IS1204-Mathematical Methods

1. Vector Algebra

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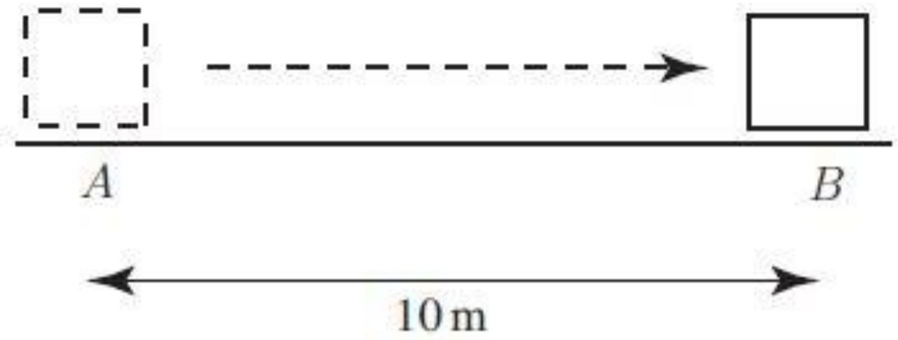
Learning Outcomes

- Define scalars and vectors
- Define vector operations
- Define a vector in component form
- Find scalar product and vector product
- Discuss applications of scalar and vector products

Outline

- Introduction: scalars and vectors
- Vector operations: addition, subtraction and multiplication
- Vectors in component form
- Scalar product and Vector product
- Applications of scalar and vector product

What are Vectors ?



Referring to the figure,

- Distance ?
- Displacement ?

Vectors and Scalars

Physical quantities can be divided into two main groups **scalar quantities** and **vector quantities**.

Scalar: Quantity having magnitude but no direction.

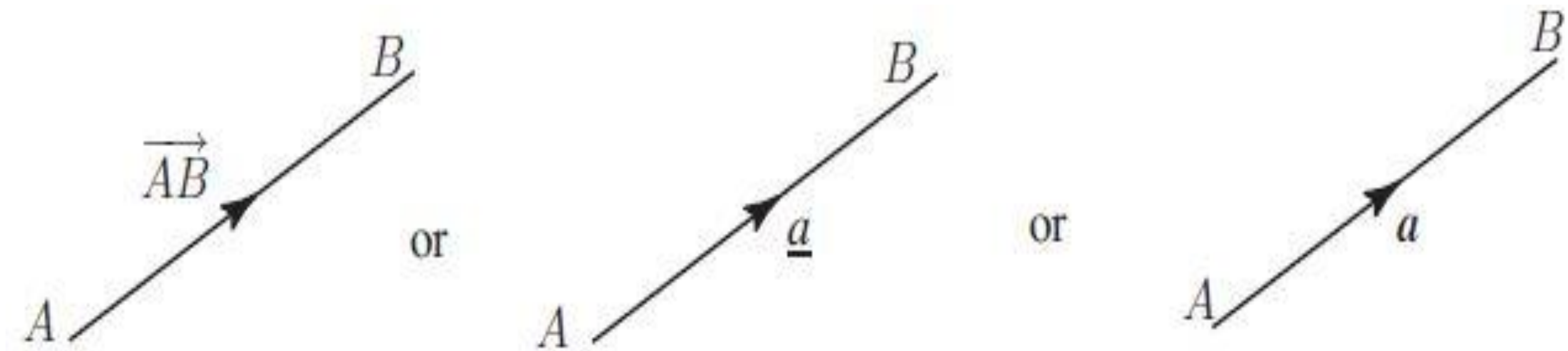
E.g. Length, Area, Volume, Mass etc…

Vector: Quantity having both magnitude and direction.

E.g. Force, Velocity, Acceleration etc….

Representation of Vectors

Vectors can be represented in different ways.



Magnitude of a vector: $\left|\overrightarrow{AB}\right|$ or $|\underline{a}|$.

Types of Vectors

a. Unit vector: A vector which has a magnitude of 1 and is given by $\hat{a} = \dfrac{\underline{a}}{|\underline{a}|}$.

b. Equal vectors: Two vectors are said to be equal vectors if they have the same magnitude and same direction.

