Viscoplastic Flow Derivation

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1 Flow of a viscoplastic fluid

Following Liu and Mei (1989): pressure at depth z in flow (normal to incline) of height h, density ρ, gravity $\mathbf{g}$, slope θ, and applied pressure $\mathbf{P}(x,t)$

$$\mathbf{p} = \rho\mathbf{g}(h-z)\cos(\theta) + \mathbf{P}(x,t)\;. \tag{1}$$

Assume ambient pressure $\mathbf{P}(x,t)$ is approximately constant.

Conservation of momentum in the x-direction requires

$$0 = \rho\mathbf{g}\sin(\theta) - \frac{\partial \mathbf{p}}{\partial x} + \frac{\partial \tau}{\partial z}\;,\; 0 \le z \le h\;. \tag{2}$$

where τ is a shear stress acting to deform the fluid.

$$\frac{\partial \mathbf{p}}{\partial x} = \mathbf{g}\frac{\partial}{\partial x}(\rho h\cos(\theta))\;. \tag{3}$$

Integrate the stresses down from the surface to z

$$\int_h^z \frac{\partial \tau}{\partial z'}dz' = \int_h^z \left(\frac{\partial \mathbf{p}}{\partial x} - \rho\mathbf{g}\sin(\theta)\right)dz'\;,\; \tau = 0, z = 0, \tag{4a}$$

$$\tau(z) = -(h-z)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)\;. \tag{4b}$$

Which can flow when the stress at the bed (z=0) is greater than the yield stress τ_y. Solve for the neutral horizon h_0 where $\tau(h_0) = \tau_y$.

$$\tau_y = \left|-(h-h_0)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)\right|\;, \tag{5a}$$

$$h_0 = h + \frac{\tau_y}{\rho\mathbf{g}|\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)|}\;. \tag{5b}$$

Which reduces in the constant-density, zero-slope case to the faux yield surface of Balmforth et al. (2007):

$$Y = h + \frac{\tau_y}{\rho\mathbf{g}|\frac{\partial h}{\partial x}|}\;. \tag{6}$$

Introduce stress-strain relationship for a Herschel-Bulkley fluid $\tau = \tau_y + K(\frac{\partial u}{\partial z})^n$ of consistency K, and power-law exponent n,

$$\tau_y + K(\frac{\partial u}{\partial z})^n = -(h-z)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right) \tag{7}$$

