Josh Andrews
ECE 484 HW 3

1

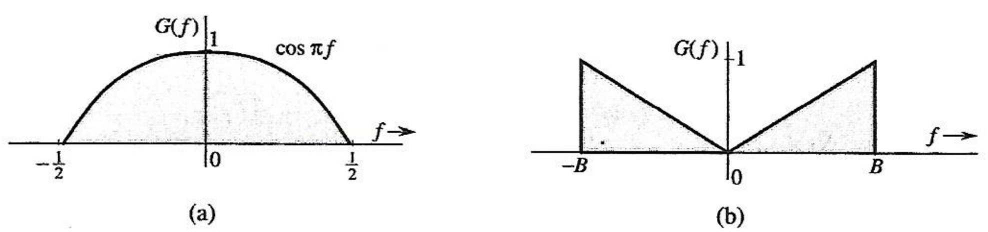
3.1-3) if $g(t) \Leftrightarrow G(f)$; show $g^*(t) \Leftrightarrow G^*(-f)$

$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt$ and take conjugate of both sides

$[G(f)]^* = \int_{-\infty}^{\infty} g^*(t) e^{j2\pi f t}\, dt$ and replace f with -f

$G^*(-f) = \int_{-\infty}^{\infty} g^*(t) e^{-j2\pi f t}\, dt \Rightarrow \boxed{g^*(t) \Longleftrightarrow G^*(-f)}$

3.1-6 From definition (3.9b), find the inverse Fourier transforms of the spectra shown in Fig. P3.1-6.



a. $G(f) = \cos \pi f \quad -\frac{1}{2} \le f \le \frac{1}{2}$

$$g(t) = \int_{-1/2}^{1/2} (\cos \pi f) e^{j2\pi f t}\, df = \int_{-1/2}^{1/2} \left(\frac{e^{j\pi f} + e^{-j\pi f}}{2}\right) e^{j2\pi f t}\, df$$

$$g(t) = \frac{1}{2}\int_{-1/2}^{1/2} e^{(j\pi + 2j\pi t)f}\, df + \frac{1}{2}\int_{-1/2}^{1/2} e^{(2j\pi t - j\pi)f}\, df$$

$$g(t) = \frac{1}{2} \cdot \left[\frac{\left[e^{(j\pi + j2\pi t)f}\right]_{-1/2}^{1/2}}{(j\pi + j2\pi t)} + \frac{\left[e^{(2j\pi t - j\pi)f}\right]_{-1/2}^{1/2}}{(j2\pi t - j\pi)}\right]$$

$$g(t) = \frac{e^{j\left(\frac{\pi + 2\pi t}{2}\right)} - e^{-j\left(\frac{\pi + 2\pi t}{2}\right)}}{2j(\pi + 2\pi t)} + \frac{e^{j\left(\frac{2\pi t - \pi}{2}\right)} - e^{-j\left(\frac{2\pi t - \pi}{2}\right)}}{2j(2\pi t - \pi)}$$

and because $\sin x = \dfrac{e^{jx} - e^{-jx}}{2j}$

$$g(t) = \frac{1}{2}\left[\frac{\sin\left(\frac{\pi + 2\pi t}{2}\right)}{\left(\frac{\pi + 2\pi t}{2}\right)} + \frac{\sin\left(\frac{2\pi t - \pi}{2}\right)}{\left(\frac{2\pi t - \pi}{2}\right)}\right] \Rightarrow \boxed{g(t) = \frac{1}{2}\left[\operatorname{Sinc}\left(\frac{2\pi t + \pi}{2}\right) + \operatorname{Sinc}\left(\frac{2\pi t - \pi}{2}\right)\right]}$$

3.1-6) b. looking at the signal, need to find the equation

will look at $\{0,B\}$ interval to find $G(f)$

$$G(f)-0=\left(\frac{1-0}{B-0}\right)(f-0) \Rightarrow G(f)=f/B$$

$g(t)=\int_{-B}^{B} G(f)\, e^{j2\pi f t}\, df$ but both signals are the same $\Rightarrow$ can look at $0 \to B$ and multiply by 2

$$g(t)=2\int_0^B \frac{f}{B} e^{j2\pi f t}\, df \Rightarrow g(t)=\frac{2}{B}\int_0^B f e^{j2\pi f t}\, df \Rightarrow$$

$$g(t)=\frac{2}{B}\left[f\left(\frac{e^{j2\pi f t}}{j2\pi t}\right)-\frac{e^{j2\pi f t}}{(j2\pi t)^2}\right]_0^B \Rightarrow g(t)=\frac{2}{B}\left[\frac{Be^{j2\pi Bt}}{j2\pi t}+\frac{e^{j2\pi Bt}-1}{(2\pi t)^2}\right]$$