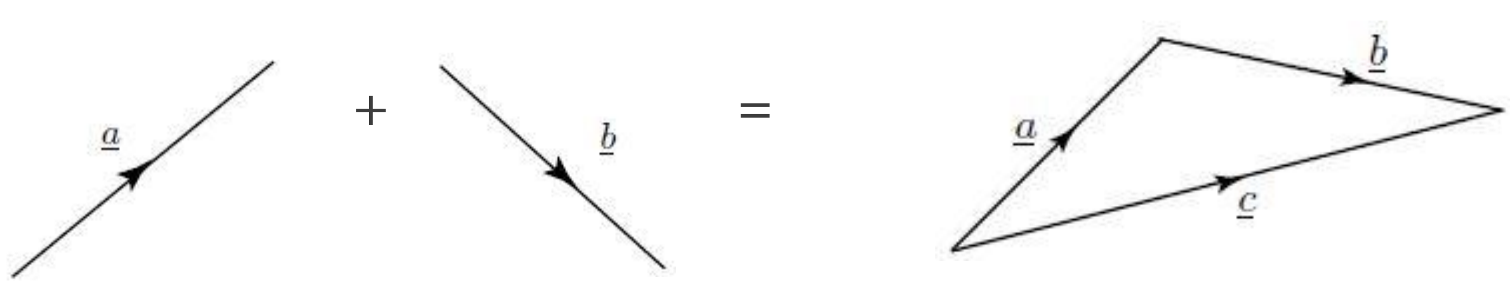
c. Zero or Null vector: A vector whose magnitude is zero.

d. Negative of a vector: The vector which has the same magnitude as the vector $\underline{a}$ but opposite in direction. It is represented by $-\underline{a}$.

Vector Operations

I. Addition of vectors

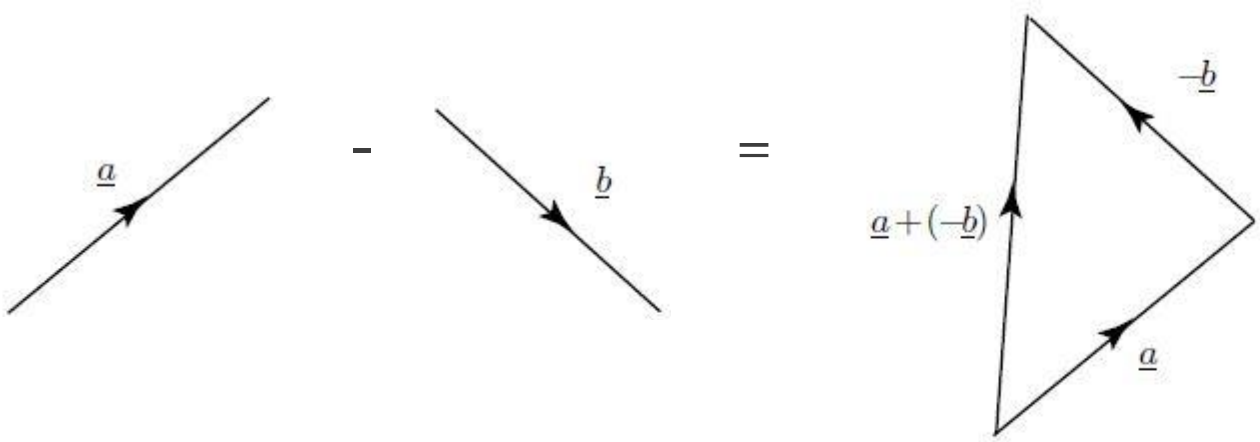
A vector whose effect is the same as a set of two vectors is called the sum or resultant of the given vectors.



Vector Operations contd..

II. Subtraction of vectors

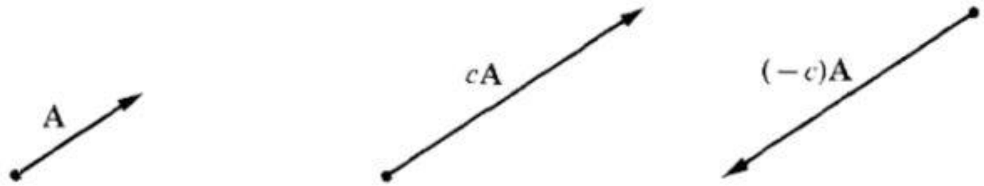
Subtraction of one vector from another is performed by adding the corresponding negative vector.



Vector Operations contd..

III. Scalar multiplication

Let $\underline{\mathbf{A}}$ be a given vector and c be a scalar. Then, the product of the vector $\underline{\mathbf{A}}$ by the scalar c is $c\underline{\mathbf{A}}$.