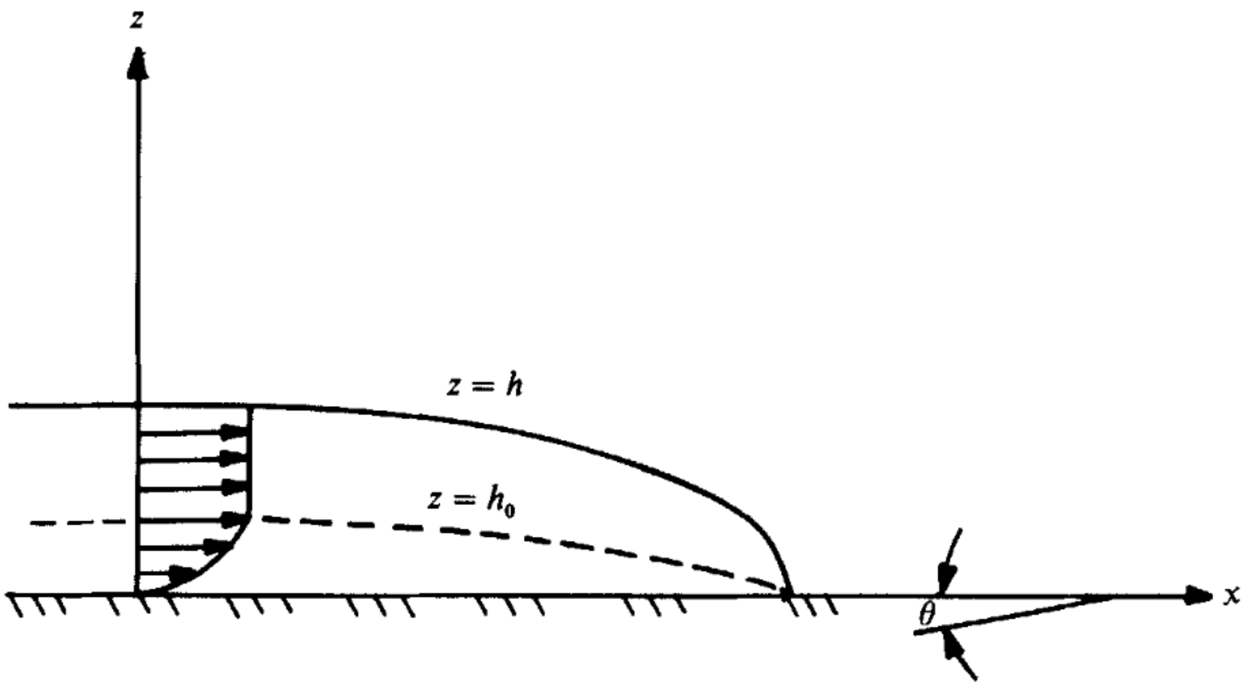

Figure 1: Schematic flow of a fluid with a finite yield stress down an inclined flow. From Liu and Mei (1989).

Substitute in yield stress:

$$K(\frac{\partial u}{\partial z})^n = -(h_0 - z)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right) \tag{8}$$

Solve for $\frac{\partial u}{\partial z}$ and integrate from 0 to z with the condition that u(0) = 0, and $\frac{\partial u}{\partial z}(h_0) = 0$:

$$u = -\frac{n}{n+1}\left((h_0 - z)^{1/n+1} - h_0^{1/n+1}\right)\left(\frac{\mathbf{g}\rho}{K}\right)^{1/n}\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)^{1/n}, \; for 0 \leq u \leq h_0\,. \tag{9}$$

Above the neutral horizon, flow has a constant velocity

$$u_p = \frac{n}{n+1}h_0^{1/n+1}\left(\frac{\mathbf{g}\rho}{K}\right)^{1/n}\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)^{1/n}, \; for h_0 \leq u \leq h\,. \tag{10}$$

Integrate over the depth of the fluid

$$Q = \int_0^{h_0} u dz + (h - h_0)u_p \tag{11a}$$

$$Q = -\frac{n(n+1)}{(n+1)(2n+1)}\left((h_0 - z)^{1/n+2} - h_0^{1/n+2}\right)\left(\frac{\mathbf{g}\rho}{K}\right)^{1/n}\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)^{1/n} \tag{11b}$$

$$+ (h - h_0)\frac{n}{n+1}h_0^{1/n+1}\left(\frac{\mathbf{g}\rho}{K}\right)^{1/n}\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)^{1/n} \tag{11c}$$

$$\tag{11d}$$

Avoid fractional powers of negative numbers by replacing

$$\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)^{1/n} = \left(\left|\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right|\right)^{1/n-1}\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right) \tag{12}$$

Rearrange and simplify to find:

$$Q = \left(\frac{\rho g}{K}\right)^{1/n} \frac{n\left|\sin\theta - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho h\cos\theta\right)\right|^{1/n-1} Y^{1/n+1}}{(n+1)(2n+1)} \left((2n+1)h - nY\right)\left(\sin\theta - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho h\cos\theta\right)\right), \tag{13a}$$

$$Y = h - \frac{\tau_y}{\rho g\left|\sin\theta - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho h\cos\theta\right)\right|}. \tag{13b}$$

Conservation of mass requires

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho Q) = 0 \tag{14}$$

which is a statement of advection for ρh with speed $q = Q/h$.

