3. Statistics of Volatility, Correlation Dependence, and Aftereffect in Prices

§ 3a. Volatility. Definition and Examples

1. Arguably, no concept in financial mathematics if as loosely interpreted and as widely discussed as '*volatility*'. A synonym to '*changeability*',[b] '*volatility*' has many definitions, and is used to denote various measures of changeability.

If $S_n = S_0 e^{H_n}$ with $H_0 = 0$ and $\Delta H_n = \sigma \varepsilon_n$, $n \geqslant 1$, where (ε_n) is Gaussian white noise ($\varepsilon_n \sim \mathcal{N}(0,1)$), then one means by volatility the natural measure of uncertainty and changeability, the *standard deviation* σ.

We recall that if $\xi \sim \mathcal{N}(\mu, \sigma^2)$ for a random variable ξ, then

$$\mathsf{P}\big(|\xi - \mu| \leqslant \sigma\big) \approx 0.68 \tag{1}$$

and

$$\mathsf{P}\big(|\xi - \mu| \leqslant 1.65\,\sigma\big) \approx 0.90. \tag{2}$$

Hence one can expect in approximately 90 % cases that the result of an observation of ξ deviates from the mean value μ by $1.65\,\sigma$ at most.

In employing the scheme '$S_n = S_{n-1} e^{h_n}$' one usually deals with small values of the h_n, so that

$$S_n \approx S_{n-1}(1 + h_n).$$

Hence, if $h_n = \sigma \varepsilon_n$, then, given the 'today' value of some price S_{n-1} we can say that its 'tomorrow' value S_n will in 90 % cases lie in the interval

$$[S_{n-1}(1 - 1.65\sigma), S_{n-1}(1 + 1.65\sigma)],$$

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[b] "Random House Webster's concise dictionary" (Random House, New York, 1993) gives the following explanation of the adjective *volatile*: "1. evaporating rapidly. 2. tending or threatening to erupt in violence; explosive. 3. *changeable*; unstable."

so that it is only in 5 % cases that S_n is larger than $S_{n-1}(1+1.65\sigma)$ and it is smaller than $S_{n-1}(1-1.65\sigma)$ in 5 % cases.

Remark 1. This explains why, in some handbooks of finance (see, for instance, [404]), one measures volatility in terms of $\nu = 1.65\,\sigma$ in place of the standard deviation σ.

2. We have already seen before that the model '$h_n = \sigma\varepsilon_n$, $n \geqslant 1$' is fairly distant from real life. It would be more realistic to use conditional Gaussian models of the kind '$h_n = \sigma_n\varepsilon_n$, $n \geqslant 1$', with a *random* sequence $\sigma = (\sigma_n)_{n\geqslant 1}$ of $\mathscr{F}_{n-1}$-measurable variables σ_n and with $\mathscr{F}_n$-measurable ε_n, where $(\mathscr{F}_n)$ is the flow of 'information' (on the price values, say; see Chapter I, § 2a for greater detail).

There is an established tradition of calling $\sigma = (\sigma_n)_{n\geqslant 1}$ (in the above model) the sequence of 'volatilities'. The random nature of the latter can be reflected by saying that 'the volatility is volatile on its own'.

Note that

$$\mathsf{E}(h_n^2 \mid \mathscr{F}_{n-1}) = \sigma_n^2, \tag{3}$$

and the sequence $H = (H_n, \mathscr{F}_n)_{n\geqslant 1}$ of the variables $H_n = h_1 + \cdots + h_n$, where $\mathsf{E}|h_n|^2 < \infty$ for $n \geqslant 1$, is a square integrable martingale with quadratic characteristic

$$\langle H\rangle_n = \sum_{k=1}^{n} \mathsf{E}(h_k^2 \mid \mathscr{F}_{k-1}), \qquad n \geqslant 1. \tag{4}$$

In view of (3),

$$\langle H\rangle_n = \sum_{k=1}^{n} \sigma_k^2; \tag{5}$$

therefore it is natural to call the quadratic characteristic

$$\langle H\rangle = (\langle H\rangle_n, \mathscr{F}_n)_{n\geqslant 1}$$

the *volatility* of the sequence H.

We note that

$$\mathsf{E}H_n^2 = \mathsf{E}\langle H\rangle_n. \tag{6}$$

3. For $ARCH(p)$ models we have

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i h_{n-i}^2 \tag{7}$$

(see Chapter II, § 3a).

Hence the *estimation* problem for the volatilities σ_n reduces to a problem of *parametric* estimation for $\alpha_0, \alpha_1, \ldots, \alpha_p$.

There exist also other, e.g., *nonparametric*, methods for obtaining estimates of volatility. For instance, if $h_n = \mu_n + \sigma_n \varepsilon_n$ for $n \geqslant 1$, where $\mu = (\mu_n)$ and $\sigma = (\sigma_n)$ are stationary sequences, then a standard estimator for σ_n is

$$\widehat{\sigma}_n = \sqrt{\frac{1}{n-1} \sum_{k=1}^{n} (h_k - \overline{h}_n)^2}\,, \tag{8}$$

where $\overline{h}_n = \dfrac{1}{n} \sum\limits_{k=1}^{n} h_k$.

It is worth noting that one can also regard the empiric volatility $\widehat{\sigma} = (\widehat{\sigma}_n)_{n \geqslant 1}$ as an index of financial statistics and to analyze it using the same methods and tools as in the case of the prices $S = (S_n)_{n \geqslant 1}$.

To this end we consider the variables

$$\widehat{r}_n = \ln \frac{\widehat{\sigma}_n}{\widehat{\sigma}_{n-1}}\,, \qquad n \geqslant 2. \tag{9}$$

Many authors and numerous observations (see, for example, [386; Chapter 10]) demonstrate that the values of the 'logarithmic returns' $\widehat{r} = (\widehat{r}_n)_{n \geqslant 2}$ oscillate rather swiftly, which indicates that the variables $\widehat{r}_n$ and $\widehat{r}_{n+1}$, $n \geqslant 2$, are negatively correlated. If we consider the example of the S&P500 Index and apply $\mathcal{R}/\mathcal{S}$-analysis to the corresponding values of $\widehat{r} = (\widehat{r}_n)_{n \geqslant 2}$ (see Chapter III, § 2a and § 4 of the present chapter), then the results will fully confirm this phenomenon of negative correlation (see [386; Chapter 10]). Moreover, we can assume in the first approximation that the $\widehat{r}_n$ are Gaussian variables, so that this negative correlation (coupled with the property of self-similarity standing out in observations) can be treated as an argument in favor of the thesis that this sequence is a *fractional noise* with Hurst parameter $\mathbb{H} < 1/2$. (According to [386], $\mathbb{H} \approx 0.31$ for the S&P500 Index.)

4. The well-known paper [44] (1973) of Black and Scholes made a significantly contribution to the understanding of the importance of the concept of volatility. This paper contains a formula for the fair (rational) price $\mathbb{C}_T$ of a standard call option (see Chapter I, § 1b). By this formula, the value of $\mathbb{C}_T$ is *independent* of μ (surprisingly, at the first glance), but *depends* on the value of the volatility σ participating in the formula describing the evolution of stock prices $S = (S_t)_{t \geqslant 0}$:

$$S_t = S_0 e^{H_t}, \qquad H_t = \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t, \tag{10}$$

where $W = (W_t)_{t \geqslant 0}$ is a standard Wiener process.

Of course, the assumption made in this model that the volatility σ in (10) is, first, a *constant* and, second, a *known* constant, is rather fantastic. Clearly, practical applications of the Black–Scholes formula require one to have at least a rough idea of the possible value of the volatility; this is necessary not only to find out the fair option prices, but also to assess the risks resulting from some or other decisions in models with prices described by formulas (9) and (10) in Chapter I, § 1b.

In this connection, we must discuss another (empirical) approach to the concept of 'volatility' that uses the Black–Scholes formula and the *real* option prices on securities markets.