Properties of vector operations:

1. $\underline{a}+\underline{b}=\underline{b}+\underline{a}$ (Commutative)

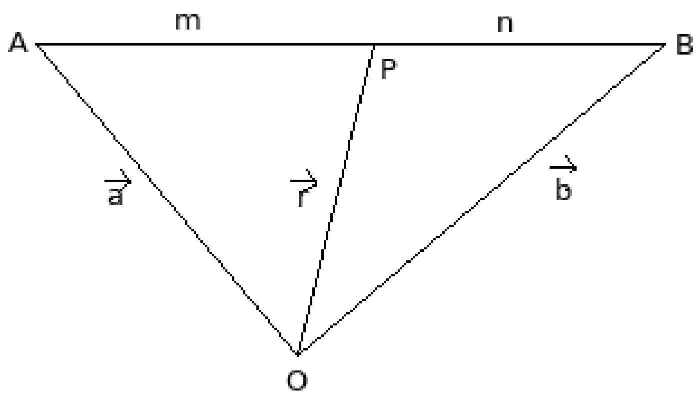
2. $(\underline{a}+\underline{b})+\underline{c}=\underline{a}+(\underline{b}+\underline{c})$ (Associative)

3. $(m+n)\underline{a}=m\underline{a}+n\underline{a}$

4. $m(\underline{a}+\underline{b})=m\underline{a}+m\underline{b}$

Ratio Formula

Let A and B be two points with position vectors a and b, C divides AB in the ratio of m:n.



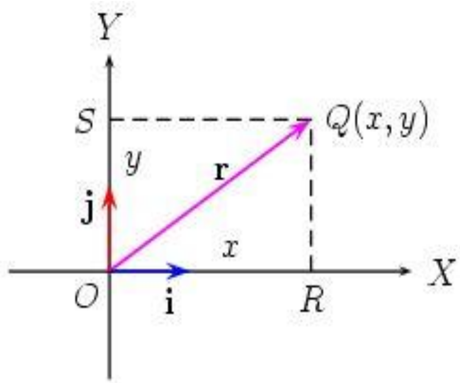
$$\overrightarrow{OP} = \frac{n\bar{a} + m\bar{b}}{n+m}$$

Example 4.1:

Prove that line joining the mid-point of two sides of a triangle is parallel to the third and half of its magnitude.

Vectors in Component Form

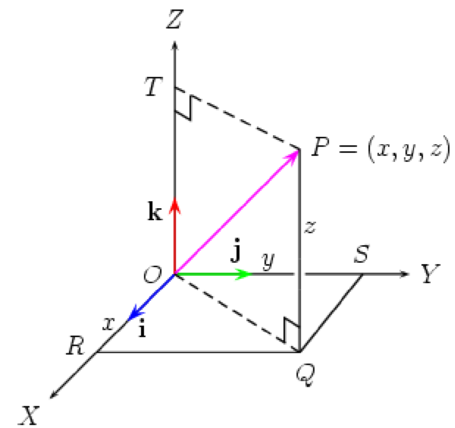
Two-Dimensional Representation



$$\overrightarrow{OQ} = \underline{r} = x\underline{i} + y\underline{j}$$

$$\left|\overrightarrow{OQ}\right| = |\underline{r}| = \sqrt{x^2 + y^2}$$

Three-Dimensional Representation



$$\overrightarrow{OP} = \underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$\left|\overrightarrow{OP}\right| = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$$

Example 4.2:

If the coordinates of the point P be (3,4,12) then find $\overrightarrow{OP}$, its magnitude and direction cosines.

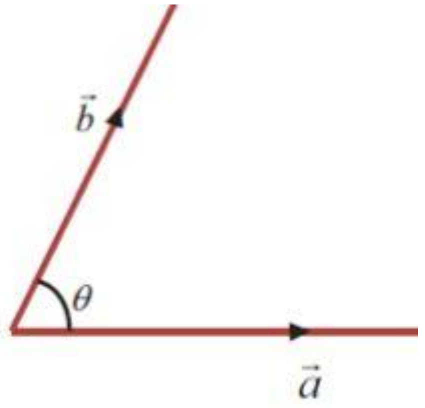
Vector Multiplication

I. Scalar product (Dot product)

Let **a** and **b** be two non-zero vectors and θ be the angle between them with $0 < \theta < \pi$.