For a multi-phase fluid, density is volume averaged over N phases,

$$\bar{\rho} = \sum_{i=1}^{N} \phi_i \rho_i \,. \tag{15}$$

2 Finite-Differences

Approximate the differential equation:

$$\frac{\partial h}{\partial t} \approx \frac{Q_r - Q_l}{\Delta x} \tag{16}$$

where Q_r and Q_l are the fluxes into and out of a grid node (with an evenly-spaced grid). Where the fluxes are approximated on the right and left with:

$$h_r^i = \frac{h(x^i) + h(x^i + \Delta x)}{2} \approx h^i + \frac{\Delta x}{2}h_x^i + \frac{\Delta x^2}{4}h_{xx}^i + O(\Delta x^3) \tag{17a}$$

$$h_{x_r}^{\,i} = \frac{h(x^i) - h(x^i + \Delta x)}{\Delta x} \approx h_x^i + \frac{\Delta x}{2}h_{xx}^i + \frac{\Delta x^2}{6}h_{xxx}^i + O(\Delta x^3) \tag{17b}$$

$$h_l^i = \frac{h(x^i) + h(x^i - \Delta x)}{2} \approx h^i - \frac{\Delta x}{2}h_x^i + \frac{\Delta x^2}{4}h_{xx}^i + O(\Delta x^3) \tag{17c}$$

$$h_{x_l}^{\,i} = \frac{h(x^i - \Delta x) - h(x^i)}{\Delta x} \approx h_x^i - \frac{\Delta x}{2}h_{xx}^i + \frac{\Delta x^2}{6}h_{xxx}^i + O(\Delta x^3) \tag{17d}$$

$$\begin{aligned} Q_r = \left(\frac{\rho g}{K}\right)^{1/n} & \frac{n(|\sin\theta - \cos\theta h_{xr}|)^{1/n-1}\left(h_r - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_{xr}|}\right)^{1/n+1}}{(n+1)(2n+1)} \\ & \cdot\left[(2n+1)h_r - n\left(h_r - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_{xr}|}\right)\right](\sin\theta - \cos\theta h_{xr}) \end{aligned} \tag{18}$$

$$\begin{aligned} Q_l = \left(\frac{\rho g}{K}\right)^{1/n} & \frac{n(|\sin\theta - \cos\theta h_{xl}|)^{1/n-1}\left(h_l - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_{xl}|}\right)^{1/n+1}}{(n+1)(2n+1)} \\ & \cdot\left[(2n+1)h_l - n\left(h_l - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_{xl}|}\right)\right](\sin\theta - \cos\theta h_{xl}) \end{aligned} \tag{19}$$

resulting in a centered 3-point stencil for the approximation. The approximation of the value of the function h on the right and left is required to accommodate the step function starting condition.

Substitute the expanded function and derivative approximations into the fluxes:

$$
\begin{aligned}
Q_r^i = &\left(\frac{\rho g}{K}\right)^{1/n} \frac{n}{(n+1)(2n+1)} (|\sin\theta - \cos\theta h_x^i + \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3)|)^{1/n-1} \\
&\cdot \left(h^i + \frac{\Delta x}{2} h_x^i + \frac{\Delta x^2}{4} h_{xx}^i + O(\Delta x^3) - \frac{\tau_y}{\rho g |\sin\theta - \cos\theta (h_x^i + \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3))|}\right)^{1/n+1} \\
&\cdot \Bigg[(2n+1)(h^i + \frac{\Delta x}{2} h_x^i + \frac{\Delta x^2}{4} h_{xx}^i + O(\Delta x^3)) \\
&- n\left(h^i + \frac{\Delta x}{2} h_x^i + \frac{\Delta x^2}{4} h_{xx}^i + O(\Delta x^3) - \frac{\tau_y}{\rho g |\sin\theta - \cos\theta (h_x^i + \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3))|}\right)\Bigg] \\
&\cdot (\sin\theta - \cos\theta (h_x^i + \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3)))
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
Q_l^i = &\left(\frac{\rho g}{K}\right)^{1/n} \frac{n}{(n+1)(2n+1)} (|\sin\theta - \cos\theta h_x^i - \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3)|)^{1/n-1} \\
&\cdot \left(h^i - \frac{\Delta x}{2} h_x^i + \frac{\Delta x^2}{4} h_{xx}^i + O(\Delta x^3) - \frac{\tau_y}{\rho g |\sin\theta - \cos\theta (h_x^i - \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3))|}\right)^{1/n+1} \\
&\cdot \Bigg[(2n+1)(h^i - \frac{\Delta x}{2} h_x^i + \frac{\Delta x^2}{4} h_{xx}^i + O(\Delta x^3)) \\
&- n\left(h^i - \frac{\Delta x}{2} h_x^i + \frac{\Delta x^2}{4} h_{xx}^i + O(\Delta x^3) - \frac{\tau_y}{\rho g |\sin\theta - \cos\theta (h_x^i - \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3))|}\right)\Bigg] \\
&\cdot (\sin\theta - \cos\theta (h_x^i - \frac{\Delta x}{2} h_{xx}^i + \frac{\Delta x^2}{6} h_{xxx}^i + O(\Delta x^3)))
\end{aligned}
\tag{21}
$$