For this definition, let $\mathbb{C}_t = \mathbb{C}_t(\sigma; T)$ be the (theoretical) value of the ask price at time $t < T$ of a standard European call-option with $f_T = (S_T - K)^+$ and with maturity time T.

The price $\mathbb{C}_t$ is theoretical. What we know in practice is the price $\widehat{\mathbb{C}}_t$ that was *actually* announced at the instant t, and we can ask for the root of the equation

$$\widehat{\mathbb{C}}_t = \mathbb{C}_t(\sigma; T). \tag{11}$$

This value of σ, denoted by $\widetilde{\sigma}_t$, is called the '*implied* volatility'; it is considered to be a good estimate for the 'actual' volatility.

It should be noted that, as regards its behavior, the 'implied volatility' is similar to the 'empirical volatility' defined (in the continuous-time case) by formulae of type (8). Its negative correlation and fractal structure are rather clearly visible (see, i.g., [386; Chapter 10]).

5. We now discuss another approach to the definition of volatility, based on the consideration of the *variation-related* characteristics of the process $H = (H_t)_{t\geqslant 0}$ defining the prices $S = (S_t)_{t\geqslant 0}$ by the formula $S_t = S_0 e^{H_t}$. The results of many statistical observations and economic arguments support the thesis that the processes $H = (H_t)_{t\geqslant 0}$ have a property of *self-similarity*, which means, in particular, that the distributions of the variables $H_{t+\Delta} - H_t$ with *distinct* values of $\Delta > 0$ are similar in certain respects (see Chapter III, § 2).

We recall that if $H = B_{\mathbb{H}}$ is a fractional Brownian motion, then

$$\mathsf{E}|H_{t+\Delta} - H_t| = \sqrt{\frac{2}{\pi}}\,\Delta^{\mathbb{H}} \tag{12}$$

for all $\Delta > 0$ and $t \geqslant 0$ and

$$\mathsf{E}|H_{t+\Delta} - H_t|^2 = \Delta^{2\mathbb{H}}. \tag{13}$$

For a strictly α-stable Lévy motion with $0 < \alpha \leqslant 2$ we have

$$\mathsf{E}|H_{t+\Delta} - H_t| = \mathsf{E}|H_\Delta| = \Delta^{1/\alpha}\mathsf{E}|H_1|. \tag{14}$$

Hence, setting $\mathbb{H} = 1/\alpha < 1$ we obtain

$$\mathsf{E}|H_{t+\Delta} - H_t| = \Delta^{\mathbb{H}}\mathsf{E}|H_1|, \tag{15}$$

which resembles formula (12) for a fractional Brownian motion.

All these formulas, together with arguments based on the Law of large numbers, suggest that it would be reasonable to introduce certain *variation-related* characteristics and to use them for testing the statistical hypothesis that the process $H = (H_t)_{t \geqslant 0}$ generating the prices $S = (S_t)_{t \geqslant 0}$ is a self-similar process of the same kind as a fractional Brownian motion or an α-stable Lévy motion.

It should be also pointed out that, from the standpoint of statistical analysis, different kinds of investors are interested in different *time intervals* and have different *time horizons*.

For instance, the *short-term* investors are eager to know the values of the prices $S = (S_t)_{t \geqslant 0}$ at times $t_k = k\Delta$, $k \geqslant 0$, with *small* $\Delta > 0$ (several minutes or even seconds). Such data are of little interest to *long-term* investors; what they value most are data on the price movements over *large* time intervals (months and even years), information on cycles (periodic or aperiodic) and their duration, information on trend phenomena, and so on.

Bearing this in mind, we shall explicitly indicate in what follows the chosen time interval Δ (taken as a unit of time, the 'characteristic' time measure of an investor) and also the interval $(a, b]$ on which we study the evolution and the 'changeability' of the financial index in question.

6. On can get a satisfactory understanding of the changeability of a process $H = (H_t)_{t \geqslant 0}$ on the time interval $(a, b]$ from the Δ-variation

$$\operatorname{Var}_{(a,b]}(H; \Delta) = \sum |H_{t_k} - H_{t_{k-1}}|, \tag{16}$$

where the sum is taken over all k such that $a \leqslant t_{k-1} < t_k \leqslant b$, and $t_k = k\Delta$.

Clearly, if a particular trajectory of $H = (H_t)_{a \leqslant t \leqslant b}$ is 'sufficiently regular' and $\Delta > 0$ is small, then the value of $\operatorname{Var}_{(a,b]}(H; \Delta)$ is close to the variation

$$\operatorname{Var}_{(a,b]}(H) \equiv \int_a^b |dH_s|, \tag{17}$$

which is by definition the supremum

$$\sup \sum |H_{t_k} - H_{t_{k-1}}| \tag{18}$$

taken over all finite partitionings $(t_0, \dots, t_n)$ of the interval $(a, b]$ such that $a = t_0 < t_1 < \cdots < t_n \leqslant b$.

In the statistical analysis of the processes $H = (H_t)_{t \geqslant 0}$ with presumably homogeneous increments, it is reasonable to consider, in place of the Δ-variations $\operatorname{Var}_{(a,b]}(H; \Delta)$, the normalized quantities

$$\nu_{(a,b]}(H; \Delta) = \frac{\operatorname{Var}_{(a,b]}(H; \Delta)}{\left[\frac{b-a}{\Delta}\right]}, \tag{19}$$

which we shall call the (empirical) Δ-*volatilities* on $(a, b]$.

It is often useful to consider the following Δ-volatility of order $\delta > 0$:

$$\nu^{(\delta)}_{(a,b]}(H;\Delta) = \frac{\operatorname{Var}^{(\delta)}_{(a,b]}(H;\Delta)}{\left[\frac{b-a}{\Delta}\right]}, \tag{20}$$

where

$$\operatorname{Var}^{(\delta)}_{(a,b]}(H;\Delta) = \sum |H_{t_k} - H_{t_{k-1}}|^{\delta} \tag{21}$$

and the summation proceeds as in (16).

We note that for a fractional Brownian motion $H = B_{\mathbb{H}}$ we have

$$\operatorname{Var}^{(2)}_{(a,b]}(H;\Delta) \xrightarrow{\mathsf{P}} \begin{cases} \infty, & 0 < \mathbb{H} < \frac{1}{2}, \\ (b-a), & \mathbb{H} = \frac{1}{2}, \\ 0, & \frac{1}{2} < \mathbb{H} \leqslant 1, \end{cases} \tag{22}$$

as $\Delta \to 0$, where "$\xrightarrow{\mathsf{P}}$" is convergence in probability.

If H is a strictly α-stable Lévy motion with $0 < \alpha < 2$, then

$$\operatorname{Var}^{(2)}_{(a,b]}(H;\Delta) \xrightarrow{\mathsf{P}} 0 \tag{23}$$

as $\Delta \to 0$.

Remark 2. One usually calls stochastic processes $H = (H_t)_{t\geqslant 0}$ with property (23) *zero-energy* processes (see, i.g., [166]). Thus, it follows from (22) and (23) that both fractional Brownian motion with $1/2 < \mathbb{H} \leqslant 1$ and strictly α-stable Lévy processes with $\mathbb{H} = 1/\alpha > 1/2$ are zero-energy processes.

7. The statistical analysis of volatility by means of $\mathcal{R}/\mathcal{S}$-analysis discussed below (see § 4) enables one to discover several remarkable and unexpected properties, which provide one with tools for the verification of some or other conjectures concerning the space-time structure of the processes $H = (H_t)_{t\geqslant 0}$ (for models with continuous time) and $H = (H_n)_{n\geqslant 0}$ (in the discrete-time case). For instance, one must definitely *discard* the conjecture of the independence of the variables h_n, $n \geqslant 1$ (generating the sequence $H = (H_n)_{n\geqslant 0}$), for many financial indexes. (In the continuous-time case one must accordingly discard the conjecture that $H = (H_t)_{t\geqslant 0}$ is a process with independent increments.)

