EE – 211 : ASSIGNMENT

Simple Polynomial Approximation to the Gaussian Q-function and Its Application

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INTRODUCTION –

This paper introduces a new polynomial approximation to the *Gaussian Q-function*, inspired by the idea that a Gaussian random variable can be approximated by a sum of uniform random variables. This innovative approach simplifies the calculation of error rates in fading and lognormal channels, where exact solutions are often difficult or impossible to obtain. Existing methods typically rely on exponential or rational functions, which can lead to large approximation errors, especially for small argument values. The proposed approximation, however, uses power functions, which makes it more suitable for analytical error rate derivations in communication systems.

One significant application of this approximation is in evaluating the average symbol error rates for *M-ary* modulation schemes, such as *M-ary* phase-shift-keying *(MPSK)* and *M-ary* pulse amplitude modulation (MPAM), in lognormal channels. Through this method, accurate explicit approximations are derived that would otherwise be challenging to achieve using previous methods. The new approximation provides results that closely match exact values for symbol error rates, demonstrating its high accuracy. This makes it a valuable tool for performance analysis in communication systems, especially in situations where closed-form solutions are not available or are computationally infeasible. By offering a more reliable and efficient approximation, this approach advances the analysis of error rates in digital communication systems under complex channel conditions.

Approximation –

The Gaussian Q-function is integral in calculating symbol error rates *(SER)* in fading channels, especially when exact solutions are unavailable. The approximation is computationally efficient and suitable for practical applications where closed-form solutions are difficult to derive.

PDF and CDF of Sum of Uniform Random Variables -

The probability density function *(PDF)* of the sum of *n* uniform random variables over the interval

(- ½, 1/2) is derived using the Laplace transform. The *PDF* is expressed as:

$$f_n(t) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} [t - (\frac{n}{2} - m)]^{n-1} \frac{1}{m!\,(n-m)!} U[t - \left(\frac{n}{2} - m\right)]$$

where *U(·)* is the unit step function.

The cumulative distribution function *(CDF)* of the sum of these random variables is given by:

$$F_n(t) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} [t - (\frac{n}{2} - m)]^{n} \frac{1}{m!\,(n-m)!} U[t - \left(\frac{n}{2} - m\right)]$$

Gaussian Approximation Using Central Limit Theorem

By the Central Limit Theorem, a sum of *n* uniform random variables approximates a Gaussian random variable with mean zero and variance *n/12*. The CDF of the Gaussian random variable is:

$$F(t) = 1 - Q\left(\frac{t}{\sqrt{n/\perp 2}}\right)$$

where *(Q(x)* is the *Gaussian Q-function*. This approximation is crucial in simplifying the evaluation of the Q-function in various communication system models.

Polynomial Approximation to the Q-function

The proposed polynomial approximation to the *Q-function* is derived by approximating the Gaussian CDF using the CDF of the sum of uniform random variables. The new approximation for the Q-function is:

$$Q_n(t) = \sum_{m=0}^{n}\sum_{p=0}^{n}(-1)^{m+p}\binom{n}{p}\binom{n}{m}\left(\frac{n}{12}\right)^{\frac{p}{2}}\left(\frac{n}{2}-m\right)^{n-p} t^p \frac{1}{m!\,(n-m)!}U(t-\frac{n}{12}\left(\frac{n}{2}-m\right))$$