Expand non-linear terms using Taylor-Series expansions about Δx and re-arrange:

$$
\begin{aligned}
Q_r^i = \left(\frac{\rho g}{K}\right)^{1/n} \frac{n}{(n+1)(2n+1)} & \Bigg[(|\sin\theta - \cos\theta h_x^i|)^{1/n-1} \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \\
& \cdot \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right](\sin\theta - \cos\theta h_x^i) \\
& + \Delta x \Bigg[(|\sin\theta - \cos\theta h_x^i|)^{1/n-2}(-cos\theta\frac{h_{xx}^i}{2}) \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \\
& \quad \cdot \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right](\sin\theta - \cos\theta h_x^i) \\
& \quad + (|\sin\theta - \cos\theta h_x^i|)^{1/n-1}(\frac{1}{n}+1) \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n} (\frac{h_x^i}{2} + \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|^2}(-\cos\theta\frac{h_{xx}^i}{2})) \\
& \quad \cdot \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right](\sin\theta - \cos\theta h_x^i) \\
& \quad + (|\sin\theta - \cos\theta h_x^i|)^{1/n-1} \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \\
& \quad \cdot \left[(2n+1)\frac{h_x^i}{2} - n\left(\frac{h_x^i}{2} + \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|^2}(-\cos\theta\frac{h_{xx}^i}{2})\right)\right](\sin\theta - \cos\theta h_x^i) \\
& \quad + (|\sin\theta - \cos\theta h_x^i|)^{1/n-1} \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right] \\
& \quad \cdot (-\cos\theta\frac{h_{xx}^i}{2}) \Bigg] \\
& + O(\Delta x^2)
\end{aligned}
\tag{22}
$$

By the symmetry of the approximation, we know the even terms will cancel, leaving:

$$
\begin{aligned}
\frac{Q_r^i - Q_l^i}{\Delta x} = & (|\sin\theta - \cos\theta h_x^i|)^{1/n-2}(-cos\theta h_{xx}^i) \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \\
& \cdot \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right](\sin\theta - \cos\theta h_x^i) \\
& + (|\sin\theta - \cos\theta h_x^i|)^{1/n-1}(\frac{1}{n}+1) \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n} (h_x^i + \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|^2}(-\cos\theta h_{xx}^i)) \\
& \cdot \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right](\sin\theta - \cos\theta h_x^i) \\
& + (|\sin\theta - \cos\theta h_x^i|)^{1/n-1} \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \\
& \cdot \left[(2n+1)\frac{h_x^i}{2} - n\left(\frac{h_x^i}{2} + \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|^2}(-\cos\theta h_{xx}^i)\right)\right](\sin\theta - \cos\theta h_x^i) \\
& + (|\sin\theta - \cos\theta h_x^i|)^{1/n-1} \left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)^{1/n+1} \left[(2n+1)h^i - n\left(h^i - \frac{\tau_y}{\rho g|\sin\theta - \cos\theta h_x^i|}\right)\right] \\
& \cdot (-\cos\theta h_{xx}^i) \\
& + O(\Delta x^2)
\end{aligned}
\tag{23}
$$

Which recovers the expanded form of the exact expression to second-order accuracy in space. Solved with 2-step Runge-Kutte gives 2nd order in time.

References

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