Simultaneously, the analysis of Δ-volatility and $\mathcal{R}/\mathcal{S}$-statistics support the thesis that the variables h_n, $n \geqslant 1$, are in fact characterized by a rather *strong aftereffect*, and this allows one to cherish hopes of a 'nontrivial' prediction of the price development.

The fractal structure in volatilities can be exposed for many financial indexes (stock and bond prices, DJIA, the S&P500 Index, and so on). It is most clearly visible in currency exchange rates. We discuss this subject in the next section.

§ 3b. Periodicity and Fractal Structure of Volatility in Exchange Rates

1. In § 1b we presented the statistics (see Fig. 29 and 30) of the number of ticks occurring during a day or a week. They unambiguously indicate

the interday inhomogeneity

and

the presence of daily cycles (periodicity).

Representing the process $H = (H_t)_{t \geqslant 0}$ with $H_t = \ln \dfrac{S_t}{S_0}$ by the formula

$$H_t = \sum h_{\tau_k} I(\tau_k \leqslant t) \tag{1}$$

(cf. formula (7) in § 1c), we can say that Fig. 29 and Fig. 30 depict only the part of the development due to the 'time-related' components of H, the instants of ticks τ_k, but give no insight in the structure of the 'phase' component, the sequence (h_{τ_k}) or the sequence of $\widetilde{h}_k = \widetilde{h}_{t_k}^{(\Delta)}$ (see the notation in § 2b).

The above-introduced concept of Δ-volatility, based on the Δ-variation $\mathrm{Var}_{(a,b]}(H; \Delta)$, enables one to gain a distinct notion of the 'intensity' of change in the processes H and $\widetilde{H}$ both with respect to the time and the phase variables.

To this end, we consider the Δ-volatility $\nu_{(a,b]}(H; \Delta)$ on the interval $(a, b]$.

We note on the onset that if $a = (k-1)\Delta$ and $b = k\Delta$, then

$$\nu_{((k-1)\Delta, k\Delta]}(H; \Delta) = |\widetilde{H}_{k\Delta} - \widetilde{H}_{(k-1)\Delta}| = |\widetilde{h}_k| \tag{2}$$

(see the notation in § 2b).

We choose as the object of our study the DEM/USD exchange rate, so that $S_t = (\mathrm{DEM/USD})_t$ and $H_t = \ln \dfrac{S_t}{S_0}$.

We set Δ to be equal to 1 hour and

$$t = 1, 2, \ldots, 24 \quad \text{(hours)}$$

in the case of the analysis of the '24-hour cycle', and we set

$$t = 1, 2, \ldots, 168 \quad \text{(hours)}$$

in the case of the 'week cycle' (the clock is set going on Monday, 0:00 GMT, so that $t = 168$ corresponds to the end of the week).

The impressive database of Olsen & Associates enables one to obtain quite reliable estimates $\widehat{\nu}_{((k-1)\Delta, k\Delta]}(H; \Delta) = \widehat{|\widetilde{h}_k|}$ for the values of $\nu_{((k-1)\Delta, k\Delta]}(H; \Delta) \equiv |\widetilde{h}_k|$ for each day of the week.

To this end let time zero be 0:00 GMT of the first Monday covered by the database. If $\Delta = 1$ h, then setting $k = 1, 2, \ldots, 24$, we obtain the intervals $(0,1], (1,2], \ldots, (23,24]$ corresponding to the intervals (of GMT)

$$(0{:}00, 1{:}00],\ (1{:}00, 2{:}00],\ \ldots,\ (23{:}00, 24{:}00].$$

As an estimate for $\widehat{\nu}_{((k-1)\Delta, k\Delta]}(H;\Delta)$ we take the arithmetic mean of the quantities $|\widetilde{H}^{(j)}_{k\Delta} - \widetilde{H}^{(j)}_{(k-1)\Delta}|$ calculated for all Mondays in the database (indexed by the integer j). In a similar way we obtain estimates of the $\widehat{\nu}_{((k-1)\Delta, k\Delta]}(H;\Delta)$ for Tuesdays ($k = 25, \ldots, 48$), $\ldots$, and for Sundays ($k = 145, \ldots, 168$).

The following charts (Fig. 36 and Fig. 37) from [427] are good illustrations of the *interday inhomogeneity* and of the *daily cycles* visible all over the week in the behavior of the Δ-volatility $\nu_{((k-1)\Delta, k\Delta]}(H;\Delta) = |\widetilde{h}_k|$ calculated for the one-hour intervals $((k-1)\Delta, k\Delta]$, $k = 1, 2, \ldots$.

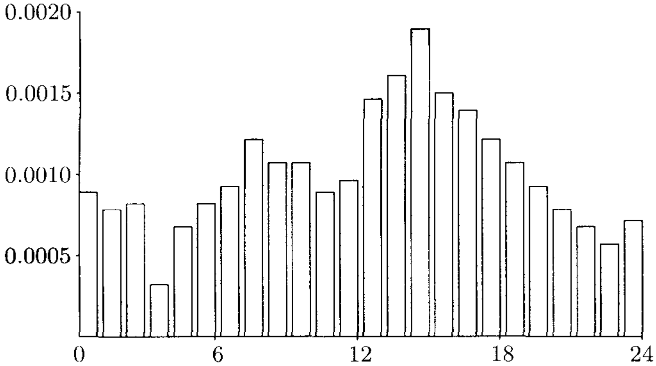

FIGURE 36. Δ-volatility of the DEM/USD cross rate during one day ($\Delta = 1$ hour), according to Reuters (05.10.1992–26.09.1993)

The above-mentioned daily periodicity of the Δ-volatility is also revealed by the analysis of its correlation properties. We devote the next section to this issue and to the discussion of the practical recommendations following from the statistical analysis of Δ-volatility.

2. We discuss now the properties of the Δ-volatility $\nu_t(\Delta) \equiv \nu_{(0,t]}(H;\Delta)$ regarded as a function of Δ for fixed t. We shall denote by $\widehat{\nu}_t(\Delta)$ its estimator $\widehat{\nu}_{(0,t]}(H;\Delta)$.

Assume that t is sufficiently large, for instance, $t = T =$ one year. We shall now evaluate $\widehat{\nu}_T(\Delta)$ for various Δ. No so long ago (see, i.g., [204], [362], [386], or [427]), the following remarkable property of the FX-market (and some other markets) was discovered: the behavior of the Δ-volatility is *highly regular*; namely,

$$\widehat{\nu}_T(\Delta) \approx C_T \Delta^{\mathbb{H}} \tag{3}$$

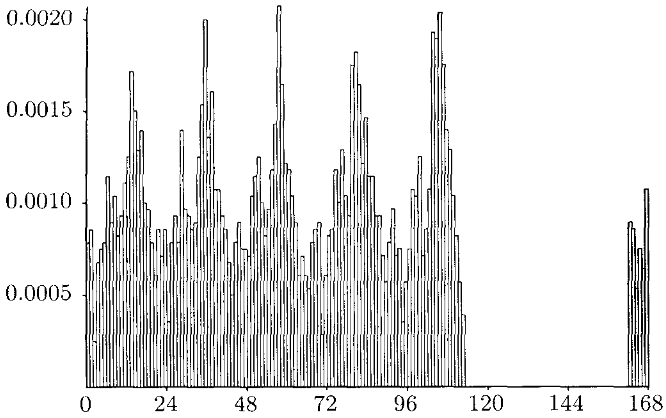

FIGURE 37. Δ-volatility of the DEM/USD cross rate during one week ($\Delta = 1$ hour), according to Reuters (05.10.1992–26.09.1993). The intervals $(0,1], \ldots, (167,168]$ correspond to the intervals $(0{:}00, 1{:}00], \ldots, (23{:}00, 24{:}00]$ of GMT

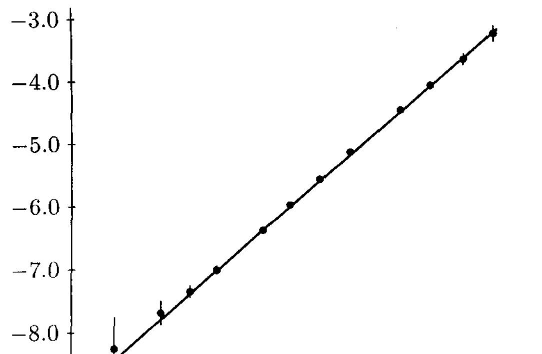

FIGURE 38. On the fractal structure of the Δ-volatility $\widehat{\nu}_T(\Delta)$. The values of $\ln \widehat{\nu}_T(\Delta)$ as a function of $\ln \Delta$ are plotted along the vertical axis

with certain constant C_T dependent on the currencies in question and with $\mathbb{H} \approx 0.585$ for the basic currencies.

