ECE484 - HW Ch3 Solutions

3.1-3

$$\mathcal{F}\{g^*(t)\} = \int_{-\infty}^{\infty} g^*(t)\, e^{-j2\pi ft}\, dt = \left(\int_{-\infty}^{\infty} g(t)\, e^{j2\pi ft}\, dt\right)^* = [G(-f)]^* \equiv G^*(-f)$$

3.1-6

(a)

$$g(t) = \int_{-1/2}^{1/2} \cos \pi f\, e^{j2\pi ft}\, df = \frac{e^{j2\pi ft}}{\pi(1-4t^2)}\left[j2t \cos \pi f + \sin \pi f\right]_{-1/2}^{1/2} = \frac{2}{\pi(1-4t^2)} \cos(\pi t)$$

(b)

$$\begin{aligned} g(t) &= \int_{-B}^{0} \left(\frac{-f}{B}\right) e^{j2\pi ft}\, df + \int_{0}^{B} \frac{f}{B} e^{j2\pi ft}\, df \\ &= \left[\frac{(j2\pi ft - 1)\, e^{j2\pi ft}}{4B\pi^2 t^2}\right]_{-B}^{0} - \left[\frac{(j2\pi ft - 1)\, e^{j2\pi ft}}{4B\pi^2 t^2}\right]_{0}^{B} \\ &= \frac{(-j2\pi Bt - 1)\, e^{-j2\pi Bt} - (j2\pi Bt - 1)\, e^{j2\pi Bt} - 2}{4B\pi^2 t^2} = \frac{2\cos 2\pi Bt + 4\pi Bt \sin 2\pi Bt - 2}{4B\pi^2 t^2} \end{aligned}$$

3.2-2 Observe $1 + \operatorname{sgn} t = 2u(t)$. Then from Table 3.1,

$$u(t) \Longleftrightarrow \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

3.3-4 From the time-shifting property, $g(t \pm T) \Longleftrightarrow G(f)e^{\pm j2\pi fT}$.. Therefore,

$$g(t+T) + g(t-T) \Longleftrightarrow G(f)e^{j2\pi fT} + G(f)e^{-j2\pi fT} = 2G(f)\cos 2\pi fT$$

(a) Let $T = 3$. Then $g(t) = \Pi\left(\frac{t}{2}\right)$ and from pair 17 of Table 3.1, $G(f) = 2\,\text{sinc}(2\pi f)$. Then

$$g(t+3) + g(t-3) \Longleftrightarrow 4\,\text{sinc}(2\pi f)\cos(6\pi f)$$

(b) Again, let $T = 3$ and then take $g(t) = \Delta\left(\frac{t}{2}\right)$. From pair 19 of Table 3.1, $G(f) = \text{sinc}^2(\pi f)$. Then

$$g(t+3) + g(t-3) \Longleftrightarrow 2\,\text{sinc}^2(\pi f)\cos(6\pi f)$$

3.3-9 The modulated signal is $g(t)\cos 2\pi f_0 t$. Multiplying by $2\cos 2\pi f_0 t$ yields

$$2g(t)\cos^2(2\pi f_0 t) = g(t)\left[1 + \cos 4\pi f_0 t\right] = g(t) + g(t)\cos 4\pi f_0 t$$

Observe that the resulting signal contains the original signal $g(t)$ and a modulated copy of the signal moved to a center frequency of $2f_0$. If the bandwidth of the original signal is $B < f_0$, then the modulated copy will not extend further than f_0 from its center frequency and a low-pass filter from $-f_0$ to f_0 will pass only $g(t)$ and filter out the modulated copy.

3.5-1

(a) The step function $u(t)$ is 0 for $t < 0$, and so any function multiplied by it also has this property. Thus the system with this impulse response is causal.

(b) The indicated impulse response is symmetric and nonzero around $t = 0$, and therefore the corresponding system is not causal.

(c) The impulse response $h(t) = e^{-a(t-t_0)}u(t-t_0)$ is a shifted version of the impulse response in part (a). If $t_0 \geq 0$ then the shift is to the right (or no shift at all in the case $t_0 = 0$) and the system is causal. If $t_0 < 0$, then the shift is to the left and the system is not causal.

(d) The indicated impulse response is symmetric and nonzero around $t = 0$, and therefore the corresponding system is not causal.

(e) The impulse response $h(t) = \text{sinc}[a(t-t_0)]$ is a shifted version of the impulse response in part (d). However, since the sinc function has infinite duration (from $-\infty$ to $+\infty$) any finite shift will still result in a noncausal system.