Scalar or dot product of **a** and **b** is defined as:

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$$



Algebraically, the dot product is defined as follows:

If $(a_1\underline{i}+a_2\underline{j}+a_3\underline{k})$ and $(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$, then

$$\underline{a}.\underline{b}=(a_1\underline{i}+a_2\underline{j}+a_3\underline{k}).(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$$

$$=a_1b_1+a_2b_2+a_3b_3$$

Example 4.3:

Find the projection of the vector $\underline{i}-2\underline{j}+\underline{k}$ on $4\underline{i}-4\underline{j}+7\underline{k}$

Properties of Scalar Product

1. $\underline{a}.\underline{b} = \underline{b}.\underline{a}$

2. $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$

3. If k is a scalar, then $k(\underline{a}.\underline{b}) = (k\underline{a}).\underline{b} = \underline{a}.(k\underline{b})$

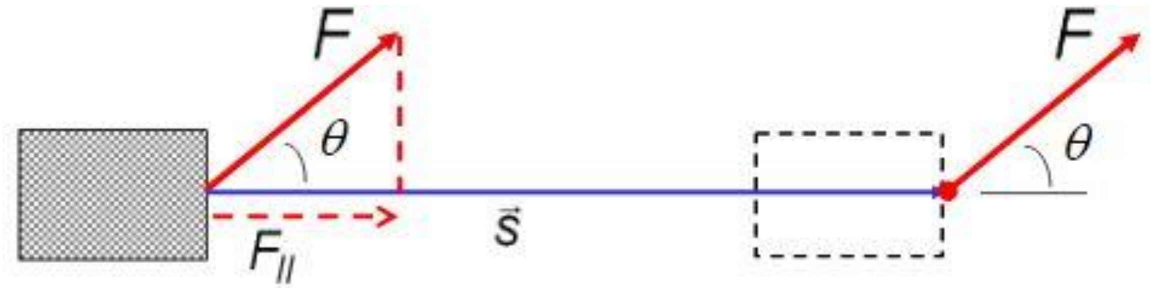
4. $\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$ and $\underline{i}.\underline{j} = \underline{i}.\underline{k} = \underline{j}.\underline{k} = 0$

5. If $\underline{a}$ and $\underline{b}$ are orthogonal, then $\underline{a}.\underline{b} = 0$

Application of Scalar Product

I. Work Done

Let a constant force ***F*** acting on an object during a displacement **s.**



$$W = (\text{Force}).(\text{Displacement})$$

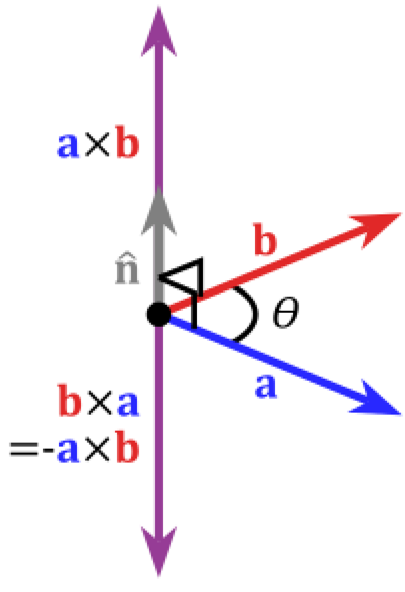
$$= \underline{F}.\underline{s}$$

II. Vector product (Cross product)

Let **a** and **b** be two non-zero vectors and θ be the angle between them with $0 < \theta < \pi$.

Vector or cross product of **a** and **b** is defined as:

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{\eta}$$



Algebraically, the cross product is defined as follows:

If $(a_1\underline{i}+a_2\underline{j}+a_3\underline{k})$ and $(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$, then

$$\underline{a}\times\underline{b}=(a_1\underline{i}+a_2\underline{j}+a_3\underline{k})\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$$

$$=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Properties of vector cross product

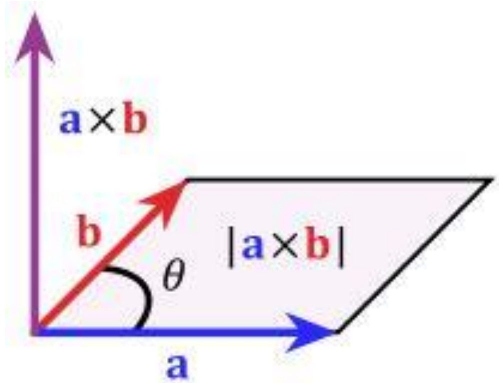
1. $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$
2. $(k\underline{a}) \times \underline{b} = k(\underline{a} \times \underline{b}) = \underline{a} \times (k\underline{b})$
3. $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$
4. $(\underline{a} + \underline{b}) \times \underline{c} = \underline{a} \times \underline{c} + \underline{b} \times \underline{c}$
5. $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$ and $\underline{i} \times \underline{j} = \underline{k}, \underline{j} \times \underline{k} = \underline{i}, \underline{k} \times \underline{i} = \underline{j}$
6. If $\underline{a}$ and $\underline{b}$ are parallel, then $\underline{a} \times \underline{b} = 0$

Applications of Vector Product

I. Area of a parallelogram

The magnitude of the cross product can be interpreted as the positive area of the parallelogram having **a** and **b** as sides.