To give (3) a more precise form we consider now the statistical data concerning $\log \widehat{\nu}_T(\Delta)$ as a function of $\ln \Delta$ with Δ ranging over a wide interval, from 10 minutes (= 600 seconds) to 2 months ($= 2 \times 30 \times 24 \times 60 \times 60 = 5\,184\,000$ seconds).

The chart in Fig. 38, which is constructed using the least squares method, shows that the empirical data cluster nicely along the straight line with slope $\mathbb{H} \cong 0.585$. Hence we can conclude from (3) that, for t large, the volatility $\nu_t(\Delta)$ regarded as a function of Δ has fractal structure with Hurst exponent $\mathbb{H} \cong 0.585$.

As shown in § 3a, we have $\mathsf{E}|H_\Delta| = \sqrt{2/\pi}\Delta^{1/2}$ for a Brownian motion $H = (H_t)_{t \geqslant 0}$, $\mathsf{E}|H_\Delta| = \sqrt{2/\pi}\Delta^{\mathbb{H}}$ for a fractional Brownian motion $H = (H_t)_{t\geqslant 0}$ with exponent $\mathbb{H}$, and $\mathsf{E}|H_\Delta| = \mathsf{E}|H_1|\Delta^{\mathbb{H}}$ with $\mathbb{H} = 1/\alpha < 1$ for a strictly α-stable Lévy process with $\alpha > 1$.

Thus, the experimentally obtained value $\mathbb{H} = 0.585 > 1/2$ supports the conjecture that the process $H = (H_t)$, $t \geqslant 0$, can be satisfactory described either by a fractional Brownian motion or by an α-stable Lévy process with $\alpha = \dfrac{1}{\mathbb{H}} \approx \dfrac{1}{0.585} \approx 1.7$.

Remark. As regards the estimates for $\mathbb{H}$ in the case of a fractional Brownian motion, see Chapter III, § 2c.6.

3. We now turn to Fig. 36. In it, the periods of maximum and minimum activity are clearly visible: 4:00 GMT (the minimum) corresponds to lunch time in Tokyo, Sydney, Singapore, and Hong Kong, when the life in the FX-market comes to a standstill. (This is nighttime in Europe and America). We have already pointed out that the maximum activity ($\approx$ 15:00) corresponds to time after lunch in Europe and the beginning of the business day in America.

The daily activity patterns during the five working days (Monday through Friday) are rather similar. Activity fades significantly on week-ends. On Saturday and most part of Sunday it is almost nonexistent. On Sunday evening, when the East Asian market begins its business day, activity starts to grow.

§ 3c. Correlation Properties

1. Again, we consider the DEM/USD exchange rate, which (as already pointed out in § 1a.4) is featured by high intensity of ticks (on the average, 3–4 ticks per minute on usual days and 15–20 ticks per minute on days of higher activity, as in July, 1994).

The above-described phenomena of periodicity in the occurrences of ticks and in Δ-volatility are visible also in the *correlation* analysis of the absolute values of the changes $|\Delta H|$. We present the corresponding results below, in subsection 3, while we start with the correlation analysis of the values of ΔH themselves.

2. Let $S_t = (\mathrm{DEM/USD})_t$ and let $H_t = \ln \dfrac{S_t}{S_0}$. We denote the results of an appropriate linear interpolation (see § 2b) by $\widetilde{S}_t$ and $\widetilde{H}_t = \ln \dfrac{\widetilde{S}_t}{S_0}$, respectively.

We choose a time interval Δ; let

$$\widetilde{h}_k = \widetilde{H}_{t_k} - \widetilde{H}_{t_{k-1}}$$

with $t_k = k\Delta$. (In § 2b we also used the notation $h_{t_k}^{(\Delta)}$; in accordance with § 3b, $|\widetilde{h}_k| = \nu_{(t_{k-1},t_k]}(H;\Delta)$.)

Let Δ be 1 minute and let $k = 1, 2, \dots, 60$. Then $\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_{60}$ is the sequence of consecutive (one-minute) increments of H over the period of *one* hour. We can assume that the sequence $\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_{60}$ is stationary (homogeneous) on this interval.

Traditionally, as a measure of the correlation dependence of stationary sequences $\widetilde{h} = (\widetilde{h}_1, \widetilde{h}_2, \dots)$, one takes their correlation function

$$\rho(k) = \frac{\mathsf{E}\,\widetilde{h}_n \widetilde{h}_{n+k} - \mathsf{E}\,\widetilde{h}_n \cdot \mathsf{E}\,\widetilde{h}_{n+k}}{\sqrt{\mathsf{D}\widetilde{h}_n \cdot \mathsf{D}\widetilde{h}_{n+k}}}, \tag{1}$$

(the *autocorrelation* function in the theory of stochastic processes).

The corresponding statistical analysis has been carried out by Olsen & Associates for the data in their (rather representative) database covering the period from January 05, 1987 through January 05, 1993 (see [204]). Using its results one can plot the following graph of the (empirical) autocorrelation function $\widehat{\rho}(k)$:

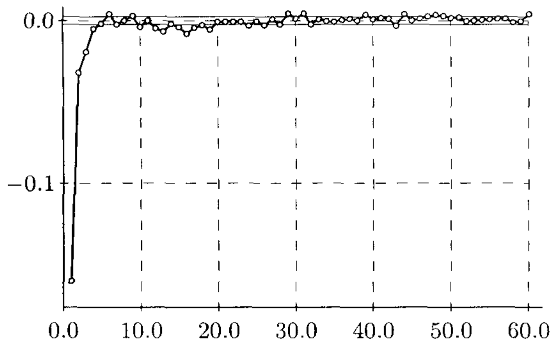

FIGURE 39. Empirical autocorrelation function $\widehat{\rho}(k)$ for the sequence of increments $\widetilde{h}_n = \widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}$ corresponding to the DEM/USD cross rate, with $t_n = n\Delta$ and $\Delta = 1$ minute

In Fig. 39 one clearly sees a *negative* correlation on the interval of approximately 4 minutes ($\widehat{\rho}(1) < 0$, $\widehat{\rho}(2) < 0$, $\widehat{\rho}(3) < 0$, $\widehat{\rho}(4) \leqslant 0$), while most of the values of the $\widehat{\rho}(k)$ with $4 < k < 60$ are close to zero.

Bearing this observation in mind, we can assume that the variables $\widetilde{h}_n$ and $\widetilde{h}_m$, are virtually uncorrelated for $|n - m| > 4$.

We note that the *phenomenon of negative correlation* on small intervals ($|n - m| \leqslant 4$ minutes) was mentioned for the first time in [189] and [191]; it has been noticed for many financial indexes (see, for instance, [145] and [192]).

