Viscoplastic Flow Derivation

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1 Flow of a viscoplastic fluid

Following **?**: pressure at depth z in flow (normal to incline) of height h, density ρ, gravity $\mathbf{g}$, slope θ, and applied pressure $\mathbf{P}(x,t)$

$$\mathbf{p} = \rho\mathbf{g}(h-z)\cos(\theta) + \mathbf{P}(x,t)\,. \tag{1}$$

Assume ambient pressure $\mathbf{P}(x,t)$ is approximately constant.

Conservation of momentum in the x-direction requires

$$0 = \rho\mathbf{g}\sin(\theta) - \frac{\partial \mathbf{p}}{\partial x} + \frac{\partial \tau}{\partial z}\,,\ 0 \le z \le h\,. \tag{2}$$

where τ is a shear stress acting to deform the fluid.

$$\frac{\partial \mathbf{p}}{\partial x} = \mathbf{g}\frac{\partial}{\partial x}(\rho h\cos(\theta))\,. \tag{3}$$

Integrate the stresses down from the surface to z

$$\int_h^z \frac{\partial \tau}{\partial z'}dz' = \int_h^z \left(\frac{\partial \mathbf{p}}{\partial x} - \rho\mathbf{g}\sin(\theta)\right)dz'\,,\ \tau = 0, z = 0, \tag{4a}$$

$$\tau(z) = -(h-z)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)\,. \tag{4b}$$

Which can flow when the stress at the bed (z=0) is greater than the yield stress τ_y. Solve for the neutral horizon h_0 where $\tau(h_0) = \tau_y$.

$$\tau_y = \left|-(h-h_0)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right)\right|\,, \tag{5a}$$

$$h_0 = h + \frac{\tau_y}{\rho\mathbf{g}|\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)|}\,. \tag{5b}$$

Which reduces in the constant-density, zero-slope case to the faux yield surface of **?**:

$$Y = h + \frac{\tau_y}{\rho\mathbf{g}|\frac{\partial h}{\partial x}|}\,. \tag{6}$$

Introduce stress-strain relationship for a Herschel-Bulkley fluid $\tau = \tau_y + K(\frac{\partial u}{\partial z})^n$ of consistency K, and power-law exponent n,

$$\tau_y + K(\frac{\partial u}{\partial z})^n = -(h-z)\mathbf{g}\rho\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h\cos(\theta)) - \sin(\theta)\right) \tag{7}$$

Substitute in yield stress:

$$K(\frac{\partial u}{\partial z})^n = -(h_0 - z)\mathbf{g}\rho \left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right) \tag{8}$$

Solve for $\frac{\partial u}{\partial z}$ and integrate from 0 to z with the condition that u(0) = 0, and $\frac{\partial u}{\partial z}(h_0) = 0$:

$$u = -\frac{n}{n+1}\left((h_0 - z)^{1/n+1} - h_0^{1/n+1} \right) \left(\frac{\mathbf{g}\rho}{K} \right)^{1/n} \left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right)^{1/n} \text{ , } for 0 \le u \le h_0 \text{ .} \tag{9}$$

Above the neutral horizon, flow has a constant velocity

$$u_p = \frac{n}{n+1} h_0^{1/n+1} \left(\frac{\mathbf{g}\rho}{K} \right)^{1/n} \left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right)^{1/n} \text{ , } for h_0 \le u \le h \text{ .} \tag{10}$$

Integrate over the depth of the fluid

$$Q = \int_0^{h_0} u dz + (h - h_0)u_p \tag{11a}$$

$$Q = -\frac{n(n+1)}{(n+1)(2n+1)}\left((h_0 - z)^{1/n+2} - h_0^{1/n+2} \right) \left(\frac{\mathbf{g}\rho}{K} \right)^{1/n} \left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right)^{1/n} \tag{11b}$$

$$+ (h - h_0)\frac{n}{n+1} h_0^{1/n+1} \left(\frac{\mathbf{g}\rho}{K} \right)^{1/n} \left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right)^{1/n} \tag{11c}$$

$$\tag{11d}$$

Avoid fractional powers of negative numbers by replacing

$$\left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right)^{1/n} = \left(\left| \frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right| \right)^{1/n-1} \left(\frac{1}{\rho}\frac{\partial}{\partial x}(\rho h \cos(\theta)) - \sin(\theta) \right) \tag{12}$$

Rearrange and simplify to find:

$$Q = \left(\frac{\rho g}{K} \right)^{1/n} \frac{n \left| \sin\theta - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho h \cos\theta\right) \right|^{1/n-1} Y^{1/n+1}}{(n+1)(2n+1)} \left((2n+1)h - nY \right) \left(\sin\theta - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho h \cos\theta\right) \right) \text{ ,} \tag{13a}$$

$$Y = h - \frac{\tau_y}{\rho g \left| \sin\theta - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho h \cos\theta\right) \right|} \text{ .} \tag{13b}$$

Conservation of mass requires

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho Q) = 0 \tag{14}$$

which is a statement of advection for ρh with speed $q = Q/h$.

For a multi-phase fluid, density is volume averaged over N phases,

$$\bar{\rho} = \sum_{i=1}^{N} \phi_i \rho_i \text{ .} \tag{15}$$

2 Conservation of energy

Assume constant specific heat capacity c_p,

$$\bar{c}_p = \left(\sum_{i=1}^{N} \phi_i \rho_i c_{p_i} \right) / \bar{\rho} \approx c_{p_{liq}} \text{ .} \tag{16}$$

Pure advection of heat follows

$$\frac{\partial}{\partial t}(\rho h T) + \frac{\partial}{\partial x}(q \rho h T) = 0 \,, \tag{17}$$

expand to find

$$\rho h \frac{\partial T}{\partial t} + T \frac{\partial}{\partial t}(\rho h) + q \rho h \frac{\partial T}{\partial t} + T \frac{\partial}{\partial x}(q \rho h) = 0 \,, \tag{18a}$$

$$\rho h (\frac{\partial T}{\partial t} + q \frac{\partial T}{\partial x}) + T (\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(q \rho h)) = 0 \,. \tag{18b}$$

Using equation 14, the above expression simplifies to

$$\frac{\partial T}{\partial t} + q \frac{\partial T}{\partial x} = 0 \,. \tag{19}$$

Heat diffusion and loss to the atmosphere through conduction/ radiation across a flat, dipping interface approximated using a linear interpolation, and latent heat of crystallization, vaporization:

$$\frac{\partial T}{\partial t} = -q \frac{\partial T}{\partial x} + \frac{1}{c_p} \frac{\partial}{\partial x} \left(\frac{k_T}{\rho} \frac{\partial T}{\partial x} \right) - \frac{\Delta x \sqrt{\left(\frac{\partial h}{\partial x} \right)^2 + 1}}{\rho c_p h} \left[e \sigma \left(T^4 - T_{atm}^4 \right) + h_{atm} \left(T - T_{atm} \right) \right] + \sum_{i=1}^{N} \frac{\rho_i L_i}{\rho c_p} \Gamma_i \,, \tag{20}$$

where k_T is the thermal diffusivity, e is the emissivity with $0 \leq e \leq 1$, sigma the Stefan-Boltzmann constant $= 5.67{\times}10^{-8}$ $Wm^{-2}K^{-4}$, T_{atm} is the ambient temperature in Kelvin, h_{atm} is the convective heat transfer coefficient of air in $Wm^{-2}K^{-1}$, L_i is the latent heat of crystallization or vaporization for phase i, and Γ_i is a volumetric reaction rate.

3 Bubble evolution