Say you have a set of random variables that vary in time $\{X_i(t)\}$. You take measurements of these variables at many points in time. The measurement results $\{x_i(t)\}$ have an additional random error $\{\delta_i(t)\}$ that are independent from one another and the random variables. Denoting $\langle\rangle$ as the time average, we know

$$\langle X_i \rangle = \langle \delta_i \rangle = \langle X_i \delta_j \rangle = 0 \tag{1}$$

$$\langle \delta_i \delta_j \rangle = \delta_{ij} \Delta_i^2 \tag{2}$$

$$\langle X_i X_j \rangle = C_{ij} \tag{3}$$

My goal is to find a good estimator for $\{X_i(t)\}$ knowing the measurement results $\{x_i(t)\}$, error variances Δ_i^2 and the covariance matrix C_{ij}. [1]

The measurements themselves are clearly not optimal as demonstrated by the maximally correlated case (e.g. $C_{ij} = 1$).

I propose to have the estimates $\{q_i(t)\}$ be a linear combination of all the measurements with coefficients determined by Δ_i & C_{ij}

$$q_i(t) = \sum_j F_{ij} x_j \tag{4}$$

Surely this isn't a new idea. Please tell me the keywords if you recognise this. Anyway, let's find the optimal matrix F that minimises the average squared errors for all variables

$$L_i = \langle (q_i - X_i)^2 \rangle \tag{5}$$

The relevant covariances in terms of the matrices F & C are

$$\langle x_i X_j \rangle = C_{ij} \Rightarrow \tag{6}$$

$$\langle q_i X_j \rangle = \sum_k \langle F_{ik} x_k X_j \rangle = \sum_k F_{ik} C_{kj} = (FC)_{ij} \tag{7}$$

and

$$\langle x_i x_j \rangle = C_{ij} + \delta_{ij}\Delta_i^2 \equiv \tilde{C}_{ij} \Rightarrow \tag{8}$$

$$\langle q_i q_j \rangle = \sum_{k,l} \langle F_{ik} x_k F_{jl} x_j \rangle = \sum_{k,l} F_{ik} F_{jl} \tilde{C}_{kl} = (F\tilde{C}F^T)_{ij} \tag{9}$$

Thus

$$L_i = \langle q_i q_i \rangle - 2\langle q_i X_i \rangle + \langle X_i X_i \rangle \tag{10}$$

$$= (F\tilde{C}F^T - 2FC + C)_{ii} \tag{11}$$

At the stationary point the derivatives with respect to all the coefficients must be zero.

$$\frac{\partial L_i}{\partial F_{ij}} = \frac{\partial}{\partial F_{ij}} \Big[\sum_{k,l} F_{ik} F_{il} \tilde{C}_{kl} - 2 \sum_k F_{ik} C_{ki} + C_{ii} \Big] \tag{12}$$

$$= \sum_{k,l} (\delta_{jk} F_{il} \tilde{C}_{kl} + F_{ik} \delta_{jl} \tilde{C}_{kl}) - 2 \sum_k \delta_{jk} C_{ki} \tag{13}$$

$$= 2 \sum_k F_{ik} \tilde{C}_{jk} - 2C_{ij} \tag{14}$$

[1] Realistically C_{ij} will have to be estimated from the measurements which may complictate things.

where I used $C_{ij} = C_{ji}$ and $\frac{\partial F_{ij}}{\partial F_{ik}} = \delta_{jk}$. Finally, I find that the matrix that minimises $L_i \, \forall i$ must satisfy

$$F\tilde{C} = C \tag{15}$$

And if $\tilde{C}$ is invertible[2]

$$F = C\tilde{C}^{-1} \tag{16}$$

Done!

You can check that in the case of perfect measurements $\tilde{C} = C$ the equation is simply solved by the identity matrix $F_{ij} = \delta_{ij}$ giving $q_i(t) = x_i(t)$ as expected.

I'm too lazy to go and apply this to more special cases.

[2] I don't know much about when it is not.

... actually I do know something about the invertibility of $\tilde{C}$. It is a nice algebra problem:

Show that no two rows or columns of $\tilde{C}$ can be equal if $\Delta_i > 0$.

Proof:

$$\tilde{C}_{ij} = C_{ij} + \delta_{ij}\Delta_i^2$$

Assume rows i and j are equal. Then $\tilde{C}_{ii} = C_{ij}$ and $\tilde{C}_{jj} = C_{ji}$. Because $C_{ij} = C_{ji}$

$$C_{ii} + \Delta_i^2 = C_{jj} + \Delta_j^2 = C_{ij}$$

We would have arrived at the same conclusion if we assumed that two columns are equal.

Now denote $C_{ii} = \sigma_i^2$, $\sigma_i > 0$. Because C is a covariance matrix, $|C_{ij}|$ cannot exceed $\sigma_i\sigma_j$. Thus

$$\sigma_i^2 + \Delta_i^2 = \sigma_j^2 + \Delta_j^2 \leq \sigma_i\sigma_j$$

If $\sigma_i \neq \sigma_j$ one of the inequalities is not satisfied. But when $\sigma_i = \sigma_j$ the inequalities require $\Delta_i = \Delta_j = 0$ which contradicts our assumption.

$\square$