There exist various explanations in the literature of this negative correlation of the increments $\Delta\widetilde{H}$ on small intervals of time. For instance, the discussion in [204] essentially reduces to the observation that the traders in the FX-market are not uniform, their interests can 'point to different directions', and they can interpret available information in different ways. Traders often widen or narrow the spread when they are given instructions to 'get the market out of balance'. Moreover, many banks overstate their spreads systematically (see [192] in this connection).

A possible 'mathematical' explanation of the phenomenon of negative values of $\mathsf{Cov}(\widetilde{h}_n, \widetilde{h}_{n+k}) = \mathsf{E}\widetilde{h}_n\widetilde{h}_{n+k} - \mathsf{E}\widetilde{h}_n\mathsf{E}\widetilde{h}_{n+k}$ for small k can be, e.g., as follows (cf. [481]).

Let $\widetilde{H}_n = \widetilde{h}_1 + \dots + \widetilde{h}_n$ with $\widetilde{h}_n = \mu_n + \sigma_n\varepsilon_n$, where the σ_n are $\mathscr{F}_{n-1}$-measurable and (ε_n) is a sequence of independent identically distributed random variables. We can also assume that the μ_n are $\mathscr{F}_{n-1}$-measurable variables. Judging by a large amount of statistical data, the 'mean values' μ_n are much smaller than σ_n (see, e.g., the table in § 2b.2) and can be set to be equal to zero for all practical purposes.

The values of $\widetilde{H}_n = \ln\dfrac{\widetilde{S}_n}{S_0}$ are in practice *not always known precisely*; it would be more realistic to assume that what we know are the values of $\widetilde{\widetilde{H}}_n = \widetilde{H}_n + \delta_n$, where (δ_n) is white noise, the noise component related to inaccuracies in our knowledge about the actual values of prices, rather to these values themselves.

We assume that (δ_n) is a sequence of independent random variables with $\mathsf{E}\delta_n = 0$ and $\mathsf{E}\delta_n^2 = C > 0$. Then, considering the sequence $\widetilde{\widetilde{h}} = (\widetilde{\widetilde{h}}_n)$ of the variables $\widetilde{\widetilde{h}}_n = \Delta\widetilde{\widetilde{H}}_n = \widetilde{h}_n + (\delta_n - \delta_{n-1})$ we obtain

$$\mathsf{E}\,\widetilde{\widetilde{h}}_n = 0, \qquad \mathsf{E}\,\widetilde{\widetilde{h}}{}_n^2 = \mathsf{E}\sigma_n^2 + 2C$$

and

$$\mathsf{E}\,\widetilde{\widetilde{h}}_n\,\widetilde{\widetilde{h}}_{n+1} = \mathsf{E}(\delta_n - \delta_{n-1})(\delta_{n+1} - \delta_n) = -C,$$
$$\mathsf{E}\,\widetilde{\widetilde{h}}_n\,\widetilde{\widetilde{h}}_{n+k} = 0, \qquad k > 1.$$

Hence the covariance function

$$\mathsf{Cov}(\widetilde{\widetilde{h}}_n, \widetilde{\widetilde{h}}_{n+k}) = \mathsf{E}\widetilde{\widetilde{h}}_n\widetilde{\widetilde{h}}_{n+k} - \mathsf{E}\widetilde{\widetilde{h}}_n \cdot \mathsf{E}\widetilde{\widetilde{h}}_{n+k}$$

(provided that $\mathsf{E}\sigma_n^2 = \mathsf{E}\sigma_1^2$, $n \geqslant 1$) can be described by the formula

$$\mathsf{Cov}(\widetilde{\widetilde{h}}_n, \widetilde{\widetilde{h}}_{n+k}) = \begin{cases} \mathsf{E}\sigma_1^2 + 2C, & k = 0, \\ -C, & k = 1, \\ 0, & k > 1. \end{cases}$$

3. To reveal cycles in volatilities by means of correlation analysis we proceed as follows.

We fix the interval Δ equal to 20 minutes. Let $t_0 = 0$ correspond to Monday, 0:00 GMT, let $t_1 = \Delta = 20$ m, $t_2 = 2\Delta = 40$ m. $t_3 = 3\Delta = 1$ h, ..., $t_{504} = 504\,\Delta = 1$ week, ..., $t_{2016} = 2016\,\Delta = 4$ weeks ($= 1$ month).

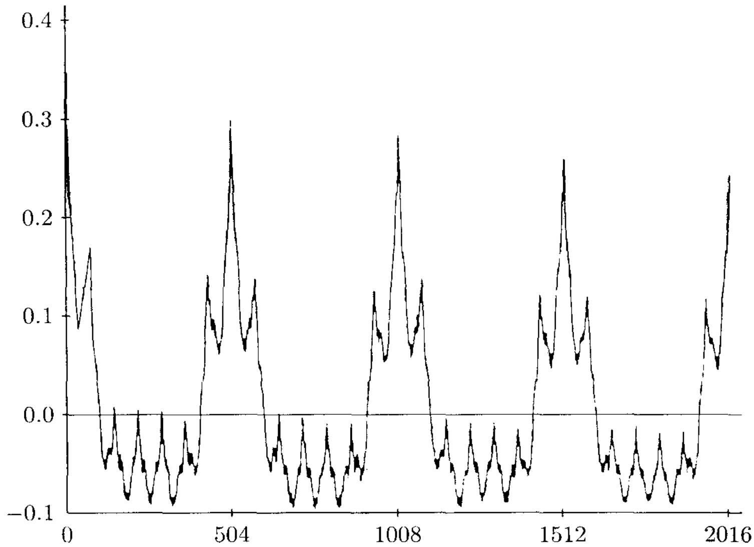

FIGURE 40. Empirical autocorrelation function $\widehat{R}(k)$ for the sequence $|\widetilde{h}_n| = |\widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}|$ corresponding to the DEM/USD cross rate (the data of Reuters; October 10, 1992–September 26, 1993; [90], [204]). The value $k = 504$ corresponds to 1 week and $k = 2016$ to 4 weeks

We set $\widetilde{h}_n = \widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}$; let

$$R(k) = \frac{\mathsf{E}|\widetilde{h}_n||\widetilde{h}_{n+k}| - \mathsf{E}|\widetilde{h}_n| \cdot \mathsf{E}|\widetilde{h}_{n+k}|}{\sqrt{\mathsf{D}|\widetilde{h}_n| \cdot \mathsf{D}|\widetilde{h}_{n+k}|}} \tag{2}$$

be the autocorrelation function of the sequence $|\widetilde{h}| = (|\widetilde{h}_1|, |\widetilde{h}_2|, \dots)$.

The graph of the corresponding empirical autocorrelation function $\widehat{R}(k)$ for $k = 0, 1, \dots, 2016$ (that is, over the period of *four* weeks) is plotted in Fig. 40. One clearly sees in it a *periodic* component in the autocorrelation function of the Δ-volatility of the sequence $|\widetilde{h}| = \big(|\widetilde{h}_n|\big)_{n\geqslant 1}$ with $|\widetilde{h}_n| = |\widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}|$, $\Delta = t_n - t_{n-1}$.

As is known, to demonstrate the full strength of the correlation methods one requires that the sequence in question be stationary. We see, however, that Δ-volatility does not have this property. Hence there arises the natural desire to 'flatten' it in one or another way, making it a stationary, homogeneous sequence.

This procedure of 'flattening' the volatility is called '*devolatization*'. We discuss it in the next section, where we are paying most attention to the concept of 'change of time', well-known in the theory of random processes, and the idea of *operational* 'θ-time', which Olsen & Associates use methodically (see [90], [204], and [362]) in their analysis of the data relating to the FX-market.

§ 3d. 'Devolatization'. Operational Time

1. We start from the following example, which is a good illustration of the main stages of '*devolatization*', a procedure of 'flattening' the volatilities.