3.2-2) use $1 \Leftrightarrow \delta(f)$ and $\operatorname{sgn} t \Leftrightarrow \frac{2}{j2\pi f}$ to show

$$u(t) \Leftrightarrow 0.5\delta(f)+\frac{1}{j2\pi f}$$

$$\operatorname{sgn}(t)=\begin{cases}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{cases} \quad \text{so } 1+\operatorname{sgn}(t)=\begin{cases}2 & t>0\\ 1 & t=0\\ 0 & t<0\end{cases} = 2u(t) \Rightarrow u(t)=\frac{1+\operatorname{sgn}(t)}{2}$$

$$u(t)=\frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t)$$

$$\downarrow \qquad \downarrow$$

$$0.5\delta(f) \quad \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{j2\pi f}\right) \Rightarrow \boxed{u(t) \Leftrightarrow 0.5\delta(f)+\frac{1}{j2\pi f}}$$

Josh Andrews Hw 3 ECE 484 ③

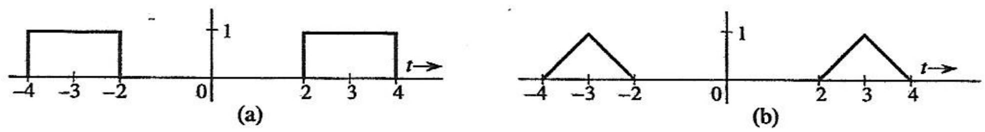
3.3-4 Use the time-shifting property to show that if $g(t) \Longleftrightarrow G(f)$, then

$$g(t+T) + g(t-T) \Longleftrightarrow 2G(f)\cos 2\pi fT$$

This is the dual of Eq. (3.37). Use this result and pairs 17 and 19 in Table 3.1 to find the Fourier transforms of the signals shown in Fig. P3.3-4.

$\Pi\left(\frac{t}{\tau}\right) \Leftrightarrow \tau\,\text{sinc}(\pi f \tau)$

$\Delta\left(\frac{t}{\tau}\right) \Leftrightarrow \frac{\tau}{2}\,\text{sinc}^2\left(\frac{\pi f \tau}{2}\right)$



because $g(t) \Leftrightarrow G(f)$ if $f(t) = g(t+T) + g(t-T)$

$$F[f(t)] = F[g(t+T)] + F[g(t-T)]$$

$$F(f) = \left(e^{j2\pi fT} G(f)\right) + \left(e^{-j2\pi fT} G(f)\right) = G(f)\left[e^{j2\pi fT} + e^{-j2\pi fT}\right]$$

$$F(f) = 2G(f)\left[\frac{e^{j2\pi fT} + e^{-j2\pi fT}}{2}\right] = \underline{2G(f)\cos(2\pi fT)}$$

<u>For Signal a</u>

rectangular pulse with $\tau = 2$ $\Rightarrow$ $\Pi\left(\frac{t}{2}\right) = 1$ but is shifted over 3 $\Rightarrow$

$\Pi\left(\frac{t+3}{2}\right)$ $\Rightarrow$ apply time shift to find transform

$\Pi\left(\frac{t+3}{2}\right) \Leftrightarrow 2e^{j6\pi f}\,\text{sinc}(2\pi f)$ and also for positive signal

$\Pi\left(\frac{t-3}{2}\right) \Leftrightarrow 2e^{-j6\pi f}\,\sin(2\pi f)$ and add together

$$\Pi\left(\frac{t+3}{2}\right) + \Pi\left(\frac{t-3}{2}\right) \Leftrightarrow 2e^{j6\pi f}\,\text{sinc}(2\pi f) + 2e^{-j6\pi f}\,\text{sinc}(2\pi f)$$

$$\Leftrightarrow 4\left[\frac{e^{j6\pi f} + e^{-j6\pi f}}{2}\right]\text{sinc}(2\pi f)$$

$$\boxed{\Pi\left(\frac{t+3}{2}\right) + \Pi\left(\frac{t-3}{2}\right) \Leftrightarrow 4\cos(6\pi f)\,\text{sinc}(2\pi f)}$$