$$Area = \left| \underline{a} \times \underline{b} \right|$$

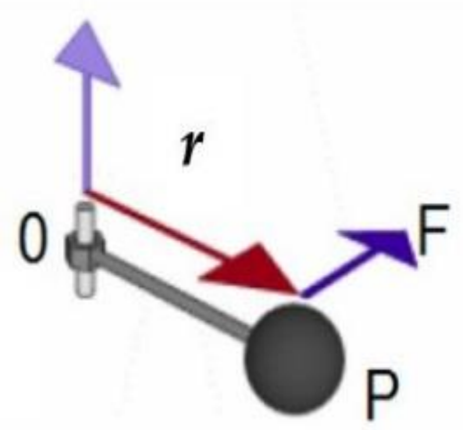


II. Moment of a force about a point

If a force $\boldsymbol{F}$ acting through a point P with position vector **r** with respect to O, then $\boldsymbol{F}$ and $\boldsymbol{r}$ lie in a plane through O.

The **torque** or **moment** of $\boldsymbol{F}$ about an axis through O perpendicular to this plane is given by

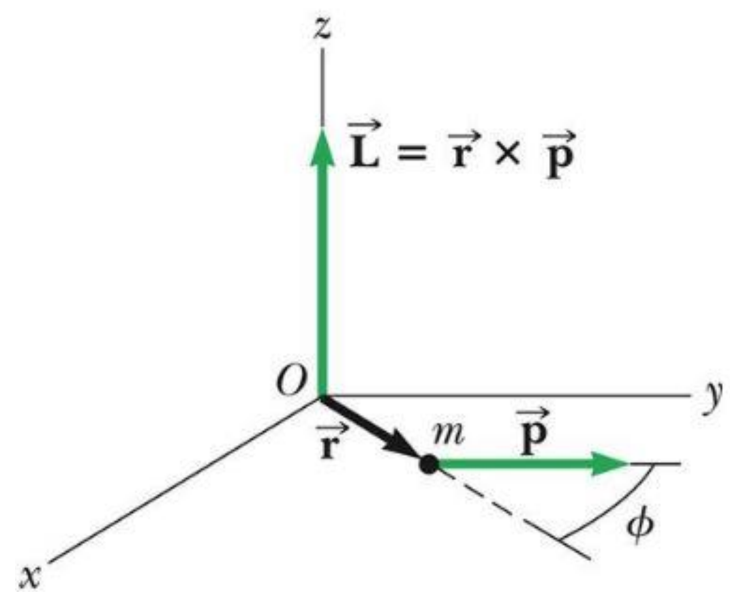
$$\boldsymbol{T} = \boldsymbol{r} \text{ x } \boldsymbol{F} = |\boldsymbol{r}|\,|\boldsymbol{F}|\, \sin\theta\,\eta$$



III. Angular momentum

Let a particle of mass *m* located at the vector position ***r*** and moving with linear momentum ***p***. The **angular momentum** of the particle is defined as;

$$L = r \times p$$



Summary

- A vector is a quantity which has magnitude as well as direction while scalar is a quantity which has only magnitude. Vector is denoted by $\overrightarrow{OP}$
- Unit coordinator vectors *i, j, k* are taken as unit vectors along axis $\overrightarrow{OP} = \underline{r} = x\underline{i} + y\,\underline{j} + z\underline{k}$
- Magnitude of a vector: $\left|\overrightarrow{OP}\right| = \left|\underline{r}\right| = \sqrt{x^2 + y^2 + z^2}$
- Unit vector of $\underline{\boldsymbol{a}}$ (non-zero vector), then $\hat{a} = \dfrac{\underline{a}}{\left|\underline{a}\right|}$.

Summary contd..

- Scalar product of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$: $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

  1. $\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$ while $\underline{i}.\underline{j} = \underline{i}.\underline{k} = \underline{j}.\underline{k} = 0$
  2. If $\underline{a}$ and $\underline{b}$ are perpendicular, then $\underline{a}.\underline{b} = 0$
  3. $\underline{a}.\underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}).(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) = a_1b_1 + a_2b_2 + a_3b_3$

- Vector product of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$: $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|\sin\theta\hat{\eta}$

  1. If $\underline{a}$ and $\underline{b}$ are parallel, then $\underline{a} \times \underline{b} = 0$
  2. $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$ and $\underline{i} \times \underline{j} = \underline{k}, \underline{j} \times \underline{k} = \underline{i}, \underline{k} \times \underline{i} = \underline{j}$
  3. $\underline{a} \times \underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Thank You..!!

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