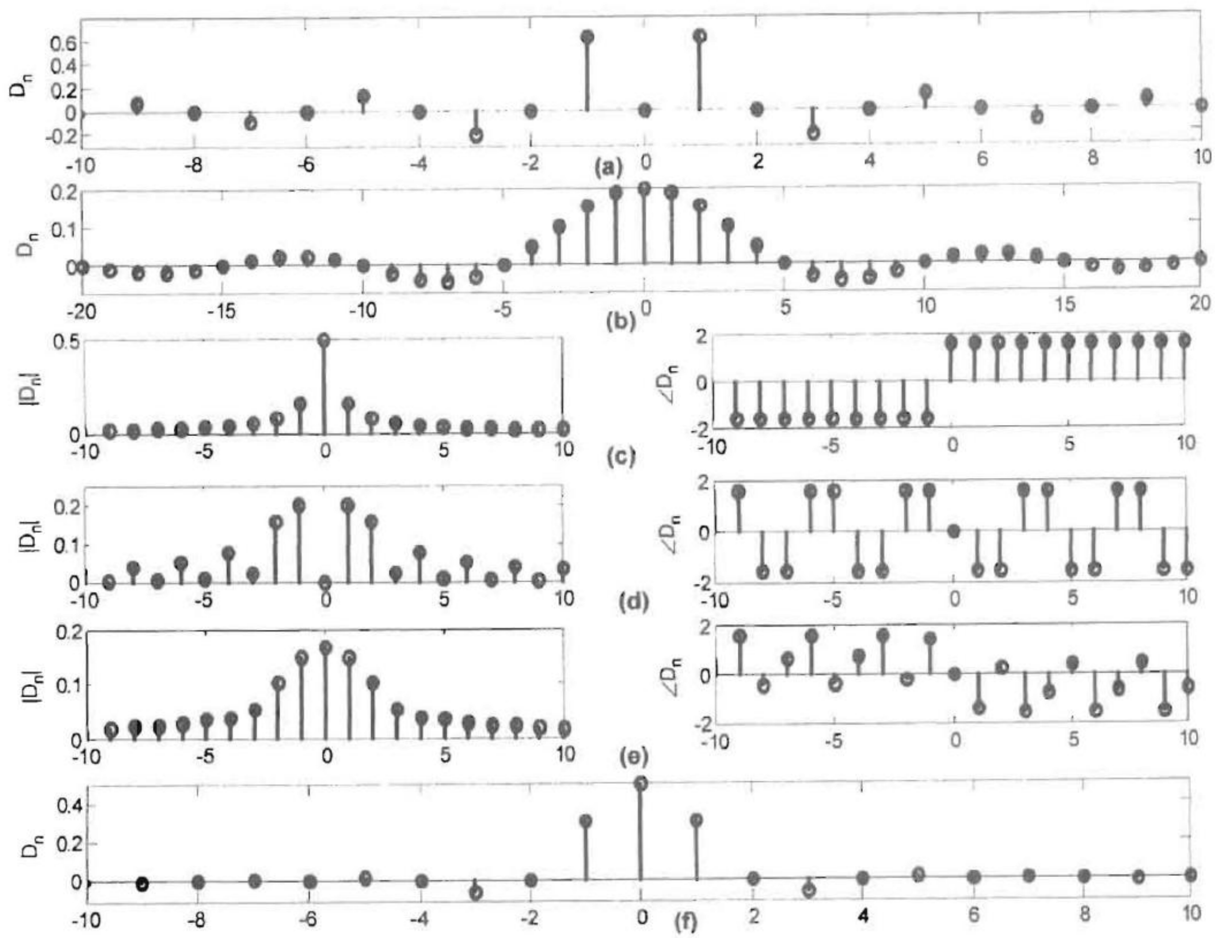

Fig. S2.9-1