3.3-4) and for signal b

have a triangular pulse $\Delta\left(\frac{t}{\tau}\right)$ with $\tau = 2$

but signals are time shifted by 3
so apply same time shifting

$$\Delta\left(\frac{t+3}{2}\right) \Leftrightarrow e^{j6\pi f}\,\mathrm{sinc}^2(\pi f)$$

$$\Delta\left(\frac{t-3}{2}\right) \Leftrightarrow e^{-j6\pi f}\,\mathrm{sinc}^2(\pi f)$$

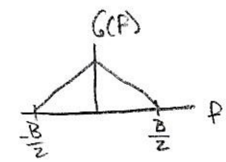
$$\Delta\left(\frac{t+3}{2}\right) + \Delta\left(\frac{t-3}{2}\right) \Leftrightarrow 2\left(\frac{e^{j6\pi f} + e^{-j6\pi f}}{2}\right)\mathrm{sinc}^2(\pi f)$$

$$\boxed{\Delta\left(\frac{t+3}{2}\right) + \Delta\left(\frac{t-3}{2}\right) \Leftrightarrow 2\cos(6\pi f)\,\mathrm{sinc}^2(\pi f)}$$

3.3-9 The process of recovering a signal $g(t)$ from the modulated signal $g(t)\cos 2\pi f_0 t$ is called **demodulation.** Show that the signal $g(t)\cos 2\pi f_0 t$ can be demodulated by multiplying it with $2\cos 2\pi f_0 t$ and passing the product through a low-pass filter of bandwidth B Hz [the bandwidth of $g(t)$]. Assume $B < f_0$.

Hint: $2\cos^2 2\pi f_0 t = 1 + \cos 4\pi f_0 t$. Recognize that the spectrum of $g(t)\cos 4\pi f_0 t$ is centered at $2f_0$ and will be suppressed by a low-pass filter of bandwidth B Hz.

modulated signal $m(t) = \underset{\text{signal}}{g(t)}\,\underset{\text{carrier}}{\cos(2\pi f_0 t)}$ and assuming



$\Rightarrow$ the demodulated signal is

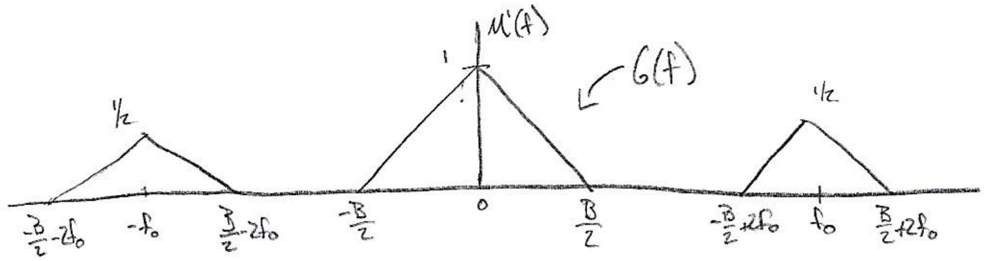
$$m'(t) = [g(t)\cos 2\pi f_0 t][2\cos 2\pi f_0 t] = g(t)\,2\cos^2(2\pi f_0 t)$$

$$m'(t) = g(t)[1 + \cos(4\pi f_0 t)] = g(t) + g(t)\cos(4\pi f_0 t)$$ and find transform

$$F(\cos(4\pi f_0 t)) = F(\cos(2\pi\, 2f_0 t)) \Rightarrow \frac{1}{2}\left[\delta(f-2f_0) + \delta(f+2f_0)\right]$$

$$M'(f) = G(f) + G(f)\left[\frac{1}{2}\left(\delta(f-2f_0) + \delta(f+2f_0)\right)\right]$$

3.3-9) cont'd $M'(f) = G(f) + \frac{1}{2} G(f-f_0) + \frac{1}{2} G(f+f_0)$ and if we plot that



and filter with B bandwidth filters out $\pm f_0$ signals

and are left with $G(f)$ $\Rightarrow Y(f) = G(f)$

and transform $\boxed{y(t) = g(t) \Rightarrow \text{Can be de-modulated by multipling with } 2\cos(2\pi f_0 t) \text{ and passing through filter}}$

3.5-1 For systems with the following impulse responses, which system is causal?