3.5-4 For $RC = 10^{-3}$, the system's frequency response is

$$H(f) = \frac{1000}{1000 + j2\pi f}$$

Hence

$$|H(f)| = \frac{1000}{\sqrt{1000^2 + (2\pi f)^2}}$$

and

$$t_d(f) = \frac{1000}{(2\pi f)^2 + 1000^2}$$

Observe from Figure 3.28 that both the amplitude response and the time delay of the circuit are decreasing functions of frequency. This monotonicity allows us to check the endpoints of the band where signal $g(t)$ is nonzero to determine whether transmission is distortionless. These frequencies are and $f_h = 101,000$ Hz, and

$$|H(f_l)| = 1.607 \times 10^{-3} \qquad t_d(f_l) = 2.58 \times 10^{-9}$$
$$|H(f_h)| = 1.576 \times 10^{-3} \qquad t_d(f_h) = 2.48 \times 10^{-9}$$

We compute the variations as

$$\Delta|H(f)| = \frac{||H(f_h)| - |H(f_l)||}{\frac{1}{2}(|H(f_h)| + |H(f_l)|)} \times 100\% = 1.95\% < 2\%$$

$$\Delta t_d(f) = \frac{|t_d(f_h) - t_d(f_l)|}{\frac{1}{2}(t_d(f_h) + t_d(f_l))} \times 100\% = 3.9\% > 1\%$$

The variations are NOT within tolerances. Thus, we cannot consider the transmission to be distortionless. Still, if we wish to make an approximate, the average amplitude response is approximately $\frac{1}{2}(|H(f_h)| + |H(f_l)|) = 1.59 \times 10^{-3}$, and the average delay is $\frac{1}{2}(t_d(f_h) + t_d(f_l)) = 6.14 \times 10^{-4}$. The resulting approximate output signal is $y(t) \approx 1.59 \times 10^{-3} g(t - 6.14 \times 10^{-4})$.

3.8-5 The system impulse is $h(t) = e^{-2t}u(t)$ and the input is $g(t) = w(t) - \cos(6\pi t + \pi/3)$. We are also given that $w(t)$ is a noise signal with PSD $S_w(f) = \Pi(f/4)$.

(a) To find the power of the output P_y we first need to compute $S_g(f) = S_w(f) + S_s(f)$ where $S_s(f)$ denotes the PSD of the signal $s(t) = -\cos(6\pi t + \pi/3)$. From Problem 3.8-1, we have $S_s(f) = \frac{1}{4}[\delta(f-3) + \delta(f+3)]$. Thus $S_g(f) = \Pi(f/4) + \frac{1}{4}[\delta(f-3) + \delta(f+3)]$. We also have the system frequency response

$$H(f) = \frac{1}{2 + j2\pi f} \quad \text{and} \quad |H(f)|^2 = \frac{1}{4 + 4\pi^2 f^2}$$

We can now obtain the output PSD as

$$S_y(f) = |H(f)|^2 S_g(f) = \frac{1}{4 + 4\pi^2 f^2}\left[\Pi(f/4) + \frac{1}{4}\delta(f-3) + \frac{1}{4}\delta(f+3)\right]$$

and the output power

$$P_y = \int_{-\infty}^{\infty} S_y(f)\, df = \int_{-2}^{2} \frac{1}{4 + 4\pi^2 f^2}\, df + \frac{1}{2}\frac{1}{4 + 36\pi^2} = \frac{\tan^{-1}(2\pi)}{2\pi} + \frac{1}{8 + 72\pi}$$

(b) The PSD of the output due to the signal is

$$S_{ys}(f) = |H(f)|^2 S_s(f) = \frac{1}{4 + 4\pi^2 f^2}\left[\frac{1}{4}\delta(f-3) + \frac{1}{4}\delta(f+3)\right]$$

and the power due to the signal is

$$P_{ys} = \int_{-\infty}^{\infty} S_{ys}(f)\, df = \frac{1}{8 + 72\pi}$$

(c) Similarly, the power due to the noise is

$$P_{yn} = \frac{\tan^{-1}(2\pi)}{2\pi}$$

(d) The SNR is

$$\text{SNR} = 10\log_{10}\frac{P_{ys}}{P_{yn}} \approx -17.2\,\text{dB}$$

Note that the negative SNR means that the power of the signal is less than the power of the noise.