Let $H_t = \int_0^t \sigma(u)\,dB_u$, where $B = (B_t)_{t\geqslant 0}$ is a standard Brownian motion and $\sigma = (\sigma(t))_{t\geqslant 0}$ is some *deterministic* function (the 'activity') characterizing the 'contribution' of the dB_u, $u \leqslant t$, to the value of H_t. We note that

$$h_n \equiv H_n - H_{n-1} = \int_{n-1}^{n} \sigma(u)\,dB_u \stackrel{\mathrm{d}}{=} \sigma_n \varepsilon_n \tag{1}$$

for each $n \geqslant 1$, where $\varepsilon_n \sim \mathscr{N}(0,1)$, $\sigma_n^2 = \int_{n-1}^{n} \sigma^2(u)\,du$, and the symbol '$\stackrel{\mathrm{d}}{=}$' means that variables coincide in distribution.

Thus, if we can register the values of the process $H = (H_t)_{t\geqslant 0}$ only at discrete instants $n = 1, 2, \dots$, then the observed sequence $h_n \equiv H_n - H_{n-1}$ has a perfectly simple structure of a *Gaussian* sequence $(\sigma_n\varepsilon_n)_{n\geqslant 1}$ of independent random variables with zero means and, in general, inhomogeneous variances (volatilities) σ_n^2.

In the discussion that follows we present a method for transforming the data so that the inhomogeneous variables σ_n^2, $n \geqslant 1$, become 'flattened'.

We set

$$\tau(t) = \int_0^t \sigma^2(u)\,du \tag{2}$$

and

$$\tau^*(\theta) = \inf\left\{t\colon \int_0^t \sigma^2(u)\,du = \theta\right\} \quad (= \inf\{t\colon \tau(t) = \theta\}), \tag{3}$$

where $\theta \geqslant 0$.

We shall assume that $\sigma(t) > 0$ for each $t > 0$, $\int_0^t \sigma^2(u)\,du < \infty$ (this property ensures that the stochastic integral $\int_0^t \sigma(u)\,dB_u$ with respect to the Brownian motion $B = (B_u)_{u\geqslant 0}$ is well defined; see Chapter III, § 3c), and let $\int_0^t \sigma^2(u)\,du \uparrow \infty$ as $t \to \infty$.

Alongside *physical* time $t \geqslant 0$, we shall also consider new, *operational* 'time' θ defined by the formula

$$\theta = \tau(t). \tag{4}$$

The 'return' from this operational time θ to physical time is described by the inverse transfomation

$$t = \tau^*(\theta). \tag{5}$$

We note that

$$\int_0^{\tau^*(\theta)} \sigma^2(u)\, du = \theta \tag{6}$$

by (3), i.e., $\tau(\tau^*(\theta)) = \theta$, so that $\tau^*(\theta) = \tau^{-1}(\theta)$ and $\tau^*(\tau(t)) = t$.

We now consider the function $\theta = \tau(t)$ performing this transformation of physical time into operational time.

Since

$$\theta_2 - \theta_1 = \int_{t_1}^{t_2} \sigma^2(u)\, du, \tag{7}$$

we see that if the activity $\sigma^2(u)$ is small, then this transformation 'compresses' the physical time (as in Fig. 41).

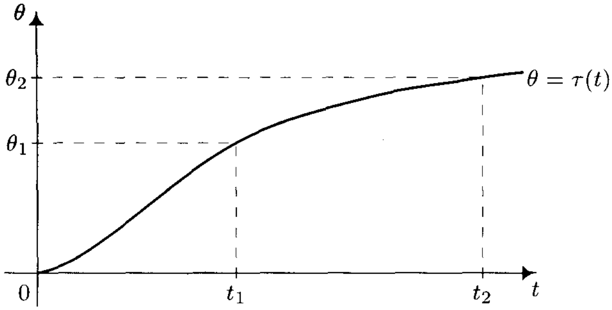

FIGURE 41. 'Compression' of a (large) time interval $[t_1, t_2]$ characterised by small activity into a (short) interval $[\theta_1, \theta_2]$ of θ-time

On the other hand, if the activity $\sigma^2(u)$ is large, then the process goes in the opposite direction: short intervals (t_1, t_2) of physical time (see Fig. 42) correspond to larger intervals (θ_1, θ_2) of operational time; time is being 'stretched'.

Now, we construct another process,

$$H^*_\theta = H_{\tau^*(\theta)}, \tag{8}$$

which proceeds in operational time. Clearly, we can 'return' from H^* to the old process by the formula

$$H_t = H^*_{\tau(t)}, \tag{9}$$

because $\tau^*(\tau(t)) = t$.

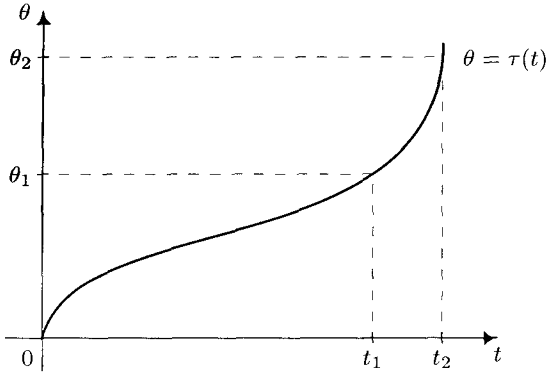

FIGURE 42. 'Stretching' a (short) interval $[t_1, t_2]$ characterized by large activity into a (long) interval $[\theta_1, \theta_2]$ of θ-time

Note that if $\theta_1 < \theta_2$, then

$$H^*_{\theta_2} - H^*_{\theta_1} = H_{\tau^*(\theta_2)} - H_{\tau^*(\theta_1)} = \int_{\tau^*(\theta_1)}^{\tau^*(\theta_2)} \sigma(u)\, dB_u$$
$$= \int_0^\infty I\bigl(\tau^*(\theta_1) < u \leqslant \tau^*(\theta_2)\bigr)\, \sigma(u)\, dB_u.$$

Hence H^* is a process with *independent increments*, $H^*_0 = 0$, and $\mathsf{E} H^*_\theta = 0$. Moreover, by the properties of stochastic integrals (see Chapter III, § 3c) we obtain

$$\mathsf{E}\bigl|H^*_{\theta_2} - H^*_{\theta_1}\bigr|^2 = \int_0^\infty I\bigl(\tau^*(\theta_1) < u \leqslant \tau^*(\theta_2)\bigr)\, \sigma^2(u)\, du$$
$$= \int_{\tau^*(\theta_1)}^{\tau^*(\theta_2)} \sigma^2(u)\, du = \theta_2 - \theta_1 \tag{10}$$

(the last equality is a consequence of (6)).

Since H^* is also a Gaussian process and has independent increments, zero mean, property (10), and continuous trajectories, it is just a *standard Brownian motion* (see Chapter III, § 3a), so that

$$H^*_\theta = \int_0^\theta \sigma^*(u)\, dH^*_u, \tag{11}$$

where $\sigma^*(u) \equiv 1$.

A comparison with the representation $H_t = \int_0^t \sigma(u)\, dB_u$, where $\sigma(u) \not\equiv 1$ in general, shows that the transition to operational time has 'flattened' the characteristic of activity $\sigma \equiv \sigma(u)$: when measured with respect to new 'time' θ, the level of activity is flat ($\sigma^*(u) \equiv 1$).

We have assumed above that $\sigma(u)$ is deterministic. However, $H_\theta^* = H_{\tau^*(\theta)}$ will be a Wiener process even if the change of time is *random*, defined by formula (3) with $\sigma(u) = \sigma(u;\omega)$, provided that $\int_0^t \sigma^2(u;\omega)\,du < \infty$ with probability one and $\int_0^t \sigma^2(u;\omega)\,du \uparrow \infty$ as $t \to \infty$. However, there exists a fundamental distinction between the cases of a *deterministic* function $\sigma = \sigma(u)$ and a *random* function $\sigma = \sigma(u;\omega)$: in the first case we can *calculate* the change $t \rightsquigarrow \theta = \tau(t)$ *in advance*, also for the 'future', which is impossible in the second case, because the 'random' changes of time corresponding to distinct realizations $\sigma = \sigma(u;\omega)$, $u \geqslant 0$, are distinct.