(a) $h(t) = e^{-at}u(t), \quad a > 0$

(b) $h(t) = e^{-a|t|}, \quad a > 0$

(c) $h(t) = e^{-a(t-t_0)}u(t-t_0), \quad a > 0, t_0 \geq 0$

(d) $h(t) = \text{sinc}\,(at), \quad a > 0$

(e) $h(t) = \text{sinc}\,[a(t-t_0)], \quad a > 0$

a. Causal, the unit step makes $h(t)=0$ for $t<0$

b. not causal, $h(t) \neq 0$ for $t<0$

c. Causal, the unit step ensures no response before t_0

d. not causal, the sinc function has non zero values before $t=0$

e. not causal, the time shift does not alter the fact that the sinc function has non-zero values before t_0

3.5-4 A bandpass signal $g(t)$ of bandwidth $B = 2000$ Hz centered at $f = 10^5$ Hz is passed through the RC filter in Example 3.16 (Fig. 3.28a) with $RC = 10^{-3}$. If over the passband, the variation of less than 2% in amplitude response and less than 1% in time delay is considered to be distortionless transmission, would $g(t)$ be transmitted without distortion? Find the approximate expression for the output signal.

$H(f) = \dfrac{1}{1 + j2\pi fRC}$ and need the magnitude $|H(f)| = \dfrac{1}{\sqrt{1+(2\pi fRC)^2}}$

$\approx \dfrac{1}{2\pi fRC}$ if $f \gg \frac{1}{RC}$

find $\Delta|H|$ at edges

$$\Delta|H| = \frac{|H(f_c - B/2)| - |H(f_c + B/2)|}{H(f_c)} = \frac{[2\pi(f_c - B/2)RC]^{-1} - [2\pi(f_c + B/2)RC]^{-1}}{[2\pi f_c RC]^{-1}}$$

$$\Delta|H| = \frac{(f_c + B/2) - (f_c - B/2)}{f_c^{-1}(f_c + B/2)(f_c - B/2)} = \frac{(f_c + B/2) - (f_c - B/2)}{f_c^{-1}(f_c + B/2)(f_c - B/2)}$$

$$\Delta|H| = \frac{f_c + B/2 - f_c + B/2}{f_c^{-1}(f_c^2 - (B/2)^2)} \approx \frac{B}{f_c} \quad \text{because } f_c^2 \gg \left(\frac{B}{2}\right)^2$$

Now plug in $B = 2000$ Hz and $f_c = 10^5$ Hz

$$\Delta|H| \approx \frac{2000}{10^5} = 0.02 \Rightarrow \underline{\text{amplitude response variation} = 2\%}$$

Now we need phase response and then derivative for time delay

$$\theta_h(f) = -\tan^{-1}(2\pi fRC) \Rightarrow t_d(f) = \frac{d\theta_h}{d(2\pi f)} = \frac{RC}{1+(2\pi fRC)^2} \approx \frac{RC}{(2\pi fRC)^2} \quad \left(\text{since } f \gg \frac{1}{RC}\right)$$

$$\Delta(t_d) = \frac{t_d(f_c - B/2) - t_d(f_c + B/2)}{t_d(f_c)}$$

$$= \frac{(f_c - B/2)^{-2} - (f_c + B/2)^{-2}}{f_c^{-2}} = \frac{2Bf_c}{f_c^{-2}(f_c^2 - \frac{B^2}{4})^2} \approx \underline{\frac{2B}{f_c}} \quad \text{since } f_c \gg \frac{B}{2}$$

Plug in to find time delay variation

$$\Delta(t_d) = \frac{2(2000)}{10^5 \text{ Hz}} = 0.04 \Rightarrow \underline{\text{time delay variation } 4\%}$$

3.5-4) Cont'd

$\boxed{\text{Because the time delay variation of 4\% is larger than the 1\% specified and therefore the signal will be transmitted with distortion}}$

To find the approximate output signal, evaluate now at the center frequency

$$|H(f_c)| \approx \frac{1}{2\pi f_c RC} = \frac{1}{2\pi(10^5)(10^{-3})} = 1.6\times10^{-3}$$

$$t_d(f_c) \approx \frac{1}{(2\pi f_c)^2 RC} = \frac{1}{10^{-3}(2\pi 10^5)^2} = 2.53\times10^{-9}\ \text{sec}$$

approximate signal

$$y(t) \approx |H(f_c)|\, g(t - t_d(f_c)) \Rightarrow \boxed{y(t) = 1.6\times10^{-3}\cdot g(t - 2.53\times10^{-9})}$$

Josh Andrews HW 3 ECE 484

3.8-5 Consider a linear system with impulse response $e^{-2t}u(t)$. The linear system input is

$$g(t) = w(t) - \cos\left(6\pi t + \frac{\pi}{3}\right)$$

in which $w(t)$ is a noise signal with power spectral density of

$$S_w(f) = \Pi\left(\frac{f}{4}\right)$$

(a) Find the total output power of the linear system.