Error Bound for the Approximation

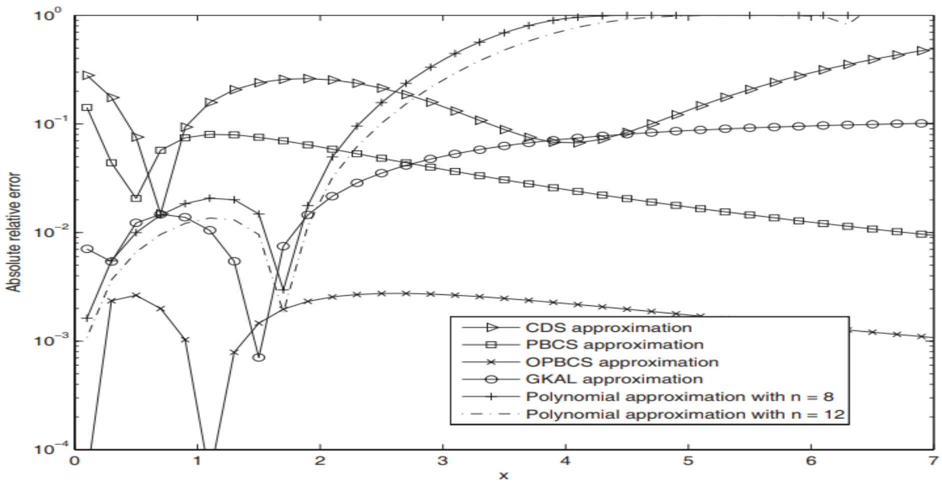
The approximation error is bounded using the *Berry-Esseen Theorem*, which provides a quantifiable bound on the error between the Gaussian Q-function and the polynomial approximation:

$$|Q(t) - Q_n(t)| = 3\sqrt{3}\,C\,(\frac{4}{\sqrt{n}})$$

where *C* is a constant with a value of 0.7655. This bound ensures the accuracy of the approximation for practical use.

Comparison with Other Approximations

The proposed polynomial approximation is compared to existing approximations such as *the GKAL, PBCS, OPBCS,* and *CDS* approximations. The results indicate that the new polynomial approximation offers smaller approximation errors for values of *t* in the range *x < 2.1*, and it outperforms other approximations like *GKAL* for values of *x < 2.9.* However, its accuracy diminishes as *x* increases beyond *2.7*.



. Comparison between the absolute relative errors of the *CDS* approximation, the *PBCS* approximation, the *OPBCS* approximation, the *GKAL* approximation and the polynomial approximations with *n = 8* and *n = 12* of the Gaussian Q function

Polynomial Approximation for Gaussian Q-Function in SER Analysis

This work develops a polynomial approximation for the Gaussian Q-function to simplify the calculation of average symbol error rate (SER) in lognormal channels, particularly for *M-ary* Pulse Amplitude Modulation (*MPAM*) signals. The average SER is expressed as:

$$P_e = \int_0^\infty P(E; y) P(y)\, dy$$

where $P(E; y)$ is the SER in an AWGN channel:

$$P(E; y) = 2(\frac{M-1}{M})\, Q\,(\sqrt{\frac{6y}{M^2-1}})$$

and $P(y)$, the *SNR PDF*, is given by:

$$P(y) = (1/\sqrt{2\pi\sigma^2 y})(\frac{10}{ln10})\exp\,(-\frac{(\log_{10} y - \mu)^2}{2\sigma^2})$$

Using the polynomial approximation for the Q-function, the *SER* simplifies to an explicit expression, avoiding the infinite integral in the original formulation. The approximation error is bounded as:

$$\left|P_e(t) - P_{e,approx}(t)\right| = 3\sqrt{3}\, C\, \frac{M-1}{2M\sqrt{n}}$$

Numerical evaluations show the approximation is accurate for moderate values of σ, with accuracy improving for higher n and larger constellation sizes M. This approach provides practical insights into system design by explicitly linking the *SER* to key parameters.

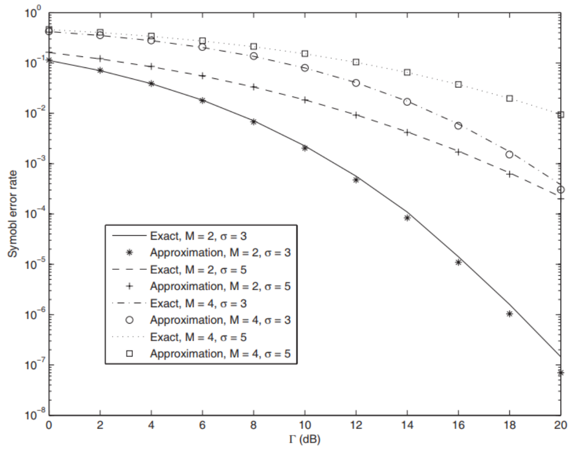


Fig. 2. Comparison of the exact SER and the approximate SER using the polynomial approximation of $Q(x)$ with $n = 8$ in lognormal channels for different values of M and σ.