2. We consider now a sequence $h = (h_n)_{n\geqslant 1}$, where $h_n = \sigma_n\varepsilon_n$ with nonflat level of activity σ_n, $n \geqslant 1$. We shall treat n as *physical* ('old') time.

We introduce the sequence of times

$$\tau^*(\theta) = \min\left\{m \geqslant 1\colon \sum_{k=1}^{m} \sigma_k^2 \geqslant \theta\right\},$$

with positive integer θ that we treat as *operational* ('new') time.

Also, let

$$h_\theta^* = \sum_{\tau^*(\theta-1)<k\leqslant\tau^*(\theta)} h_k$$

for $\theta = 1, 2, \dots$, where $\tau^*(0) = 0$.

We note that $\mathsf{E}h_0^* = 0$ and the corresponding variances are

$$\mathsf{D}h_\theta^* = \mathsf{D}\Bigg[\sum_{\tau^*(\theta-1)<k\leqslant\tau^*(\theta)} h_k\Bigg] = \sum_{\tau^*(\theta-1)<k\leqslant\tau^*(\theta)} \sigma_k^2 \approx 1,$$

because the values of the σ_k^2 are usually fairly small (see the table in § 2b.2).

Thus, we can say that the transition to new 'time' θ transforms the inhomogeneous sequence $h = (h_n)_{n\geqslant 1}$ into an (almost) homogeneous sequence $h^* = (h_\theta^*)_{\theta\geqslant 1}$.

If the σ_n are random variables, that is $\sigma_n = \sigma_n(\omega)$, and our aim is to calculate the change of time for all instants (including the future), then we can implement the above-discuss idea of 'devolatization' by replacing the $\sigma_n^2(\omega)$ by their mean values $\mathsf{E}\sigma_n^2(\omega)$ or, in practice, by *estimates* of these mean values.

We see from the representation $h_n = \sigma_n\varepsilon_n$ that if the σ_n are $\mathscr{F}_{n-1}$-measurable, then $\mathsf{E}h_n^2 = \mathsf{E}\sigma_n^2$. Hence if time n of GMT falls, for example, on Monday, then we can take an estimate for $\mathsf{E}\sigma_n^2$ equal to the arithmetic mean of the values of h_k^2 over all k such that $((k-1)\Delta, k\Delta]$ corresponds to the same time interval of some other Monday covered by the database.

The change of time (2) is defined in terms of $\sigma(u)$ squared, which, of course, is not the only possible way to change time $t \rightsquigarrow \theta = \tau(t)$. For example, we could use the values of $|\sigma(u)|$ in place of $\sigma^2(u)$.

3. This is just the change of time used by Olsen & Associates ([90], [360]–[362]). They say that this kind of 'devolatization' captures *periodicity* better and the behavior pattern of the autocorrelation function of the 'devolatized' sequence $(|\widetilde{h}^*|)$ for the DEM/USD exchange rate is more 'smooth'.

Referring to [90] for the details, we shall present only the result of their statistical analysis of the properties of the autocorrelation function of the sequence $|\widetilde{h}^*|$.

As in § 3c.3, we assume that Δ is equal to 20 minutes, $t_n = n\Delta$, and $\widetilde{h}_n = \widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}$.

In Fig. 42 (§ 3c) we plotted the graph of the empirical estimate $\widehat{R}(k)$ of the autocorrelation function

$$R(k) = \frac{\mathsf{E}|\widetilde{h}_n||\widetilde{h}_{n+k}| - \mathsf{E}|\widetilde{h}_n| \cdot \mathsf{E}|\widetilde{h}_{n+k}|}{\sqrt{\mathsf{D}(|\widetilde{h}_n|) \cdot \mathsf{D}(|\widetilde{h}_{n+k}|)}}, \tag{12}$$

in which the periodic structure is clearly visible.

In [90], after 'devolatization' and the transition to new, operational 'time' θ the authors plot a graph (see Fig. 43) of the *empirical* correlation function $\widehat{R}^*(\theta)$, $\theta \geqslant 0$, for the sequence $|h^*| = (|h^*_\theta|)_{\theta \geqslant 1}$. This graph is rather interesting for further analysis. In the same paper one can find a graph (see Fig. 44) of the transition $\tau^*(\theta) \colon \theta \rightsquigarrow t$ from operational time θ to real time t (just in our case of the DEM/USD cross rate; new time is normalized so that a week of physical time corresponds to a week of operational time).

It is clear from Fig. 44 that the function $t = \tau^*(\theta)$ is approximately linear during the five business days, while on the week-end, when the transactions in the FX-market fade, large intervals of physical time correspond to small intervals of operational time. (Only the latter are in fact interesting for business trading).

4. It should be noted that the above method of 'devolatizing' in order to eliminate the periodic component is not the only one used in the analysis of the FX-market. For instance, we can point out the papers [7], [13], and [306], where most diversified techniques, linear and nonlinear regression analysis, Fourier transformation, neural networks are used to find similar patterns in financial time series. We also point out the paper [297] by I. L. Legostaeva and this author, which relates to the same range of problems. In it (in connection with the study of the Wolf numbers characterizing solar activity), to analyze the 'trend' component $a = (a_k)$ of a sequence $\xi = (\xi_k)$ with an additive 'white noise' $\eta = (\eta_k)$ $(\xi_k = a_k + \eta_k)$, the authors use a *minimax approach* suitable for the study of a much broader class of 'trend' sequences $a = (a_k)_{k \geqslant 1}$ than the standard polynomials classes of regression analysis. (See also [45], [338], and [416].)

5. For conclusion we present a graph of the periodic component of the 'activity' (see § 3b) corresponding to the CHF/USD cross rate, which is isolated by means of 'devolatization' (see [90]; cf. also Fig. 37 in § 3b).

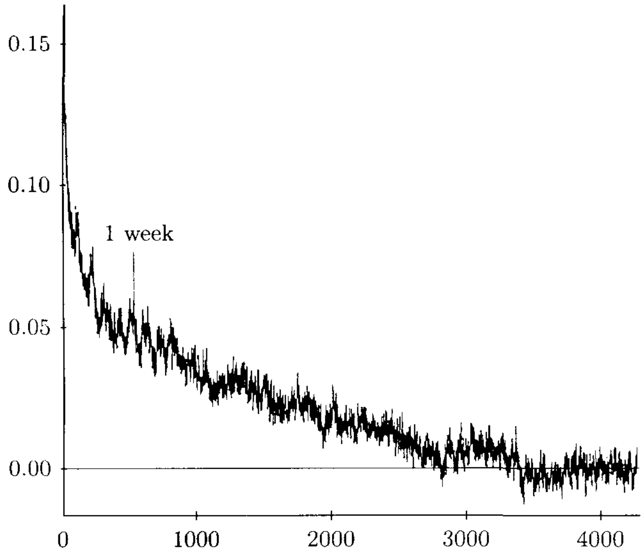

FIGURE 43. Empirical autocorrelation function $\widehat{R}^*(\theta)$ for the sequence $|\widetilde{h}^*| = (|\widetilde{h}^*_\theta|)_{\theta \geqslant 1}$ of the absolute values of 'devolatized' variables considered with respect to *operational* 'time' θ, with interval $\Delta\theta = 20$ m; the case of the DEM/USD cross rate ([90])