(b) Find the output power of the signal component due to the sinusoidal input.

(c) Find the output power of the noise component.

(d) Determine the output signal-to-noise ratio (SNR) in decibels.

$h(t) = e^{-2t}u(t)$

$H(f) = \frac{1}{2+j2\pi f}$ $\Updownarrow \mathcal{L}$

a) First need find transform of $g(t)$ and $g(t) = w(t) - \cos(6\pi t + \pi/3)$

$g(t) = w(t) - \cos(3 \cdot 2\pi + \pi/3)$

$\Rightarrow S_x(f) = 1/2\left(\delta(f-3)e^{j\pi/3} + \delta(f+3)e^{-j\pi/3}\right)$

and know $S_w(f)$ $\Rightarrow S_g(f) = S_w(f) + S_x(f) = \Pi\left(\frac{f}{4}\right) + 1/2\left(\delta(f-3)e^{j\pi/3} + \delta(f+3)e^{-j\pi/3}\right)$

$S_g(f) = \Pi(f/4) + 1/2\left[(1/2 + j\sqrt{3}/2)\delta(f-3) + (1/2 - j\sqrt{3}/2)\delta(f+3)\right]$ $\Rightarrow$ factor and ignore imaginary terms

$S_g(f) = \Pi(f/4) + 1/4\left[\delta(f-3) + \delta(f+3)\right]$ and to find output power Need PSD

$S_y(f) = |H(f)|^2 S_g(f) = \left|\frac{1}{2+j2\pi f}\right|^2 \cdot \left(\Pi(f/4) + 1/4\left[\delta(f-3) + \delta(f+3)\right]\right)$

$= \left(\frac{1}{4+4\pi^2 f^2}\right)\left(\Pi(f/4) + 1/2\left[\delta(f-3) + \delta(f+3)\right]\right)$ inverse $= P_y$

$P_y = \int_{-\infty}^{\infty} S_y(f)df = \int_{-2}^{2} \frac{1}{4+4\pi^2 f^2}df + \frac{1}{4}\int_{-\infty}^{\infty}\frac{\delta(f-3)}{4+4\pi^2 f^2}df + \frac{1}{4}\int_{-\infty}^{\infty}\frac{\delta(f+3)}{4+4\pi^2 f^2}$

$= \frac{\tan^{-1}(2\pi) - \tan^{-1}(-2\pi)}{4\pi} + \frac{2}{16+144\pi^2}$

$$\boxed{P_y = 226 \text{ mW}}$$

3.8-5 b) $\Rightarrow S_{yx}(f) = |H(f)|^2 S_x(f) = \left(\frac{1}{4+4\pi^2 f^2}\right)\left(\frac{1}{4}\left[\delta(f-3)+\delta(f+3)\right]\right)$

- inverse is P_{yx}

$$\Rightarrow P_{yx} = \frac{1}{4}\left[\frac{1}{4+4\pi(3)^2} + \frac{1}{4+4\pi(-3)^2}\right] = \frac{2}{16+144\pi^2}$$

$$\boxed{P_{yx} = 1.4 \text{ mW}}$$

c) Power of noise, found from $S_{yw}(f) = |H(f)|^2 S_w(f)$

$$P_{yw} = \int_{-}^{\infty}\left(\frac{1}{4+4\pi^2 f^2}\right)\Pi\left(\frac{f}{4}\right)df = \int_{-2}^{2}\frac{1}{4+4\pi^2 f^2}\,df = \left[\frac{\tan^{-1}(\pi f)}{4\pi}\right]_{-2}^{2}$$

$$P_{yw} = \frac{2\tan^{-1}(2\pi)}{4\pi}$$

$$\boxed{P_{yw} = 225 \text{ mW}}$$

- given approximated P values, checking $P_{yx} + P_{yw} = P_y$ ✓

d)

$$SNR_{dB} = 10\log_{10}\left(\frac{P_{yx}}{P_{yw}}\right) = 10\log_{10}\left(\frac{1.4}{225}\right)$$

$$\boxed{SNR_{dB} = -22.06 \text{ dB}}$$