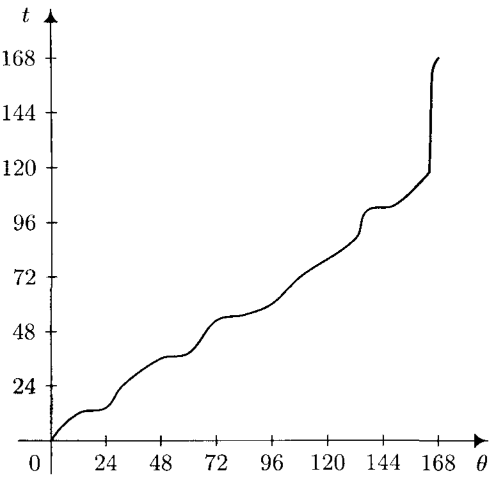

FIGURE 44. Graph of the transition $t = \tau^*(\theta)$ from operational time θ (plotted along the horizontal axis) to physical time (along the vertical axis). Time is measured in hours; 168 hours form 1 week ([90])

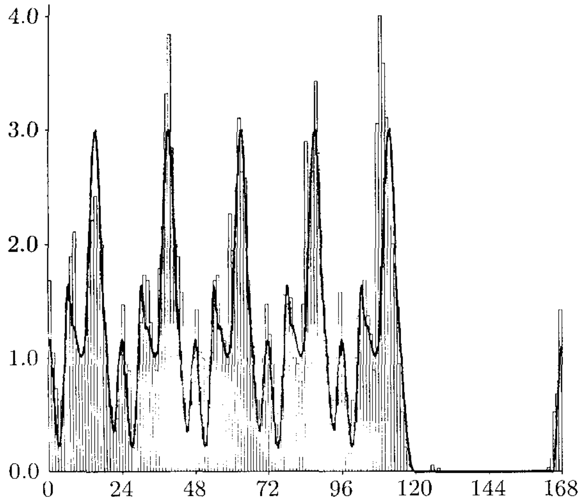

FIGURE 45. The thick curve is the periodic component of the 'activity' corresponding to the CHF/USD cross rate (over the period of 168 hours= 1 week)

In Fig. 45 we clearly see the geography-related structure of the periodic component (with 24-hour cycle during the week) arising from the differences in business time between three major FX-markets: East Asia, Europe, and America. In [D4] one can find an interesting representation of this component as the sum of three periodic components corresponding to these three markets. This can be helpful if one wants to take the factor of periodicity more consistently into account in the predictions of the evolution of exchange rates.

§ 3e. 'Cluster' Phenomenon and Aftereffect in Prices

1. We assumed in our initial scheme that the exchange rates (prices) $S = (S_t)_{t\geqslant 0}$ and their logarithms $H_t = \ln \dfrac{S_t}{S_0}$ could be described by stochastic processes with discrete intervention of chance:

$$S_t = S_0 + \sum_{n\geqslant 1} s_{\tau_n} I(\tau_n \leqslant t) \tag{1}$$

and

$$H_t = \sum_{n\geqslant 1} h_{\tau_n} I(\tau_n \leqslant t). \tag{2}$$

After that, we proceeded to their continuous modifications $\widetilde{S} = (\widetilde{S}_t)_{t\geqslant 0}$, $\widetilde{H} = (\widetilde{H}_t)_{t\geqslant 0}$ and, finally, to the variables $\widetilde{h}_n = \widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}$, where $t_n - t_{n-1} \equiv \Delta$

(= const). It was for these variables $\widetilde{h}_n$ and for Δ equal to 1 minute that we discovered the *negative* autocovariance $\widetilde{\rho}(k) = \mathsf{E}\widetilde{h}_n\widetilde{h}_{n+k} - \mathsf{E}\widetilde{h}_n\,\mathsf{E}\widetilde{h}_{n+k}$ for small values of k $(k = 1, 2, 3, 4)$). For k large the autocovariance is close to zero, therefore we can assume for all practical purposes that the variables $\widetilde{h}_n$ and $\widetilde{h}_{n+k}$ are *uncorrelated* for such k.

Of course we are *far from saying* that they are *independent*; our analysis of the empirical autocovariance function $\widehat{R}(k)$ in § 3c (for the DEM/USD cross rate, as usual) showed that this was not the case.

The next step ('devolatization') enables us to flatten the level of 'activity' by means of the transition to new, operational time that takes into account the different intensity of changes in the values of the process $\widetilde{H} = (\widetilde{H}_t)_{t\geqslant 0}$ on different intervals.

As is clear from the statistical analysis of the sequence $(|\widetilde{h}_n|)_{n\geqslant 1}$ considered with respect to operational time θ, the autocorrelation function $\widehat{R}^*(\theta)$

1) is *rather large for small* θ;
2) *decreases fairly slowly with the growth of* θ.

It is maintained in [90] that for periods of about one month the behavior of $\widehat{R}^*(\theta)$ can be fairly well described by a *power* function, rather than an exponential; that is, we have

$$\widehat{R}^*(\theta) \sim k\theta^{-\alpha} \quad \text{as} \quad \theta \to \infty, \tag{3}$$

rather than

$$\widehat{R}^*(\theta) \sim k\exp(-\theta^{\beta}) \quad \text{as} \quad \theta \to \infty, \tag{4}$$

as could be expected and which holds in many popular models of financial mathematics (for instance, *ARCH* and *GARCH*; see [193] and [202] for greater detail).

This property of the slow decrease of the empirical autocorrelation function $\widehat{R}^*(\theta)$ has important practical consequences. It means that we have indeed a strong aftereffect in prices; 'prices remember their past', so to say. Hence one may hope to be able to predict price movements. To this end, one must, of course, learn to produce sequences $h = (h_n)_{n\geqslant 1}$ that have at least correlation properties similar to the ones observed in practice. (See [89], [360], and Chapter II, § 3b in this connection).

2. The fact that the autocorrelation is fairly *strong* for small θ is a convincing explanation for the *cluster property* of the 'activity' measured in terms of the volatility of $|\widetilde{h}_n|$.

This property, known since B. Mandelbrot's paper [322] (1963), essentially means the kind of behavior when *large values* of volatility are usually followed also by *large values* and *small values* are followed by *small* ones.

That is, if the variation $|\widetilde{h}_n| = |\widetilde{H}_{t_n} - \widetilde{H}_{t_{n-1}}|$ is large, then (with probability sufficiently close to one) the next value, $|\widetilde{h}_{n+1}|$, will also be large. while if $|\widetilde{h}_n|$ is small, then (with probability close to one) the next value will also be small. This property is clearly visible in Fig. 46; it can be observed in practice for many financial indexes.

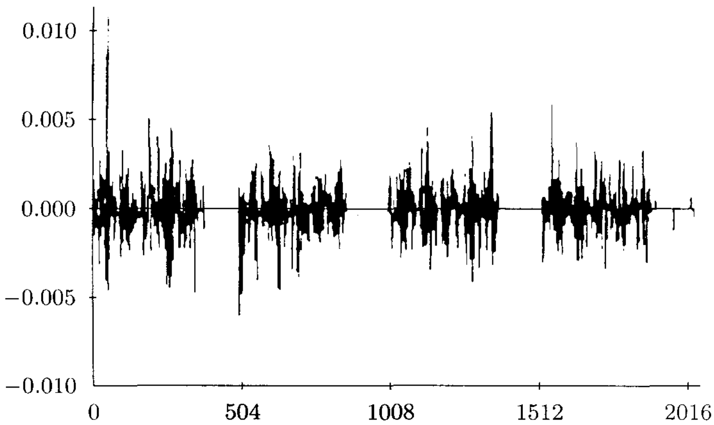

FIGURE 46. Cluster property of the variables $\widetilde{h}_k = \widetilde{h}_{t_k}^{(\Delta)}$ for the DEM/USD cross rate (the data of Reuters; October 5, 1992–November 2, 1992; [427]). The interval Δ is 20 minutes, the mark 504 corresponds to one week, and 2016 corresponds to four weeks. The clusters of large and small values of $|\widetilde{h}_k|$ are clearly visible.

We note that this cluster property is also well captured by $\mathcal{R}/\mathcal{S}$-analysis, which we discuss in the next section.